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Ill-posedness of the Hydrostatic Euler and Navier–Stokes Equations

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Abstract

We prove that the linearization of the hydrostatic Euler equations at certain parallel shear flows is ill-posed. The result also extends to the hydrostatic Navier–Stokes equations with a small viscosity.

1. Introduction

The hydrostatic approximation arises naturally when studying flows where the depth of the region of interest is small compared to horizontal dimensions. Examples include atmospheric and oceanic flows, boundary layers, and blood flow. Nevertheless, the equations of fluid motion resulting from this approximation, without regularizing "viscosity" terms, do not appear to be well analyzed. In Lions' book [14], this is posed as a challenge to analysts.

In a widely cited paper, Oliger and Sundström [16] showed that the imposition of open boundary conditions on the hydrostatic Euler equations is problematic. Without open boundaries, Grenier [9] and Brenier [2,3] proved results on well-posedness as well as convergence of solutions of the full Euler equations to the hydrostatic limit, but only under the assumption that the profile of the horizontal velocity is convex. They note the connection between this hypothesis and Rayleigh's stability criterion. Brenier [2] also notes that convexity cannot be expected to persist for all times. Grenier [9,10] gives examples where solutions of the Euler and Navier–Stokes equations do not converge to the hydrostatic limit, but these results do not show ill-posedness of the hydrostatic equations themselves. Bresch et al. [4] prove a well-posedness result for the hydrostatic Navier–Stokes equations, but with a viscosity that is isotropic, i.e. equal in the horizontal and vertical directions. This assumption is asymptotically inconsistent with the derivation of the hydrostatic approximation, as we shall see below. With proper asymptotic scalings, only the second derivative in the vertical direction remains in the viscosity term. Indeed, the viscosity in [4] is intended to represent not the viscosity of the liquid, but rather an eddy viscosity.

In this note, we shall give an elementary proof demonstrating the existence of velocity profiles such that the linearization of the hydrostatic Euler and Navier–Stokes equations is ill-posed. This ill-posedness is in essence not new, since it is equivalent to a known long wave instability of the Euler equations. The implication of ill-posedness for the hydrostatic equations seems to have gone unnoticed in the literature. In particular, velocity profiles leading to ill-posedness exist in any neighborhood of the rest state.

The result shown here seems to indicate that the quest for a general existence theorem for the hydrostatic Euler equations is basically hopeless, unless the equations are regularized by "viscosity" terms. This situation is rather discomforting, since the viscosity that is used in the study of atmospheric and oceanic flows is of uncertain size and physical origin (it does not represent the viscosity of the fluid, but rather represents a crude way to account for the effects of turbulence on unresolved scales, interaction with unresolved topography, etc.).

It is interesting that HONG and HUNTER [12] do not find any analogue of the ill-posedness reported here, even though the only difference in their equations is that they prescribe the pressure rather than the vertical velocity at the upper boundary! Therefore the Hadamard instability discussed here depends on boundary conditions; see also the well-posedness results of OLEINIK [15] and XIN and ZHANG [20] for the viscous case.

The ill-posedness result is based on finding unstable eigenvalues; in contrast, there are no eigenvalues for the case with one free surface studied in [12]. Below, we shall consider the case of two free surfaces. In this case, the linearization at any nonconstant parallel flow is ill-posed. For the full Euler equations, this implies a long wave instability of liquid sheets which have an internally nonuniform velocity. We shall show that an analogous instability also exists in axisymmetric jets. The dependence of well-posedness on boundary conditions is particularly unsettling for the study of problems where the real world system actually has no well-defined boundary, such as oceanic or atmospheric currents.

2. The hydrostatic Euler and Navier–Stokes equations

We consider the Euler or Navier–Stokes equations in the strip $-\infty < x < \infty$, $-\varepsilon < y < \varepsilon$. The Euler equations have the form

$$\begin{aligned} u_t + uu_x + vu_y &= -p_x, \\ v_t + uv_x + vv_y &= -p_y, \\ u_x + v_y &= 0. \end{aligned} \tag{1}$$

At the boundaries of the strip, we assume nonpenetration, that is $v = 0$.

