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Homogenization and Enhancement for the G—Equation

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Abstract

We consider the so-called G-equation, a level set Hamilton–Jacobi equation used as a sharp interface model for flame propagation, perturbed by an oscillatory advection in a spatio-temporal periodic environment. Assuming that the advection has suitably small spatial divergence, we prove that, as the size of the oscillations diminishes, the solutions homogenize (average out) and converge to the solution of an effective anisotropic first-order (spatio-temporal homogeneous) level set equation. Moreover, we obtain a rate of convergence and show that, under certain conditions, the averaging enhances the velocity of the underlying front. We also prove that, at scale one, the level sets of the solutions of the oscillatory problem converge, at long times, to the Wulff shape associated with the effective Hamiltonian. Finally, we also consider advection depending on position at the integral scale.

1. Introduction

We study the limit, as $\varepsilon \to 0$, of the solution to the level-set equation

$$\begin{cases} (i) \quad u_t^\varepsilon = |Du^\varepsilon| + \left\langle V\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right), Du^\varepsilon \right\rangle \quad \text{in} \quad \mathbb{R}^N \times (0, T) \\ (ii) \quad u^\epsilon = u_0 \quad \text{on} \quad \mathbb{R}^N \times \{0\}. \end{cases} \tag{1.1}$$

Equation (1.1)(i) is referred to as the G-equation, and is used as a model for flame propagation in turbulent fluids [23,24]. In that setting, the level sets of the function u^ϵ represent the evolving flame surface and $-V$ is the fluid velocity field.

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At points where u^ϵ is differentiable and $|Du^\epsilon| \neq 0$, the level sets of u^ϵ move with normal velocity

$$\nu = 1 - \left\langle V\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right), \hat{n}\right\rangle ,$$

where $\hat{n} = -Du^\epsilon/|Du^\epsilon|$ is the exterior normal vector of the front. When $V \equiv 0$, level sets move with constant speed $s_L = 1$, which is called the laminar speed of flame propagation.

We assume that the vector field $V \in \mathcal{C}^{0,1}(\mathbb{R}^{N+1}; \mathbb{R}^N)$ is $\mathbb{Z}^{N+1}$-periodic in both x and t, that is, for all $(x,t) \in \mathbb{R}^{N+1}$, $k \in \mathbb{Z}^N$ and $s \in \mathbb{Z}$,

$$V(x+k, t+s) = V(x,t). \tag{1.2}$$

Our first result says that there exists a positively homogeneous of degree one, convex and continuous Hamiltonian $\bar{H}$, such that, as $\epsilon \to 0$, the u^ε's converge locally uniformly in $\mathbb{R}^N \times [0,\infty)$ to the solution $\bar{u}$ of the initial value problem

$$\begin{cases} \bar{u}_t = \bar{H}(D\bar{u}) & \text{in } \mathbb{R}^N \times [0,\infty), \\ \bar{u} = u_0 & \text{on } \mathbb{R}^N \times \{0\}. \end{cases} \tag{1.3}$$

Although V is bounded, we do not assume that $|V| < 1$, and, hence, the Hamiltonian

$$H(x,t,p) = |p| + \langle V(x,t), p\rangle$$

is not coercive in $|p|$ at every point (x,t). This lack of coercivity is the main mathematical challenge in the analysis. If either $|V| < 1$ or the nonlinearity $|Du|$ were replaced with $|Du|^2$, then H would be coercive in $|p|$ and the problem would be within the scope of the theory developed in Ref. [14,19]. There are, however, relatively few homogenization results about noncoercive Hamiltonians [1,3,4,7,11, 17], and none of them deals with the particular structure considered here.

The following simple example shows that, in the absence of coercivity, some additional assumptions about the divergence of V are necessary in order for the u^ϵ's to have a local uniform limit. To this end, let $V = V(x)$ be a smooth $\mathbb{Z}^N$-periodic vector field such that, in the cube $Q_1 = (-\frac{1}{2}, \frac{1}{2})^N$, $V(x) = -10x$, if $|x| < 1/6$, and $V(x) = 0$ if $|x| \geqq 1/3$. It is known that the u^ϵ's have the control representation

$$u^\epsilon(x,t) = \sup(u_0(X_x(t))), \tag{1.4}$$

where the supremum is over all functions $X_x \in W^{1,\infty}([0,t]; \mathbb{R}^N)$ such that $X_x(0) = x$ and $X_x'(s) = \kappa(s) + V(X_x(s))$ with the controls $\kappa(\cdot)$ satisfying $|\kappa| \leqq 1$. If $u_0(x) = \langle p, x\rangle$ with $|p| > 0$, we see easily that $\lim_{\varepsilon\to 0}|u^\epsilon(0,t)| = 0$ for all $t > 0$. However, $\liminf_{\epsilon\to 0} u^\epsilon(x_\epsilon, t) > 0$ if $t > 0$ and $\{x_\epsilon\}_\epsilon$ is any point satisfying $|x_\epsilon| = \epsilon/2$. Roughly speaking, the problem with such a vector field V is that it traps the trajectories which start at the lattice points. If the divergence of V is sufficiently small, however, it is reasonable to expect that the controls are strong enough to overcome such traps.

We assume that V has "small divergence", in the sense that, for all $t \in \mathbb{R}$,

$$\alpha(t) = \frac{1}{c_I} - \|\mathrm{div}_x V(\cdot, t)\|_{L^N(Q_1)} \geqq 0 \quad \text{and} \quad \alpha^* = \int_0^1 \alpha(s)\,\mathrm{d}s > 0, \tag{1.5}$$

where c_I is the isoperimetric constant in the cube Q_1 (see, for instance, [15]), that is, the smallest constant such that, for all measurable subsets E of Q_1,

$$(|E| \wedge |Q_1 \backslash E|)^{(N-1)/N} \leqq c_I \mathrm{Per}(E, Q_1),$$

and also the optimal constant for the Poincaré inequality

$$\|f - \langle f \rangle\|_{L^{1^*}(Q_1)} \leqq c_I \|Df\|_{L^1(Q_1)},$$

for $f \in W^{1,1}(Q_1)$, $1^* = N/(N-1)$ and $\langle f \rangle = \int_{Q_1} f(x)\,\mathrm{d}x$.

To state the main results, we introduce some additional notation. Throughout the paper we use Q_1^+ and $BUC(\bar{U})$ to denote, respectively, the space-time cube $Q_1^+ = Q_1 \times [0, 1] \subset \mathbb{R}^{N+1}$ and the space of bounded uniformly continuous functions on $\bar{U}$, and we write

$$\langle V \rangle = \int_{Q_1^+} V(x, t)\,\mathrm{d}x\,\mathrm{d}t \qquad \text{and} \qquad \langle x \mathrm{div} V \rangle = \int_{Q_1^+} x \mathrm{div} V(x, t)\,\mathrm{d}x\,\mathrm{d}t.$$

We have:

Theorem 1.1. *Assume that $V \in \mathcal{C}^{0,1}(\mathbb{R}^{N+1}; \mathbb{R}^N)$ satisfies (1.2) and (1.5) and that $u_0 \in \mathcal{C}^0(\mathbb{R}^N)$ is bounded. There exists a positively homogeneous of degree one, Lipschitz continuous, convex Hamiltonian $\bar{H} : \mathbb{R}^N \to \mathbb{R}$ such that, if $u^\varepsilon \in \mathcal{C}^0(\mathbb{R}^N \times [0, +\infty))$ and $\bar{u} \in \mathcal{C}^0(\mathbb{R}^N \times [0, +\infty))$ are the solutions to the initial value problems (1.1) and (1.3), respectively, with initial datum u_0, then, as $\varepsilon \to 0$, the u^ε's converge locally uniformly in $\mathbb{R}^N \times [0, T]$ to $\bar{u}$. Moreover, for all $P \in \mathbb{R}^N$,*

$$\bar{H}(P) \geqq |P| \int_0^1 \left(1 - c_I \|\mathrm{div} V(\cdot, t)\|_{L^N(Q_I)}\right) \mathrm{d}t + \langle\, \langle V \rangle + \langle x \mathrm{div} V \rangle, P \rangle. \tag{1.6}$$

Finally, the convex map $P \to \bar{H}(P) - \langle\, \langle V \rangle + \langle x \mathrm{div} V \rangle, P\, \rangle$ is coercive.

For Lipschitz continuous initial datum u_0, we can actually estimate the convergence rate as $\epsilon \to 0$. We have:

Theorem 1.2. *Assume that $u_0 \in C^{0,1}(\mathbb{R}^N)$ and let $u^\epsilon, \bar{u} \in BUC(\mathbb{R}^N \times [0, T])$, for all $T > 0$, be, respectively, the solutions to (1.1) and (1.3). Then, for all $T > 0$, there exists a positive constant C that depends only on T, N, V and the Lipschitz constant of u_0, such that, for all $(x, t) \in \mathbb{R}^N \times [0, T]$,*

$$|u(x, t) - u^\epsilon(x, t)| \leqq C\epsilon^{1/3}.$$

In the case that, for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$,

$$\operatorname{div}_x V(x,t) = 0, \tag{1.7}$$

we derive some additional properties of the function $\bar{H}$. To simplify the statement we also assume, without any loss of generality (see Lemma 3.1 below), that

$$\int_{Q_1} V(x,t)\,\mathrm{d}x = 0 \qquad \forall t \in \mathbb{R}. \tag{1.8}$$

Then, according to Theorem 1.1, the averaged Hamiltonian $\bar{H} : \mathbb{R}^N \to \mathbb{R}$ satisfies, for all $P \in \mathbb{R}^N$,

$$\bar{H}(P) \geqq |P|.$$

We establish here a necessary and sufficient condition to have the strict inequality

$$\bar{H}(P) > |P|,$$

in which case we have enhancement of the speed due to averaging.

Recall that, since $\bar{H}$ is homogeneous of degree one, the level sets of $\bar{u}$ move with speed $\bar{H}(\hat{n})$ in the direction of the normal vector $\hat{n} = -D\bar{u}/|D\bar{u}|$. Therefore, we refer to the situation $\bar{H}(P) > |P|$ as "enhancement", because it implies that such velocity fields lead to faster propagation of interfaces compared to the case $V \equiv 0$.

First, we state the result in the case where V depends only on x. We have:

Theorem 1.3. *Assume that $V \in C^{0,1}(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic,* $\operatorname{div} V = 0$ *and* $\langle V \rangle = 0$*, and let $P \in \mathbb{R}^N \backslash \{0\}$. Then $\bar{H}(P) = |P|$ if and only if, for all $x \in \mathbb{R}^N$, $\langle V(x), P\rangle = 0$. In particular, if $N = 2$, then $\bar{H}(P) = |P|$ if and only if the stream function E associated to V is of the form $E = \tilde{E}(\langle P, \cdot\rangle)$ for some $\tilde{E} : \mathbb{R} \to \mathbb{R}$, that is, V is a shear drift in the direction orthogonal to P.*

When V is also time dependent, the characterization of equality $\bar{H}(P) = |P|$ is provided by

Theorem 1.4. *Assume that, for all $t \in \mathbb{R}$,* $\operatorname{div}_x V(\cdot,t) = 0$ *and* $\int_{Q_1} V(x,t)\,\mathrm{d}x = 0$*, and fix $P \in \mathbb{R}^N \backslash \{0\}$. Then $\bar{H}(P) = |P|$ if and only if there exists $\hat{z} \in BV_{loc}(\mathbb{R})$ such that $\hat{z}' \geqq -|P|$ in the sense of distribution and the function $z(x,t) = \hat{z}(\frac{\langle P,x\rangle}{|P|} + t)$ is $\mathbb{Z}^{N+1}$-periodic and satisfies, for all $t \in \mathbb{R}$, in the sense of distributions*

$$\operatorname{div}\left((z(\cdot,t) + \langle P, \cdot\rangle)V(\cdot,t)\right) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.9}$$

We continue with some observations about these results. Theorem 1.3 yields that, if $N = 2$, $\bar{H}(P) = |P|$ and V is not constant, then $P = (P_1, P_2)$ must be a rational direction, that is, either $P_2 = 0$ or $P_1/P_2 \in \mathbb{Q}$, since V is $\mathbb{Z}^2$-periodic. For Theorem 1.4, observe that, if z is not constant, then $P/|P|$ must be a rational vector.

Also, (1.9) is equivalent to saying (see Lemma 3.2 below) that, for any fixed $t > 0$, the map $x \to z(x,t) + \langle P, x\rangle$ is constant along the flow of the differential equation $X'(s) = V(X(s), t)$.

We remark that it is possible to construct nontrivial examples of time-dependent flows for which $\bar{H}(P) = |P|$. Indeed when $N = 2$ for any smooth, $\mathbb{Z}^1$-periodic (E_1, E_2) such that $E_1(0) = 0$, let $E(x_1, x_2, t) = E_1(x_1 + t)E_2(x_2)$, $V = \nabla^{\perp} E$ and $P = (1, 0)$. Then $\bar{H}(P) = |P|$ because the map $\hat{z}(s) = [s] - s$, where $[s]$ stands for the integer part of s, satisfies the condition of Theorem 1.4. For more analysis and numerical computation of $\bar{H}$ for specific flow structures, we refer to [12, 13, 20–22].

The next result of the paper is about the long time behavior of the solution to (1.1) with $\varepsilon = 1$ and, in particular, the convergence, as $t \to \infty$, of its zero level set to the Wulff-shape associated with the effective $\bar{H}$, which is given by

$$\mathcal{W} = \{y \in \mathbb{R}^N : \langle P, y\rangle + \bar{H}(P) \geqq 0 \quad \text{for all} \quad P \in \mathbb{R}^N\}. \tag{1.10}$$

We consider the initial value problem

$$\begin{cases} u_t = |Du| + \langle V(x,t), Du\rangle & \text{in} \quad \mathbb{R}^N \times (0, \infty), \\ u = u_0 \;\; \text{on} \;\; \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.11}$$

and set, for all $t \geqq 0$,

$$K(t) = \{x \in \mathbb{R}^N : u(x,t) \geqq 0\}.$$

Recall that, in the language of front propagation (see, for example, [6]), the family of closed sets $(K(t))_{t \geqq 0}$ is a solution of the front propagation problem

$$\nu = 1 - \langle V(x,t), \hat{n}\rangle$$

starting from $K(0) = K_0$.

We have:

Theorem 1.5. *Let K_0 be a non-empty compact subset of $\mathbb{R}^N$. There exist $C > 0$ and $T > 0$ such that, for all $t \geqq T$,*

$$K(t) \subset (t + C)\mathcal{W}. \tag{1.12}$$

Moreover, there exists a constant $C_0 > 0$, independent of K_0, such that, if K_0 contains a cube of side length C_0, then there exist $C > 0$ and $T > 0$ such that, for all $t \geqq T$,

$$(t - Ct^{2/3})\mathcal{W} \subset K(t). \tag{1.13}$$

We note that we do not know whether the size condition on K_0 is actually necessary.

