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Two-phase Entropy Solutions of a Forward–Backward Parabolic Equation

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Abstract

This article deals with the Cauchy problem for a forward–backward parabolic equation, which is of interest in physical and biological models. Considering such an equation as the singular limit of an appropriate pseudoparabolic third-order regularization, we consider the framework of entropy solutions, namely weak solutions satisfying an additional entropy inequality inherited by the higher order equation. Moreover, we restrict the attention to two-phase solutions, that is solutions taking values in the intervals where the parabolic equation is well-posed, proving existence and uniqueness of such solutions.

1. Introduction

In this paper we study the forward–backward parabolic equation in one space dimension

$$u_t = (\phi(u))_{xx} \quad \text{in } \mathbb{R} \times (0, T] =: S_T, \tag{1}$$

subject to the initial condition

$$u(x, 0) = u_0(x) \quad \text{for } x \in \mathbb{R}. \tag{2}$$

We suppose that the function ϕ is nonmonotone and piecewise linear, namely (see Fig. 1):

$$\phi(u) = \begin{cases} \phi_-(u) & \text{for} \quad u \leqq b, \\ \phi_0(u) & \text{for} \quad b < u < c, \\ \phi_+(u) & \text{for} \quad u \geqq a, \end{cases}$$

where

$$\phi_\pm(u) := \alpha_\pm u + \beta_\pm, \quad \phi_0(u) := \frac{A(u-b) - B(u-c)}{c-b}.$$

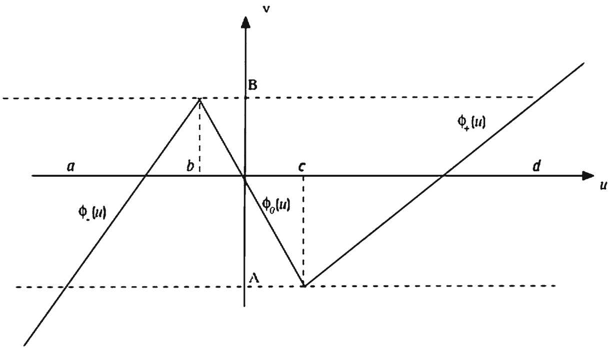

Fig. 1. The function ϕ

Here $-\infty < b < c < \infty$, $\alpha_\pm > 0$, $\beta_\pm \in \mathbb{R}$, and $A := \phi_+(c) < \phi_-(b) =: B$. We also denote by $a \in (-\infty, b)$ and $d \in (c, \infty)$ the roots of the equations $\phi_-(u) = A$, respectively $\phi_+(u) = B$. Since u can take values in the interval (b, c), the Cauchy problem (1), (2) is ill-posed.

Equation (1) with a cubic ϕ arises in the theory of phase transitions. In this context the function u represents the phase field, whose values characterize the difference between the two phases; the half-lines $(-\infty, b)$ and (c, ∞) correspond to stable phases and the interval (b, c) to an unstable phase (for example see [4]). Therefore the sets

$$S^- := \{(u, \phi_-(u)) | u \in (-\infty, b)\} \equiv \{(s^-(v), v) | v \in (-\infty, B)\}$$

and

$$S^+ := \{(u, \phi_+(u)) | u \in (c, \infty)\} \equiv \{(s^-(v), v) | v \in (c, \infty)\}$$

are referred to as the *stable branches*, and $S^0 := \{(u, \phi(u)) | u \in (b, c)\}$ as the *unstable branch* of the graph of ϕ.

With a (nonmonotone) function ϕ of a different shape, Equation (1) also arises in mathematical models of population dynamics [20,21,30], oceanography [1], image processing [22], and gradient systems associated with nonconvex functionals ([2,29] and references therein).