The hydrostatic approximation arises in "thin" domains, that is if ε is small relative to horizontal lengths relevant to the flow. It is then natural to rescale y and

v with ε. The resulting equations are

$$\begin{aligned} u_t + uu_x + vu_y &= -p_x, \\ \varepsilon^2(v_t + uv_x + vv_y) &= -p_y, \\ u_x + v_y &= 0. \end{aligned} \tag{2}$$

In the hydrostatic limit, ε is set to 0. The pressure p is then a function $P(x)$ of x only, and we have the reduced system

$$\begin{aligned} u_t + uu_x + vu_y &= -P'(x), \\ u_x + v_y &= 0, \end{aligned} \tag{3}$$

posed on the strip $-1 < y < 1$.

We shall also consider an analogous reduction for the Navier–Stokes equations. In this case, we shall assume that the viscosity is also a small parameter, of order ε^2. The Navier–Stokes equations then have the form

$$\begin{aligned} u_t + uu_x + vu_y &= \varepsilon^2(u_{xx} + u_{yy}) - p_x, \\ v_t + uv_x + vv_y &= \varepsilon^2(v_{xx} + v_{yy}) - p_y, \\ u_x + v_y &= 0. \end{aligned} \tag{4}$$

The boundary conditions are $u = v = 0$.

After rescaling v and y with ε and setting $\varepsilon = 0$, we now obtain the reduced system

$$\begin{aligned} u_t + uu_x + vu_y &= u_{yy} - P'(x), \\ u_x + v_y &= 0. \end{aligned} \tag{5}$$

3. Linear stability of parallel shear flow

A solution of the Euler equations is given by $u = U(y)$, $v = p = 0$, where $U(y)$ is any function. We linearize the hydrostatic Euler equations at this solution and end up with the linearized problem

$$\begin{aligned} u_t + U(y)u_x + U'(y)v &= -P'(x), \\ u_x + v_y &= 0. \end{aligned} \tag{6}$$

We satisfy the incompressibility condition by introducing a streamfunction,

$$u = \psi_y, \quad v = -\psi_x, \tag{7}$$

and we differentiate the first equation of (7) with respect to y to eliminate the unknown pressure. The result is

$$\psi_{yyt} + U(y)\psi_{yyx} - U''(y)\psi_x = 0, \tag{8}$$

subject to the boundary conditions

$$\psi(x, -1, t) = \psi(x, 1, t) = 0. \tag{9}$$

The following theorem shows that, for certain velocity profiles $U(y)$ the linearized equations are ill-posed in the sense of Hadamard.

Theorem 1. *The Equation* (8)*, with the boundary conditions* (9)*, has solutions of the form*

$$\psi(x, y, t) = \chi(y)\exp(i\alpha(x - ct)), \tag{10}$$

where c is given by the equation

$$\int_{-1}^{1} (U(y) - c)^{-2}\,\mathrm{d}y = 0. \tag{11}$$

If the roots of (11) *are complex, this implies Hadamard instability, since the growth rate is α Im c, and c is independent of α.*

For the proof, we insert (10) in (8). The result is the hydrostatic Orr–Sommerfeld equation

$$(U(y) - c)\chi''(y) - U''(y)\chi(y) = 0, \tag{12}$$

with the boundary conditions $\chi(-1) = \chi(1) = 0$. We note that this equation is independent of α.

We see by inspection that $\chi_1(y) = U(y) - c$ is a solution of the differential equation for any value of c. We can then use reduction of order to find the general solution. If we set $\chi(y) = q(y)(U(y) - c)$, we find

$$(U(y) - c)q''(y) + 2U'(y)q'(y) = 0. \tag{13}$$

This has the general solution

$$q(y) = k_1 + k_2 \int (U(y) - c)^{-2}\,\mathrm{d}y. \tag{14}$$

We note that this solution is simply the leading term in HEISENBERG's [11] long wave expansion for the solution of the Orr–Sommerfeld equation. If we impose the boundary condition $\chi(-1) = 0$, we find that

$$\chi(y) = K(U(y) - c)\int_{-1}^{y} (U(z) - c)^{-2}\,\mathrm{d}z \tag{15}$$

for some constant K. The other boundary condition $\chi(1) = 0$ then leads to the eigenvalue relation (11).

One must now show that the profiles leading to complex roots of (11) actually exist. It was pointed out by HEISENBERG [11] and later by ROSENBLUTH and SIMON [18] that the existence of nonreal solutions of (11) is sufficient for long wave instability of the Euler equations; however, they did not exhibit specific profiles for which this condition is satisfied. We shall give a very simple argument showing the existence of a class of such profiles. Using different arguments, researchers have shown in the literature that specific profiles lead to complex eigenvalues, for example the profile $U(y) = \tanh(y/L)$ for large enough L [5], and piecewise linear profiles (see [7, p. 147], [17, p. 388]).