The final result of the paper is about homogenization when V also depends on x at the integral scale, that is, we are interested in the behavior as $\varepsilon \to 0$ of the solutions to the initial value problem

$$\begin{cases} u^{\varepsilon}_t = |Du^{\varepsilon}| + \langle V(x, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}), Du^{\varepsilon}\rangle & \text{in} \quad \mathbb{R}^N \times (0, T) \\ u^{\varepsilon} = u_0 \quad \text{on} \quad \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.14}$$

where $V : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is smooth, bounded, and $\mathbb{Z}^N$-periodic with respect to the last two variables. That is, for all $(x, y, s) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$,

$$V(x, y + k, s + h) = V(x, y, s), \tag{1.15}$$

is divergence free in the fast variable, that is, for all $(x, y, s) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$,

$$\operatorname{div}_y V(x, y, s) = 0, \tag{1.16}$$

and satisfies, for all $x \in \mathbb{R}^N$, the "smallness" condition

$$\left| \int_0^1 \int_{Q_1} V(x, y, s)\, \mathrm{d}y\, \mathrm{d}s \right| < 1. \tag{1.17}$$

The homogenized initial value problem is

$$\begin{cases} \bar{u}_t = \bar{H}(x, D\bar{u}) & \text{in} \quad \mathbb{R}^N \times (0, T), \\ \bar{u} = u_0 \quad \text{on} \quad \mathbb{R}^N. \end{cases} \tag{1.18}$$

We have:

Theorem 1.6. *Assume* (1.15), (1.16) *and* (1.18). *There exists* $\bar{H} \in C^0(\mathbb{R}^N \times \mathbb{R}^N)$, *which is positively homogeneous of degree one and convex with respect to the second variable, such that, for any initial condition* $u_0 \in BUC(\mathbb{R}^N)$, *the solution* u^ε *to* (1.14) *converges, as* $\varepsilon \to 0$, *locally uniformly in* $\mathbb{R}^N \times [0, T]$ *to the solution* $\bar{u}$ *of* (1.18). *Moreover* $\bar{H}$ *satisfies, for all* $(x, P) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$\bar{H}(x, P) \geqq |P| + \langle \int_0^1 \int_{Q_1} V(x, y, s)\, \mathrm{d}y\, \mathrm{d}s, P \rangle. \tag{1.19}$$

The paper is organized as follows. Theorem 1.1 and Theorem 1.2 are proved in Section 2. In Section 3 we prove Theorem 1.3 and Theorem 1.4, while Theorem1.5 is proved in Section 4. In Section 5, we prove an extension of Theorem 1.1 to the case where $V = V(x, x/\varepsilon, t/\varepsilon)$ has large-scale spatial variation. The Appendix contains a proof of Lemma 2.3, which plays an important role in the proof of Theorem 1.1.

About the time this paper was completed, we learned about a similar but less general homogenization result obtained by different methods in Ref. [25]. In particular, it is proved in Ref. [25] that homogenization takes place for time-independent advection satisfying $V = V_1 + V_2$ with $\operatorname{div} V_1 = 0$ and $|V_2| < 1$.

Finally, we remark that throughout the paper we will need some basic results from the theory of viscosity solutions, such as comparison principles, representation formulae, etc. All such results can be found, for instance, in Ref. [5].

2. Homogenization

We begin with some preliminary discussion and results to set the necessary background for the proofs of Theorem 1.1 and Theorem 1.2. First, we recall that for any $\lambda > 0$ and any $P \in \mathbb{R}^N$, the "penalized" cell problem

$$v_{\lambda,t} + \lambda v_\lambda = |Dv_\lambda + P| + \langle V, Dv_\lambda + P\rangle \quad \text{in} \quad \mathbb{R}^{N+1}, \tag{2.1}$$

has a unique $\mathbb{Z}^{N+1}$-periodic solution $v_\lambda \in BUC(\mathbb{R}^{N+1})$, which is actually Hölder continuous and satisfies, for all $(x, t) \in \mathbb{R}^{N+1}$, the bound

$$-\lambda^{-1}|P|(1 + \|V\|_\infty) \leqq v_\lambda \leqq \lambda^{-1}|P|(1 + \|V\|_\infty). \tag{2.2}$$

We also recall that, in the periodic setting, homogenization is equivalent to proving that the (λv_λ)'s converge uniformly in $\mathbb{R}^{N+1}$, as $\lambda \to 0$, to some constant $\bar{c}(P)$ and that $\bar{c} \in \mathcal{C}^0(\mathbb{R}^N)$. In this case, $\bar{H}(P) = \bar{c}(P)$.

In view of (2.2), to prove the convergence of the λv_λ's, we need to control their oscillations. We have:

Lemma 2.1. *For all $P \in \mathbb{R}^N$ and $\lambda \in (0, 1]$,*

$$\text{osc}(\lambda v_\lambda) \leqq C|P|\lambda, \tag{2.3}$$

where

$$C = 4(1 + \|V\|_\infty)(2^{1/N}{\alpha^*}^{-1}N + 3).$$

Before we present the proof of Lemma 2.1, which is one of the most important parts of the paper, we point out its main consequence:

Corollary 2.2. *Let C be the constant given by Lemma 2.1. There exists some $\bar{H}(P) \in \mathbb{R}$ such that*

$$\|\lambda v_\lambda - \bar{H}(P)\|_\infty \leqq C|P|\lambda.$$

Proof. The maps $\lambda \to \lambda \min v_\lambda$ and $\lambda \to \lambda \max v_\lambda$ are, respectively, nonincreasing and nondecreasing. For the sake of completeness we present a formal proof, which can be easily made rigorous using viscosity solutions arguments. Since the two claims are proved similarly, we present details only for the former.

To this end, for $0 < \lambda < \mu$, let (x, t) be a maximum of $v_\mu - v_\lambda$. Then, at least formally, at (x, t), we have $D_{x,t}v_\mu = D_{x,t}v_\lambda$, and

$$v_{\mu,t} + \mu v_\mu \leqq |Dv_\mu + P| + \langle V, Dv_\mu + P\rangle$$

and

$$v_{\lambda,t} + \lambda v_\lambda \geqq |Dv_\lambda + P| + \langle V, Dv_\lambda + P\rangle.$$

It follows that

$$\mu v_\mu(x, t) \leqq \lambda v_\lambda(x, t).$$

Let (y, s) be a minimum point of λv_λ. Then

$$\begin{aligned}\mu v_\mu(y,s) &\leqq \mu(v_\mu(x,t) - v_\lambda(x,t) + v_\lambda(y,s)) \\ &\leqq \lambda v_\lambda(x,t) - \mu v_\lambda(x,t) + \mu v_\lambda(y,s) \\ &\leqq \lambda v_\lambda(x,t) + \lambda(v_\lambda(y,s) - v_\lambda(x,t)) \leqq \lambda v_\lambda(y,s),\end{aligned}$$

and, hence,

$$\min \mu v_\mu \leqq \min \lambda v_\lambda.$$

The above remark combined with Lemma 2.1 implies that the λv_λ's converge uniformly to some constant $\bar{H}(P)$ and that

$$\lambda \min v_\lambda \leqq \bar{H}(P) \leqq \lambda \max v_\lambda.$$

□

We continue with the following lemma:

Proof of Lemma 2.1. Without any loss of generality, we may assume that V is smooth. Indeed, if the result holds for any smooth V, then it also holds by approximation for any $V \in C^{0,1}$.

Recalling (2.2), $w_\lambda = \lambda v_\lambda$ satisfies, in the viscosity sense,

$$w_{\lambda,t} - |Dw_\lambda(x,t)| - \langle V(x,t), Dw_\lambda(x,t)\rangle \geqq -C_0\lambda \qquad \text{in } \mathbb{R}^{N+1}, \tag{2.4}$$

where

$$C_0 = 2(1 + \|V\|_\infty)|P|.$$

It follows that $(x,t) \to w_\lambda(x,t) + C_0\lambda t$ is a viscosity super-solution of the level-set initial value problem

$$\begin{cases} (i) \quad z_t = |Dz| + \langle V, Dz\rangle \quad \text{in} \quad \mathbb{R}^N \times (0,\infty), \\ (ii) \quad z = w_\lambda \quad \text{on} \quad \mathbb{R}^N \times \{0\}. \end{cases} \tag{2.5}$$

The standard comparison of viscosity solutions then implies that, for all $(x,t) \in \mathbb{R}^{N+1}$,

$$w_\lambda(x,t) + C_0\lambda t \geqq z(x,t). \tag{2.6}$$

The next step is to understand the evolution of the perimeter of the level-sets of z. For this we need the following result, which is proved in the Appendix.

We have:

Lemma 2.3. *Assume that $V \in C^{1,1}(\mathbb{R}^{N+1})$ and let $z \in BUC(\mathbb{R}^{N+1})$ be a solution of (2.5)(i). Then, for any level $\theta \in (\,\inf z(\cdot,0), \sup z(\cdot,0)\,)$ such that*

$$\{z(\cdot,0)=\theta\}=\partial\{z(\cdot,0)>\theta\}=\partial\{z(\cdot,0)<\theta\} \quad \text{and} \quad |\{z(\cdot,0)=\theta\}|=0,$$

and for all $t>0$, we have

$$\{z(\cdot,t)=\theta\}=\partial\{z(\cdot,t)>\theta\}=\partial\{z(\cdot,t)<\theta\}$$

as long as $\{z(\cdot,t)<\theta\}\neq\emptyset$. Moreover the sets $\{z(\cdot,t)>\theta\}$ and $\{z(\cdot,t)<\theta\}$ are locally of finite perimeter, and $|\{z(\cdot,t)=\theta\}|=0$. Finally, for any compactly supported $\varphi\in\mathcal{C}^0(\mathbb{R}^N)$, the maps $t\to I(t)=\int_{\{z(\cdot,t)>\theta\}}\varphi(x)\,\mathrm{d}x$ and $t\to J(t)=\int_{\{z(\cdot,t)<\theta\}}\varphi(x)\,\mathrm{d}x$ are absolutely continuous and satisfy, for almost all $t>0$,

$$\frac{d}{\mathrm{d}t}I(t)=\int_{\{z(\cdot,t)=\theta\}}\varphi(x)(1-\langle V(x,t),\nu(x,t)\rangle)d\mathcal{H}^{N-1}(x),$$

and

$$\frac{d}{\mathrm{d}t}J(t)=-\int_{\{z(\cdot,t)=\theta\}}\varphi(x)(1+\langle V(x,t),\nu(x,t)\rangle)d\mathcal{H}^{N-1}(x),$$

where $\nu(x,t)$ denotes in the former identity the measure theoretic outward unit normal to $\{z(\cdot,t)>\theta\}$ at $x\in\partial\{z(\cdot,t)>\theta\}$, while in the latter is the measure theoretic outward unit normal to $\{z(\cdot,t)<\theta\}$ at $x\in\partial\{z(\cdot,t)<\theta\}$.

Continuing with the ongoing proof, suppose that there exists $\theta\in\mathbb{R}$ with

$$\inf w_\lambda(\cdot,0)<\theta<\sup w_\lambda(\cdot,0),$$

such that

$$\{w_\lambda(\cdot,0)=\theta\}=\partial\{w_\lambda(\cdot,0)>\theta\}=\partial\{w_\lambda(\cdot,0)<\theta\} \quad \text{and} \quad |\{w_\lambda(\cdot,0)=\theta\}|=0,$$

and such that

$$|\{x\in Q_1 : w_\lambda(x,0)<\theta\}|<1/2.$$

For all $t\geqq 0$, set

$$\rho(t)=|\{x\in Q_1 : z(x,t)<\theta\}|\,.$$

Let $[0,T)$ be the maximal interval on which $\rho(t)<1/2$ for any $t\in[0,T)$. Note that $T>0$ because $\rho(0)<1/2$. We claim that, for all $0\leqq t_1\leqq t_2<T$,

$$\rho(t_2)-\rho(t_1)\leqq-\int_{t_1}^{t_2}\alpha(s)\rho(s)^{(N-1)/N}\,\mathrm{d}s. \tag{2.7}$$

Indeed, fix a positive integer R and let $Q_R=\left(-\frac{R}{2},\frac{R}{2}\right)^N$. The space periodicity of z gives

$$\rho(t)=R^{-N}\,|\{x\in Q_R : z(x,t)<\theta\}|\,.$$

For $h > 0$ small, let $\chi_h \in C^{0,1}(\mathbb{R}^N; [0,1])$ be such that

$$\chi_h = 1 \text{ in } Q_R \qquad \text{and} \qquad \chi_h = 0 \text{ in } \mathbb{R}^N \backslash Q_{R+h},$$

and, for any $t \in [0,T]$, set

$$\rho_{R,h}(t) = R^{-N} \int_{\{z(\cdot,t)<\theta\}} \chi_h(x)\,\mathrm{d}x,$$

and note that,

$$\lim_{h\to 0} \rho_{R,h}(t) = \rho(t).$$

It follows from Lemma 2.3 that, for almost all $t \in (0,T)$,

$$\frac{d}{\mathrm{d}t}\rho_{R,h}(t) = -R^{-N} \int_{\{z(\cdot,t)=\theta\}} \chi_h(x)(1 + \langle V(x,t), \nu(x,t)\rangle) d\mathcal{H}^{N-1}(x).$$

Moreover

$$\int_{\{z(\cdot,t)=\theta\}} \chi_h(x) d\mathcal{H}^{N-1}(x) \geqq \mathcal{H}^{N-1}(\{z(\cdot,t)=\theta\} \cap Q_R),$$

and, in view of the spatial periodicity of z,

$$\mathcal{H}^{N-1}(\{z(\cdot,t)=\theta\} \cap Q_R) \geqq R^N \mathcal{H}^{N-1}(\{z(\cdot,t)=\theta\} \cap Q_1).$$

The isoperimetric inequality in the box Q_1 and the fact that $|\{z(\cdot,t)<\theta\} \cap Q_1| < 1/2$ give

$$\mathcal{H}^{N-1}(\{z(\cdot,t)=\theta\} \cap Q_1) \geqq \frac{1}{c_I} |\{z(\cdot,t)<\theta\} \cap Q_1|^{(N-1)/N}.$$

Using once more the space periodicity of z we get

$$\begin{aligned} |\{z(\cdot,t)<\theta\} \cap Q_1|^{(N-1)/N} &= (R+1)^{-(N-1)} |\{z(\cdot,t)<\theta\} \cap Q_{R+1}|^{(N-1)/N} \\ &\geqq (R(R+1)^{-1})^{N-1} \left(\rho_{R,h}(t)\right)^{(N-1)/N}. \end{aligned}$$