Quite a few regularizations have been proposed and investigated for problems of this kind (for example see [1–3,11,32]). Among them, the pseudoparabolic regularization

$$u_t = (\phi(u))_{xx} + \varepsilon u_{xxt} \quad (\varepsilon > 0) \tag{3}$$

arises, if nonequilibrium effects are taken into account (see [3]). Remarkably, it gives rise to a class of *viscous entropy inequalities* satisfied by classical solutions of (3) for any $\varepsilon > 0$ (see Section 3). Analogue to the case of hyperbolic conservation laws (see [7]), solutions of (3) with initial data (2) are expected to converge as $\varepsilon \to 0$ to some solution of the Cauchy problem (1), (2), also satisfying a suitable limiting *entropy inequality*.

Results of this kind have been proved in [24] for the initial-boundary value problem with homogeneous Dirichlet conditions, and in [23,25] for Neumann conditions. However, due to the nonmonotone character of ϕ, the situation is more cumbersome with respect to the case of scalar hyperbolic conservation laws. Firstly, it turns out that weak entropy *measure-valued solutions* are obtained by such a limiting procedure. Secondly, no uniqueness is known within this class.

Early nonuniqueness results for forward–backward parabolic equations were proved in [14,26] (for a more recent result concerning the Perona–Malik equation, see [33]). It can be argued that the class of solutions considered in these cases is too wide, so that some narrower class of well-posedness, defined by additional constraints, should be considered. The weak entropy measure-valued solutions discussed in [23–25] could be considered in this case. To date, uniqueness within this class has not been proved.

This motivates our investigation, which concerns the well-posedness of problem (1), (2) within a more restricted class of solutions (see Definition 1). Solutions of this class, whose choice was suggested in [8], are of physical interest, for they describe the transition between stable phases. In some respects, they can be regarded as the counterpart of piecewise smooth solutions in the theory of hyperbolic conservation laws. Like the latter, they exhibit an interface which evolves according to the Rankine–Hugoniot condition, obeying admissibility conditions which follow from the entropy inequality. Such conditions can be viewed as prescriptions to select *admissible jumps* between the stable branches of ϕ. As already pointed out in [23], this gives rise to a *hysteresis loop* typical of first-order phase transitions (see [4]). A numerical exploration of such solutions is performed in [16].

In this paper we prove the existence and uniqueness of such solutions, which we call *two-phase entropy solutions*, for the Cauchy problem (1), (2) (the proof of similar results for the Neumann initial-boundary value problem was outlined in [17]; for the Riemann problem they have been studied in [13]). Results are stated in Section 2 and proved in Section 4 for existence, in Section 5 for uniqueness. In Section 3 we discuss the entropy formulation of the problem in the light of the existing literature.

2. Results

To study the existence of solutions of the Cauchy problem (1), (2), we make the following assumptions on the initial data u_0:

$$(A_1)\quad \begin{cases} \text{(i)} & u_0(\mathbb{R}_-) \subseteq (-\infty, b],\ u_0(\mathbb{R}_+) \subseteq [c, \infty)\,; \\ \text{(ii)} & u_0 \in H^{2,\infty}(\mathbb{R}_-) \cap H^{2,\infty}(\mathbb{R}_+),\ \lim_{x\to\pm\infty} u_0'(x) = 0\,; \\ \text{(iii)} & \lim_{\eta\to 0^+} \phi(u_0)(-\eta) = \lim_{\eta\to 0^+} \phi(u_0)(\eta). \end{cases}$$

Here and below we set

$$H^{k,\infty}(I) := \{u \in C^k(I) \mid \|u\|_{k,\infty} < \infty\} \quad (I \subseteq \mathbb{R})$$

with

$$\|u\|_{k,\infty} := \sum_{j=0}^{k} \sup_{x\in I} |u^{(j)}(x)| \quad (k \in \{0\} \cup \mathbb{N}).$$

To study uniqueness the following assumption will be also needed:

$$(A_2) \quad \begin{cases} \text{The functions } \phi(u_0) - A,\ \phi(u_0) - B \text{ change sign at most} \\ \text{a finite number of times in any compact subset of } \mathbb{R}. \end{cases}$$

