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Deformations of Annuli with Smallest Mean Distortion

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Abstract

We determine the extremal mappings with smallest mean distortion for mappings of annuli. As a corollary, we find that the Nitsche harmonic maps are Dirichlet energy minimizers among all homeomorphisms $h : \mathbb{A}(r, R) \to \mathbb{A}(r', R')$. However, outside the Nitsche range of the modulus of the annuli, within the class of homeomorphisms, no such energy minimizers exist. In this case we identify the BV-limits of minimizers.

1. Introduction

A homeomorphism $f = f(z)$ between planar domains Ω and Ω' has *finite distortion* if

– f lies in the Sobolev space $W^{1,1}_{loc}(\Omega, \Omega')$ of functions whose first derivatives are locally integrable, and
– f satisfies the distortion inequality $|f_{\bar{z}}| \leqq k(z)|f_z|$, $0 \leqq k(z) < 1$ almost everywhere in Ω.

Such mappings are generalizations of quasiconformal homeomorphisms where one works with the stronger assumption $\|k(z)\|_\infty \leqq k < 1$. Mappings of finite distortion have found considerable interest in geometric function theory and the mathematical theory of elasticity. In [4], we initiated the study of extremal problems for these mappings and furthered these studies in [2] to extremal mappings between Riemann surfaces and their degenerations. A comprehensive overview of the theory of mappings of finite distortion in two-dimensions can be found in [3, 14].

The Jacobian determinant of a mapping f of finite distortion is non-negative almost everywhere, since

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$$J(z, f) = |f_z|^2 - |f_{\bar{z}}|^2 \geqq (1 - k^2)|f_z|^2 \geqq 0$$

However, a key property of mappings of finite distortion is that $J(z, f)$ may vanish only in the trivial situation where $f_z = f_{\bar{z}} = 0$. The distortion function of particular interest to us in this article is defined by the rule

$$\mathbb{K}(z, f) = \frac{|f_z|^2 + |f_{\bar{z}}|^2}{|f_z|^2 - |f_{\bar{z}}|^2} = \frac{\|Df\|^2}{J(z, f)} \quad \text{if } J(z, f) > 0$$

and we conveniently set $\mathbb{K}(z, f) = 1$ if $f_z = f_{\bar{z}} = 0$. Notice that then $\mathbb{K}(z, f) \geqq 1$ and we have the equality $\mathbb{K}(z, f) \equiv 1$ if and only if f is conformal, by the Looman–Menchoff theorem.

The classical formulations of the extremal Grötsch and Teichmüller problems are concerned with finding mappings $\Omega \to \Omega'$ in some class (for instance, with free or prescribed boundary values) which have smallest L^∞-norm of the distortion function, thus "extremal quasiconformal mappings". In this article we shall investigate mappings in some class which minimize integral means of the distortion function $\mathbb{K}(z, f)$.

The case of bounded simply connected domains, without boundary data, is trivial; the extremals are the conformal mappings of Ω onto Ω' asserted to exist by the Riemann mapping theorem (the simply connected case where the boundary data is prescribed is solved in [4]). For the free boundary problem, we consider the first nontrivial case where there are conformal invariants; namely doubly connected domains and, in particular, annuli.

Given two annuli

$$\mathbb{A} = \{z : r < |z| < R\} \quad \text{and} \quad \mathbb{A}' = \left\{\zeta : r' < |\zeta| < R'\right\}$$

we shall consider homeomorphisms of finite distortion $f : \mathbb{A} \to \mathbb{A}'$. Here note that $|f|$ extends continuously to $\overline{\mathbb{A}}$, with values r' and R' on the boundary of $\mathbb{A}$. We shall normalize our mappings in the obvious way so that

$$|f(z)| = r' \quad \text{for } |z| = r \qquad \text{and} \qquad |f(z)| = R' \quad \text{for } |z| = R$$

Let $\mathcal{F} = \mathcal{F}(\mathbb{A}, \mathbb{A}')$ denote the family of all normalised homeomorphisms $f : \mathbb{A} \to \mathbb{A}'$ of finite distortion. Since $\mathbb{A}$ and $\mathbb{A}'$ are certainly diffeomorphic $\mathcal{F} \neq \emptyset$.

There are two integral means of the distortion function $\mathbb{K}(z, f)$ which concern us in this work. First, for $f \in \mathcal{F}$, the average

$$\mathcal{K}_f := \frac{1}{|\mathbb{A}|} \iint_{\mathbb{A}} \mathbb{K}(z, f)\, \mathrm{d}z$$

and secondly the weighted average

$$\mathcal{K}_f^* := \frac{1}{\mu(\mathbb{A})} \iint_{\mathbb{A}} \mathbb{K}(z, f)\, \mathrm{d}\mu(z)$$

where we have used the notation $\mathrm{d}\mu(z) = \mathrm{d}z/|z|^2$ and $\mathrm{d}z = \mathrm{d}x\, \mathrm{d}y$.

The minimization problems we address here are to evaluate the following infima:

$$\inf\left\{\mathcal{K}_f : f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')\right\} \tag{1}$$
$$\inf\left\{\mathcal{K}_f^* : f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')\right\} \tag{2}$$

Further, we should decide if the infimum is attained and, in that case, prove uniqueness (up to the obvious rotational symmetry of the annuli).

The concept of the conformal modulus will prove useful in formulating our results. For the annulus $\mathbb{A}$ we have the equivalent definitions

$$\mathrm{Mod}(\mathbb{A}) = 2\pi \log \frac{R}{r} = \iint_{\mathbb{A}} \frac{\mathrm{d}z}{|z|^2} = \mu(\mathbb{A}) \tag{3}$$

Every topological annulus is conformally equivalent to a round annulus $\mathbb{A}$, and we can set $\mathrm{Mod}(\Omega) = \mathrm{Mod}(\mathbb{A})$.

For the weighted averages we have then

Theorem 1. *Among all mappings $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$ the infimum of*

$$\iint_{\mathbb{A}} \mathbb{K}(z, f)\, \mathrm{d}\mu(z)$$

is attained at the power function

$$f^\alpha(z) = r' r^{-\alpha} |z|^{\alpha-1} z, \quad \textit{where } \alpha = \frac{\mathrm{Mod}(\mathbb{A}')}{\mathrm{Mod}(\mathbb{A})}$$

Furthermore, we have

$$\frac{1}{\mu(\mathbb{A})} \iint_{\mathbb{A}} \mathbb{K}(z, f^\alpha)\, \mathrm{d}\mu(z) = \frac{1}{2}\left(\frac{\mathrm{Mod}(\mathbb{A}')}{\mathrm{Mod}(\mathbb{A})} + \frac{\mathrm{Mod}(\mathbb{A})}{\mathrm{Mod}(\mathbb{A}')}\right) = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) \tag{4}$$

This extremal map is unique up to rotations of the annuli.

The minimization of the average $\mathcal{K}_f$ for mappings $f : \mathbb{A} \to \mathbb{A}'$ of finite distortion is more subtle. The infimum is attained only when the image annulus $\mathbb{A}'$ is not too fat when compared with $\mathbb{A}$. This special restriction on the moduli of the annuli is explicitly described by the Nitsche bound - we describe in the next section how these bounds arise. In terms of the radii, these bounds read as

$$\frac{R'}{r'} \leqq \frac{R}{r} + \sqrt{\frac{R^2}{r^2} - 1}, \quad \text{or equivalently} \quad \frac{R}{r} \geqq \frac{1}{2}\left(\frac{R'}{r'} + \frac{r'}{R'}\right) \tag{5}$$

In terms of the moduli, $\gamma = \mathrm{Mod}(\mathbb{A})$ and $\sigma = \mathrm{Mod}(\mathbb{A}')$, the inequality has the form

$$\mathrm{e}^{\gamma/2\pi} \geqq \cosh(\sigma/2\pi)$$

Assuming that the Nitsche bound (5) holds, we can find (uniquely) a real parameter $\omega \geqq -r^2$ such that

$$\frac{R'}{r'} = \frac{R + \sqrt{R^2 + \omega}}{r + \sqrt{r^2 + \omega}} \tag{6}$$

