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Asymptotic Stability of the Wave Equation on Compact Manifolds and Locally Distributed Damping: A Sharp Result

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Abstract

Let $(M, \mathbf{g})$ be a n-dimensional ($n \geqq 2$) compact Riemannian manifold with boundary where $\mathbf{g}$ denotes a Riemannian metric of class C^∞. This paper is concerned with the study of the wave equation on $(M, \mathbf{g})$ with locally distributed damping, described by

$$u_{tt} - \Delta_{\mathbf{g}} u + a(x)\, g(u_t) = 0, \quad \text{on } M \times]0, \infty[\,, u = 0 \text{ on } \partial M \times]0, \infty[\,,$$

where ∂M represents the boundary of M and $a(x)\, g(u_t)$ is the damping term. The main goal of the present manuscript is to generalize our previous result in Cavalcanti et al. (Trans AMS 361(9), 4561–4580, 2009), treating the conjecture in a more general setting and extending the result for n-dimensional compact Riemannian manifolds $(M, \mathbf{g})$ with boundary in two ways: (i) by reducing arbitrarily the region $M_* \subset M$ where the dissipative effect lies (this gives us a totally sharp result with respect to the boundary measure and interior measure where the damping is effective); (ii) by controlling the existence of subsets on the manifold that can be left without any dissipative mechanism, namely, *a precise part of radially symmetric subsets*. An analogous result holds for compact Riemannian manifolds without boundary.

1. Introduction

Let $(M, \mathbf{g})$ be an n-dimensional ($n \geqq 2$) compact Riemannian manifold with smooth boundary ∂M where $\mathbf{g}$ denotes a Riemannian metric of class C^∞. We let ν

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denote the outward unit normal vector field along the boundary ∂M. We denote by ∇ the Levi–Civita connection on M and by Δ the Laplace–Beltrami operator on M. This paper addresses uniform stabilization of solutions of the following damped problem:

$$\begin{cases} u_{tt} - \Delta u + a(x)\, g(u_t) = 0 & \text{in } M \times]0, \infty[\,, \\ u = 0 & \text{on } \partial M \times]0, \infty[\,, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & x \in M, \end{cases} \tag{1.1}$$

where $a(x) \geqq a_0 > 0$ on an open proper subset M_* (to be defined later) of M and, in addition, g is a monotonically increasing function such that $k|s| \leqq |g(s)| \leqq K|s|$ for all $|s| \geqq 1$. The term $a(x)g(u_t)$ is the nonlinear damping term.

The results presented here for Riemannian manifolds with boundary can be adapted for Riemannian manifolds without boundary. We explain the technical details later.

1.1. Literature overview

The literature offers a rich body of results regarding the wave equation subject to a locally distributed damping in the Euclidean setting; for instance, see [1,6,9,17,18,20–24,26,29,32] and a long list of references therein. On the other hand, the problem on compact Riemannian manifolds is less developed than the problem on Euclidean spaces. Among the important works, it is worth mentioning [2–5,8,12,14,19,25,30]. RAUCH AND TAYLOR [25] are among the pioneers in investigating the long time behaviour of weak solutions of the Cauchy problem for the linear wave equation on a compact manifold $(M, \mathbf{g})$ without boundary with a dissipative term, which is described by the equation

$$\begin{cases} u_{tt} - \Delta u + 2a(x)\, u_t = 0 & \text{in } M \times]0, \infty[\,, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & x \in M. \end{cases} \tag{1.2}$$

Assuming that a is a bounded nonnegative function on M such that $a \in C^\infty$, we say that the *Rauch–Taylor condition* holds if there exists a time $T_0 > 0$ such that any ray of the geometric optics with length greater than T_0 meets the open set $\{x \in M; a(x) > 0\} \times \mathbb{R}$. In this case it was established by RAUCH AND TAYLOR [25] that the energy

$$E(t) = \frac{1}{2} \int_M \left(|u_t|^2 + |\nabla u|^2 \right) \mathrm{d}x$$

decays exponentially. An analogous result was settled by BARDOS et al. [2] for Riemannian manifolds with boundary. In that work, the authors presented sharp sufficient conditions for the observation, control and stabilization of the linear wave equation on a compact Riemannian manifold $(M, \mathbf{g})$ with boundary. In particular, when one considers the equation

$$\begin{cases} u_{tt} - \Delta u + 2a(x)\, u_t = 0 & \text{in } M \times]0, \infty[\,, \\ u = 0 & \text{on } \partial M \times]0, \infty[\,, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & x \in M, \end{cases}$$

$a \in C^\infty$, and $a(x) > 0$ in some nonempty open subset ω of M, they proved that the exponential decay holds if and only if a similar condition on the ray of geometric optics for a Riemannian manifold with boundary is satisfied. Although the results in [2] are always stated in the framework of boundary damping, the result mentioned above is a direct consequence of their proof (see the appendix of [15]). A canonical example of an open subset ω verifying the geometric control condition is when ω is a neighbourhood of the boundary of a Euclidean domain. A canonical example where the condition on the ray of geometric optics is not satisfied is when there exists a periodic ray of the geometric optic that does not intercept the damping area (for instance, a flat disc with a damping area that does not contain a pair of antipodal boundary points). The intuitive idea behind these kinds of results is that if every ray of geometric optics remains at least a well-defined proportion of time in the damping area during its traveling, then the energy decays exponentially.

Related to Problem (1.2) on compact Riemannian manifolds without boundary, it is worth mentioning the recent result due to Christianson [8]. Assuming that $u^0 = 0$ and, in addition, that $a(x) > 0$ outside a neighbourhood of a closed hyperbolic geodesic γ, he proved the following energy estimate

$$E(t) \leqq Ce^{-t/C}||u^1||^2_{H^\varepsilon(M)}, \quad t \geqq 0,$$

for some $C > 0$ and for all ε. The proofs of the above mentioned works are based on microlocal analysis and, although they are refined in sharpness, it seems that they do not extend to nonlinear problems.

1.2. On previous results and methodology

This work is concerned with answering the following question: Which are the "smallest" open sets M_* such that if the damping is effective on M_*, then the optimal decay rate of the energy holds? In this subsection we present some results in this direction for compact surfaces in $\mathbb{R}^3$ without boundary (See [4,5]) giving typical examples and making remarks about the technical tools we use.

The microlocal analysis does not seem to be suitable for treating nonlinear problems. In order to overcome this limitation, we use the multiplier method. A multiplier we use is given by $\langle q, \nabla u\rangle$, where q is a smooth vector field on M and $\langle\cdot,\cdot\rangle$ stands for the Riemannian metric. If q is "well behaved" in a domain $V \subset M$, V can be left without damping. Well behaved means, in general, satisfying some differential equation on V. Therefore, the existence of such a V and q depends on the geometry of V. Let us give some examples.

Let M be a compact surface without boundary embedded in $\mathbb{R}^3$ and let ν be the outward normal vector field. Consider $x^0 \in \mathbb{R}^3\backslash M$ and the function $m : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $m(x) = x - x^0$. Denote

$$M_1 := \{x \in M; m(x)\cdot\nu(x) > 0\} \;\; M_0 = M\backslash M_1.$$

In [4], the authors of the present manuscript proved that umbilical domains compactly contained in M_0 can be left without damping (See Fig. 1). More generally, if the principal curvatures k_1 and k_2 satisfy $|k_1(x) - k_2(x)| < \varepsilon$ for ε small enough

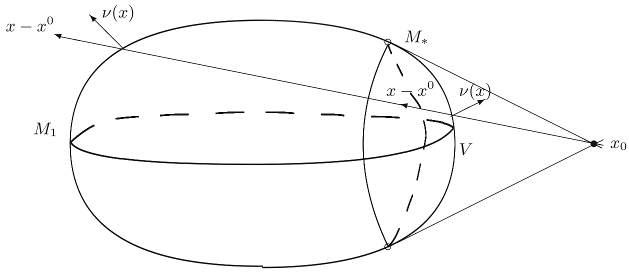

Fig. 1. The observer is at x_0. The subset M_0 is the "visible" part of M and M_1 is its complement. The subset $M_* \supset M \backslash V$ is an open set and the damping is effective there

on V, then V can be left without damping. For this result, the vector field q on V is defined as the projection of the vector field $m(x) := x - x^0$ on the tangent bundle of V, and it is extended smoothly outside V.

In [5] the authors of the present work proved that for every $p \in M$, there exists a neighborhood V sufficiently small that it can be left without damping. The vector field q used here is the gradient of a smooth function f that satisfies $\inf\limits_{x \in V} \nabla f(x) > 0$ and $Hess(f) \approx \mathbf{g}$ (such a function always exists for a sufficiently small neighborhood) on V and it is smoothly extended outside V.

Let $V_1, \ldots, V_k$ be domains that can be left without damping, as described above, with the respective vector fields $q_1, \ldots, q_k$. Suppose that their closures are pairwise disjoint and let $M_* \supset M \backslash \left(\cup_{i=1}^k V_k\right)$ be an open set. If the damping is effective on M_*, that is, if $a(x) \geqq a_0 > 0$ on M_*, then uniform and optimal decay rates of the energy hold. The vector field q defined here is given by q_i on V_i and it is smoothly extended outside $\cup_{i=1}^k V_k$.

As a particular case, observe that if we consider x_1 and x_2 opposite with respect to the center of a sphere and sufficiently far from each other, the damping can be made effective in an arbitrarily small neighborhood of the meridian. This generalizes the result due to BARDOS et al. [2] when $M = S^2$ is a sphere (See Fig. 2).

In [5] the authors also proved that we can choose $V_1, \ldots, V_k$ in such a way that their closures are pairwise disjoint and they cover almost all M. In other words, for every $\varepsilon > 0$, there exist $M_* \supset M \backslash \left(\cup_{i=1}^k V_i\right)$ with $meas(M_*) < \varepsilon$ such that if the damping is effective on M_*, then uniform and optimal decay rates of the energy hold. Although the result is sharp with respect to the volume where the damping acts, we do not have any control about the regions that can be left free of damping. The connected components of V can be extremely small. See Fig. 3.

Figure 4 illustrates a more general case, where small regions and umbilical regions are left without damping. Moreover, the area where the damping is effective can be made arbitrarily small.

The hypothesis of the damping region in terms of the rays of the geometric optics has a close relationship with the hypothesis of the existence of a "nice" vector field q: If the damping is strategically distributed, then the optimal decay rate of the energy holds. In one hand, the results in terms of the ray of the geometric optics are more general than our results for the linear case. But our results also

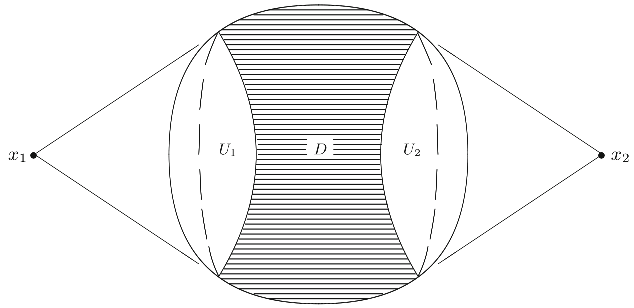

Fig. 2. The observers x_1 and x_2 symmetric with respect to the center of the sphere. When they are positioned sufficiently far from each other, the area where the damping is effective can be made arbitrarily small

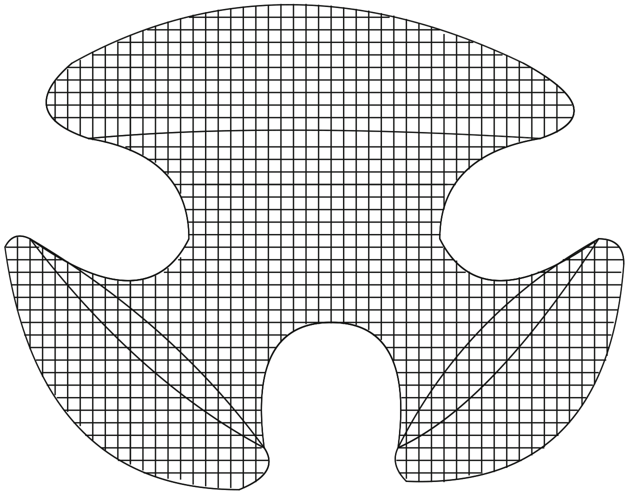

Fig. 3. The non-dissipative area (*in white*) is arbitrarily large while the demarcated area (*in black*) contains dissipative effects and can be considered arbitrarily small, both totally distributed on M

consider the nonlinear case and give explicitly examples of regions that can be left without damping, which can be a difficult task if we use the hypothesis on the ray of geometric optics on a Riemannian manifold.

In our opinion, there are plenty of space left for further studies about the relationships between these two different kind of hypothesis.

1.3. The main goal

The main goal of the present manuscript is to generalize the results presented in subsection 1.2 (See [4,5]) for n-dimensional compact Riemannian manifolds $(M, \mathbf{g})$ with or without boundary. We proceed as follows:

(1) We prove that for every $x \in M$ (including the case $x \in \partial M$), there exists a neighborhood that can be left without damping;

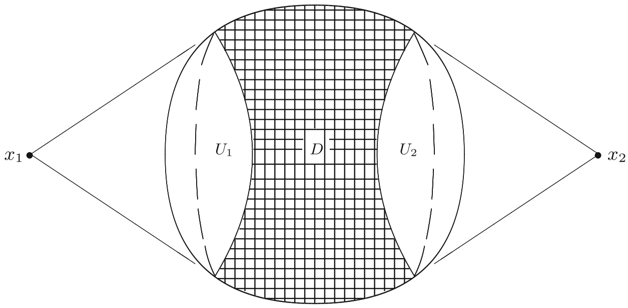

Fig. 4. A compact surface M without boundary is considered such that $U_1 \cup U_2 \cup D$ contains M. While regions U_1 and U_2 are *umbilical* and free of dissipation the region D contains dissipative effects but the superficial measure of the *white part* of it can be considered arbitrarily large and the complementary part (*in black*) possesses superficial measure arbitrarily small. This can always be done for a *finite* number of observers located at $x_1, \dots, x_n$, with corresponding *disjoint umbilical regions* $U_1, \dots, U_n$ and a dissipative area D such that the union $\cup_{i=1}^{n} U_i \cup D$ covers the whole surface M

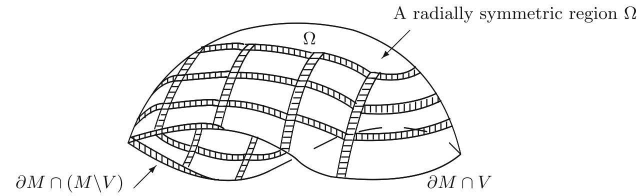

Fig. 5. The *demarcated region* $M \backslash V$ (*in black*) illustrates the damped region on the compact manifold M with boundary ∂M, which can be considered *as small as desired*. Ω is radially symmetric region without damping. The measure of $\partial M \cap (M \backslash V)$ can also be arbitrarily small

(2) We prove that a very precise portion of radially symmetric domains can be left without damping;
(3) Let $\varepsilon > 0$ and $V_1, \dots, V_k$ be domains as in (i) and (ii) which closures are pairwise disjoint. We prove that there exist a $V \supset \cup_{i=1}^{k} V_i$ that can be left without damping and such that $meas(V) \geqq meas(M) - \varepsilon$ and $meas(V \cap \partial M) \geqq meas(\partial M) - \varepsilon$. In particular several radially symmetric domains can be left without damping in a similar way as in Fig. 4.

