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Asymptotic Behavior of Solutions to a Model for the Flow of a Reacting Fluid

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Abstract

We give a complete analysis of solutions of a model for the flow of a multispecies reacting fluid occupying a thin cylinder whose walls may be semipermeable with respect to some or all of the chemical species. We prove the global existence of solutions and establish a number of time-independent a priori bounds sufficient to determine the corresponding time-asymptotic steady-state. We then derive necessary conditions and sufficient conditions ensuring that this steady-state reflects complete combustion, that is, that at least one of the reactant species is depleted.

1. Introduction

We give a complete analysis of solutions of a model for the flow of a multispecies reacting fluid occupying a thin cylinder whose walls may be semipermeable with respect to some or all of the chemical species. We prove the global existence of solutions and establish a number of time-independent a priori bounds sufficient to determine the corresponding time-asymptotic steady-state. We then derive necessary conditions and sufficient conditions ensuring that this steady-state reflects complete combustion, that is, that at least one of the reactant species is depleted.

More specifically, we consider a multispecies fluid occupying the unit interval and described at each point by the fluid density, velocity, temperature, and species mass fractions. Under certain conditions a chemical reaction will occur and may result in the influx or efflux of a specified mass of certain of the species through the cylinder walls at the point of the reaction. We give a careful derivation of the fluid equations corresponding to mass balance, Newton's second law, energy balance, and elementary chemical kinetics, resulting in what is essentially the Navier-Stokes

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system for heat-conducting, chemically reacting flow. This derivation is carried out in Section 2.

Of particular importance is the construction of the correct physical entropy and the proof of the most basic of our a priori bounds showing that the space integral of entropy is bounded independently of time. This is the essential starting point for the derivation of time-independent a priori bounds in stronger norms leading to the proof of global existence of fairly general large-energy solutions. Once the correct entropy has been constructed, the overall analysis follows somewhat familiar lines, although there is considerable technical complexity involved in adapting existing ideas to the present context. One of these deserves special mention, namely the surprising fact, first observed by Kazhikov and Shelukhin [10], that for the equations of nonreacting flow, *pointwise* bounds for density and temperature can be derived from the rather weak L^1 bound for entropy. The proof uses neither maximum principles nor Sobolev estimates but rather exploits a certain dissipative mechanism resulting from the compressive effect of pressure applied along particle trajectories. The pointwise bounds first given in [10] were time-dependent, however, and therefore gave no information about time-asymptotic behavior. A significant improvement was made by Chen [3] who by amplifying the analysis of [10] derived pointwise estimates for density and temperature which are in fact independent of time, now for a simpler model of reacting flow. (Similar time-independent pointwise bounds for the reduced case of barotropic, nonreacting flow have been given by a number of authors; see [11], for example, and the references contained therein). In the present paper, we adapt Chen's analysis to the present context, again with considerable increase in technical complexity, thereby achieving time-independent pointwise control of density and temperature. This in turn enables us to derive a number of parabolic L^2 bounds for higher derivatives of velocity and temperature. These incorporate initial layer effects corresponding to nonsmooth initial data and to some extent follow the analysis of Hoff [8] for nonreacting flows. There results the global existence of solutions corresponding to quite general large, nonsmooth initial data. This existence result is stated in Theorem 2.1 in Section 2 and is proved in Section 3.

Next, we show that the various time-independent a priori bounds derived in the existence theory are sufficiently strong to imply the time-asymptotic compactness of solutions in various spatial norms. In particular, we show that as time tends to infinity, velocity tends to zero, temperature to a constant, and density and mass fractions to certain functions of space, balanced so that the asymptotic pressure is constant. These results are stated precisely in Theorem 2.2 in Section 2 and are proved in Section 4. Finally in Theorem 2.3 we analyze the steady-state solution in a fairly representative case and give necessary conditions and sufficient conditions showing that if the total initial energy is sufficiently large depending on the system parameters, then the fluid undergoes complete combustion, that is, that in the time asymptotic state at least one reactant species is depleted. Theorem 2.3 is stated below in Section 2 and is proved in Section 5.

The model we consider is based on standard considerations for the thermodynamics of multicomponent ideal fluids and on standard models of compressible, viscous fluid flow. Complete discussions are given in Callen [2, Appendix D.6] and Williams [12, Appendix C]; see also Giovangigli [6]. The model is fairly general in that a wide range of chemical reactions is allowed and the cylinder walls may be semipermeable with respect to some or all of the chemical species. There are two shortcomings of the model that should be noted, however. First, the viscosity and heat conduction coefficients are taken to be positive constants, whereas arguments based on scaling limits of the Boltzmann equation suggest that these should be positive powers of the temperature (see Williams [12, pp. 640 and 642] or Boltzmann [1, p. 176], for example). This dependence could be accommodated in the entropy bound discussed above, but the rather intricate argument leading from the entropy bound to pointwise estimates for density and temperature would fail, as would the subsequent derivation of parabolic estimates for higher derivatives of temperature and velocity, at least in the absence of overly severe restrictions on the initial data. We point out that, while there is a significant literature for the case that these coefficients depend on density (which would be the predicted dependence when the flow is assumed to be isentropic or isothermal) a completely satisfactory theory of viscous compressible flow in the regularity class of the present paper and with the predicted temperature dependence of viscosity and heat conduction coefficients remains open (but see the remarks below concerning related results in [5]). The other weakness in the model is the assumption that fluid particles of different species have the same velocities, whereas, since their molecular weights may differ, so too should their accelerations; hence their velocities, when species-independent forces are present. The difficulty in identifying a viscosity for such a model is discussed briefly but left unresolved in [12, p. 614]. A satisfactory model incorporating different velocities and accelerations for different species would clearly require a level of complexity significantly greater than that of the present work.

We call special attention to the results of Feireisl et al. [5], who consider a similar model for multispecies reacting flow in the more realistic case of three space dimensions. Viscosity and heat-conduction coefficients are allowed to depend on temperature in a certain way, but species diffusion must be included. Quite general large-energy initial data is considered, and the authors prove the global existence of weak solutions satisfying a certain entropy-rate inequality in place of the energy equation. The analysis includes a number of difficult and intricately-related weak convergence arguments for approximate solutions, issues which do not arise in the one-dimensional case considered here. Indeed, the restriction to one space dimension enables us to show that even large-energy solutions acquire some limited regularity in positive time. This fact makes the weak convergence analysis of [5] unnecessary and also allows for the determination of the time-asymptotic behavior and the derivation of nearly sharp criteria for complete combustion. We also point out that, while it may be possible to adapt certain of the arguments of [5] to the present case in order to accommodate some dependence of viscosity and heat conduction coefficients on temperature, we are doubtful that the specific dependence predicted by scaling limits of solutions of the Boltzmann equation could be included.

Our results extend those of Chen et al. [4] in which a simpler chemical reaction was considered and the a priori bounds supporting the existence theory were time-dependent, so that a determination of the time-asymptotic behavior could not be made. In the present paper we allow much more general chemical reactions, we allow the container wall to be semipermeable, and most important, by careful modeling and mathematical analysis, we are able to establish time-independent bounds which do in fact enable a determination of the asymptotic behavior.

There is by now a substantial literature on the modeling of multicomponent reacting fluids and related mathematical problems. The reader may consult [6] and [12] for references to the scientific and engineering literature and [5] for references to the mathematics literature.

2. Derivation of the model and statement of results

In this section we give a careful derivation of the model under consideration, a detailed list of our assumptions about the system parameters, and precise statements of our results on existence, large-time behavior, and complete combustion.

We consider a mixture of chemical species $\mathcal{M}_1, \ldots, \mathcal{M}_J$ occupying a thin cylinder along [0, 1] and which under appropriate conditions undergo the reaction

$$\sum_{j=1}^{J} \nu'_j \mathcal{M}_j \rightharpoonup \sum_{j=1}^{J} \nu''_j \mathcal{M}_j \tag{2.1}$$

in which ν'_j and ν''_j are nonnegative integers representing the number of particles of species $\mathcal{M}_j$ present respectively before and after the reaction. Since mass is conserved,

$$\sum_{j=1}^{J} \nu'_j w_j = \sum_{j=1}^{J} \nu''_j w_j \tag{2.2}$$

where w_j are the respective molecular weights (mass per mole number). We will allow for the possibility that the walls of the cylinder are semipermeable in the sense that, when a reaction does occur, μ'_j particles of species $\mathcal{M}_j$ enter the tube and μ''_j particles exit the tube at the point of the reaction, μ'_j and μ''_j being nonnegative integers. For example, in the simplest (conceptual if not practical) realization of a hydrogen cell, we may imagine a conducting cylinder split by a plane containing the axis of symmetry, with the two halves oppositely charged and separated by an electrical insulator. The tube is filled with an oxygen-rich mixture and when a reaction occurs, charged particles enter the tube from its walls, thus effecting the well-known reaction $2H_2 + 0_2 \rightharpoonup 2H_2 0$.

The individual species in the tube will be assumed to be ideal polytropic fluids having the same velocity $u(x,t)$ and temperature $\theta(x,t)$ at each point (x,t). The composite density will be denoted $\rho(x,t)$ and the mass fractions by $z^j(x,t)$, $j = 1, \ldots, J$. We shall derive evolution equations for the unknown functions ρ, u, θ, and $z = (z^1, \ldots, z^J)$ corresponding to the balance of mass, momentum, and energy and elementary chemical kinetics. First we introduce composite state functions $P(\rho, \theta, z)$ and $e(\theta, z)$ representing pressure and specific internal energy, and a cumulative reaction rate function $F(\theta, z)$ as follows. Recall that, if N_j moles of species $\mathcal{M}_j$ are confined at equilibrium in a volume V, then the static pressure and energy will be

$$P_j = \frac{RN_j\theta}{V} \quad \text{and} \quad U_j = c_j N_j \theta$$

where R is the universal gas constant and c_j is a constant specific heat (usually denoted c_{Vj}). If the volume is a segment of length $\mathrm{d}x$ with unit cross-sectional area, then $V = \mathrm{d}x$ and $N_j w_j = z^j \rho \mathrm{d}x$, so that

$$P_j = \frac{Rz^j \rho\theta}{w_j} \quad \text{and} \quad U_j = \frac{c_j}{w_j} z^j \rho\theta \mathrm{d}x.$$

The macroscopic pressure is then

$$P = \sum_{j=1}^{J} P_j = \frac{R\rho\theta}{w(z)} \quad \text{where } \frac{1}{w(z)} = \sum_{j=1}^{J} \frac{z^j}{w_j} \tag{2.3}$$

and the composite energy is $\rho e \mathrm{d}x$ where e is the composite specific internal energy

$$e = c(z)\theta \quad \text{where } c(z) = \sum_{j=1}^{J} \frac{c^j}{w_j} z^j. \tag{2.4}$$

The cumulative rate function F is described as follows. If $x_k(t)$, $k = 1, 2$, are the positions at time t of two fixed fluid particles with $x_1 < x_2$, then the number of reactions occurring in the set

$$\{(x, s) : 0 \leqq s \leqq t \quad \text{and} \quad x_1(s) \leqq x \leqq x_2(s)\}$$

divided by Avogadro's number is

$$\int_0^t \int_{x_1(s)}^{x_2(s)} \rho F \mathrm{d}x \mathrm{d}s.$$

We shall assume that there is such function F, that F is determined by θ and z, and that $F = 0$ when either $\theta \leqq \theta_{ig}$, where θ_{ig} is a positive ignition temperature below which the reaction (2.1) cannot occur, or when any "reactant species", that is, a species $\mathcal{M}_j$ for which $a_j < 0$, is depleted.

