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Gas Flows with Several Thermal Nonequilibrium Modes

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Abstract

We study gas flows with any finite number of thermal nonequilibrium modes. The equations describing such flows are a hyperbolic system with several relaxation equations. An important feature is entropy increase dictated by physics for any irreversible process. Under physical assumptions we obtain properties of thermodynamic variables relevant to stability. By the energy method we prove global existence and uniqueness for the Cauchy problem when the initial data are small perturbations of constant equilibrium states. We give a precise formulation of the fundamental solution for the linearized system, and use it to obtain large time behavior of solutions to the nonlinear system. In particular, we show that the entropy increases but stays bounded. The resulting asymptotic picture of nonequilibrium flows is in a pointwise sense both in space and in time.

1. Introduction

Many theoretical studies of gas flow adopt the local thermodynamic equilibrium model, namely the well-known Euler equations for compressible fluids, which take the following form in one space dimension:

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + p_x = 0, \\ &\left(e + \frac{1}{2}u^2\right)_t + (pu)_x = 0. \end{aligned} \tag{1.1}$$

Here, we have used Lagrangian coordinates, and v, u, p and e are, respectively, the specific volume, velocity, pressure and internal energy of the gas. Other commonly used thermodynamic variables are the temperature T and the entropy s. The basic thermodynamic laws state that among the thermodynamic variables only two are independent. If we choose for that role s and v, which in turn are specified functions of x and t, then the local rate per unit mass at which e is increasing at (x, t) is

$$e_t(x, t) = T(s(x, t), v(x, t))s_t(x, t) - p(s(x, t), v(x, t))v_t(x, t), \tag{1.2}$$

see [1]. Equation (1.1) is supplemented by equations of state, usually those for the perfect gas. Therefore, it is a system of three equations in three unknowns: two thermodynamic variables and the velocity.

However, at very high velocity and temperature, the assumption of local thermodynamic equilibrium, which leads to (1.1), is inadequate. The simplest nonequilibrium flow, with only one nonequilibrium mode, was studied in a previous paper, [11]. There we discussed thermodynamic properties of the flow, established global existence, and obtained the flow's large time behavior. In this paper we extend the study to flows with any finite number of nonequilibrium modes. This is an important step towards the real gas situation: For diatomic gases such as common air, the number of nonequilibrium modes in fact depends on the temperature range. If the temperature is high enough for the upper vibrational states of the molecules to be appreciably populated, the coupling between vibration and rotation must be included since the two effects are of the same order of magnitude.

Besides its significance in physics, the system describing the flow is an interesting subject in the theory of partial differential equations. As is to be seen in (1.7), this is a hyperbolic system with relaxation. Global existence of solutions to a general hyperbolic relaxation system was established in [10 and 2] under assumptions so strong that they preclude any nonequilibrium flow. In fact, their assumptions were originally introduced for hyperbolic-parabolic systems of conservation laws, [7], and they imply that the solution decays in time. This is true for viscous flow when heat conduction is present, as described by the Navier–Stokes equations for compressible fluids. The situation in nonequilibrium flow is different because at least part of the solution cannot decay in time. It is then important to establish a general theory under a weak dissipation assumption. At the same time, there is a belief that conclusions for systems with one rate equation may not be true for systems with more than one rate equation. The intrinsic difficulty associated with systems with several rate equations comes from the fact that the relaxation time scale of each rate equation is independent of the other. This is indeed the case in physics as far as small departure from equilibrium states is concerned. An important issue is then to study the coupling among the different modes. Studying the equations of gas flow with several nonequilibrium modes helps us to identify what is in common for systems with one rate equation and those with several. This may lead to some weak form of dissipation assumption.

We now derive the equations for flows with a fixed number of nonequilibrium modes. We assume that the flow is everywhere in instantaneous translational equilibrium. We use subscript "1" to denote thermodynamic quantities related to the translational mode, and i, $2 \leqq i \leqq m$, for those related to the nonequilibrium modes. For example,

$$e = \sum_{i=1}^{m} e_i, \quad s = \sum_{i=1}^{m} s_i, \tag{1.3}$$

where e_1 and s_1 denote the specific translational energy and entropy, respectively, of the molecules, while e_i and s_i, $2 \leqq i \leqq m$, are the energy and entropy, respectively, of the ith mode, which is a nonequilibrium mode. Therefore, e is the total specific internal energy, and s is the total specific entropy. Similarly, T_i, $1 \leqq i \leqq m$, denotes the temperature associated with the ith mode of the molecules. For a nonequilibrium state we must have some i, $2 \leqq i \leqq m$, such that $T_i \neq T_1$. Equivalently, a state is in equilibrium if and only if

$$T_i = T_1, \quad 2 \leqq i \leqq m. \tag{1.4}$$

The translational mode and the nonequilibrium modes obey different thermodynamic equations. Among the thermodynamic variables related to the translational mode, only two are independent, while among those related to the ith mode, $2 \leqq i \leqq m$, only one is independent. If we choose s_1 and v as the independent variables for the translational mode, and s_i as the independent variable for the ith mode, $2 \leqq i \leqq m$, then similar to (1.2) we have

$$\begin{aligned} (e_1)_t(x,t) &= T_1(s_1(x,t), v(x,t))(s_1)_t(x,t) - p(s_1(x,t), v(x,t))v_t(x,t), \\ (e_i)_t(x,t) &= T_i(s_i(x,t))(s_i)_t(x,t), \quad 2 \leqq i \leqq m. \end{aligned} \tag{1.5}$$

Here we note that the energies of internal structure are volume-independent. Equations (1.3)–(1.5) imply

$$e_t(x,t) = T_1(s_1, v)s_t + \sum_{i=2}^{m} [T_i(s_i) - T_1(s_1, v)](s_i)_t - p(s_1, v)v_t. \tag{1.6}$$

From above we see that the flow is completely determined by $m + 1$ independent thermodynamic variables and the velocity. To represent the flow, the system of $m+2$ equations can be chosen as the conservation of mass, momentum, energy, and $m-1$ rate equations. Here each of the rate equations describes how an internal structure relaxes to its local equilibrium value. We assume that each T_i can only slightly deviate from T_1. In Lagrangian coordinates the system reads

$$\begin{aligned} & v_t - u_x = 0, \\ & u_t + p_x = 0, \\ & \left(e + \frac{1}{2}u^2\right)_t + (pu)_x = 0, \\ & (e_i)_t = \frac{E_i - e_i}{\tau_i}, \quad 2 \leqq i \leqq m, \end{aligned} \tag{1.7}$$

where

$$\tau_i = \tau_i(v, e_1) > 0, \quad 2 \leqq i \leqq m, \tag{1.8}$$

is the local relaxation time for the ith mode, and $E_i = E_i(v, e_1)$ is the local equilibrium value of e_i. That is, if we express e_i as a function of T_i,

$$e_i = \omega_i(T_i), \quad 2 \leqq i \leqq m, \tag{1.9}$$

then

$$E_i = \omega_i(T_1), \quad 2 \leqq i \leqq m. \tag{1.10}$$

The relaxation time τ_i sets the time scale for e_i to relax to E_i. Both τ_i and E_i are known functions of the translational mode. System (1.7) is supplemented by appropriate equations of state. For further discussion on nonequilibrium flow, the readers are referred to [9] and references therein.

We regard all the thermodynamic variables related to the translational mode as functions of v and e_1, for example, $T_1 = T_1(v, e_1)$, $s_1 = s_1(v, e_1)$, and so on. We also introduce the following notation for the pressure:

$$p = p(v, e_1) = \bar{p}(v, s_1) = \tilde{p}(v, T_1). \tag{1.11}$$

The basic assumptions in this paper are dictated by physics:

$$\begin{aligned} \tilde{p}_v &= \frac{\partial}{\partial v}\tilde{p}(v, T_1) < 0, \quad (T_1)_{e_1} = \frac{\partial}{\partial e_1} T_1(v, e_1) > 0, \\ p_{e_1} &= \frac{\partial}{\partial e_1} p(v, e_1) \neq 0, \quad \omega_i'(T_1) > 0, \quad 2 \leqq i \leqq m. \end{aligned} \tag{1.12}$$

By direct calculation and using (1.5) and (1.10), (1.12) implies

$$\begin{aligned} &c_f^2 \equiv -\bar{p}_v = p p_{e_1} - p_v = -\tilde{p}_v + \frac{p_{e_1}^2 T_1}{(T_1)_{e_1}} > 0; \\ &(E_i)_{e_1} = \omega_i'(T_1)(T_1)_{e_1} > 0, \quad 2 \leqq i \leqq m; \\ &a_i \equiv \omega_i'(T_1) T_1 p_{e_1} \neq 0, \quad p_{e_1} a_i > 0, \quad 2 \leqq i \leqq m; \\ &b \equiv \frac{p_{e_1} \sum_{i=2}^m a_i}{c_f^2 - \sum_{i=2}^m (E_i)_{e_1} \tilde{p}_v} > 0. \end{aligned} \tag{1.13}$$

Here the derivation of the first equation can be found, for example, in Section 9 of [4]. We further define

$$c^2 \equiv \frac{c_f^2}{1+b} > 0. \tag{1.14}$$

The positive quantities c_f and c are called the frozen speed of sound and the equilibrium speed of sound, respectively. Their physical meaning will be discussed in Section 2.

To simplify our notation we introduce new variables

$$\chi_i = E_i - e_i, \quad 2 \leqq i \leqq m, \tag{1.15}$$

namely the departures of nonequilibrium internal energies from their local equilibrium values. We also let

$$\|\cdot\|_l \equiv \|\cdot\|_{H^l}, \quad \|\cdot\| \equiv \|\cdot\|_{L^2}, \tag{1.16}$$

where the norms are with respect to the space variable x.

We are now ready to state the main results of the paper. Consider the Cauchy problem of (1.7) with initial data

$$(v, u, e_1, \dots, e_m)(x, 0) = (v_0, u_0, e_{1,0}, \dots, e_{m,0})(x). \tag{1.17}$$

Here the initial function $(v_0, u_0, e_{1,0}, \dots, e_{m,0})$ is a small perturbation of a constant equilibrium state $(v^*, u^*, e_1^*, \dots, e_m^*)$. By their physical meaning we have

$$v^* > 0, \quad e_i^* > 0, \quad 1 \leqq i \leqq m.$$

Without loss of generality we take $u^* = 0$. Since the constant state is an equilibrium state, we have

$$e_i^* = E_i^* \quad \text{or} \quad \mathrm{T}_\mathrm{i}^* = \mathrm{T}_1^*, \quad 2 \leqq \mathrm{i} \leqq \mathrm{m}, \tag{1.18}$$

by (1.4). Here we use the superscript "*" to denote thermodynamic variables taking their values at the constant state. Therefore, $E_i^* = E_i(v^*, e_1^*)$, and so on. Our first theorem is on global existence of solutions.

Theorem 1.1. *Let* (1.12) *hold, and* v^*, e_1^*, ..., e_m^* *be positive constants such that* (1.18) *is satisfied. Let* $l \geqq 2$ *be an integer. Then there exist positive constants* ε *and* C*, such that if*

$$\|(v_0 - v^*, u_0, e_{1,0} - e_1^*, \dots, e_{m,0} - e_m^*)\|_l \leqq \varepsilon,$$

the Cauchy problem (1.7), (1.17) *has a unique global solution* $(v, u, e_1, \dots, e_m)(x, t)$*, satisfying*

$$(v - v^*, u, e_1 - e_1^*, \dots, e_m - e_m^*) \in C^0([0, \infty); H^l) \cap C^1([0, \infty); H^{l-1}), \tag{1.19}$$
$$p_x,\ u_x \in L^2([0, \infty); H^{l-1}), \quad \chi_i \in L^2([0, \infty); H^l), \quad 2 \leqq i \leqq m,$$

and the following energy inequality

$$\begin{aligned} &\sup_{t \geqq 0} \|(v - v^*, u, e_1 - e_1^*, \dots, e_m - e_m^*)\|_l^2(t) \\ &\quad + \int_0^\infty \left[\|p_x\|_{l-1}^2(t) + \|u_x\|_{l-1}^2(t) + \sum_{i=2}^m \|\chi_i\|_l^2(t) \right] \mathrm{d}t \\ &\leqq C \|(v_0 - v^*, u_0, e_{1,0} - e_1^*, \dots, e_{m,0} - e_m^*)\|_l^2. \end{aligned} \tag{1.20}$$

Our next theorem is on the large time behavior of the solution.