Because of scale invariance in the target, there is no loss of generality in assuming, for the purpose of identifying a minimizer, that

$$r' = r + \sqrt{r^2 + \omega} \quad \text{and} \quad R' = R + \sqrt{R^2 + \omega} \tag{7}$$

We now recall the complex harmonic Nitsche map [20] defined by

$$h^\omega : \mathbb{A}' \to \mathbb{A}, \quad h^\omega(\zeta) = \frac{1}{2}\left(\zeta - \frac{\omega}{\overline{\zeta}}\right),$$

and its inverse

$$f^\omega : \mathbb{A} \to \mathbb{A}', \quad f^\omega(z) = z + z\left(1 + \frac{\omega}{|z|^2}\right)^{1/2} \tag{8}$$

Theorem 2. *Under the Nitsche condition at* (5), *for each homeomorphism* $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$ *we have*

$$\frac{1}{|\mathbb{A}|}\iint_{\mathbb{A}} \mathbb{K}(z, f)\, \mathrm{d}z \geqq \frac{1}{|\mathbb{A}|}\iint_{\mathbb{A}} \mathbb{K}(z, f^\omega)\, \mathrm{d}z = \frac{R\sqrt{R^2+\omega} - r\sqrt{r^2+\omega}}{R^2 - r^2} \tag{9}$$

The extremal map $f = f^\omega$ *is unique up to rotations of the annuli.*

There is a surprising conformally invariant formulation of the lower bound, namely

$$\frac{1}{|\mathbb{A}|}\iint_{\mathbb{A}} \mathbb{K}(z, f)\, \mathrm{d}z \geqq \frac{\tanh(\gamma/\pi)}{\tanh(\sigma/2\pi)} + \frac{\tanh(\sigma/\pi)}{\sinh(\gamma/2\pi)} \tag{10}$$

where $\gamma = \mathrm{Mod}(\mathbb{A})$ and $\sigma = \mathrm{Mod}(\mathbb{A}')$; for a proof see the calculation following (49) below. This bound is surprising in as much as the right-hand side depends only on the modulus of $\mathbb{A}$ and does not depend on the fact that $\mathbb{A}$ is assumed to be a "round" annulus. The left-hand side would appear to depend on this since it is not invariant under precomposition with the conformal map, as a Jacobian derivative term would appear after the change of variables.

We are also interested in the extremal problem outside the Nitsche bound (5). It is important to look more closely at the critical inverse Nitsche map, corresponding to $\omega = -r^2$, which we denote by

$$f^\#(z) = z + z\left(1 - \frac{r^2}{|z|^2}\right)^{1/2}, \quad r \leqq |z| \leqq R \tag{11}$$

This maps $\mathbb{A}$ onto the *critical annulus*

$$\mathbb{A}^\# = \left\{\zeta : r < |\zeta| < R + \sqrt{R^2 - r^2}\right\} \tag{12}$$

From (9) with $\omega = -r^2$ we find the mean distortion of $f^{\#}$ to be

$$\mathcal{K}_{f^{\#}} = \frac{R}{\sqrt{R^2 - r^2}}$$

where, by Theorem 2,

$$\mathcal{K}_{f^{\#}} = \inf \left\{ \mathcal{K}_f : f \in \mathcal{F}(\mathbb{A}, \mathbb{A}^{\#}) \right\}$$

The inverse to $f^{\#}$ is the critical Nitsche harmonic map

$$h^{\#}(\zeta) = \frac{1}{2}\left(\zeta + \frac{r^2}{\overline{\zeta}}\right)$$

Its Jacobian determinant vanishes identically on the inner boundary of $\mathbb{A}^{\#}$,

$$|h^{\#}_{\zeta}|^2 - |h^{\#}_{\overline{\zeta}}|^2 \equiv \frac{1}{4}\left(1 - \frac{r^4}{|\zeta|^4}\right) = 0 \quad \text{for } |\zeta| = r$$

Further,

$$|Df^{\#}| = |f^{\#}_z| + |f^{\#}_{\overline{z}}| = \frac{|f^{\#}|}{\sqrt{|z|^2 - r^2}} \in L^2_{\text{weak}}(\mathbb{A})$$

but $|Df^{\#}| \notin L^2(\mathbb{D})$. Thus we see various degenerations of our extremals at the critical case, suggesting that there are no extremal homeomorphisms beyond this range of the moduli.

Suppose next that the target annulus is fatter than the critical one. In other words, let

$$\mathbb{A}' = \left\{ \zeta : r' < |\zeta| < R' \right\} \supsetneq \mathbb{A}^{\#} \tag{13}$$

where $0 < r' < r < R \leqq R' = R + \sqrt{R^2 - r^2}$. Thus

$$\frac{R'}{r'} > \frac{R}{r} + \sqrt{\frac{R^2}{r^2} - 1} \tag{14}$$

Theorem 3. *Under the fatness condition* (14), *the infimum at* (1) *is not attained by any homeomorphism* $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$.

The sequences $\{f_n\} \subset \mathcal{F}(\mathbb{A}, \mathbb{A}')$ minimizing the mean distortion will be bounded in $W^{1,1}(\mathbb{A}, \mathbb{A}')$ but the derivatives will lack equi-integrability. Consequently, $\{f_n\}$ will contain no subsequence converging weakly in $W^{1,1}(\mathbb{A}, \mathbb{A}')$. However, the derivatives of f_n will converge in the sense of measures to a limit. The precise description of the limit measure will be given in Section 9.

The minimization of the integral means of the distortion functions of homeomorphisms $f : \mathbb{A} \to \mathbb{A}'$ turns out to be equivalent to the Dirichlet type problems for the inverse mappings $h = f^{-1} : \mathbb{A}' \to \mathbb{A}$. If a homeomorphism $f \in W^{1,1}_{loc}(\mathbb{A}, \mathbb{A}')$

has integrable distortion, then $h \in W^{1,2}(\mathbb{A}', \mathbb{A})$ and we can consider the two energy functionals

$$E[h] := \iint_{\mathbb{A}'} \|Dh(\zeta)\|^2 \,\mathrm{d}\zeta = \iint_{\mathbb{A}} \mathbb{K}(z, f)\,\mathrm{d}z < \infty$$

$$F[h] := \iint_{\mathbb{A}'} \frac{\|Dh(\zeta)\|^2}{|h|^2} \,\mathrm{d}\zeta = \iint_{\mathbb{A}} \frac{\mathbb{K}(z, f)}{|z|^2}\,\mathrm{d}z < \infty$$

In general, the converse is not true, simply because the inverse of a homeomorphism $h \in W^{1,2}(\mathbb{A}', \mathbb{A})$ need not even belong to the Sobolev class $W^{1,1}_{loc}(\mathbb{A}', \mathbb{A})$. It has bounded variation but fails to be absolutely continuous on lines, see [10,11] for related questions and results.

In this paper we overcome such subtleties by proving a so called correction lemma. Accordingly, for every homeomorphism $h \in W^{1,2}(\mathbb{A}', \mathbb{A})$ we can construct a homeomorphism $\tilde{h} \in W^{1,2}(\mathbb{A}', \mathbb{A})$, with $E[\tilde{h}] \leqq E[h]$ and $F[\tilde{h}] \leqq F[h]$, whose inverse lies in $\mathcal{F}(\mathbb{A}, \mathbb{A}')$. For details see Lemma 7 below. As a consequence, the minimization problems for $\mathcal{K}_f$ and $\mathcal{K}^*_f$ are equivalent to the corresponding minimization problems for $E[h]$ and $F[h]$ with $h = f^{-1}$.