Remark 1. In view of assumption (A_1), there exist finite limits

$$\lim_{\eta\to 0^+} u_0(\pm\eta) =: u_0(0^\pm), \qquad \lim_{\eta\to 0^+} u_0'(\pm\eta) =: u_0'(0^\pm).$$

Since

$$u_0(0^-) \leqq b < c \leqq u_0(0^+),$$

the initial datum u_0 has a jump discontinuity at the origin. On the other hand, $\phi(u_0)$ belongs to $C(\mathbb{R})$, with

$$\phi(u_0)(0) := \alpha_- u_0(0^-) + \beta_- = \alpha_+ u_0(0^+) + \beta_+. \tag{4}$$

In the following, we denote by $C^{2,1}(Q)$ the set of functions $u \in C(Q)$ such that $u_x, u_{xx}, u_t \in C(Q)$ $(Q \subseteq S_T)$ and by $C^l([0,\tau])$ ($l > 0$ noninteger, $\tau \in (0,T]$) the Banach space of functions $u \in C^{[l]}([0,\tau])$ with norm

$$\|u\|^{(l)}_{(0,\tau)} := \|u\|_{[l],\infty} + \langle u^{([l])}\rangle^{(l-[l])}_{(0,\tau)},$$

where $\|u\|_{k,\infty}$ ($k \geqq 0$ integer) is defined as above and

$$\langle u\rangle^{(\sigma)}_{(0,\tau)} := \sup_{s,t\in(0,\tau), s\neq t} \frac{|u(s)-u(t)|}{|s-t|^\sigma} \quad (\sigma \in (0,1)).$$

Inspired by [8,23,25], we make the following definition.

Definition 1. *By a* two-phase entropy solution *of the Cauchy problem (1), (2) in $S_\tau := \mathbb{R} \times (0,\tau]$ ($\tau \in (0,T]$) we mean any couple of functions $\xi = \xi(t)$, $u = u(x,t)$ such that:*

(i) $\xi \in C^{\frac{3}{2}}([0,\tau])$, $\xi(0) = 0$, *and there exists at most a finite number of intervals $(\tau', \tau'') \subseteq (0,\tau]$ such that $\xi'(t) \neq 0$ for any $t \in (\tau', \tau'')$;*

(ii) $u \in L^\infty(S_\tau) \cap C\left(\bar{A}^+_\tau \backslash \gamma\right) \cap C\left(\bar{A}^-_\tau \backslash \gamma\right)$, *where*

$$A^\pm_\tau := \left\{(x,t) \in S_\tau \;\middle|\; \pm(x - \xi(t)) > 0\right\},$$
$$\gamma := \{(\xi(t),t) \mid t \in [0,\tau]\},$$

and for any $t \in (0,\tau]$ there exist finite the limits

$$\lim_{\eta\to 0^+} u(\xi(t) \pm \eta, t) =: u(\xi(t)^\pm, t).$$

Moreover,

$$u\left(A^-_\tau\right) \subseteq (-\infty, b], \quad u\left(A^+_\tau\right) \subseteq [c, \infty); \tag{5}$$

(iii) $u \in C^{2,1}\left(A_\tau^+\right) \cap C^{2,1}\left(A_\tau^-\right)$, $u_x \in L^\infty(S_\tau)$, *and for any* $t \in (0, \tau]$ *there exist finite the limits*

$$\lim_{\eta\to 0^+} u_x(\xi(t) \pm \eta, t) =: u_x(\xi(t)^\pm, t);$$

(iv) *u is a classical solution of the problem*

$$\begin{cases} u_t = (\phi(u))_{xx} & \text{in } A_\tau^- \cup A_\tau^+ \\ u(x,0) = u_0(x) & x \in \mathbb{R}\backslash\{0\}, \end{cases}$$

such that

$$\lim_{x\to\pm\infty} u_x(x,t) = 0 \quad (t \in (0,\tau]); \tag{6}$$

(v) *for any* $t \in (0, \tau]$ *there holds:*

$$\begin{aligned} \phi(u)(\xi(t)^+, t) &:= \lim_{\eta\to 0^+} \phi(u)(\xi(t) + \eta, t) \\ &= \lim_{\eta\to 0^+} \phi(u)(\xi(t) - \eta, t) =: \phi(u)(\xi(t)^-, t); \end{aligned} \tag{7}$$