Theorem 1.2. *Let* (1.12) *hold, and* v^*, e_1^*, ..., e_m^* *be positive constants such that* (1.18) *is satisfied. Let the initial data* $(v_0, u_0, e_{1,0}, \dots, e_{m,0})$ *be a perturbation of* *the constant state* $(v^*, 0, e_1^*, \dots, e_m^*)$*, satisfying*

$$
\begin{aligned}
&(v_0 - v^*, u_0, e_{1,0} - e_1^*, \dots, e_{m,0} - e_m^*) \in H^6(\mathbb{R}),\\
&(v_0 - v^*, u_0, e_{1,0} - e_1^*, \dots, e_{m,0} - e_m^*)(x) = O(1)(x^2+1)^{-\frac{3}{4}},\\
&(v_0', u_0', e_{1,0}', \dots, e_{m,0}')(x) = O(1)(x^2+1)^{-\frac{3}{4}},\\
&s_0''(x) = O(1)(x^2+1)^{-\frac{1}{4}}, \qquad (1.21)\\
&\|(v_0 - v^*, u_0, e_{1,0} - e_1^*, \dots, e_{m,0} - e_m^*)\|_6 + \sup_{x\in\mathbb{R}} \Big\{ (x^2+1)^{\frac{3}{4}} \Big(|v_0 - v^*| + |u_0| \\
&\quad + \sum_{i=1}^m |e_{i,0} - e_i^*| + |v_0'| + |u_0'| + \sum_{i=1}^m |e_{i,0}'| \Big)(x) + (x^2+1)^{\frac{1}{4}} |s_0''|(x) \Big\} \equiv \varepsilon_0 \ll 1,
\end{aligned}
$$

where s_0 *is the initial entropy. Then for all* $x \in \mathbb{R}$, $t \geqq 0$, *the solution of* (1.7)*,* (1.17) *has the following property:*

$$
\begin{aligned}
&(p - p^*, u)(x,t)\\
&\quad = O(1)\varepsilon_0 \Big\{ (t+1)^{-\frac{1}{2}} \Big[\exp\Big(-\frac{(x + c^*(t+1))^2}{\nu(t+1)} \Big) + \exp\Big(-\frac{(x - c^*(t+1))^2}{\nu(t+1)} \Big) \Big]\\
&\qquad + [(x + c^*(t+1))^2 + t + 1]^{-\frac{3}{4}} + [(x - c^*(t+1))^2 + t + 1]^{-\frac{3}{4}} \Big\}, \qquad (1.22)\\
&(\chi_2, \dots, \chi_m, p_x, u_x)(x,t)\\
&\quad = O(1)\varepsilon_0 (t+1)^{-\frac{1}{2}} \Big\{ (t+1)^{-\frac{1}{2}} \Big[\exp\Big(-\frac{(x + c^*(t+1))^2}{\nu(t+1)} \Big)\\
&\qquad + \exp\Big(-\frac{(x - c^*(t+1))^2}{\nu(t+1)} \Big) \Big]\\
&\qquad + [(x + c^*(t+1))^2 + t + 1]^{-\frac{3}{4}} + [(x - c^*(t+1))^2 + t + 1]^{-\frac{3}{4}} \Big\},\\
&\| ((\chi_2)_x, \dots, (\chi_m)_x, p_{xx}, u_{xx})(\cdot, t) \|_{L^\infty} = O(1)\varepsilon_0 (t+1)^{-\frac{3}{2}},\\
&s(x,t) - s^* = O(1)\varepsilon_0 (x^2+1)^{-\frac{3}{4}}, \quad s_x(x,t) = O(1)\varepsilon_0 (x^2+1)^{-\frac{3}{4}},
\end{aligned}
$$

where p^*, c^* *and* s^* *are the pressure, equilibrium speed of sound and entropy evaluated at the constant state, and* $\nu > 0$ *is a constant depending only on the constant state. Here the equilibrium speed of sound c, is defined by* (1.14)*,* (1.13) *and* (1.11)*.*

From (1.22) we see that indeed the solution contains three parts: The entropy wave does not decay. The perturbations of pressure and velocity decay at the rate $(t+1)^{-1/2}$. And the departures of internal structures from their local equilibrium values are higher order terms.

The plan for this paper is as follows: In Section 2, we study thermodynamic properties of nonequilibrium flows, and derive the Chapman–Enskog expansion for (1.7). In Section 3, we prove Theorem 1.1 by energy estimates. In Section 4, we present the fundamental solution of the linearized system. And finally in Section 5, we prove Theorem 1.2.

2. Thermodynamic properties and Chapman-Enskog expansion

Besides their own interests in physics, the thermodynamic properties of nonequilibrium flows are the foundation for establishing the global existence of solutions and for studying the large time behavior.

First we study the entropy s. Notice that by (1.15) the system (1.7) for nonequilibrium flows can be written as

$$w_t + f(w)_x = r(w), \tag{2.1}$$

where

$$w = \begin{pmatrix} v \\ u \\ e + \frac{1}{2}u^2 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}, \quad f(w) = \begin{pmatrix} -u \\ p \\ pu \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad r(w) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \chi_2/\tau_2 \\ \vdots \\ \chi_m/\tau_m \end{pmatrix}. \tag{2.2}$$

Proposition 2.1. *Let (1.12) hold. If the entropy s is regarded as a function of w defined in (2.2), then $-s$ is strictly convex. Moreover, the Hessian H of $-s$ with respect to w is a symmetrizer of (2.1), (2.2) in the following sense: Hf' is symmetric for all w under consideration, while Hr' is symmetric and semi-negative definite on the equilibrium manifold $T_i = T_1$, $2 \leqq i \leqq m$.*

Proof. Let

$$H = -\nabla^2 s(w) = \begin{pmatrix} H_1 & H_2 \\ H_2^t & H_3 \end{pmatrix}, \tag{2.3}$$

where

$$\begin{aligned} H_1 &= \begin{pmatrix} -(s_1)_{vv} & u(s_1)_{e_1 v} & -(s_1)_{e_1 v} \\ u(s_1)_{ve_1} & (s_1)_{e_1} - u^2(s_1)_{e_1 e_1} & u(s_1)_{e_1 e_1} \\ -(s_1)_{ve_1} & u(s_1)_{e_1 e_1} & -(s_1)_{e_1 e_1} \end{pmatrix}, \\ H_2 &= \begin{pmatrix} (s_1)_{e_1 v} & \cdots & (s_1)_{e_1 v} \\ -u(s_1)_{e_1 e_1} & \cdots & -u(s_1)_{e_1 e_1} \\ (s_1)_{e_1 e_1} & \cdots & (s_1)_{e_1 e_1} \end{pmatrix} \in \mathbb{R}^{3\times(m-1)}, \\ H_3 &= H_{31} - \operatorname{diag}\left(s_2''(e_2), \ldots, s_m''(e_m)\right) \in \mathbb{R}^{(m-1)\times(m-1)}, \end{aligned} \tag{2.4}$$

and H_{31} is the matrix whose entries are all $-(s_1)_{e_1 e_1}$. Here in (2.4) $s_1 = s_1(v, e_1)$ and $s_i = s_i(e_i)$, $2 \leqq i \leqq m$. Using the assumption (1.12), by direct calculation we can show that H_1 is positive definite. We can also show that

$$\det H = \prod_{i=2}^{m} \left[-s_i''(e_i)\right] \det H_1. \tag{2.5}$$

From (1.5) we have

$$s_i'(e_i) = \frac{1}{T_i}, \quad 2 \leqq i \leqq m.$$

Therefore, by (1.9) and (1.12),

$$-s_i''(e_i) = \frac{1}{T_i^2 \omega_i'(T_i)} > 0, \quad 2 \leqq i \leqq m. \tag{2.6}$$

Equation (2.5) implies det $H > 0$. Similarly, we can show that all leading principal submatrices of H have positive determinants. This implies that H is positive definite, or $-s$ is strictly convex.

Note that from (2.2) and (1.11),

$$f' = \begin{pmatrix} A_1 & A_2 \\ 0_{(m-1)\times 3} & 0_{(m-1)\times(m-1)} \end{pmatrix}, \tag{2.7}$$

where

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ p_v & -up_{e_1} & p_{e_1} \\ up_v & -u^2 p_{e_1} + p & up_{e_1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \cdots & 0 \\ -p_{e_1} & \cdots & -p_{e_1} \\ -up_{e_1} & \cdots & -up_{e_1} \end{pmatrix} \in \mathbb{R}^{3\times(m-1)}.$$

Equations (2.3) and (2.7) give us

$$Hf' = \begin{pmatrix} H_1 A_1 & H_1 A_2 \\ H_2^t A_1 & H_2^t A_2 \end{pmatrix}.$$

It is a classical result for the Euler equations (1.1) that $H_1 A_1$ is symmetric. By direct calculation we have $H_2^t A_2 = 0_{(m-1)\times(m-1)}$. Using (1.5) we also have $A_1^t H_2 = H_1 A_2 \in \mathbb{R}^{3\times(m-1)}$, where in $H_1 A_2$ the first and the third rows are zero, while the entries in the second row are all the same as $-p_{e_1}/T_1$. Therefore, Hf' is symmetric.

On the equilibrium manifold $T_i = T_1, 2 \leqq i \leqq m$, we have by (1.15), (1.9) and (1.10),

$$\chi_i = 0, \quad 2 \leqq i \leqq m. \tag{2.8}$$

Hence by (2.2),

$$r' = \begin{pmatrix} 0_{3\times 3} & 0_{3\times(m-1)} \\ R_1 & R_2 \end{pmatrix}, \tag{2.9}$$

with

$$R_1 = \begin{pmatrix} \frac{(E_2)_v}{\tau_2} & -\frac{u(E_2)_{e_1}}{\tau_2} & \frac{(E_2)_{e_1}}{\tau_2} \\ \vdots & \vdots & \vdots \\ \frac{(E_m)_v}{\tau_m} & -\frac{u(E_m)_{e_1}}{\tau_m} & \frac{(E_m)_{e_1}}{\tau_m} \end{pmatrix} \in \mathbb{R}^{(m-1)\times 3},$$

$$R_2 = \begin{pmatrix} -\frac{(E_2)_{e_1}}{\tau_2} & \cdots & -\frac{(E_2)_{e_1}}{\tau_2} \\ \cdots & \cdots & \cdots \\ -\frac{(E_m)_{e_1}}{\tau_m} & \cdots & -\frac{(E_m)_{e_1}}{\tau_m} \end{pmatrix} - \text{diag}\left(\frac{1}{\tau_2}, \ldots, \frac{1}{\tau_m}\right) \in \mathbb{R}^{(m-1)\times(m-1)}.$$

Using (1.10), (1.5), (2.4) and (2.6) we have

$$\begin{pmatrix} R_1 & R_2 \end{pmatrix} = \operatorname{diag}\left(\frac{\omega_2'(T_1)}{\tau_2}, \ldots, \frac{\omega_m'(T_1)}{\tau_m}\right)(-T_1^2)(H_2^t H_3). \tag{2.10}$$

Equations (2.3), (2.9) and (2.10), together with the fact that H_3 is symmetric, give us

$$Hr' = \begin{pmatrix} H_2 \\ H_3 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \end{pmatrix} = -T_1^2 \begin{pmatrix} H_2 \\ H_3 \end{pmatrix} \operatorname{diag}\left(\frac{\omega_2'(T_1)}{\tau_2}, \ldots, \frac{\omega_m'(T_1)}{\tau_m}\right) \begin{pmatrix} H_2^t & H_3^t \end{pmatrix} \tag{2.11}$$

on the equilibrium manifold. Clearly, the right-hand side of (2.11) is symmetric and semi-negative definite under the assumption (1.12). □

Next we study the frozen speed of sound c_f and the equilibrium speed of sound c, as defined in (1.13) and (1.14). This is to be done by examining connections between the Equation (1.7) for nonequilibrium flows and the Euler equations (1.1). In particular, we examine two important limits associated with a nonequilibrium flow: the frozen flow and the equilibrium flow.

The frozen flow is the limit as all internal structures become "frozen", that is, as $\tau_i \to \infty$, $2 \leqq i \leqq m$. Here recall that τ_i is the time scale for the ith mode to relax to its local equilibrium value. For the frozen flow, all the rate equations in (1.7) take the form $(e_i)_t = 0$, hence are replaced by the algebraic equations

$$e_i(x,t) = e_i(x,0) \equiv e_i(x), \quad 2 \leqq i \leqq m.$$

By (1.3), Equation (1.7) becomes the Euler equations (1.1), with

$$e(x,t) = e_1(x,t) + e_I(x), \tag{2.12}$$

where $e_I(x) = \sum_{i=2}^m e_i(x)$ is given. From (1.11) and (2.12),

$$p = p(v, e_1) = p(v, e - e_I).$$

The sound speed of the frozen flow is

$$\left[p\frac{\partial}{\partial e}p(v, e-e_I) - \frac{\partial}{\partial v}p(v, e-e_I)\right]^{1/2} = (pp_{e_1} - p_v)^{1/2},$$

which is exactly the frozen speed of sound c_f defined in (1.13).

Similarly, the equilibrium flow is the limit as all internal processes take place infinitely rapidly, namely as $\tau_i \to 0$, $2 \leqq i \leqq m$. In general, we expect $(e_i)_t$ to stay finite. This implies

$$E_i = e_i, \quad 2 \leqq i \leqq m. \tag{2.13}$$

Again, Equation (1.7) is reduced to the Euler equations (1.1). However, by (1.3) and (2.13) the internal energy has a different expression

$$e = e_1 + \sum_{i=2}^m E_i(v, e_1). \tag{2.14}$$

Equation (2.14) defines e_1 implicitly as a function of v and e, denoted as

$$e_1 = e_1^{(r)}(v, e). \tag{2.15}$$

Correspondingly, as a function of v and e_1, p can be regarded as a function of v and e. That is,

$$p = p^{(r)}(v, e) = p(v, e_1^{(r)}(v, e)). \tag{2.16}$$

Denote the sound speed of the equilibrium flow as c,

$$c^2 = p\frac{\partial}{\partial e} p^{(r)}(v, e) - \frac{\partial}{\partial v} p^{(r)}(v, e). \tag{2.17}$$

We want to show that c defined by (2.17) is in fact the same as in (1.14).

For our convenience we introduce the following identities relating the thermodynamic variables for the translational mode:

$$\begin{aligned} &\tilde{p}_v = p_v - (T_1)_v p_{e_1}/(T_1)_{e_1}, \quad T_1 p_{e_1} = p(T_1)_{e_1} - (T_1)_v, \\ &\bar{p}_v = \tilde{p}_v - \frac{p_{e_1}^2 T_1}{(T_1)_{e_1}}. \end{aligned} \tag{2.18}$$

These identities are derived from (1.5), see [3 or 4] for details. We now differentiate (2.14) with respect to v and e, respectively. These give us

$$\left(e_1^{(r)}\right)_v = -\frac{\sum_{i=2}^m (E_i)_v}{1 + \sum_{i=2}^m (E_i)_{e_1}}, \quad \left(e_1^{(r)}\right)_e = \frac{1}{1 + \sum_{i=2}^m (E_i)_{e_1}}. \tag{2.19}$$

Substituting (2.16) and (2.19) into (2.17), and using (1.13), (1.10) and (2.18), we have

$$\begin{aligned} c^2 &= p p_{e_1} \left(e_1^{(r)}\right)_e - p_v - p_{e_1} \left(e_1^{(r)}\right)_v \\ &= c_f^2 - \frac{p_{e_1}}{1 + \sum_{i=2}^m (E_i)_{e_1}} \sum_{i=2}^m \left[p(E_i)_{e_1} - (E_i)_v\right] \\ &= c_f^2 - \frac{p_{e_1}}{1 + \sum_{i=2}^m (E_i)_{e_1}} \sum_{i=2}^m \omega_i'(T_1) T_1 p_{e_1} \\ &= \frac{c_f^2 - \sum_{i=2}^m \omega_i'(T_1)(T_1)_{e_1} \tilde{p}_v}{1 + \sum_{i=2}^m \omega_i'(T_1)(T_1)_{e_1}}. \end{aligned} \tag{2.20}$$

Using a_i and b as defined in (1.13), together with (2.18), we have

$$\begin{aligned} c^2 &= c_f^2 \Big/ \left\{1 + \sum_{i=2}^m \omega_i'(T_1)(T_1)_{e_1} \left(1 + \tilde{p}_v/c_f^2\right) \Big/ \left[1 - \sum_{i=2}^m \omega_i'(T_1)(T_1)_{e_1} \tilde{p}_v/c_f^2\right]\right\} \\ &= c_f^2 \Big/ \left\{1 + \sum_{i=2}^m a_i p_{e_1} \Big/ \left[c_f^2 - \sum_{i=2}^m \omega_i'(T_1)(T_1)_{e_1} \tilde{p}_v\right]\right\} = \frac{c_f^2}{1 + b}. \end{aligned}$$

Therefore, c defined in (2.17) is the same as in (1.14).