Combining all the above we obtain

$$R^{-N} \int_{\{z(\cdot,t)=\theta\}} \chi_h(x) d\mathcal{H}^{N-1}(x) \geqq \frac{1}{c_I}(R(R+1)^{-1})^{N-1} \left(\rho_{R,h}(t)\right)^{(N-1)/N}.$$

Next we estimate the integral

$$\int_{\{z(\cdot,t)=\theta\}} \chi_h(x) \langle V(x,t), \nu(x,t)\rangle d\mathcal{H}^{N-1}(x).$$

For some constant k depending only on N we have:

$$
\begin{aligned}
&-\int_{\{z(\cdot,t)=\theta\}} \chi_h(x)\langle V(x,t), \nu(x,t)\rangle d\mathcal{H}^{N-1}(x) \\
&\quad = -\int_{\{z(\cdot,t)<\theta\}} \operatorname{div}(\chi_h V)(x,t)\,\mathrm{d}x \\
&\quad \leqq -\int_{\{z(\cdot,t)<\theta\}} \chi_h(x)\operatorname{div} V(x,t)\,\mathrm{d}x + \|D\chi_h\|_\infty \|V\|_\infty\, |Q_{R+h}\setminus Q_R| \\
&\quad \leqq \left(\int_{Q_{R+1}} |\operatorname{div} V(x,t)|^N \ \mathrm{d}x\right)^{1/N} \left(\int_{\{z(\cdot,t)<\theta\}} \chi_h(x)^{N/(N-1)}\,\mathrm{d}x\right)^{(N-1)/N} \\
&\qquad + kR^{N-1}\|V\|_\infty \\
&\quad \leqq (R+1)R^{N-1}\|\operatorname{div} V(\cdot,t)\|_{L^N(Q_1)} \left(\rho_{R,h}(t)\right)^{(N-1)/N} + kR^{N-1}\|V\|_\infty.
\end{aligned}
$$

Hence, for almost all $t \in (0,T)$, we get

$$
\begin{aligned}
\frac{d}{\mathrm{d}t}\rho_{R,h}(t) \leqq &-\left(\frac{1}{c_I}\left(\frac{R}{R+1}\right)^{N-1} - \frac{R+1}{R}\|\operatorname{div} V(\cdot,t)\|_{L^N(Q_1)}\right)(\rho_{R,h}(t))^{(N-1)/N} \\
&+\frac{k}{R}\|V\|_\infty.
\end{aligned}
$$

Integrating first over $[t_1, t_2]$ and then letting $h \to 0$ and $R \to +\infty$ gives (2.7).

Since $\alpha(t) \geqq 0$, $\rho(t)$ is non-increasing on $[0,T)$. Hence $T = +\infty$. Integrating (2.7) over $(0,t)$ we obtain, for every $t \geqq 0$, that

$$
\rho(t) \leqq \left[\rho^{1/N}(0) - \frac{1}{N}\int_0^t \alpha(s)\,\mathrm{d}s\right]_+^N,
$$

where $[s]_+ = \max\{s, 0\}$.

From the assumption $\rho(0) < 1/2$, it follows that

$$
t^* = 1 + \frac{N}{2^{1/N}\alpha^*} \geqq \inf\left\{t : \int_0^t \alpha(s)\,\mathrm{d}s \geqq \frac{N}{2^{1/N}}\right\},
$$

and, hence, $\rho = 0$ in $[t^*, \infty)$. The continuity and the spatial periodicity of z then yield that

$$
z \geqq \theta \quad \text{in } \mathbb{R}^N \times [t^*, \infty).
$$

Let k be an integer in the interval $[t^*, t^*+1]$. The space-time periodicity of w_λ and (2.6) give

$$
\begin{aligned}
\inf_{t\in[0,1]} \inf_{x\in\mathbb{R}^N} w_\lambda(x,t) &= \inf_{t\in[k,k+1]} \inf_{x\in\mathbb{R}^N} w_\lambda(x,t) \\
&\geqq \inf_{t\in[k,k+1]} \inf_{x\in\mathbb{R}^N} z(x,t) - C_0\lambda(t^*+2) \\
&\geqq \theta - C_0\lambda(t^*+2). \qquad (2.8)
\end{aligned}
$$

It follows that, if $\theta \in \mathbb{R}$ is such that

$$|\{x \in Q_1 : w_\lambda(x,0) < \theta\}| < 1/2, \tag{2.9}$$

then

$$\inf_{t\in[0,1]} \inf_{x\in\mathbb{R}^N} w_\lambda(x,t) \geqq \theta - C\lambda \tag{2.10}$$

where

$$C = C_0(t^* + 2).$$

Now we derive an upper bound. To this end, suppose that $\theta \in \mathbb{R}$ with

$$\inf w_\lambda(\cdot,0) < \theta < \sup w_\lambda(\cdot,0),$$

such that

$$\{w_\lambda(\cdot,0)=\theta\} = \partial\{w_\lambda(\cdot,0) > \theta\} = \partial\{w_\lambda(\cdot,0) < \theta\} \quad \text{and} \quad |\{w_\lambda(\cdot,0)=\theta\}| = 0,$$

and such that

$$|\{x \in Q_1 : w_\lambda(x,0) > \theta\}| < 1/2. \tag{2.11}$$

The claim is that

$$\max_{x,t} w_\lambda(x,t) \leqq \theta + C\lambda, \tag{2.12}$$

where $C = 2|P|(1 + \|V\|_\infty)(2 + N2^{-1/N}(\alpha^*)^{-1})$. Indeed, arguing by contradiction, we assume that

$$\max_{x,t} w_\lambda(x,t) > \theta + C\lambda.$$

Then, by continuity and periodicity of w_λ, there is some $\tau \in [0,1]$ such that

$$|\{x \in Q_1 : w_\lambda(x,\tau) > \theta + C\lambda\}| > 0.$$

Let z satisfy (2.5-(i)) with initial condition $z = w_\lambda$ on $\mathbb{R}^N \times \{\tau\}$. As before, we have, for all $(x,t) \in \mathbb{R}^N \times [\tau, +\infty)$,

$$w_\lambda(x,t) \geqq z(x,t) - C_0\lambda(t-\tau).$$

Set

$$\rho(t) = |\{x \in Q_1 : z(x,t) > \theta + C\lambda\}| \qquad t \geqq \tau\ , \tag{2.13}$$

and observe that ρ is continuous with $\rho(\tau) > 0$. Then, arguing as before, we find that, for all $\tau \leqq t_1 \leqq t_2$,

$$\rho(t_2) - \rho(t_1) \geqq \int_{t_1}^{t_2} \alpha(s)\,(\min\{\rho(s), 1-\rho(s)\})^{(N-1)/N}\,\mathrm{d}s. \tag{2.14}$$

Since $\alpha(t) \geqq 0$ for all $t \geqq 0$, it follows that ρ is nondecreasing on $[\tau, +\infty)$. We claim that there is some $T \leqq t^* = \tau + 1 + N/(2^{1/N}\alpha^*)$ such that $\rho \geqq 1/2$ for $t \geqq T$. Indeed, otherwise, $\rho < 1/2$ on $[\tau, t^*]$ and integrating (2.14) over $[\tau, t^*]$ gives

$$\rho(t^*) \geqq \left(\frac{1}{N}\int_{\tau}^{t^*} \alpha(s)\,\mathrm{d}s\right)^N > \frac{1}{2},$$

which contradicts our assumption. Let now k be an integer in $[T, T+1]$. The time-periodicity of w_λ yields

$$\begin{aligned} 1/2 \leqq \rho(k) &= |\{x \in Q_1 : z(x,k) > \theta + C\lambda\}| \\ &\leqq |\{x \in Q_1 : w_\lambda(x,k) + C_0\lambda(k-\tau) > \theta + C\lambda\}| \\ &= |\{x \in Q_1 : w_\lambda(x,0) + C_0\lambda(k-\tau) > \theta + C\lambda\}|. \end{aligned}$$

It follows from $k \leqq (t^* + 1)$ that

$$C_0(k-\tau) \leqq C \quad \text{and } |\{x \in Q_1 : w_\lambda(x,0) > \theta\}| \geqq 1/2,$$

which contradicts the definition of θ, and, hence, (2.12) holds.

Finally, set

$$\bar{\theta} = \sup\left\{\theta \in \mathbb{R} : |\{x \in Q_1 \ : \ w_\lambda(x,0) < \theta\}| < \frac{1}{2}\right\}. \tag{2.15}$$

In view of the above, using (2.10) and (2.12), we get

$$\min_{t\in[0,1]}\ \min_{x\in\mathbb{R}^N} w_\lambda(x,t) \geqq \bar{\theta} - C\lambda \tag{2.16}$$

and

$$\max_{t\in[0,1]}\ \max_{x\in\mathbb{R}^N} w_\lambda(x,t) \leqq \bar{\theta} + C\lambda, \tag{2.17}$$

where $C = 2|P|(1 + \|V\|_\infty)(3 + N2^{-1/N}/\alpha^*)$. It follows that $\mathrm{osc}(w_\lambda) \leqq 2C\lambda$, and, therefore, (2.3). □

We proceed with the following proof.

Proof of Theorem 1.1. Let $\bar{H}(P)$ be defined by Corollary 2.2. The fact that the map $P \to \bar{H}(P)$ is positively homogeneous, convex and Lipschitz continuous follows easily from the properties of (2.1) and the comparison principle of viscosity solutions.

To prove (1.6), first we perturb (2.1) by a vanishing viscosity, that is, for $\eta > 0$ we consider

$$v^\eta_{\lambda,t} + \lambda v^\eta_\lambda - \eta\Delta v^\eta_\lambda = |Dv^\eta_\lambda + P| + \langle V, Dv^\eta_\lambda + P\rangle \qquad \text{in } \mathbb{R}^{N+1}, \tag{2.18}$$

which has a unique $\mathbb{Z}^{N+1}$-periodic solution $v^\eta_\lambda \in BUC(\mathbb{R}^{N+1})$ which is at least in $C^1(\mathbb{R}^{N+1})$ and converges uniformly, as $\eta \to 0$, to v_λ.

Integrating (2.18) over Q_1^+ and using the periodicity, we find

$$\int_{Q_1^+} \lambda v_\lambda^\eta \, dx \, dt = \int_{Q_1^+} |Dv_\lambda^\eta + P| \, dx \, dt + \int_{Q_1^+} \langle V, Dv_\lambda^\eta \rangle \, dx \, dt + \langle \langle V \rangle, P \rangle.$$

Set

$$\xi(x) = \langle P, x \rangle, \quad \langle v_\lambda^\eta(t) \rangle = \int_{Q_1} v_\lambda^\eta(x,t) \, dx \quad \text{and} \quad \langle \xi \rangle = \int_{Q_1} \xi(x) \, dx.$$

Since both V and v_λ are $\mathbb{Z}^{N+1}$-periodic, for each $t \in [0,1]$, we have

$$\begin{aligned}
&\int_{Q_1} \langle V(x,t), Dv_\lambda^\eta(x,t) \rangle \, dx \\
&\quad = -\int_{Q_1} (v_\lambda^\eta(x,t) - \langle v_\lambda^\eta(t) \rangle) \operatorname{div} V(x,t) \, dx \\
&\quad = -\int_{Q_1} (v_\lambda^\eta(x,t) - \langle v_\lambda^\eta(t) \rangle + \xi(x) - \langle \xi \rangle) \operatorname{div} V(x,t) \, dx \\
&\qquad + \int_{Q_1} (\xi(x) - \langle \xi \rangle) \operatorname{div} V(x,t) \, dx.
\end{aligned}$$

Applying, for each $t \in [0,1]$, Hölder's and Poincaré's inequalities yields

$$\begin{aligned}
&\int_{Q_1} (v_\lambda^\eta(x,t) - \langle v_\lambda^\eta \rangle_t + \xi(x) - \langle \xi \rangle) \operatorname{div} V(x,t) \, dx \\
&\leqq \left(\int_{Q_1} |v_\lambda^\eta(x,t) - \langle v_\lambda^\eta \rangle_t + \xi(x) - \langle \xi \rangle|^{N/(N-1)} \, dx \right)^{(N-1)/N} \\
&\quad \times \left(\int_{Q_1} |\operatorname{div} V(x,t)|^N \, dx \right)^{1/N} \\
&\leqq c_I \left(\int_{Q_1} |Dv_\lambda^\eta(x,t) + P| \, dx \right) \|\operatorname{div} V(\cdot,t)\|_{L^N(Q_I)},
\end{aligned}$$

and, since

$$\int_{Q_1^+} (\xi(x) - \langle \xi \rangle) \operatorname{div} V(x,t) \, dx \, dt = \langle \langle x \operatorname{div} V \rangle, \; P \rangle,$$

$$\begin{aligned}
\int_{Q_1^+} \lambda v_\lambda^\eta(x,t) \, dx \, dt \geqq \int_0^1 (1 - c_I \|\operatorname{div} V(\cdot,t)\|_{L^N(Q_I)}) \int_{Q_1} |Dv_\lambda^\eta(x,t) + P| \, dx \, dt \\
+ \langle \langle V \rangle + \langle x \operatorname{div} V \rangle, P \rangle. \qquad (2.19)
\end{aligned}$$

Finally, in view of (1.5), we have, for all $t \geqq 0$,

$$1 - c_I \|\operatorname{div} V(\cdot,t)\|_{L^N(Q_I)} \geqq 0,$$

while the periodicity of v_λ implies, again for all $t \geqq 0$, that

$$|P| = \left| \int_{Q_1} (Dv_\lambda^\eta(x,t) + P) \, dx \right| \leqq \int_{Q_1} |Dv_\lambda^\eta(x,t) + P| \, dx.$$

Letting $\eta \to 0$ and then $\lambda \to 0$ in (2.19) we obtain (1.6). □

Error estimates for the periodic homogenization of coercive Hamilton–Jacobi equations were obtained earlier in Ref. [10]. Although the proof of Theorem 1.2 is almost the same as the one of the analogous result in Ref. [10], we present it here for the sake of completeness. To simplify the presentation below we denote by C constants that may change from line to line but depend only on u_0, $\|Du_0\|_\infty$ and V.

We have:

Proof of Theorem 1.2. For all $(x, t, P) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, set

$$H(x, t, P) = |P| + \langle V(x, t), P\rangle.$$

To avoid any technical difficulties due to the unboundedness of the domain, we first assume that u_0 is $(M\epsilon)\mathbb{Z}^N$-periodic for some positive integer M, which implies that u^ϵ and $\bar{u}$ are also $(M\epsilon)\mathbb{Z}^N$-periodic, and we obtain the estimate with constant independent of M. Then we use the finite speed of the propagation property of the averaged initial value problem, to remove the restriction on u_0.