(vi) *the* Rankine–Hugoniot condition

$$\xi'(t) = -\frac{\alpha_+ u_x(\xi(t)^+, t) - \alpha_- u_x(\xi(t)^-, t)}{u(\xi(t)^+, t) - u(\xi(t)^-, t)} \tag{8}$$

holds for any $t \in (0, \tau]$*;*

(vii) *for any* $t \in (0, \tau]$ *the* entropy conditions

$$\begin{cases} (a) & \xi'(t) \geqq 0 \ \text{ if } \ \phi(u)(\xi(t), t) = A, \\ (b) & \xi'(t) \leqq 0 \ \text{ if } \ \phi(u)(\xi(t), t) = B, \\ (c) & \xi'(t) = 0 \ \text{ if } \ \phi(u)(\xi(t), t) \in (A, B) \end{cases} \tag{9}$$

are satisfied.

The curve γ *is called the interface. A two-phase entropy solution in* S_T *is said to be global.*

Remark 2. In view of requirement (ii) in Definition 1, u has a jump discontinuity at any point of the curve γ. However, $\phi(u)$ can be made continuous in S_τ by setting (see 7):

$$\phi(u)(\xi(t), t) := \alpha_- u(\xi(t)^-, t) + \beta_- = \alpha_+ u(\xi(t)^+, t) + \beta_+ \quad (t \in (0, \tau]). \tag{10}$$

The definition of $\phi(u)$ can be further extended at $t = 0$ setting

$$\phi(u)(x, 0) := \phi(u_0)(x) \quad \text{for any } x \in \mathbb{R};$$

then $\phi(u) \in C\left(\overline{S_\tau}\right)$, as is easily seen. In particular, this implies

$$\lim_{t\to 0} \phi(u)(\xi(t), t) = \phi(u_0)(0),$$

thus

$$\lim_{t\to 0} u(\xi(t)^\pm, t) = u_0(0^\pm) \tag{11}$$

(see 4 and 10).

Remark 3. If u satisfies the properties (ii)–(iv) of Definition 1, its restrictions $u\big|_{A_\tau^\pm}$ satisfy in the classical sense the following problems:

$$\begin{cases} w_t = \alpha_\pm w_{xx} & \text{in } A_\tau^\pm \\ w(\xi(t),t) = u(\xi(t)^\pm,t) & t \in (0,\tau] \\ w(x,0) = u_0(x) & x \in \mathbb{R}_\pm. \end{cases}$$

Observe that equality (11) is the compatibility condition of order zero for the above problem (for example see [15]).

Remark 4. If u satisfies (ii)–(iv) above, the Rankine–Hugoniot condition is satisfied if and only if u is a weak solution of Equation (1) in S_τ (see Section 3).

The jump discontinuity of u and the continuity of $\phi(u)$ across the curve γ imply $\phi(u)(\xi(t),t) \in [A,B]$ for any $t \in [0,\tau]$. Further restrictions on two-phase solutions derive from the Rankine–Hugoniot and entropy conditions. In particular, the latter imply that at any fixed $(x,t) \in S_\tau$ we can jump between stable phases only when $\phi(u)(x,t)$ takes one of the values A, B. Observe that the assumptions made on the sign of ξ' (see (i), (vii) in Definition 1) imply that, at any fixed point $\bar{x} \in \mathbb{R}$, only a finite number of changes of phase can take place in the interval of existence $(0,\tau]$.

Our first result deals with local existence of two-phase solutions.

