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Erratum: Least Supersolution Approach to Regularizing Free Boundary Problems

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The paper [2], *Least supersolution approach to regularizing free boundary problems*. Arch. Rational Mech. Anal. 191 (2009), no. 1, 97–141, deals with a class of regularizing elliptic free boundary problems that are models in combustion theory. More specifically, we study the limit free boundary problem arising from passing the limit as $\varepsilon \to 0$ of the following family of semilinear equations

$$\Delta u = \beta_\varepsilon(u) F(\nabla u) \quad \text{in } \Omega \tag{SE_ε}$$

where F is a Lipchitz continuous function bounded away from 0 and infinity.

The least supersolution approach is used to construct solutions whose level sets satisfy some geometric properties which are preserved when passing to the limit. These results combined with a detailed analysis of the global profiles which arise as blow-ups, yields regularity results for the free boundary of the limit problem.

An important geometric property of the least supersolutions is that they have a uniform linear growth rate away from the ε-level surfaces. This is the content of the Theorem (3.1) in the cited paper. The idea of the proof is that if the uniform linear growth rate is violated, one is able to construct barriers that are still supersolutions of the problem but are strictly smaller inside the domain, providing a contradiction. In the original proof of Theorem (3.1), there is flaw in the construction of these barriers $\overline{w}_\varepsilon$. The technical point is that restricted to $B_{d_\varepsilon}(y_\varepsilon)$ they do not extend superharmonically by zero across their free boundaries $\partial B_{d_\varepsilon/4}(y_\varepsilon)$ inside $B_{d_\varepsilon/4}(y_\varepsilon)$.

In this paper, we correct the proof of Theorem (3.1) by using new barriers $\overline{w}_\varepsilon$ that are more suitable for the problem (SE_ε). These new barriers were already constructed in the original paper in Proposition (6.4) to control the height decay of the least supersolutions.

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This way, the new version of Theorem (3.1) will hold for the least supersolutions of the (SE_ε) problems instead of the slightly more general (E_ε) problems. As a consequence, all the geometric properties obtained in sections 3 and Theorem (4.1) (the passage to the limit) will be only proven for the least supersolutions of (SE_ε). This change is minor and has no effect whatsoever in our results since the study of the limit problem (classification of global profiles and regularity theory) was developed just for the limits of the (SE_ε) problems.

We present here a slightly more complete statement of Proposition (6.4) by specifying the range of the parameter with respect to the size of the ball. This can be directly read from the proof. The constants appearing in the next statement are defined and explicitly computed there.

Proposition 6.4. *There exist universal constants $T_0 > 0$, $\kappa_2 > 0$ and $0 < \kappa_1 < 1$ such that for any $\eta > 0$ given, we can find for any $0 < \delta < T_0\eta$ a radially symmetric function $\Theta_\delta \in C^1(\mathbb{R}^N) \cap W^{2,\infty}_{loc}(\mathbb{R}^N)$ satisfying*

(i) $\Theta_\delta \equiv \frac{\delta}{4}$ *in* $B_{\kappa_1\eta}$
(ii) $\Theta_\delta \geqq \kappa_2\eta$ *in* $\mathbb{R}^N \backslash B_\eta$
(iii) Θ_δ *is a viscosity supersolution of* (SE_δ) *in* $\mathbb{R}^N$

Remark 0.1. Although the constant $T_0 > 0$ does not appear in the original statement of Proposition (6.4) in [2], it was computed in its proof as $T_0 = \frac{\sqrt{2A_0}}{10\kappa_3} = \frac{\sqrt{2A_0}}{10}\kappa_1$.

Remark 0.2. (Typos in the proof of Proposition (6.4)). We take advantage of this opportunity to correct some typos in the original proof of Proposition (6.4). The first one is that the constant κ_4 should be $\kappa_4 = \sqrt{2A_0}/2(N-2)$ instead of $\kappa_4 = \sqrt{2A_0}/2(N-2)L$. The second one is that when $N = 2$, the function Γ should be defined as $\Gamma(r) = 3/4 + \sqrt{2A_0}(L + 1/\sqrt{2A_0})\log\left(\frac{r}{L+1/\sqrt{2A_0}}\right)$ instead of $\Gamma(r) = 3/4 + \sqrt{2A_0}(L + 1/\sqrt{2A_0})\log\left(\frac{r}{L+\sqrt{2A_0}}\right)$. Also, in the last two paragraphs of the proof it is said that Θ and Θ_ε are solutions of the equations $\Delta u = \beta(u)F(\nabla u)$ and $\Delta u = \beta_\varepsilon(u)F(\nabla u)$, respectively. These are typos. They are in fact supersolutions of these equations as claimed in the (original) statement of Proposition (6.4) (iii). This Proposition will be used to provide the new proof of Theorem (3.1) that follows below.

Here, we state the new (slightly different) version of Theorem (3.1) along with its new corrected proof.