Let $v_\lambda = v_\lambda(\cdot, \cdot; P) \in BUC(\mathbb{R}^{N+1})$ be the $\mathbb{Z}^{N+1}$-periodic solution to (2.1) and recall that the map

$$P \to \lambda v_\lambda(\cdot, \cdot; P) \text{ is } (1 + \|V\|_\infty)\text{-Lipschitz continuous.} \tag{2.20}$$

To simplify statements henceforward, we say the f is L-Lipschitz continuous if it is Lipschitz continuous with constant at most L.

Fix $T > 0$ and consider $\Phi : \mathbb{R}^N \times [0, T] \times \mathbb{R}^N \times [0, T] \to \mathbb{R}$ given by

$$\Phi(x, t, y, s) = u^\varepsilon(x, t) - u(y, s) - \varepsilon v_\lambda\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \frac{x-y}{\varepsilon^\beta}\right) - \frac{|x-y|^2}{2\varepsilon^\beta} - \frac{(t-s)^2}{2\varepsilon} - \delta s,$$

where $\beta \in (0, 1)$, $\lambda \in (0, 1)$ and $\delta > 0$ are to be chosen later.

Since Φ is periodic in the space variables, it has a maximum at some point $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$. The main part of the proof consists in showing that either $\hat{t} = 0$ or $\hat{s} = 0$ for a suitable choice of λ and δ.

We argue by contradiction and assume that $\hat{t} > 0$ and $\hat{s} > 0$. Since $\bar{u}$ is Lipschitz continuous and (2.20) holds, by standard arguments from the theory of viscosity solutions, we have

$$\frac{|\hat{x} - \hat{y}|}{\varepsilon^\beta} \leqq C\left(1 + \frac{\varepsilon^{1-\beta}}{\lambda}\right) \quad \text{and} \quad \frac{|\hat{t} - \hat{s}|}{\varepsilon} \leqq C. \tag{2.21}$$

We claim that

$$\frac{\hat{t} - \hat{s}}{\varepsilon} \leqq \bar{H}\left(\frac{\hat{x} - \hat{y}}{\varepsilon^\beta}\right) + C\left(\frac{\varepsilon^{1-\beta}}{\lambda} + \lambda\right). \tag{2.22}$$

Indeed for $\alpha, \beta > 0$ small, let $(x_\alpha, t_\alpha, y_\alpha, r_\alpha, z_\alpha)$ be a maximum point of Ψ_1 given by

$$\Psi_1(x,t,y,r,z) = u^\varepsilon(x,t) - \varepsilon v_\lambda\left(y, r, \frac{z-\hat{y}}{\varepsilon^\beta}\right) - \frac{|x-\hat{y}|^2}{2\varepsilon^\beta} - \frac{(t-\hat{s})^2}{2\varepsilon}$$
$$-\frac{1}{2\alpha}\left(|\varepsilon y - x|^2 + |z-x|^2 + |\varepsilon r - t|^2\right)$$
$$-\frac{\beta}{2}(|x-\hat{x}|^2 + (t-\hat{t})^2).$$

Since $(\hat{x}, \hat{t})$ is the unique maximum point of $\Psi_1(x, t, x/\varepsilon, t/\varepsilon, x)$, we have that $(x_\alpha, t_\alpha, y_\alpha, r_\alpha, z_\alpha)$ converges to $(\hat{x}, \hat{t}, \hat{x}/\varepsilon, \hat{t}/\varepsilon, \hat{x})$ as $\alpha \to 0$, with

$$\lim_{\alpha\to 0^+} \frac{1}{2\alpha}\left(|\varepsilon y_\alpha - x_\alpha|^2 + |z_\alpha - x_\alpha|^2 + |\varepsilon r_\alpha - t_\alpha|^2\right) = 0, \tag{2.23}$$

while (2.20) implies that

$$\frac{|z_\alpha - x_\alpha|}{\alpha} \leqq C\frac{\varepsilon^{1-\beta}}{\lambda}. \tag{2.24}$$

From the equation satisfied by u^ε we have

$$\begin{aligned}&\frac{t_\alpha - \hat{s}}{\varepsilon} + \frac{t_\alpha - \varepsilon r_\alpha}{\alpha} + \beta(t_\alpha - \hat{t})\\ &\leqq H\left(\frac{x_\alpha}{\varepsilon}, \frac{t_\alpha}{\varepsilon}, \frac{x_\alpha - \hat{y}}{\varepsilon^\beta} + \frac{x_\alpha - \varepsilon y_\alpha}{\alpha} + \frac{x_\alpha - z_\alpha}{\alpha} + \beta(x_\alpha - \hat{x})\right),\end{aligned} \tag{2.25}$$

while from the equation satisfied by v_λ we also have

$$\frac{t_\alpha - \varepsilon r_\alpha}{\alpha} + \lambda v_\lambda\left(y_\alpha, r_\alpha, \frac{z_\alpha - \hat{y}}{\varepsilon^\beta}\right) \geqq H\left(y_\alpha, r_\alpha, \frac{x_\alpha - \varepsilon y_\alpha}{\alpha} + \frac{z_\alpha - \hat{y}}{\varepsilon^\beta}\right).$$

Using the bound on the oscillation of the λv_λ's in Lemma 2.1, we get

$$\bar{H}\left(\frac{z_\alpha - \hat{y}}{\varepsilon^\beta}\right) + C\lambda\frac{|z_\alpha - \hat{y}|}{\varepsilon^\beta} \geqq H\left(y_\alpha, r_\alpha, \frac{x_\alpha - \varepsilon y_\alpha}{\alpha} + \frac{z_\alpha - \hat{y}}{\varepsilon^\beta}\right) - \frac{t_\alpha - \varepsilon r_\alpha}{\alpha}.$$

Combining the above inequality with (2.25) and using the regularity of H gives

$$\begin{aligned}\bar{H}(\frac{z_\alpha - \hat{y}}{\varepsilon^\beta}) + C\lambda\frac{|z_\alpha - \hat{y}|}{\varepsilon^\beta} \geqq\ & \frac{t_\alpha - \hat{s}}{\varepsilon} + \beta(t_\alpha - \hat{t})\\ & - C\left(\frac{|z_\alpha - x_\alpha|}{\varepsilon^\beta} + \frac{|z_\alpha - x_\alpha|}{\alpha} + \beta|x_\alpha - \hat{x}|\right)\\ & - C\left(\left|\frac{x_\alpha}{\varepsilon} - y_\alpha\right| + \left|\frac{t_\alpha}{\varepsilon} - r_\alpha\right|\right)\left(\frac{|x_\alpha - \varepsilon y_\alpha|}{\alpha} + \frac{|z_\alpha - \hat{y}|}{\varepsilon^\beta}\right).\end{aligned}$$

Using (2.23) and (2.24) we now let $\alpha \to 0$ to get

$$\bar{H}\left(\frac{\hat{x} - \hat{y}}{\varepsilon^\beta}\right) + C\lambda\frac{|\hat{x} - \hat{y}|}{\varepsilon^\beta} \geqq \frac{\hat{t} - \hat{s}}{\varepsilon} - C\frac{\varepsilon^{1-\beta}}{\lambda}.$$

Recalling (2.21) we obtain

$$\bar{H}\left(\frac{\hat{x}-\hat{y}}{\varepsilon^{\beta}}\right)+C\left(\lambda+\varepsilon^{1-\beta}+\frac{\varepsilon^{1-\beta}}{\lambda}\right)\geqq\frac{\hat{t}-\hat{s}}{\varepsilon},$$

and, since $\lambda\in(0,1)$, we finally get (2.22).

We now show that

$$\frac{\hat{t}-\hat{s}}{\varepsilon}-\delta\geqq\bar{H}\left(\frac{\hat{x}-\hat{y}}{\varepsilon^{\beta}}\right)-C\frac{\varepsilon^{1-\beta}}{\lambda}. \tag{2.26}$$

To this end, for $\alpha,\beta>0$, we consider

$$\Psi_2(y,s,z)=\bar{u}(y,s)+\varepsilon v_\lambda\left(\frac{\hat{x}}{\varepsilon},\frac{\hat{t}}{\varepsilon},\frac{\hat{x}-z}{\varepsilon^{\beta}}\right)+\frac{|\hat{x}-y|^2}{2\varepsilon^{\beta}}+\frac{(\hat{t}-s)^2}{2\varepsilon}+\delta s$$
$$+\frac{|z-y|^2}{2\alpha}+\frac{\beta}{2}(|y-\hat{y}|^2+|s-\hat{s}|^2),$$

which has a minimum at some point $(y_\alpha,s_\alpha,z_\alpha)$. Using (2.20) and the fact that $(\hat{y},\hat{s})$ is the unique minimum of the map $(y,s)\to\Psi_2(y,s,y)$, we find that $(y_\alpha,s_\alpha,z_\alpha)$ converges to $(\hat{y},\hat{s},\hat{y})$ as $\alpha\to 0$, with

$$\frac{|z_\alpha-y_\alpha|}{\alpha}\leqq C\frac{\varepsilon^{1-\beta}}{\lambda}\qquad\text{and}\qquad\lim_{\alpha\to 0}\frac{|z_\alpha-y_\alpha|^2}{\alpha}=0.$$

Since $\bar{u}$ solves (1.3) we have

$$\frac{\hat{t}-s_\alpha}{\varepsilon}-\delta\geqq\bar{H}\left(\frac{\hat{x}-y_\alpha}{\varepsilon^{\beta}}-\frac{y_\alpha-z_\alpha}{\alpha}-\beta(y_\alpha-\hat{y})\right),$$

and, in view of the Lipschitz continuity of $\bar{H}$, letting $\alpha\to 0^+$ gives (2.26).

Combining (2.22) and (2.26) we obtain

$$\delta-C\left(\frac{\varepsilon^{1-\beta}}{\lambda}+\lambda\right)\leqq 0,$$

which for $\lambda=\varepsilon^{(1-\beta)/2}$ and $\delta=3C\varepsilon^{(1-\beta)/2}$ gives a contradiction. So either $\hat{t}=0$ or $\hat{s}=0$.

Next we estimate $\max_{x,y,s}\Phi(x,0,y,s)$ and $\max_{x,t,y}\Phi(x,t,y,0)$. We have:

$$\begin{aligned}
&\max_{x,y,s}\Phi(x,0,y,s)\\
&\quad\leqq\max_{x,y,s}\left\{u_0(x)-u_0(y)+Cs+C\frac{\varepsilon^{1-\beta}}{\lambda}|x-y|-\frac{|x-y|^2}{2\varepsilon^{\beta}}-\frac{s^2}{2\varepsilon}-\delta s\right\}\\
&\quad\leqq\max_{x,y}\left\{C|y-x|+C\varepsilon-\frac{|x-y|^2}{2\varepsilon^{\beta}}\right\}\\
&\quad\leqq C\left(\varepsilon^{\beta}+\varepsilon\right)\leqq C\epsilon^{\beta},
\end{aligned}$$

and, similarly,

$$\max_{x,y,s} \Phi(x,t,y,0) \leqq C\,\varepsilon^{\beta}.$$

Therefore, for any $(x,t) \in \mathbb{R}^N \times [0,T]$, we have

$$u^{\varepsilon}(x,t) - \bar{u}(x,t) \leqq \delta t + C\epsilon^{\beta} \leqq CT\varepsilon^{(1-\beta)/2} + C\varepsilon^{\beta}.$$

Choosing $\beta = 1/3$ finally gives

$$u^{\varepsilon}(x,t) - u(x,t) \leqq C\varepsilon^{1/3}.$$

This reverse inequality can be obtained in the same way. □

3. Enhancement of speed

To prove Theorem 1.3 and Theorem 1.4 we need three results; we formulate these as Lemmas, here, but present their proofs at the end of the section.

We begin with:

Lemma 3.1. *Let V and $\bar{H}$ be as in Theorem 1.1. Then, for any $\mathbb{Z}$-periodic $c \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^N)$, the averaged Hamiltonian associated to $V - c$ is $\bar{H} - \langle \int_0^1 c(s)\,\mathrm{d}s, \cdot\rangle$.*

The second preliminary result is:

Lemma 3.2. *The divergence zero condition (1.9) is equivalent to the fact that, for any fixed time t, the map $x \to z(x,t) + \langle P, x\rangle$ is constant along the flow of the ODE $X'(s) = V(X(s),t)$.*

To state the final result, we recall the notion of an ε-mollifier. To this end, let $\phi \in \mathcal{C}_c^{\infty}(\mathbb{R}^{N+1};[0,1])$ be such that $\phi(0) = 1$ and $\int_{\mathbb{R}^{N+1}} \phi = 1$ and define the ε-mollifier ϕ_ε by $\phi_\varepsilon = \varepsilon^{-(N+1)}\phi(x/\varepsilon, t/\varepsilon)$. Then $\int_{\mathbb{R}^{N+1}} \phi_\varepsilon = 1$ and, for any $f \in L^1_{loc}(\mathbb{R}^{N+1})$, $f_\varepsilon = f * \phi_\varepsilon$ is a smooth approximation of f.

For the rest of the section we assume that, for each $t \in \mathbb{R}$,

$$\mathrm{div}_x V(\cdot,t) = 0 \quad \text{and} \quad \int_{Q_1} V(x,t)\,\mathrm{d}x = 0; \tag{3.1}$$

recall that the average zero condition is actually not a restriction, since we can always replace V by $V - \int_{Q_1} V(x,t)\,\mathrm{d}x$, a fact, which, in view of Lemma 3.1, simply adds a translation to the effective Hamiltonian.

The final preliminary result is:

Lemma 3.3. *Assume (3.1) and fix $P \in \mathbb{R}^N$. There exists a bounded, $\mathbb{Z}^{N+1}$-periodic $z \in BV_{loc}(\mathbb{R}^N \times \mathbb{R})$ such that, for all $\varepsilon > 0$ the smooth functions $z_\varepsilon = z * \phi_\varepsilon$'s satisfy in $\mathbb{R}^{N+1}$*

$$z_{\varepsilon,t} + \bar{H}(P) \geqq |Dz_\varepsilon + P| + \int_{\mathbb{R}^{N+1}} (z(y,s) + \langle P, y\rangle)\langle D\phi_\varepsilon(\cdot - y, \cdot - s), V(y,s)\rangle\,\mathrm{d}y\,\mathrm{d}s. \tag{3.2}$$

We proceed with Theorem 1.3, which is a straightforward consequence of Theorem 1.4. We have:

Proof of Theorem 1.3. If $\bar{H}(P) = |P|$, then let

$$z(x,t) = \hat{z}\left(\frac{\langle P, x\rangle}{|P|} + t\right) \quad \text{and} \quad z_1(x) = \int_0^1 z(x,s)\,\mathrm{d}s,$$

where $\hat{z}$ is given by Theorem 1.4.