We can now derive the evolution equations for ρ, u, θ, and z. First, the increase over a time interval $\mathrm{d}t$ in the mass of fluid occupying a segment $(x, x + \mathrm{d}x)$ moving with the fluid is the net influx of mass per reaction times the number of reactions occurring over that space-time interval; this product is $[\sum_{j=1}^{J} (\mu'_j - \mu''_j) w_j] \rho F \mathrm{d}x \mathrm{d}t$. It then follows in the usual way that

$$\rho_t + (\rho u)_x = a\rho F \tag{2.5}$$

where

$$a = \sum_{j=1}^{J} (\mu'_j - \mu''_j) w_j \equiv a' - a''.$$

Exactly the same considerations applied to the density ρz^j of the jth species show that

$$(\rho z^j)_t + (\rho z^j u)_x = a_j \rho F$$

where

$$a_j = (\nu_j'' - \nu_j' + \mu_j' - \mu_j'')\, w_j.$$

Applying (2.5), we then find that

$$z_t^j + z_x^j u = (a_j - a z^j) F. \tag{2.6}$$

Of course, the mass fractions z^j should be nonnegative and sum to one. This will be true if it is true initially and if F satisfies the condition that, if $a_j < 0$ for some j, then $F(\theta, z) = 0$ whenever $z^j = 0$: first, it is clear that this condition implies the persistence of the nonnegativity of a_j when $a_j < 0$, by (2.6). For other j we have that $\dot{z}^j \geqq -aFz^j$, again by (2.6), where the dot denotes the convective derivative $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$. Thus z^j remains nonnegative for these j as well. Also, if we define $\zeta = \sum_{j=1}^J z^j$, then $\dot{\zeta} = aF(1-\zeta)$ since $\sum a_j = a$ by (2.2), and therefore $\zeta(x,t) = 1$ for all (x,t) if $\zeta(x,0) = 1$ for all x.

Next, while it is true that chemical reactions do not increase or decrease momentum overall [12, p. 609], there may be a net change in both momentum and energy due to the influx or efflux of particles at the time and location of a reaction. We shall assume that particles which exit the tube do so at the point of reaction and with the local velocity and temperature u and θ, and that particles which enter do so at the point of reaction and with velocity and temperature $\bar{u}$ and $\bar{\theta}$, where $\bar{u}$ may be either u or a constant and $\bar{\theta}$ may be either θ or a constant. The change over a time interval $\mathrm{d}t$ in the momentum $\rho u \mathrm{d}x$ of the fluid occupying an interval $(x, x+\mathrm{d}x)$ moving with the fluid is therefore the usual contribution $(\varepsilon u_x - P)|_x^{x+\mathrm{d}x}\mathrm{d}t$, where ε is a positive viscosity constant, plus the term

$$\left[\sum_{j=1}^J \left(\mu_j' w_j \bar{u} - \mu_j'' w_j u\right)\right] \rho F \mathrm{d}x\mathrm{d}t = \left(a'\bar{u} - a''u\right)\rho F \mathrm{d}x\mathrm{d}t.$$

We thus conclude that

$$(\rho u)_t + (\rho u^2 + P)_x = \varepsilon u_{xx} + (a'\bar{u} - a''u)\rho F.$$

Similarly, the change over a time interval $\mathrm{d}t$ in the total energy $\rho[e(\theta,z) + \frac{1}{2}u^2]$ of the fluid occupying an interval $(x, x+\mathrm{d}x)$ moving with the fluid is the sum of four terms: the impulse applied by the viscous and pressure terms $(\varepsilon u u_x - uP)|_x^{x+\mathrm{d}x}\mathrm{d}t$, a heat-conduction term $\lambda\theta_x|_x^{x+\mathrm{d}x}\mathrm{d}t$, where λ is a positive thermal conductivity constant, the heat of the reaction $\rho Q \mathrm{d}x\mathrm{d}t$, where $Q = Q(\theta, z)$ will be assumed to satisfy $|Q(\theta,z)| \leqq CF(\theta,z)$ for a constant C, and a term measuring the net influx or efflux of kinetic and thermal energy at points of reaction. Elementary considerations show that the latter contributions are

$$\tfrac{1}{2}\sum_{j=1}^{J}\left(\mu'_j w_j \bar{u}^2 - \mu''_j w_j u^2\right)\rho F \mathrm{d}x\mathrm{d}t + \sum_{j=1}^{J}\left(\mu'_j c_j \bar{\theta} - \mu''_j c_j \theta\right)\rho F \mathrm{d}x\mathrm{d}t.$$

Combining all these terms and defining

$$\kappa' = \sum \mu'_j c_j \quad \text{and} \quad \kappa'' = \sum \mu''_j c_j$$

we then conclude that

$$(\rho E)_t + (\rho E u)_x + (uP)_x = \lambda\theta_{xx} + \rho Q + \left[\tfrac{1}{2}\left(a'\bar{u}^2 - a''u^2\right) + \left(\kappa'\bar{\theta} - \kappa''\theta\right)\right]\rho F,$$

where $E = e(\theta, z) + \frac{1}{2}u^2$.

We now combine the above equations and notations into a single system:

$$\rho_t + (\rho u)_x = a\rho F, \tag{2.7}$$

$$(\rho u)_t + (\rho u^2 + P(\rho, \theta, z))_x = \varepsilon u_{xx} + (a'\bar{u} - a''u)\rho F, \tag{2.8}$$

$$\begin{aligned} &\left(\rho\left(e(\theta, z) + \tfrac{1}{2}u^2\right)\right)_t + \left(\rho\left(e(\theta, z) + \tfrac{1}{2}u^2\right)u\right)_x + (uP(\rho, \theta, z))_x \\ &\quad = \lambda\theta_{xx} + \varepsilon(uu_x)_x + \rho Q(\theta, z) + \left[\tfrac{1}{2}(a'\bar{u}^2 - a''u^2) + (\kappa'\bar{\theta} - \kappa\theta)\right]\rho F(\theta, z), \end{aligned} \tag{2.9}$$

$$z^j_t + uz^j_x = (a_j - az^j)F(\theta, z), \tag{2.10}$$

which is to be solved for $x \in (0, 1)$ and $t \geqq 0$ subject to boundary conditions

$$u(0, t) = u(1, t) = \theta_x(0, t) = \theta_x(1, t) = 0 \tag{2.11}$$

and initial conditions

$$(\rho, u, \theta, z)(x, 0) = (\rho_0, u_0, \theta_0, z_0)(x). \tag{2.12}$$

In the above system ε and λ are positive constants,

$$\begin{aligned} &a = a' - a'', \quad a' = \sum \mu'_j w_j, \quad a'' = \sum \mu''_j w_j, \\ &a_j = (\nu''_j - \nu'_j - \mu''_j + \mu'_j)w_j, \\ &\kappa' = \sum \mu'_j c_j, \quad \kappa'' = \sum \mu''_j c_j, \end{aligned}$$

where w_j and c_j are positive constants and μ'_j, μ''_j, ν'_j and ν''_j are nonnegative integers satisfying (2.2), and the functions P and e are as defined in (2.3) and (2.4).

We shall make the following assumptions:

Either $\bar{u}$ is a constant or $\bar{u} = u$ and $a \leqq 0$. (2.13)

Either $\bar{\theta}$ is a constant or $\bar{\theta} = \theta$ and $\kappa' - \kappa'' \leqq 0$. (2.14)

There is a j such that $a_j < 0$. (2.15)

There is a j such that $a_j \geqq 0$ and if $a < 0$ then there is a j such that $a_j > 0$. (2.16)

F and Q are Lipschitz functions of θ and z on $(0, \infty) \times V$, where V is a neighborhood of the set $\{z : z^j \geqq 0 \text{ for all } j \text{ and } \sum z^j = 1\}$, and there are constants C and C' such that (2.17)

$$|Q(\theta, z)| \leqq CF(\theta, z) \leqq C'$$

on $(0, \infty) \times V$.

There is a nonnegative function $h = h(\theta)$ defined on $(0, \infty)$ and a constant C such that, if $G^j(\theta, z) \equiv (a_j - az^j)F(\theta, z)$, then (2.18)

$$\left|\frac{\partial G^j}{\partial \theta}(\theta, z)\right| \leqq C\frac{h(\theta)^{1/2}}{\theta}$$

and

$$\xi \cdot \nabla_z G(\theta, z)\, \xi \leqq -h(\theta)\, |\xi|^2,\ (\theta, z) \in (0, \infty) \times V \text{ and } \xi \in \mathbf{R}^J.$$

There is a positive temperature θ_{ig} such that $F(\theta, z) = 0$ for $\theta \leqq \theta_{ig}$; and if $a_j < 0$ for a particular j, then $F(\theta, z) = 0$ when $z^j = 0$. (2.19)

$$\sum_{j=1}^{J} (\nu_j'' - \nu_j' + \mu_j' - \mu_j'') = 0. \tag{2.20}$$

Some comments on these hypotheses are in order. First, a is the net mass entering the system as a result of a single reaction, so that the assumption (2.13) disallows the net influx of momentum and kinetic energy except at a given, fixed constant velocity and (2.14) disallows the net influx of thermal energy except at a given, fixed temperature. The assumptions (2.15) and (2.16) stipulate that there are both reactant species and product species occurring in the reaction (2.1) (taking into account possible influx or efflux of species).

Next, we illustrate the assumptions (2.17)–(2.19) by examining the simple but representative case that there is a single reactant species $\mathcal{M}_1$ and a single product species $\mathcal{M}_2$ and that there is no influx or efflux of particles at all (this is the case considered in [4], for example). In this case $a = 0$ and $a_1 < 0 < a_2 = -a_1$, and the equations (2.10) for z reduce to

$$\dot{z}_1 = a_1 F$$
$$\dot{z}_2 = -a_1 F.$$

The reaction in this situation is typically modeled by an equation $\dot{z}^1 = -h(\theta)z^1$ where h is a nonnegative Arrhenius function vanishing on $(0, \theta_{ig})$ and satisfying

$h = \exp(-\text{const.}/\theta)$ for $\theta >> \theta_{ig}$. In this case

$$F(\theta, z) = -\frac{h(\theta)z^1}{a_1} \quad \text{and} \quad G(\theta, z) = \begin{bmatrix} -h(\theta)z^1 \\ h(\theta)(1 - z^2) \end{bmatrix}$$

in our notation. The conditions in (2.18) and (2.19) are then easily checked.

Finally, the assumption (2.20) is closely related to our stipulation that the reaction (2.1) goes forward, that is, in the direction indicated, in all thermodynamical regimes under consideration. Indeed, a condition similar to but weaker than (2.20) is typically postulated as a consequence of the fact that the physical entropy cannot decrease in a chemical reaction. In the present paper the logic is reserved: we shall *assume* (2.20) and then apply it in Lemma 3.1 below to derive a time-independent bound for the entropy. See [12, pp. 529–531] for further discussion of the underlying physics.

Concerning the initial data $(\rho_0, u_0, \theta_0, z_0)$ we assume that there is a constant C_0 such that

$$C_0^{-1} \leqq \rho_0 \leqq C_0 \quad \text{almost everywhere,} \tag{2.21}$$

$$\|u_0\|_{L^4([0,1])} \leqq C_0, \tag{2.22}$$

$$\theta_0 \geqq C_0^{-1} \quad \text{and} \quad \|\theta_0\|_{L^2([0,1])} \leqq C_0, \tag{2.23}$$

$$\|z_0\|_{H^1([0,1])} \leqq C_0, \tag{2.24}$$

$$z_0^j(x) \in [0, 1] \quad \text{and} \quad \sum_{j=1}^{J} z_0^j(x) = 1, \quad x \in [0, 1], \tag{2.25}$$

and

$$\sum_{\{j:a_j \geqq 0\}} \int_0^1 \rho_0 z_0^j \mathrm{d}x \geqq C_0^{-1}. \tag{2.26}$$

In particular, there are no smallness conditions on the initial data.