Proposition 2.2. *Let* (1.12) *hold. The equilibrium speed of sound and the frozen speed of sound satisfy*

$$0 < c < c_f. \tag{2.21}$$

Proof. Inequality (2.21) is straightforward by (1.13) and (1.14). □

The last topic in this section is the Chapman–Enskog expansion of (1.7). The expansion gives us the connection between a nonequilibrium flow and a viscous flow with zero heat conduction. We first derive the equation for χ_i defined by (1.15). From (1.10), (1.7), (1.3), (2.18) and (1.13) we have

$$\begin{aligned}(\chi_i)_t &= (E_i)_t - (e_i)_t = \omega_i'(T_1)\left[(T_1)_v v_t + (T_1)_{e_1}(e_i)_t\right] - \frac{\chi_i}{\tau_i} \\ &= \omega_i'(T_1)\left[(T_1)_v - (T_1)_{e_1} p\right] u_x - \omega_i'(T_1)(T_1)_{e_1}\sum_{j=2}^{m}\frac{\chi_j}{\tau_j} - \frac{\chi_i}{\tau_i} \\ &= -a_i u_x - \omega_i'(T_1)(T_1)_{e_1}\sum_{j=2}^{m}\frac{\chi_j}{\tau_j} - \frac{\chi_i}{\tau_i}, \qquad 2 \leqq i \leqq m.\end{aligned} \tag{2.22}$$

For the equilibrium flow we set $\chi_i = 0$, $2 \leqq i \leqq m$, which is (2.13). For the Chapman–Enskog expansion we use the next order correction, and set the fastest decaying term $(\chi_i)_t$ in (2.22) as zero. This gives us

$$\frac{\chi_i}{\tau_i} + \omega_i'(T_1)(T_1)_{e_1}\sum_{j=2}^{m}\frac{\chi_j}{\tau_j} = -a_i u_x, \quad 2 \leqq i \leqq m. \tag{2.23}$$

Sum up (2.23) for all i and simplify. We have

$$\sum_{j=2}^{m}\frac{\chi_j}{\tau_j} = \frac{-\sum_{j=2}^{m} a_j u_x}{1 + \sum_{j=2}^{m}\omega_j'(T_1)(T_1)_{e_1}}. \tag{2.24}$$

Substituting (2.24) into (2.23) gives us

$$\frac{\chi_i}{\tau_i} = \left[\frac{\sum_{j=2}^{m} a_j \omega_i'(T_1)(T_1)_{e_1}}{1 + \sum_{j=2}^{m}\omega_j'(T_1)(T_1)_{e_1}} - a_i\right] u_x, \quad 2 \leqq i \leqq m. \tag{2.25}$$

From (1.13) we have

$$a_j \omega_i'(T_1) = a_i \omega_j'(T_1), \quad 2 \leqq i, j \leqq m. \tag{2.26}$$

Using (2.26), Equation (2.25) can be simplified as

$$\frac{\chi_i}{\tau_i} = \frac{-a_i u_x}{1 + \sum_{j=2}^{m}\omega_j'(T_1)(T_1)_{e_1}}, \quad 2 \leqq i \leqq m. \tag{2.27}$$

Therefore, the total internal energy up to the same order correction is

$$e = e_1 + \sum_{i=2}^{m} E_i + \frac{u_x \sum_{i=2}^{m} a_i \tau_i}{1 + \sum_{i=2}^{m}\omega_i'(T_1)(T_1)_{e_1}}, \tag{2.28}$$

where we have used (1.3), (1.15) and (2.27). Analogous to (2.14) for the equilibrium flow, we define $e_1^{(r)}(v, e)$ by

$$e = e_1^{(r)} + \sum_{i=2}^{m} E_i(v, e_1^{(r)}). \tag{2.29}$$

Subtracting (2.29) from (2.28), then up to the same order of accuracy we have

$$e_1 - e_1^{(r)} + \sum_{i=2}^{m} (E_i)_{e_1} \left(e_1 - e_1^{(r)} \right) = - \frac{u_x \sum_{i=2}^{m} a_i \tau_i}{1 + \sum_{i=2}^{m} \omega_i'(T_1)(T_1)_{e_1}},$$

which implies

$$e_1 - e_1^{(r)} = - \left\{ \frac{\sum_{i=2}^{m} a_i \tau_i}{\left[1 + \sum_{i=2}^{m} \omega_i'(T_1)(T_1)_{e_1}\right]^2} \right\} (v, e_1^{(r)}) u_x. \tag{2.30}$$

Up to the same order of accuracy we also have

$$p = p(v, e_1) = p(v, e_1^{(r)}) - \left\{ \frac{p_{e_1} \sum_{i=2}^{m} a_i \tau_i}{\left[1 + \sum_{i=2}^{m} \omega_i'(T_1)(T_1)_{e_1}\right]^2} \right\} (v, e_1^{(r)}) u_x. \tag{2.31}$$

Substituting (2.31) into the first three equations of (1.7) and by (1.13) and (1.12), we obtain the second order Chapman–Enskog expansion for the nonequilibrium flow:

$$\begin{aligned} & v_t - u_x = 0, \\ & u_t + p_x^{(r)} = (\mu u_x)_x, \\ & \left(e + \frac{1}{2} u^2 \right)_t + (p^{(r)} u)_x = (\mu u u_x)_x, \end{aligned} \tag{2.32}$$

where

$$p^{(r)} = p^{(r)}(v, e) = p\left(v, e_1^{(r)}(v, e) \right), \tag{2.33}$$

$$\mu = \mu(v, e) = \left\{ \frac{p_{e_1} \sum_{i=2}^{m} a_i \tau_i}{\left[1 + \sum_{i=2}^{m} \omega_i'(T_1)(T_1)_{e_1}\right]^2} \right\} \left(v, e_1^{(r)}(v, e) \right) > 0. \tag{2.34}$$

System (2.32) is the Navier–Stokes equations with zero heat conduction, where μ plays the role of viscosity. This system has been studied in details in [5]. In particular, we found that the absence of heat conduction implies the nondecay of part of the solution. We referred to (2.32) as a system of composite type.

The Chapman–Enskog expansion suggests that the system (1.7) for nonequilibrium flows is also of composite type. This has been verified for the case of one nonequilibrium mode, see [11]. For the case of more than one nonequilibrium modes the conclusion stays true. This can be easily illustrated by a special solution: Let u be a constant, p be a positive constant, $v = v(x) > 0$, and $e_i = E_i = \omega_i(T_1(x))$, $2 \leqq i \leqq m$, where $T_1(x)$ is determined by $v(x)$ and p. Such a special solution to (1.7) is both a frozen flow and an equilibrium flow. Clearly, if $v(x)$ and $e_i(x)$ are perturbations of constants, the perturbations do not decay in time. For a generic nonequilibrium solution, the entropy in fact increases in time, as to be seen in the next section.

3. Global existence of solution

In this section we prove Theorem 1.1, hence establish the existence of solutions global in time. The approach is the energy estimate, based on the thermodynamic properties studied in Section 2. From the discussion in Section 2, we understand that part of the solution does not decay in time. The other part, however, does. Therefore, to perform the energy estimate it is crucial to separate different parts of the solution according to their decay rates. This is done mainly based on our knowledge of the fundamental solution of the linearized system, which is to be discussed in the next section. A correct way to separate different parts of the solution is to use s for the nondecaying portion, p and u for the leading term of the decaying portion, and χ_i, $2 \leqq i \leqq m$, for the higher order terms.

We now derive the equations for the entropy s and the pressure p, respectively. From (1.3), (1.5)–(1.7), (1.13) and (1.15) we have

$$\begin{aligned} s_t &= \frac{1}{T_1}(e_t + pv_t) + \frac{1}{T_1}\sum_{i=2}^{m}(T_1 - T_i)(s_i)_t \\ &= \frac{1}{T_1}\sum_{i=2}^{m}(T_1 - T_i)\frac{1}{T_i}(e_i)_t = \sum_{i=2}^{m}\left(\frac{1}{T_i} - \frac{1}{T_1}\right)\frac{\chi_i}{\tau_i}, \\ p_t &= p_v v_t + p_{e_1}(e_1)_t = p_v u_x + p_{e_1}\left[-uu_t - (pu)_x - \sum_{i=2}^{m}\frac{\chi_i}{\tau_i}\right] \\ &= -u_x c_f^2 - p_{e_1}\sum_{i=2}^{m}\frac{\chi_i}{\tau_i}. \end{aligned}$$

Together with (1.7) and (2.22) we have

$$\begin{aligned} & p_t + c_f^2 u_x = -p_{e_1}\sum_{j=2}^{m}\frac{\chi_j}{\tau_j}, \\ & u_t + p_x = 0, \\ & (\chi_i)_t + a_i u_x = -\frac{\chi_i}{\tau_i} - \omega_i'(T_1)(T_1)_{e_1}\sum_{j=2}^{m}\frac{\chi_j}{\tau_j}, \qquad 2 \leqq i \leqq m, \\ & s_t = \sum_{j=2}^{m}\left(\frac{1}{T_j} - \frac{1}{T_1}\right)\frac{\chi_j}{\tau_j}. \end{aligned} \tag{3.1}$$

Notice that the right-hand side of the entropy equation is

$$\sum_{i=2}^{m}\frac{T_1 - T_i}{T_i T_1 \tau_i}\left[\omega_i(T_1) - \omega_i(T_i)\right] > 0$$

by (1.9), (1.10), (1.12) and (1.15). Therefore, the entropy increases in time, characterizing an irreversible process. The thermodynamic variables p, s and χ_i, $2 \leqq i \leqq m$, are functions of v and e_i, $1 \leqq i \leqq m$. Under the assumption (1.12), by direct calculation we can verify that the Jacobian is nonzero. Therefore, we are free to use p, s and χ_i, $2 \leqq i \leqq m$, or v and e_i, $1 \leqq i \leqq m$, as independent variables. That is, we can use (3.1) or (1.7) as needed.

According to Proposition 2.1, the system (1.7) is symmetrizable. Local existence and uniqueness for the Cauchy problem are classical for such a system, see [6] and references therein. To prove Theorem 1.1 all we need is to prove the following a priori estimate.

Proposition 3.1. *Let* (1.12) *hold, and* v^*, $e_1^*, \ldots, e_m^*$ *be positive constants such that* (1.18) *is satisfied. Let* $l \geqq 2$ *be an integer and* $t_0 > 0$ *be a constant. Suppose that* $(v, u, e_1, \ldots, e_m)(x, t)$ *is a solution to* (1.7) *and* (1.17)*, satisfying* (1.19) *with* $[0, \infty)$ *replaced by* $[0, t_0]$*. For* $0 \leqq t \leqq t_0$ *define*

$$N_l^2(t) \equiv \sup_{0 \leqq t' \leqq t} \left\| \left(v - v^*, u, e_1 - e_1^*, \ldots, e_m - e_m^*\right) \right\|_l^2 (t') + \int_0^t \left(\|p_x\|_{l-1}^2 + \|u_x\|_{l-1}^2 + \sum_{i=2}^m \|\chi_i\|_l^2 \right)(t')\, \mathrm{d}t', \tag{3.2}$$

in particular,

$$N_l(0) \equiv \left\| \left(v_0 - v^*, u_0, e_{1,0} - e_1^*, \ldots, e_{m,0} - e_m^*\right) \right\|_l .$$

Then there exist positive constants ε *and* C*, independent of* t_0*, such that if* $N_l(t_0) \leqq \varepsilon$*, then*

$$N_l(t_0) \leqq C N_l(0). \tag{3.3}$$

Proof. Let C be a universal positive constant independent of t_0. As in the Introduction we use the superscript "*" to label the thermodynamic variables at the constant equilibrium state $(v^*, e_1^*, \ldots, e_m^*)$. Also let $u^* = 0$. Set

$$S(w) = -s + s^* + (\nabla s)^* (w - w^*), \tag{3.4}$$

where w is defined in (2.2), and the gradient operator ∇ is with respect to w. By Proposition 2.1, $-s$ is strictly convex with respect to w. Hence for $N_l(t_0) \leqq \varepsilon$, where ε is small and independent of t_0, S is equivalent to $|w - w^*|^2$, or to

$$\left| \left(v - v^*, u, e_1 - e_1^*, \ldots, e_m - e_m^*\right) \right|^2 . \tag{3.5}$$

From (3.4), (2.1) and (2.2) we have

$$S(w)_t = -s_t + (\nabla s)^* w_t = -s_t - (\nabla s)^* \left[f(w) - f(w^*) \right]_x , \tag{3.6}$$

where we have noticed that by (1.3), (1.5) and (1.18),

$$\left(\frac{\partial s}{\partial e_i} \right)^* = \left(-\frac{1}{T_1} + \frac{1}{T_i} \right)^* = 0, \quad 2 \leqq i \leqq m.$$