Lemma 1. *Assume that $U(y)$ is an odd continuous function and that $U(y)^{-2}$ is integrable. Then there are purely imaginary roots of* (11).

Note that this assumption prevents U from being smooth at 0; we shall address this point below. We look for an imaginary eigenvalue $c = i\beta$. Equation (11) then becomes

$$\int_{-1}^{1} \frac{1}{U(y)^2 - \beta^2 + 2i\beta U(y)}\,\mathrm{d}y = 0. \tag{16}$$

The imaginary part of this vanishes as a result of symmetry. The real part is negative if β is large, but by the Lebesgue dominated convergence theorem it converges to the positive limit

$$\int_{-1}^{1} U(y)^{-2}\,\mathrm{d}y \tag{17}$$

as $\beta \to 0$. Consequently, there exists a nonzero β such that $c = \pm i\beta$ is a solution of (11).

Let $U(y)$ be a given function as specified in the lemma above. The requirement that U^{-2} is integrable prevents U from being smooth at the origin. We can, however, find a sequence U_n such that U_n is smooth and U_n converges uniformly to U. Let now β_0 be such that

$$\mathrm{Re}\left[\int_{-1}^{1} \frac{1}{U(y)^2 - \beta_0^2 + 2i\beta_0 U(y)}\,\mathrm{d}y\right] > 0. \tag{18}$$

Then it follows that also

$$\mathrm{Re}\left[\int_{-1}^{1} \frac{1}{U_n(y)^2 - \beta_0^2 + 2i\beta_0 U_n(y)}\,\mathrm{d}y\right] > 0. \tag{19}$$

for large enough n. As above, we conclude the existence of a $\beta > \beta_0$ for which

$$\int_{-1}^{1} \frac{1}{U_n(y)^2 - \beta^2 + 2i\beta U(y)}\,\mathrm{d}y = 0. \tag{20}$$

For the viscous case, we obtain the linearized equation

$$\psi_{yyt} + U(y)\psi_{yyx} - U''(y)\psi_x = \psi_{yyyy}, \tag{21}$$

and the hydrostatic Orr–Sommerfeld equation becomes

$$(U(y) - c)\chi''(y) - U''(y)\chi(y) = -\frac{i}{\alpha}\chi''''(y). \tag{22}$$

The boundary conditions are $\chi(-1) = \chi'(-1) = \chi(1) = \chi'(1) = 0$. We note that the assumptions on U above allow the no-slip condition $U(-1) = U(1) = 0$ to be satisfied so that the profile U is an admissible initial condition for the Navier–Stokes equation. It is a routine application of matched asymptotics (see, for example [19] for a discussion of the rigorous justification of formal expansions) to show that if the inviscid problem has a purely imaginary eigenvalue $c = i\beta$, where α and β have the same sign, then the viscous problem has an eigenvalue close to $i\beta$ if α is large. Hence the hydrostatic equation remains ill-posed if viscosity is included. We note that E and Engquist [8] prove the nonexistence of global smooth solutions for the hydrostatic Navier–Stokes equations. The linear ill-posedness would suggest that for general initial data smooth solutions do not exist even locally.

4. Remarks on the nonlinear problem

The ill-posedness discussed in the preceding section concerns the linearization at a given shear flow profile. It is not obvious what this implies for the nonlinear problem. One simple consequence is that the nonlinear problem cannot have a solution which depends smoothly on the initial data.

We shall consider the nonlinear hydrostatic Euler equations (4) with periodic boundary conditions in the x direction: $u(x+2\pi, y, t) = u(x, y, t)$ and the flow rate constraint

$$\int_{-1}^{1} u(x, y, t)\, \mathrm{d}y = 0. \tag{23}$$

We formulate the equations in terms of the stream function, which leads to

$$\psi_{yyt} + \psi_y \psi_{yyx} - \psi_x \psi_{yyy} = 0. \tag{24}$$

We impose the initial condition

$$\psi(x, y, 0) = \Psi(y) + \varepsilon \sum_{n=1}^{\infty} \exp(-\sqrt{n}) Re[\exp(inx)\chi(y)], \tag{25}$$

where

$$\Psi(y) = \int_{-1}^{y} U(\xi)\, d\xi, \tag{26}$$

$U(y)$ is a profile as discussed in the previous section, and $\chi(y)$ is the eigenfunction corresponding to the unstable eigenvalue. If $U(y)$ is C^∞, then $\psi(x, y, 0)$ is a C^∞ function.