The solutions we obtain will be weak solutions in the usual sense: if we write the equations in the system (2.7)–(2.10) in the form

$$v_t^j + f^j(v)_x = (b^{jk}(v)v_x^k)_x + g^j(v),$$

we then say that v is a weak solution provided that v is suitably integrable and

$$\int_0^1 v^j(x,\cdot)\psi(x,\cdot)\mathrm{d}x \,\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_0^1 (v^j\psi_t + (f^j - b^{jk}v_x^k)\psi_x + g^j\psi)\mathrm{d}x\mathrm{d}t \tag{2.27}$$

for all $t_2 \geqq t_1 \geqq 0$ and all $\psi \in C^1([0, 1] \times [0, \infty))$, with the exception that, when $v^j = \rho u$, (2.27) is required to hold only for ψ satisfying the boundary conditions $\psi(0, t) = \psi(1, t) = 0$.

The following theorem gives our main result concerning the existence and regularity of solutions of (2.7)–(2.12):

Theorem 2.1. *Assume that the system (2.7)–(2.10) satisfies the conditions (2.13)–(2.20) described above and that initial data $(\rho_0, u_0, \theta_0, z_0)$ is given satisfying (2.21)–(2.26). Then there is a global weak solution (ρ, u, θ, z) to the initial-boundary value problem (2.7)–(2.12) and there is a constant C depending on the system parameters in (2.7)–(2.10) and on C_0 such that*

$$C^{-1} \leqq \rho(x,t) \leqq C, \quad t \geqq 0,\ a.a.\ x \in (0,1), \tag{2.28}$$

$$\theta(x,t) \geqq C^{-1}\mathrm{e}^{-Ct}, \quad t \geqq 0,\ x \in [0,1], \tag{2.29}$$

$$z^j(x,t) \in [0,1] \quad \textit{and} \quad \sum_j z^j(x,t) = 1,\ t \geqq 0,\ x \in [0,1], \tag{2.30}$$

and such that the following regularity conditions and energy estimates hold:

$$\rho \in C([0,\infty);\ L^p([0,1])),\ p \in [1,\infty), \tag{2.31}$$

u *and* θ *are Hölder continuous on* $[0,1] \times [\tau,\infty)$ *for every* $\tau > 0$ (2.32)
with Hölder norms bounded by $C\max\{1,\tau\}^{-\beta}$, *where* β *is a universal positive constant,*

z *is Hölder continuous on* $[0,1] \times [0,\infty)$ *with Hölder norm bounded by* C, (2.33)

$u(\cdot,t) \to u_0$ *strongly in* L^p *as* $t \to 0$ *for* $p \in [1,4)$ (2.34)
and $\theta(\cdot,t) \rightharpoonup \theta_0$ *weakly in* L^2 *as* $t \to 0$,

$$\begin{aligned}&\sup_{t\geqq 0}\int_0^1 \left[\tfrac{1}{2}\rho u^2 + \rho e(\theta,z) + \rho(\theta - \log\theta + 1) + |z_x|^2 + \theta^2 + u^4\right]\mathrm{d}x \\ &+ \int_0^\infty\int_0^1 \left[\left(1+\frac{1}{\theta}\right)u_x^2 + \left(1+\frac{1}{\theta^2}\right)\theta_x^2 + u^2u_x^2 + h(\theta)|z_x|^2 + F\right]\mathrm{d}x\mathrm{d}t \leqq C,\end{aligned} \tag{2.35}$$

and

$$\sup_{t>0}\int_0^1 \left[\sigma(t)u_x^2 + \sigma^2\theta_x^2\right]\mathrm{d}x + \int_0^\infty\int_0^1 \left[\sigma\dot{u}^2 + \sigma^2\dot{\theta}^2 + \sigma^2\dot{u}_x^2 + \sigma^3\dot{\theta}_x^2\right]\mathrm{d}x\mathrm{d}t \leqq C, \tag{2.36}$$

where $\sigma = \min\{1,t\}$.

Theorem 2.1 is proved in Section 3.

The time-independent bounds in (2.28), (2.35), and (2.36) enable us to derive the following results concerning the large-time behavior of solutions:

Theorem 2.2. *Assume in addition to the hypotheses of Theorem 2.1 that one of the following conditions holds:* $\bar{u} = 0, \bar{u} = u$, *or there is a constant* $\tilde{C}$ *such that* $|\nabla_z F(\theta,z)| \leqq \tilde{C}h(\theta)$ *for* (θ,z) *as in (2.18). Define*

$$\begin{aligned}E_\infty = &\int_0^1 \rho_0\left(e_0 + \tfrac{1}{2}u_0^2\right)\mathrm{d}x \\ &+ \int_0^\infty\int_0^1 \left(\rho Q + \left[\tfrac{1}{2}(a'\bar{u}^2 - a''u^2) + (\kappa'\bar{\theta} - \kappa''\theta)\right]\rho F\right)\mathrm{d}x\mathrm{d}t\end{aligned}$$

and

$$M_\infty = \int_0^1 \rho_0 \, \mathrm{d}x + \int_0^\infty \int_0^1 a\rho F \, \mathrm{d}x\mathrm{d}t$$

(*which are finite by the bounds in* Theorem 2.1). *Then there is a function* $z_\infty \in H^1([0,1])$ *and a bi-Lipschitz homeomorphism* X_∞ *of* $[0,1]$ *such that, if*

$$P_\infty = RE_\infty \left[\int_0^1 c(z_\infty(x)) w(z_\infty(x)) \, \mathrm{d}x \right]^{-1}$$

where $c(z)$ *and* $w(z)$ *are as in* (2.3) *and* (2.4)*, and if*

$$\theta_\infty = \frac{P_\infty}{RM_\infty} \int_0^1 w(z_\infty(x)) \, \mathrm{d}x,$$

then as $t \to \infty$*,*

$$\rho(\cdot,t) \to \rho_\infty(\cdot) \equiv \frac{P_\infty}{R\theta_\infty} w(z_\infty(\cdot)) \text{ in } L^p([0,1]),\ p \in [1,\infty), \tag{2.37}$$

$$u(\cdot,t) \to 0 \quad \text{in } H_0^1([0,1]), \tag{2.38}$$

$$\theta(\cdot,t) \to \theta_\infty \quad \text{in } H^1([0,1]), \tag{2.39}$$

$$z(\cdot,t) \to z_\infty \quad \text{uniformly on } [0,1], \tag{2.40}$$

and

$$X(\cdot,t) \to X_\infty \quad \text{uniformly on } [0,1], \tag{2.41}$$

where $X(y,t)$ *is defined by* $\frac{dX}{\mathrm{d}t} = u(X,t)$, $X(y,0) = y$.

Theorem 2.2 is proved in Section 4.

Of particular interest are the conditions that guarantee that complete combustion has occurred, that is, that in the asymptotic state, some reactant species has been depleted. (Recall that $\mathcal{M}_j$ is a reactant species if $a_j < 0$). In view of the evolution equations (2.10) for z, this should be true when the asymptotic temperature θ_∞ is greater than the ignition temperature θ_{ig}, and the latter should hold when the total initial energy in the system is sufficiently large. The precise statements for a representative case are given in the following theorem, in which we denote by $\underline{c}$ and $\overline{c}$ the min and max over k of c_k/w_k.

Theorem 2.3. *In addition to the hypotheses of* Theorem 2.2 *assume the following:* $a'' = \kappa'' = 0$; $\bar{u}$ *and* $\bar{\theta}$ *are constants;* $h(\theta) > 0$ *for* $\theta > \theta_{ig}$*;* Q *is a multiple of* F*, so that the energy equation* (2.9) *may be written*

$$(\rho E)_t + (\rho E u + uP)_x = \lambda \theta_{xx} + \varepsilon (u u_x)_x + q\rho F \tag{2.42}$$

for a constant q*; and finally* $F(\theta, z) > 0$ *when* $\theta > \theta_{ig}$ *and* $z^j > 0$ *for all reactant species* $\mathcal{M}_j$*. Let* $\mathcal{E}_0 = \int_0^1 (\rho E)(x,0)\mathrm{d}x$ *be the total initial energy and let* $E_\infty, \rho_\infty, \theta_\infty$*, and* z_∞ *be as in* Theorem 2.2.

(a) *If*

$$\mathcal{E}_0 > \bar{c}\,\theta_{ig} \int_0^1 \rho_0 \left(1 - \frac{a}{a_j} z_0^j\right) \mathrm{d}x$$

for some reactant species $\mathcal{M}_j$, *then* $\theta_\infty > \theta_{ig}$, ρ_∞ *and* z_∞ *are constants, and some reactant species* $\mathcal{M}_k$ *becomes depleted, that is* $z_\infty^k \equiv 0$.

(b) *If for some reactant species* $\mathcal{M}_j$, $z_\infty^j \equiv 0$ *but* $z_0^j \not\equiv 0$, *then* $\theta_\infty \geqq \theta_{ig}$ *and*

$$\mathcal{E}_0 \geqq \underline{c}\theta_{ig} \int_0^1 \rho_0 \left(1 - \frac{a}{a_j} z_0^j\right) \mathrm{d}x + \frac{q}{a_j} \int_0^1 \rho_0 z_0^j \,\mathrm{d}x.$$

Theorem 2.3 is proved in Section 5. The gap between the sufficient condition in (a) and the necessary condition in (b) can be closed in certain special cases: see the discussion at the end of Section 1 of [4].

3. Existence: proof of Theorem 2.1

In this section we prove the global existence stated in Theorem 2.1. The major part of the analysis consists in the derivation of a priori bounds for smooth local-in-time approximate solutions. The first of these, given in Lemma 3.1, are uncontingent bounds corresponding to the balance of mass and energy and to the monotonicity of entropy. We then apply these bounds in Lemmas 3.2 and 3.3 to obtain pointwise estimates for density and temperature and L^2 bounds for certain higher order derivatives of velocity and temperature. These a priori bounds suffice to show that the approximate solutions can be extended to all time and provide the compactness needed to obtain the solutions of Theorem 2.1 in the limit as the smoothing parameter tends to zero.

More specifically, we approximate the initial data $(\rho_0, u_0, \theta_0, z_0)$ by smooth functions $(\rho_0^\delta, u_0^\delta, \theta_0^\delta, z_0^\delta)$ satisfying the conditions (2.21)–(2.26) with a constant C_0 which is independent of δ. We can then show as in HOFF and TSYGANOV [9] that there is a smooth solution $(\rho^\delta, u^\delta, \theta^\delta, z^\delta)$ defined for some positive time $\bar{T}$ possibly depending on δ. In particular, ρ^δ, u^δ, θ^δ, and z^δ are C^2 in t, ρ^δ and z^δ are C^2 in x, and u^δ and θ^δ are C^3 in x. It follows that ρ^δ and θ^δ are strictly positive for small time and that the boundary conditions (2.11) hold in a strict pointwise sense. Also, the argument given in the discussion following (2.6) applies to show that $(z^\delta)^j(x,t) \in [0,1]$ and that $\sum_j (z^\delta)^j(x,t) = 1$. Our immediate aim is to show that this approximate solution satisfies the estimates (2.28)–(2.36) with a constant C which is independent of δ and $\bar{T}$. For the time being we suppress the superscript δ.