Since V is independent of x, z_1 also satisfies (1.9) in the sense of distributions and, since z is periodic, z_1 is actually a constant. In this case (1.9) reduces to $\langle V(x), P\rangle = 0$ for all $x \in \mathbb{R}^N$.

When $N = 2$ and $\langle V\rangle = 0$, there exists a $\mathbb{Z}^2$-periodic stream function $E : \mathbb{R}^2 \to \mathbb{R}$ such that $V = (-\frac{\partial E}{\partial x_2}, \frac{\partial E}{\partial x_1})$. In this case, if $P = (P_1, P_2)$ and $q = (-P_1, P_2)$, $\langle V, P\rangle = 0$ in $\mathbb{R}^2$ becomes

$$0 = \langle DE(x), q\rangle.$$

If q is an irrational direction, then the map $t \to x + tq$ is dense in $\mathbb{R}^2/Z^2$. Since E is constant along this trajectory, E is constant and therefore V is identically equal to 0. Otherwise, $t \to x + tq$ is periodic and E has to be constant along this trajectory. This means that $E = \tilde{E}(\langle\cdot, P\rangle)$ for some smooth periodic map $\tilde{E} : \mathbb{R} \to \mathbb{R}$, and, hence, V is a shear advection.

Conversely, if $\langle V, P\rangle = 0$ in $\mathbb{R}^N$, then $v_\lambda = |P|/\lambda$ is the unique solution to (2.1), and $\lambda v_\lambda = |P|$ clearly converges uniformly to $\bar{H}(P) = |P|$. □

We turn now to the proof.

Proof of Theorem 1.4. Assume that, for some $P \in \mathbb{R}^N\backslash\{0\}$, $\bar{H}(P) = P$ and let z be given by Lemma 3.3. We first prove that z is a function of only $\langle P, x\rangle$ and t. More precisely, we claim that there exists $\tilde{z} \in BV_{loc}(\mathbb{R}^2)$ such that

$$z = \tilde{z}(\langle P, x\rangle, t) \text{ and } \tilde{z}_s(s,t) \geqq -1 \text{ in the sense of distributions.} \tag{3.3}$$

To this end, let ϕ_ε and $z_\varepsilon = z * \phi_\varepsilon$ be as in Lemma 3.3; note that z_ε is $\mathbb{Z}^{N+1}$-periodic. Then, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,

$$\int_{Q_1}\int_{\mathbb{R}^{N+1}} (z(y,s) + \langle P, y\rangle)\langle D\phi_\varepsilon(x-y, t-s), V(y,s)\rangle\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x = 0. \tag{3.4}$$

Indeed the periodicity of z and V gives,

$$\begin{aligned}&\int_{Q_1}\int_{\mathbb{R}^{N+1}} (z(y,s) + \langle P, y\rangle)\langle D\phi_\varepsilon(x-y, t-s), V(y,s)\rangle\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\\ &= \int_{\partial Q_1}\int_{\mathbb{R}^{N+1}} (z(y,s) + \langle P, y\rangle)\phi_\varepsilon(x-y,t-s)\langle V(y,s), \nu_x\rangle\,\mathrm{d}y\,\mathrm{d}s\, d\mathcal{H}^{N-1}(x)\end{aligned}$$

$$= \int_{\mathbb{R}^{N+1}} \int_{\partial Q_1} (z(x-y,t-s) + \langle P, x-y\rangle)\phi_\varepsilon(y,s)$$
$$\times \langle V(x-y,t-s), \nu_x\rangle d\mathcal{H}^{N-1}(x)\,\mathrm{d}y\,\mathrm{d}s$$
$$= \int_{\mathbb{R}^{N+1}} \int_{\partial Q_1} \langle P, x-y\rangle\phi_\varepsilon(y,s)\langle V(x-y,t-s), \nu_x\rangle d\mathcal{H}^{N-1}(x)\,\mathrm{d}y\,\mathrm{d}s.$$

while (3.1) yields

$$\int_{\mathbb{R}^{N+1}} \int_{\partial Q_1} \langle P, x-y\rangle\phi_\varepsilon(y,s)\langle V(x-y,t-s), \nu_x\rangle d\mathcal{H}^{N-1}(x)\,\mathrm{d}y\,\mathrm{d}s$$
$$= \int_{\mathbb{R}^{N+1}} \int_{Q_1} \mathrm{div}_x\left(\langle P, x-y\rangle\phi_\varepsilon(y,s)V(x-y,t-s)\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$= \int_{\mathbb{R}^{N+1}} \int_{Q_1} \langle P, V(x-y,t-s)\rangle\phi_\varepsilon(y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s = 0,$$

and, hence, (3.4) holds.

Next we integrate (3.2) over $Q_1 \times (0,1)$. Using (3.4) and the periodicity of $z_\varepsilon(\cdot,t)$ we get

$$|P| = \bar{H}(P) \geqq \int_0^1 \int_{Q_1} |Dz_\varepsilon(y,t) + P|\,\mathrm{d}y\,\mathrm{d}t \geqq \int_0^1 \left| \int_{Q_1} (Dz_\varepsilon(y,t) + P)\,\mathrm{d}y \right| \mathrm{d}t$$
$$= |P|.$$

It follows that, for all $(x,t) \in \mathbb{R}^{N+1}$, there exists $\theta(x,t) \geqq -1$ such that $Dz_\varepsilon(x) = \theta(x,t)P$. Thus z_ε is of the form $z_\varepsilon(x,t) = \tilde{z}_\varepsilon(\langle x, P\rangle, t)$, with $\tilde{z}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\tilde{z}_{\varepsilon,s}(s,t) \geqq -1$.

Passing to the limit $\varepsilon \to 0$, we also find that $z = \tilde{z}(\langle x, P\rangle, t)$ for some map $\tilde{z} \in BV_{loc}(\mathbb{R}^2, \mathbb{R})$ satisfying $\tilde{z}_s(s,t) \geqq -1$ in the sense of distributions, whence (3.3) holds.

Next we claim that $\tilde{z}$ satisfies, in the sense of distributions,

$$\tilde{z}_t(s,t) = \tilde{z}_s(s,t)|P| \quad \text{in } \mathbb{R}^2, \tag{3.5}$$

and that (1.9) holds. Note that this proves the "if" part, since (3.5) implies the existence of a map $\hat{z} \in BV_{loc}(\mathbb{R}, \mathbb{R})$ such that

$$\tilde{z}(s,t) = \hat{z}(|P|^{-1}s + t).$$

Moreover we have $\hat{z}'(s) \geqq -|P|$ in the sense of distributions because $\tilde{z}_s(s,t) \geqq -1$ in the same sense. Finally, $z(x,t) = \hat{z}\left(\frac{\langle P, z\rangle}{|P|} + t\right)$ is periodic in space and time.

We continue with the proofs of (3.5) and (1.9). If z is constant, then (3.5) is obvious and (1.9) just follows from (3.2) when $\varepsilon \to 0$.

Next, we assume that z is not constant. In this case $z = \tilde{z}(\langle \cdot, P\rangle, t)$ is $\mathbb{Z}^{N+1}$-periodic and not constant. Therefore P has to be a rational direction. So, up to a rational change of coordinates, we may assume without loss of generality that $P = \theta e_1$ for some $\theta > 0$, while V is still $\mathbb{Z}^{N+1}$-periodic.

Using the notation $x = (x_1, x')$ for each vector of $\mathbb{R}^N$ with $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$, for a fixed $(x_1, t) \in \mathbb{R}^2$ we integrate (3.2) over the cube $\{x_1\} \times Q_1' \times \{t\}$, where $Q_1' = (-1/2, 1/2)^{N-1}$, and obtain

$$\begin{aligned}&\tilde{z}_{\varepsilon,t}(\langle P, x\rangle, t) + |P| \geqq (\tilde{z}_{\varepsilon,s}(\langle P, x\rangle, t) + 1)|P| \\ &\quad + \int_{Q_1'} \int_{\mathbb{R}^{N+1}} (\tilde{z}(\theta(x_1 - y_1), t - s) + \theta(x_1 - y_1))\langle D\phi_\varepsilon(y, s), \\ &\quad V(x - y, t - s)\rangle \, \mathrm{d}y \, \mathrm{d}s \, \mathrm{d}x'. \end{aligned} \tag{3.6}$$

It turns out that the last integral in the right-hand side of the above inequality vanishes. Indeed

$$\begin{aligned}&\int_{Q_1'} \int_{\mathbb{R}^{N-1}} \langle D\phi_\varepsilon(y, s), V(x - y, t - s)\rangle \, \mathrm{d}y' \, \mathrm{d}x' \\ &\quad = \int_{\mathbb{R}^{N-1}} \frac{\partial \phi_\varepsilon}{\partial x_1}(y, s) \int_{Q_1'} V_1(x - y, t - s) \, \mathrm{d}x' \, \mathrm{d}y' \\ &\qquad + \sum_{j=2}^{N} \int_{Q_1'} \int_{\mathbb{R}^{N-1}} \phi_\varepsilon(y, s) \frac{\partial V_j}{\partial x_j}(x - y, t - s) \, \mathrm{d}y' \, \mathrm{d}x'.\end{aligned}$$

The periodicity of V yields, for any $j = 2, \ldots, N$,

$$\int_{\mathbb{R}^{N-1}} \phi_\varepsilon(y, s) \int_{Q_1'} \frac{\partial V_j}{\partial x_j}(x - y, t - s) \, \mathrm{d}x' \, \mathrm{d}y' = 0,$$

while the divergence free condition and, again, the periodicity give

$$\begin{aligned}\frac{\partial}{\partial x_1} \int_{Q_1'} V_1(x - y, t - s) \, \mathrm{d}x' &= \int_{Q_1'} \frac{\partial V_1}{\partial x_1}(x - y, t - s) \, \mathrm{d}x' \\ &= -\sum_{j=2}^{N} \int_{Q_1'} \frac{\partial V_1}{\partial x_j}(x - y, t - s) \, \mathrm{d}x' = 0.\end{aligned}$$

On the other hand,

$$\int_{-1/2}^{1/2} \int_{Q_1'} V_1(x - y, t - s) \, \mathrm{d}x' \, \mathrm{d}x_1 = \int_{Q_1} V_1(x, t - s) \, \mathrm{d}x = 0,$$

and, hence, for all $x_1 \in \mathbb{R}$,

$$\int_{Q_1'} V_1(x - y, t - s) \, \mathrm{d}x' = 0.$$

Therefore

$$\int_{Q_1'} \int_{\mathbb{R}^{N-1}} \langle D\phi_\varepsilon(y, s), V(x - y, t - s)\rangle \, \mathrm{d}y' \, \mathrm{d}x' = 0,$$

which, going back to (3.6), proves that

$$\tilde{z}_{\varepsilon,t}(\langle P, x\rangle, t) \geqq \tilde{z}_{\varepsilon,s}(\langle P, x\rangle, t)|P| \quad \text{in } \mathbb{R}^{N+1}.$$

Since $\tilde{z}_\varepsilon$ is periodic, integrating the above inequality over $(-1/2, 1/2) \times (0, 1)$ shows that in fact it must be an equality. Letting $\varepsilon \to 0$ then gives (3.5).

To prove (1.9), we first combine (3.2), (3.3) and (3.5) to get, for all $(x, t) \in \mathbb{R}^{N+1}$,

$$0 \geqq \int_{\mathbb{R}^{N+1}} (z(y, s) + \langle P, y\rangle)\langle D\phi_\varepsilon(x - y, t - s), V(y, s)\rangle \,\mathrm{d}y \,\mathrm{d}s.$$

Averaging over the cube Q_1 we see that, as a matter of fact, equality must hold for all $(x, t) \in \mathbb{R}^{N+1}$.

Integrating the resulting equality against any compactly supported smooth function $\psi : \mathbb{R}^N \to \mathbb{R}$ we get

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^{N+1}} \psi(x)(z(y, s) + \langle P, y\rangle)\langle D\phi_\varepsilon(x - y, t - s), V(y, s)\rangle \,\mathrm{d}y \,\mathrm{d}s \,\mathrm{d}x = 0,$$

and after integrating again by parts and letting $\varepsilon \to 0$, we obtain, for all $t \in \mathbb{R}$,

$$\int_{\mathbb{R}^N} \langle D\psi(x), (z(x, t) + \langle P, x\rangle)V(x, t)\rangle \,\mathrm{d}x = 0,$$

which is exactly (1.9).

To prove the "only if" part let $\hat{z}$ be as claimed. Since $\hat{z} \in BV_{loc}(\mathbb{R}, \mathbb{R})$, we may assume without loss of generality that $\hat{z}$, and, hence, $z(x, t) = \hat{z}\left(\frac{\langle P,x\rangle}{|P|} + t\right)$ are lower semi-continuous.

We show next that z satisfies, in the viscosity sense,

$$\partial_t z + |P| \geqq |Dz + P| + \langle V, Dz + P\rangle \qquad \text{in } \mathbb{R}^{N+1}. \tag{3.7}$$

To this end, let ϕ be a smooth test function such that $z \geqq \phi$ with equality at $(\bar{x}, \bar{t})$. It follows from equality $z(x, t) = \hat{z}\left(\frac{\langle P,x\rangle}{|P|} + t\right)$ that $D\phi(\bar{x}, \bar{t}) = \theta P/|P|$ where $\theta = \phi_t(\bar{x}, \bar{t})$.

Since $\hat{z}' \geqq -|P|$ in the sense of distributions, it follows that $\theta \geqq -|P|$. Moreover recalling (1.9) and Lemma 3.2, we have, for a fixed t, that the function $x \to V(x, t) + \langle P, x\rangle$ is constant under the flow of the ODE $X'(s) = V(X(s), t)$.

Let now X be a solution with $X(0) = \bar{x}$ and $t = \bar{t}$. Then, for any $s \in \mathbb{R}$,

$$\begin{aligned}\phi(\bar{x}, \bar{t}) + \langle P, \bar{x}\rangle = z(\bar{x}, \bar{t}) + \langle P, \bar{x}\rangle &= z(X(s), \bar{t}) + \langle P, X(s)\rangle \\ &\geqq \phi(X(s), \bar{t}) + \langle P, X(s)\rangle,\end{aligned}$$

and, therefore,

$$0 = \frac{d}{\mathrm{d}s}_{|s=0} \left[\phi(X(s), \bar{t}) + \langle X(s), P\rangle\right] = \langle D\phi(\bar{x}, \bar{t}) + P, V(\bar{x}, \bar{t})\rangle.$$

Combining the above relations gives

$$\phi_t(\bar{x}, \bar{t}) + |P| = |D\phi(\bar{x}, \bar{t}) + P| + \langle V(\bar{x}, \bar{t}), D\phi(\bar{x}, \bar{t}) + P\rangle,$$

that is, z is a super-solution of (3.7).