For this purpose, we will construct an intrinsic multiplier that will play an important role when establishing the desired uniform decay rates of the energy. Fix $\epsilon > 0$. This multiplier is given by $\langle \nabla f, \nabla u \rangle$, where $f : M \to \mathbb{R}$ is a smooth function such that its Hessian $\nabla^2 f$ is closely related to $\mathbf{g}$ on an open subset $V \subset M$ that satisfies $meas(V) \geqq meas(M) - \epsilon$, $meas(V \cap \partial M) \geqq meas(\partial M) - \epsilon$ and $\langle \nabla f, \nu \rangle < 0$ on $V \cap \partial M$. This construction will be clarified during the proof. In addition, because of the radially symmetric region (Fig. 5), we are not sure if the uniqueness results due to TRIGGIANI AND YAO [30] can be employed to our case, so that new unique continuation arguments are required (see Theorem 5.1).

Our presentation will be focused on the problem defined on Riemannian manifolds with smooth boundary, although the result also holds for Riemannian manifolds without boundary. In the further case the initial condition has zero average, and if the solution u of the equation is such that $u(\cdot, t)$ does not have zero average for every t, then we can add an additional term u in the equation (see [5]) and the rest of the proof is analogous.

Our paper is organized as follows. Section 2 is concerned with the statement of the problem. We also introduce some notation and present the main result. In Section 3 we present the preliminaries in Differential Geometry that we need in this work. Sections 4, 5, 6 and 7 are devoted to the proof of the main result.

2. Statement of the problem and the main result

Let $(M, \mathbf{g})$ be an n-dimensional compact Riemannian manifold with boundary. For $\epsilon > 0$, we shall prove that there exist an open subset $V \subset M$ and smooth functions $\alpha, f : M \to \mathbb{R}$ such that $meas(V) \geqq meas(M) - \epsilon/2$, $meas(V \cap \partial M) \geqq meas(\partial M) - \epsilon/2$, $\nabla\alpha|_V \equiv 0$ and such that α and f satisfy

$$C \int_0^T \int_V \left[u_t^2 + |\nabla u|^2\right] \mathrm{d}M\, \mathrm{d}t \leqq \int_0^T \int_V \left(\frac{\Delta f}{2} - \alpha\right) u_t^2\, \mathrm{d}M\, \mathrm{d}t$$
$$+ \int_0^T \int_V \left[\nabla^2 f(\nabla u, \nabla u) + \left(\alpha - \frac{\Delta f}{2}\right)|\nabla u|^2\right] \mathrm{d}M\, \mathrm{d}t,$$

for some positive constant C and, furthermore,

$$\langle \nabla f, \nu \rangle < 0 \text{ on } \partial M \cap V.$$

Moreover if $V_1, \ldots, V_k$ are radially symmetric to be presented in Lemma 6.2 with pairwise disjoint closures, we can choose V in such a way that $V \supset (\cup_{i=1}^k V_i)$. In what follows, M_* will be an open set containing $M \backslash V$ and satisfying $meas(M_*) < \epsilon$.

In this paper, we investigate the stability properties of function $u(x, t)$ which solves the following damped problem:

$$\begin{cases} u_{tt} - \Delta u + a(x)\, g(u_t) = 0 & \text{on } M \times]0, \infty[\,, \\ u = 0 & \text{on } \partial M \times]0, \infty[\\ u(0) = u^0, \quad u_t(0) = u^1, & x \in M, \end{cases} \tag{2.1}$$

where the feedback function g satisfies the following assumptions:

Assumption 2.1. (i) $g(s)$ is continuous and monotone increasing,
(ii) $g(s)\, s > 0$ for $s \neq 0$,
(iii) $k\,|s| \leqq g(s) \leqq K\,|s|$ for $|s| > 1$,

where k and K are two positive constants.

In addition, to obtain the stabilization of problem (2.1), we shall need the following geometrical assumption:

Assumption 2.2. We assume that $a \in L^\infty(M)$ is a nonnegative function such that

$$a(x) \geqq a_0 > 0, \quad \text{almost everywhere on } M_*, \tag{2.2}$$

where M_* is an open set of M that contains $M \backslash V$.

In the sequel we define $\Sigma = M \times]0, T[$ and we set

$$H_0^1(M) := \{v \in H^1(M); v|_{\partial M} = 0\},$$

which is a Hilbert space with the topology endowed by $H^1(M)$. A summary of the geometric tools necessary is presented in Section 3, including Sobolev spaces on Riemannian manifolds. For details about this subject we refer the reader to Taylor's book [28].

The condition $v|_{\partial M} = 0$ is required in order to guarantee the validity of the Poincaré inequality,

$$||h||^2_{L^2(M)} \leqq (\lambda_1)^{-1} ||\nabla h||^2_{L^2(M)}, \quad \text{for all } h \in H_0^1(M), \tag{2.3}$$

where λ_1 is the first eigenvalue of the Laplace–Beltrami operator for the Dirichlet problem.

We observe that the problem (2.1) can be written in the following form

$$\frac{\mathrm{d}U}{\mathrm{d}t} + AU = G(U),$$

where

$$A = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}$$

is a maximal monotone operator and $G(\cdot)$ represents a locally Lipschitz perturbation. So, making use of standard semigroup arguments we have the following result:

Theorem 2.1. (i) *Under the conditions above, problem* (2.1) *is well posed in the space $H_0^1(M) \times L^2(M)$, that is, for any initial data $\{u^0, u^1\} \in H_0^1(M) \times L^2(M)$, there exists a unique weak solution of* (2.1) *in the class*

$$u \in C(\mathbb{R}_+; H_0^1(M)) \cap C^1(\mathbb{R}_+; L^2(M)). \tag{2.4}$$

(ii) *In addition, the velocity term of the solution has the following regularity*:

$$u_t \in L^2_{loc}\big(\mathbb{R}_+; L^2\,(M)\big), \tag{2.5}$$

(consequently, $g\,(u_t) \in L^2_{loc}\,\big(\mathbb{R}_+; L^2\,(M)\big)$ by Assumption 2.1.

Furthermore, if $\{u^0, u^1\} \in \{H_0^1(M) \cap H^2\,(M) \times H_0^1(M)\}$ then, the solution has the following regularity

$$u \in L^\infty\big(\mathbb{R}_+; H_0^1(M) \cap H^2\,(M)\big) \cap W^{1,\infty}\big(\mathbb{R}_+; H_0^1(M)\big) \cap W^{2,\infty}\big(\mathbb{R}_+; L^2\,(M)\big).$$

Supposing that u is the unique global weak solution of problem (2.1), we define the corresponding energy functional by

$$E(t) = \frac{1}{2}\int_M \left[|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right] \mathrm{d}M. \tag{2.6}$$

For every solution of (2.1) in the class (2.4) the following identity holds

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2}\int_M a(x)\, g(u_t)u_t \,\mathrm{d}M\,\mathrm{d}t, \quad \text{for all } t_2 > t_1 \geqq 0, \tag{2.7}$$

and therefore the energy is a non-increasing function of the time variable t.

Before stating our stability result, we will define some needed functions. For this purpose, we are following the ideas first introduced in Lasiecka and Tataru [13]. For the reader's comprehension we will repeat them briefly. Let h be a concave, strictly increasing function, with $h(0) = 0$, and such that

$$h\,(s\,g(s))) \geqq s^2 + g^2(s), \ \text{ for } |s| \leqq 1. \tag{2.8}$$

Note that such a function can be straightforwardly constructed, given the hypotheses on g in Assumption 2.1. With this function, we define

$$\beta(.) = h\left(\frac{.}{meas\,(\Sigma_1)}\right). \tag{2.9}$$

As β is monotone increasing, then $cI+\beta$ is invertible for all $c \geqq 0$. For L a positive constant, we set

$$p(x) = (cI+\beta)^{-1}(Lx)\,, \tag{2.10}$$

where the function p is easily seen to be positive, continuous and strictly increasing with $p(0) = 0$. Finally, let

$$q(x) = x - (I+p)^{-1}(x)\,. \tag{2.11}$$

We can now proceed to state our stability result.

Theorem 2.2. *Assume that Assumptions 2.1 and Assumption 2.2 are in place. Let u be the weak solution of the problem (2.1). With the energy $E(t)$ defined as in (2.6), there then exists a $T_0 > 0$ such that*

$$E(t) \leqq S\left(\frac{t}{T_0} - 1\right), \quad \forall t > T_0, \tag{2.12}$$

with $\lim_{t\to\infty} S(t) = 0$, where the contraction semigroup $S(t)$ is the solution of the differential equation

$$\frac{d}{\mathrm{d}t}S(t) + q(S(t)) = 0, \quad S(0) = E(0), \tag{2.13}$$

(where q is as given in (2.11)). Here, the constant L (from definition (2.10)) will depend on $meas(\Sigma)$, and the constant c (from definition (2.10)) is taken here to be $c \equiv \frac{k^{-1}+K}{meas(\Sigma)(1+||a||_\infty)}$.

Remark 2.1. If the feedback is linear, for example, $g(s) = s$, then, under the same assumptions as in Theorem 2.2, we have that the energy of problem (2.1) decays exponentially with respect to the initial energy, that is, there exist two positive constants $C > 0$ and $k > 0$ such that

$$E(t) \leqq Ce^{-kt} E(0), \qquad t > 0. \tag{2.14}$$

As another example, we can consider $g(s) = s^{\ell}$, $\ell > 1$ at the origin. Since the function $s^{\frac{\ell+1}{2}}$ is convex for $\ell \geqq 1$, then solving

$$S_t + S^{\frac{\ell+1}{2}} = 0, \tag{2.15}$$

we obtain the following polynomial decay rate:

$$E(t) \leqq C(E(0))[E(0)^{\frac{-\ell+1}{2}} + t(\ell - 1)]^{\frac{2}{-\ell+1}}.$$

We can find more interesting explicit decay rates in CAVALCANTI et al. [7].

3. Preliminaries in differential geometry

The theory regarding differential calculus of tensor fields on Riemannian manifolds can be found in [27].

Let $(M, \mathbf{g})$ be an n-dimensional, $n \geqq 2$, compact Riemannian manifold, with smooth boundary or without boundary, with smooth metric $\mathbf{g}(\cdot, \cdot) = \langle \cdot, \cdot \rangle$ and norm $|\cdot|$. The tangent space of M at x is denoted by $T_x M$. Fix a coordinate system $(x_1, \ldots, x_n)$ and let $(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$ be the coordinate vector fields. If $\mathbf{g}_{ij} = \left\langle \partial/\partial x_i, \partial/\partial x_j \right\rangle$, we have that

$$\mathbf{g}(X, Y) = \langle X, Y \rangle = \sum_{i,j=1}^{n} \mathbf{g}_{ij} b_i c_j, \quad X = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i} \in T_x M, \tag{3.16}$$

$$|X| = (\mathbf{g}(X, X))^{\frac{1}{2}}. \tag{3.17}$$

Let $T_x^* M$ be the space of linear forms on $T_x M$. The Riemannian metric induces natural isomorphism $\iota : T_x M \to T_x^* M$ given by $v \to \langle v, \cdot \rangle$. For $v \in T_x M$ we denote $v^\flat := \iota(v)$ and similarly for $\varphi \in T_x^* M$ we denote $\varphi^\sharp := \iota^{-1}(\varphi)$. ι and ι^{-1} are called musical isomorphisms.

Let $T_x^{m,s} M$ be the space of tensors of type (m, s) on $T_x M$. If $m = 0$, then we simply denote $T_x^s M := T_x^{0,s} M$. The musical isomorphisms allow us to identify $T_x^{m,s} M$ and $T_x^{m+s} M$ in the following fashion: $\Psi \in T_x^{m,s} M$ is identified with $\tilde{\Psi} \in T_x^{m+s} M$ which is defined as

$$\tilde{\Psi}(v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+s}) = \Psi(v_1^\flat, \ldots, v_m^\flat, v_{m+1}, \ldots, v_{m+s}).$$

Denote the tangent bundle of M by TM, the cotangent bundle of M by T^*M and the tensor bundle of type (m, s) by $T^{m,s}M$. Let ∇ denote the *Levi–Civita* connection of M. Consider a vector field X on M. ∇_X is a differential operator that, when operated on a C^k, $k \geqq 1$, tensor field of type (m, s), gives a C^{k-1} tensor field of type (m, s). If f is a C^1 function on M (tensor field of type $(0, 0)$), then $\nabla_X f := \mathrm{d}f(X) = X(f)$. If Y is another vector field on M, then $\nabla_X Y$ is the covariant derivative of Y with respect to X. Other covariant derivatives are defined in such a way that the "product rule" holds. If φ is a one-form on M, then $\nabla_X \varphi$ is defined as

$$(\nabla_X \varphi)(Y) := X(\varphi(Y)) - \varphi(\nabla_X Y).$$

It is not difficult to prove that $\nabla_X \varphi$ is well defined, that is, if $x \in M$ then $[(\nabla_X \varphi)(x)](Y)$ depends only on $Y(x)$. If Ψ is a tensor field of type (m, s) on M, $\varphi_1, \ldots, \varphi_m$ are one-forms on M and $X, Y_1, \ldots, Y_s$ are vector fields on M, then $\nabla_X \Psi$ is defined as

$$\begin{aligned}
&(\nabla_X \Psi)(\varphi_1, \ldots, \varphi_m, Y_1, \ldots, Y_s) := X(\Psi(\varphi_1, \ldots, \varphi_m, Y_1, \ldots, Y_s))\\
&-\Psi(\nabla_X \varphi_1, \ldots, \varphi_m, Y_1, \ldots, Y_s) - \Psi(\varphi_1, \nabla_X \varphi_2, \ldots, \varphi_m, Y_1, \ldots, Y_s) \ldots\\
&\ldots - \Psi(\varphi_1, \ldots, \nabla_X \varphi_m, Y_1, \ldots, Y_s) - \Psi(\varphi_1, \ldots, \varphi_m, \nabla_X Y_1, \ldots, Y_s) \ldots\\
&\ldots - \Psi(\varphi_1, \ldots, \varphi_m, Y_1, \ldots, \nabla_X Y_s).
\end{aligned}$$

Likewise, it is not difficult to prove that $(\nabla_X \Psi)(x) : T_x^{m,s} M \to T_x^{m,s} M$ is well defined.

Let Ψ, Ψ_1 and Ψ_2 be tensor fields of type (m, s) on M, $c \in \mathbb{R}$ and X a vector field on M. Then covariant derivatives have the following properties:

(1) $\nabla_X (c\Psi_1 + \Psi_2) = c\nabla_X \Psi_1 + \nabla_X \Psi_2$.
(2) $\nabla_X (T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2)$

where $\otimes$ denotes the tensor product.

∇ is also a differential operator that operates on C^k, $k \geqq 1$, tensor fields of type (m, s) on M, and gives as a result a C^{k-1} tensor field of type $(m, s+1)$. If Ψ is a tensor field of type (m, s) on M, then $\nabla \Psi$ is called the *covariant differential of* Ψ and it is defined as

$$(\nabla \Psi)(\varphi_1, \ldots, \varphi_m, X, Y_1, \ldots, Y_s) = (\nabla_X \Psi)(\varphi_1, \ldots, \varphi_m, Y_1, \ldots, Y_s)$$

where $\varphi_1, \ldots, \varphi_m$ are one-forms on M and $X, Y_1, \ldots, Y_s$ are vector fields on M. It is not difficult to see that $\nabla \Psi$ is a well defined tensor field of type $(m, s+1)$.

Let Ψ a tensor field of type $(0, s)$ on M. The divergent $\mathrm{div}\Psi$ of Ψ is a tensor field of type $(0, s-1)$ defined as

$$(\mathrm{div}\Psi)(v_2, \ldots, v_s)(x) := \sum_{i=1}^{n} (\nabla_{e_i} \Psi)(e_i, v_2, \ldots, v_s)(x)$$

where $x \in M$ and $(e_1, \ldots, e_n)$ is an orthonormal basis of $T_x M$.