Lemma 3.1. *Let* (ρ, u, θ, z) *be the local-in time-smooth solution described above. Then*

$$C^{-1} \leqq \int_0^1 \rho(x,t)\,\mathrm{d}x \leqq C, \quad t \leqq \bar{T}, \tag{3.1}$$

$$\int_0^1 \left(\tfrac{1}{2}\rho u^2 + \rho e\right)(x,t)\,\mathrm{d}x \leqq C, \quad t \leqq \bar{T}, \tag{3.2}$$

$$\int_0^{\bar{T}} \int_0^1 \rho\,(F + |Q|)\,\mathrm{d}x\mathrm{d}t \leqq C, \tag{3.3}$$

and

$$\sup_{0 \leqq t \leqq \bar{T}} \int_0^1 S(x,t)\,\mathrm{d}x + \int_0^{\bar{T}} \int_0^1 \left(\frac{\theta_x^2}{\theta^2} + \frac{u_x^2}{\theta}\right)\mathrm{d}x\mathrm{d}t \leqq C, \tag{3.4}$$

where

$$S(\rho,u,\theta,z) = \frac{R}{w(z)}(\rho\log\rho - \rho + 1) + c(z)\rho(\theta - \log\theta + 1) + \tfrac{1}{2}\rho u^2.$$

In all cases the constant C *is as described in the statement of Theorem 2.1 and in particular is independent of* δ *and* $\bar{T}$.

Proof. Combining (2.7) and (2.10) we obtain that

$$(z^j\rho)_t + (z^j\rho u)_x = a_j\rho F,$$

so that

$$\int_0^1 (z^j\rho)(x,t)\,\mathrm{d}x - a_j\int_0^t\int_0^1 \rho F\,\mathrm{d}x\mathrm{d}s = \int_0^1 z_0^j\rho_0\,\mathrm{d}x.$$

The bound (3.3) then follows by choosing j such that $a_j < 0$ (see (2.15)) and the upper bound in (3.1) then follows from (2.7). The lower bound in (3.1) is obtained by summing over j such that $a_j \geqq 0$ and applying (2.26), and the energy estimate (3.2) follows by integrating (2.9) and noting that, in all cases, $a'\bar{u}^2 - a''u^2 \leqq C$ and $\kappa'\bar{\theta} - \kappa''\theta \leqq C$.

To prove (3.4) we first compute from (2.7) and (2.8) that

$$\rho\frac{\mathrm{d}u}{\mathrm{d}t} + P_x = \varepsilon u_{xx} + a'(\bar{u} - u)\rho F, \tag{3.5}$$

where $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$, and therefore that

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\frac{u^2}{2} + uP_x = \varepsilon u u_{xx} + a'(\bar{u}u - u^2)\rho F. \tag{3.6}$$

Subtracting this from (2.9) we obtain

$$\rho\frac{\mathrm{d}e}{\mathrm{d}t} + u_xP = \lambda\theta_{xx} + \varepsilon u_x^2 + \rho Q + B\rho F \tag{3.7}$$

where B is determined by u, θ, and z and satisfies $B \geqq -C\theta$. A straightforward calculation based on (2.7) and (3.7) then shows that

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{R\log\rho}{w(z)} - c(z)\log\theta\right] + \frac{\lambda\theta_{xx}}{\theta} + \frac{\varepsilon u_x^2}{\theta} + \frac{\rho Q}{\theta}$$
$$= \left[\frac{aR}{w(z)} - \frac{B}{\theta}\right]\rho F + R\rho\log\rho\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{w(z)} + \rho(1-\log\theta)\frac{\mathrm{d}c(z)}{\mathrm{d}t}. \quad (3.8)$$

We note that if functions $g(x,t)$ and $G(x,t)$ satisfy $\rho\frac{\mathrm{d}g}{\mathrm{d}t} = G$, then $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 g\,\mathrm{d}x = \int_0^1(G + a\rho F g)\mathrm{d}x$. Applying this in (3.6) and (3.8) and combining with an elementary bound for $\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\frac{R}{w(z)}(-\rho+1) - \rho c(z)\right]\mathrm{d}x$ based on (3.1) and (3.3), we then find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 S(x,t)\mathrm{d}x + \int_0^1\left[\frac{\lambda\theta_x^2}{\theta^2} + \frac{\varepsilon u_x^2}{\theta} + \rho Q\left(\frac{1}{\theta} - 1\right)\right](x,t)\mathrm{d}x$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}0(1) + \int_0^1 \rho F\left[\tilde{B} - \frac{B}{\theta} + \frac{aR}{w(z)}\log\rho - ac(z)\log\theta\right]\mathrm{d}x$$
$$+ \int_0^1\left[R\rho\log\rho\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{w(z)} - \rho\log\theta\frac{\mathrm{d}c(z)}{\mathrm{d}t}\right]\mathrm{d}x, \quad (3.9)$$

where $\tilde{B}$ is determined by u, θ, and z and satisfies $\tilde{B} \leqq C$, and where $0(1)$ denotes a term which is bounded by C by (3.1)–(3.3). Computing $\frac{\mathrm{d}c(z)}{\mathrm{d}t}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{w(z)}$ from (2.3), (2.4), and (2.10) and applying (3.7), we then find that the right side of (3.9) is bounded by

$$\frac{\mathrm{d}}{\mathrm{d}t}0(1) + \int_0^1 \rho F\left[\left(\sum\frac{a_j}{w_j})\right)R\log\,\rho - \left(\sum\frac{c_j a_j}{w_j}\right)\log\theta\right]\mathrm{d}x.$$

The first term in the integral is zero, by (2.20). To bound the second term we note that if this term is not zero for a given fixed t, then there is an x_1 such that $\theta(x_1,t) \geqq \theta_{ig}$, and in any case there is an x_2 such that $\theta(x_2,t) \leqq C$, by (3.1) and (3.2). The term in question is therefore either zero or is bounded by

$$C\left[1 + \left(\int_0^1 \frac{\theta_x 2}{\theta^2}\mathrm{d}x\right)^{1/2}\right]\int_0^1 \rho F\,\mathrm{d}x,$$

which by (3.3) is $\frac{\mathrm{d}}{\mathrm{d}t}0(1)$ plus a term which can be absorbed into the left side of (3.9). The entropy estimate (3.4) thus follows from (3.9). □

In the following lemma we apply the results of Lemma 3.1 to derive pointwise bounds for density and temperature:

Lemma 3.2. *Let (ρ, u, θ, z) be the smooth solution of (2.7)–(2.11) on $[0,1]\times[0,\bar{T}]$ described at the beginning of this section. Then there is a positive number η and for each $T \leqq \bar{T}$ there is a positive number $\underline{\theta}(T)$ such that*

$$C^{-1} \leqq \rho(x,t) \leqq \begin{cases} Ce^{Ct}, & t \leqq \bar{T}, \\ C \;\; \textit{if} \displaystyle\int_{t/2}^t\int\frac{\theta_x^2}{\theta^2}\,\mathrm{d}x\mathrm{d}t \leqq \eta \;\; \textit{and} \;\; t \leqq \bar{T}, \end{cases} \quad (3.10)$$

and

$$\theta(x,t) \geqq \underline{\theta}(T), \quad t \leqq T \leqq \bar{T}. \tag{3.11}$$

Here η and $\underline{\theta}(T)$ depend on the same quantities as C in Theorem 2.1 *and $\underline{\theta}(T)$ may depend additionally on T. Both are independent of δ and $\bar{T}$.*

Proof. First we define $f(x,t) = (\varepsilon u_x - P(\rho,\theta,z))(x,t)$ and compute from (2.8) that, for $y, y_0 \in [0,1]$,

$$\left.\int_{X(y_0,\cdot)}^{X(y,\cdot)} (\rho u)(x,\cdot)\mathrm{d}x\right|_0^t = \int_0^t [f\,(X(y,s),s) - f(X(y_0,s),s)]\,\mathrm{d}s + \int_0^t \int_{X(y_0,s)}^{X(y,s)} (a'\bar{u} - a''u)\rho F \mathrm{d}x\mathrm{d}s, \tag{3.12}$$

where $X(y,t)$ is the integral curve of u originating from y:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = u(X,t) \\ X(y,0) = y. \end{cases} \tag{3.13}$$

Next, we introduce a renormalized density as follows. First if $a \geqq 0$ we choose j such that $a_j < 0$ and define

$$g(X(y,t),t) = \log\left[\frac{(z^j a - a_j)(X(y,t),t)}{(z^j a - a_j)(y,0)}\right].$$

If $a < 0$ we choose j such that $a_j > 0$ and define

$$g(X(y,t),t) = \log\left[\frac{(a_j - az^j)(X(y,t),t)}{(a_j - az^j)(y,0)}\right].$$

Then in either case,

$$-C \leqq g \leqq C \quad \text{and} \quad g_t + ug_x = -aF \tag{3.14}$$

by (2.10). It follows that if $A(x,t) = \rho(x,t)\mathrm{e}^{\,g(x,t)}$, then

$$A_t + (Au)_x = 0, \tag{3.15}$$

and therefore that, for any y,

$$\begin{aligned} \varepsilon \log A(X(y,\cdot),\cdot)|_0^t &= -\int_0^t \varepsilon u_x(X(y,s),s)\mathrm{d}s \\ &= -\int_0^t [f(X(y,s),s) + P(X(y,s),s)]\,\mathrm{d}s \end{aligned} \tag{3.16}$$

(we abbreviate $P(\rho(x,t),\cdots)$ by $P(x,t)$). Substituting from (3.12), we then obtain

$$\varepsilon\log A(X(y,t),t)=\varepsilon\log A(y,0)-\int_{X(y_0,\cdot)}^{X(y,\cdot)}(\rho u)(x,\cdot)\,\mathrm{d}x\Big|_0^t-\int_0^t P(X(y,s),s)\,\mathrm{d}s$$
$$-\int_0^t f(X(y_0,s),s)\,\mathrm{d}s+\int_0^t\int_{X(y_0,s)}^{X(y,s)}(a'\bar u-a''u)\rho F\,\mathrm{d}x\mathrm{d}s. \tag{3.17}$$

We now choose y_0 to control the second-last term on the right (y_0 will be depend on t but will be independent of y). To do this we define

$$W(y,t)=\int_0^t f(X(y,s),s)\mathrm{d}s+\int_0^y(\rho_0u_0)(x)\mathrm{d}x$$

then compute W_t and W_y and apply these to derive an expression for the time derivative of the integral on the left below. After a somewhat lengthy computation we obtain

$$\int_0^1\frac{A(y,0)}{A(X(y,\cdot),\cdot)}W(y,\cdot)\mathrm{d}y\,\Big|_0^t=-\int_0^t\int_0^1(P+\rho u^2)\mathrm{d}x\mathrm{d}s+I(t) \tag{3.18}$$

where

$$I(t)=-\int_0^t\int_0^1\rho F(a''u-a'\bar u)\,[X(x,t;s)-x]\,\mathrm{d}x\mathrm{d}s \tag{3.19}$$

and where $X(x,t;s)$ is as in (3.13) but with the second condition replaced by $X(x,s;s)=x$. Observe that from (3.13) and (3.15),

$$\frac{\partial X(y,t)}{\partial y}=\frac{A(y,0)}{A(X(y,t),t)}, \tag{3.20}$$

so that the integral on the left side of (3.18) at time t is $W(y_0,t)$ for some $y_0(t)$ which we now fix. Then from the definition of W,

$$\int_0^t f(X(y_0,s),s)\mathrm{d}s=\int_0^1\int_0^y(\rho_0u_0)(\xi)\mathrm{d}\xi\mathrm{d}y-\int_0^{y_0}(\rho_0u_0)(\xi)\mathrm{d}\xi$$
$$-\int_0^t\int_0^1(P+\rho u^2)\mathrm{d}x\mathrm{d}s+I(t).$$