Integrate (3.6) over $\mathbb{R} \times [0, t]$ for $0 \leqq t \leqq t_0$ and use (3.1). We have

$$\int_{-\infty}^{\infty} S(w(x,t))\,dx + \int_0^t \int_{-\infty}^{\infty} \sum_{i=2}^{m} \left(\frac{1}{T_i} - \frac{1}{T_1}\right) \frac{\chi_i}{\tau_i}(x,t')\,\mathrm{d}x\mathrm{d}t'$$
$$= \int_{-\infty}^{\infty} S(w(x,0))\,\mathrm{d}x. \tag{3.7}$$

By (1.9), (1.10), (1.15) and (1.12), we have

$$\left(\frac{1}{T_i} - \frac{1}{T_1}\right)\frac{\chi_i}{\tau_i} = \frac{\left[\omega_i^{-1}(E_i) - \omega_i^{-1}(e_i)\right]\chi_i}{T_1 T_i \tau_i} \geqq \frac{\chi_i^2}{C}, \qquad 2 \leqq i \leqq m. \tag{3.8}$$

Substituting (3.8) into (3.7) and using the fact that S is equivalent to (3.5), we obtain the energy estimate for the solution:

$$\left\|\left(v - v^*, u, e_1 - e_1^*, \ldots, e_m - e_m^*\right)\right\|^2(t) + \int_0^t \sum_{i=2}^{m} \|\chi_i\|^2(t')\,dt' \leqq C N_0^2(0). \tag{3.9}$$

Next we perform energy estimates for the derivatives of the solution. For $1 \leqq k \leqq l$, taking the kth derivative of (3.1) with respect to x, we have

$$\begin{aligned}
&\left(\partial_x^k p\right)_t + \partial_x^k\left(c_f^2 u_x\right) = -\partial_x^k\left(p_{e_1}\sum_{j=2}^{m}\frac{\chi_j}{\tau_j}\right),\\
&\left(\partial_x^k u\right)_t + \partial_x^{k+1} p = 0,\\
&\left(\partial_x^k \chi_i\right)_t + \partial_x^k(a_i u_x) = -\partial_x^k\left[\frac{\chi_i}{\tau_i} + \omega_i'(T_1)(T_1)_{e_1}\sum_{j=2}^{m}\frac{\chi_j}{\tau_j}\right], \quad 2 \leqq i \leqq m,\\
&\left(\partial_x^k s\right)_t = \sum_{j=2}^{m}\partial_x^k\left[\left(\frac{1}{T_j} - \frac{1}{T_1}\right)\frac{\chi_j}{\tau_j}\right].
\end{aligned} \tag{3.10}$$

Define

$$\begin{aligned}
&K = c_f^2 - \sum_{j=2}^{m}\omega_j'(T_1)(T_1)_{e_1}\tilde{p}_v,\\
&\eta_1 = \frac{1 + \sum_{j=2}^{m}\omega_j'(T_1)(T_1)_{e_1}}{K}, \quad \eta_2 = \frac{p_{e_1}}{K}, \quad \eta_3 = -\frac{(T_1)_{e_1}\tilde{p}_v}{T_1 K},\\
&\zeta_i = \frac{1}{T_1\omega_i'(T_1)}, \quad \xi_i = 1 + \sum_{j=i}^{m}\omega_j'(T_1)(T_1)_{e_1}, \quad 2 \leqq i \leqq m.
\end{aligned} \tag{3.11}$$

Multiply the first equation in (3.10) by $\eta_1\partial_x^k p - \eta_2\sum_{j=2}^{m}\partial_x^k\chi_j$, the second one by $\partial_x^k u$, the last one by $\partial_x^k s$, and the one for χ_i by $-\eta_2\partial_x^k p + \zeta_i\partial_x^k\chi_i - \eta_3\sum_{j=2}^{m}\partial_x^k\chi_j$.

Sum up all these equations for $i = 2, \ldots, m$, and use (3.11), (1.13) and (2.18). We have

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left[\eta_1\left(\partial_x^k p\right)^2 - 2\eta_2\partial_x^k p\sum_{j=2}^m \partial_x^k\chi_j + \left(\partial_x^k u\right)^2 + \sum_{i=2}^m \zeta_i\left(\partial_x^k\chi_i\right)^2\right.\\
&\quad\left. - \eta_3\left(\sum_{j=2}^m \partial_x^k\chi_j\right)^2 + \left(\partial_x^k s\right)^2\right] + \sum_{i=2}^m \frac{\zeta_i}{\tau_i}\left(\partial_x^k\chi_i\right)^2\\
&= -\left(\partial_x^k p\,\partial_x^k u\right)_x + \left(\eta_1\partial_x^k p - \eta_2\sum_{j=2}^m\partial_x^k\chi_j\right)\left[c_f^2\partial_x^{k+1}u - \partial_x^k\left(c_f^2 u_x\right)\right]\\
&\quad + \sum_{i=2}^m\left(\eta_2\partial_x^k p - \zeta_i\partial_x^k\chi_i + \eta_3\sum_{j=2}^m\partial_x^k\chi_j\right)\left[\partial_x^k(a_i u_x) - a_i\partial_x^{k+1}u\right] + \frac{1}{2}(\eta_1)_t\\
&\quad \times\left(\partial_x^k p\right)^2 - (\eta_2)_t\partial_x^k p\sum_{j=2}^m\partial_x^k\chi_j + \frac{1}{2}\sum_{i=2}^m(\zeta_i)_t\left(\partial_x^k\chi_i\right)^2 - \frac{1}{2}(\eta_3)_t\left(\sum_{j=2}^m\partial_x^k\chi_j\right)^2\\
&\quad + \left(\eta_1\partial_x^k p - \eta_2\sum_{j=2}^m\partial_x^k\chi_j\right)\left[p_{e_1}\partial_x^k\sum_{j=2}^m\frac{\chi_j}{\tau_j} - \partial_x^k\left(p_{e_1}\sum_{j=2}^m\frac{\chi_j}{\tau_j}\right)\right]\\
&\quad + \eta_2\partial_x^k p\left[\partial_x^k\left(\sum_{i=2}^m\omega_i'(T_1)(T_1)_{e_1}\sum_{j=2}^m\frac{\chi_j}{\tau_j}\right) - \sum_{i=2}^m\omega_i'(T_1)(T_1)_{e_1}\partial_x^k\sum_{j=2}^m\frac{\chi_j}{\tau_j}\right]\\
&\quad + \sum_{i=2}^m\zeta_i\partial_x^k\chi_i\left\{\omega_i'(T_1)(T_1)_{e_1}\partial_x^k\sum_{j=2}^m\frac{\chi_j}{\tau_j} - \partial_x^k\left[\omega_i'(T_1)(T_1)_{e_1}\sum_{j=2}^m\frac{\chi_j}{\tau_j}\right]\right\}\\
&\quad - \eta_3\sum_{j=2}^m\partial_x^k\chi_j\left\{\sum_{i=2}^m\omega_i'(T_1)(T_1)_{e_1}\partial_x^k\sum_{j=2}^m\frac{\chi_j}{\tau_j} - \partial_x^k\left[\sum_{i=2}^m\omega_i'(T_1)(T_1)_{e_1}\sum_{j=2}^m\frac{\chi_j}{\tau_j}\right]\right\}\\
&\quad + \sum_{i=2}^m\zeta_i\partial_x^k\chi_i\left[\frac{\partial_x^k\chi_i}{\tau_i} - \partial_x^k\left(\frac{\chi_i}{\tau_i}\right)\right] + \partial_x^k s\sum_{j=2}^m\partial_x^k\left[\left(\frac{1}{T_j} - \frac{1}{T_1}\right)\frac{\chi_j}{\tau_j}\right].
\end{aligned} \tag{3.12}$$

We claim that the left-hand side of (3.12) can be written as

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left\{\eta_1\left(\partial_x^k p - \frac{\eta_2}{\eta_1}\sum_{i=2}^m\partial_x^k\chi_i\right)^2 + \left(\partial_x^k u\right)^2 + \left(\partial_x^k s\right)^2 + \sum_{i=2}^m\zeta_i\frac{\xi_{i+1}}{\xi_i}\right.\\
&\quad\left.\times\left[\partial_x^k\chi_i - \frac{(T_1)_{e_1}\omega_i'(T_1)}{\xi_{i+1}}\sum_{j=i+1}^m\partial_x^k\chi_j\right]^2\right\} + \sum_{i=2}^m\frac{\zeta_i}{\tau_i}\left(\partial_x^k\chi_i\right)^2.
\end{aligned} \tag{3.13}$$

To verify the claim, by (3.11), (2.18) and (1.13) we only need to show

$$\sum_{i=2}^{m} \zeta_i \frac{\xi_{i+1}}{\xi_i} \left[\partial_x^k \chi_i - \frac{(T_1)_{e_1} \omega_i'(T_1)}{\xi_{i+1}} \sum_{j=i+1}^{m} \partial_x^k \chi_j \right]^2$$
$$= \sum_{i=2}^{m} \zeta_i \left(\partial_x^k \chi_i \right)^2 - \frac{(T_1)_{e_1}}{T_1 \xi_2} \left(\sum_{j=2}^{m} \partial_x^k \chi_j \right)^2 . \tag{3.14}$$

For $m = 2$, (3.14) is true by the definition of ζ_i in (3.11). For $m > 2$, (3.14) is proved by induction: If (3.14) is true for $m - 1$, then for m the left-hand side of (3.14) is equal to

$$\frac{\zeta_2 \xi_3}{\xi_2} \left[\partial_x^k \chi_2 - \frac{(T_1)_{e_1} \omega_2'(T_1)}{\xi_3} \sum_{j=3}^{m} \partial_x^k \chi_j \right]^2 + \sum_{i=3}^{m} \zeta_i \left(\partial_x^k \chi_i \right)^2 - \frac{(T_1)_{e_1}}{T_1 \xi_3} \left(\sum_{j=3}^{m} \partial_x^k \chi_j \right)^2 ,$$

which can be simplified as the right-hand side of (3.14), using (3.11).

On the right-hand side of (3.12) we use (1.7) to replace the derivatives with respect to t by those with respect to x. Integrate (3.12) over $\mathbb{R} \times [0, t]$ for $0 \leqq t \leqq t_0$ and use (3.13). For $1 \leqq k \leqq l$ and small $N_l(t_0)$ we obtain

$$\frac{1}{2} \int_{-\infty}^{\infty} \left\{ \eta_1 \left(\partial_x^k p - \frac{\eta_1}{\eta_2} \sum_{i=2}^{m} \partial_x^k \chi_i \right)^2 + \left(\partial_x^k u \right)^2 + \left(\partial_x^k s \right)^2 \right.$$
$$\left. + \sum_{i=2}^{m} \zeta_i \frac{\xi_{i+1}}{\xi_i} \left[\partial_x^k \chi_i - \frac{(T_1)_{e_1} \omega_i'(T_1)}{\xi_{i+1}} \sum_{j=i+1}^{m} \partial_x^k \chi_j \right]^2 \right\} (x, t) \, \mathrm{d}x$$
$$+ \int_0^t \int_{-\infty}^{\infty} \left[\sum_{i=2}^{m} \frac{\zeta_i}{\tau_i} \left(\partial_x^k \chi_i \right)^2 \right] (x, t') \, \mathrm{d}x \mathrm{d}t' \leqq C \left[N_k^2(0) + N_l^3(t_0) \right], \tag{3.15}$$

where we have used (1.9), (1.10) and (1.15). By (3.11) and (1.12), inequality (3.15) implies

$$\left\| \left(\partial_x^k p, \partial_x^k u, \partial_x^k \chi_2, \ldots, \partial_x^k \chi_m, \partial_x^k s \right) \right\|^2 (t) + \int_0^t \sum_{i=2}^{m} \left\| \partial_x^k \chi_i \right\|^2 (t') \, \mathrm{d}t'$$
$$\leqq C \left[N_k^2(0) + N_l^3(t_0) \right]. \tag{3.16}$$

Next we apply ∂_x^{k-1} to the third equation of (3.1) with $i = 2$, and multiply the result by $\partial_x^k u / a_2$. Using (1.7), this gives us

$$\begin{aligned}\left(\partial_x^k u\right)^2 =& -\left(\frac{1}{a_2}\,\partial_x^k u\,\partial_x^{k-1}\chi_2\right)_t - \left(\frac{1}{a_2}\partial_x^k p\,\partial_x^{k-1}\chi_2\right)_x + \left(\frac{1}{a_2}\partial_x^{k-1}\chi_2\right)_x\partial_x^k p\\ &+ \partial_x^k u\,\partial_x^{k-1}\chi_2\left\{\left[\left(\frac{1}{a_2}\right)_v - \left(\frac{1}{a_2}\right)_{e_1} p\right]u_x - \left(\frac{1}{a_2}\right)_{e_1}\sum_{j=2}^m \frac{\chi_j}{\tau_j}\right\}\\ &+ \left[\left(\partial_x^k u\right)^2 - \frac{1}{a_2}\partial_x^k u\,\partial_x^{k-1}(a_2 u_x)\right]\\ &- \frac{1}{a_2}\partial_x^k u\,\partial_x^{k-1}\left[\frac{\chi_2}{\tau_2} + \omega_2'(T_1)(T_1)_{e_1}\sum_{j=2}^m\frac{\chi_j}{\tau_j}\right]. \end{aligned} \tag{3.17}$$

Integrate this equation over $\mathbb{R}\times[0,t]$ for $0 \leqq t \leqq t_0$. For $1 \leqq k \leqq l$ and small $N_l(t_0)$ we then have

$$\begin{aligned}\int_0^t \|\partial_x^k u\|^2(t')\,dt' \leqq& C\left[\|\partial_x^k u\|(t)\,\|\partial_x^{k-1}\chi_2\|(t) + N_k^2(0) + N_l^3(t_0)\right]\\ &+ C\int_0^t\int_{-\infty}^{\infty}\left[\left|\partial_x^k\chi_2\,\partial_x^k p\right| + \left|\partial_x^k u\right|\sum_{j=2}^m\left|\partial_x^{k-1}\chi_j\right|\right](x,t')\,\mathrm{d}x\mathrm{d}t'. \end{aligned} \tag{3.18}$$

Applying (3.16) and (3.9) to it, (3.18) is simplified as

$$\int_0^t \|\partial_x^k u\|^2(t')\,dt' \leqq C\left[N_k^2(0) + N_l^3(t_0)\right] + C\int_0^t\int_{-\infty}^{\infty}\left|\partial_x^k\chi_2\,\partial_x^k p\right|(x,t')\,\mathrm{d}x\mathrm{d}t'. \tag{3.19}$$