Theorem 0.3. (Linear growth away from level set ε). *Let u_ε be the least supersolution of (SE_ε). There exists a universal constant $C_3 > 0$ such that if $x_0 \in B^\star_\varepsilon \cap \Omega^+_\varepsilon$*

$$u_\varepsilon(x_0) \geqq C_3 d_\varepsilon(x_0)$$

Proof. Let us prove by contradiction. If this is not the case, for $\varepsilon > 0$ small enough, there exists $y_\varepsilon \in B^\star_\varepsilon \cap \Omega^+_\varepsilon$ such that $u_\varepsilon(y_\varepsilon) << d_\varepsilon(y_\varepsilon) = d_\varepsilon$. The idea it is to construct an admissible supersolution (in $\mathcal{S}_\varepsilon$) strictly below u_ε in some point providing a contradiction. Since, $y_\varepsilon \in B_\varepsilon^\star \cap \Omega_\varepsilon^+$, we have $B_{d_\varepsilon}(y_\varepsilon) \subset \Omega_\varepsilon^+$ and thus

$$\Delta u_\varepsilon = 0 \text{ in } \quad B_{d_\varepsilon}(y_\varepsilon)$$

By Harnack inequality, there exists a universal constant $C > 0$ such that

$$u_\varepsilon \leqq C u_\varepsilon(y_\varepsilon) << d_\varepsilon \text{ in } B_{d_\varepsilon/2}(y_\varepsilon)$$

Since $B_{d_\varepsilon}(y_\varepsilon) \subset \Omega_\varepsilon^+$, we have $u_\varepsilon > \varepsilon$ in $B_{d_\varepsilon}(y_\varepsilon)$. In particular, $\varepsilon < u_\varepsilon(y_\varepsilon) << d_\varepsilon$. Thus, if $\varepsilon > 0$ is small enough (say $0 < \varepsilon < \varepsilon_0$), we have

$$\max\{\varepsilon, C u_\varepsilon(y_\varepsilon)\} \leqq \frac{1}{8}\min\{\kappa_2, T_0\} d_\varepsilon. \tag{0.1}$$

For $0 < \varepsilon < \varepsilon_0$, the estimate (0.1) implies that $\varepsilon < T_0 d_\varepsilon/4 =: T_0 \eta_\varepsilon$. This way, by applying Proposition (6.4) with $\eta = \eta_\varepsilon = d_\varepsilon/4$ (i.e., taking $B_{d_\varepsilon/4}(y_\varepsilon)$ as B_η) and $\delta = \varepsilon$ we can find a radially symmetric function $\Theta_\varepsilon \in C^1(\mathbb{R}^N)$ satisfying also because (0.1) the following properties (see Fig. 1):

(a) $\Theta_\varepsilon \equiv \frac{\varepsilon}{4}$ in $B_{\kappa_1 d_\varepsilon/4}(y_\varepsilon)$. In particular, $\Theta_\varepsilon = \frac{\varepsilon}{4} < \varepsilon < u_\varepsilon$ inside $B_{\kappa_1 d_\varepsilon/4}(y_\varepsilon)$.
(b) $\Theta_\varepsilon \geqq \kappa_2 d_\varepsilon/4 \geqq 2C u_\varepsilon(y_\varepsilon)$ in $\mathbb{R}^N \setminus B_{d_\varepsilon/4}(y_\varepsilon)$ and $u_\varepsilon \leqq C u_\varepsilon(y_\varepsilon)$ in $B_{d_\varepsilon/2}(y_\varepsilon)$. In particular, $\Theta_\varepsilon > u_\varepsilon$ along $\partial B_{d_\varepsilon/2}(y_\varepsilon)$
(c) Θ_ε is a viscosity supersolution to (SE_ε) in $\mathbb{R}^N$.

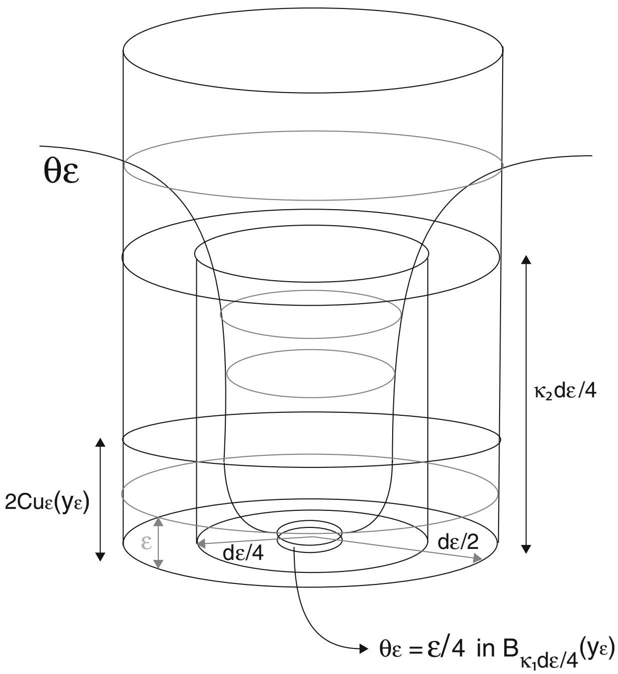

Fig. 1. Schematic representation of the barriers Θ_ε

This way, let us define

$$\overline{w}_\varepsilon = \begin{cases} \min\{u_\varepsilon, \Theta_\varepsilon\} & \text{in } \overline{B_{d_\varepsilon/2}(y_\varepsilon)} \\ u_\varepsilon & \text{in } \Omega \backslash \overline{B_{d_\varepsilon/2}(y_\varepsilon)} \end{cases} \tag{0.2}$$

Then, by (b) and (c), we observe that $\overline{w}_\varepsilon \in \mathcal{S}_\varepsilon$ (see Proposition 2.8 in [1] for example). Also by (a), we have $\overline{w}_\varepsilon \equiv \Theta_\varepsilon \equiv \frac{\varepsilon}{4} < \varepsilon < u_\varepsilon$ in $B_{\kappa_1 d_\varepsilon/4}(y_\varepsilon)$ and $B_{\kappa_1 d_\varepsilon/4}(y_\varepsilon) \subset B_{d_\varepsilon/2}(y_\varepsilon)$, a contradiction. This finishes the proof. □

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