Then it is easy to check that $\tilde{z}(x,t) = z(x,t) + \frac{|P|}{\lambda} + \|z\|_\infty$ is a super-solution of (2.1). It then follows from the comparison principle that

$$v_\lambda \leqq z + \lambda^{-1}|P| + \|z\|_\infty.$$

Recalling that λv_λ converges uniformly, as $\lambda \to 0$, to $\bar{H}(P)$, we obtain $\bar{H}(P) \leqq |P|$. Since the reverse inequality always holds, the proof of the implication is complete. □

We continue with the proofs of the lemmas.

Proof of Lemma 3.1. Fix $\lambda > 0$ and $P \in \mathbb{R}^N$, let v_λ be the solution of (2.1) and let us recall that λv_λ converges uniformly, as $\lambda \to 0$, to $\bar{H}(P)$.

Set $\bar{c} = \int_0^1 c(s)\,\mathrm{d}s$ and consider $w_\lambda \in BUC(\mathbb{R}^{N+1})$ given by

$$w_\lambda(x,t) = v_\lambda(x - \int_0^t c(s)\,\mathrm{d}s, t) - \langle \int_0^t (c(s) - \bar{c})\,\mathrm{d}s + \frac{\bar{c}}{\lambda}, P\rangle + 2\|c\|_\infty |P|.$$

It follows that w_λ is a super-solution of

$$z_{\lambda,t} + \lambda z_\lambda = |Dz_\lambda + P| + \langle V - c, Dz_\lambda + P\rangle \quad \text{in } \mathbb{R}^{N+1}. \tag{3.8}$$

We present only a formal proof, which can be easily justified using viscosity solution arguments. To this end, observe that it is immediate from the definition of w_λ that

$$w_{\lambda,t} + \lambda w_\lambda \geqq -\langle Dv_\lambda, c(t)\rangle + v_{\lambda,t} - \langle c - \bar{c}, P\rangle + \lambda v_\lambda - \langle \bar{c}, P\rangle,$$

while

$$|Dw_\lambda + P| + \langle V - c, Dw_\lambda + P\rangle = |Dv_\lambda + P| + \langle V - c, Dv_\lambda + P\rangle.$$

The comparison principle now gives $w_\lambda \geqq z_\lambda$ where z_λ is the solution of (3.8).

Since the λz_λ's and the λw_λ's converge uniformly, as $\lambda \to 0$, to the averaged Hamiltonian $\bar{H}_c(P)$ associated to $V - c$ and to $\bar{H}(P) - \langle \bar{c}, P\rangle$ respectively, we get $\bar{H}_c(P) \leqq \bar{H}(P) - \langle \bar{c}, P\rangle$. The opposite inequality is proved similarly by considering $-c$ instead of c. □

Proof of Lemma 3.2. Let $X_x(\cdot)$ be the solution of $X'(s) = V(X(s), t)$ with initial condition x at time $s = 0$. Then, for any $h \in \mathbb{R}$ and $\psi : C_c^\infty(\mathbb{R}^N)$ the divergence zero property of V yields

$$\int_{\mathbb{R}^N} \psi(x)(z(X_x(h), t) + \langle P, X_x(h)\rangle)\,\mathrm{d}x = \int_{\mathbb{R}^N} \psi(X_x(-h))(z(x,t) + \langle P, x\rangle)\,\mathrm{d}x.$$

Therefore, in view of (1.9),

$$\begin{aligned} &\frac{d}{dh}_{|h=0} \int_{\mathbb{R}^N} \psi(x)(z(X_x(h), t) + \langle P, X_x(h)\rangle)\,\mathrm{d}x \\ &\quad = \int_{\mathbb{R}^N} \langle D\psi(x), V(x,t)\rangle (z(x,t) + \langle P, x\rangle)\,\mathrm{d}x = 0, \end{aligned}$$

Applying this last equality to the test function $\psi \circ X_x(-s)$ we get

$$\begin{aligned}\frac{d}{dh}_{|h=0} &\int_{\mathbb{R}^N} \psi(x)(z(X_x(s+h),t) + \langle P, X_x(s+h)\rangle)\,\mathrm{d}x \\ &= \frac{d}{dh}_{|h=0} \int_{\mathbb{R}^N} \psi(X_x(-s))(z(X_x(h),t) + \langle P, X_x(h)\rangle)\,\mathrm{d}x = 0.\end{aligned}$$

Hence $\mathrm{d}s \int_{\mathbb{R}^N} \psi(x)(z(X^x(s),t) + \langle P, X^x(s)\rangle)\,\mathrm{d}x$ is constant in time, which means that $z(\cdot,t)$ is constant along the flow. □

Proof of Lemma 3.3. For $\lambda > 0$, let v_λ be the solution of of (2.1) and set

$$z_\lambda(x,t) = v_\lambda(x,t) - v_\lambda(0,0).$$

It follows from Lemma 2.1 and Corollary 2.2 that, for some $C > 0$ independent of λ,

$$\|z_\lambda\|_\infty \leqq C|P|.$$

Next we show that the z_λ's are also bounded in $BV_{loc}(\mathbb{R}^{N+1})$. Indeed, for $\alpha > 0$, consider the $\mathbb{Z}^{N+1}$-periodic solution $v_{\lambda,\alpha}$ to

$$v_{\lambda,\alpha,t} + \lambda v_{\lambda,\alpha} = \alpha \Delta v_{\lambda,\alpha} + |Dv_{\lambda,\alpha} + P| + \langle V, Dv_{\lambda,\alpha} + P\rangle \quad \text{in } \mathbb{R}^{N+1}, \tag{3.9}$$

which is at least in $C^{1,1}$ and, moreover, converges uniformly, as $\alpha \to 0$, to v_λ.

Integrating (3.9) over a cylinder of the form $Q_R \times (-R,R)$ for some positive integer R, we obtain, using the periodicity, that

$$\begin{aligned}\int_{Q_R\times(-R,R)} \lambda v_{\lambda,\alpha} \geqq &\int_{Q_R\times(-R,R)} |Dv_{\lambda,\alpha} + P| \\ &+ \int_{Q_R\times(-R,R)} \langle V(x,t), Dv_{\lambda,\alpha}(x,t) + P\rangle \,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Since, in view of (3.1),

$$\int_{Q_R} \langle V(x,t), Dv_{\lambda,\alpha}(x,t) + P\rangle\,\mathrm{d}x = 0,$$

it follows that

$$\int_{Q_R\times(-R,R)} |Dv_{\lambda,\alpha} + P| \leqq 2R^{N+1}\|\lambda v_{\lambda,\alpha}\|_\infty. \tag{3.10}$$

Let $\Phi \in C_c^\infty(Q_R \times (-R,R); \mathbb{R}^{N+1})$ be such that $|\Phi(x,t)| \leqq 1$ for all $(x,t) \in \mathbb{R}^{N+1}$. From (3.9) and (3.10) we get

$$\begin{aligned}\int_{Q_R\times(-R,R)} v_{\lambda,\alpha} \mathrm{div}_{x,t}\Phi\,\mathrm{d}x\,\mathrm{d}t &= -\int_{Q_R\times(-R,R)} [v_{\lambda,\alpha,t}\Phi + \langle Dv_{\lambda,\alpha}, \Phi\rangle]\,\mathrm{d}x\,\mathrm{d}t \\ = -\int_{Q_R\times(-R,R)} &[(\lambda v_{\lambda,\alpha} - \alpha\Delta v_{\lambda,\alpha} - |Dv_{\lambda,\alpha} + P| \\ - \langle V, Dv_{\lambda,\alpha} + P\rangle)&\Phi + \langle Dv_{\lambda,\alpha}, \Phi\rangle]\,\mathrm{d}x\,\mathrm{d}t \\ \leqq C_R(P)\|\lambda v_{\lambda,\alpha}\|_\infty &+ \alpha \int_{Q_R\times(-R,R)} v_{\lambda,\alpha}\Delta\Phi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $C_R(P)$ depends only on N, R, $\|V\|_\infty$ and P.

Letting $\alpha \to 0$ yields

$$\int_{Q_R\times(-R,R)} v_\lambda \mathrm{div}_{x,t}\Phi\,\mathrm{d}x\,\mathrm{d}t \leqq C_R(P)\|\lambda v_\lambda\|_\infty \leqq C_R(P)(\bar{H}(P)+C\lambda), \quad (3.11)$$

which in turn implies, in view of the assumptions on Φ, that the z_λ's are bounded in L^∞ and in BV_{loc} uniformly with respect to λ. Hence the z_λ's converge, up to a subsequence and in L^1_{loc}, to some $\mathbb{Z}^{N+1}$-periodic $z \in BV_{loc}(\mathbb{R}^{N+1})$.

Let $\varepsilon > 0$, ϕ_ε and z_ε as in statement of the lemma, set $z_{\lambda,\varepsilon} = \phi_\varepsilon * z_\lambda$, and $z_{\lambda,\alpha,\varepsilon} = \phi_\varepsilon * z_{\lambda,\alpha}$ and fix $(x,t) \in \mathbb{R}^{N+1}$.

It follows from (3.9) that

$$\begin{aligned} z_{\lambda,\alpha,\varepsilon,t} + \lambda z_{\lambda,\alpha,\varepsilon} + \lambda v_{\lambda,\alpha}(0,0) &\geqq \alpha\Delta z_{\lambda,\alpha,\varepsilon} + |Dz_{\lambda,\alpha,\varepsilon}+P| \\ &+ \int_{\mathbb{R}^{N+1}} \phi_\varepsilon(x-y,t-s)\langle V(y,s), Dz_{\lambda,\alpha}(y,s)+P\rangle\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

Integrating by parts and using (3.1) we find

$$\begin{aligned} z_{\lambda,\alpha,\varepsilon,t} + \lambda z_{\lambda,\alpha,\varepsilon} + \lambda v_{\lambda,\alpha}(0,0) &\geqq \alpha\Delta z_{\lambda,\alpha,\varepsilon} + |Dz_{\lambda,\alpha,\varepsilon}+P| \\ &+ \int_{\mathbb{R}^{N+1}} (z_{\lambda,\alpha}(y,s)+\langle P,y\rangle)\langle D\phi_\varepsilon(x-y,t-s), V(y,s)\rangle\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

Letting first $\alpha \to 0$ and then $\lambda \to 0$ gives (3.2). □

4. Convergence to the Wulff shape

We begin by recalling some important facts from the theory of front propagation. The first is (see, for instance [6]), that the family of sets $(K(t))_{t\geqq 0}$ is independent of the choice of u_0 as long as $K_0 = \{x \in \mathbb{R}^n : u_0(x) \geqq 0\}$. The second (again see [6] and the references therein), which we will use repetitively in the sequel, is the following superposition principle of the geometric flow. If $(K_0^\theta)_{\theta\in\Theta}$ is an arbitrary family of non-empty closed subsets of $\mathbb{R}^N$, with corresponding solution $(K^\theta(t))_{t\geqq 0}$, then the solution starting from $\overline{\bigcup_{\theta\in\Theta} K_0^\theta}$ is given by $(\overline{\bigcup_{\theta\in\Theta} K^\theta(t)})_{t\geqq 0}$. This can be seen either by using the control representation of the geometric flow or the stability and comparison properties of viscosity solutions. A consequence is the well known inclusion principle. If $K_0 \subset K_0'$ are two non-empty closed subsets of $\mathbb{R}^N$, then the corresponding solutions $(K(t))_{t\geqq 0}$ and $(K'(t))_{t\geqq 0}$ satisfy $K(t) \subset K'(t)$ for all $t \geqq 0$.

We also remark that the convergence, as $t \to \infty$, of the level sets of geometric equations without spatio-temporal inhomogeneities was considered in Ref. [18]. The results of Ref. [18] do not, however, apply to the problem at hand.

The proof of Theorem 1.5 is long. We formulate two important steps as separate lemmas which we prove at the end of the section.

The first step regards some "controllability" estimates. We have:

Lemma 4.1. *Assume* (3.1). *There exist a positive integer n_0 and $T > 0$ such that, for all $x \in \mathbb{R}^N$, the solution $(\hat{K}(t))_{t\geqq 0}$ of the front propagation problem starting from the set $x + \{k \in \mathbb{Z}^N : |k| \leqq n_0\}$ contains $Q_1(x)$ at time T.*

The second step regards some growth properties for fronts.

Lemma 4.2. *There exist a positive integer $\bar{R}$ and positive constants r and T_1 such that, if the initial compact set $K_0 \subset \mathbb{R}^N$ contains a set of the form $Q_{\bar{R}}(k)$ for some $k \in \mathbb{Z}^N$, then the solution $(K(t))_{t \geqq 0}$ of the front propagation problem starting from K_0 satisfies, for all $t \geqq 0$,*

$$Q_{rt}(k) \subset K(t + T_1) \qquad \forall t \geqq 0.$$

We can now continue with the proof.

Proof of Theorem 1.5. We first show (1.12). It is well known (see [6]) that the characteristic function $\mathbf{1}_{K(t)}$ of $K(t)$ is a solution to the geometric equation

$$\partial_t u = |Du| + \langle V, Du \rangle \qquad \text{in } \mathbb{R}^N \times (0, +\infty). \tag{4.1}$$

Next, fix a direction $\nu \in \mathbb{R}^N$ with $|\nu| = 1$. For $\lambda > 0$, let v_λ be the solution of

$$v_{\lambda,t} + \lambda v_\lambda = |Dv_\lambda + \nu| + \langle V, Dv_\lambda + \nu \rangle \qquad \text{in } \mathbb{R}^{N+1},$$

and recall that there is some constant $\bar{C}$ independent of ν and λ such that

$$\operatorname{osc}(v_\lambda) \leqq \bar{C} \qquad \text{and} \qquad \|\lambda v_\lambda - \bar{H}(\nu)\|_\infty \leqq \bar{C}\lambda.$$

It is immediate that

$$z(x, t) = v_\lambda(x, t) - v_\lambda(0) + \langle \nu, x \rangle + (\bar{H}(\nu) + \bar{C}\lambda)t + C,$$

with

$$C = \bar{C} - \min_{y \in K_0} \langle \nu, x \rangle + 1,$$

is a super-solution of (4.1). Since (4.1) geometric, it is immediate that $\max(z, 0)$ is also a super-solution. Moreover, we clearly have $\max(z(\cdot, 0), 0) \geqq \mathbf{1}_{K_0}$.