Let us examine some important examples of differential operators. If f is a function of class C^1 on M, then ∇f is a one-form defined as

$$\nabla f(X) = \nabla_X f = X(f) = \mathrm{d}f(X) = \left\langle (\nabla f)^\sharp, X \right\rangle.$$

The usual gradient of f can be identified with ∇f because $(\nabla f)^\sharp = \text{grad} f$. In what follows, we denote the gradient of f by ∇f if there is no possibility of misunderstanding.

Another important tensor field is the Hessian of a C^2 function on M. It is defined as $\nabla^2 f := \nabla(\nabla f)$. It is well known that $\nabla^2 f$ is a symmetric tensor field of type $(0, 2)$ on M. Notice that

$$\nabla^2 f(X, Y) = \langle \nabla_Y(\nabla f), X\rangle, \quad \text{for all } X, Y \in T_x M,\ x \in M. \tag{3.18}$$

The divergent of a vector field Y is defined as $\text{div}X := \text{div}X^\flat$. If $x \in M$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_x M$, then

$$\text{div}X = \langle \nabla_{e_i} X, e_i\rangle. \tag{3.19}$$

If f is a C^1 function defined on M, then

$$\text{div}(fX) = f\,\text{div}X + X(f). \tag{3.20}$$

The Laplacian Δf of a C^2 function f on M is defined as

$$\Delta f = \text{div}\nabla f. \tag{3.21}$$

Δ is the *Laplace–Beltrami operator*.

If H, X, Y are vector fields on M, then

$$\nabla H(X, Y) := \nabla H^\flat(X, Y) = \langle \nabla_X H, Y\rangle, \quad \text{for all } X, Y \in T_x M,\ x \in M. \tag{3.22}$$

For any function f and vector field H on M, the following identity holds on each $x \in M$ (see [14, p. 22])

$$\langle \nabla f, \nabla(H(f))\rangle = \nabla H(\nabla f, \nabla f) + \frac{1}{2}\Big[\text{div}(|\nabla f|^2 H) - |\nabla f|^2 \text{div}H\Big]. \tag{3.23}$$

In what follows we shall denote by $\chi(M)$ the set of all smooth vector fields on M. Analogously, we will denote by $\Gamma^\infty(T^k M)$ the set of all smooth tensor fields of type $(0, k)$.

For each $x \in M$, $T_x^k M$ is an inner product space defined as follows. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_x M$. For any $\phi_1, \phi_2 \in T_x^k M$, $x \in M$, the inner product is given by

$$\langle \phi_1, \phi_2\rangle_{T_x^k M} = \sum_{i_1,\ldots,i_k=1}^{n} \phi_1(e_{i_1}, \ldots, e_{i_k})\phi_2(e_{i_1}, \ldots, e_{i_k}). \tag{3.24}$$

In particular, for $k = 1$, we have $\langle \phi_1, \phi_2\rangle_{T_x^1 M} = \mathbf{g}(\phi_1^\sharp, \phi_2^\sharp)$ for all $\phi_1, \phi_2 \in T_x^1 M$.

In view of (3.24), $\Gamma^\infty(T^k M)$ are inner product spaces endowed with the following inner product

$$\langle \psi_1, \psi_2\rangle_{\Gamma^\infty(T^k M)} = \int_M \langle \psi_1, \psi_2\rangle_{T^k M}\ \mathrm{d}M, \quad \psi_1, \psi_2 \in \Gamma^\infty(T^k M), \tag{3.25}$$

where $\mathrm{d}M$ is the volume element of M in the metric $\mathbf{g}$. We denote by $L^2(M, \Gamma^\infty(T^kM))$ the completions of $\Gamma^\infty(T^kM)$ in the inner product given by (3.25). In addition, $L^2(M)$ is the completion of $C^\infty(M)$ with the usual inner product

$$(f_1, f_2)_{L^2(M)} = \int_M f_1(x) f_2(x)\,\mathrm{d}x, \quad f_1, f_2 \in C^\infty(M). \tag{3.26}$$

The Sobolev space $H^k(M)$ is the completion of $C^\infty(M)$ with respect to the norm $||\cdot||$,

$$||f||^2_{H^k(M)} = \sum_{i=1}^{k} ||\nabla^i f||^2_{L^2(M,\Gamma^\infty(T^iM))} + ||f||^2_{L^2(M)}, \quad f \in C^\infty, \tag{3.27}$$

where $\nabla^i f$ is the i^{th} covariant differential of f in the metric $\mathbf{g}$ and $||\cdot||_{L^2(M,\Gamma^\infty(T^iM))}$ are the corresponding norms, induced by the inner products (3.24) and (3.25). For details on Sobolev spaces on Riemannian manifolds, we refer the reader to TAYLOR [28].

Remark 3.1. In order to simplify the notation, we denote the L^2-norm, without distinguishing whether the argument of the norm is a function or tensor field of type $(0, k)$.

We collect below a few formulas to be invoked in the sequel (See [28]).
Divergence or Gauss theorem: If $X \in H^1(M, \chi(M))$ and ν is the outward normal vector field of ∂M, then

$$\int_M \mathrm{div} X \;\mathrm{d}M = \int_{\partial M} \langle X, \nu\rangle \;\mathrm{d}\partial M. \tag{3.28}$$

Green's Theorem 1: If $H \in H^1(M, \chi(M))$ and $q \in H^1(M)$ then,

$$\int_M (\mathrm{div} H) q \;\mathrm{d}M = -\int_M \langle H, \nabla q\rangle \;\mathrm{d}M + \int_{\partial M} (\langle H, \nu\rangle) q \;\mathrm{d}\partial M. \tag{3.29}$$

Green's Theorem 2: If $f \in H^2(M)$ and $q \in H^1(M)$, then,

$$\int_M (\Delta f) q \;\mathrm{d}M = -\int_M \langle \nabla f, \nabla q\rangle \;\mathrm{d}M + \int_{\partial M} (\partial_\nu f) q \;\mathrm{d}\partial M. \tag{3.30}$$

We conclude this subsection by presenting some remarks about differentiable manifolds with boundary.

Let M be a differentiable manifold with boundary (Fig. 6). We introduce some notation and conventions here.

The interior of M will be denoted by $\mathrm{int}(M)$ and the boundary of M by ∂M. An open set $V \subset M$ is an open set of the topological space M. Therefore it can intercept the boundary. The boundary ∂V of V can be written as the disjoint union $\partial V = \partial_1 V \cup \partial_2 V$ where $\partial_1 V := \partial V \cap \mathrm{int}(M)$ and $\partial_2 V := \partial V \cap \partial M$.

We say that an open subset $V \subset M$ has smooth boundary $\overline{\partial_1 V}$ if $\overline{\partial_1 V}$ is a smooth hypersurface of M with smooth boundary $\overline{\partial_1 V} \cap \partial M$. Therefore the term "smooth"

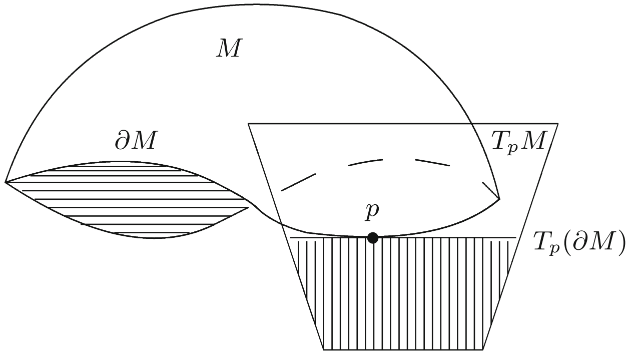

Fig. 6. A compact manifold M with boundary ∂M

ignores $\partial_2 V$. A typical example of an open subset of $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n;\, x_n \geqq 0\}$ with smooth boundary is $V := \{(x_1, \ldots, x_n) \in \mathbb{R}^n;\, x_{n-1} \geqq 0,\, x_n \geqq 0\}$.

We state a classical result which can be proven using the existence of a tubular neighborhood for some submanifolds.

Proposition 3.1. *Let M be a differentiable manifold with boundary. Suppose that $V \subset M$ is an open subset with smooth boundary $\overline{\partial_1 V}$ which intercepts ∂M transversally. Let M_* be an arbitrary open subset of M such that $M_* \supset M \backslash V$. Then there exists an open subset $W \subset M$ with smooth boundary $\overline{\partial_1 W}$ which intercepts ∂M transversally such that $M \backslash V \subset\subset W \subset\subset M_*$.*

From now on, and in order to obtain the decay rate estimate given in (2.12), we will work with regular solutions of problem (2.1). So, by using standard arguments of density, the same decay rate estimate remains true for weak solutions.

4. A fundamental identity

This section is devoted to proving (4.37) and to explaining how it is used to determine V.

Proposition 4.1. *Let $(M, \mathbf{g})$ be a n-dimensional compact manifold and H a vector field of class C^1. Then for every regular solution u of problem* (1.1) *we have the following identity*

$$\left[\int_M u_t\, \langle H, \nabla u\rangle\, \mathrm{d}M\right]_0^T + \frac{1}{2}\int_0^T \int_M (divH) \left\{|u_t|^2 - |\nabla u|^2\right\} \mathrm{d}M\, \mathrm{d}t$$
$$+ \int_0^T \int_M \nabla H(\nabla u, \nabla u)\, \mathrm{d}M\, \mathrm{d}t + \int_0^T \int_M a(x)\, g(u_t)\, \langle H, \nabla u\rangle\, \mathrm{d}M\, \mathrm{d}t$$
$$= \int_0^T \int_{\partial M} \partial_\nu u\, \langle H, \nabla u\rangle\, \mathrm{d}\partial M\, \mathrm{d}t + \frac{1}{2}\int_0^T \int_{\partial M} \langle H, \nu\rangle \left[u_t^2 - |\nabla u|^2\right]\, \mathrm{d}\partial M\, \mathrm{d}t. \tag{4.31}$$

Proof. Multiplying the equation of (1.1) by the multiplier $H(u) = \langle \nabla u, H \rangle$ and integrating on $M \times]0, T[$, we obtain

$$0 = \int_0^T \int_M (u_{tt} - \Delta u + a(x) g(u_t)) \langle H, \nabla u \rangle \, \mathrm{d}M \, \mathrm{d}t. \tag{4.32}$$

Next, we will estimate some terms on the right-hand side of identity (4.32). Taking (3.20), (3.23), (3.28), (3.29) and (3.30) into account, we obtain

$$\begin{aligned}
&\int_0^T \int_M (-\Delta u) \langle H, \nabla u \rangle \, \mathrm{d}M \, \mathrm{d}t \\
&= \int_0^T \int_M \langle \nabla u, \nabla(\langle H, \nabla u \rangle) \rangle \, \mathrm{d}M \, \mathrm{d}t - \int_0^T \int_{\partial M} \partial_\nu u \, \langle H, \nabla u \rangle \, \mathrm{d}\partial M \, \mathrm{d}t \\
&= \int_0^T \int_M \nabla H(\nabla u, \nabla u) \, \mathrm{d}M \, \mathrm{d}t \\
&\quad + \int_0^T \int_M \left[\frac{1}{2} \mathrm{div}(|\nabla u|^2 H) - \frac{1}{2} \mathrm{div} H |\nabla u|^2 \right] \mathrm{d}M \, \mathrm{d}t \\
&\quad - \int_0^T \int_{\partial M} \partial_\nu u \, \langle H, \nabla u \rangle \, \mathrm{d}\partial M \, \mathrm{d}t \\
&= \int_0^T \int_M \nabla H(\nabla u, \nabla u) \, \mathrm{d}M \, \mathrm{d}t + \int_0^T \int_M \frac{1}{2} \Big\langle H, \nabla \big[|\nabla u|^2 \big] \Big\rangle \mathrm{d}M \, \mathrm{d}t \\
&\quad - \int_0^T \int_{\partial M} \partial_\nu u \, \langle H, \nabla u \rangle \, \mathrm{d}\partial M \, \mathrm{d}t \\
&= \int_0^T \int_M \nabla H(\nabla u, \nabla u) \, \mathrm{d}M \, \mathrm{d}t - \frac{1}{2} \int_0^T \int_M \mathrm{div} H |\nabla u|^2 \mathrm{d}M \, \mathrm{d}t \\
&\quad - \int_0^T \int_{\partial M} \partial_\nu u \, \langle H, \nabla u \rangle \, \mathrm{d}\partial M \, \mathrm{d}t + \frac{1}{2} \int_0^T \int_{\partial M} \langle H, \nu \rangle \, |\nabla u|^2 \, \mathrm{d}\partial M \, \mathrm{d}t
\end{aligned} \tag{4.33}$$

and, integrating by parts and considering (3.20) and (3.29), we obtain

$$\begin{aligned}
&\int_0^T \int_M (u_{tt} + a(x) \, g(u_t)) \langle H, \nabla u \rangle \, \mathrm{d}M \, \mathrm{d}t \\
&= \left[\int_M u_t \, \langle H, \nabla u \rangle \right]_0^T - \int_0^T \int_M u_t \, \langle H, \nabla u_t \rangle \, \mathrm{d}M \, \mathrm{d}t \\
&\quad + \int_0^T \int_M a(x) \, g\,(u_t) \, \langle H, \nabla u \rangle \, \mathrm{d}M \, \mathrm{d}t \\
&= \left[\int_M u_t \, \langle H, \nabla u \rangle \right]_0^T - \frac{1}{2} \int_0^T \int_M \Big\langle H, \nabla (u_t^2) \Big\rangle \mathrm{d}M \, \mathrm{d}t \\
&\quad + \int_0^T \int_M a(x) \, g\,(u_t) \, \langle H, \nabla u \rangle \, \mathrm{d}M \, \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \left[\int_M u_t \, \langle H, \nabla u \rangle \right]_0^T + \frac{1}{2}\int_0^T \int_M (\mathrm{div} H) u_t^2 \, \mathrm{d}M \, \mathrm{d}t \\
&\quad + \int_0^T \int_M a(x)\, g\,(u_t)\, \langle H, \nabla u \rangle \, \mathrm{d}M \, \mathrm{d}t - \frac{1}{2}\int_0^T \int_{\partial M} \langle H, \nu \rangle \,(u_t)^2 \, \mathrm{d}\partial M \, \mathrm{d}t.
\end{aligned}
\tag{4.34}
$$

Combining (4.32), (4.33) and (4.34), we deduce (4.31), which concludes the proof of Proposition 4.1 □

Employing (4.31) with $H = \nabla f$ where $f : M \to \mathbb{R}$ is a C^∞ function to be determined later, from (3.22) and (3.18), we infer

$$
\begin{aligned}
&\left[\int_M u_t \, \langle \nabla f, \nabla u \rangle \, \mathrm{d}M \right]_0^T + \frac{1}{2}\int_0^T \int_M \Delta f \left\{ u_t^2 - |\nabla u|^2 \right\} \mathrm{d}M \, \mathrm{d}t \\
&\quad + \int_0^T \int_M \nabla^2 f(\nabla u, \nabla u) \, \mathrm{d}M \, \mathrm{d}t + \int_0^T \int_M a(x)\, g(u_t)\, \langle \nabla f, \nabla u \rangle \, \mathrm{d}M \, \mathrm{d}t \\
&= \int_0^T \int_{\partial M} \partial_\nu u \, \langle \nabla f, \nabla u \rangle \, \mathrm{d}\partial M \, \mathrm{d}t + \frac{1}{2}\int_0^T \int_{\partial M} \underbrace{\langle \nabla f, \nu \rangle}_{=\partial_\nu f} \left[u_t^2 - |\nabla u|^2\right] \mathrm{d}\partial M \, \mathrm{d}t.
\end{aligned}
\tag{4.35}
$$

We have the following identity:

Lemma 4.2. *Let u be a regular solution to problem (1.1) and $\alpha \in C^1(M)$. Then*

$$
\begin{aligned}
\left[\int_M u_t \, \alpha \, u \, \mathrm{d}M \right]_0^T &= \int_0^T \int_M \alpha \, u_t^2 \, \mathrm{d}M \, \mathrm{d}t - \int_0^T \int_M \alpha \, |\nabla u|^2 \, \mathrm{d}M \, \mathrm{d}t \\
&\quad - \int_0^T \int_M \langle \nabla u, \nabla \alpha \rangle \, u \, \mathrm{d}M \, \mathrm{d}t - \int_0^T \int_M a(x)\, g(u_t)\, \alpha \, u \, \mathrm{d}M \, \mathrm{d}t \\
&\quad + \int_0^T \int_{\partial M} \partial_\nu u \, \alpha \, u \, d\partial M \, \mathrm{d}t.
\end{aligned}
\tag{4.36}
$$

Proof. Multiplying the first equation of (1.1) by $\alpha \, u$ and integrating by parts we obtain the desired result. □

Combining (4.36) and (4.35), we deduce

$$
\begin{aligned}
&\int_0^T \int_M \left(\frac{\Delta f}{2} - \alpha\right) u_t^2 \mathrm{d}M \, \mathrm{d}t \\
&\quad + \int_0^T \int_M \left[\nabla^2 f(\nabla u, \nabla u) + \left(\alpha - \frac{\Delta f}{2}\right) |\nabla u|^2 \right] \mathrm{d}M \, \mathrm{d}t \\
&= - \left[\int_M u_t \, \langle \nabla f, \nabla u \rangle \, \mathrm{d}M \right]_0^T - \left[\int_M u_t \, \alpha \, u \, \mathrm{d}M \right]_0^T \\
&\quad - \int_0^T \int_M a(x)\, g(u_t)\, \alpha \, u \, \mathrm{d}M \, \mathrm{d}t - \int_0^T \int_M a(x)\, g(u_t)\, \langle \nabla f, \nabla u \rangle \, \mathrm{d}M \, \mathrm{d}t
\end{aligned}
$$

$$+\int_0^T \int_{\partial M} \partial_\nu u \, \langle \nabla f, \nabla u \rangle \, \mathrm{d}\partial M \ \mathrm{d}t + \frac{1}{2} \int_0^T \int_{\partial M} \langle \nabla f, \nu \rangle \left[u_t^2 - |\nabla u|^2 \right] \mathrm{d}\partial M \ \mathrm{d}t$$
$$-\int_0^T \int_M \langle \nabla u, \nabla \alpha \rangle \, u \ \mathrm{d}M \, \mathrm{d}t + \int_0^T \int_{\partial M} \partial_\nu u \, \alpha \underbrace{u}_{=0} \ \mathrm{d}\partial M \ \mathrm{d}t. \tag{4.37}$$

We observe that since we are working with regular solutions, then, $u_t = 0$ on ∂M.

Now we use the fact that $u|_{\partial M} = 0$ (and therefore $\nabla u = \langle \nabla u, \nu \rangle \, \nu = (\partial_\nu u)\nu$) in (4.37) in order to get

$$\begin{aligned}
&\int_0^T \int_M \left(\frac{\Delta f}{2} - \alpha \right) u_t^2 \mathrm{d}M \, \mathrm{d}t \\
&\quad + \int_0^T \int_M \left[\nabla^2 f(\nabla u, \nabla u) + \left(\alpha - \frac{\Delta f}{2} \right) |\nabla u|^2 \right] \mathrm{d}M \, \mathrm{d}t \\
&= - \left[\int_M u_t \ \langle \nabla f, \nabla u \rangle \ \mathrm{d}M \right]_0^T - \left[\int_M u_t \, \alpha \, u \, \mathrm{d}M \right]_0^T \\
&\quad - \int_0^T \int_M a(x) \, g(u_t) \, \alpha \, u \, \mathrm{d}M \, \mathrm{d}t - \int_0^T \int_M a(x) \, g(u_t) \, \langle \nabla f, \nabla u \rangle \ \mathrm{d}M \, \mathrm{d}t \\
&\quad + \frac{1}{2} \int_0^T \int_{\partial M} \langle \nabla f, \nu \rangle \, |\nabla u|^2 \ \mathrm{d}\partial M \ \mathrm{d}t - \int_0^T \int_M \langle \nabla u, \nabla \alpha \rangle \, u \ \mathrm{d}M \, \mathrm{d}t.
\end{aligned} \tag{4.38}$$

Remark 4.1. This is the precise moment in which the properties of the function f will play an important role. Note that all we need is to find an open subset $V \subset M$ with smooth boundary $\overline{\partial_1 V}$ which intercepts ∂M transversally and smooth functions $\alpha, f : M \to \mathbb{R}$ such that $\nabla \alpha|_V \equiv 0$ and

$$\begin{aligned}
C \int_0^T \int_V \left[u_t^2 + |\nabla u|^2 \right] \mathrm{d}M \, \mathrm{d}t &\leqq \int_0^T \int_V \left(\frac{\Delta f}{2} - \alpha \right) u_t^2 \, \mathrm{d}M \, \mathrm{d}t \\
&\quad + \int_0^T \int_V \left[\nabla^2 f(\nabla u, \nabla u) + \left(\alpha - \frac{\Delta f}{2} \right) |\nabla u|^2 \right] \mathrm{d}M \, \mathrm{d}t,
\end{aligned} \tag{4.39}$$

for some positive constants C and α and

$$\langle \nabla f, \nu \rangle < 0 \quad \text{on } \partial M \cap V. \tag{4.40}$$

Assuming, for a moment, that (4.39) and (4.40) hold, from (4.38) yields

$$\begin{aligned}
2C \int_0^T E(t) \, \mathrm{d}t \leqq C^* &\int_0^T \int_{M \setminus V} \left[u_t^2 + |\nabla u|^2 \right] \mathrm{d}M \, \mathrm{d}t \\
&+ \left| \left[\int_M u_t \ \langle \nabla f, \nabla u \rangle \ \mathrm{d}M \right]_0^T \right| + \left| \left[\int_M u_t \, \alpha \, u \, \mathrm{d}M \right]_0^T \right| \\
&+ \left| \int_0^T \int_M a(x) \, g(u_t) \, \alpha \, u \, \mathrm{d}M \, \mathrm{d}t \right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^T\int_M a(x)\,g(u_t)\,\langle\nabla f,\nabla u\rangle\;\mathrm{d}M\,\mathrm{d}t\right|\\
&+\left|\int_0^T\int_{M\backslash V}\langle\nabla u,\nabla\alpha\rangle\,u\;\mathrm{d}M\,\mathrm{d}t\right|\\
&+\frac{1}{2}\left|\int_0^T\int_{\partial M\cap(M\backslash V)}\langle\nabla f,\nu\rangle\,|\nabla u|^2\;\mathrm{d}\partial M\;\mathrm{d}t\right| \qquad (4.41)
\end{aligned}
$$

where C^* is a constant that depends on C, α and f.

The inequality (4.41) is controlled by considering a standard procedure in the Euclidean setting. We shall adapt a similar procedure for the Riemannian case later. The main idea behind this is to consider the dissipative area, namely M_*, containing the set $M\backslash V$ as stated in (2.2). It is important to observe that the volume of $M_*(M_*\cap\partial M)$ can be made arbitrarily small, because the volume of $V(\partial_2 V\subset\partial M)$ can be made arbitrarily close to the volume of $M(\partial M)$.

5. A unique continuation theorem

In this section we prove the following unique continuation theorem for the wave equation on Riemannian manifolds:

Theorem 5.1. *Let M be a compact Riemannian manifold, eventually with boundary. Let $u\in C^0(0,T;H_0^1(M))\cap C^1(0,T;L^2(M))$ be the weak solution of the wave equation*

$$
\begin{cases}
u_{tt}-\Delta u=0 & \text{in } M\times\,]0,\infty[\\
u=0 & \text{on } \partial M\times\,]0,\infty[\\
u(0)=u^0\in H_0^1(M),\;\; u_t(0)=u^1\in L^2(M). &
\end{cases}
$$

Suppose that there exists an open set $V\subset M$ with smooth boundary $\overline{\partial_1 V}$ which intercepts ∂M transversally, and smooth functions $\alpha, f : M\to\mathbb{R}$ such that $\nabla\alpha|_V\equiv 0$ and conditions (4.39) and (4.40) hold.

If $u\equiv 0$ in a neighborhood M_* of $M\backslash V$, then $u\equiv 0$.

Proof. It is sufficient to work with regular solutions since by standard density arguments the result follows for weak solutions. Considering $a(x)\equiv 0$ in (1.1), from (4.38), (4.39), (4.40) and making use of the identity of the energy, we infer

$$
\begin{aligned}
2CE_0T &= 2C\int_0^T E(t)\,\mathrm{d}t = C\int_0^T\int_M\left(u_t^2+|\nabla u|^2\right)\mathrm{d}M\;\mathrm{d}t\\
&= C\int_0^T\int_V\left(u_t^2+|\nabla u|^2\right)\mathrm{d}M\;\mathrm{d}t + C\int_0^T\int_{M\backslash V}\left(u_t^2+|\nabla u|^2\right)\mathrm{d}M\;\mathrm{d}t\\
&\leqq C^*\int_0^T\int_{M\backslash V}\left(u_t^2+|\nabla u|^2\right)\mathrm{d}M\;\mathrm{d}t\\
&\quad -\left[\int_M u_t\,\langle\nabla f,\nabla u\rangle\;\mathrm{d}M+\int_M u_t\,\alpha\,u\,\mathrm{d}M\right]_0^T - \int_0^T\int_M\langle\nabla u,\nabla\alpha\rangle\,u\,\mathrm{d}M\,\mathrm{d}t\\
&\quad +\frac{1}{2}\int_0^T\int_{\partial M\cap(M\backslash V)}\langle\nabla f,\nu\rangle\,|\nabla u|^2\,\mathrm{d}\partial M\;\mathrm{d}t. \qquad (5.42)
\end{aligned}
$$

Let us find upper bounds of the last three terms of the right-hand side of inequality (5.42). Notice that

$$\left| \int_M u_t \, \langle \nabla f, \nabla u \rangle \, \mathrm{d}M + \int_M u_t \, u \, \alpha \, \mathrm{d}M \right| \leqq C_1 \int_M u_t^2 + |\nabla u|^2 \, \mathrm{d}M$$

due to the boundedness of $|\nabla f|$ and α, the Cauchy–Schwartz inequality, the inequality $ab \leqq a^2/2 + b^2/2$ and the Poncaré inequality. Therefore

$$\left[\int_M u_t \, \langle \nabla f, \nabla u \rangle \, \mathrm{d}M + \int_M u_t \, u \, \alpha \, \mathrm{d}M \right]_0^T \leqq 2 \, C_1 \, E_0. \tag{5.43}$$

Analogously we have that

$$\int_0^T \int_M \langle \nabla u, \nabla \alpha \rangle \; u \; \mathrm{d}M \; \mathrm{d}t \leqq C_2 \int_0^T \int_{M \backslash V} |\nabla u|^2 \; \mathrm{d}M \; \mathrm{d}t \tag{5.44}$$

due to the condition $\nabla \alpha = 0$ on V.

Now we control the last term of the right-hand side of (5.42). First of all, we construct a cut off vector field similar to the one used in Lions [15] (see Lemma 2.3) for the *Euclidean* setting.

Let $W \subset M$ be an open set with smooth boundary $\overline{\partial_1 W}$ which intercepts ∂M transversally and $M \backslash V \subset\subset W \subset\subset M_*$ (See Proposition 3.1).

Let $\eta_\partial : \partial M \to \mathbb{R}$ be a smooth cut-off function such that

$$\begin{cases} \eta_\partial(x) = 1 & \text{if } \; x \in \partial M \cap (M \backslash V) \\ \eta_\partial(x) = 0 & \text{if } \; x \in \partial M \cap (M \backslash W) \\ \eta_\partial(x) \in [0, 1] & \text{otherwise.} \end{cases}$$

This function exists because of the existence of a tubular neighborhood of $\partial M \cap (M \backslash V)$ that does not intercept $\partial M \cap (M \backslash W)$.

Let H_∂ be a vector field on ∂M defined as $H_\partial(x) := \eta_\partial(x).\nu(x)$. Then H_∂ satisfies

$$\begin{cases} H_\partial(x) = \nu(x) & \text{if } \; x \in \partial M \cap (M \backslash V) \\ H_\partial(x) = 0 & \text{if } \; x \in \partial M \cap (M \backslash W) \\ \langle H_\partial(x), \nu(x) \rangle \geqq 0 & \text{otherwise.} \end{cases}$$

Now we extend H_∂ into the whole manifold M, which gives a vector field similar to Lions' vector field defined in [15] (see Lemma 2.3) for the *Euclidean* setting. In order to do that, consider a small tubular neighborhood

$$U_\varepsilon := \{x \in M; \operatorname{dist}(x, \partial M) < \varepsilon\}$$

where $\varepsilon > 0$. For every $x \in U_\varepsilon$, let $\pi(x)$ be the point in ∂M that minimizes the distance from x to ∂M. We can choose ε small enough in order to state

$$\{x \in U_\varepsilon; \pi(x) \in \partial M \cap W\} \subset M_* \tag{5.45}$$

because $\partial M \cap W \subset\subset M_*$. Now define a smooth cutoff function $\eta : M \to \mathbb{R}$ such that

$$\begin{cases} \eta(x) = 1 & \text{if } x \in \partial M \\ \eta(x) = 0 & \text{if } x \notin U_\varepsilon \\ \eta(x) \in [0,1] & \text{otherwise.} \end{cases}$$

We also define a smooth vector field H_U in U_ε which is the parallel transport of H_∂ along the minimizing geodesic. Finally, define the vector field H as

$$\begin{cases} H_\nu(x) = \eta(x)\, H_U(x) & \text{if } x \in U_\varepsilon \\ H_\nu(x) = 0 & \text{otherwise.} \end{cases}$$

Notice that H_ν is a smooth extension of H_∂ to the whole manifold, and it vanishes outside M_* because of condition (5.45). Now we are ready to control the last term of the right-hand side of (5.42).

First of all, notice that

$$\begin{aligned} \int_0^T \int_{\partial M \cap (M \setminus V)} \langle \nabla f, \nu \rangle\, |\nabla u|^2 \, \mathrm{d}\partial M \ \mathrm{d}t &\leqq C_3 \int_0^T \int_{\partial M \cap (M \setminus V)} |\nabla u|^2 \, \mathrm{d}\partial M \ \mathrm{d}t \\ &\leqq C_3 \int_0^T \int_{\partial M} \langle H_\nu, \nu \rangle\, (\partial_\nu u)^2 \, \mathrm{d}\partial M \ \mathrm{d}t \end{aligned} \tag{5.46}$$

for some positive constant C_3 because $u \equiv 0$ on ∂M, $a \equiv 0$ in M and $\langle H_\nu, \nu \rangle = 1$ on $\partial M \cap (M \setminus V)$. Equation (4.31) becomes

$$\begin{aligned} \frac{1}{2} \int_0^T \int_{\partial M} \langle H_\nu, \nu \rangle\, (\partial_\nu u)^2 \, \mathrm{d}\partial M \ \mathrm{d}t &= \left[\int_{U_\varepsilon} u_t \ \langle H_\nu, \nabla u \rangle \ \mathrm{d}M \right]_0^T \\ &\quad + \frac{1}{2} \int_0^T \int_{U_\varepsilon} (\operatorname{div} H_\nu) \left\{ |u_t|^2 - |\nabla u|^2 \right\} \mathrm{d}M \, \mathrm{d}t \\ &\quad + \int_0^T \int_{U_\varepsilon} \nabla H_\nu (\nabla u, \nabla u) \, \mathrm{d}M \, \mathrm{d}t. \end{aligned} \tag{5.47}$$

Considering that the energy is constant and that $H_\nu \equiv 0$ outside W, we have that

$$\begin{aligned} &\frac{1}{2} \int_0^T \int_{\partial M} \langle H_\nu, \nabla u \rangle\, (\partial_\nu u)^2 \, \mathrm{d}\partial M \ \mathrm{d}t \\ &\quad \leqq C_4\, E_0 + C_4 \int_0^T \int_{M_*} \left(u_t^2 + |\nabla u|^2 \right) \, \mathrm{d}M \ \mathrm{d}t \end{aligned} \tag{5.48}$$

for some positive constant C_4.