Theorem 1. *Assume hypothesis* (A_1). *Suppose that either* $\phi(u_0)(0) \in (A,B)$ *or* $\alpha_+ u_0'(0^+) \neq \alpha_- u_0'(0^-)$. *Then there exists* $\tau \in (0,T]$ *such that the Cauchy problem* (1), (2) *has a two-phase entropy solution in* S_τ.

To prove the above theorem, we address two different auxiliary problems, which we describe as the moving boundary problem and the steady boundary problem. As the name suggests, the first arises when the interface moves; it is a free boundary problem formally similar to a two-phase Stefan problem (concerning the wide literature on the Stefan problem, see in particular [9,10]). With respect to the classical case, the main difference is that different values of the unknown are prescribed on either side of the interface. For this reason, we address the problem by a different technique, based on an iterative procedure (see Section 4.1).

On the other hand, the second auxiliary problem arises when $\xi' \equiv 0$; formally, it amounts to solve a parabolic problem with discontinuous diffusivity (see Section 4.2). After studying such problems, we show that for small times the solution of either problem, depending on the assumption satisfied by initial data, is indeed a two-phase entropy solution. Hence Theorem 1 follows.

Under more restrictive assumptions on the initial data, the same methods allow one to prove global existence. This is the content the following theorem (actually, the global solution mentioned in the statement is that of the steady boundary problem; see Section 4.3).

Theorem 2. *Let assumption* (A_1) *be satisfied and* $\phi(u_0)(x) \in [A,B]$ *for any* $x \in \mathbb{R}$. *Then there exists a global two-phase entropy solution of the Cauchy problem* (1), (2).

The following uniqueness result will be proved in Section 5.

Theorem 3. *Let assumptions (A_1)–(A_2) be satisfied. Then there exists at most one two-phase entropy solution of the Cauchy problem* (1), (2).

3. Entropy formulation

It is the purpose of this section to motivate Definition 1, concerning in particular the entropy conditions (9). Let us first make the following definitions.

Definition 2. *Let $u_0 \in L^\infty(\mathbb{R})$. By a* weak solution *to problem (1), (2) in S_τ, we mean any couple $u \in L^\infty(S_\tau)$, $w \in L^\infty(S_\tau) \cap L^2((0,\tau); H^1_{loc}(\mathbb{R}))$ such that $w = \phi(u)$ and*

$$\iint_{S_\tau} \{u\psi_t - w_x\psi_x\}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0(x)\psi(x,0)\,\mathrm{d}x = 0$$

for any $\psi \in C^{1,1}(\overline{S}_\tau)$ with compact support, $\psi(\cdot,\tau) = 0$ in $\mathbb{R}$.

Definition 3. *Let $u_0 \in L^\infty(\mathbb{R})$. By a weak entropy solution of (1), (2) in S_τ we mean any weak solution of the problem such that the* entropy inequality

$$\iint_{S_\tau} \left\{G(u)\psi_t - g(w)w_x\psi_x - g'(w)(w_x)^2\psi\right\}\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\mathbb{R}} G(u_0)(x)\psi(x,0)\,\mathrm{d}x \geqq 0 \tag{12}$$

holds for any $g \in C(\mathbb{R})$ nondecreasing and any ψ as above, $\psi \geqq 0$, where

$$G(u) := \int_0^u g(\phi(s))\,\mathrm{d}s + k \quad (k \in \mathbb{R}). \tag{13}$$

The above definition of weak entropy solution can be motivated directly as follows. Should Equation (1) admit a classical solution u, then multiplying the equation by $g(w)$ with $g \in C^1(\mathbb{R})$, $g' \geqq 0$, $w = \phi(u)$ one finds that

$$(G(u))_t = g(w)w_{xx} = (g(w)w_x)_x - g'(w)(w_x)^2.$$

Subsequently, multiplying the above inequality by ψ as in Definition 3 and integrating by parts, one obtains (12) with the equality sign. If applied to the Cauchy problem for (3), with solution given by a couple u^ε, $w^\varepsilon := \phi(u^\varepsilon) + \varepsilon u_t$ $(\varepsilon > 0)$, the same calculation gives the inequality

$$\iint_{S_T} \left\{G(u^\varepsilon)\psi_t - g(w^\varepsilon)w^\varepsilon_x\psi_x - g'(w^\varepsilon)(w^\varepsilon_x)^2\psi\right\}\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\mathbb{R}} G(u_0)(x)\psi(x,0)\,\mathrm{d}x \geqq 0, \tag{14}$$

which can be regarded as a viscous entropy inequality. Hence only inequality could be expected, if u were obtained by the "vanishing viscosity" method (in this connection, see Remark 5).