Applying ∂_x^{k-1} to the second equation of (3.1) and multiplying the result by $\partial_x^k p$, we also have

$$\begin{aligned}\left(\partial_x^k p\right)^2 =& -\left(\partial_x^k p\,\partial_x^{k-1}u\right)_t + \left(\partial_x^{k-1}p_t\,\partial_x^{k-1}u\right)_x + \partial_x^{k-1}\left(c_f^2 u_x\right)\partial_x^k u\\ &+ \partial_x^k u\,\partial_x^{k-1}\left(p_{e_1}\sum_{j=2}^m\frac{\chi_j}{\tau_j}\right). \end{aligned} \tag{3.20}$$

Integrate (3.20) over $\mathbb{R}\times[0,t]$ with $0 \leqq t \leqq t_0$. Again, for $1 \leqq k \leqq l$ and small $N_l(t_0)$ we have

$$\begin{aligned}\int_0^t \|\partial_x^k p\|^2(t')\,\mathrm{d}t' \leqq& C\left[\|\partial_x^k p\|(t)\,\|\partial_x^{k-1}u\|(t) + N_k^2(0) + N_l^3(t_0)\right]\\ &+ C\int_0^t\left(\|\partial_x^k u\|^2 + \sum_{j=2}^m\|\partial_x^{k-1}\chi_j\|^2\right)(t')\,\mathrm{d}t'. \end{aligned}$$

Substitute (3.16), (3.9) and (3.19) into the right-hand side. The above inequality becomes

$$\int_0^t \|\partial_x^k p\|^2(t')\,\mathrm{d}t' \leqq C\left[N_k^2(0) + N_l^3(t_0)\right] + C\int_0^t\int_{-\infty}^{\infty}\left|\partial_x^k\chi_2\,\partial_x^k p\right|(x,t')\,\mathrm{d}x\mathrm{d}t',$$

which can be further simplified as

$$\int_0^t \|\partial_x^k p\|^2(t')\,\mathrm{d}t' \leqq C\left[N_k^2(0) + N_l^3(t_0)\right] + C\int_0^t \|\partial_x^k \chi_2\|^2(t')\,\mathrm{d}t'$$
$$\leqq C\left[N_k^2(0) + N_l^3(t_0)\right], \tag{3.21}$$

using (3.16). With (3.16) and (3.21), inequality (3.19) becomes

$$\int_0^t \|\partial_x^k u\|^2(t')\,\mathrm{d}t' \leqq C\left[N_k^2(0) + N_l^3(t_0)\right]. \tag{3.22}$$

Sum up (3.16), (3.21) and (3.22) for $1 \leqq k \leqq l$. Together with (3.9) we have

$$\left\|(v - v^*, u, e_1 - e_1^*, \ldots, e_m - e_m^*)\right\|_l^2(t)$$
$$+ \int_0^t \left(\|p_x\|_{l-1}^2 + \|u_x\|_{l-1}^2 + \sum_{i=2}^m \|\chi_i\|_l^2\right)(t')\,\mathrm{d}t'$$
$$\leqq C\left[N_l^2(0) + N_l^3(t_0)\right].$$

The definition of $N_l(t)$ in (3.2) then implies

$$N_l^2(t_0) \leqq C\left[N_l^2(0) + N_l^3(t_0)\right].$$

Therefore, there exists a small $\varepsilon > 0$, independent of t_0, such that $N_l(t_0) \leqq \varepsilon$ implies

$$N_l(t_0) \leqq C N_l(0).$$

□

4. Fundamental solution of linearized system

In this section we consider the linearized system of (3.1) around a constant state $(v^*, e_1^*, \ldots, e_m^*)$ that is an equilibrium state. As in the Introduction, the superscript "*" is used to label the thermodynamic variables related to it. The system reads:

$$\begin{aligned}
&p_t + \left(c_f^*\right)^2 u_x = -p_{e_1}^* \sum_{j=2}^m \frac{\chi_j}{\tau_j^*},\\
&u_t + p_x = 0,\\
&(\chi_i)_t + a_i^* u_x = -\frac{\chi_i}{\tau_i^*} - \omega_i'(T_1^*)(T_1)_{e_1}^* \sum_{j=2}^m \frac{\chi_j}{\tau_j^*}, \quad 2 \leqq i \leqq m,\\
&s_t = 0,
\end{aligned} \tag{4.1}$$

where c_f and a_i are defined by (1.13). Notice that the linearized entropy equation is so because of (1.18). The purpose of this section is to obtain the fundamental solution for the Cauchy problem of (4.1). This is crucial to the study of large time behavior for the nonlinear system (1.7). Besides, the fundamental solution itself is important to the theory of partial differential equations. In [11] a general theory of fundamental solutions to hyperbolic balance laws was established under stability and dissipation assumptions. A special case was discussed in an earlier paper, [8]. In this section we apply the general theory in [11] to (4.1), and give an explicit formulation of its fundamental solution.

Observe that in (4.1) the entropy equation is decoupled from the rest, and its fundamental solution for the Cauchy problem is simply the Dirac δ-function. Therefore, we only need to consider the other $m+1$ equations, which are dissipative due to the relaxation. We write these equations as

$$w_t + Aw_x = Bw, \tag{4.2}$$

where

$$\begin{aligned} w &= (p, u, \chi_2, \dots, \chi_m)^t, \\ A &= \begin{pmatrix} 0 & \left(c_f^*\right)^2 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & a_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_m^* & 0 & \cdots & 0 \end{pmatrix}, \qquad (4.3) \\ B &= \begin{pmatrix} 0 & 0 & -\frac{p_{e_1}^*}{\tau_2^*} & \cdots & -\frac{p_{e_1}^*}{\tau_m^*} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\frac{1+\omega_2'(T_1^*)(T_1)_{e_1}^*}{\tau_2^*} & \cdots & -\frac{\omega_2'(T_1^*)(T_1)_{e_1}^*}{\tau_m^*} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & -\frac{\omega_m'(T_1^*)(T_1)_{e_1}^*}{\tau_2^*} & \cdots & -\frac{1+\omega_m'(T_1^*)(T_1)_{e_1}^*}{\tau_m^*} \end{pmatrix}. \end{aligned}$$

The Green's function $G(x, t)$ for the Cauchy problem of (4.2) is the solution matrix satisfying the initial condition

$$G(x, 0) = \delta(x)I, \tag{4.4}$$

where δ is the Dirac δ-function, and I is the $(m+1) \times (m+1)$ identity matrix. The fundamental solution is $G(x-y, t-t')$.

In [11] Green's function has been found for systems in the form (4.2) under the following assumptions:

Assumption 4.1. There exists a symmetric and positive definite matrix A_0 such that A_0A is symmetric, and A_0B is symmetric and semi-negative definite.

Assumption 4.2. Any eigenvector of A is not in the null space of B.

Our first step is to verify these assumptions with A and B given in (4.3). Let

$$A_0 = \begin{pmatrix} A_{01} & A_{02} \\ A_{02}^t & A_{03} \end{pmatrix}, \tag{4.5}$$

where

$$A_{01} = \begin{pmatrix} \left[1 + \sum_{i=2}^m \omega_i'(T_1^*)(T_1)_{e_1}^*\right]/K^* & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{02} = \begin{pmatrix} -\frac{p_{e_1}^*}{K^*} & \cdots & -\frac{p_{e_1}^*}{K^*} \\ 0 & \cdots & 0 \end{pmatrix}_{2\times(m-1)} \tag{4.6}$$

$$A_{03} = \mathrm{diag}\left(\frac{1}{T_1^*\omega_2'(T_1^*)}, \ldots, \frac{1}{T_1^*\omega_m'(T_1^*)}\right) + \frac{(T_1)_{e_1}^*\tilde{p}_v^*}{T_1^*K^*}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}_{(m-1)\times(m-1)}$$

with K and $\tilde{p}_v$ defined in (3.11) and (1.12), respectively. By direct calculation we have

$$A_0A = \mathrm{diag}\left(\tilde{A}, 0_{(m-1)\times(m-1)}\right), \quad \tilde{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$A_0B = \frac{1}{T_1^*}\mathrm{diag}\left(0, 0, -\frac{1}{\omega_2'(T_1^*)\tau_2^*}, \ldots, -\frac{1}{\omega_m'(T_1^*)\tau_m^*}\right),$$

where we have used (1.13) and (2.18). Clearly, A_0A is symmetric, and A_0B is symmetric and semi-negative definite by (1.12). Also, A_0 is symmetric and positive definite since it has Cholesky decomposition, upon which we have (3.13). Thus Assumption 4.1 is verified. From (4.3), A has eigenvalues

$$\lambda_1 = -c_f^*, \quad \lambda_2 = 0, \quad \lambda_3 = c_f^*, \tag{4.7}$$

where λ_1 and λ_3 are simple, while λ_2 has multiplicity $m-1$. That is, $m_1 = m_3 = 1$ and $m_2 = m - 1$, where m_i is the multiplicity of λ_i, $1 \leqq i \leqq 3$. Denote the left eigenvectors of A associated with λ_i as $l_j^{(i)}$, and the corresponding right eigenvectors as $r_j^{(i)}$, $1 \leqq j \leqq m_i$, satisfying

$$l_j^{(i)}A = \lambda_i l_j^{(i)}, \quad Ar_j^{(i)} = \lambda_i r_j^{(i)}, \quad l_j^{(i)}r_{j'}^{(i')} = \delta_{ii'}\delta_{jj'},$$
$$1 \leqq i, i' \leqq 3, \quad 1 \leqq j \leqq m_i, \quad 1 \leqq j' \leqq m_{i'}.$$

We then have

$$l_1^{(1)} = (1, -c_f^*, 0, \ldots, 0), \quad l_1^{(3)} = (1, c_f^*, 0, \ldots, 0), \tag{4.8}$$

$$l_1^{(2)} = \left(-\frac{a_2^*}{\left(c_f^*\right)^2}, 0, 1, 0, \ldots, 0\right), \ldots, l_{m-1}^{(2)} = \left(-\frac{a_m^*}{\left(c_f^*\right)^2}, 0, \ldots, 0, 1\right),$$

and

$$r_1^{(1)} = \frac{1}{2}\left(1, -\frac{1}{c_f^*}, \frac{a_2^*}{\left(c_f^*\right)^2}, \ldots, \frac{a_m^*}{\left(c_f^*\right)^2}\right)^t, \quad r_1^{(3)} = \frac{1}{2}\left(1, \frac{1}{c_f^*}, \frac{a_2^*}{\left(c_f^*\right)^2}, \ldots, \frac{a_m^*}{\left(c_f^*\right)^2}\right)^t,$$

$$r_i^{(2)} = \mathbf{e}_{i+2}^t, \quad 1 \leqq i \leqq m-1, \tag{4.9}$$

where $\{\mathbf{e}_i\}_{i=1}^{m+1}$ is the standard basis of the row vector space $\mathbb{R}^{m+1}$. From (4.9), (1.13) and (1.12) it is clear that none of the $r_j^{(i)}$ is in the null space of B. Therefore, Assumption 4.2 is also satisfied.

We are now ready to apply Theorem 3.6 in [11] to (4.2), (4.3): For $x \in \mathbb{R}, t \geqq 0$, the Green's function G of (4.2), (4.3) has the property

$$\begin{aligned}\frac{\partial^l}{\partial x^l}G(x,t) = \frac{\partial^l}{\partial x^l}&\left[\sum_{j=1}^{m'} \frac{1}{\sqrt{4\pi\mu_j^{(r)}t}}\exp\left(-\frac{(x-\lambda_j^{(r)}t)^2}{4\mu_j^{(r)}t}\right)P_j^{(r)}\right]\\ &+O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}\sum_{j=1}^{m'}\exp\left(-\frac{(x-\lambda_j^{(r)}t)^2}{Ct}\right)\\ &+\sum_{j=1}^{3}\sum_{k=1}^{m_j}e^{-\mu_{jk}t}\sum_{i=0}^{l}\delta^{(l-i)}(x-\lambda_j t)P_{jk}^{(i)}(t),\end{aligned} \tag{4.10}$$

where $l \geqq 0$ is any integer; the constants m', $\lambda_j^{(r)}$ and $\mu_j^{(r)}$ and the constant projections $P_j^{(r)}$ are determined from A and B via Procedure 3.2 in [11]; $C > 0$ is a constant; $m_1 = m_3 = 1$ and $m_2 = m - 1$ are the multiplicities of λ_j, $1 \leqq j \leqq 3$, given in (4.7);

$$\mu_{j1} = -l_1^{(j)}Br_1^{(j)} = \frac{1}{2(c_f^*)^2}\sum_{i=2}^{m}\frac{p_{e_1}^* a_i^*}{\tau_i^*} = \frac{(p_{e_1}^*)^2 T_1^*}{2(c_f^*)^2}\sum_{i=2}^{m}\frac{\omega_i'(T_1^*)}{\tau_i^*} \equiv \bar{\mu}, \quad j=1,3 \tag{4.11}$$

by (4.8), (4.9), (4.3) and (1.13); $\mu_{2k} > 0$, $1 \leqq k \leqq m-1$, are the eigenvalues of

$$\mu_2 = -l^{(2)}Br^{(2)}, \tag{4.12}$$

where

$$l^{(2)} = \begin{pmatrix} l_1^{(2)} \\ \vdots \\ l_{m-1}^{(2)} \end{pmatrix}, \quad r^{(2)} = \left(r_1^{(2)}, \dots, r_{m-1}^{(2)}\right); \tag{4.13}$$

and $P_{jk}^{(i)}(t)$ are $(m+1)\times(m+1)$ polynomial matrices in t with degrees not more than i. In particular, $P_{j1}^{(0)} = r_1^{(j)}l_1^{(j)}$, $j = 1, 3$, while $P_{2k}^{(0)}$ are subprojections of $r^{(2)}l^{(2)}$.