Joining (5.42), (5.43), (5.44) and (5.48) and choosing a sufficiently big T, we have that

$$E_0 \leqq C_5 \int_0^T \int_{M_*} \left(u_t^2 + |\nabla u|^2 \right) \, \mathrm{d}M \ \mathrm{d}t \tag{5.49}$$

for some positive constant C_5, which settles the theorem. □

Remark 5.1. It is important to note that Theorem 5.1 can be extend for ultra-weak solutions, for instance, those in the class $u \in C^0([0,T], L^2(M)) \cap C^1([0,T], H^{-1}(M))$, where $H^{-1}(M)$ means $(H_0^1(M))'$. This corresponds to initial data taken in $\{u^0, u^1\} \in L^2(M) \times H^{-1}(M)$. Indeed, it is sufficient to observe that given $\{u^0, u^1\} \in L^2(M) \times H^{-1}(M)$ there exists $\{u_i^0, u_i^1\} \subset H_0^1(M) \times L^2(M)$ such that $u_i^0 \to u^0$ in $L^2(M)$ and $u_i^1 \to u^1$ in $H^{-1}(M)$. So, considering, for each $i \in \mathbb{N}$, the sequence of problems

$$\begin{cases} u_i'' - \Delta u_i = 0 \ \text{ in } M \times (0,T) \\ u_i = 0 \ \text{ on } M_* \times (0,T) \\ u_i(0) = u_i^0, \ u_i'(0) = u_i^1, \end{cases} \tag{5.50}$$

it is not difficult to prove that there exists $u \in C^0([0,T], L^2(M)) \cap C^1([0,T], H^{-1}(M))$ such that

$$u_i \to u \ \text{ in } C^0([0,T]; L^2(M)), \tag{5.51}$$

$$u_i' \to u' \ \text{ in } C^0([0,T]; H^{-1}(M)), \tag{5.52}$$

where u is the ultra-weak solution to problem

$$\begin{cases} u'' - \Delta u = 0 \ \text{ in } M \times (0,T) \\ u = 0 \ \text{ on } M_* \times (0,T) \\ u(0) = u^0, \ u'(0) = u^1. \end{cases} \tag{5.53}$$

The proof of the above statement can be found in Lions--Magenes [16] (see Chap. 3, Theorem 9.3, section 9.5)

Then, assuming that, for each $i \in \mathbb{N}$, the conditions (4.39) and (4.40) hold, where the constants C and α are independent of i (see 6.64) we can apply Theorem 5.1 to problem 5.50 to conclude that $u_i \equiv 0$. From this fact, and considering the convergence (5.51), we deduce that $u \equiv 0$ as desired.

6. Construction of f

Fix $\epsilon > 0$. The next sections are devoted to the construction of a smooth function $f : M \to \mathbb{R}$ as well as an open subset $V \subset M$ with smooth boundary $\overline{\partial_1 V}$ which intercepts ∂M transversally such that $meas(V) > meas(M) - \epsilon$, $meas(\partial_2 V) > meas(\partial M) - \epsilon$ and the inequalities (4.39) and (4.40) hold. First of all, we construct these functions locally. Afterwards we glue them together. We can put radially symmetric open sets satisfying some conditions inside V (and outside the damping region).

6.1. Construction of a function satisfying (4.39) and (4.40) locally

The general idea of the construction of a function f satisfying (4.39) and (4.40) locally is similar to the one presented in [5]. We split the the construction of such a function f into three cases: a neighborhood of an interior point of M, a radially symmetric domain and a neighborhood of a boundary point of M.

6.1.1. Construction of a function satisfying (4.39) in a neighborhood of a interior point of M

Lemma 6.1. *Let M^n be a compact n-dimensional Riemannian manifold with Riemanian metric $\mathbf{g}$ of class C^2. Fix $p \in \operatorname{int}(M)$. Then there exists a neighborhood V_p of p with smooth boundary ∂V_p, a smooth function $f : V_p \to \mathbb{R}$ and positive constants α and C such that* (4.39) *holds for every regular solution u to problem* (1.1).

Proof. Fix $p \in \operatorname{int}(M)$. We begin with an orthonormal basis $(e_1, \ldots, e_n)$ of $T_p M$. Put a normal coordinate system $(x_1, \ldots, x_n)$ in a neighborhood $\widetilde{V}_p$ of p such that $\partial/\partial x_i(p) = e_i(p)$ for every $i = 1, \ldots, n$. It is well known that in this coordinate system we have that $\Gamma_{ij}^k(p) = 0$, where Γ_{ij}^k are the Christoffel symbols with respect to $(x_1, \ldots, x_n)$ (See, for instance, [10]).

The Hessian with respect to $(x_1, \ldots, x_n)$ is given by

$$\nabla^2 f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial f}{\partial x_k}.$$

The Laplacian of f is the trace of the Hessian with respect to the metric $\mathbf{g}$. If $\mathbf{g}_{ij}$ denote the components of the Riemannian metric with respect to $(x_1, \ldots, x_n)$ and $\mathbf{g}^{ij}$ are the components of the inverse matrix of $\mathbf{g}_{ij}$, then the Laplacian of f is given by

$$\Delta f = \sum_{i,j} \mathbf{g}^{ij} \nabla^2 f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Consider the function $f : \widetilde{V}_p \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{2} \sum_{i=1}^{n} x_i^2.$$

It is immediately evident that $\Delta f(p) = n$. Moreover $\nabla^2 f(p) = \mathbf{g}(p)$, which implies that

$$\nabla^2 f(p)(v, v) = |v|_p^2.$$

We are interested in finding a neighborhood $V_p \subset \widetilde{V}_p$ of p and a strictly positive constant C such that

$$\begin{aligned} &C \int_0^T \int_{V_p} \left(|\nabla u|^2 + u_t^2\right) \mathrm{d}M\, \mathrm{d}t \\ &\quad \leqq \int_0^T \int_{V_p} \left[\nabla^2 f(\nabla u, \nabla u) + \left(\alpha - \frac{\Delta f}{2}\right) |\nabla u|^2 + \left(\frac{\Delta f}{2} - \alpha\right) u_t^2\right] \mathrm{d}M.\, \mathrm{d}t \end{aligned} \tag{6.54}$$

for some $\alpha \in \mathbb{R}$. We claim that if we consider $\alpha = \frac{n}{2} - \frac{1}{2}$ and $C = 1/4$ (or any $C \in (0, 1/4]$), we obtain the desired inequality, which means that it is enough to prove that there exists $V_p \subset \widetilde{V}_p$ verifying

$$\int_0^T \int_{V_p} \nabla^2 f(\nabla u, \nabla u) + \left(\frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2}\right) |\nabla u|^2 \,\mathrm{d}M \,\mathrm{d}t \geqq 0 \tag{6.55}$$

and

$$\int_0^T \int_{V_p} \left(\frac{\Delta f}{2} - \frac{n}{2} + \frac{1}{4}\right) u_t^2 \,\mathrm{d}M \,\mathrm{d}t \geqq 0. \tag{6.56}$$

In order to prove the existence of a subset $V_p \subseteq \widetilde{V}_p$ where (6.55) holds, let κ be the smooth field of symmetric bilinear form on $\widetilde{V}_p$ defined as

$$\kappa(X, Y) = \nabla^2 f(X, Y) + \left(\frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2}\right) \mathbf{g}(X, Y)$$

where X and Y are vector fields on $\widetilde{V}_p$. It is clearly a positive definite bilinear form on p, since $\nabla^2 f(p)(X, Y) = \mathbf{g}(p)(X, Y)$ and

$$\kappa(p)(X, Y) = \frac{1}{4}\mathbf{g}(p)(X, Y).$$

Therefore, there exists a neighborhood $\widehat{V}_p$ such that κ is positive definite and

$$\int_0^T \int_{\widehat{V}_p} \nabla^2 f(\nabla u, \nabla u) + \left(\frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2}\right) |\nabla u|^2 \,\mathrm{d}M \,\mathrm{d}t \geqq 0.$$

To prove the existence of $V_p \subset \widetilde{V}_p$ such that (6.56) holds is easier. It is enough to notice that at p we have that

$$\left(\frac{\Delta f(p)}{2} - \frac{n}{2} + \frac{1}{4}\right) = \frac{1}{4}$$

and the existence of $V_p \subset \widetilde{V}_p$ such that (6.56) holds is immediately evident. Therefore $V_p \subset \widetilde{V}_p$ is a neighborhood of p such that (6.54) holds is settled. $\square$

Now we consider a radially symmetric subset of M. We say that an open set $V \subset M$ is radially symmetric with respect to $p \in V$ if the expression of the metric in polar coordinates $(r, \theta) = (r, \theta_1, \theta_2, \ldots, \theta_{n-1})$ centered in p is given by $ds^2 = dr^2 + Q^2(r)d\theta^2$. A particular case is the unit sphere, where $Q(r) = \sin r$. Another particular case is the hyperbolic space with constant curvature -1, where $Q(r) = \sinh r$. Q is always differentiable at the origin and $Q'(0) = 1$.

Lemma 6.2. *Let M^n be a compact n-dimensional Riemannian manifold with Riemmanian metric $\mathbf{g}$ of class C^2. Let $\tilde{V} \subset \mathrm{int} M$ be a radially symmetric open subset with respect to $p \in M$. Then there exist a precisely definable subset of $V \subset \tilde{V}$, a function $f \in C^\infty(V)$ and positive constants α and C such that* (4.39) *holds for every regular solution u to problem* (1.1).

Proof. We are interested in finding a radially symmetric differentiable function $f : V \to \mathbb{R}$ with respect to p, such that its Hessian is conformal (proportional) to the Riemannian metric. Using calculations with respect to a polar coordinate system centered in p, we have that

$$\begin{aligned} \nabla^2 f\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) &= \frac{\partial^2 f}{\partial r^2}, \\ \nabla^2 f\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) &= 0, \\ \nabla^2 f\left(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right) &= 0, \end{aligned} \tag{6.57}$$

if $i \neq j$ and

$$\nabla^2 f\left(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_i}\right) = f'(r)\, Q(r)\, Q'(r). \tag{6.58}$$

Let $F = f'$. In order for $\nabla^2 f$ to be conformal to the Riemannian metric, we compare (6.57) and (6.58) and get

$$\frac{F'}{F} = \frac{Q'}{Q},$$

which can be solved by $F = Q$. Therefore $\nabla^2 f = Q' \langle \cdot, \cdot \rangle$.

Now we are interested in Riemannian manifolds M such that

$$\begin{aligned} &C \int_0^T \int_V \left(|\nabla u|^2 + u_t^2\right) \mathrm{d}M\, \mathrm{d}t \\ &\leqq \int_0^T \int_V \left[\nabla^2 f(\nabla u, \nabla u) + \left(\alpha - \frac{\Delta f}{2}\right) |\nabla u|^2 + \left(\frac{\Delta f}{2} - \alpha\right) u_t^2\right] \mathrm{d}M\ \mathrm{d}t \end{aligned} \tag{6.59}$$

holds for some $\alpha, C \in \mathbb{R}$. Hence it is enough to prove that

$$\left(1 - \frac{n}{2}\right) Q'(r) + \alpha - C \geqq 0 \tag{6.60}$$

and

$$\frac{n}{2} Q'(r) - \alpha - C \geqq 0 \tag{6.61}$$

on V. We must have

$$\alpha \geqq C + \left(\frac{n}{2} - 1\right) \tag{6.62}$$

and

$$\alpha \leqq \frac{n}{2} - C \tag{6.63}$$

due to (6.60), (6.61) and $Q'(0) = 1$. Combining (6.62) and (6.63) we have that

$$C \in (0, 1/2] \quad \text{and} \quad \alpha \in [n/2 - 1 + C, n/2 - C]. \tag{6.64}$$

Now taking C and α as in (6.64), we have that (4.39) is satisfied in any open set such that

$$Q'(r) \in \left[\frac{2}{n}(\alpha + C), \frac{\alpha - C}{\frac{n}{2} - 1}\right] \tag{6.65}$$

(if $n = 2$, $Q'(r)$ does not need to satisfy any upper bound). □

Remark 6.1. It is clear that it is interesting to choose a very small C in order to pick a larger V. As particular cases where (4.39) is satisfied we have:

(1) Euclidean domains: In this case, $Q(r) = r$ and $Q'(r) \equiv 1$. Therefore (4.39) holds for any Euclidean domain.
(2) For surfaces: In this case we can pick a subset V with $\inf_{x \in V} Q'(r) \geqq \varepsilon > 0$. For instance, any compactly contained open set in a semi-sphere satisfies this condition. Two dimensional hyperbolic spaces also satisfy this condition.

The following theorem generalizes Lemmas 6.1 and 6.2. Its proof proceeds as in the proof of Lemma 6.2. It is not used in its full force in this work. For this work, the lemmas above are enough.

Theorem 6.3. *Let M^n be a compact n-dimensional Riemannian manifold with Riemmanian metric $\mathbf{g}$ of class C^2 and consider a smooth function $f : M \to \mathbb{R}$. Suppose that $\alpha, C > 0$ and $V \subset \operatorname{int} M$ are such that $C \in (0, 1/2]$, $\alpha \in [n/2 - 1 + C, n/2 - C]$ and*

$$\frac{2}{n}(\alpha + C)\, \mathbf{g}(p)(v, v) \leqq \nabla^2 f(p)(v, v) \leqq \frac{\alpha - C}{\frac{n}{2} - 1} \mathbf{g}(p)(v, v)$$

for every $p \in V$ and $v \in T_p M$. Then (4.39) holds for every regular solution u to problem (1.1).

6.1.2. Construction of a function satisfying (4.39) and (4.40) in a neighborhood of a boundary point of M First of all, we begin with a technical lemma.

Lemma 6.4. *Let M be a compact Riemannian manifold with boundary. Then there exist a Riemannian manifold $\tilde{M}$ and an isometric immersion $f : M \to \tilde{M}$ such that $f(M) \subset\subset \operatorname{int}(\tilde{M})$.*

Proof. Let V be a small open tubular neighborhood of ∂M in M. Then $M \backslash V$ is diffeomorphic to M. Put a Riemannian metric $\tilde{\mathbf{g}}_1$ on $M \backslash V$ in such a way that it is isometric to M. Finally, extend $\tilde{\mathbf{g}}_1$ to a Riemannian metric $\tilde{\mathbf{g}}$ on M. Now set $\tilde{M} = (M, \tilde{\mathbf{g}})$. □

Lemma 6.5. *Let M^n be a compact n-dimensional Riemannian manifold with boundary with Riemmanian metric $\mathbf{g}$ of class C^2. Fix $p \in \partial M$. Then there exists a neighborhood V_p of p with smooth boundary $\overline{\partial_1 V_p}$ which intercepts ∂M transversally, a smooth function $f : V_p \to \mathbb{R}$ and positive constants α and C such that* (4.39) *and* (4.40) *holds for every regular solution u of Problem* (1.1).

Proof. Let $p \in \partial M$. We can extend M into a Riemannian manifold $\tilde{M}$ as in Lemma 6.4. Now we follow a construction similar to that made before.

Fix an orthonormal basis $(e_1, \dots, e_n)$ of $T_p\tilde{M}$ such that the subspace $T_p\partial M \subset T_p\tilde{M}$ is spanned by $\{x_2, \dots, x_n\}$ and e_1 points inside M. Put a normal coordinate system $(x_1, \dots, x_n)$ in a neighborhood $\widetilde{V}_p \subset \tilde{M}$ of p such that $\partial/\partial x_i(p) = e_i(p)$ for every $i = 1, \dots, n$.