To establish a link between Definition 2 and Definition 1 we need the following lemma. By $[h]$ we denote the jump of any function h across the interface γ—namely,

$$[h] = [h](\xi(t), t) := h(\xi(t)^+, t) - h(\xi(t)^-, t),$$

where the limits

$$h(\xi(t)^\pm, t) := \lim_{\eta \to 0^+} h(\xi(t) \pm \eta, t)$$

are supposed to be finite. The standard proof is omitted.

Lemma 1. *Let (u, w) be a weak entropy solution of problem (1), (2) in S_τ. Suppose the following:*

(i) $u \in C^{2,1}\left(A_\tau^+\right) \cap C^{2,1}\left(A_\tau^-\right)$, $u \in C^{1,1}(\omega \cap \bar{A}_\tau^+) \cap C^{1,1}(\omega \cap \bar{A}_\tau^-)$ *for any compact subset $\omega \subseteq S_\tau$, where $A_\tau^\pm$ denote the subsets of S_τ introduced in Definition 1;*

(ii) $w(\cdot, t) \in C(\mathbb{R})$ *for any $t \in (0, \tau]$.*

Set

$$\gamma := \partial A_\tau^+ \cap \partial A_\tau^- \equiv \{(\xi(t), t) \mid t \in [0, \tau]\}$$

with $\xi \in C^1((0, \tau])$, $\xi(0) = 0$. Then both the Rankine–Hugoniot condition

$$\xi' = -\frac{[w_x]}{[u]} \tag{15}$$

and the entropy condition

$$\xi'[G(u)] \geqq -g(w)[w_x] \tag{16}$$

hold on $\gamma \setminus \{(0, 0)\}$.

The following result is proved arguing as for hyperbolic conservation laws (for example see [28])—namely, selecting admissible directions of propagation of the interface by a proper choice of g in (16).

Proposition 1. *Let (u, w) be a weak entropy solution of problem (1), (2) in S_τ satisfying the assumptions of Lemma 1 and the invariance conditions (5). Then both the Rankine–Hugoniot condition (8) and the entropy conditions (9) are satisfied.*

Proof. Inequality (8) follows immediately from (5) and (15). Concerning (9), choose

$$g(s) = g_k(s) := \operatorname{sgn}(s - k) \quad (s \in \mathbb{R}), \tag{17}$$

so that

$$G(u) = \int_0^u \operatorname{sgn}(\phi(s) - k)\, \mathrm{d}s \quad (k \in \mathbb{R})$$

(see (13)). From (15), (16) we obtain:

$$\xi' \, \{[G(u)] - g(w)[u]\} \geqq 0 \quad \text{on } \gamma \backslash \{(0,0)\}$$

for any nondecreasing g. Then the choice (17) gives

$$\xi'(t) \int_{u(\xi(t)^-,t)}^{u(\xi(t)^+,t)} \{\text{sgn}\, (\phi(s) - k) - \text{sgn}\, (w(\xi(t),t) - k)\} \, \mathrm{d}s \geqq 0 \tag{18}$$

for any $t \in (0, \tau]$ and any $k \in \mathbb{R}$. Observe that $u(\xi(t)^-, t) < u(\xi(t)^+, t)$ by assumption (5) ($t \in (0, \tau]$).