To obtain the explicit form of G, we follow the reduction process, Procedure 3.2 in [11], to compute the leading term in (4.10). From (4.3) B has the eigenvalue zero with multiplicity 2. Let

$$R^0 = \begin{pmatrix} I_{2\times 2} \\ 0_{(m-1)\times 2}\end{pmatrix}, \quad L^0 = \begin{pmatrix} 1 & 0 & -\frac{p_{e_1}^*}{1+\sum_{i=2}^m \omega_i'(T_1^*)(T_1)_{e_1}^*} & \dots & -\frac{p_{e_1}^*}{1+\sum_{i=2}^m \omega_i'(T_1^*)(T_1)_{e_1}^*} \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix},$$

$$Q_0 = R^0L^0 = \begin{pmatrix} L^0 \\ 0_{(m-1)\times(m+1)}\end{pmatrix}. \tag{4.14}$$

Here the columns of R^0 are the right eigenvectors of B corresponding to the eigenvalue zero, and the rows of L^0 are the left eigenvectors. Clearly, $L^0 R^0 = I_{2\times 2}$, and Q_0 is the eigenprojection of B associated with the eigenvalue zero.

The directions of heat kernels in (4.10), $\lambda_j^{(r)}$, are the distinct eigenvalues of

$$L^0 A R^0 = \begin{pmatrix} 0 & (c^*)^2 \\ 1 & 0 \end{pmatrix}, \tag{4.15}$$

where we have used (2.20) and (1.13). That is,

$$\lambda_1^{(r)} = -c^*, \quad \lambda_2^{(r)} = c^*. \tag{4.16}$$

The fact that both $\lambda_1^{(r)}$ and $\lambda_2^{(r)}$ are simple implies that

$$m' = 2, \tag{4.17}$$

$P_j^{(r)}$ is the eigenprojection of $Q_0 A Q_0$ corresponding to $\lambda_j^{(r)}$ when restricted to the range of Q_0, and

$$\mu_j^{(r)} = \mathrm{tr}\left(-ASAP_j^{(r)}\right), \quad j = 1, 2, \tag{4.18}$$

where S is the value at zero of the reduced resolvent of B with respect to the eigenvalue zero.

To compute $P_j^{(r)}$ we write (4.15) in spectral decomposition:

$$L^0 A R^0 = -c^* \begin{pmatrix} 1/2 \\ -1/(2c^*) \end{pmatrix} \begin{pmatrix} 1 & -c^* \end{pmatrix} + c^* \begin{pmatrix} 1/2 \\ 1/(2c^*) \end{pmatrix} \begin{pmatrix} 1 & c^* \end{pmatrix},$$

which implies

$$P_j^{(r)} = R^0 \begin{pmatrix} 1/2 \\ 1/(2\lambda_j^{(r)}) \end{pmatrix} \begin{pmatrix} 1 & \lambda_j^{(r)} \end{pmatrix} L^0, \quad j = 1, 2.$$

Using (4.14) we have

$$P_j^{(r)} = \begin{pmatrix} 1/2 & \lambda_j^{(r)}/2 & \tilde{l} \\ 1/(2\lambda_j^{(r)}) & 1/2 & \tilde{l}/(\lambda_j^{(r)}) \\ 0_{(m-1)\times 1} & 0_{(m-1)\times 1} & 0_{(m-1)\times(m-1)} \end{pmatrix}, \quad j = 1, 2, \tag{4.19}$$

$$\tilde{l} = -\frac{p_{e_1}^*}{2[1 + \sum_{i=2}^m \omega_i'(T_1^*)(T_1)_{e_1}^*]} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times(m-1)}.$$

Let $\nu_1, \ldots, \nu_\rho$ be all the nonzero (hence negative) eigenvalues of B. Let the corresponding eigenprojections be $Q_1, \ldots, Q_\rho$. Then S in (4.18) is

$$S = \sum_{j=1}^{\rho} \frac{1}{\nu_j} Q_j.$$

This implies $Q_0 + S = (Q_0 + B)^{-1}$, with Q_0 defined in (4.14). Therefore,

$$S = (Q_0 + B)^{-1} - Q_0. \tag{4.20}$$

Substituting (4.20) into (4.18), we have for $j = 1, 2$,

$$\begin{aligned}\mu_j^{(r)} &= -\mathrm{tr}\left(A(Q_0 + B)^{-1} A P_j^{(r)}\right) + \mathrm{tr}\left(A Q_0 A P_j^{(r)}\right)\\ &= -\mathrm{tr}\left((Q_0 + B)^{-1} A P_j^{(r)} A\right) + \mathrm{tr}\left(Q_0 A P_j^{(r)} A\right).\end{aligned} \tag{4.21}$$

From (4.14) and (4.3) we have,

$$\begin{aligned}Q_0 + B &= \begin{pmatrix} I_{2\times 2} & B_1 \\ 0_{(m-1)\times 2} & B_2 \end{pmatrix},\\ B_1 &= -p_{e_1}^* \begin{pmatrix} \frac{1}{\tau_2^*} + \beta_1 & \cdots & \frac{1}{\tau_m^*} + \beta_1 \\ 0 & \cdots & 0 \end{pmatrix},\\ \beta_1 &= \left[1 + \sum_{i=2}^{m} \omega_i'(T_1^*)(T_1)_{e_1}^*\right]^{-1},\\ B_2 &= -\mathrm{diag}\left(\frac{1}{\tau_2^*}, \ldots, \frac{1}{\tau_m^*}\right) - (T_1)_{e_1}^* \begin{pmatrix} \omega_2'(T_1^*) \\ \vdots \\ \omega_m'(T_1^*) \end{pmatrix} \begin{pmatrix} \frac{1}{\tau_2^*} & \cdots & \frac{1}{\tau_m^*} \end{pmatrix}.\end{aligned} \tag{4.22}$$

We can verify that

$$\begin{aligned}(Q_0 + B)^{-1} &= \begin{pmatrix} I_{2\times 2} & B_3 \\ 0_{(m-1)\times 2} & B_2^{-1} \end{pmatrix},\\ B_3 &= p_{e_1}^* \beta_1 \begin{pmatrix} -(1 + \tau_2^*) + \beta_1\beta_2 & \cdots & -(1 + \tau_m^*) + \beta_1\beta_2 \\ 0 & \cdots & 0 \end{pmatrix},\\ \beta_2 &= \sum_{i=2}^{m} \tau_i^* \omega_i'(T_1^*)(T_1)_{e_1}^*,\\ B_2^{-1} &= -\mathrm{diag}\left(\tau_2^*, \ldots, \tau_m^*\right) + \beta_1 (T_1)_{e_1}^* \begin{pmatrix} \tau_2^*\omega_2'(T_1^*) \\ \vdots \\ \tau_m^*\omega_m'(T_1^*) \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}_{1\times(m-1)}.\end{aligned} \tag{4.23}$$

From (1.13), (2.20), (4.3), (4.16) and (4.19) we also have for $j = 1, 2$,

$$\begin{aligned}A P_j^{(r)} A &= \begin{pmatrix} \frac{1}{2}(c_f^*)^2 & \frac{1}{2}(c_f^*)^2 \lambda_j^{(r)} & 0_{1\times(m-1)} \\ \frac{1}{2}\lambda_j^{(r)} & \frac{1}{2}\left(\lambda_j^{(r)}\right)^2 & 0_{1\times(m-1)} \\ \tilde{r} & \tilde{r}\lambda_j^{(r)} & 0_{(m-1)\times(m-1)} \end{pmatrix},\\ \tilde{r} &= \frac{1}{2}\left(a_2^*, \ldots, a_m^*\right)^t.\end{aligned} \tag{4.24}$$

Substituting (4.23), (4.24) and (4.14) into (4.21) gives us

$$\mu_j^{(r)} = \sum_{i=2}^{m} \frac{1}{2} a_i^* p_{e_1}^* \beta_1 (\tau_i^* - \beta_1 \beta_2), \quad j = 1, 2. \tag{4.25}$$

Using the definitions of β_1 and β_2 in (4.22) and (4.23), (4.25) can be simplified as

$$\mu_j^{(r)} = \frac{p_{e_1}^* \sum_{i=2}^m a_i^* \tau_i^*}{2\left[1 + \sum_{i=2}^m \omega_i'(T_1^*)(T_1)_{e_1}^*\right]^2} = \frac{1}{2}\mu^*, \quad j = 1, 2, \tag{4.26}$$

where μ is exactly the one defined in (2.34).

We summarize the discussion in this section as the following theorem.

Theorem 4.1. *Let* (1.12) *hold, and* v^*, $e_1^*, \ldots, e_m^*$ *be positive constants such that* (1.18) *is satisfied. Let* $l \geqq 0$ *be any integer. For* $x \in \mathbb{R}$, $t \geqq 0$, *the Green's function* G *of* (4.2), (4.3), *which are* (4.1) *without the entropy equation, has the following property:*

$$\begin{aligned}
&\frac{\partial^l}{\partial x^l} G(x,t) \\
&= \frac{\partial^l}{\partial x^l}\left\{\frac{1}{\sqrt{2\pi\mu^* t}}\left[\exp\left(-\frac{(x+c^*t)^2}{2\mu^* t}\right)P_1^{(r)} + \exp\left(-\frac{(x-c^*t)^2}{2\mu^* t}\right)P_2^{(r)}\right]\right\} \\
&\quad + O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{l+1}{2}}\left[\exp\left(-\frac{(x+c^*t)^2}{Ct}\right) + \exp\left(-\frac{(x-c^*t)^2}{Ct}\right)\right] \\
&\quad + \sum_{i=0}^{l}\left\{e^{-\bar{\mu}t}\left[\delta^{(l-i)}(x+c_f^* t)P_1^{(i)}(t) + \delta^{(l-i)}(x-c_f^* t)P_3^{(i)}(t)\right]\right. \\
&\quad \left. + \sum_{j=1}^{m-1} e^{-\mu_{2j}t}\delta^{(l-i)}(x)P_{2j}^{(i)}(t)\right\}.
\end{aligned} \tag{4.27}$$

Here the positive constants μ^* *and* $\bar{\mu}$ *are given by* (4.26) *and* (4.11), *respectively, and* μ_{2j}, $1 \leqq j \leqq m-1$, *are the eigenvalues of* μ_2 *defined by* (4.8), (4.9) (4.12) *and* (4.13); $C > 0$ *is a constant; the constant projections* $P_j^{(r)}$, $j = 1, 2$, *are given by* (4.16) *and* (4.19); $P_1^{(i)}$, $P_3^{(i)}$ *and* $P_{2j}^{(i)}$, $1 \leqq j \leqq m-1$, *are* $(m+1) \times (m+1)$ *polynomial matrices in* t *with degrees not more than* i. *In particular,* $P_j^{(0)} = r_1^{(j)} l_1^{(j)}$, $j = 1, 3$, *while* $P_{2j}^{(0)}$, $1 \leqq j \leqq m-1$, *are subprojections of* $r^{(2)} l^{(2)}$; *see* (4.8), (4.9) *and* (4.13). *We recall that* c *is the equilibrium speed of sound, given by* (1.14) *or* (2.17).

Recall that $\{\mathbf{e}_i\}_{i=1}^{m+1}$ is the standard basis of the row vector space $\mathbb{R}^{m+1}$. From (4.14) we have $\mathbf{e}_k Q_0 = 0$, $3 \leqq k \leqq m+1$. Applying Theorem 3.9 in [11] we obtain the following refinement of G_k, the kth row of G, $3 \leqq k \leqq m+1$.

Theorem 4.2. *Under the same assumptions and same notations as in Theorem 4.1, for $3 \leqq k \leqq m+1$, the kth row of G, denoted as G_k, has the following estimate:*

$$\begin{aligned}
&\frac{\partial^l}{\partial x^l} G_k(x,t) \\
&= \frac{\partial^{l+1}}{\partial x^{l+1}} \left\{ \frac{1}{\sqrt{2\pi\mu^* t}} \left[\exp\left(-\frac{(x+c^*t)^2}{2\mu^* t} \right) \eta_{k1} + \exp\left(-\frac{(x-c^*t)^2}{2\mu^* t} \right) \eta_{k2} \right] \right\} \\
&\quad + O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{l}{2}-1} \left[\exp\left(-\frac{(x+c^*t)^2}{Ct} \right) + \exp\left(-\frac{(x-c^*t)^2}{Ct} \right) \right] \\
&\quad + \sum_{i=0}^{l} \left\{ e^{-\bar{\mu} t} \left[\delta^{(l-i)}(x+c_f^* t) \left(P_1^{(i)} \right)_k (t) + \delta^{(l-i)}(x-c_f^* t) \left(P_3^{(i)} \right)_k (t) \right] \right. \\
&\quad \left. + \sum_{j=1}^{m-1} e^{-\mu_{2j} t} \delta^{(l-i)}(x) \left(P_{2j}^{(i)} \right)_k (t) \right\}, \qquad (4.28)
\end{aligned}$$

where η_{k1} and η_{k2} are constant row vectors in $\mathbb{R}^{m+1}$, and $(P_1^{(i)})_k$ is the kth row of $P_1^{(i)}$, etc.

5. Large time behavior

As the final section we now prove Theorem 1.2. This is to combine Theorems 4.1 and 4.2, Duhamel's principle, evolution of elementary waves, and weighted energy estimates together. Since the fundamental solution described in Theorems 4.1 and 4.2 has the same structure as the one for one nonequilibrium mode, [11], the focus of this section is on the weighted energy estimate.