Consider the function $f : \widetilde{V}_p \to \mathbb{R}$ defined by

$$f(x) = x_1 + \frac{1}{2}\sum_{i=1}^{n} x_i^2.$$

It is immediately evident that $\Delta f(p) = n$, $\langle \nabla f(p), \nu(p)\rangle = -1$ and $\nabla^2 f(p) = \mathbf{g}(p)$. Moreover, using the same argument as used in Lemma 6.1, we can find a neighborhood such that (4.39) is satisfied. Finally, we can further restrict this neighborhood into a neighborhood $\widehat{V}_p \subset \tilde{M}$ in order to satisfy (4.40) in such a way that $V_p := \widehat{V}_p \cap M$ has smooth boundary $\overline{\partial_1 V_p}$ that intercepts ∂M transversally. □

6.2. A function that satisfies inequality (4.39) and (4.40) in a wide domain

The main aim of this section is to prove the following theorem:

Theorem 6.6. *Let $(M^n, \mathbf{g})$ be a compact n-dimensional Riemannian manifold with boundary. Fix $\epsilon > 0$. Then there exist an open subset $V \subset M$ and smooth functions $\alpha, f : M \to \mathbb{R}$ such that $meas(V) \geqq meas(M) - \epsilon$, $meas(V \cap \partial M) \geqq meas(\partial M) - \epsilon$, $\nabla\alpha|_V \equiv 0$ and*

$$C\int_0^T\int_V [u_t^2 + |\nabla u|^2]\mathrm{d}M\,\mathrm{d}t \leqq \int_0^T\int_V \left(\frac{\Delta f}{2} - \alpha\right) u_t^2\,\mathrm{d}M\,\mathrm{d}t + \int_0^T\int_V \left[\nabla^2 f(\nabla u, \nabla u) + \left(\alpha - \frac{\Delta f}{2}\right)|\nabla u|^2\right]\mathrm{d}M\,\mathrm{d}t,$$

for some positive constant C. In addition,

$$\langle \nabla f, \nu\rangle < 0 \text{ on } \partial M \cap V.$$

Moreover, if M contains radially symmetric subsets, then we can choose V in such a way that a precise part of these radially symmetric subsets is contained in V.

We begin by proving some preliminary results. The following lemma is classical and can be found in [31] (see the proof of Lemma 1.9 there).

Lemma 6.7. *Let M be a topological space which is locally compact, Hausdorff and has countable basis. Then there exist a increasing sequence of open sets $(V_i)_{i\in\mathbb{N}}$ such that*:

(1) $M = \cup_{i=1}^{\infty} V_i$.
(2) $\bar{V}_i \subset V_{i+1}$.
(3) $\bar{V}_i$ *is compact.*

Given a compact Riemannian manifold M, eventually with boundary, the injectivity radius $\operatorname{inj}(V)$ of a subset $V \subset\subset M\backslash\partial M$ is given by $\inf\limits_{x\in U} \operatorname{inj}(x)$, where $\operatorname{inj}(x)$ is the injectivity radius of x in M.

Let $V \subset\subset M\backslash\partial M$ be an open subset. We want to define a mollifier smoothing $f_\varepsilon : V \to \mathbb{R}$ of a locally summable function $f : M \to \mathbb{R}$. The bump function $\eta : V \to \mathbb{R}$ is defined similarly as in the Euclidean case:

$$\widehat{\eta}(x, y, \varepsilon) = \begin{cases} \exp\left(\frac{1}{\left(\frac{\operatorname{dist}(x,y)}{\varepsilon}\right)^2 - 1}\right) & \text{if } \operatorname{dist}(x, y) < \varepsilon < \operatorname{inj}(V) \\ 0 & \text{if } \operatorname{dist}(x, y) \geqq \varepsilon. \end{cases}$$

The function $\widehat{\eta}$ is clearly C^∞. We normalize $\widehat{\eta}$ and get

$$\eta(x, y, \varepsilon) = \frac{\widehat{\eta}(x, y, \varepsilon)}{\int_M \widehat{\eta}(x, y, \varepsilon)\, \mathrm{d}M(y)}.$$

Notice that η is also smooth. We define the mollifier smoothing $f_\varepsilon : V \to \mathbb{R}$ by

$$f_\varepsilon(x) = \int_M \eta(x, y, \varepsilon) f(y)\, \mathrm{d}M. \tag{6.66}$$

Lemma 6.8. *Let M be a compact Riemannian manifold, eventually with boundary. Let $f : M \to \mathbb{R}$ be a locally summable function, $V \subset\subset M\backslash\partial M$ be an open subset and $\varepsilon < \operatorname{inj}(V)$ be a strictly positive number. Then the mollifier smoothing $f_\varepsilon : V \to \mathbb{R}$ defined by* (6.66) *is a smooth function.*

Proof. The theorem holds because a Riemannian manifold behaves like Euclidean domains inside the injectivity radius. For the complete proof, see [11]. □

Lemma 6.9. *Let M be a Riemannian manifold and consider two subsets A and B such that* $\operatorname{dist}(A, B) > 0$. *Suppose that $\bar{A}$ and $\bar{B}$ are compact. Then there exist open subsets $O_A \supset\supset A$ and $O_B \supset\supset B$ with smooth boundaries such that* $\operatorname{dist}(O_A, O_B) > 0$. *Moreover there exists a smooth (cut-off) function $\rho : M \to \mathbb{R}$ such that $\rho|_{O_A} \equiv 1$, $\rho|_{O_B} \equiv 0$ and $\rho(M) \subset [0, 1]$.*

Proof. Let $\varepsilon \in (0, \operatorname{dist}(A, B)/3)$ such that $A_\varepsilon := \{x \in M; \operatorname{dist}(x, A) < \varepsilon\}$ and $B_\varepsilon := \{x \in M; \operatorname{dist}(x, B) < \varepsilon\}$ has compact closures and the injectivity radius ε. Observe that A_ε and B_ε are open subsets of M.

Notice that $\widehat{\rho} : M \to \mathbb{R}$ defined by

$$\widehat{\rho}(x) = \frac{\frac{d(x,B_\varepsilon)-d(x,A_\varepsilon)}{d(x,B_\varepsilon)+d(x,A_\varepsilon)} + 1}{2}$$

is continuous, $\widehat{\rho}(x) = 1$ if $x \in A_\varepsilon$, $\widehat{\rho}(x) = 0$ if $x \in B_\varepsilon$ and $\widehat{\rho}(M) \subset [0, 1]$.

We are going to built open sets O_A by O_B with smooth boundary such that $A \subset\subset O_A \subset\subset A_\varepsilon$ and $B \subset\subset O_B \subset\subset B_\varepsilon$.

Consider the mollifier smoothing $\widehat{\rho}_\varepsilon : A_\varepsilon \to \mathbb{R}$ of $\widehat{\rho}$, which is a smooth function. Notice that $\widehat{\rho}_\varepsilon(x) \neq 1$ for every $x \in \partial A_\varepsilon$. Let $s \in (\sup_{x\in\partial A_\varepsilon} \widehat{\rho}_\varepsilon(x), 1)$ a regular value of $\widehat{\rho}_\varepsilon$. The classical Sard's Theorem states that the inverse image of a regular value is a smooth embedded hypersurface of A_ε (in our case it is compact and without boundary). Finally take set $O_A := {\rho_\varepsilon}^{-1}((s, 1])$. We can define O_B in the same way.

It remains to prove the existence of a smooth (cut-off) function ρ. Let $\lambda <$ dist(O_A, O_B) be a positive number such that $(\partial O_A)_\lambda := \{x \in M; \operatorname{dist}(x, \partial O_A) < \lambda\}$ is a tubular neighborhood of $\partial O_A \subset M$. Consider a non-increasing smooth function $\tilde{\rho} : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{cases} \tilde{\rho}(x) = 1 & \text{if } x \leqq 1/3; \\ \tilde{\rho}(x) = 0 & \text{if } x \geqq 2/3; \\ \tilde{\rho}(x) \in [0, 1] & \text{if } x \in [1/3, 2/3]. \end{cases}$$

Set $\tilde{\rho}_\lambda(x) := \tilde{\rho}(x/\lambda)$. Define

$$\rho(x) = \begin{cases} 1 & \text{if } x \in O_A; \\ 0 & \text{if } x \in M\backslash(O_A \cup (\partial O_A)_\lambda); \\ \alpha_\lambda(\operatorname{dist}(x, O_A)) & \text{otherwise}. \end{cases}$$

Then ρ is in fact a smooth (cut-off) function satisfying the stated properties. □

Lemma 6.10. *The set O_A constructed in Lemma 6.9 has a finite number of components and the closure of each component is a Riemannian manifold with smooth boundary.*

Proof. Denote the set of the components of O_A by $\{O_\lambda\}_{\lambda\in\Lambda}$. Choose a point x_λ from the boundary of the connected component O_λ of O_A. The set $\{x_\lambda, \lambda \in \Lambda\}$ does not have an accumulation point because the boundary of O_A is the inverse image of a regular point. Then $\{x_\lambda, \lambda \in \Lambda\}$ is a finite set. Therefore O_A has a finite number of components and the closure of each component is a Riemannian manifold with smooth boundary □

Now we prove the main theorem of this section:

Proof of Theorem 6.6. First of all, extend M to a Riemannian manifold $\tilde{M}$ as in Lemma 6.4. For every $p \in M$, we choose the following neighborhoods $\widehat{W}_p$ of p, functions $f_p \in C^\infty(\widehat{W}_p)$ and constants α_p and C_p:

(1) If $p \in \text{int}(M)$, then we can choose choose $\widehat{W}_p = V$, $f \in C^\infty(V)$, $\alpha_p = n/2 - 1/2$ and $C_p = 1/4$ as in Lemma 6.1.
(2) If $p \in \text{int}(M)$ is the center of a radially symmetric domain V_p in which we do not want to have the damping acting, then we can choose a radially symmetric neighborhood $\widehat{W}_p$ which is a little bit larger that the neighborhood V_p of p, that is, $\bar{V}_p \subset\subset \widehat{W}_p$ (We can do this due to the flexibility of the constant C). In addition, we choose the function $f_p = f$ and constants $\alpha_p = \alpha$ and $C_p = C$ as in the proof of Lemma 6.2.
(3) If $p \in \partial M$, then choose the open neighborhood $\widehat{W}_p = \widehat{V}_p \subset \text{int}(\tilde{M})$, $f_p = C^\infty(V)$ and constants α_p and C_p as in the proof of Lemma 6.5.

In (2), we should be careful in order to choose $\widehat{W}_p$ in such a way that $\overline{(\widehat{W}_x)} \cap \overline{(\widehat{W}_y)} = \emptyset$ of $x \neq y$.

Using the compactness of M, we can choose a finite covering $\{\widehat{W}_i\}_{i=1}^k$ of M. For $i = 1, \ldots k$, denote the respective functions by $f_i : \widehat{W}_i \to \mathbb{R}$, the respective constants by α_i and set $C = \min\{C_1, \ldots, C_k\}$. We choose $\widehat{W}_i$ in such a way that every neighborhood of the radially symmetric domains is in the finite covering. Moreover, we put them before the other domains, that is, $\{\widehat{W}_i\}_{i=1}^l$ are the neighborhoods of the radially symmetric domains.

Denote $B = (\bigcup_{i=1}^k \partial \widehat{W}_i \cup \partial M) \cap M$. Notice that $M - B$ is an open subset of M. Denote the points of $M - B$ which are in $\widehat{W}_1$ by W_1. For $i = 2, \ldots, k$, denote the points of $M - B$ which are in $\widehat{W}_i - \cup_{l=1}^{i-1} \widehat{W}_l$ by W_i. Observe that we have the disjoint union $M - B = \cup_{i=1}^k W_i$. Moreover, without loss of generality, we can suppose that $W_i \neq \emptyset$, $i = 1, \ldots, k$. We claim that W_i, $i = 1, \ldots, k$, are open subsets of M; in fact, $M - B$ is an open subset and it can be written as a countable union of connected components. Each connected component is either completely contained in $\widehat{W}_i$ or it does not intercept $\widehat{W}_i$. Therefore, each W_i is a union of connected components of $M - B$. For the sake of simplicity, we keep writing $f_i : W_i \to \mathbb{R}$ instead of $f_i|_{W_i}$. Observe that for $i = 1, \ldots, l$, $W_i = \widehat{W}_i$ are the neighborhood's radially symmetric domains.

Fix $i \in \{1, \ldots, k\}$. Using Lemma 6.7, we can find an open set $\widehat{V}_i \subset\subset W_i$ such that $meas(W_i \backslash \widehat{V}_i) < \epsilon/k$. If W_i is a neighborhood of a boundary point of M, then we can suppose further that $meas(\partial M \cap (W_i \backslash \widehat{V}_i)) < \epsilon/k$. Notice that $\text{dist}(\widehat{V}_i, B) = d_i > 0$ due to the compactness of B and $\overline{\widehat{V}_i}$. Using Lemma 6.9 there exist open subsets $V_i \supset\supset \widehat{V}_i$ and $O_i \supset\supset M - W_i$ with smooth boundaries and a smooth (cut-off) function $\rho_i : M \to \mathbb{R}$ such that $\rho_i|_{V_i} \equiv 1$, $\rho_i|_{O_i} \equiv 0$ and $\rho_i(M) \subset [0, 1]$. Moreover, if V_i is a neighborhood of a boundary point, we can further assume that $\overline{\partial_1 V_i}$ intercepts ∂M transversally. In the case of a radially symmetric domain, we can assume without loss of generality that V_i is the original radially symmetric domain.

Now set $\rho = \sum_{i=1}^k \rho_i$ and $V = \bigcup_{i=1}^k V_i$. We can see that

(1) $meas(M) - meas(V) = \sum_{i=1}^k meas(W_i - V_i) \leqq \sum_{i=1}^k meas(W_i - \widehat{V}_i) < \epsilon$ which implies that $meas(V) > meas(M) - \epsilon$;
(2) Analogously, we have that $meas(\partial M \cap V) > meas(\partial M) - \epsilon$;
(3) $\rho|_V \equiv 1$.

Now we are in position to construct α and f. Define

$$f(x)=\begin{cases} f_i(x)\rho(x) & \text{if } \ x\in W_i\\ 0 & \text{if } \ x\in B\end{cases}$$

and

$$\alpha(x)=\begin{cases} \alpha_i\,\rho(x) & \text{if } \ x\in W_i\\ 0 & \text{if } \ x\in B.\end{cases}$$

Notice that f is smooth because $f_i\,\rho_i$ is smooth for every $i=1,\dots,k$ and $f=\sum_{i=1}^k f_i\,\rho_i$. Likewise α is smooth. Using property (3) for ρ, above, it is not difficult to see that α and f are smooth and satisfies all the required conditions, which settles the theorem. □

7. Controlling Equation (4.41)

We will denote

$$\chi=\left[\int_M u_t\,\langle\nabla f,\nabla u\rangle\;\mathrm{d}M\right]_0^T+\left[\int_M u_t\,\alpha\,u\,\mathrm{d}M\right]_0^T. \tag{7.67}$$

Next we will estimate some terms in (4.41). Let us denote:

$$R_1:=\max_{x\in M}|\nabla f(x)| \tag{7.68}$$

and

$$R_2:=\max_{x\in M}|\alpha(x)|. \tag{7.69}$$

Estimate for $I_1:=\int_0^T\int_M a(x)\,g(u_t)\,\langle\nabla f,\nabla u\rangle\,\mathrm{d}M\,\mathrm{d}t$.