If $w(\xi(t), t) = A$, let us choose $k \in (A, B)$. Then

$$w(\xi(t), t) - k = A - k < 0 \Rightarrow \text{sgn}\, (w(\xi(t), t) - k) = -1,$$

thus

$$\text{sgn}\, (\phi(s) - k) - \text{sgn}\, (w(\xi(t), t) - k) = \text{sgn}\, (\phi(s) - k) + 1 \geqq 0 \tag{19}$$

for any $s \in [u(\xi(t)^-, t),\ u(\xi(t)^+, t)] = [a, c]$ ($t \in (0, \tau]$). Since $\text{sgn}\, (\phi(\cdot) - k) + 1$ does not identically vanish in $[a, c]$, by inequality (18) we obtain $\xi'(t) \geqq 0$. This proves condition (9)(a).

Condition (9)(b) similarly follows choosing $k \in (A, B)$, which gives, since $w(\xi(t), t) = B$,

$$\text{sgn}\, (\phi(s) - k) - \text{sgn}\, (w(\xi(t), t) - k) = \text{sgn}\, (\phi(s) - k) - 1 \leqq 0 \tag{20}$$

for any $s \in [u(\xi(t)^-, t),\ u(\xi(t)^+, t)] = [b, d]$ ($t \in (0, \tau]$).

Finally, if $w(\xi(t), t) \in (A, B)$, we choose first $k \in (w(\xi(t), t), B)$, then $k \in (A, w(\xi(t), t))$. Hence both (19) and (20) hold, which implies $\xi'(t) = 0$ ($t \in (0, \tau]$). This proves condition (9)(c), and completes the proof. □

In view of the above proposition, it is easy to give conditions ensuring that a weak entropy solution be a two-phase entropy solution; we leave their formulation to the reader.

Remark 5. It is natural to ask whether weak entropy solutions can be obtained as the "vanishing viscosity" limit of a sequence of classical solutions of the Cauchy problem (3), (2). It can be checked that results analogous to those proved in [19] for the Neumann initial-boundary value problem (see also [8,17]) hold in the present case, too. This allows one to associate with the sequence of "viscous solutions" a family of Young measures, which is a natural candidate as a weak entropy measure-valued solution of the problem.

Unfortunately, no results concerning the structure of such Young measures, like those in [24], are known for the Cauchy problem (1), (2). Therefore the above question to our knowledge is open. If such results were available, two-phase entropy solutions would be a particular case of the weak entropy measure-valued solutions obtained by the approximating procedure. However, even in this case their existence would not follow from general existence results like those in [24].

4. Existence

In this section we prove Theorem 1, following the outline in Section 2.

4.1. Moving boundary problem

For any $C \in [A, B]$ define $\kappa_- \in (-\infty, b]$, $\kappa_+ \in [c, \infty)$ by the equalities

$$\alpha_-\kappa_- + \beta_- = \alpha_+\kappa_+ + \beta_+ = C. \tag{21}$$

Definition 4. *Let $C \in [A, B]$. By a solution in S_τ ($\tau \in (0, T]$) of the moving boundary problem at the value C, we mean any couple of functions $\xi = \xi(t)$, $u = u(x, t)$ such that:*

(i) $\xi \in C^{\frac{3}{2}}([0, \tau])$, $\xi(0) = 0$, $\xi'(t) \neq 0$ *for any* $t \in (0, \tau]$ *;*
(ii) *u satisfies requirements (ii)–(iv) of Definition 1;*
(iii) *for any $t \in (0, \tau]$ there holds:*

$$u(\xi(t)^{\pm}, t) = \kappa_{\pm}, \tag{22}$$

$$\xi'(t) = -\frac{\alpha_+ u_x(\xi(t)^+, t) - \alpha_- u_x(\xi(t)^-, t)}{\kappa_+ - \kappa_-}. \tag{23}$$