Corollary 1. *The absolute minimum of the homogeneous energy $F[h]$ is attained (uniquely up to a rotation of the annuli) by a power stretching.*

Corollary 2. *Within the Nitsche range (5) for the annuli $\mathbb{A}$ and $\mathbb{A}'$ the minimum of the harmonic energy $E[h]$ is obtained (uniquely up to rotation) by a Nitsche map*

$$h(\zeta) = \lambda\left(\zeta - \frac{\omega}{\bar{\zeta}}\right), \quad \omega \in \mathbb{R},\ \lambda > 0$$

Corollary 3. *Outside the Nitsche range (5) for the annuli $\mathbb{A}$ and $\mathbb{A}'$ the infimum of $E[h]$ is not attained by any homeomorphism $h \in W^{1,2}(\mathbb{A}', \mathbb{A})$.*

In studying $E[h]$ outside the Nitsche range we shall be led to look at a minimizing sequence $\{h^n\} \subset W^{1,2}(\mathbb{A}', \mathbb{A})$. Its limit will exist in the Sobolev space $W^{1,2}(\mathbb{A}', \mathbb{A})$. Moreover, although the limit fails to be a homeomorphism, no cavity or crack will occur within the solid body of the ring $\mathbb{A}$ under such deformation of $\mathbb{A}'$. For this reason we shall accept such a limit as a legitimate minimizer of the Dirichlet integral.

Both energy integrals $E[h]$ and $F[h]$ are invariant under a conformal change of the independent variable. Thus, given any topological annulus $\Omega \subset \mathbb{C}$ of the same modulus as $\mathbb{A}'$, we may consider homeomorphisms $g : \Omega \to \mathbb{A}$ in the Sobolev class $W^{1,2}(\Omega, \mathbb{A})$. To each such g there corresponds a mapping $h = g \circ \chi : \mathbb{A}' \to \mathbb{A}$, where $\chi : \mathbb{A} \to \Omega$ is a conformal transformation. The corresponding energy integrals for $g \in W^{1,2}(\Omega, \mathbb{A})$ reduce to

$$E[g] = \iint_{\Omega} \|Dg\|^2 = \iint_{\mathbb{A}'} \|Dh\|^2 = E[h]$$

and

$$F[g] = \iint_{\Omega} \frac{\|Dg\|^2}{|g|^2} = \iint_{\mathbb{A}'} \frac{\|Dh\|^2}{|h|^2} = F[h]$$

Of particular interest in elasticity theory is to know how to deform, with minimal energy, a given topological annulus onto a given round annulus. Corollaries 1, 2 and 3 provide us with the solution to this problem. For example, as a result of our computation we find the minimum energy required to deform a given annulus onto a punctured disk.

Corollary 4. *Let* $h \in W^{1,2}(\Omega, \mathbb{D}\backslash\{0\})$ *be homeomorphism from a topological annulus* Ω *onto the punctured unit disk* $\mathbb{D}\backslash\{0\}$. *Then*

$$E[h] = \iint_{\mathbb{A}'} \|Dh\|^2 \geqq \pi \frac{\mathrm{e}^{\sigma/\pi}+1}{\mathrm{e}^{\sigma/\pi}-1} = \pi \coth(\sigma/2\pi)$$

with $\sigma = \mathrm{Mod}(\Omega)$. *The infimum is attained by the harmonic map*

$$h(\zeta) = C\frac{|\chi(\zeta)|^2-1}{\overline{\chi(\zeta)}}$$

where $\chi : \Omega \to \mathbb{A}' = \{w : 1 < |w| < e^{\sigma/2\pi}\}$ *is a conformal map of* Ω *onto a round annulus and the constant*

$$C = \left(e^{\sigma/\pi} + e^{-\sigma/\pi} - 2\right)^{-1/2}$$

The conformally invariant nature of the minimization problems described above strongly suggests natural generalizations also within the theory of Teichmüller spaces of Riemann surfaces. The study of minimal deformations of annuli described here does indeed prove an important tool, and we refer to [2] for such further developments. Other papers studying the problem of minimizing mean and related distortional functionals are [19], where the classical Teichmüller problem is discussed, and [15], where the Grötsch type problems with certain densities and some applications in materials science are considered.

2. The Nitsche problem

We have talked about the Nitsche bounds above and observed a connection with harmonic mappings. It is worthwhile to flesh this connection out a little—the survey [18] has more detail. In his studies of minimal surfaces Nitsche [20] was led, in 1962, to the following problem concerning harmonic homeomorphisms between annuli. For each ρ we can define the function

$$N(\rho) = \{\sup R : \text{there is a harmonic homeomorphism } h : \mathbb{A}(\rho,1) \to \mathbb{A}(R,1)\}$$

Nitsche showed [20] the strict inequality $N(\rho) < 1$ for every $\rho < 1$ and conjectured - for various reasons - that

$$N(\rho) \stackrel{?}{=} \frac{2\rho}{\rho^2+1}$$

The Nitsche examples above give harmonic homeomorphisms whenever $R \leqq 2\rho/(\rho^2+1)$.

Various estimates for $N(\rho)$ were given by Lyzzaik [17] and Weitsman [22] after the problem was raised again by Schober [21]. As far as we are aware the current best general estimate is

$$N(\rho) \leqq \frac{1}{1+\rho^2 \log^2(\rho)/2}$$

obtained by Kalaj [16]—there are other estimates for ρ close to 1.

Our results show, as noted in Corollary 3, that outside the Nitsche range there is no homeomorphic energy minimizer (such a map would be harmonic). Nitsche's question asks somewhat more. One should prove that outside the Nitsche range there are not even any homeomorphic stationary points $h : \mathbb{A}(\rho, 1) \to \mathbb{A}(R, 1)$ for the energy functional.

Further information on planar harmonic mappings can be found in Duren's recent book [6].

3. Polar coordinates

Given the symmetry of the problem we are considering, we frequently view functions $f = f(z)$ of one complex variable $z = \rho e^{i\theta}$ as functions of the polar coordinates $0 < \rho < \infty$, $0 \leqq \theta < 2\pi$. Then the Cauchy–Riemann derivatives of f are

$$f_z = \frac{1}{2}\, e^{-i\theta}\left(f_\rho - \frac{i}{\rho}\, f_\theta\right), \quad f_{\bar z} = \frac{1}{2}\, e^{i\theta}\left(f_\rho + \frac{i}{\rho}\, f_\theta\right)$$

Hence

$$|f_z|^2 + |f_{\bar z}|^2 = \frac{1}{2}\left(|f_\rho|^2 + \rho^{-2}|f_\theta|^2\right),$$
$$J(z, f) = |f_z|^2 - |f_{\bar z}|^2 = \frac{1}{\rho}\, \Im(f_\theta\, \overline{f_\rho}) \tag{15}$$

which together yield

$$\mathbb{K}(z, f) = \frac{\rho |f_\rho|^2 + \rho^{-1}|f_\theta|^2}{2\Im(f_\theta \overline{f_\rho})} \tag{16}$$

We naturally expect extremal mappings between annuli to be radial stretchings of the form

$$f(\rho e^{i\theta}) = F(\rho) e^{i\theta}$$

with $F : [r, R] \to [r', R']$ increasing, $0 < r < R$ and $0 < r' < R'$. A remarkable feature of radial mappings is that

$$\frac{f_\theta}{\rho f} = \frac{i}{\rho} \in i\mathbb{R}, \quad \frac{f_\rho}{\rho f} = \frac{\dot F}{\rho F} \in \mathbb{R}$$

At points where F is differentiable we calculate the distortion function for such mappings as

$$\mathbb{K}(z, f) = \frac{1}{2}\left(\frac{\rho \dot{F}}{F} + \frac{F}{\rho \dot{F}}\right) \tag{17}$$

For the inverse Nitsche map (8) we see that $F(\rho) = \rho + \sqrt{\rho^2 + \omega}$ and hence $\dot{F}(\rho) = F(\rho)/\sqrt{\rho^2 + \omega}$, so that

$$\mathbb{K}(z, f^\omega) = \frac{1}{2}\left(\frac{\rho}{\sqrt{\rho^2 + \omega}} + \frac{\sqrt{\rho^2 + \omega}}{\rho}\right)$$

$$\iint_{\mathbb{A}} \mathbb{K}(z, f^\omega) = \pi \int_r^R \frac{\rho^2}{\sqrt{\rho^2 + \omega}} + \sqrt{\rho^2 + \omega} = \pi\left(R\sqrt{R^2 + \omega} - r\sqrt{r^2 + \omega}\right)$$

Which establishes the second half of (9).