Assume that this initial value problem has a solution, which we represent as $\psi(x, y, t) = \Psi(y) + \psi^\varepsilon(x, y, t)$. We put (24) into the form

$$\psi_t^\varepsilon = L\psi^\varepsilon + N(\psi^\varepsilon), \tag{27}$$

where

$$\begin{aligned} L\psi^\varepsilon &= -D^{-1}(\Psi' \psi^\varepsilon_{yyx} - \Psi''' \psi^\varepsilon_x), \\ N(\psi^\varepsilon) &= -D^{-1}(\psi^\varepsilon_y \psi^\varepsilon_{yyx} - \psi^\varepsilon_x \psi^\varepsilon_{yyy}). \end{aligned} \tag{28}$$

Here D^{-1} stands for the inverse of $\partial^2/\partial y^2$ with Dirichlet boundary conditions.

With these notations, we have the following result which rules a smooth dependence of ψ^ε on ε.

Theorem 2. *Let $U(y)$ be one of the Hadamard unstable profiles discussed in the preceding section. Assume that X,Y and Z are Banach spaces of periodic functions on $\mathbb{R} \times (-1, 1)$, and that L maps Y continuously into X and N satisfies a bound of the form*

$$\|N(\psi)\|_X \leqq \|\psi\|_Z \|\psi\|_Y. \tag{29}$$

Then it is not possible that ψ^ε is a differentiable function of ε as an element of $L^p((0, \tau), Y)$ for some $\tau > 0$ and that ψ^ε tends to zero in $L^q((0, \tau), Z)$, where $1/p + 1/q = 1$.

For the proof, let $\phi^\varepsilon = \psi^\varepsilon/\varepsilon$, and $\chi = \lim_{\varepsilon\to 0}\phi^\varepsilon$. We find that

$$\phi_t^\varepsilon = L\phi^\varepsilon + \frac{1}{\varepsilon}N(\psi^\varepsilon), \tag{30}$$

and

$$\frac{1}{\varepsilon}\|N(\psi^\varepsilon)\|_X \leqq \|\psi^\varepsilon\|_Z\|\phi^\varepsilon\|_Y. \tag{31}$$

Letting $\varepsilon \to 0$, we therefore find

$$\chi_t = L\chi, \tag{32}$$

in the function space $L^1((0,\tau), X)$. That is, χ is a solution of the linearized problem for a finite interval $(0,\tau)$. This solution, however, does not exist.

We can argue similarly for the hydrostatic Navier–Stokes equations if we augment the equations by a forcing term to compensate for the fact that the profile $U(y)$ is not a solution.

5. The role of boundary conditions

In Hong and Hunter [12], the difference is that the condition $p = 0$ is imposed on one of the boundaries (actually at infinity, but it does not make a difference) rather than $v = 0$. This condition corresponds to a free surface rather than a wall. We shall now consider the case where this condition is imposed on both boundaries. We shall consider the Euler equation in a domain bounded by two free surfaces, $-h_1(x,t) < y < h_2(x,t)$. The evolution of the free surfaces is determined by a kinematic free surface condition, and the pressure is zero at the free surfaces.

In the hydrostatic approximation, h_1 and h_2 get rescaled along with y and v. The pressure is equal to zero in the hydrostatic limit, and we find

$$\begin{aligned} u_t + uu_x + vu_y &= 0,\\ u_x + v_y &= 0. \end{aligned} \tag{33}$$

This system, however, is underdetermined without the imposition of a boundary condition. To resolve this difficulty, we impose the constraint

$$\int_{-h_1}^{h_2} v_t + uv_x + vv_y \,\mathrm{d}y = 0, \tag{34}$$

which arises from integrating the vertical momentum equation and imposing the pressure boundary condition.

We now linearize at a shear flow $u = U(y)$ and flat surfaces $h_1 = h_2 = 1$. With ψ denoting the stream function of the perturbation as before, we find

$$\psi_{yt} + U(y)\psi_{yx} - U'(y)\psi_x = 0, \tag{35}$$

with the constraint

$$\int_{-1}^{1} \psi_{xt} + U(y)\psi_{xx}\,\mathrm{d}y = 0. \tag{36}$$

Theorem 3. *The Equation* (35) *with the constraint* (36) *has solutions of the form* $\psi(x, y, t) = \exp(i\alpha(x - ct))\chi(y)$, *where* c *satisfies*