Substituting into (3.17) we then obtain that

$$\varepsilon\log A(X(y,t),t)=\varepsilon\log A(y,0)-\int_{X(y_0,\cdot)}^{X(y,\cdot)}(\rho u)(x,\cdot)\mathrm{d}x\Big|_0^t-\int_0^t P(X(y,s),s)\mathrm{d}s$$
$$-\int_0^1\int_0^y(\rho_0u_0)(\xi)\mathrm{d}\xi\mathrm{d}y+\int_0^{y_0}(\rho_0u_0)(\xi)\mathrm{d}\xi$$
$$+\int_0^t\int_0^1(P+\rho u^2)\mathrm{d}x\mathrm{d}s$$
$$+\int_0^t\int_{X(y_0,s)}^{X(y,s)}(a'\bar u-a''u)\rho F\,\mathrm{d}x\mathrm{d}s-I(t)$$

for any $y \in [0, 1]$. Taking the exponential and recalling that $A = \rho e^g$, we then obtain

$$\begin{aligned}
&\rho(X(y,t),t)\exp\left[\varepsilon^{-1}\int_0^t P(X(y,s),s)\mathrm{d}s\right]\\
&= \rho_0(y)\exp\left[\varepsilon^{-1}B(X(y,t),t)+\varepsilon^{-1}\int_0^t\int_0^1(P+\rho u^2)\mathrm{d}x\mathrm{d}s-\varepsilon^{-1}I(t)\right], \quad (3.21)
\end{aligned}$$

where B is 0(1) by Lemma 3.1 (and is different from the B occurring in the proof of Lemma 3.1). Multiplying by $\frac{R\theta}{w(z)}$, we then get $\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\exp[\varepsilon^{-1}\int_0^t P(X(y,s),s)ds]$ on the left. Integrating with respect to t and substituting back into (3.19) we obtain finally

$$\begin{aligned}
\rho(X(y,t),t) &= \frac{\widetilde{D}(X(y,t),t)e^{\int_0^t \mu(s)\mathrm{d}s - I(t)/\varepsilon}}{1+\int_0^t(\theta D)(X(y,s),s)e^{\int_0^s \mu(\tau)d\tau - I(s)/\varepsilon}\mathrm{d}s}\\
&= \tilde{D}(\cdot,t)e^{-I(t)/\varepsilon}\left[e^{-\int_0^t\mu}+\int_0^t(\theta D)(\cdot,s)e^{-\int_s^t\mu - I(s)/\varepsilon}\mathrm{d}s\right]^{-1} \quad (3.22)
\end{aligned}$$

for functions D and $\widetilde{D}$ which by Lemma 3.1 satisfy $C^{-1} \leqq D, \widetilde{D} \leqq C$ and where

$$\mu(t) = \varepsilon^{-1}\int_0^1(P+\rho u^2)(x,t)\mathrm{d}x.$$

We check that there are positive constants $\underline{\mu}$ and $\bar{\mu}$ depending on the same quantities as C in Theorem 2.1 such that

$$\underline{\mu} \leqq \mu(t) \leqq \bar{\mu}. \quad (3.23)$$

The upper bound here is immediate from (3.2) and the definition of P, and the lower bound follows from the estimate in (3.4) for $\int_0^1 \rho(\theta - \log\theta + 1)\,\mathrm{d}x$ via Jensen's inequality and (3.1). We also note that, from (3.3), (2.17), and the definition (3.19) of I,

$$|I(t)| \leqq C, \quad 0 \leqq t \leqq \bar{T}. \quad (3.24)$$

It then follows from (3.22) that

$$\rho(X(y,t),t) \geqq \frac{C^{-1}}{e^{-\underline{\mu}t}+\int_0^t\theta(X(y,s),s)e^{\underline{\mu}(s-t)}\mathrm{d}s}.$$

Thus if we denote by $M_\theta(t)$ and $M_\rho(t)$ the suprema over x of $\theta(\cdot,t)$ and $\rho(\cdot,t)$ and by $m_\theta(t)$ and $m_\rho(t)$ the infima over x of $\theta(\cdot,t)$ and $\rho(\cdot,t)$, then

$$m_\rho(t) \geqq \frac{C^{-1}}{e^{-\underline{\mu}t}+\int_0^t M_\theta(s)e^{\underline{\mu}(s-t)}\mathrm{d}s}. \quad (3.25)$$

Next from (3.1) and (3.2) we have that for each t there is point $x_0 \in [0,1]$ such that $\theta(x_0,t) \leqq C$. Thus

$$\left|\theta^{1/2}(x,t)-\theta^{1/2}(x_0,t)\right| \leqq \left(\int_0^1 \theta \mathrm{d}x\right)^{1/2}\left(\int_0^1 \frac{\theta_x^2}{\theta^2}\mathrm{d}x\right)^{1/2}$$
$$\leqq C m_\rho^{-1/2}\left(\int \frac{\theta_x^2}{\theta^2}\mathrm{d}x\right)^{1/2}$$

by (3.2). It follows that

$$M_\theta(t) \leqq C + m_\rho(t)^{-1}\int_0^1 \frac{\theta_x^2}{\theta^2}(x,t)\mathrm{d}x$$

and

$$m_\theta(t) \geqq C - m_\rho(t)^{-1}\int_0^1 \frac{\theta_x^2}{\theta^2}(x,t)\mathrm{d}x.$$

Substituting the first of these into (3.25) and applying a Gronwall estimate to m_ρ^{-1}, we find that $m_\rho \geqq C^{-1}$ for all t, which proves the lower bound in (3.10). The first of the upper bounds in (3.10) follows by applying (3.23) and (3.24) in (3.22) and discarding the integral in the denominator. To obtain the other upper bound in (3.10) we again apply (3.23) and (3.24) in (3.22) and then the above bound for m_θ and the lower bound for ρ already proved to obtain

$$\rho(X(y,t),t)^{-1} \geqq C^{-1}\int_0^t m_\theta(s)\mathrm{e}^{-\int_s^t \mu(\tau)\mathrm{d}\tau}\mathrm{d}s$$
$$\geqq C^{-1}\left[1 - C\int_0^t \mathrm{e}^{\bar{\mu}(s-t)}\int_0^1 \frac{\theta_x^2}{\theta^2}(x,s)\mathrm{d}x\mathrm{d}s\right].$$

Now by (3.4),

$$C\int_0^t\int_0^1 \mathrm{e}^{\bar{\mu}(s-t)}\frac{\theta_x^2}{\theta^2}\mathrm{d}x\mathrm{d}s \leqq C\mathrm{e}^{-\bar{\mu}t/2}\int_0^{t/2}\int_0^1 \frac{\theta_x^2}{\theta^2}\mathrm{d}x\mathrm{d}s + C\int_{t/2}^t\int_0^1 \frac{\theta_x^2}{\theta^2}\mathrm{d}x\mathrm{d}s$$
$$\leqq \tfrac{1}{2} + C\int_{t/2}^t\int_0^1 \frac{\theta_x^2}{\theta^2}\mathrm{d}x\mathrm{d}s$$

if t is large, depending on $0(1)$ constants. Thus

$$\rho(X(y,t),t)^{-1} \geqq C^{-1} - C\int_{t/2}^t\int_0^1 \frac{\theta_x^2}{\theta^2}\mathrm{d}x\mathrm{d}s \ \geqq C^{-1}$$

provided $\int_{t/2}^t\int_0^1 \frac{\theta_x^2}{\theta^2}\mathrm{d}x\mathrm{d}s \leqq \eta$ for η as described in the statement of the lemma. This proves (3.11).

To prove the lower bound (3.11) for θ we apply (3.7) to obtain an evolution equation for θ^{-1}, multiply by $2k\theta^{-(2k-1)}$, and integrate over $[0,1]\times[0,t]$ to obtain an estimate for $\|\theta(\cdot,t)^{-1}\|_{L^{2k}}$. The bound (3.11) then follows in the limit as $k \to \infty$. The details are similar to those occurring in the proof of Lemma 2.4 of [10] and so are omitted. $\square$

Lemma 3.3. *Let* (ρ, u, θ, z) *be the smooth solution of* (2.7)–(2.11) *on* $[0, 1]\times[0, \bar{T}]$ *described at the beginning of this section. Then there is a constant* C *as in* Theorem 2.1 *which is independent of* $\bar{T}$ *and* δ *but which (in this lemma only) may depend additionally on an upper bound for* ρ *on* $[0, 1] \times [0, \bar{T}]$ *such that*

$$\sup_{0\leqq t\leqq \bar{T}} \int_0^1 \left[\tfrac{1}{2}\rho u^2 + \rho e(\theta, z) + \rho(\theta - \log\theta + 1) + |z_x|^2 + \theta^2 + u^4\right](x, t)\mathrm{d}x$$
$$+ \int_0^{\bar{T}} \int_0^1 \left[\left(1 + \frac{1}{\theta}\right) u_x^2 + \left(1 + \frac{1}{\theta^2}\right)\theta_x^2 + u^2 u_x^2 + h(\theta)|z_x|^2 + F\right]\mathrm{d}x\mathrm{d}t \leqq C \tag{3.26}$$

and

$$\sup_{0<t\leqq \bar{T}} \int_0^1 \left[\sigma u_x^2 + \sigma^2 \dot{u}^2 + \sigma^2\theta_x^2 + \sigma^3\dot{\theta}^2\right](x, t)\mathrm{d}x$$
$$+ \int_0^{\bar{T}} \int_0^1 \left[\sigma\dot{u}^2 + \sigma^2\dot{\theta}^2 + \sigma^2\dot{u}_x^2 + \sigma^3\dot{\theta}_x^2\right]\mathrm{d}x\mathrm{d}t \leqq C, \tag{3.27}$$

where $\sigma(t) = \min\{1, t\}$.

Proof. The derivations are straightforward but lengthy and parallel those of similar estimates in [8]. We therefore give only a brief outline of the various steps in their correct order. First we differentiate (2.10) with respect to x, multiply by z_x^j, and integrate. Applying the hypotheses (2.18) and (2.19) we obtain

$$\sup_{0\leqq t\leqq \bar{T}} \int_0^1 |z_x(x, t)|^2\mathrm{d}x + \int_0^{\bar{T}} \int_0^1 h(\theta)|z_x|^2\mathrm{d}x\mathrm{d}t \leqq C. \tag{3.28}$$

Next, we multiply (3.5) by u^3, (3.7) by θ, and (3.5) again by θ, integrate, and then add appropriate multiples of the resulting three equations. Estimating a very large number of lower-order terms, making use of (3.28) and the pointwise bounds in Lemma 3.2, we obtain

$$\sup_{0\leqq t\leqq \bar{T}} \int_0^1 \left(\rho[\theta^2 + \theta u^2 + u^4]\right)(x, t)\,\mathrm{d}x + \int_0^{\bar{T}} \int_0^1 (u^2 u_x^2 + \theta_x^2)\,\mathrm{d}x\mathrm{d}t \leqq C.$$

A bound for $\iint u_x^2\mathrm{d}x\mathrm{d}t$ can then be derived easily by multiplying (3.5) by u and integrating. There remain four additional estimates: $H^1((0, 1))$ bounds for u and θ and $L^2((0, 1))$ bounds for $\dot{u}$ and $\dot{\theta}$ together with the corresponding parabolic space-time bounds. Again, the derivations are standard (although the two H^1 bounds are coupled) but the details are exceptionally lengthy and in any case parallel similar estimates in [8]. We therefore omit the details. □

Proof of Theorem 2.1. We now let $(\rho^\delta, u^\delta, \theta^\delta, z^\delta)$ be the approximate smooth solution described at the beginning of this section. The fairly strong a priori estimates of Lemmas 3.1–3.3 then apply to show in the usual way that $(\rho^\delta, u^\delta, \theta^\delta, z^\delta)$ can be extended as a solution on all of $[0, 1] \times [0, \infty)$. The first contingency in (3.10) gives an upper bound for ρ^δ on $[0, 1] \times [0, \bar{T}]$, now for any $\bar{T} > 0$, which may depend on $\bar{T}$ but which is independent of δ. The constant C in Lemma 3.3 then depends on $\bar{T}$ in the same way but is independent of δ. (This $\bar{T}$ dependence will be removed shortly.)