4. Invariant integrals

For certain nonlinear differential expressions, defined for diffeomorphisms $f : \mathbb{A} \to \mathbb{A}'$, their integral mean does not depend on the specific choice of the diffeomorphism f and we call them *invariant integrals*. They play a fundamental role in our computations in a way similar to that played by the integral means of null Lagrangians in the polyconvex calculus of variations [5]. As a typical example consider the expression $J(z, f)|f|^{-2}$, with integral mean

$$\iint_{\mathbb{A}} \frac{J(z, f)\mathrm{d}z}{|f|^2} = \iint_{\mathbb{A}'} \frac{\mathrm{d}\zeta}{|\zeta|^2} = \mathrm{Mod}(\mathbb{A}')$$

depending only on $\mathbb{A}'$.

This identity generalizes, as an inequality, to all homeomorphisms $f : \mathbb{A} \to \mathbb{A}'$ in the Sobolev class $W^{1,1}_{loc}(\mathbb{A}, \mathbb{A}')$, namely

$$\iint_{\mathbb{A}} \frac{J(z, f)\mathrm{d}z}{|f|^2} \leqq \iint_{\mathbb{A}'} \frac{\mathrm{d}\zeta}{|\zeta|^2} = \mathrm{Mod}(\mathbb{A}') \tag{18}$$

Precisely, there is a set $\mathbb{E} \subset \mathbb{A}$ of measure zero such that

$$\iint_{\mathbb{A}} \frac{J(z, f)\mathrm{d}z}{|f|^2} = \iint_{f(\mathbb{A}\setminus\mathbb{E})} \frac{\mathrm{d}\zeta}{|\zeta|^2}$$

The reader is referred to [7, 3.1.4, 3.1.8, 3.2.5], for a proof. For more explicit statements see, for example [8,9 and 11].

We shall introduce the following nonlinear differential expressions defined for homeomorphisms $f : \mathbb{A} \to \mathbb{A}'$ in the Sobolev class $W^{1,1}_{loc}(\mathbb{A}, \mathbb{A}')$,

$$\mathcal{J}_f = \frac{J(z,f)}{|f|^2} \quad \left[\mathcal{J}_f = \frac{\dot{F}}{\rho F} \quad \text{for } f(z) = F(|z|)\frac{z}{|z|} \right] \tag{19}$$

$$\mathcal{A}_f = \Re \frac{f_\rho(z)}{|z| f(z)} \quad \left[\mathcal{A}_f = \frac{\dot{F}}{\rho F} \quad \text{for } f(z) = F(|z|)\frac{z}{|z|} \right] \tag{20}$$

$$\mathcal{B}_f = \Im \frac{f_\theta(z)}{|z|^2 f(z)} \quad \left[\mathcal{B}_f = \frac{1}{\rho^2} \quad \text{for } f(z) = F(|z|)\frac{z}{|z|} \right] \tag{21}$$

With the next lemma we show that these give rise to invariant integrals.

Lemma 1. *The following integrals are independent of the mapping* $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$*,*

$$\iint_{\mathbb{A}} \mathcal{A}_f \, \mathrm{d}z = \mathrm{Mod}(\mathbb{A}') \tag{22}$$

$$\iint_{\mathbb{A}} \mathcal{B}_f \, \mathrm{d}z = \mathrm{Mod}(\mathbb{A}) \tag{23}$$

More generally, for every continuous function $u : [r, R] \to \mathbb{R}$

$$\iint_{\mathbb{A}} \mathcal{B}_f(z) u(|z|) \mathrm{d}z = \iint_{\mathbb{A}} \frac{u(|z|)}{|z|^2} \mathrm{d}z = 2\pi \int_r^R \frac{u(\rho)\mathrm{d}\rho}{\rho} \tag{24}$$

In addition, for every $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$ *we have the estimate*

$$\iint_{\mathbb{A}} \frac{J(z,f)\mathrm{d}z}{|f|^2} \leqq \iint_{\mathbb{A}'} \frac{\mathrm{d}\zeta}{|\zeta|^2} = \mathrm{Mod}(\mathbb{A}') \tag{25}$$

where equality holds whenever f *belongs to the class* $W^{1,2}_{loc}(\mathbb{A}, \mathbb{A}')$.

Proof. Note first that

$$|f|_\rho = \frac{1}{2|f|}\left(|f|^2\right)_\rho = |f| \frac{f_\rho \overline{f} + f \overline{f_\rho}}{2 f \overline{f}} = |f| \Re \left(\frac{f_\rho}{f} \right)$$

Therefore

$$\mathcal{A}_f(z) = \frac{|f|_\rho}{\rho |f|}$$

By Fubini's theorem

$$\iint_{\mathbb{A}} \mathcal{A}_f(z) \mathrm{d}z = \int_0^{2\pi} \left(\int_r^R \frac{|f|_\rho}{|f|} \mathrm{d}\rho \right) \mathrm{d}\theta = \int_0^{2\pi} \left(\log R' - \log r' \right) \, \mathrm{d}\theta = \mathrm{Mod}(\mathbb{A}')$$

For (23) and (24), from the increment of the argument we have

$$\int_0^{2\pi} \frac{f_\theta(\rho e^{i\theta})}{f(\rho e^{i\theta})} \mathrm{d}\theta = 2\pi i, \quad \text{for almost every } \rho \in (r, R) \tag{26}$$

Hence by Fubini's theorem

$$\iint_{\mathbb{A}} \mathcal{B}_f(z) u(|z|) \mathrm{d}z = \Im \int_r^R \frac{u(\rho)}{\rho} \int_0^{2\pi} \frac{f_\theta(\rho \mathrm{e}^{i\theta})}{f(\rho \mathrm{e}^{i\theta})} \mathrm{d}\theta \, \mathrm{d}\rho = 2\pi \int_r^R \frac{u(\rho)}{\rho} \mathrm{d}\rho$$

Finally we remark that a homeomorphism $f \in W^{1,2}_{loc}(\mathbb{A}, \mathbb{A}')$ satisfies the Lusin condition N and therefore (25) holds as an equality in this case. □

5. Invariant lower bounds

The invariant integrals provide powerful tools for extremal problems on mappings of finite distortion. As a particular example in this section we will show how to obtain optimal integral mean estimates for the distortion functions.

For this purpose, let $f : \mathbb{A} \to \mathbb{A}'$ be a given mapping of finite distortion, where $\mathbb{A} = \{z : r < |z| < R\}$ and $\mathbb{A}' = \{z : r' < |z| < R'\}$. We begin with pointwise lower bounds for the distortion function $\mathbb{K}(z, f)$.

Lemma 2. *For every parameter $\lambda \geqq -r^2$ we have*

$$\mathbb{K}(z, f) \geqq \frac{-\lambda \, |f_\rho(z)|^2}{2|z|^2 \, J(z, f)} + \frac{\sqrt{|z|^2 + \lambda}}{|z|} \tag{27}$$

or, equivalently,

$$\mathbb{K}(z, f) \geqq \frac{\lambda |f_\theta(z)|^2}{2(|z|^2 + \lambda)|z|^2 J(z, f)} + \frac{|z|}{\sqrt{|z|^2 + \lambda}} \tag{28}$$

almost everywhere on the set $\mathbb{A}^+ = \{z \in \mathbb{A} : J(z, f) > 0\}$. For $z \in \mathbb{A}^+$, equality holds if and only if

$$f_\theta(z) = i\sqrt{|z|^2 + \lambda} \; f_\rho(z) \tag{29}$$

Proof. These estimates are equivalent to the obvious inequality

$$|\sqrt{|z|^2 + \lambda} \; f_\rho + i f_\theta|^2 \geqq 0, \quad \rho^2 \geqq -\lambda \tag{30}$$

once we take into account the identities (15) and (16). □

For later purposes we note that for every mapping of finite distortion

$$f_\rho = f_\theta = 0 \quad \text{outside } \mathbb{A}^+ \tag{31}$$

This observation is useful in integrating the pointwise bounds.