By the Cauchy–Schwarz inequality, taking (7.68) into account and considering the inequality $ab\leqq\frac{a^2}{4\vartheta}+\vartheta b^2$, where ϑ is a positive number, we obtain

$$|I_1|\leqq\frac{||a||_{L^\infty(M)}R_1^2}{\vartheta}\int_0^T\int_M a(x)|g(u_t)|^2\mathrm{d}M\,\mathrm{d}t+2\vartheta\int_0^T E(t)\;\mathrm{d}t. \tag{7.70}$$

Estimate for $I_2=\int_0^T\int_M a(x)\,\alpha\,g(u_t)\,u\,dM\;\,dt$.

Similarly we infer

$$|I_2|\leqq\frac{||a||_{L^\infty(M)}R_2^2\lambda_1^{-1}}{16\vartheta}\int_0^T\int_M a(x)|g(u_t)|^2\,\mathrm{d}M\,\mathrm{d}t+2\vartheta\int_0^T E(t)\;\mathrm{d}t, \tag{7.71}$$

where λ_1 comes from the Poincaré inequality given in (2.3).

Choosing ϑ sufficiently small and inserting (5.46), (4.31), (7.67), (7.70) and (7.71) into (4.41) yields

$$\int_0^T E(t)\,\mathrm{d}t \leqq |\chi| + C_1 \int_0^T \int_M a(x)\,(g(u_t))^2 \mathrm{d}M\,\mathrm{d}t$$
$$+C_1 \int_0^T \int_{M\backslash V} [|\nabla u|^2 + a(x)\,u_t^2]\,\mathrm{d}M\,\mathrm{d}t$$
$$+C_1 \int_0^T \int_{M\backslash V} |u|^2\,\mathrm{d}M\;\mathrm{d}t \tag{7.72}$$

where

$$C_1 := C_1 \left\{C, ||a||_{L^\infty(M)}, \lambda_1^{-1}, R_1, R_2, a_0^{-1}, ||H||_{W^{1,\infty}(M)}\right\}.$$

Now we are going to control the quantity $\int_0^T \int_{M\backslash V} |\nabla u|^2\,\mathrm{d}M\,\mathrm{d}t$ in terms of the damping term $\int_0^T \int_M [a(x)\,|g(u_t)|^2 + a(x)\,u_t^2]\,\mathrm{d}M\,\mathrm{d}t$ (The last term will be controlled afterwards). For this purpose we have to built a "cut-off" function η_ε on a specific neighborhood of $M\backslash V$. First of all, define $\tilde{\eta} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{\eta}(x) = \begin{cases} 1 & \text{if } \; x \leqq 0 \\ (x-1)^2 & \text{if } \; x \in [1/2, 1] \\ 0 & \text{if } \; x > 1 \end{cases}$$

and it is defined on $(0, 1/2)$ in such a way that $\tilde{\eta}$ is a non-increasing function of class C^1. For $\varepsilon > 0$, set $\tilde{\eta}_\varepsilon(x) := \tilde{\eta}(x/\varepsilon)$. It is straightforward that there exists a constant C which does not depend on ε such that

$$\frac{|\tilde{\eta}_\varepsilon'(x)|^2}{\tilde{\eta}_\varepsilon(x)} \leqq \frac{C}{\varepsilon^2}$$

for every $x < \varepsilon$.

Let $M_* \supset M\backslash V$ be an open subset of M. Let $\tilde{M}$ an extension of M as in Lemma 6.4. Notice that we can also extend V and M_* to open subsets $\tilde{V}$ and $\tilde{M}_*$ of $\tilde{M}$, respectively, in such a way that $\tilde{V}$ has smooth boundary and $\tilde{V} \subset \tilde{M}_*$.

Now let $\varepsilon > 0$ such that

$$\tilde{\omega}_\varepsilon := \{x \in \tilde{M}; \operatorname{dist}(x, \partial\tilde{V}) < \varepsilon\} \subset\subset \tilde{M}_*$$

is a tubular neighborhood of $\partial\tilde{V}$ and $\omega_\varepsilon := \tilde{\omega}_\varepsilon \cup M\backslash V$ is contained in M_*. Define $\eta_\varepsilon : \tilde{M} \to \mathbb{R}$ as

$$\eta_\varepsilon(x) = \begin{cases} \tilde{\eta}_\varepsilon(d(x, \tilde{M}\backslash\tilde{V})) & \text{if } x \; \in \omega_\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward that η_ε is a function of class C^1 on M due to the smoothness of $\partial(\tilde{M}\backslash\tilde{V})$ and $\partial\omega_\varepsilon$. Notice also that

$$\frac{|\nabla\eta_\varepsilon(x)|^2}{\eta_\varepsilon(x)} = \frac{|\tilde{\eta}_\varepsilon'(\mathrm{d}(x, \omega_\varepsilon))|^2}{\tilde{\eta}_\varepsilon(\mathrm{d}(x, \omega_\varepsilon))} \leqq \frac{C}{\varepsilon^2} \tag{7.73}$$

for every $x \in \omega_\varepsilon$. In particular, $\frac{|\nabla\eta_\varepsilon|^2}{\eta_\varepsilon} \in L^\infty(\tilde{\omega}_\varepsilon)$.

Taking $\xi = \eta_\varepsilon$ in the identity (4.36) we obtain

$$\int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |\nabla u|^2 \mathrm{d}M\,\mathrm{d}t = -\left[\int_{\omega_\varepsilon} u_t u \eta_\varepsilon\,\mathrm{d}M\right]_0^T + \int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |u_t|^2\,\mathrm{d}M$$
$$-\int_0^T \int_{\omega_\varepsilon} u\,\langle \nabla u, \nabla \eta_\varepsilon\rangle\,\mathrm{d}M\,\mathrm{d}t$$
$$-\int_0^T \int_{\omega_\varepsilon} a(x)\,g(u_t) u \eta_\varepsilon\,\mathrm{d}M\,\mathrm{d}t. \tag{7.74}$$

Next we will estimate terms on the right-hand side of (7.74).

Estimate for $K_1 := \int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |u_t|^2\,\mathrm{d}M\,\mathrm{d}t$

From (2.2), since $\eta_\varepsilon \leqq 1$ and $\omega_\varepsilon \subset M_*$, where the damping lies, we deduce

$$K_1 \leqq a_0^{-1} \int_0^T \int_M a(x)\,u_t^2\,\mathrm{d}M\ \mathrm{d}t. \tag{7.75}$$

Estimate for $K_2 := -\int_0^T \int_{\omega_\varepsilon} a(x)\,g(u_t) u \eta_\varepsilon\,\mathrm{d}M\,\mathrm{d}t.$

The Cauchy–Schwarz inequality, the inequality $ab \leqq \frac{1}{4\vartheta} a^2 + \vartheta b^2$ and (2.3) yield

$$|K_2| \leqq \frac{\lambda_1^{-1} ||a||_{L^\infty(M)}}{4\vartheta} \int_0^T \int_M a(x)\,|g(u_t)|^2\,\mathrm{d}M + 2\vartheta \int_0^T E(t)\ \mathrm{d}t, \tag{7.76}$$

where ϑ is a positive constant.

Estimate for $K_3 := \int_0^T \int_{\omega_\varepsilon} u\,\langle \nabla u, \nabla \eta_\varepsilon\rangle\,dM\,dt.$

Considering (7.73) and applying Cauchy–Schwarz inequality, we can write

$$|K_3| \leqq \frac{1}{2} \int_0^T \left[\int_{\omega_\varepsilon} \eta_\varepsilon |\nabla u|^2\,\mathrm{d}M + \int_{\omega_\varepsilon} \frac{|\nabla \eta_\varepsilon|^2}{\eta_\varepsilon} |u|^2\,\mathrm{d}M\right]\mathrm{d}t$$
$$\leqq \frac{1}{2} \int_0^T \left[\int_{\omega_\varepsilon} \eta_\varepsilon |\nabla u|^2\,\mathrm{d}M + \frac{M}{\varepsilon^2} \int_{\omega_\varepsilon} |u|^2\,\mathrm{d}M\right]\mathrm{d}t. \tag{7.77}$$

Combining (7.74)–(7.77) we arrive at the following inequality

$$\frac{1}{2} \int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |\nabla u|^2\,\mathrm{d}M\,\mathrm{d}t \leqq |Y| + \frac{\lambda_1^{-1} ||a||_{L^\infty(M)}}{4\vartheta} \int_0^T \int_M a(x)\,|g(u_t)|^2\,\mathrm{d}M$$
$$+2\vartheta \int_0^T E(t)\ \mathrm{d}t + \frac{M}{2\varepsilon^2} \int_0^T \int_{\omega_\varepsilon} |u|^2\,\mathrm{d}M\ \mathrm{d}t,$$
$$+a_0^{-1} \int_0^T \int_M a(x)\,u_t^2\,\mathrm{d}M\ \mathrm{d}t. \tag{7.78}$$

where

$$Y := -\left[\int_{\omega_\varepsilon} u_t u \eta_\varepsilon\,\mathrm{d}M\right]_0^T. \tag{7.79}$$

Thus, combining (7.78) and (7.72), keeping in mind that

$$\frac{1}{2}\int_0^T\int_{M\backslash V}|\nabla u|^2\,\mathrm{d}M\,\mathrm{d}t \leqq \frac{1}{2}\int_0^T\int_{\omega_\varepsilon}\eta_\varepsilon|\nabla u|^2\,\mathrm{d}M\,\mathrm{d}t$$

and choosing ϑ small enough, we deduce

$$\begin{aligned}\int_0^T E(t)\,\,\mathrm{d}t \leqq |\chi| + C_1|Y| + C_2\int_0^T\int_M[a(x)\,|g(u_t)|^2 + a(x)\,|u_t|^2]\,\mathrm{d}M\,\mathrm{d}t\\ +\frac{MC_2}{\varepsilon^2}\int_0^T\int_{\omega_\varepsilon}|u|^2\,\mathrm{d}M\,\,\mathrm{d}t,\end{aligned}\tag{7.80}$$

where $C_2 = C_2(C_1, \lambda_1^{-1}, ||a||_{L^\infty(M)}, a_0^{-1})$.

On the other hand, from (7.67), (7.79) and (2.7) the following estimate holds

$$\begin{aligned}|\chi| + 2C_2|Y| &\leqq C_3(E(0) + E(T))\\ &= C_3\left[2\,E(T) + \int_0^T\int_M a(x)\,g(u_t)\,u_t\,\mathrm{d}M\right],\end{aligned}\tag{7.81}$$

where C_3 is a positive constant which depends also on R_1 and R_2.

Then, (7.80) and (7.81) yield

$$\begin{aligned}T\,E(T) &\leqq \int_0^T E(t)\,\,\mathrm{d}t\\ &\leqq C\,E(T) + C\left[\int_0^T\int_M[a(x)\,|g(u_t)|^2 + a(x)\,|u_t|^2]\,\mathrm{d}M\,\mathrm{d}t\right]\\ &\quad +C\int_0^T\int_{\omega_\varepsilon}|u|^2\,\mathrm{d}M\,\,\mathrm{d}t,\end{aligned}\tag{7.82}$$

where C is a positive constant which depends on $a_0, \lambda_1, R_1, R_2, ||a||_{L^\infty(M)}, n$ and $\frac{M}{\varepsilon^2}$.

Our aim is to absorb the last term on the right-hand side of (7.82). In order to do this, let us consider the following lemma, where T_0 is a positive constant which is sufficiently large for our purpose.

Lemma 7.1. *Under the hypothesis of Theorem 2.2 and for all $T > T_0$, there exists a positive constant $C(T_0, E(0))$ such that if (u, u_t) is the solution of (1.1) with regular initial data, we have*

$$\int_0^T\int_M|u|^2\,\mathrm{d}M\,\,\mathrm{d}t \leqq C(T_0, E(0))\left\{\int_0^T\int_M\left(a(x)\,g^2(u_t))+a(x)u_t^2\right)\mathrm{d}M\,\,\mathrm{d}t\right\}.\tag{7.83}$$

Proof. We argue by contradiction. For simplicity we shall denote $u' := u_t$. Let us suppose that (7.83) is not verified and let $\{u_k(0), u_k'(0)\}$ be a sequence of initial data where the corresponding solutions $\{u_k\}_{k\in\mathbb{N}}$ of (1.1) with $E_k(0)$, assumed uniformly bounded in k, verifies

$$\lim_{k\to+\infty} \frac{\displaystyle\int_0^T ||u_k(t)||^2_{L^2(M)}\,\mathrm{d}t}{\displaystyle\int_0^T\int_M \left(a(x)\,g^2(u_k') + a(x)\,u_k'^2\right)\mathrm{d}M\;\mathrm{d}t} = +\infty, \tag{7.84}$$

that is

$$\lim_{k\to+\infty} \frac{\displaystyle\int_0^T\int_M \left(a(x)\,g^2(u_k') + a(x)\,u_k'^2\right)\mathrm{d}M\;\mathrm{d}t}{\displaystyle\int_0^T ||u_k(t)||^2_{L^2(M)}\,\mathrm{d}t} = 0. \tag{7.85}$$

Since $E_k(t) \leqq E_k(0) \leqq L$, where L is a positive constant, we obtain a subsequence, still denoted by $\{u_k\}$ from now on, which verifies the convergence:

$$u_k \rightharpoonup u \text{ weakly in } H^1(\Sigma_T), \tag{7.86}$$

$$u_k \rightharpoonup u \text{ weak star in } L^\infty(0,T;W), \tag{7.87}$$

$$u_k' \rightharpoonup u' \text{ weak star in } L^\infty(0,T;L^2(M)). \tag{7.88}$$

Employing compactness results we also deduce that

$$u_k \to u \text{ strongly in } L^2(0,T;L^2(M)). \tag{7.89}$$

At this point we will divide our proof into two cases, namely, $u \neq 0$ and $u = 0$.
(i) Case (I): $u \neq 0$.
We also observe that from (7.85) and (7.89) we have

$$\lim_{k\to+\infty}\int_0^T\int_M \left(a(x)\,g^2(u_k') + a(x)\,u_k'^2\right)\mathrm{d}M\;\mathrm{d}t = 0 \tag{7.90}$$

Passing to the limit in the equation, when $k \to +\infty$, we get,

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{on } M \times (0,T) \\ u_t = 0 & \text{on } M_* \times (0,T), \end{cases} \tag{7.91}$$

and for $u_t = \mu$, we obtain, in the distributional sense

$$\begin{cases} \mu_{tt} - \Delta \mu = 0 & \text{on } M \times (0,T), \\ \mu = 0 & \text{on } M_* \times (0,T). \end{cases}$$

From uniqueness results given by Remark 5.1 (after Theorem 5.1) we conclude that $\mu \equiv 0$, that is, $u_t = 0$. Returning to (7.91) we obtain the following elliptic equation for almost everywhere $t \in (0,T)$ given by

$$\begin{cases} \Delta u = 0 & \text{on } M \\ u_t = 0 & \text{on } M, \end{cases}$$

which implies that $u = 0$, which is a contradiction.