The standard comparison gives, for all $(x, t) \in \mathbb{R}^N \times [0, +\infty)$,

$$\max(z(x, t), 0) \geqq \mathbf{1}_{K(t)}(x),$$

which, in turn, implies that, for all $t \geqq 0$,

$$K(t) \subset \{x \in \mathbb{R}^N : z(x, t) \geqq 1\}.$$

Recalling that $\bar{H}(\nu) \geqq |\nu| = 1$, we find, for all $(x, t) \in \mathbb{R}^N \times [1, \infty)$,

$$z(x, t) \leqq \langle \nu, x \rangle + (\bar{H}(\nu) + \bar{C}\lambda)t + C + \bar{C} \leqq \langle \nu, x \rangle + \bar{H}(\nu)((1 + \bar{C}\lambda)t + C + \bar{C}),$$

and, therefore, for all $t \geqq 0$,

$$K(t) \subset \{x \in \mathbb{R}^N : \langle \nu, x \rangle + \bar{H}(\nu)((1 + \bar{C}\lambda)t + C + \bar{C} - 1) \geqq 0\} \qquad \forall t \geqq 1.$$

Letting $\lambda \to 0$, we get, for all $t \geqq 0$ and a new positive constant C,

$$K(t) \subset \{x \in \mathbb{R}^N : \langle \nu, x \rangle + \bar{H}(\nu)(t + C) \geqq 0\} \qquad \forall t \geqq 1.$$

Taking the intersection of the right-hand side over all ν we obtain, by the definition of $\mathcal{W}$, that, for all $t \geqq 0$,

$$K(t) \subset (t + C)\mathcal{W}.$$

The proof of (1.13) is more intricate. To this end, let $\bar{R}$, r and T_1 be defined by Lemma 4.2 and let $(K(t))_{t \geqq 0}$ be the solution of the front propagation problem starting at $K_0 \subset \mathbb{R}^N$, a compact set which contains $Q_{\bar{R}}(k)$ for some $k \geqq \mathbb{Z}^N$. Then, for all $t \geqq 0$,

$$B_{rt} \subset K(t + T_1). \tag{4.2}$$

Recall that $u^\varepsilon(x, t) = \mathbf{1}_{K(t/\varepsilon)}(x/\varepsilon)$ is the solution to

$$\begin{cases} u^\varepsilon_t = |Du^\varepsilon| + \langle V(x/\varepsilon, t/\varepsilon), Du^\varepsilon \rangle & \text{in } \mathbb{R}^N \times (0, \infty), \\ u^\varepsilon(x, 0) = \mathbf{1}_{K_0}(x/\varepsilon). \end{cases} \tag{4.3}$$

and, in view of (4.2), for all $(x, t) \in \mathbb{R}^{N+1}$,

$$u^\varepsilon(x, t) \geqq \mathbf{1}_{B_{r(t/\varepsilon - T_1)}}(x/\varepsilon) = \mathbf{1}_{B_{r(t-\varepsilon T_1)}}(x).$$

Next, fix $\delta \in (0, 1)$ such that $T_1 + \delta/(r\varepsilon)$ is an integer and, for $w_\delta(x) = (\delta - |x|) \vee 0$, let w^ε_δ be the solution of (4.3) with initial datum w_δ. Then we know from Theorem 1.2 that there exists a constant $\bar{C} > 0$ such that, for all $t \in (0, 1)$,

$$\|w^\varepsilon_\delta - \bar{w}_\delta\|_\infty \leqq \bar{C}\varepsilon^{1/3}$$

where $\bar{w}_\delta$ is the solution of the homogenized problem

$$\begin{cases} \bar{w}_{\delta,t} = \bar{H}(D\bar{w}_\delta) & \text{in } \mathbb{R}^N \times (0, 1), \\ \bar{w}_\delta(\cdot, 0) = w_\delta & \text{on } \mathbb{R}^N. \end{cases}$$

Note that the constant $\bar{C}$ is independent of δ because the w_δ's have Lipschitz constants which are bounded uniformly in δ.

From the Lax-Oleinik formula, $\bar{w}_\delta$ is given, for $(x, t) \in \mathbb{R}^N \times [0, \infty)$ by

$$\bar{w}_\delta(x, t) = \sup_{y \in \mathcal{W}} w_\delta(x - ty).$$

Since $w_\delta \leqq \mathbf{1}_{B_\delta} \leqq u^\varepsilon(\cdot, \varepsilon T_1 + \delta/r)$, it follows, from the time-periodicity of V and the choice of δ, that, for all $(x, t) \in \mathbb{R}^N \times [0, 1])$,

$$w^\varepsilon_\delta(x, t) \leqq u^\varepsilon(x, t + \varepsilon T_1 + \delta/r).$$

Hence, for all $t \in [0, 1]$,

$$\{w^\varepsilon_\delta(\cdot, t) \geqq \delta/2\} \subset \{u^\varepsilon(\cdot, t + \varepsilon T_1 + \delta/r) \geqq \delta/2\},$$

while

$$\{w^\varepsilon_\delta(\cdot, t) \geqq \delta/2\} \supset \{\bar{w}_\delta(\cdot, t) \geqq \delta/2 + \bar{C}\varepsilon^{1/3}\} \supset \{\sup_{y \in \mathcal{W}} w_\delta(\cdot - ty) \geqq \delta/2 + \bar{C}\varepsilon^{1/3}\}.$$

If we choose $\delta/2 - \bar{C}\varepsilon^{1/3} > 0$, then

$$\{\sup_{y\in\mathcal{W}} w_\delta(\cdot - ty) \geqq \delta/2+\bar{C}\varepsilon^{1/3}\} \supset \{x : \sup_{y\in\mathcal{W}} (\delta - |x-ty|) \geqq \delta/2+\bar{C}\varepsilon^{1/3}\} \supset t\mathcal{W}.$$

Therefore, for all $t \in [0, 1]$,

$$\varepsilon K((t+\varepsilon T_1+\delta/r)/\varepsilon) = \{u^\varepsilon(\cdot, t+\varepsilon T_1+\delta/r) \geqq \delta/2\} \supset t\mathcal{W},$$

that is, for all $t \in [0, 1/\varepsilon]$,

$$K(t+T_1+\delta/(r\varepsilon)) \supset t\mathcal{W}. \tag{4.4}$$

Finally, for t sufficiently large, choose $\varepsilon = 1/(t - 4\bar{C}t^{2/3}/r)$ and $\delta = (n-T_1)r\varepsilon$ where n is the integer part of $4\bar{C}t^{2/3}/r + 1$. Then $n = T_1 + \delta/(r\varepsilon)$ is an integer, $\delta/2 - \bar{C}\varepsilon^{1/3}$ is positive and, applying inclusion (4.4) to $t - n$ which belongs to $[0, 1/\varepsilon]$, we get

$$K(t) \supset (t - Ct^{2/3})\mathcal{W}$$

for some new constant C. □

We conclude the section with the proofs of the two lemmas used in the proof.

Proof of Lemma 4.1. Fix $x \in \mathbb{R}^N$ and let $(K(t))_{t\geqq 0}$ be the solution of the front propagation $V_{x,t} = 1 - V(x, t)$ starting from $K_0 = \{x\} + \mathbb{Z}^N$, that is, $K(t) = \{y \in \mathbb{R}^N : u(y, t) = 0\}$ where u is the solution to

$$\begin{cases} \partial_t u = |Du| + \langle V, Du\rangle & \text{in } \mathbb{R}^N \times (0, \infty) \\ u = -d_{K_0} & \text{on } \mathbb{R}^N, \end{cases}$$

where d_{K_0} is the distance function to the set K_0.

Then, for each $t > 0$, the set $K(t)$ is $\mathbb{Z}^N$-periodic and a non-empty interior (because it has an interior ball property, as explained in the Appendix). So $\rho(t) = |K(t) \cap Q_1(x)|$ is positive for positive time and, following the computation in the proof of Theorem 1.1, it satisfies for all $t_2 > t_1 \geqq 0$,

$$\rho(t_2) - \rho(t_1) \geqq \frac{1}{c_I}\int_{t_1}^{t_2} (\min\{\rho(t), 1-\rho(t)\})^{(N-1)/N}\, \mathrm{d}t.$$

Hence there exists a time T depending only on N such that $|K(T) \cap Q_1(x)| = 1$. This means that $Q_1(x) \subset K(T)$.

It follows from the finite speed of propagation that there exists a positive integer n_0 such that the solution $\hat{K}(t)$ starting from $\{x + k \in \mathbb{Z}^N : |k| \leqq n_0\}$ coincides with $K(t)$ on $Q_1(x) \times [0, T]$. Then $Q_1(x) \subset \hat{K}(T)$. □

Proof of Lemma 4.2. Consider the solution $\bar{u}$ to

$$\begin{cases} \bar{u}_t = \bar{H}(D\bar{u}) & \text{in } \mathbb{R}^N \times (0,\infty), \\ \bar{u}(x,0) = (-|x|) \vee (-1) & \text{on } \mathbb{R}^N \times \{0\}. \end{cases}$$

Since, $\bar{H}(p) \geqq |p|$, given $0 < \theta_1 < \theta_0 < 1$ and $\delta > 0$ small, there exists $\bar{t} \in (0,1)$ such that

$$\{\bar{u}(\cdot,0) \geqq -\theta_0\} + B(0,\delta) \subset \{\bar{u}(\cdot,\bar{t}) \geqq -\theta_1\}.$$

The fact that the solution u^ϵ of

$$\begin{cases} \partial_t u^\epsilon = |Du^\epsilon| + \langle V(\frac{x}{\epsilon},\frac{t}{\epsilon}), Du^\epsilon\rangle & \text{in } \mathbb{R}^N \times (0,\infty), \\ u^\epsilon(x,0) = (-|x|) \vee (-1) & \text{on } \mathbb{R}^N \times \{0\}, \end{cases}$$

converges, as $\varepsilon \to 0$, locally uniformly to $\bar{u}$, yields an $\epsilon \in (0,\delta/4)$ such that

$$\{u^\epsilon(\cdot,0) \geqq -\theta_0\} + B(0,\frac{\delta}{2}) \subset \{u^\epsilon(\cdot,\bar{t}) \geqq -\theta_0\}. \tag{4.5}$$

Next, fix n_0 and T as in Lemma 4.1. Choose ϵ such that $T_0 = \frac{\bar{t}}{\epsilon}$ is an integer and

$$Q_{n_0+1}(0) \subset B\left(0, \frac{\delta}{2\varepsilon}\right), \tag{4.6}$$

and set

$$\tilde{K}_0 = \{x : u^\epsilon(\varepsilon x, 0) \geqq -\theta_0\}.$$

The solution of the front propagation problem starting from $\tilde{K}_0$ is given by $\tilde{K}(t) = \{x : u^\epsilon(\epsilon x, \epsilon t) \geqq -\theta_0\}$. From (4.5) we have

$$\tilde{K}_0 + B\left(0, \frac{\delta}{2\epsilon}\right) \subset \tilde{K}(T_0),$$

while from (4.6), for any $k \in \mathbb{Z}^N$ with $|k| \leqq n_0$, we have $\tilde{K}_0 + k \subset \tilde{K}(T_0)$.

The periodicity of V also implies that the solution of the front propagation problem starting from $\tilde{K}_0 + k$ is just $\tilde{K}(t) + k$ while the solution starting from $\tilde{K}(T_0)$ is $K(t + T_0)$.

From the inclusion principle we get, for all $k \in \mathbb{Z}^N$ with $|k| \leqq n_0$ and all $t \geqq 0$,

$$\tilde{K}(t) + k \subset \tilde{K}(t + T_0). \tag{4.7}$$

Then Lemma 4.1 implies that, for all $t \geqq 0$,

$$\tilde{K}(t) + Q_1 \subset \tilde{K}(t + T_0 + T).$$

In particular, by induction, we get, for all positive integers n and all $t \geqq 0$,

$$\tilde{K}(t) + Q_n \subset \tilde{K}(t + n(T_0 + T)).$$

Choose a positive integer M such that, for all $t \in [0, T_0 + T]$, $\tilde{K}(t) \subset Q_M$. Then, for all positive integers n such that $n \geqq M$ and all $t \in [0, T_0 + T]$,

$$Q_{n-M} \subset \tilde{K}(t) + Q_n \subset \tilde{K}(t + n(T_0 + T)),$$

and, hence, there exist $r > 0$ and $T_1 > 0$ such that, for all $t \geqq 0$,

$$Q_{rt} \subset \tilde{K}(t + T_1).$$

Finally, choose a positive integer $\bar{R}$ such that $\tilde{K}_0 \subset Q_{\bar{R}}$. Then, for any compact initial set K_0 such that $Q_{\bar{R}}(k) \subset K_0$ for some $k \in \mathbb{Z}^N$, we have $\tilde{K}_0 + k \subset K_0$. Therefore the solution of the front propagation problem $K(t)$ starting from K_0 satisfies, for all $t \geqq 0$,

$$Q_{rt}(k) \subset \tilde{K}(t + T_1) + k \subset K(t + T_1).$$

□

5. Homogenization for x-dependent velocities at scale one

Before we begin the proof, we remark that, since we are able to prove only that $\bar{H}$ is continuous with respect to the x-variable, uniqueness of the solution to Eq. (1.18) could be an issue. This is not, however, the case because, in view of (1.17) and (1.19), $\bar{H}$ is coercive. Note that this is the reason we do not consider V, which also depends on a slow time variable, because then the coercivity of the averaged Hamiltonian would no longer ensure a comparison principle for the limit problem.

Proof of Theorem 1.6. For any fixed $(x, P) \in \mathbb{R}^N \times \mathbb{R}^N$, let $v_\lambda^{P,x} = v_\lambda^{P,x}(y, t)$ be the solution to

$$v_{\lambda,t}^{P,x} + \lambda v_\lambda^{P,x} = |Dv_\lambda^{P,x} + P| + \langle V(x, y, t), Dv_\lambda^{P,x} + P\rangle \qquad \text{in } \mathbb{R}^N \times \mathbb{R}.$$

From Lemma 2.1 and Corollary 2.2 we know that there exist $\bar{H}(x, P)$ and, for any $M > 0$, a constant $C_M > 0$ independent of λ and P, such that, for all $x \in \mathbb{R}^N$ such that $|x| \leqq M$,

$$\|\bar{H}(x, P) - \lambda v_\lambda^{P,x}\|_\infty \leqq C|P|\lambda. \tag{5.1}$$

Arguing as in the proof of Theorem 1.1, one easily checks that $\bar{H}$ is positively homogeneous of degree one and convex in P, and that (1.19) holds.

To complete the proof, it only remains to show that $\bar{H}$ is continuous in (x, P). To this end, observe that the standard comparison arguments imply $\bar{H}$ is $(1 + \|V\|_\infty)$-Lipschitz continuous in P.