Lemma 3. *For every $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$ we have*

$$\iint_{\mathbb{A}^+} \frac{|f_\rho(z)|^2}{|z|^2 J(z, f)} \geqq \operatorname{Mod}(\mathbb{A}') \tag{32}$$

The equality occurs if and only if almost everywhere on $\mathbb{A}^+$ we have

$$\Re\left(\frac{f_\rho}{f}\right) = \left|\frac{f_\rho}{f}\right| \tag{33}$$

and

$$\Im\left(\frac{f_\theta}{f}\right) = k \tag{34}$$

for some constant $k > 0$.

Proof. We start with formula (22) and obtain

$$\operatorname{Mod}(\mathbb{A}') = \iint_{\mathbb{A}} \Re \frac{f_\rho(z)}{|z| f(z)} \leqq \iint_{\mathbb{A}^+} \frac{|f_\rho(z)|}{|z|\,|f(z)|} \tag{35}$$

Combining the estimate with Hölder's inequality gives

$$\operatorname{Mod}(\mathbb{A}') \leqq \iint_{\mathbb{A}^+} \frac{|f_\rho|}{|z|\,|f|} \leqq \left(\iint_{\mathbb{A}^+} \frac{|f_\rho|^2}{|z|^2 J(z, f)}\right)^{1/2} \left(\iint_{\mathbb{A}^+} \frac{J(z, f)}{|f|^2}\right)^{1/2} \tag{36}$$

A glance at (18) shows that we have proved the claim (32).

To achieve equality it is necessary that almost everywhere on $\mathbb{A}^+$

$$\Re\left(\frac{f_\rho}{f}\right) = \left|\frac{f_\rho}{f}\right|$$

and, since we have used Hölder's inequality, that almost everywhere on $\mathbb{A}^+$

$$\frac{J(z, f)}{|f(z)|^2} = k \frac{|f_\rho(z)|^2}{|z|^2 J(z, f)}$$

for some constant $k > 0$. The latter condition takes the form

$$k\,|f|\,|f_\rho| = |z| J(z, f) = \Im\left(f_\theta \overline{f_\rho}\right)$$

which together with (33) reduces to (34). Conversely, under the conditions (33) and (34) we have equality at (32). □

Lemma 4. *For every $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$ and for every parameter $\lambda \geqq -r^2$ we have*

$$\iint_{\mathbb{A}^+} \frac{|f_\theta(z)|^2}{|z|^2\,(|z|^2 + \lambda)\, J(z, f)} \geqq \frac{4\pi^2}{\operatorname{Mod}(\mathbb{A}')} \log^2 \frac{R + \sqrt{R^2 + \lambda}}{r + \sqrt{r^2 + \lambda}} \tag{37}$$

Equality occurs if and only if almost everywhere on $\mathbb{A}^+$ we have

$$\Im\left(\frac{f_\theta}{f}\right) = \left|\frac{f_\theta}{f}\right| \tag{38}$$

and

$$\Re\left(\frac{f_\rho}{f}\right) = \frac{\alpha}{\sqrt{|z|^2+\lambda}} \tag{39}$$

for some constant $\alpha > 0$.

Proof. Let us apply (24) with the auxiliary function $u(\rho) = \rho\ \left(\rho^2+\lambda\right)^{-1/2}$. That gives

$$2\pi \int_r^R \frac{\mathrm{d}\rho}{\sqrt{\rho^2+\lambda}} = \iint_{\mathbb{A}} \Im\left(\frac{f_\theta(z)}{f(z)}\right) \frac{\mathrm{d}z}{|z|\sqrt{|z|^2+\lambda}}$$

Clearly this quantity is less than or equal to

$$\iint_{\mathbb{A}^+} \frac{|f_\theta|}{|z|\sqrt{|z|^2+\lambda}\,|f|} \leqq \left(\iint_{\mathbb{A}^+} \frac{|f_\theta|^2}{|z|^2(|z|^2+\lambda)J(z,f)}\right)^{1/2} \left(\iint_{\mathbb{A}} \frac{J(z,f)}{|f(z)|^2}\right)^{1/2}$$

According to (25) the last integral is bounded by $\mathrm{Mod}(\mathbb{A}')$. On the other hand,

$$2\pi \int_r^R \frac{\mathrm{d}\rho}{\sqrt{\rho^2+\lambda}} = 2\pi \log \frac{R+\sqrt{R^2+\lambda}}{r+\sqrt{r^2+\lambda}}$$

Hence we have the inequality (37). Finally, for the equality to hold there, in complete analogy with the proof of Lemma 3, we see that it occurs if and only if both (38) and (39) are true. □

6. Proof of Theorem 1

The idea in proving Theorem 1 is to find combinations of invariant integrals involving $|z|^{-2}\mathbb{K}$, and then to apply the integral estimates of the previous section. In fact, using (15) and (16) we may write

$$\frac{\mathbb{K}(z,f)}{|z|^2} = \frac{|f_\rho|^2}{2|z|^2 J(z,f)} + \frac{|f_\theta|^2}{2|z|^4 J(z,f)} \tag{40}$$

on the set $\mathbb{A}^+ = \{z \in \mathbb{A} : J(z,f) > 0\}$. As we observed before, outside $\mathbb{A}^+$ we have $f_\rho = f_\theta = 0$. Hence

$$2\iint_{\mathbb{A}} \frac{\mathbb{K}(z,f)}{|z|^2} \geqq \iint_{\mathbb{A}^+} \frac{|f_\rho|^2}{|z|^2 J(z,f)} + \iint_{\mathbb{A}^+} \frac{|f_\theta|^2}{|z|^4 J(z,f)} \tag{41}$$

For the first integral on the right-hand side we use Lemma 3 while for the second we apply Lemma 4, with the parameter $\lambda = 0$. Consequently,

$$2\iint_{\mathbb{A}} \frac{\mathbb{K}(z,f)}{|z|^2} \geqq \mathrm{Mod}(\mathbb{A}') + \frac{\mathrm{Mod}(\mathbb{A})^2}{\mathrm{Mod}(\mathbb{A}')} \tag{42}$$

This is precisely the desired estimate (4). Furthermore, it is a quick calculation to verify that the equality occurs for the power function

$$f^{\alpha}(z) = \frac{r'}{r^{\alpha}}|z|^{\alpha-1}z, \quad \alpha = \frac{\mathrm{Mod}(\mathbb{A}')}{\mathrm{Mod}(\mathbb{A})}$$

As for the uniqueness of the extremal mappings, if for $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$ the equality holds in the lower bound (42), then by the above proof f satisfies (33) and (34) and (38) and (39), with $\lambda = 0$. The combination of these identities reads simply

$$f_{\theta} = i\,k\,f \quad \text{and} \quad \rho\, f_{\rho} = \alpha\, f \tag{43}$$

where α and k are positive constants.

If we integrate f_{θ}/f over the circle $|z| = \rho$ in $\mathbb{A}$ we find that $k = 1$, see (26). Thus the general solution to the system of PDE's (43) is

$$f(z) = Cz|z|^{\alpha-1},$$

which is unique up to the rotations of the annuli as claimed. The proof of Theorem 1 is complete. □

7. Proof of Theorem 2

The proof of Theorem 2 is similar to that of Theorem 1, but now we make use of Lemma 2 and integrate the pointwise bounds it provides for the distortion function.