(ii) Case (II): $u = 0$.
Defining

$$c_k := \left[\int_0^T \int_M |u_k|^2 \mathrm{d}M \ \mathrm{d}t \right]^{1/2}, \tag{7.92}$$

and

$$\overline{u}_k := \frac{1}{c_k} u_k, \tag{7.93}$$

we obtain

$$\int_0^T \int_M |\overline{u}_k|^2 \mathrm{d}M \ \mathrm{d}t = \int_0^T \int_M \frac{|u_k|^2}{c_k^2} \mathrm{d}M \ \mathrm{d}t = \frac{1}{c_k^2} \int_0^T \int_M |u_k|^2 \mathrm{d}M \ \mathrm{d}t = 1. \tag{7.94}$$

Setting

$$\overline{E}_k(t) := \frac{1}{2} \int_M |\overline{u}_k'|^2 \, \mathrm{d}M + \frac{1}{2} \int_M |\nabla \overline{u}_k|^2 \, \mathrm{d}M,$$

we deduce automatically that

$$\overline{E}_k(t) = \frac{E_k(t)}{c_k^2}. \tag{7.95}$$

Recalling (7.82) we obtain, for T large enough that

$$E(T) \leqq \hat{C} \left[\int_0^T \int_M (a(x)\, g^2(u_t) + a(x)\, u_t^2) \, \mathrm{d}M \ \mathrm{d}t + \int_0^T \int_M |u|^2 \, \mathrm{d}M \ \mathrm{d}t \right],$$

and employing the identity $E(T) - E(0) = -\int_0^T \int_M a(x)\, g(u_t)\, u_t \, \mathrm{d}M \ \mathrm{d}t$, we can write

$$E(t) \leqq E(0) \leqq \tilde{C} \left[\int_0^T \int_M (a(x)\, g^2(u_t) + a(x)\, u_t^2) \, \mathrm{d}M \ \mathrm{d}t + \int_0^T \int_M |u|^2 \, \mathrm{d}M \ \mathrm{d}t \right],$$

for all $t \in (0, T)$, with T large enough. The last inequality and (7.95) give us

$$\overline{E}_k(t) := \frac{E_k(t)}{c_k^2} \leqq \tilde{C} \left[\frac{\int_0^T \int_M (a(x)\, g^2(u_k') + a(x)\, u_k'^2)}{\int_0^T \int_M |u_k|^2 \, \mathrm{d}M \ \mathrm{d}t} + 1 \right]. \tag{7.96}$$

From (7.85) and (7.96) we conclude that there exists a positive constant $\hat{V}$ such that

$$\overline{E}_k(t) := \frac{E_k(t)}{c_k^2} \leqq \hat{C}, \quad \text{for all } t \in [0, T] \text{ and for all } k \in \mathbb{N},$$

that is,

$$\frac{1}{2}\int_M |\overline{u}_k'|^2\,\mathrm{d}M + \frac{1}{2}\int_\Omega |\nabla \overline{u}_k|^2\,\mathrm{d}M \leqq \hat{C}, \quad \text{for all } t \in [0, T] \text{ and for all } k \in \mathbb{N}. \tag{7.97}$$

For a subsequence $\{\overline{u}_k\}$, we obtain

$$\overline{u}_k \rightharpoonup \overline{u} \text{ weak star in } L^\infty(0, T; V), \tag{7.98}$$

$$\overline{u}_k' \rightharpoonup \overline{u}' \text{ weak star in } L^\infty(0, T; L^2(M)), \tag{7.99}$$

$$\overline{u}_k \to \overline{u} \text{ strongly in } L^2(0, T; L^2(M)). \tag{7.100}$$

We observe that from (7.85) we deduce

$$\lim_{k\to+\infty}\int_0^T\int_M \frac{a(x)\,g^2(u_k')}{c_k^2}\,\mathrm{d}M\ \mathrm{d}t = 0 \ \text{ and } \lim_{k\to+\infty}\int_0^T\int_M a(x)\,|\overline{u}_k'|^2\,\mathrm{d}M\ \mathrm{d}t = 0. \tag{7.101}$$

In addition $\overline{u}_k$ satisfies the equation

$$\overline{u}_k'' - \Delta\overline{u}_k + a(x)\,\frac{g(u_k')}{c_k} = 0 \quad \text{on } M \times (0, T).$$

Passing to the limit when $k \to +\infty$ and taking the above convergence into account, we obtain

$$\begin{cases} \overline{u}'' - \Delta\overline{u} = 0 & \text{on } M \times (0, T), \\ \overline{u}' = 0 & \text{on } M_* \times (0, T). \end{cases} \tag{7.102}$$

Then, $\mu = \overline{u}_t$ verifies, in the distributional sense

$$\begin{cases} \mu_{tt} - \Delta\,\mu = 0 & \text{on } M \\ \mu = 0 & \text{on } M_*. \end{cases}$$

Applying the same idea used in case $u \neq 0$ we have that $\mu = \overline{u}_t = 0$. Returning to (7.102) we obtain, for almost everywhere $t \in (0, T)$ that

$$\begin{cases} \Delta\,\overline{u} = 0 & \text{on } M \\ \overline{u}_t = 0 & \text{on } M, \end{cases}$$

from which we deduce that $\overline{u} = 0$, which is a contradiction in view of (7.94) and (7.100). The lemma is proved. □

Inequalities (7.82) and (7.83) lead us to the following result.

Proposition 5.2.2. *For $T > 0$ large enough, the solution $[u, u_t]$ of (2.1) satisfies*

$$E(T) \leqq C\int_0^T\int_M \left[a(x)\,|u_t|^2 + a(x)\,|g\,(u_t)|^2\right]\mathrm{d}M\,\mathrm{d}t \tag{7.103}$$

where the constant $C = C(T_0, E(0), C, a_0, \lambda_1, R_1, R_2, ||a||_{L^\infty(M)}, n, \frac{M}{\varepsilon^2})$.

7.1. Conclusion of Theorem 3.1

In what follows we will proceed exactly as in Lasiecka and Tataru's work [13] (see Lemma 3.2 and Lemma 3.3 of the referred paper) adapted to our context. Let $\Sigma := M \times (0, T)$,

$$\Sigma_\alpha = \{(t, x) \in \Sigma / \ \ |u_t| > 1 \ \text{ almost everywhere}\},$$
$$\Sigma_\beta = \Sigma \backslash \Sigma_\alpha.$$

Then using hypothesis (iii) in Assumption 2.1, we obtain

$$\int_{\Sigma_\alpha} a(x) \left([g(u_t)]^2 + (u_t)^2\right) \mathrm{d}\Sigma_\alpha \leqq \left(k^{-1} + K\right) \int_{\Sigma_\alpha} a(x) g(u_t)\, u_t \mathrm{d}\Sigma_\alpha. \tag{7.104}$$

Moreover, from (2.8)

$$\int_{\Sigma_\beta} a(x) \left([g(u_t)]^2 + (u_t)^2\right) \mathrm{d}\Sigma_\beta \leqq (1 + ||a||_\infty) \int_{\Sigma_\beta} h\,(a(x) g(u_t)\, u_t)\, \mathrm{d}\Sigma_\beta. \tag{7.105}$$

Then by Jensen's inequality

$$\begin{aligned}
&(1 + ||a||_\infty) \int_{\Sigma_\beta} h\,(g(u_t)\, u_t)\, \mathrm{d}\Sigma_\beta \\
&\quad \leqq (1 + ||a||_\infty) meas\,(\Sigma)\, h \left(\frac{1}{meas\,(\Sigma)} \int_\Sigma a(x) g(u_t)\, u_t \mathrm{d}\Sigma\right) \\
&\quad = (1 + ||a||_\infty) meas\,(\Sigma)\, \beta \left(\int_\Sigma a(x) g(u_t)\, u_t \mathrm{d}\Sigma\right),
\end{aligned} \tag{7.106}$$

where $\beta(s) = h(\frac{s}{meas(\Sigma)})$ is defined in (2.9). Thus

$$\begin{aligned}
\int_\Sigma a(x) \left([g(u_t)]^2 + (u_t)^2\right) \mathrm{d}\Sigma \leqq \left(k^{-1} + K\right) &\int_\Sigma a(x) g(u_t)_t\, \mathrm{d}\Sigma_1 \\
&+ (1 + ||a||_\infty) meas\,(\Sigma)\, \beta \\
&\times \left(\int_{\Sigma_1} a(x) g(u_t)\, u_t \mathrm{d}\Sigma\right).
\end{aligned} \tag{7.107}$$

Splicing, together, (7.103) and (7.107), we have

$$\begin{aligned}
E(T) \leqq (1 + ||a||_\infty) C \Bigg[&\frac{K_0}{(1 + ||a||_\infty)} \int_\Sigma g(u_t)\, u_t \mathrm{d}\Sigma_1 \\
&+ meas\,(\Sigma)\, \beta \left(\int_\Sigma a(x)\, g(u_t)\, u_t \mathrm{d}\Sigma\right)\Bigg],
\end{aligned} \tag{7.108}$$

where $K_0 = k^{-1} + K$. Setting

$$L = \frac{1}{C\, meas\,(\Sigma)\,(1 + ||a||_\infty)},$$
$$c = \frac{M_0}{meas\,(\Sigma)\,(1 + ||a||_\infty)},$$

we obtain

$$p\,[E(T)] \leqq \int_\Sigma a(x)\, g\,(u_t)\, u_t\, \mathrm{d}\Sigma = E(0) - E(T), \tag{7.109}$$

where the function p is as defined in (2.10). To finish the proof of Theorem 3.1, we invoke the following result from I. LASIECKA et al. [13]:

Lemma B. *Let p be a positive, increasing function such that $p(0) = 0$. Since p is increasing we can define an increasing function q, $q(x) = x - (I + p)^{-1}(x)$. Consider a sequence s_n of positive numbers which satisfies*

$$s_{m+1} + p(s_{m+1}) \leqq s_m.$$

Then $s_m \leqq S(m)$, where $S(t)$ is a solution of the differential equation

$$\frac{d}{\mathrm{d}t} S(t) + q(S(t)) = 0, \quad S(0) = s_0.$$

Moreover, if $p(x) > 0$ for $x > 0$, then $\lim_{t\to\infty} S(t) = 0$.

With this result in mind, we replace T (resp. 0) in (7.109) with $m(T + 1)$ (resp. mT) to obtain

$$E(m(T+1)) + p\,(E(m(T+1))) \leqq E(mT), \quad \text{for } m = 0, 1, \dots. \tag{7.110}$$

Applying Lemma B with $s_m = E(mT)$ thus results in

$$E(mT) \leqq S(m), \quad m = 0, 1, \dots. \tag{7.111}$$

Finally, using the dissipativity of $E(t)$ inherent in the relation (2.7), we have for $t = mT + \tau, 0 \leqq \tau \leqq T$,

$$E(t) \leqq E(mT) \leqq S(m) \leqq S\left(\frac{t-\tau}{T}\right) \leqq S\left(\frac{t}{T} - 1\right) \quad \text{for } t > T, \tag{7.112}$$

where we have used above the fact that $S(.)$ is dissipative. The proof of Theorem 2.2 is now completed.

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References

1. Alabau-Boussouira, F.: Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems. *Appl. Math. Optim.* **51**(1), 61–105 (2005)
2. Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.* **30**(5), 1024–1065 (1992)
3. Bellassoued, M.: Decay of solutions of the elastic wave equation with a localized dissipation. *Annales de la Faculté des Sciences de Toulouse* **XII**(3), 267–301 (2003)
4. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Fukuoka, R., Soriano, J.A.: Uniform stabilization of the wave equation on compact surfaces and locally distributed damping. *Methods Appl. Anal.* **15**(4), 405–426 (2008)
5. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Fukuoka, R., Soriano, J.A.: Uniform stabilization of the wave equation on compact surfaces and locally distributed damping—a sharp result. *Trans. AMS* **361**(9), 4561–4580 (2009)
6. Cavalcanti, M.M., Oquendo, H.P.: Frictional versus viscoelastic damping in a semilinear wave equation. *SIAM J. Control Optim.* **42**(4), 1310–1324 (2003)
7. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Lasiecka, I.: Wellposedness and optimal decay rates for wave equation with nonlinear boundary damping-source interaction. *J. Differ. Equ.* **236**, 407–459 (2007)
8. Christianson, H.: Semiclassical non-concentration near hyperbolic orbits. *J. Funct. Anal.* **246**(2), 145–195 (2007)
9. Dafermos, C.M.: Asymptotic behavior of solutions of evolution equations. Nonlinear evolution equations (Proc. Sympos., Univ. Wisconsin, Madison, Wis., 1977), Publ. Math. Res. Center Univ. Wisconsin, vol. 40. Academic Press, New York, 103–123, 1978
10. do Carmo, M.P.: *Riemannian Geometry. Mathematics: Theory & Applications*. Birkhäuser Boston, Inc., Boston, 1992
11. Fukuoka, R.: Mollifier smoothing of tensor fields on differentiable manifolds and applications to Riemannian Geometry. http://arxiv.org/abs/math.DG/0608230
12. Hitrik, M.: Expansions and eigenfrequencies for damped wave equations. *Journées équations aux Dérivées Partielles (Plestin-les-Grèves, 2001)*, Exp. No. VI, 10 pp., Univ. Nantes, Nantes, 2001
13. Lasiecka, I., Tataru, D.: Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping. *Differ. Integral Equ.* **6**, 507–533 (1993)
14. Lasiecka, I., Triggiani, R., Yao, P.F.: Inverse/observability estimates for second order hyperbolic equations with variable coefficients. *J. Math. Anal. Appl.* **235**, 13–57 (1999)
15. Lions, J.L.: Controlabilité exacte, perturbations et stabilisation de systèmes distribués, tome 1, Masson, 1988
16. Lions, J.L., Magenes, E.: *Problèmes Aux Limites Non Homogènes et Applications*, Vol. 1. Dunod, Paris, 1968
17. Liu, K.: Localy distributed control and damping for conservative systems. *SIAM J. Control Optim.* **35**(5), 1574–1590 (1997)
18. Martinez, P.: A new method to obtain decay rate estimates for dissipative systems with localized damping. *Rev. Mat. Complutense* **12**(1), 251–283 (1999)
19. Miller, L.: Escape function conditions for the observation, control, and stabilization of the wave equation. *SIAM J. Control Optim.* **41**(5), 1554–1566 (2002)
20. Nakao, M.: Decay of solutions of the wave equation with a local nonlinear dissipation. *Math. Ann.* **305**, 403–417 (1996)
21. Nakao, M.: Decay of solutions of the wave equation with local degenerate dissipation. *Isr. J. Math.* **95**, 25–42 (1996)
22. Nakao, M.: Decay and global existence for nonlinear wave equations with localized dissipations in general exterior domains. In: *New Trends in the Theory of Hyperbolic Equations*. Oper. Theory Adv. Appl., 159. Birkhäuser, Basel, 213–299, 2005

23. Nakao, M.: Energy decay for the wave equation with boundary and localized dissipations in exterior domains. *Math. Nachr.* **278**(7–8), 771–783 (2005)
24. Rauch, J., Taylor, M.: Exponential decay of solutions to hyperbolic equations in bounded domains. *Indiana Univ. Math. J.* **24**, 79–86 (1974)
25. Rauch, J., Taylor, M.: Decay of solutions to n ondissipative hyperbolic systems on compact manifolds. *Comm. Pure Appl. Math.* **28**(4), 501–523 (1975)
26. Rauch, J.: Qualitative behavior of dissipative wave equations on bounded domains. *Arch. Rational Mech. Anal.* **62**(1), 77–85 (1976)
27. Spivak, M.: *A Comprehensive Introduction to Differential Geometry*, Vols. 1–5. Publish or Perish, California, 1999
28. Taylor, M.: *Partial Differential Equations, 1, 2*. Springer, Berlin, 1991
29. Toundykov, D.: Optimal decay rates for solutions of nonlinear wave equation with localized nonlinear dissipation of unrestricted growth and critical exponents source terms under mixed boundary. *Nonlinear Anal. T. M. A.* **67**(2), 512–544 (2007)
30. Triggiani, R., Yao, P.F.: Carleman estimates with no lower-order terms for general Riemannian wave equations. Global uniqueness and observability in one shot. *Appl. Math. Optim.* **46**, 331–375 (2002). Special issue dedicated to J. L. Lions
31. Warner, F.W.: *Foundations of Differentiable Manifolds and Lie Groups*. Scott, Foresman and Company, Glenview, 1971
32. Zuazua, E.: Exponential decay for the semilinear wave equation with locally distributed damping. *Comm. Partial Differ. Equ.* **15**(2), 205–235 (1990)

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