Next, fix $M > 0$, $x_1, x_2 \in \mathbb{R}^N$ with $|x_1|, |x_2| \leqq M$. Once again the standard comparison of viscosity solutions (see [5]) gives

$$\|v_\lambda^{P,x_1} - v_\lambda^{P,x_2}\|_\infty \leqq C_M|P|\frac{|x_1 - x_2|^{\lambda/L}}{\lambda} \tag{5.2}$$

where $L = \sup_{|x| \leqq M} \|D_{y,s}V(x, \cdot, \cdot)\|_\infty$ and C_M depends only on M.

Combining (5.1) with (5.2), we obtain, for all $x_1, x_2 \in \mathbb{R}^N$ such that $|x_1|, |x_2| \leqq M$ and $|x_1 - x_2| \leqq 1$,

$$|\bar{H}(x_1, P) - \bar{H}(x_2, P)| \leqq C_M |P| \omega(|x_1 - x_2|)$$

with (the modulus) ω given, for $r \in (0, 0]$, by

$$\omega(r) = \inf_{\lambda \in (0,1]} (\lambda^{-1} r^{\lambda/L} + \lambda).$$

The proof of the continuity of $\bar{H}$ is now complete. □

6. Appendix

We present here the proof of Lemma 2.3.

Proof of Lemma 2.3. Fix $\theta \in (\inf z(\cdot, 0), \sup z(\cdot, 0))$ and let $K_\theta(t) = \{z(\cdot, t) \geqq \theta\}$. Since z is a solution to (2.5)-(i), $K_\theta(t)$ is given by

$$K_\theta(t) = \left\{ x \in \mathbb{R}^N : \begin{array}{l} \exists \xi : [0, t] \to \mathbb{R}^N \text{ absolutely continuous such that } \xi(t) = x, \\ z(\xi(0), 0) \geqq \theta, \ |\xi'(s) + V(\xi(s), s)| \leqq 1 \text{ almosteverywhere } s \in (0, t) \end{array} \right\},$$

that is, $K_\theta(t)$ is the reachable set for the controlled system

$$\xi'(s) = f(\xi(s), s, \alpha(s)), \qquad |\alpha(s)| \leqq 1,$$

where $f(x, s, \alpha) = \alpha + V(x, s)$ and $\alpha \in \mathbb{R}^N$ are such that $|\alpha| \leqq 1$.

Since V is of class $\mathcal{C}^{1,1}$, it follows from Ref. [9] that, for any $0 < \tau < T$, there exists a constant $r = r(\tau, T) > 0$ such that, for all $t \in [\tau, T]$, the set $K_\theta(t)$ has the interior ball property of radius r, that is,

$$\text{for all } x \in \partial K_\theta(t) \quad \text{there exists } y \in \mathbb{R}^N \quad \text{such that } x \in \overline{B}(y, r) \subset K_\theta(t),$$

where $\overline{B}(y, r)$ stands for the closed ball of radius r centered at y, and, hence, for all $t > 0$, $K_\theta(t)$ is a set of finite perimeter (see, for instance [2]).

Note that [9] deals with time independent dynamics, but the proofs can be easily adapted to the time-dependent framework considered here.

Next set $d(x, t) = d_{K_\theta(t)}(x)$. It follows that d is Lipschitz continuous in (x, t) and, moreover,

$$d_t = -1 + \langle V(x - dDd, t)\ , Dd(x, t) \rangle \quad \text{a.e in } \{d > 0\}. \tag{6.1}$$

It is, of course, clear that $x \to d(x, t)$ is $1-$Lipschitz continuous. The $\|V\|_\infty +$ 1-Lipschitz continuity of the map $t \to d(x, t)$ comes from the above representation formula of $K_{\theta,t}$.

To check (6.1), recall that, since the map $P \to |P| + \langle V, P \rangle$ is convex, z is a sub-solution of (2.5)(i) with an equality (see [8] and [16]), that is, for any test function φ and any local maximum point (x, t) of $z - \varphi$, we have

$$\varphi_t = |D\varphi| + \langle V, D\varphi \rangle.$$

The invariance property of the geometric equation (2.5)-(i) (see [6]) then implies that the map $(x, t) \to \mathbf{1}_{K_\theta(t)}(x)$ is also a sub-solution of (2.5)(i) with equality.

Assume now that d is differentiable at some point $(\bar{x}, \bar{t})$ and let $\bar{y}$ be the unique projection of $\bar{x}$ onto $K_\theta(\bar{t})$ and note that $w(y, t) := \mathbf{1}_{K_\theta(t)}(y) + d(y + \bar{x} - \bar{y}, t)$ has a local maximum at $(\bar{y}, \bar{t})$. Indeed, if $y \notin K_\theta(t)$, then, for all (y, t) sufficiently close to $(\bar{y}, \bar{t})$,

$$w(y, t) = d(y + \bar{x} - \bar{y}, t) \leqq 1 + d(\bar{x}, \bar{t}) = w(\bar{y}, \bar{t}),$$

and, if $y \in K_{\theta,t}$, then

$$w(y, t) = 1 + d(y + \bar{x} - \bar{y}, t) \leqq 1 + |y + \bar{x} - \bar{y} - y| = 1 + |\bar{x} - \bar{y}| = w(\bar{y}, \bar{t}).$$

Since $(x, t) \to \mathbf{1}_{K_\theta(t)}(x)$ is a sub-solution of (2.5)(i) with equality and $|Dd(\bar{x}, \bar{t})| = 1$ we obtain

$$-d_t(\bar{x}, \bar{t}) = 1 + \langle V(\bar{y}, \bar{t}) \ , \ -Dd(\bar{x}, \bar{t})\rangle,$$

and (6.1) holds because $\bar{y} = \bar{x} - d(\bar{x}, \bar{t})Dd(\bar{x}, \bar{t})$.

Next we assume that θ is such that $\partial\{z(\cdot, 0) \geqq \theta\} = \{z(\cdot, 0) = \theta\}$, which is true for almost all $\theta \in (\inf z(\cdot, 0), \sup z(\cdot, 0))$. It then follows from Ref. [6] that, for all $t \geqq 0$, $\partial\{z(\cdot, t) \geqq \theta\} = \{z(\cdot, t) = \theta\}$ for any $t \geqq 0$.

We now prove that, for all $\varphi \in C_c^\infty(\mathbb{R}^N)$ and all $t \in (0, T)$,

$$\int_{K_\theta(t)} \varphi(x)\,\mathrm{d}x - \int_{K_\theta(0)} \varphi(x)\,\mathrm{d}x \\ = \int_0^t \int_{\partial K_\theta(s)} \varphi(x)(1 - \langle V(x, s), \nu(x, s)\rangle) d\mathcal{H}^{N-1}(x)\,\mathrm{d}s. \qquad (6.2)$$

To this end, for $h > 0$ small, let $\zeta_h : \mathbb{R} \to [0, 1]$ be such that $\zeta_h(\rho) = 1$ if $\rho \leqq 0$, $\zeta_h(\rho) = 1 - \rho/h$ if $\rho \in [0, h]$ and $\zeta_h(\rho) = 0$ if $\rho \geqq h$.

Multiplying (6.1) by $\varphi\zeta_h'(d)$ (which makes sense because the sets $\{d(\cdot, s) = 0\} = \partial K_\theta(t)$ and $\{d(\cdot, s) = h\}$ have a zero measure since $K_\theta(t)$ and $\{d(\cdot, s) \leqq h\}$ have the interior ball property) and integrating over $\mathbb{R}^N \times (0, t)$ gives

$$\int_{\mathbb{R}^N} \varphi(x)\zeta_h(d(x, t))\,\mathrm{d}x - \int_{\mathbb{R}^N} \varphi(x)\zeta_h(d(x, 0))\,\mathrm{d}x \\ = \int_0^t \frac{-1}{h} \int_{\{0<d(\cdot,s)<h\}} \varphi(x)(-1 + \langle V(x - d(x, s)Dd(x, s), s), \\ Dd(x, s)\rangle)\,\mathrm{d}x\,\mathrm{d}s. \qquad (6.3)$$

Note that

$$\lim_{h\to 0}\left[\int_{\mathbb{R}^N}\varphi(x)\zeta_h(d(x,t))\,\mathrm{d}x-\int_{\mathbb{R}^N}\varphi(x)\zeta_h(d(x,0))\,\mathrm{d}x\right]$$
$$=\int_{K_\theta(t)}\varphi(x)\,\mathrm{d}x-\int_{K_\theta(0)}\varphi(x)\,\mathrm{d}x.$$

We now concentrate on the right hand side of (6.3). Since $|Dd| = 1$ almost everywhere in $\{d > 0\}$, the co-area formula implies

$$\int_0^t \frac{-1}{h}\int_{\{0<d(\cdot,s)<h\}}\varphi(x)(-1+\langle V(x-d(x,s)Dd(x,s),s),Dd(x,s)\rangle)\,\mathrm{d}x\,\mathrm{d}s$$
$$=\int_0^t\frac{1}{h}\int_0^h\int_{\{d(\cdot,s)=\sigma\}}\varphi(x)(1-\langle V(x-\sigma\nu^\sigma(x,s),s),\nu^\sigma(x,s)\rangle)$$
$$\times\, d\mathcal{H}^{N-1}(x)\,\mathrm{d}\sigma,\ \mathrm{d}s$$

where $\nu^\sigma(x,s)$ is the measure theoretic outward unit normal to the set $\{d(\cdot,s) < \sigma\}$, which has a finite perimeter since it satisfies the interior ball property with radius $r+\sigma$. □

In order to complete the proof of (6.2) we just need to use the following lemma:

Lemma 6.1. *Let E be a closed subset of $\mathbb{R}^N$ with the interior ball property of radius $r > 0$. Then, for all compactly supported in x $\Phi \in C(\mathbb{R}^N \times \mathcal{S}^{N-1})$,*

$$\lim_{\sigma\to 0}\int_{\{d_E(\cdot)=\sigma\}}\Phi(x,\nu^\sigma(x))d\mathcal{H}^{N-1}(x)=\int_{\partial E}\Phi(x,\nu(x))d\mathcal{H}^{N-1}(x),$$

where $d_E(x)$ stands for the distance of x to E and $\nu^\sigma(x)$ (resp. $\nu(x)$) is the measure theoretic outward unit normal to $\{d_E(\cdot) < \sigma\}$ (resp. to E) at $x \in \partial E$.

Proof. Set $E_\sigma = \{d_E(\cdot) \leqq \sigma\}$ and denote by Π_σ the projection of ∂E_σ onto ∂E. It is known that Π_σ is uniquely defined for $\mathcal{H}^{N-1}$—almost everywhere point $x \in \partial E_\sigma$. Let $\mu_\sigma = \mathcal{H}^{N-1}\lfloor\partial E_\sigma$ and $\bar{\mu}_\sigma = \Pi_\sigma\sharp\mu_\sigma$, $\mu_0 = \bar{\mu}_0 = \mathcal{H}^{N-1}\lfloor\partial E$.

The first claim is that $\bar{\mu}_\sigma$ is absolutely continuous with respect to $\bar{\mu}_0$. For this, let us first recall that, since E has the interior ball property of radius r, the map Π_σ^{-1} is well-defined on $F_\sigma := \{x \in \partial E\ ;\ \exists y \in \partial E_\sigma \text{ with } |x-y| = \sigma\}$ and that Π_σ^{-1} is Lipschitz continuous, with constant at most $(r+\sigma)/r$, on F_σ (see [2]). So, if Z is a Borel subset of ∂E, then

$$\bar{\mu}_\sigma(Z)=\mathcal{H}^{N-1}(\Pi_\sigma^{-1}(Z))\leqq \mathrm{Lip}(\Pi_\sigma^{-1})\mathcal{H}^{N-1}(Z)=\frac{r+\sigma}{r}\bar{\mu}_0(Z).$$

In particular, $\bar{\mu}_\sigma$ is absolutely continuous with respect to $\bar{\mu}_0$ and, if $f_\sigma = \dfrac{d\bar{\mu}_\sigma}{d\bar{\mu}_0}$, then f_σ is bounded by $\frac{r+\sigma}{r}$ $\mathcal{H}^{N-1}$—almost everywhere in ∂E.

Next we note that, for any $\Phi \in C(\mathbb{R}^N \times \mathcal{S}^{N-1})$,

$$\int_{\partial E_\sigma}\Phi(x,\nu^\sigma(x))d\mathcal{H}^{N-1}(x)=\int_{\partial E}\Phi(y+\sigma\nu(y),\nu(y))f_\sigma(y)d\mathcal{H}^{N-1}(y).\quad(6.4)$$

Indeed, the definitions of $\bar{\mu}_\sigma$ and f_σ, give

$$\begin{aligned}\int_{\partial E} &\Phi(y+\sigma\nu(y),\nu(y)) f_\sigma(y) d\mathcal{H}^{N-1}(y)\\ &= \int_{\partial E} \Phi(y+\sigma\nu(y),\nu(y)) d(\Pi_\sigma \sharp \mu_\sigma)(y)\\ &= \int_{\partial E_\sigma} \Phi(\Pi_\sigma(x)+\sigma\nu(\Pi_\sigma(x)),\nu(\Pi_\sigma(x))) d\mu_\sigma(x),\end{aligned}$$

which implies (6.4) because $\nu(\Pi_\sigma(x)) = \nu^\sigma(x)$ and $\Pi_\sigma(x) + \sigma\nu(\Pi_\sigma(x)) = x$.

In view of (6.4), to complete the proof, it only remains to show that the (f_σ)'s converge, as $\sigma \to 0$, to 1 in $L^1(\partial E, \mathcal{H}^{N-1})$.

Applying (6.4) with $\Phi = 1$ gives

$$\mathrm{Per}(E_\sigma) = \int_{\partial E} f_\sigma(y) d\mathcal{H}^{N-1}(y).$$

Since $f_\sigma \leqq \frac{r+\sigma}{r}$ $\mathcal{H}^{N-1}$−almost everywhere in ∂E, the lower-continuity of the perimeter implies that $\lim_{\sigma\to 0} \mathrm{Per}(E_\sigma) = \mathrm{Per}(E)$. Using again the inequality $f_\sigma \leqq r^{-1}r+\sigma$ which holds $\mathcal{H}^{N-1}$−almost everywhere in ∂E, we obtain, in the limit $\sigma \to 0$,

$$\begin{aligned}\int_{\partial E} |1 - f_\sigma(y)| d\mathcal{H}^{N-1}(y) \leqq &\left|1 - \frac{r+\sigma}{r}\right| \mathrm{Per}(\partial E)\\ &+ \int_{\partial E} \left(\frac{r+\sigma}{r} - f_\sigma(y)\right) d\mathcal{H}^{N-1}(y) \to 0.\end{aligned}$$

□

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