We first consider the case $-r^2 \leqq \lambda < 0$. Since $\mathbb{K}(z, f) \geqq 1$ we can write

$$\begin{aligned}\iint_{\mathbb{A}} \mathbb{K}(z, f) &\geqq \iint_{\mathbb{A}^+} \mathbb{K}(z, f) + \iint_{\mathbb{A}\setminus\mathbb{A}^+} \frac{\sqrt{|z|^2+\lambda}}{|z|} \\ &\geqq \frac{-\lambda}{2} \iint_{\mathbb{A}^+} \frac{|f_{\rho}(z)|^2}{|z|^2 J(z, f)} + \iint_{\mathbb{A}} \frac{\sqrt{|z|^2+\lambda}}{|z|}\end{aligned}$$

Here we may appeal to Lemma 3 for the first integral and compute the second explicitly. This argument gives the first part of

Lemma 5. *Suppose* $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$ *and* $-r^2 \leqq \lambda < 0$. *Then*

$$\begin{aligned}\iint_{\mathbb{A}} \mathbb{K}(z, f) \geqq \frac{-\lambda}{2}\mathrm{Mod}(\mathbb{A}') &+ \pi\left[R\sqrt{R^2+\lambda} - r\sqrt{r^2+\lambda}\right] \\ &+\pi\lambda \log\frac{R+\sqrt{R^2+\lambda}}{r+\sqrt{r^2+\lambda}}\end{aligned}$$

Furthermore, the equality occurs in this estimate if and only if

$$f(z) = C\frac{z}{|z|}\left(|z| + \sqrt{|z|^2+\lambda}\right)$$

where C *is any non-zero complex number.*

Proof. It remains to prove the uniqueness part of the claim. For the equality in the above integral estimate we must have $\mathbb{A} = \mathbb{A}^+$ and $f_\theta(z) = i\sqrt{|z|^2 + \lambda}\; f_\rho(z)$. In addition, since we used Lemma 3, it is necessary to add the conditions (33) and (34). Together these requirements reduce to the simple necessary and sufficient condition

$$\frac{f_\theta}{f} = i\,k \quad \text{and} \quad \frac{f_\rho}{f} = \frac{k}{\sqrt{\rho^2 + \lambda}} \tag{44}$$

for some constant $k > 0$. Integrating over a circle gives $k = 1$. Thus the general solution to the system of PDE's (44) is

$$f(z) = C\frac{z}{|z|}\left(|z| + \sqrt{|z|^2 + \lambda}\right)$$

where C is any non-zero complex number. □

Next consider the case $\lambda > 0$ and proceed by using the lower bound (28) at Lemma 2, obtaining

$$\iint_{\mathbb{A}} \mathbb{K}(z, f) \geqq \frac{\lambda}{2}\iint_{\mathbb{A}^+} \frac{|f_\theta(z)|^2}{|z|^2(|z|^2 + \lambda)\, J(z, f)} + \iint_{\mathbb{A}} \frac{|z|}{\sqrt{|z|^2 + \lambda}} \tag{45}$$

In analogy to Lemma 5 one obtains

Lemma 6. *For every* $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$ *and for every* $\lambda > 0$ *we have*

$$\iint_{\mathbb{A}} \mathbb{K}(z, f) \geqq \frac{2\lambda\pi^2}{\operatorname{Mod}(\mathbb{A}')}\log^2\frac{R + \sqrt{R^2 + \lambda}}{r + \sqrt{r^2 + \lambda}}$$
$$+\pi\left[R\sqrt{R^2 + \lambda} - r\sqrt{r^2 + \lambda}\right] - \pi\lambda\log\frac{R + \sqrt{R^2 + \lambda}}{r + \sqrt{r^2 + \lambda}}$$

The equality holds if and only if

$$f(z) = C\frac{z}{|z|}\left(|z| + \sqrt{|z|^2 + \lambda}\right) \tag{46}$$

where C *is any non-zero complex number.*

Proof. We apply Lemma 4 to the first integral in (45) and compute the second integral explicitly. This proves the lower bound. For the equality, as before, we must have $\mathbb{A} = \mathbb{A}^+$ and $f_\theta(z) = i\sqrt{|z|^2 + \lambda}\; f_\rho(z)$. Further, this time we must add the conditions (38) and (39). These requirements reduce to the same set of equations as at (44). Thus the equality occurs if and only if f takes the form (46). □

We are now ready for the proof of Theorem 2. Assuming that the target annulus $\mathbb{A}' = \{z : r' < |z| < R'\}$ satisfies the relative Nitsche bounds (5), we found at (6) a unique parameter $\omega \geqq -r^2$ such that

$$r' = r + \sqrt{r^2 + \omega} < R' = R + \sqrt{R^2 + \omega}$$

We will then use the above integral estimates with the special choice $\lambda = \omega$:

If $-r^2 \leqq \omega < 0$ then the lower bound at Lemma 5 reads as

$$\iint_{\mathbb{A}} \mathbb{K}(z, f) \geqq \pi \left(R\sqrt{R^2+\omega} - r\sqrt{r^2+\omega} \right) \tag{47}$$

For the case $\omega > 0$ we use the lower bound at Lemma 6. However, a manipulation shows that the lower bound also now attains precisely the same form (47). In other words, we have shown that under the Nitsche bound (5)

$$\iint_{\mathbb{A}} \mathbb{K}(z, f) \geqq \iint_{\mathbb{A}} \mathbb{K}(z, f^{\omega}) = \pi \left(R\sqrt{R^2+\omega} - r\sqrt{r^2+\omega} \right)$$

where

$$f^{\omega}(z) = \frac{z}{|z|} \left(|z| + \sqrt{|z|^2+\omega} \right)$$

is the Nitsche map. Moreover, according to Lemmas 5 and 6 the mappings f^{ω} are the only minimizers, up to rotation of the rings. Hence we have completed the proof of Theorem 2. □

In (10) we indicated a conformally invariant formulation of Theorem 2, that

$$\frac{1}{|\mathbb{A}|} \iint_{\mathbb{A}} \mathbb{K}(z, f)\, \mathrm{d}z \geqq \frac{\tanh(\gamma/\pi)}{\tanh(\sigma/2\pi)} + \frac{\tanh(\sigma/\pi)}{\sinh(\gamma/2\pi)} \tag{48}$$

To see this form of the lower bound in Theorem 2, we may solve for ω in terms of R, r and $a = R'/r'$ from (6),

$$a = \frac{R + \sqrt{R^2+\omega}}{r + \sqrt{r^2+\omega}}, \quad \text{so } \omega = \frac{4a(R-ra)(Ra-r)}{(a^2-1)^2} \tag{49}$$

Then

$$\begin{aligned}
&\frac{R\sqrt{R^2+\omega} - r\sqrt{r^2+\omega}}{R^2-r^2} \\
&= \frac{R\sqrt{R^2 + \frac{4a(R-ra)(Ra-r)}{(a^2-1)^2}} - r\sqrt{r^2 + \frac{4a(R-ra)(Ra-r)}{(a^2-1)^2}}}{R^2-r^2} \\
&= \frac{R\sqrt{R^2(a^2-1)^2 + 4a(R-ra)(Ra-r)} - r\sqrt{r^2(a^2-1)^2 + 4a(R-ra)(Ra-r)}}{(a^2-1)(R^2-r^2)} \\
&= \frac{R^2|1 - 2ra/R + a^2| - r^2|1 - 2Ra/r + a^2|}{(a^2-1)(R^2-r^2)}
\end{aligned}$$

Now $1-2ra/R+a^2 \geqq 0$, and the Nitsche bound at (5) gives $1-2Ra/r+a^2 \leqq 0$ so

$$\begin{aligned}\frac{R\sqrt{R^2+\omega}-r\sqrt{r^2+\omega}}{R^2-r^2} &= \frac{R^2(1-2ra/R+a^2)+r^2(1-2Ra/r+a^2)}{(a^2-1)(R^2-r^2)}\\ &= \frac{(a-1)^2(R^2+r^2)+2a(R-r)^2}{(a^2-1)(R^2-r^2)}\\ &= \frac{(a-1)(R^2+r^2)}{(a+1)(R^2-r^2)}+\frac{2(R-r)}{(a-1/a)(R+r)}\\ &= \frac{(a-1)(b^2+1)}{(a+1)(b^2-1)}+\frac{2(b-1)}{(a-1/a)(b+1)}\end{aligned}$$

with $b = R/r$. Then the result follows from the definition of the moduli at (3).

8. Beyond the Nitsche bound

We are now given two round annuli $\mathbb{A}$ and $\mathbb{A}'$ with inner and outer radii r, R and r', R', respectively. Moreover, we assume that the relative fatness condition (14) is satisfied.

Here it will be convenient to use the normalization

$$r = 1, \quad r' < 1 < R' = R + \sqrt{R^2-1} \tag{50}$$

Suppose $f : \mathbb{A} \to \mathbb{A}'$ is a homeomorphism of finite distortion. Then we may use Lemma 5, this time with the choice $\lambda = -1$. As a result we get

$$2\iint_{\mathbb{A}} \mathbb{K}(z, f) \geqq \operatorname{Mod}(\mathbb{A}') + 2\pi R\sqrt{R^2-1} - 2\pi \log\left(R+\sqrt{R^2-1}\right) \tag{51}$$

or equivalently,

$$\iint_{\mathbb{A}} \mathbb{K}(z, f) \geqq \iint_{\mathbb{A}} \mathbb{K}(z, f^{\#}) + \pi \log \frac{1}{r'} \tag{52}$$

where

$$f^{\#}(z) = \frac{z}{|z|}\left(|z| + \sqrt{|z|^2-1}\right)$$

is the critical inverse Nitsche map from (11).