We now obtain the solution (ρ, u, θ, z) of Theorem 2.1 in the limit as $\delta \to 0$. First observe that the uniform H^1 estimate for z^δ in (3.28) gives a $C^{1/2}$ bound for $z^\delta(\cdot, t)$ which is uniform in δ and t. Also, $z_t^\delta = \dot{z}^\delta - u^\delta z_x^\delta$, $\dot{z}^\delta = 0(1)$, and

$$\int_{t_1}^{t_2} \int (|z_x^\delta| u^\delta)^2 \mathrm{d}x\mathrm{d}t \leqq C \int_{t_1}^{t_2} \|u^\delta(\cdot, t)\|_\infty^2 \mathrm{d}t$$
$$\leqq C|t_2 - t_1|^{1/2} \int \int (u^\delta u_x^\delta)^2 \mathrm{d}x\mathrm{d}t \leqq C|t_2 - t_1|^{1/2}.$$

The family $\{z^\delta\}$ is therefore uniformly bounded and equicontinuous on $[0, 1] \times [0, \infty)$ so that there is a sequence $\delta \to 0$ such that z^δ converges uniformly on compact sets in $[0, 1] \times [0, \infty)$ to a function z which is Hölder continuous on $[0, 1] \times [0, \infty)$. Similar arguments applied away from $t = 0$ to $\{u^\delta\}$ and $\{\theta^\delta\}$ show that $u^\delta \to u$ and $\theta^\delta \to \theta$ uniformly on compact sets in $[0, 1] \times (0, \infty)$ for limiting functions u and θ and for a further subsequence $\delta \to 0$.

Now fix $t > 0$. Then since $u^\delta(\cdot, t) \to u(\cdot, t)$ uniformly, $u_x^\delta(\cdot, t) \to u_x(\cdot, t)$ in $\mathcal{D}'((0, 1))$. On the other hand the uniform bound in (3.27) for $\|u_x^\delta(\cdot, t)\|_{L^2}$ shows that every subsequence of the aforementioned sequence has in turn a further subsequence for which $u_x^\delta(\cdot, t)$ has a weak-L^2 limit, which necessarily is $u_x(\cdot, t)$. It follows that $u(\cdot, t) \in H_0^1$ for $t > 0$ and that $u_x^\delta(\cdot, t) \rightharpoonup u_x(\cdot, t)$ weakly in $L^2((0, 1))$ for $t > 0$. A stronger result holds for the sequence $\theta_x^\delta(\cdot, t)$: the various estimates in (3.26) and (3.27) applied in (3.7) show that $\theta_{xx}^\delta(\cdot, t)$ is bounded in $L^2((0, 1))$ independently of δ. It follows that, for a further subsequence $\delta \to 0$, $\theta_x^\delta(\cdot, t) \to \theta_x(\cdot, t)$ strongly in $L^2((0, 1))$ for $t > 0$.

We now show that the constant C in Lemma 3.3 may be taken to be independent of time (the above compactness arguments requiring only independence of δ). First, applying the strong convergence of θ^δ and θ_x^δ in the bound (3.4), we find that $\int_0^\infty \int_0^1 (\theta_x/\theta)^2 \mathrm{d}x\mathrm{d}t \leqq C$, where C is exactly as described in the statement of Theorem 2.1. There is therefore a time $\bar{T}$ which we now fix such that $\int_{\bar{T}/2}^\infty \int_0^1 (\theta_x/\theta)^2 \mathrm{d}x\mathrm{d}t \leqq \eta/4$, where η is as in (3.10). Now for each $k = 0, 1, \dots$, $\int_{2^{k-1}\bar{T}}^{2^k\bar{T}} \int \int_0^1 (\theta_x^\delta/\theta^\delta)^2 \mathrm{d}x\mathrm{d}t \to \int_{2^{k-1}\bar{T}}^{2^k\bar{T}} \int_0^1 (\theta_x/\theta)^2 \mathrm{d}x\mathrm{d}t$, so that by taking further subsequences and applying a diagonal process, we obtain a sequence $\delta \to 0$ such that $\int_{2^{k-1}\bar{T}}^{2^k\bar{T}} \int_0^1 (\theta_x^\delta/\theta^\delta)^2 \mathrm{d}x\mathrm{d}t \leqq \eta/2$ for every such k and δ. It follows that $\int_{t/2}^t \int_0^1 (\theta_x^\delta/\theta^\delta)^2 \mathrm{d}x\mathrm{d}t \leqq \eta$ for every $t \geqq \bar{T}$ and every δ. We can therefore invoke both contingencies in (3.10) and conclude that there is an upper bound for ρ^δ on $[0, 1] \times [0, \infty)$ which is uniform in δ. In particular, the bounds in (3.26) and (3.27) hold for a constant C which is now independent of both $\bar{T}$ and δ.

Next, we show that there is a further sequence $\delta \to 0$ such that $\rho^\delta(\cdot, t)$ converges strongly in $L^2((0, 1))$ for every $t \geqq 0$. To prove this we define X^δ as in (3.13) with $u = u^\delta$ and $f^\delta = \varepsilon u_x^\delta - P^\delta$, as in Lemma 3.2. Applying (3.26) and (3.27) we then obtain uniform bounds for $f^\delta(X^\delta(y, t), t)$ as functions of y and t in $H^1([0, 1] \times [\tau, T])$ for $0 < \tau < T$. It follows that, modulo a further subsequence, $f^\delta(X^\delta(\cdot, \cdot), \cdot)$ is strongly convergent in $L^2([0, 1] \times (\tau, T])$ for each such τ, T, and is strongly convergent in $L^1([0, 1] \times (0, T))$ for $T > 0$. We then apply this convergence in (3.16) to obtain the strong L^2 convergence of $\log A^\delta(X^\delta(\cdot, t), t)$ for each $t \geqq 0$, where $A^\delta = \rho^\delta \mathrm{e}^{g^\delta}$. There is therefore a sequence $\delta \to 0$ such that $\rho^\delta(X^\delta(\cdot, t), t)$ is strongly convergent in $L^2((0, 1))$ for each $t \geqq 0$. Convergence of the sequence $\rho^\delta(\cdot, t)$ then follows from this Lagrangean convergence exactly as in the proof of Theorem 1 in [7].

The pointwise bounds and Hölder continuity properties stated in Theorem 2.1 are clearly retained for the limiting solution (ρ, u, θ, z) and the estimates in (2.35) and (2.36) hold because L^2 norms are nonincreasing under weak limits. Also, the modes of convergence described above clearly suffice to show that (ρ, u, θ, z) is a weak solution of (2.7)–(2.12) in the required sense.

There remain the assertions in (2.31) and (2.34) concerning continuity in time into $L^p((0, 1))$. We shall examine the arguments for ρ and u, the result for θ being weaker and easier. First we again let $L(y, t) = \log A(X(y, t), t)$ where $A = \rho \mathrm{e}^g$ and X are as in (3.13), say for the approximate smooth solution but with δ suppressed. Applying (3.15) we then obtain

$$\begin{aligned}\int_0^1 |L(y, t) - L(y, 0)|\mathrm{d}y &\leqq C \int_0^t \int_0^1 |u_x(X(y, s), s)|\mathrm{d}y\mathrm{d}s \\ &\leqq C \int_0^t \int_0^1 |u_x(x, s)|\mathrm{d}x\mathrm{d}s \\ &\leqq Ct^{1/2}\end{aligned}$$

again by (3.15), which shows that $A(X(y, t), t)dX = A_0(y)\mathrm{d}y$. It then follows that

$$\int_0^1 |(\rho \mathrm{e}^g)(x, t) - (\rho \mathrm{e}^g)(x, 0)|\mathrm{d}x \leqq Ct^{1/2}.$$

This same estimate holds for the limiting solution, and since z is Hölder continuous on $[0, 1] \times [0, \infty)$, we can conclude that $\rho(\cdot, t) \to \rho_0$ in L^1 as $t \to 0$, hence in L^p for $p \in [0, \infty)$. Exactly the same argument proves the continuity at positive times.

Finally to prove (2.34) we note that, since u is locally Hölder continuous on $[0, 1] \times (0, \infty)$ we need to examine only the continuity at $t = 0$. To do this we apply the weak form of the momentum equation (2.8) to a test function $\varphi \in \mathcal{D}((0, 1))$ and apply the various bounds in (2.35) and (2.36) to obtain that

$$\begin{aligned}&\left| \int_0^1 [(\rho u)(x, t) - (\rho_0 u_0)(x)]\, \varphi(x)\mathrm{d}x \right| \\ &\quad \leqq C \left[t + t^{1/2} \left(\int_0^t \int_0^1 u_x^2 \, \mathrm{d}x\mathrm{d}s \right)^{1/2} \right] \|\varphi_x\|_{L^2} + Ct\|\varphi\|_{L^2}. \qquad (3.29)\end{aligned}$$

It follows that $(\rho u)(\cdot, t) \to \rho_0 u_0$ in $\mathcal{D}'((0,1))$ as $t \to 0$. But since $(\rho u)(\cdot, t)$ is uniformly bounded in L^2, this convergence is weak in L^2 as well, from which we conclude that $(\rho^{1/2} u)(\cdot, t) \rightharpoonup \rho_0^{1/2} u_0$ weakly in $L^2((0,1))$ as $t \to 0$ by the time-continuity of ρ proved above. If we now fix t and let φ tend to $u(\cdot, t)$ in (3.29) and apply the bound in (2.36) for $\|u_x(\cdot, t)\|_{L^2}$, we find that

$$\begin{aligned}\int (\rho u^2)(x,t)\mathrm{d}x &\leqq \int (\rho_0 u_0)(x) u(x,t)\mathrm{d}x + C\left[t^{1/2} + \left(\int_0^t \int_0^1 u_x^2\right)^{1/2}\right] + Ct \\ &\leqq \tfrac{1}{2}\int (\rho u^2)(x,t)\mathrm{d}x + \tfrac{1}{2}\int \frac{(\rho_0^2 u_0^2)(x)}{\rho(x,t)}\mathrm{d}x \\ &\quad + C\left[[t^{1/2} + \left(\int_0^t \int_0^1 u_x^2\right)^{1/2}\right] + Ct\end{aligned}$$

so that

$$\limsup_{t\to 0} \tfrac{1}{2}\int (\rho u^2)(x,t)\mathrm{d}x \leqq \tfrac{1}{2}\int (\rho_0 u_0^2)(x)\mathrm{d}x.$$