Notice that the superscript "*" denotes the constant equilibrium state. We write the system (3.1) without the entropy equation as

$$\tilde{w}_t + A\tilde{w}_x = B\tilde{w} + \tilde{g}, \qquad (5.1)$$

where

$$\tilde{w} = (p - p^*, u, \chi_2, \ldots, \chi_m)^t, \qquad (5.2)$$

$$\begin{aligned}
\tilde{g} &= (\tilde{g}_1, \ldots, \tilde{g}_{m+1})^t \qquad (5.3)\\
&= \begin{pmatrix}
\left[(c_f^*)^2 - c_f^2 \right] u_x + \sum_{j=2}^m \left(\frac{p_{e_1}^*}{\tau_j^*} - \frac{p_{e_1}}{\tau_j} \right) \chi_j \\
0 \\
(a_2^* - a_2) u_x + \left(\frac{1}{\tau_2^*} - \frac{1}{\tau_2} \right) \chi_2 + \sum_{j=2}^m \left[\frac{\omega_2'(T_1^*)(T_1)_{e_1}^*}{\tau_j^*} - \frac{\omega_2'(T_1)(T_1)_{e_1}}{\tau_j} \right] \chi_j \\
\vdots \\
(a_m^* - a_m) u_x + \left(\frac{1}{\tau_m^*} - \frac{1}{\tau_m} \right) \chi_m + \sum_{j=2}^m \left[\frac{\omega_m'(T_1^*)(T_1)_{e_1}^*}{\tau_j^*} - \frac{\omega_m'(T_1)(T_1)_{e_1}}{\tau_j} \right] \chi_j
\end{pmatrix},
\end{aligned}$$

and A and B are the same as in (4.3). Here the linear part of (5.1) has the Green's function given in Theorems 4.1 and 4.2. Introduce a linear transformation to diagonalize $P_1^{(r)}$ and $P_2^{(r)}$ in (4.27):

$$w = (w_1, \dots, w_{m+1})^t = L^{(r)}\tilde{w}, \quad \tilde{w} = R^{(r)}w, \tag{5.4}$$

where

$$L^{(r)} = \begin{pmatrix} 1 & -c^* & 2\tilde{l} \\ 1 & c^* & 2\tilde{l} \\ 0_{(m-1)\times 1} & 0_{(m-1)\times 1} & I_{(m-1)\times(m-1)} \end{pmatrix}, \tag{5.5}$$
$$R^{(r)} = \left[L^{(r)}\right]^{-1} = \begin{pmatrix} 1/2 & 1/2 & -2\tilde{l} \\ -1/(2c^*) & 1/(2c^*) & 0_{1\times(m-1)} \\ 0_{(m-1)\times 1} & 0_{(m-1)\times 1} & I_{(m-1)\times(m-1)} \end{pmatrix},$$

and $\tilde{l}$ is the same as given in (4.19). Under such a linear transformation (5.1)–(5.3) becomes

$$w_t + L^{(r)}AR^{(r)}w_x = L^{(r)}BR^{(r)}w + g, \tag{5.6}$$

where

$$g = (g_1, \dots, g_{m+1})^t = L^{(r)}\tilde{g}. \tag{5.7}$$

Denote the Green's function of the linear part of (5.6) as G. Then G is the one given in Theorems 4.1 and 4.2, multiplied by $L^{(r)}$ from the left and $R^{(r)}$ from the right. That is, for $x \in \mathbb{R}$ and $t \geqq 0$, we have

$$G(x,t) = D(x,t) + H(x,t), \tag{5.8}$$

where for $l \geqq 0$,

$$\begin{aligned}
&\partial_x^l D(x,t) = \partial_x^l \left[\frac{1}{\sqrt{2\pi\mu^* t}} \sum_{i=1}^{2} \exp\left(-\frac{(x-c_i t)^2}{2\mu^* t}\right) P_i^{(r)}\right], \\
&\partial_x^l H(x,t) = O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{l+1}{2}} \sum_{i=1}^{2} \exp\left(-\frac{(x-c_i t)^2}{Ct}\right) \\
&\qquad + \sum_{j=0}^{l} \left[\sum_{i=1,3} e^{-\bar{\mu}t} \delta^{(l-j)}(x-d_i t) P_i^{(j)}(t) + \sum_{i=1}^{m-1} e^{-\mu_{2i}t} \delta^{(l-j)}(x) P_{2i}^{(j)}(t)\right], \\
&c_{1,2} = \mp c^*, \quad d_{1,3} = \mp c_f^*, \\
&P_i^{(r)} = \mathbf{e}_i^t \mathbf{e}_i, \quad i = 1, 2,
\end{aligned} \tag{5.9}$$

$\{\mathbf{e}_i\}_{i=1}^{m+1}$ is the standard basis in the row vector space $\mathbb{R}^{m+1}$, and μ^*, $\bar{\mu}$, μ_{2i} ($1 \leqq i \leqq m-1$), and C are the same as in Theorem 4.1, while $P_i^{(j)}$ ($i = 1, 3$) and $P_{2i}^{(j)}$

$(1 \leqq i \leqq m-1)$ are $(m+1)\times(m+1)$ polynomial matrices in t with degrees not more than j. Also, for $3 \leqq k \leqq m+1$, the kth row of G is

$$G_k(x,t) = H_k(x,t) = H_{ka}(x,t) + H_{kb}(x,t), \tag{5.10}$$

where H_k is the kth row of H, and so on,

$$\begin{aligned}
\partial_x^l H_{ka}(x,t) &= \partial_x^{l+1}\left[\frac{1}{\sqrt{2\pi\mu^* t}}\sum_{i=1}^{2}\exp\left(-\frac{(x-c_i t)^2}{2\mu^* t}\right)\eta_{ki}\right],\\
\partial_x^l H_{kb}(x,t) &= O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l}{2}-1}\sum_{i=1}^{2}\exp\left(-\frac{(x-c_i t)^2}{Ct}\right)\\
&\quad+\sum_{j=0}^{l}\left[\sum_{i=1,3} e^{-\bar{\mu}t}\delta^{(l-j)}(x-d_i t)\left(P_i^{(j)}\right)_k(t)\right.\\
&\quad\left.+\sum_{i=1}^{m-1} e^{-\mu_{2i}t}\delta^{(l-j)}(x)\left(P_{2i}^{(j)}\right)_k(t)\right],
\end{aligned} \tag{5.11}$$

and η_{ki} are constant row vectors in $\mathbb{R}^{m+1}$, $i = 1, 2$, and $3 \leqq k \leqq m+1$.

Let

$$w_{m+2} = s - s^*, \tag{5.12}$$

and

$$g_{m+2} = \sum_{i=2}^{m}\left(\frac{1}{T_i} - \frac{1}{T_1}\right)\frac{\chi_i}{\tau_i}. \tag{5.13}$$

By Duhamel's principle, (5.6) and the last equation of (3.1), we have for $l \geqq 0$,

$$\begin{aligned}
\partial_x^l w_i(x,t) &= \int_{-\infty}^{\infty} G_i(x-y,t)\partial_y^l w(y,0)\,\mathrm{d}y \\
&\quad+\int_0^t\int_{-\infty}^{\infty} G_i(x-y,t-t')\partial_y^l g(y,t')\,\mathrm{d}y\mathrm{d}t', \quad 1 \leqq i \leqq m+1,\\
\partial_x^l w_{m+2}(x,t) &= \partial_x^l w_{m+2}(x,0) + \int_0^t \partial_x^l g_{m+2}(x,t')\,\mathrm{d}t'.
\end{aligned} \tag{5.14}$$

Introduce the following notations: Let ν be any fixed constant such that

$$\nu > \max\{2\mu^*, C\}, \tag{5.15}$$

where μ^* and C are the same as in (5.9) and (5.11). Define

$$\begin{aligned}
\tilde{\psi}_i(x,t) &\equiv \left[|x-c_i(t+1)|^3 + (t+1)^2\right]^{-\frac{1}{2}}, \quad i = 1, 2, \\
\phi_i(x,t) &\equiv (t+1)^{-\frac{1}{2}}\exp\left(-\frac{(x-c_i(t+1))^2}{\nu(t+1)}\right) + [(x-c_i(t+1))^2 + t + 1]^{-\frac{3}{4}}\\
&\quad+\tilde{\psi}_j(x,t), \quad i,j = 1,2, \quad \text{and} \quad j \neq i,\\
\phi_0(x) &\equiv (x^2+1)^{-\frac{3}{4}},
\end{aligned} \tag{5.16}$$

where c_i, $i = 1, 2$, are given in (5.9). Also let

$$M(t) \equiv \sup_{0 \leqq t' \leqq t} \max_{i=1,2} \left\{ \|(w_i \phi_i^{-1})(\cdot, t')\|_{L^\infty} + \|(w_{ix} \phi_i^{-1})(\cdot, t')\|_{L^\infty} (t'+1)^{\frac{1}{2}} + \sum_{l=2}^{4} \|\partial_x^l w_i(\cdot, t')\|_{L^\infty} (t'+1)^{\frac{5-l}{2}} \right\}$$
$$+ \sup_{0 \leqq t' \leqq t} \max_{3 \leqq i \leqq m+1} \left\{ \left\| w_i(\cdot, t') \left(\sum_{j=1}^{2} \phi_i(\cdot, t') \right)^{-1} \right\|_{L^\infty} (t'+1)^{\frac{1}{2}} + \left\| w_{ix}(\cdot, t') \left(\sum_{j=1}^{2} \tilde{\psi}_j(\cdot, t') \right)^{-1} \right\|_{L^\infty} (t'+1)^{\frac{1}{2}} + \sum_{l=2}^{4} \left\| \partial_x^l w_i(\cdot, t') \right\|_{L^\infty} (t'+1)^{\frac{5-l}{2}} \right\}$$
$$+ \sup_{0 \leqq t' \leqq t} \left\{ \left\| (\phi_0^{-1} w_{m+2})(\cdot, t') \right\|_{L^\infty} + \left\| (\phi_0^{-1} \partial_x w_{m+2})(\cdot, t') \right\|_{L^\infty} + \left\| (\phi_0^{-\frac{1}{3}} \partial_x^2 w_{m+2})(\cdot, t') \right\|_{L^\infty} \right\}. \tag{5.17}$$

Then for $x \in \mathbb{R}$ and $t \geqq 0$, we have the following:

(i) For $i = 1, 2$,

$$|w_i(x,t)| \leqq M(t)\phi_i(x,t), \ |w_{ix}(x,t)| \leqq M(t)(t+1)^{-\frac{1}{2}}\phi_i(x,t), \quad |\partial_x^l w_i(x,t)| \leqq M(t)(t+1)^{-\frac{5-l}{2}}, \quad 2 \leqq l \leqq 4. \tag{5.18}$$

(ii) For $3 \leqq i \leqq m+1$,

$$|w_i(x,t)| \leqq M(t)(t+1)^{-\frac{1}{2}} \sum_{j=1}^{2} \phi_j(x,t),$$
$$|w_{ix}(x,t)| \leqq M(t)(t+1)^{-\frac{1}{2}} \sum_{j=1}^{2} \tilde{\psi}_j(x,t), \tag{5.19}$$
$$\left| \partial_x^l w_i(x,t) \right| \leqq M(t)(t+1)^{-\frac{5-l}{2}}, \quad 2 \leqq l \leqq 4.$$

(iii)

$$|w_{m+2}(x,t)| \leqq M(t)\phi_0(x), \quad |\partial_x w_{m+2}(x,t)| \leqq M(t)\phi_0(x), \quad |\partial_x^2 w_{m+2}(x,t)| \leqq M(t)\phi_0^{\frac{1}{3}}(x). \tag{5.20}$$

We now perform the weighted energy estimate. This is to make use of the fact that w_i consists of waves along the equilibrium acoustic directions to obtain needed decay rates in a neighborhood of the particle path. For $l \geqq 1$, $t \geqq 0$ and $\varepsilon > 0$ we define

$$\tilde{N}_l^2(t;\varepsilon) \equiv \int_{-\varepsilon t}^{\varepsilon t} (\varepsilon^2 x^2 + 1)^{-\frac{3}{2}} \left[\left(\partial_x^l u\right)^2 + \left(\partial_x^l p\right)^2 + \sum_{i=2}^m \left(\partial_x^l \chi_i\right)^2 \right](x,t)\,\mathrm{d}x$$
$$+ \int_{t/2}^t \int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2 x^2+1)^{-\frac{3}{2}} \left\{ \sum_{j=1}^l \left[\left(\partial_x^j u\right)^2 + \left(\partial_x^j p\right)^2 \right] \right.$$
$$\left. + \sum_{i=2}^m \sum_{j=0}^l \left(\partial_x^j \chi_i\right)^2 \right\}(x,t')\,\mathrm{d}x\mathrm{d}t'. \qquad (5.21)$$

Lemma 5.1. *Suppose $2 \leqq k \leqq 5$ and $\varepsilon > 0$ is small. Under the assumptions of Theorem 1.2, for $t \geqq 0$ we have the following recursive relation:*

$$\tilde{N}_k^2(t;\varepsilon) = O(1)\left[M(t)^2 + \varepsilon_0^2\right](t+1)^{-5+k}\varepsilon^{-1}(\varepsilon^4 t^2+1)^{-\frac{3}{2}}$$
$$+ O(1)\varepsilon_0^2 e^{-\varepsilon t/2} + O(1)\tilde{N}_{k-1}^2(t;\varepsilon), \qquad (5.22)$$

where $M(t)$ is defined in (5.17) and ε_0 in (1.21).