In the next section we show that the lower bound (51) and (52) is optimal: we will construct a minimizing sequence which attains the bound in the limit. However, no single homeomorphism $f : \mathbb{A} \to \mathbb{A}'$ of finite distortion can achieve the bound. Otherwise, if $f \in \mathcal{F}(\mathbb{A}, \mathbb{A}')$ would satisfy (51) with an equality, then Lemma 5 would show that such an f has to be of the form

$$f(z) = C\frac{z}{|z|}\left(|z| + \sqrt{|z|^2-1}\right) = Cf^{\#}(z)$$

for some rotation C. But then $\mathbb{A}' = f^{\#}(\mathbb{A})$, which is in contradiction with the fatness condition (14). Hence Theorem 3 follows once we have constructed the minimizing sequence.

We will complete this section with a remark which suggests how to construct a minimizing sequence: The target annulus $\mathbb{A}'$ splits into a union of two annuli,

$$\mathbb{A}'_0 = \left\{\zeta : r' < |\zeta| < 1\right\} \text{ and } \mathbb{A}'_{\#} = \left\{\zeta : 1 < |\zeta| < R'\right\}$$

where $\mathbb{A}'_{\#}$ is the critical Nitsche annulus, with $f^{\#}(\mathbb{A}) = \mathbb{A}'_{\#}$. The additional term in (52) is no other than the infimum of the integrals

$$\iint_{\mathcal{A}} \mathbb{K}(z, \phi)\, \mathrm{d}z$$

over all annuli $\mathcal{A} = \{z : 1 < |z| < \sigma\}$ and over all $\phi \in \mathcal{F}(\mathcal{A}, \mathbb{A}'_0)$ with the fixed target $\mathbb{A}'_0 = \{\zeta : r' < |\zeta| < 1\}$. Indeed, by Theorem 1 we see that

$$\iint_{\mathcal{A}} \frac{\mathbb{K}(z, \phi)}{|z|^2} = \left(\frac{\operatorname{Mod}(\mathbb{A}'_0)}{2\operatorname{Mod}(\mathcal{A})} + \frac{\operatorname{Mod}(\mathcal{A})}{2\operatorname{Mod}(\mathbb{A}'_0)}\right)\operatorname{Mod}(\mathcal{A}) > \frac{\operatorname{Mod}(\mathbb{A}'_0)}{2} = \pi \log \frac{1}{r'}$$

This suggests that in order to reach the right hand side of (52) we must shrink $\mathcal{A}$ to the unit circle and, by Theorem 1, for ϕ we may take the power functions.

9. The minimizing sequence and its BV-limit

The Sobolev space $W^{1,1}(\mathbb{A}, \mathbb{A}')$ appears to be the natural domain of definition for homeomorphisms $f : \mathbb{A} \to \mathbb{A}'$ of finite distortion. However, when studying the extremal problem beyond the Nitsche bound we found that the lack of weak compactness in $W^{1,1}(\mathbb{A}, \mathbb{A}')$ prevents the minimizing sequences from converging to a mapping with a minimal integral mean distortion. Therefore we should expect that the weak star closure of this Sobolev space plays a role here.

Let us briefly recall the space $BV(\Omega, \mathbb{C})$ of functions $f : \Omega \to \mathbb{C}$ of bounded variation; the interested reader may wish to look at [1] for a complete BV-theory. The Banach space $BV(\Omega, \mathbb{C})$ consists of those integrable functions defined in a domain $\Omega \subset \mathbb{C}$ whose distributional derivatives f_z and $f_{\bar{z}}$ are complex Radon measures, bounded linear functionals on $C_0(\Omega)$. The BV-norm is given by

$$\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + [f_z]_\Omega + [f_{\bar{z}}]_\Omega$$

where the latter two symbols denote the total variation of the measures f_z and $f_{\bar{z}}$, respectively.

Every bounded sequence $\{f^n\}_{n=1}^\infty$ in $BV(\Omega, \mathbb{C})$ contains a subsequence $\{f^{n_j}\}$ converging to an $f \in BV(\Omega, \mathbb{C})$ in the weak star topology. This means that $f^{n_j} \to f$ in $L^1(\Omega)$ and for every test function $\phi \in C_0(\Omega)$ we have

$$\langle f_z^{n_j}, \phi\rangle \to \langle f, \phi\rangle, \quad \langle f_{\bar{z}}^{n_j}, \phi\rangle \to \langle f_{\bar{z}}, \phi\rangle$$

The angular brackets here stand for the duality action of measures on $C_0(\Omega)$.

Let us now return to the problem of minimizing the integral mean distortion

$$\inf \left\{ \frac{1}{|\mathbb{A}|} \iint_{\mathbb{A}} \mathbb{K}(z, f)\, \mathrm{d}z \; : \; f \in \mathcal{F}(\mathbb{A}, \mathbb{A}') \right\} \tag{53}$$

in the case of a fat target (14). Suppose $\{f^n\} \subset \mathcal{F}(\mathbb{A}, \mathbb{A}')$ is a minimizing sequence. By Hölder's inequality

$$\iint_{\mathbb{A}} |Df^n| \leqq \iint_{\mathbb{A}} \sqrt{\mathbb{K}(z, f^n) J(z, f^n)} \leqq \left[\iint_{\mathbb{A}} \mathbb{K}(z, f^n)\right]^{\frac{1}{2}} \left[\iint_{\mathbb{A}} J(z, f^n)\right]^{\frac{1}{2}} \tag{54}$$

which shows that $\{f^n\}$ is a bounded sequence in $W^{1,1}(\mathbb{A}, \mathbb{A}')$. Thus $\{f^n\}$ contains a subsequence, again denoted by $\{f^n\}$, converging in the weak star topology to a mapping f of bounded variation in $\mathbb{A}$.

We shall discuss this phenomenon in more detail for the mappings $f^n : \mathbb{C} \to \mathbb{C}$ defined by the rule

$$f^n(z) = \begin{cases} \frac{z}{|z|}\left(|z| + \sqrt{|z|^2 - 1}\right) & \text{for } |z| \geqq r_n \\ r' z |z|^{n-1} & \text{for } 1 \leqq |z| \leqq r_n \\ r' z & \text{for } |z| \leqq 1 \end{cases} \tag{55}$$

Here $0 < r' < 1$ is fixed and the radii $r_n > 1$ are uniquely determined from the equation

$$r_n + \sqrt{r_n^2 - 1} = r' \, (r_n)^n, \tag{56}$$

Elementary analysis shows that

$$\lim_{n\to\infty} (r_n)^n = \frac{1}{r'} \quad \text{and} \quad \lim_{n\to\infty} n(r_n - 1) = \log \frac{1}{r'} \tag{57}$$

The pointwise limit of $\{f^n(z)\}$ exhibits a discontinuity along the unit circle,

$$f^n(z) \;\to\; f(z) = \begin{cases} \frac{z}{|z|}\left(|z| + \sqrt{|z|^2 - 1}\right) & \text{for } |z| > 1 \\ r' z & \text{for } |z| \leqq 1 \end{cases} \tag{58}$$

Outside the unit disk f is the critical inverse Nitsche map.

Computing the derivatives of f^n one has $f^n_z = 1 + \frac{1}{2}\left(\frac{|z|}{\sqrt{|z|^2-1}} + \frac{\sqrt{|z|^2-1}}{|z|}\right)$ for $|z| > r_n$, $f^n_z = \frac{n+1}{2} r' |z|^{n-1}$ for $1 < |z| < r_n$ and $f^n_z = r'$ for $|z| < 1$ and also $f^n_{\bar{z}} = \frac{z^2}{2|z|^3\sqrt{|z|^2-1}}$ for $|z| > r_n$, $f^n_{\bar{z}} = \frac{n-1}{2} r' |z|^{n-3} z^2$ for $1 < |z| < r_n$, and $f^n_{\bar{z}} = 0$ for $|z| < 1$. These formulas show that the sequence $\{f^n\}$ is bounded in $W^{1,1}(\Omega, \mathbb{C})$ for every bounded domain $\Omega \subset \mathbb{C}$. Thus f is its weak star limit.