These facts prove that $(\rho^{1/2} u)(\cdot, t) \to \rho_0^{1/2} u_0$ strongly in L^2. The pointwise bounds and strong continuity in time for ρ then apply again to show that $u(\cdot, t) \to u_0$ strongly in L^2, hence in L^p for $p \in [1, 4)$ as $t \to 0$, by the L^4 estimate in (2.35). This completes the proof of Theorem 2.1. $\square$

4. Large time behavior: proof of Theorem 2.2

In this section we prove the results stated in Theorem 2.2 concerning the large-time behavior of the solution (ρ, u, θ, z) constructed in Theorem 2.1. Important use will be made of the fact that there is a system of integral curves $X(y, t)$ of the velocity field u, exactly as described in (3.13). This is an easy consequence of the bound $\int_0^T \|u(\cdot, t)\|_{L^\infty}\mathrm{d}t < \infty$ for $T > 0$, which in turn follows in a straightforward way from the estimates in (2.35) and (2.36). The conservation of renormalized mass (3.15) and the pointwise bounds for ρ in (2.28) then apply to show that

$$C^{-1} \leqq \left|\frac{\partial X}{\partial y}(y,t)\right| \leqq C. \tag{4.1}$$

The overall outline of the proof is as follows: First we show that u tends to zero, θ to its spatial average, and $z(X(y,t), t)$ to a certain function of y. These facts then enable us to prove that $\int_0^1 \rho e\,\mathrm{d}x$ tends to a constant, from which we conclude that θ tends to a constant as well. The arguments for density and pressure are somewhat more complicated: First we show that $f \equiv \varepsilon u_x - P$ tends to its spatial average and therefore that P also tends to its spatial average. Then by integrating the mass equation (3.15) and applying the previous conclusions about θ and z, we find that P tends to a constant and therefore that $\rho(X(y,t), t)$ converges to a certain function of y. To complete the proof we then show that the trajectories $X(y, t)$ have time-asymptotic limits $X_\infty(y)$ and therefore that $\rho(x, t)$ converges to a function of x.

Proof of Theorem 2.2. We define a Lagrangian velocity v by $v(y,t) = u(X(y,t),t)$ and observe that $v_t = \dot{u}$ and $C^{-1}|u_x| \leqq |v_y| \leqq C|u_x|$. It then follows from the bounds in (2.35) and (2.36) for u_x and $\dot{u}_x$ that $\|v_y(\cdot,t)\|_{L^2}^2 \in (BV \cap L^1)([1,\infty))$ and so approaches zero as $t \to \infty$. The same is therefore true for $\|u_x(\cdot,t)\|_{L^2}$, and this proves (2.38). A similar argument applies to $\theta(\cdot,t)$ and shows that

$$\theta(\cdot,t) - \bar{\theta}(t) \to 0 \text{ in } H^1((0,1)) \text{ as } t \to \infty \tag{4.2}$$

where

$$\bar{\theta}(t) = \int_0^1 \theta(x,t)\mathrm{d}x.$$

To obtain the convergence of z we note that, for y fixed, each $z^j(X(y,t),t)$ is bounded and monotone in t by the equations (2.10). It follows that

$$z(X(y,t),t) \to \eta(y) \text{ pointwise in } y \text{ as } t \to \infty \tag{4.3}$$

for a function $\eta = (\eta^1,\ldots,\eta^J)$.

Next, we examine the large-time behavior of the total energy. Integrating in (2.9), we find that

$$\int_0^1 (\rho E)(x,t)\mathrm{d}x = \int_0^1 \rho_0 E_0 \mathrm{d}x + \int_0^t \int_0^1 \left[\rho Q + \frac{1}{2}\left(a'\bar{u}^2 - a''u^2\right) + (\kappa'\bar{\theta} - \kappa''\theta)\right]\rho F \mathrm{d}x\mathrm{d}s.$$

(It is clear that this relation holds for the smooth solutions $(\rho^\delta, u^\delta, \theta^\delta, z^\delta)$ of Section 3 and that equality is retained in the limit as $\delta \to 0$.) The various bounds in Theorem 2.1 apply to show that the integrand in the second integral on the right here is in $L^1([0,1]\times[0,\infty))$ and therefore that the right side converges as $t \to \infty$, say to E_∞. Then since $u(\cdot,t) \to 0$ in H_0^1 as $t \to \infty$,

$$\int_0^1 (\rho e)(x,t)\mathrm{d}x \to E_\infty \quad \text{as } t \to \infty. \tag{4.4}$$

Next, recalling the notations $A = \rho \mathrm{e}^g$ in (3.14) and (3.15) and writing $g = g(z)$, we obtain that

$$\begin{aligned}\int_0^1 (\rho e)(x,t)\mathrm{d}x &= \int_0^1 (A\mathrm{e}^{-g(z)}c(z)\theta)(x,t)\mathrm{d}x \\ &= \bar{\theta}(t)\int_0^1 (A\mathrm{e}^{-g(z)}c(z))(x,t)\mathrm{d}x + o_t(1)\end{aligned}$$

where $o_t(1) \to 0$ as $t \to \infty$; and since $A(X(y,t),t)dX = A_0(y)\mathrm{d}y = \rho_0(y)\mathrm{d}y$,

$$\begin{aligned}\int_0^1 (\rho e)(x,t)\mathrm{d}x &= \bar{\theta}(t)\int_0^1 \rho_0(y)\mathrm{e}^{-g(z(X(y,t),t))}c(z(X(y,t),t))\mathrm{d}y + o_t(1) \\ &= \bar{\theta}(t)\int_0^1 \rho_0(y)\mathrm{e}^{-g(\eta(y))}c(z(\eta(y)))\mathrm{d}y + o_t(1)\end{aligned} \tag{4.5}$$

by (4.2). Comparing (4.2)–(4.4), we conclude that

$$\theta(\cdot,t) \to \theta_\infty \equiv E_\infty \Big/ \int_0^1 \rho_0(y)\mathrm{e}^{-g(\eta(y))} c(z(\eta(y)))\mathrm{d}y \quad \text{in } H^1((0,1)) \text{ as } t \to \infty. \tag{4.6}$$

(We will derive another expression for θ_∞ below.)

Next, we show than the pressure converges as $t \to \infty$. To do this we define $\tilde{f}(y,t) = f(X(y,t),t)$, where $f = \varepsilon u_x - P$ as in the proof of Lemma 3.2. Straightforward estimates based on (2.35) and (2.36) show that

$$\int_1^\infty \int_0^1 \left(\tilde{f}_t^2 + \tilde{f}_y^2\right) \mathrm{d}y\mathrm{d}t \leqq C.$$

We shall apply this to prove that f tends to its spatial average as $t \to \infty$. First let ζ be a smooth function on $[0,1]$ which is positive on $(0,1)$ and which vanishes at the endpoints and define

$$b(t) = \int_0^1 \zeta(X(y,t))\rho_0^{-1}(y)\mathrm{e}^{g(X(y,t),t)} \tilde{f}_y(y,t)^2 \mathrm{d}y.$$

We estimate the variation $\int_1^\infty |b'(t)|\mathrm{d}t$ directly; the dominant term is

$$2\int_1^\infty \left| \int_0^1 \zeta(X(y,t))\rho_0^{-1}(y)\mathrm{e}^{g(X(y,t),t)} \tilde{f}_y(y,t)\tilde{f}_{yt}(y,t)\mathrm{d}y \right| \mathrm{d}t,$$

which by (2.8) and the fact that $\rho \mathrm{e}^g dX = \rho_0 \mathrm{d}y$ may be written

$$2\int_1^\infty \left| \int_0^1 \zeta(X(y,t))(\rho^{-1}\left[\rho\dot{u} + (a''u - a'\bar{u})\rho F\right])(X(y,t),t)\tilde{f}_{yt}(y,t)\mathrm{d}y \right| \mathrm{d}t.$$

We integrate by parts in y, noting that the boundary terms are zero (because $\zeta(0) = \zeta(1) = 0$) to find that

$$\begin{aligned} \operatorname{Var} b|_{[1,\infty)} &\leqq C(\zeta)\left(\int_1^\infty \int_0^1 \tilde{f}_t^2 \mathrm{d}y\mathrm{d}t\right)^{1/2} \left(\int_1^\infty \int_0^1 [\dot{u}^2 + (1+u^2)F + \dot{u}_x^2 \right. \\ &\qquad \left. + u_x^2 + (\bar{u}^2 + u^2)F_x^2]\mathrm{d}x\mathrm{d}t\right)^{1/2} \\ &\leqq C(\zeta) \end{aligned}$$

(the extra hypotheses of Theorem 2.2 are applied here to bound the $\bar{u}^2 F_x^2$ term). Thus $b \in (BV \cap L^1)[1,\infty)$ and therefore tends to zero as $t \to \infty$. Changing variables, we then have that

$$b(t) = \int_0^1 \zeta(x)A(x,t)^{-1} f_x(x,t)^2 \mathrm{d}x \to 0 \quad \text{as } t \to \infty.$$

An easy argument then shows that

$$\| f(\cdot,t) - \int_0^1 f(x,t)\mathrm{d}x \|_{L^1} \to 0 \quad \text{as } t \to \infty,$$

and since

$$\int_0^1 f(x,t)\mathrm{d}x = -\int_0^1 (P(\rho,\theta,z))(x,t)\mathrm{d}x \equiv -\bar{P}(t)$$

and $u_x(\cdot,t) \to 0$ in L^2, we conclude that

$$\left(\frac{R\rho\theta}{w(z)}\right)(\cdot,t) - \bar{P}(t) \to 0 \text{ in } L^1((0,1)) \quad \text{as } t \to \infty. \tag{4.7}$$

The next step is to show that $\bar{P}(t)$ tends to a constant as $t \to \infty$. To see this we first note that, by (4.7) and (4.1),

$$\left(\frac{R\rho\theta}{c(z)}\right)(X(\cdot,t),t) - \bar{P}(t) \to 0 \text{ in } L^1((0,1)) \quad \text{as } t \to \infty,$$

so that by (4.3) and (4.6),

$$\begin{aligned}
&\frac{1}{A(X(\cdot,t),t)} - \frac{R\theta_\infty}{\bar{P}(t)w(\eta(\cdot))\mathrm{e}^{g(\eta(\cdot))}} \\
&\quad = \frac{1}{(\rho \mathrm{e}^{g(z)})(X(\cdot,t),t)\bar{P}(t)}\left[\bar{P}(t) - \frac{R\rho\theta}{w(z)}\,(X(\cdot,t),t)\right] + o_t(1) \\
&\quad = o_t(1)
\end{aligned}$$

where $o_t(1)$ now denotes a term which tends to zero in $L^1((0,1))$ as $t \to \infty$. Integrating and applying the fact that

$$\int_0^1 \frac{\mathrm{d}y}{A(X(y,t),t)} = \int_0^1 \frac{\mathrm{d}y}{\rho_0(y)}$$

we then find that

$$\frac{R\theta_\infty}{\bar{P}(t)}\int_0^1 \left[w(\eta(y))\mathrm{e}^{g(\eta(y))}\right]^{-1}\mathrm{d}y \to \int_0^1 \frac{\mathrm{d}y}{\rho_0(y)}$$

so that

$$\bar{P}(t) \to P_\infty \equiv \frac{R\theta_\infty \int_0^1 \left[w(\eta(y))\mathrm{e}^{g(\eta(y))}\right]^{-1}\mathrm{d}y}{\int_0^1 \rho_0(y)^{-1}\mathrm{d}y} \tag{4.8}$$

and therefore from (4.7) that

$$\left(\frac{R\rho\theta}{w(z)}\right)(\cdot,t) \to P_\infty \text{ in } L^1((0,1)) \quad \text{as } t \to \infty.$$