$$\int_{-1}^{1} (U(y) - c)^2 \, dy. \tag{37}$$

Unless U *is constant, the roots of this equation are always complex.*

The ansatz for ψ yields the equation

$$(U(y) - c)\chi'(y) - U'(y)\chi(y) = 0, \tag{38}$$

with the constraint

$$\int_{-1}^{1} (U(y) - c)\chi(y) \, dy = 0. \tag{39}$$

This yields the eigenfunction $\chi(y) = U(y) - c$ and the eigenvalue relation

$$\int_{-1}^{1} (U(y) - c)^2 \, dy. \tag{40}$$

The Cauchy–Schwarz inequality implies that

$$\left(\int_{-1}^{1} U(y) \, dy\right)^2 \leqq 2 \int_{-1}^{1} U(y)^2 \, dy, \tag{41}$$

with equality only if U is constant. It is an immediate consequence that c must be complex. Hence the hydrostatic equations are ill-posed in the same manner as above, for any non-constant velocity profile.

The eigenvalue relation just obtained is the same which arises in the long wave limit of the full Euler equations.

6. Long wave instability of jets

In the preceding section, we showed that liquid sheets with a nonconstant velocity profile are always unstable. It is natural to expect an analogous instability for an axisymmetric jet. Stability of jets is extensively discussed in the literature, see, for example the monographs of LIN [13] and YARIN [21], but the focus is usually on a jet of uniform velocity which is destabilized by capillarity or by Kelvin–Helmholtz instability at the interface. Little seems to be known about inviscid stability or instability of jets with nonuniform internal velocities. DEBLER and YU [6] analyze inviscid linear stability of jets with nonuniform speed, but they miss the instability found here because they consider only axisymmetric perturbations.

We consider a jet occupying the region

$$r = \sqrt{y^2 + z^2} < \varepsilon h(x, \phi, t), \tag{42}$$

where ϕ is the polar angle in the (y, z)-plane, and ε is small. The Euler equations are

$$\begin{aligned} u_t + uu_x + vu_y + wu_z &= -p_x, \\ v_t + uv_x + vv_y + wv_z &= -p_y, \\ w_t + uw_x + vw_y + ww_z &= -p_z, \\ u_x + v_y + w_z &= 0. \end{aligned} \tag{43}$$

The boundary condition on the surface of the jet is $p = 0$.

In the long wave (hydrostatic) approximation, we scale y, z, v, w with ε and p with ε^2; the only reduction arising from this is the pressure drops out of the axial momentum equation. We linearize at a quadratic velocity profile $U(r) = 1 - r^2$, $r < 1$ and look for disturbances proportional to $\exp(i\alpha(x - ct))$. The linearized equations become

$$\begin{aligned} (U - c)i\alpha u + vU_y + wU_z &= 0, \\ (U - c)i\alpha v &= -p_y, \\ (U - c)i\alpha w &= -p_z, \\ i\alpha u + v_y + w_z &= 0. \end{aligned} \tag{44}$$

We can combine these equations into the single equation

$$\frac{\Delta p}{(U - c)^2} + \nabla p \cdot \nabla \frac{1}{(U - c)^2} = 0. \tag{45}$$

Here the operations ∇ and Δ refer only to the y and z variables.

We look for snakelike perturbations of the form $p(r)\exp(i\phi)$. This leads to the eigenvalue problem

$$p'' + \frac{1}{r}p' - \frac{1}{r^2}p - \frac{2U'}{U - c}p' = 0. \tag{46}$$

The substitution

$$p(r) = rq(r^2/(1 - c)) \tag{47}$$

transforms this equation to

$$\rho(1 - \rho)q''(\rho) + 2q'(\rho) + q(\rho) = 0. \tag{48}$$

This is a hypergeometric equation (see, for example [1]). We thus find the solution

$$p(r) = r\,{}_2F_1\left(-\frac{1}{2} - \frac{\sqrt{5}}{2}, -\frac{1}{2} + \frac{\sqrt{5}}{2}, 2, \frac{r^2}{1 - c}\right). \tag{49}$$

The boundary condition $p(1) = 0$, thus, reduces to finding the zeros of

$${}_2F_1\left(-\frac{1}{2} - \frac{\sqrt{5}}{2}, -\frac{1}{2} + \frac{\sqrt{5}}{2}, 2, z\right). \tag{50}$$

One of the roots is at $z = 2.39779 + 1.35603i$; we therefore have a complex eigenvalue $c = 0.68401 + 0.17870i$.

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