Proof. Similar to the proof of Proposition 3.1 and under the same notations, we multiply the first equation in (3.10) by $\eta_1 \partial_x^k p - \eta_2 \sum_{j=2}^m \partial_x^k \chi_j$, the second one by $\partial_x^k u$, and the one for χ_i by $-\eta_2 \partial_x^k p + \zeta_i \partial_x^k \chi_i - \eta_3 \sum_{j=2}^m \partial_x^k \chi_j$. Sum up the results with $i = 2, \ldots, m$. We have (3.12) without the term $\frac{1}{2}\partial_t(\partial_x^k s)^2$ on the left-hand side and the last term on the right-hand side. Use the corresponding terms of (3.13) to replace the left-hand side. Multiply the equation by the weight $e^{-\varepsilon(t-t')}(\varepsilon^2 x^2+1)^{-\frac{3}{2}}$, and integrate it over $[-\varepsilon t, \varepsilon t] \times [t/2, t]$. After integration by parts, applying (1.20) with $l = 6$, together with (1.21), (5.2), (5.4), (5.5), (5.18) and (5.19), and using (1.7) and (3.1) to convert the derivatives with respect to t into derivatives with respect to x, we arrive at

$$\int_{-\varepsilon t}^{\varepsilon t} (\varepsilon^2 x^2+1)^{-\frac{3}{2}} \left\{ \eta_1 \left(\partial_x^k p - \frac{\eta_2}{\eta_1} \sum_{i=2}^m \partial_x^k \chi_i \right)^2 \right.$$
$$\left. + \sum_{i=2}^m \zeta_i \frac{\xi_{i+1}}{\xi_i} \left[\partial_x^k \chi_i - \frac{(T_1)_{e_1}\omega_i'(T_1)}{\xi_{i+1}} \sum_{j=i+1}^m \partial_x^k \chi_j \right]^2 + \left(\partial_x^k u\right)^2 \right\}(x,t)\,\mathrm{d}x$$
$$+ \int_{t/2}^t \int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2 x^2+1)^{-\frac{3}{2}} \sum_{i=2}^m \left(\partial_x^k \chi_i\right)^2 (x,t')\,\mathrm{d}x\mathrm{d}t'$$
$$= O(1)e^{-\varepsilon t/2}\varepsilon_0^2 + O(1)\varepsilon^{-1}(\varepsilon^4 t^2+1)^{-\frac{3}{2}}\left[M(t)^2+\varepsilon_0^2\right](t+1)^{-5+k}$$

$$+ O(1)(\varepsilon+\varepsilon_0)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}$$
$$\times\left[\left(\partial_x^k p\right)^2+\left(\partial_x^k u\right)^2\right](x,t')\,\mathrm{d}x\mathrm{d}t'+O(1)\varepsilon_0\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}$$
$$\times\left[\sum_{j=1}^{k-1}\left(\partial_x^j u\right)^2+\sum_{i=2}^{m}\sum_{j=0}^{k-1}\left(\partial_x^j\chi_i\right)^2\right](x,t')\,\mathrm{d}x\mathrm{d}t'. \tag{5.23}$$

The left-hand side of (5.23) can be replaced by

$$\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^k p\right)^2+\left(\partial_x^k u\right)^2+\sum_{i=2}^{m}\left(\partial_x^k\chi_i\right)^2\right](x,t)\,\mathrm{d}x$$
$$+\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\sum_{i=2}^{m}\left(\partial_x^k\chi_i\right)^2(x,t')\,\mathrm{d}x\mathrm{d}t'. \tag{5.24}$$

Similarly, multiply (3.17) and (3.20) by the weight and integrate. We have

$$\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left(\partial_x^k u\right)^2(x,t')\,\mathrm{d}x\mathrm{d}t'$$
$$=O(1)\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^k u\right)^2+\left(\partial_x^{k-1}\chi_2\right)^2\right](x,t)\,\mathrm{d}x+O(1)\varepsilon_0^2e^{-\varepsilon t/2}$$
$$+O(1)\varepsilon^{-1}(\varepsilon^4t^2+1)^{-\frac{3}{2}}\left[M(t)^2+\varepsilon_0^2\right](t+1)^{-5+k}$$
$$+O(1)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\sum_{i=2}^{m}\left(\partial_x^{k-1}\chi_i\right)^2(x,t')\,\mathrm{d}x\mathrm{d}t'$$
$$+O(1)(\varepsilon+\varepsilon_0)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}$$
$$\times\left[\left(\partial_x^k p\right)^2+\sum_{j=1}^{k-1}\left(\partial_x^j u\right)^2+\sum_{i=2}^{m}\sum_{j=0}^{k-2}\left(\partial_x^j\chi_i\right)^2\right](x,t')\,\mathrm{d}x\mathrm{d}t'$$
$$+O(1)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left|\partial_x^k p\,\partial_x^k\chi_2\right|(x,t')\,\mathrm{d}x\mathrm{d}t' \tag{5.25}$$

and

$$\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left(\partial_x^k p\right)^2(x,t')\,\mathrm{d}x\mathrm{d}t'$$
$$=O(1)\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^k p\right)^2+\left(\partial_x^{k-1}u\right)^2\right](x,t)\,\mathrm{d}x+O(1)\varepsilon_0^2e^{-\varepsilon t/2}$$
$$+O(1)\varepsilon^{-1}(\varepsilon^4t^2+1)^{-\frac{3}{2}}\left[M(t)^2+\varepsilon_0^2\right](t+1)^{-\frac{11}{2}+k}$$
$$+O(1)(\varepsilon+\varepsilon_0)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\sum_{j=1}^{k-1}\left(\partial_x^j u\right)^2\right.$$

$$+\sum_{i=2}^{m}\sum_{j=0}^{k-2}\left(\partial_x^j\chi_i\right)^2\Bigg](x,t')\,\mathrm{d}x\mathrm{d}t' + O(1)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}$$
$$\times\left[\left(\partial_x^k u\right)^2+\sum_{i=2}^{m}\left(\partial_x^{k-1}\chi_i\right)^2\right](x,t')\,\mathrm{d}x\mathrm{d}t'. \tag{5.26}$$

Substitute (5.25) into (5.26) and simplify. We then have

$$\begin{aligned}
&\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left(\partial_x^k p\right)^2(x,t')\,\mathrm{d}x\mathrm{d}t'\\
&= O(1)\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^k p\right)^2+\left(\partial_x^k u\right)^2\right](x,t)\,dx + O(1)\varepsilon_0^2e^{-\varepsilon t/2}\\
&\quad + O(1)\varepsilon^{-1}(\varepsilon^4t^2+1)^{-\frac{3}{2}}\left[M(t)^2+\varepsilon_0^2\right](t+1)^{-5+k}+O(1)\tilde{N}_{k-1}^2(t;\varepsilon)\\
&\quad + O(1)\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left(\partial_x^k\chi_2\right)^2(x,t')\,\mathrm{d}x\mathrm{d}t'.
\end{aligned}\tag{5.27}$$

Substitute (5.27) into (5.25). The right-hand side of (5.25) can now be replaced by the right-hand side of (5.27). Such a result and (5.27) further simplify the right-hand side of (5.23). Together with (5.24) we have

$$\begin{aligned}
&\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^k p\right)^2+\left(\partial_x^k u\right)^2+\sum_{i=2}^{m}\left(\partial_x^k\chi_i\right)^2\right](x,t)\,\mathrm{d}x\\
&\qquad+\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\sum_{i=2}^{m}\left(\partial_x^k\chi_i\right)^2(x,t')\,\mathrm{d}x\mathrm{d}t'\\
&\quad= O(1)\varepsilon_0^2e^{-\varepsilon t/2}+O(1)\varepsilon^{-1}(\varepsilon^4t^2+1)^{-\frac{3}{2}}\left[M(t)^2+\varepsilon_0^2\right](t+1)^{-5+k}\\
&\qquad+O(1)(\varepsilon+\varepsilon_0)\tilde{N}_{k-1}^2(t;\varepsilon).
\end{aligned}\tag{5.28}$$

Combine (5.28) with (5.27) and (5.25), whose right-hand side is now the same as that of (5.27). We obtain

$$\begin{aligned}
&\int_{-\varepsilon t}^{\varepsilon t}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^k p\right)^2+\left(\partial_x^k u\right)^2+\sum_{i=2}^{m}\left(\partial_x^k\chi_i\right)^2\right](x,t)\,\mathrm{d}x\\
&\quad+\int_{t/2}^{t}\int_{-\varepsilon t}^{\varepsilon t} e^{-\varepsilon(t-t')}(\varepsilon^2x^2+1)^{-\frac{3}{2}}\left[\left(\partial_x^k p\right)^2+\left(\partial_x^k u\right)^2+\sum_{i=2}^{m}\left(\partial_x^k\chi_i\right)^2\right](x,t')\,\mathrm{d}x\mathrm{d}t'\\
&= O(1)\varepsilon_0^2e^{-\varepsilon t/2}+O(1)\left[M(t)^2+\varepsilon_0^2\right]\varepsilon^{-1}(\varepsilon^4t^2+1)^{-\frac{3}{2}}(t+1)^{-5+k}\\
&\quad+O(1)\tilde{N}_{k-1}^2(t;\varepsilon).
\end{aligned}$$

This immediately gives (5.22). □

We have the following lemmas under the assumptions of Theorem 1.2. The proofs of these lemmas are similar to those for the case of $m=2$, [11]. This is because the Green's function G has the same structure for $m=2$ and for $m>2$, (5.8)–(5.11).

Lemma 5.2. *Let* $2 \leqq l \leqq 5$ *and* $1 \leqq i \leqq m+1$*. For* $t \geqq 0$ *we have*

$$\left\| \left(w_{m+2} \partial_x^l w_i \right) (\cdot, t) \right\|_{L^\infty} = O(1) M(t) \left[M(t) + \varepsilon_0 \right] (t+1)^{-\sigma}, \tag{5.29}$$

where

$$\sigma = \begin{cases} 2 & \text{if } l = 2, 3, \\ \frac{7}{4} & \text{if } l = 4, \\ \frac{3}{4} & \text{if } l = 5. \end{cases} \tag{5.30}$$

Lemma 5.3. *For* $i = 1, 2$*,* $x \in \mathbb{R}$ *and* $t \geqq 0$ *we have*

$$\int_{-\infty}^{\infty} G_i(x-y, t) \partial_y^l w(y, 0)\, dy = O(1)\varepsilon_0 \begin{cases} (t+1)^{-\frac{l}{2}} \phi_i(x, t) & \text{for } l = 0, 1, \\ (t+1)^{-\frac{5-l}{2}} & \text{for } 2 \leqq l \leqq 4. \end{cases} \tag{5.31}$$

Lemma 5.4. *For* $3 \leqq k \leqq m+1$*,* $x \in \mathbb{R}$ *and* $t \geqq 0$ *we have*

$$\begin{aligned} &\int_{-\infty}^{\infty} G_k(x-y, t) \partial_y^l w(y, 0)\, dy \\ &= O(1)\varepsilon_0 \begin{cases} (t+1)^{-\frac{1}{2}} \sum_{i=1}^{2} \phi_i(x, t) & \text{for } l = 0, \\ (t+1)^{-\frac{1}{2}} \sum_{i=1}^{2} \tilde{\psi}_i(x, t) & \text{for } l = 1, \\ (t+1)^{-\frac{5-l}{2}} & \text{for } 2 \leqq l \leqq 4. \end{cases} \end{aligned} \tag{5.32}$$

Lemma 5.5. *For* $0 \leqq l \leqq 2$ *and* $x \in \mathbb{R}$ *we have*

$$\partial_x^l w_{m+2}(x, 0) = O(1)\varepsilon_0 \begin{cases} \phi_0(x) & \text{for } l = 0, 1, \\ \phi_0^{\frac{1}{3}}(x) & \text{for } l = 2. \end{cases} \tag{5.33}$$

Lemma 5.6. *Let* $l = 0, 1$ *and* $i = 1, 2$*. For* $x \in \mathbb{R}$ *and* $t \geqq 0$ *we have*

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} G_i(x-y, t-t') \partial_y^l g(y, t')\, \mathrm{d}y \mathrm{d}t' \\ &= O(1) \left[M(t)^2 + \varepsilon_0^2 \right] (t+1)^{-\frac{l}{2}} \phi_i(x, t). \end{aligned} \tag{5.34}$$

Lemma 5.7. *For* $3 \leqq k \leqq m+1$*,* $x \in \mathbb{R}$ *and* $t \geqq 0$ *we have*

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} G_k(x-y, t-t') \partial_y^l g(y, t')\, \mathrm{d}y \mathrm{d}t' \\ &= O(1) \left[M(t)^2 + \varepsilon_0^2 \right] (t+1)^{-\frac{1}{2}} \begin{cases} \sum_{i=1}^{2} \phi_i(x, t) & \text{for } l = 0, \\ \sum_{i=1}^{2} \tilde{\psi}_i(x, t) & \text{for } l = 1. \end{cases} \end{aligned} \tag{5.35}$$

Lemma 5.8. *Let* $2 \leqq l \leqq 4$ *and* $1 \leqq i \leqq m+1$. *For* $x \in \mathbb{R}$ *and* $t \geqq 0$ *we have*

$$\int_0^t \int_{-\infty}^{\infty} G_i(x-y, t-t')\partial_y^l g(y,t')\,\mathrm{d}y\mathrm{d}t' = O(1)\left[M(t)^2 + \varepsilon_0^2\right](t+1)^{-\frac{5-l}{2}}. \tag{5.36}$$

Lemma 5.9. *For* $0 \leqq l \leqq 2$, $x \in \mathbb{R}$ *and* $t \geqq 0$ *we have*

$$\int_0^t \partial_x^l g_{m+2}(x,t')\,\mathrm{d}t' = O(1)M(t)^2\phi_0(x). \tag{5.37}$$

We are now ready to complete the stability analysis. Equation (5.14) and Lemmas 5.3–5.9 imply that

$$|\partial_x^l w_i(x,t)| \leqq C\left[\varepsilon_0 + M(t)^2\right](t+1)^{-\frac{l}{2}}\phi_i(x,t), \quad i=1,2, \quad l=0,1,$$

$$|\partial_x^l w_i(x,t)| \leqq C\left[\varepsilon_0 + M(t)^2\right](t+1)^{-\frac{1}{2}}$$

$$\times \begin{cases} \sum_{j=1}^2 \phi_j(x,t) & \text{for } l=0 \\ \sum_{j=1}^2 \tilde{\psi}_j(x,t) & \text{for } l=1 \end{cases}, \quad 3 \leqq i \leqq m+1,$$

$$|\partial_x^l w_i(x,t)| \leqq C\left[\varepsilon_0 + M(t)^2\right](t+1)^{-\frac{5-l}{2}}, \quad 1 \leqq i \leqq m+1, \quad 2 \leqq l \leqq 4,$$

$$|\partial_x^l w_{m+2}(x,t)| \leqq C\left[\varepsilon_0 + M(t)^2\right] \begin{cases} \phi_0(x) & \text{for } l=0,1 \\ \phi_0^{\frac{1}{3}}(x) & \text{for } l=2 \end{cases}.$$

These inequalities and (5.17) imply that

$$M(t) \leqq C\left[\varepsilon_0 + M(t)^2\right].$$

If $M(t)$ is sufficiently small, we have

$$M(t) \leqq C\varepsilon_0. \tag{5.38}$$

By the continuity argument, if ε_0 is sufficiently small, we have (5.38) for all $t \geqq 0$. Substituting (5.38) into (5.18)–(5.20) and using (5.2), (5.4), (5.5), (5.16), (5.9) and (5.12), we obtain (1.22).

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