It follows that $(\frac{R\rho\theta}{w(z)})(X(\cdot,t),t) \to P_\infty$ in L^1 as well, so that by (4.3) and (4.6),

$$\rho(X(\cdot,t),t) \to \frac{P_\infty w(\eta(\cdot))}{R\theta_\infty} \text{ in } L^1((0,1)) \quad \text{as } t \to \infty. \tag{4.9}$$

To obtain the convergence of $\rho(\cdot,t)$ we therefore need to show that the integral curves $X(y,t)$ converge as $t\to\infty$. To see this we apply (3.15), (4.3), and (4.9) to obtain

$$X(y,t)=\int_0^y \frac{A_0(\xi)}{A(X(\xi,t),t)}\mathrm{d}\xi=\int_0^y \frac{\rho_0(\xi)\mathrm{e}^{-g(z(X(\xi,t),t))}}{\rho(X(\xi,t),t)}\,\mathrm{d}\xi$$
$$\to \frac{R\theta_\infty}{P_\infty}\int_0^y \frac{\rho_0(\xi)\mathrm{e}^{-g(\eta(\xi))}}{w(\eta(\xi))}\mathrm{d}\xi\equiv X_\infty(y). \tag{4.10}$$

Thus there is a bi-Lipschitz homeomorphism X_∞ of $[0,1]$ such that

$$X(y,t)\to X_\infty(y)\text{ uniformly in } y \quad \text{as } t\to\infty. \tag{4.11}$$

This enables us to improve the convergence in (4.3) as follows. First let $t_j\to\infty$ be a given sequence. Then by the uniform H^1 bound for $z(\cdot,t)$ in (2.35) there is a subsequence t_{j_k} such that $z(x,t_{j_k})$ converges uniformly in x, say to $g(x)$, so that $z(X_\infty(y),t_{j_k})\to g(X_\infty(y))$ uniformly in y. Also, this same uniform H^1 bound shows that

$$\left|z(X(y,t_{j_k}),t_{j_k})-z(X_\infty(y),t_{j_k})\right|\leqq C\left|X(y,t_{j_k})-X_\infty(y)\right|^{1/2},$$

so that $z(X(y,t_{j_k}),t_{j_k})\to g(X_\infty(y))$ uniformly in y as $t\to\infty$. But by (4.3), $z(X(y,t_{j_k}),t_{j_k})\to\eta(y)$, so that in fact $g=\eta\circ X_\infty^{-1}$. Since this limit is independent of the sequence t_{j_k}, $z(X(y,t),t)\to\eta(y)$ uniformly in y and therefore by (4.10)

$$z(\cdot,t)\to z_\infty\equiv\eta\circ X_\infty^{-1}\quad\text{uniformly as } t\to\infty. \tag{4.12}$$

It then follows from this, (4.6), (4.7), and (4.8) that

$$\rho(\cdot,t)\to\frac{P_\infty}{R\theta_\infty}\,w(z_\infty(\cdot))\quad\text{in } L^1 \text{ as } t\to\infty. \tag{4.13}$$

This convergence holds also in L^p for $p\in[1,\infty)$ by the uniform pointwise bound for ρ in (2.28).

The results (2.37)–(2.41) of Theorem 2.2 thus follow from (4.6), (4.10), (4.12), and (4.13). To complete the proof we therefore need to show that the expressions in (4.6) and (4.8) for θ_∞ and P_∞ are equivalent to those in the statement of the theorem. To see this we first compute $\frac{\partial X_\infty}{\partial y}$ from (4.10) and then make the change of variable $h=X_\infty^{-1}(x)$ in (4.6) to obtain the required expression for P_∞. We then compute from (2.7) and (4.13) that

$$\begin{aligned}M_\infty&\equiv\int_0^1\rho_0(x)\mathrm{d}x+\int_0^\infty\int_0^1 a(\rho F)(x,t)\mathrm{d}x\mathrm{d}t\\&=\lim_{t\to\infty}\int_0^1\rho(x,t)\mathrm{d}x\\&=\frac{P_\infty}{R\theta_\infty}\int_0^1 w(z_\infty(x))\mathrm{d}x,\end{aligned}$$

which gives the required expression for θ_∞. □

5. Complete combustion: proof of Theorem 2.3

In this section we derive the conditions described in Theorem 2.3 for complete combustion to occur, that is, for there to be at least one reactant species which is depleted in the time-asymptotic state. We begin with the following facts.

Lemma 5.1. *Assume that the hypotheses and notations of Theorem 2.3 are in force and let $\rho_\infty, \theta_\infty$, and z_∞ be as in* Theorem 2.2. *Then if $a_j \neq 0$,*

$$\theta_\infty = \frac{\int_0^1 \rho_0 (E_0 - \frac{q}{a_j} z_0^j)\mathrm{d}x + \dfrac{q}{a_j}\int_0^1 \rho_\infty z_\infty^j \mathrm{d}x}{\int_0^1 \rho_\infty c(z_\infty)\mathrm{d}x} \tag{5.1}$$

and

$$\int_0^1 \rho_\infty \mathrm{d}x = \int_0^1 \rho_0 \left(1 - \frac{a}{a_j} z_0^j\right)\mathrm{d}x + \frac{a}{a_j}\int_0^1 \rho_\infty z_\infty^j \mathrm{d}x. \tag{5.2}$$

Also,

$$\text{if } \theta_\infty > \theta_{ig}, \text{ then } z_\infty \text{ and } \rho_\infty \text{ are constants and there is a reactant species } \mathcal{M}_j \text{ such that } z_\infty^j \equiv 0; \tag{5.3}$$

$$\text{if } z_0^j \not\equiv 0 \text{ but } z_\infty^j \equiv 0 \text{ for some reactant species } \mathcal{M}_j, \text{ then } \theta_\infty \geqq \theta_{ig}. \tag{5.4}$$

Proof. First from (2.7) and (2.10),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho z^j \mathrm{d}x = a_j \int \rho F \mathrm{d}x \tag{5.5}$$

so that by (2.42),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho \left(E - \frac{q}{a_j} z^j\right)\mathrm{d}x = 0 \tag{5.6}$$

and therefore

$$\int_0^1 \left[\rho\left(E - \frac{q}{a_j} z^j\right)\right](x,t)\mathrm{d}x = \int_0^1 \rho_0 \left(E_0 - \frac{q}{a_j} z_0^j\right)\mathrm{d}x.$$

To prove (5.1) we take the limit as $t \to \infty$, apply the convergence results of Theorem 2.2, then solve the resulting equation for θ_∞.

To prove (5.2) we again apply (2.7) and (2.10) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho \,\mathrm{d}x = a \int \rho F \mathrm{d}x = \frac{a}{a_j}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho z^j \mathrm{d}x;$$

we then integrate and take the limit as $t \to \infty$ to compute $\int_0^1 \rho_\infty \mathrm{d}x$.

Now assume that $\theta_\infty > \theta_{ig}$. Then by the hypotheses of the theorem there is a constant C and a time t_0 such that $h(x,t) \geqq C^{-1}$ for all x and for $t \geqq t_0$. It then follows from (2.35) that

$$\int_0^\infty \int_0^1 |z_x|^2 \mathrm{d}x\mathrm{d}t \leqq C. \tag{5.7}$$

On the other hand the uniform bound in (2.35) for $\int_0^1 |z_x|^2 \mathrm{d}x$ and the convergence (2.40) show that $z_x(\cdot,t) \rightharpoonup (z_\infty)_x$ weakly in L^2 as $t \to \infty$. Thus if $\varphi \in L^2$,

$$\int_0^1 z_x^j(x,t)\varphi(x)\mathrm{d}x \to \int_0^1 (z_\infty^j)_x(x)\varphi(x)\mathrm{d}x$$

as $t \to \infty$. The bounded convergence theorem then applies to show that for $T > 0$,

$$\int_0^T \int_0^1 z_x^j(t+s,x)\varphi(x)\mathrm{d}x\mathrm{d}s \to T\int_0^1 (z_\infty^j)_x(x)\varphi(x)\mathrm{d}x$$

as $t \to \infty$. The left side here is bounded by $CT^{1/2}\|\varphi\|_{L^2}$ by (5.7), however, so that

$$\left|\int_0^1 (z_\infty^j)_x \varphi \mathrm{d}x\right| \leqq CT^{-1/2} \to 0 \quad \text{as } T \to \infty.$$

This proves that $(z_\infty)_x = 0$ and therefore that z_∞ is a constant. It then follows from (2.37) that $\rho(\cdot,t)$ tends to a constant ρ_∞ in L^p as $t \to \infty$. To complete the proof of (5.3) we note that if $\mathcal{M}_j$ is a reactant species, then $a_j < 0 < a$ and the evolution equation (2.10) for z^j shows that z^j is nonincreasing along integral curves of u. Thus if $z_\infty^j > 0$ for all reactant species and if $\theta_\infty > \theta_{ig}$, then by our hypotheses on F, $F(x,t) \geqq C^{-1} > 0$ for some C and for all x and all t sufficiently large. But then (2.10) would apply to show that some z^j becomes negative in finite time, which is impossible. There therefore must be some reactant species for which $z_\infty^j \equiv 0$. This proves (5.3).

Finally to prove (5.4) we suppose that, for some reactant species $\mathcal{M}_j$, $z_\infty^j \equiv 0$ but $z^j(y,0) > 0$ for some y. Then $z^j(X(y,t),t) > 0$ for all t by (2.10) since $F(\theta,z) \geqq -Cz^j$ by (2.17) and (2.19). On the other hand if $\theta_\infty < \theta_{ig}$, then there would be a time t_0 such that $F(x,t) = 0$ for all $t \geqq t_0$ and all x. But then $z^j(X(y,t),t) = z^j(X(y,t_0),t_0) > 0$ for all $t \geqq t_0$ and therefore $z_\infty^j = z^j(X(y,t_0),t_0) > 0$, contradicting our supposition that $z_\infty^j \equiv 0$. This proves that $\theta_\infty \geqq \theta_{ig}$. □

Proof of Theorem 2.3. To prove (a) we first note that, if $\mathcal{M}_j$ is a reactant species, then $q/a_j < 0$ and so by (5.5),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho z^j \mathrm{d}x \leqq 0. \tag{5.8}$$

Thus by (5.1) and (5.2),

$$\theta_\infty \geqq \mathcal{E}_0 / \int_0^1 \rho_\infty c(z_\infty)\mathrm{d}x$$
$$\geqq \mathcal{E}_0/\bar{c} \int_0^1 \rho_0 \left(1 - \frac{a}{a_j} z_0^j\right) \mathrm{d}x.$$

Therefore if the hypothesis of (a) holds, then $\theta_\infty > \theta_{ig}$ and the conclusions of (a) follow from (5.3).

To prove (b) we let j be as in the hypothesis and apply (5.1), (5.2), and the assumption that $\theta_\infty \geqq \theta_{ig}$ to obtain

$$\theta_{ig} \leqq \frac{\mathcal{E}_0 - \frac{q}{a_j}\int_0^1 \rho_0 z_0^j \mathrm{d}x}{\int_0^1 \rho_\infty c(z_\infty)\mathrm{d}x} \leqq \frac{\mathcal{E}_0 - \frac{q}{a_j}\int_0^1 \rho_0 z_0^j \mathrm{d}x}{\underline{c}\int_0^1 \rho_0 (1 - \frac{a}{a_j}) z_0^j \mathrm{d}x},$$

since $z_\infty^j \equiv 0$. The conclusion in (b) then follows. □

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