Next we examine the singular part of the measures f_z and $f_{\bar{z}}$ using (57).

$$f_z^{\text{sing}} = \pi(1 - r')\, \mathrm{d}\nu, \quad f_{\bar{z}}^{\text{sing}} = \pi(1 - r') \frac{z}{\bar{z}} \mathrm{d}\nu \tag{59}$$

where $\mathrm{d}\nu$ is unit mass uniformly distributed on the unit circle.

Next, we fix a radius $R > 1$ and define $R' = R + \sqrt{R^2 - 1}$. Each f^n maps the annulus $\mathbb{A} = \{z : 1 < |z| < R\}$ homeomorphically onto the annulus

$$\mathbb{A}' = \left\{\zeta : r' < |\zeta| < R'\right\}$$

From the above formulae on the derivatives we may easily determine the distortion function of f^n. This turns out as

$$\mathbb{K}(z, f^n) = \begin{cases} \frac{1}{2}\left(\frac{|z|}{\sqrt{|z|^2-1}} + \frac{\sqrt{|z|^2-1}}{|z|}\right) & \text{for } |z| > r_n \\ \frac{1}{2}(n + 1/n) & \text{for } 1 < |z| < r_n \\ 1 & \text{for } |z| < 1 \end{cases} \tag{60}$$

The limit of the integral means of the distortion functions is now easy to determine,

$$\begin{aligned} \lim_{n\to\infty} \iint_{\mathbb{A}} \mathbb{K}(z, f^n)\, \mathrm{d}z &= \iint_{\mathbb{A}} \mathbb{K}(z, f)\, \mathrm{d}z + \lim_{n\to\infty} \iint_{1<|z|<r_n} \mathbb{K}(z, f^n)\, \mathrm{d}z \\ &= \iint_{\mathbb{A}} \mathbb{K}(z, f)\, \mathrm{d}z + \pi \log \frac{1}{r'} \end{aligned}$$

where the last equality follows from formula (57). The right hand side coincides with that in inequality (52), as $f(z) = f^{\#}(z)$ for all $z \in \mathbb{A}$. In other words, the sequence $\{f^n\}$ is a minimizing sequence for the problem (53).

10. Correction lemma

In order to apply the above results to energy minimization problems for the inverse mappings and to Corollaries 1–3 we need to establish the following

Lemma 7. *Let $h : \Omega' \to \Omega$ be a homeomorphism of finite Dirichlet energy,*

$$E[h] = \iint_{\Omega} \|Dh(\zeta)\|^2 \mathrm{d}\zeta = \iint_{\Omega} \left(|h_\zeta|^2 + |h_{\bar\zeta}|^2\right) \mathrm{d}\zeta < \infty$$

Then there exists a homeomorphism $\tilde h : \Omega' \to \Omega$ such that

- *$E[\tilde h] \leqq E[h]$*
- *The inverse $\tilde f = \tilde h^{-1}$ belongs to $W^{1,1}(\Omega, \Omega')$ and has finite distortion*
- *We have the identity*

$$\iint_{\Omega} \mathbb{K}(z, \tilde f) \mathrm{d}z = E[\tilde h]$$

Proof. We express Ω as a locally finite countable union of closed convex domains $\overline{\Omega_j} \subset \Omega$, $j = 1, 2, \dots$ with pairwise disjoint interiors. Through the homeomorphism h we have a decomposition

$$\Omega' = \bigcup_{j=1}^{\infty} \overline{\Omega'_j}$$

into a countable union of closed Jordan domains $\overline{\Omega'_j} = h^{-1}(\overline{\Omega_j})$. In each Ω'_j we solve the Dirichlet problem for $h_j \in C(\overline{\Omega'_j})$,

$$\begin{cases} \Delta h_j = 0 & \text{in } \Omega'_j \\ h_j(\zeta) = h(\zeta) & \text{on } \partial\Omega'_j \end{cases}$$

Since h lies in the Royden algebra $C(\overline{\Omega'_j}) \cap W^{1,2}(\Omega'_j)$ and Ω'_j is a Jordan domain, we are reduced via the Riemann mapping to the unit disk and the Poisson integral. This results in the well known energy estimates

$$\iint_{\Omega'_j} \|Dh_j\|^2 \leqq \iint_{\Omega'_j} \|Dh\|^2$$

Next we observe that $h_j : \partial\Omega'_j \to \partial\Omega_j$ is a homeomorphism and the domain Ω_j is convex. Thus by the theorem of Rado–Kneser–Choquet, see [6], each h_j is a homeomorphism of $\overline{\Omega'_j}$ onto $\overline{\Omega_j}$. Hence we may consider $f_j = h_j^{-1} : \overline{\Omega_j} \to \overline{\Omega'_j}$ and define $\tilde{f} : \Omega \to \Omega'$ and $\tilde{h} : \Omega' \to \Omega$ by the rules

$$\tilde{f}(z) = f_j(z) \quad \text{for } z \in \overline{\Omega_j} \tag{61}$$

$$\tilde{h}(\zeta) = h_j(\zeta) \quad \text{for } z \in \overline{\Omega'_j} \tag{62}$$

It follows that $\tilde{f}$ and $\tilde{h}$ are well defined homeomorphisms inverse to each other.

We then prove that $\tilde{f} \in W^{1,1}(\Omega, \Omega')$. For this note that we may argue as in (54) to locally estimate the L^1-norms of the derivative,

$$\iint_{\Omega_j} \|D\tilde{f}\| \leqq \left[\iint_{\Omega_j} \mathbb{K}(z, f_j)\right]^{1/2} \left[\iint_{\Omega_j} J(z, f_j)\right]^{1/2}$$

Summing up and using the Schwarz inequality yields

$$\iint_{\Omega} \|D\tilde{f}\| \leqq \left[\sum_{j=1}^{\infty} \iint_{\Omega_j} \mathbb{K}(z, \tilde{f})\right]^{1/2} \left[\sum_{j=1}^{\infty} \iint_{\Omega_j} J(z, \tilde{f})\right]^{1/2}$$

On the other hand, since f_j and h_j are smooth we may change variables as in [4] to obtain

$$\iint_{\Omega_j} \mathbb{K}(z, f_j)\mathrm{d}z = \iint_{\Omega'_j} \|Dh_j\|^2 \leqq \iint_{\Omega'_j} \|Dh\|^2$$

Thus

$$\iint_{\Omega} \|Df_j\| \leqq \left[\sum_{j=1}^{\infty} \iint_{\Omega_j} \|Dh\|^2\right]^{1/2} \left[\sum_{j=1}^{\infty} |\Omega_j'|\right]^{1/2} \leqq \sqrt{E[h]}\sqrt{|\Omega'|}$$

Since $\tilde{f}$ is continuous on Ω the above proves $\tilde{f} \in W^{1,1}(\Omega, \Omega')$. This map has integrable distortion since

$$\iint_{\Omega} \mathbb{K}(z, \tilde{f}) = \sum_{j=1}^{\infty} \iint_{\Omega_j} \mathbb{K}(z, f_j) \leqq \iint_{\Omega'} \|Dh\|^2$$

Finally we appeal to the recent result of HENCL–KOSKELA–ONNINEN [12] which tells us that a homeomorphism $\tilde{f} : \Omega \to \Omega'$ of integrable distortion in the Sobolev class $W^{1,1}(\Omega, \Omega')$ has its inverse $\tilde{h} \in W^{1,2}(\Omega', \Omega)$ and satisfies the identity

$$\iint_{\Omega} \mathbb{K}(z, \tilde{f})\, \mathrm{d}z = \iint_{\Omega'} \|D\tilde{h}\|^2$$

This completes the proof of the Lemma. □

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