Clearly, equalities (21)–(23) imply (7)–(8), whereas condition (i) above implies condition (i) of Definition 1. Hence any solution of the moving boundary problem is a two-phase entropy solution of the Cauchy problem for (1), if it also satisfies the entropy conditions (9). Since by Definition 4 $\xi' \neq 0$ in $(0, \tau]$, either (a) or (b) of condition (9) holds; thus either $C = A$ or $C = B$ must be chosen. Therefore we can exhibit two-phase entropy solutions by constructing solutions of the moving boundary problem with $C = A$ (respectively, $C = B$) such that $\xi' \geqq 0$ ($\xi' \leqq 0$, respectively) in $(0, \tau]$. Conversely, any two-phase entropy solution such that $\xi' < 0$ (respectively, $\xi' > 0$) in $(0, \tau]$ is a solution of the moving boundary problem at the value B (respectively, at the value A).

Remark 6. It is convenient for further developments to write the moving boundary problem in a slightly different form. Let $\xi = \xi(t)$, $u = u(x, t)$ be a solution of the problem; set

$$v(y, t) := u(\xi(t) + y, t). \tag{24}$$

In view of Definitions 1 and 4, the moving boundary problem amounts to find a couple $\xi = \xi(t)$, $v = v(y, t)$ such that:

(i) $\xi \in C^{\frac{3}{2}}([0, \tau])$, $\xi(0) = 0$, $\xi'(t) \neq 0$ for any $t \in (0, \tau]$;
(ii) $v \in L^\infty(S_\tau) \cap C\left(\overline{Q_\tau^+} \setminus \{y = 0\}\right) \cap C\left(\overline{Q_\tau^-} \setminus \{y = 0\}\right)$, where

$$Q_\tau^{\pm} := \mathbb{R}_{\pm} \times (0, \tau] \quad (\tau \in (0, T]),$$

and for any $t \in (0, \tau]$ there exist finite the limits

$$\lim_{\eta \to 0^+} v(\pm\eta, t) =: v(0^\pm, t).$$

Moreover,

$$v\left(Q_\tau^-\right) \subseteq (-\infty, b], \quad v\left(Q_\tau^+\right) \subseteq [c, \infty)\,; \tag{25}$$

(iii) $v \in C^{2,1}\left(Q_\tau^+\right) \cap C^{2,1}\left(Q_\tau^-\right)$, $v_y \in L^\infty(S_\tau)$, and for any $t \in (0, \tau]$ there exist finite the limits

$$\lim_{\eta \to 0^+} v_y(\pm\eta, t) =: v_y(0^\pm, t)\,;$$

(iv) v is a classical solution of the problem

$$\begin{cases} v_t = \{\phi(v)\}_{yy} + \xi' v_y & \text{in } Q_\tau^- \cup Q_\tau^+ \\ v(y, 0) = u_0(y) & y \in \mathbb{R}\setminus\{0\}, \end{cases} \tag{26}$$

such that

$$\lim_{y \to \pm\infty} v_y(y, t) = 0 \quad (\tau \in (0, T])\,;$$

(v) for any $t \in (0, \tau]$

$$v(0^\pm, t) = \kappa_\pm\,; \tag{27}$$

(vi) for any $t \in (0, \tau]$

$$\xi'(t) = -\frac{\alpha_+ v_y(0^+, t) - \alpha_- v_y(0^-, t)}{\kappa_+ - \kappa_-}. \tag{28}$$

Concerning existence of solutions to the moving boundary problem, we shall prove the following result (as always in the following equalities, either upper or lower signs must be chosen).

Theorem 4. *Let assumption* (A_1) *be satisfied; suppose* $u_0(0^\pm) = \kappa_\pm$, *and* $\alpha_+ u_0'(0^+) \neq \alpha_- u_0'(0^-)$. *Then for some* $\tau \in (0, T]$ *there exists a solution* u *in* S_τ *of the moving boundary problem with* C *given by* (21).

Remark 7. Condition (22) and assumption (i) of the above theorem obviously imply (11); they are equivalent to $\phi(u)(\xi(t), t) = C$ for any $t \in [0, \tau]$, whence $\phi(u) \in C(\overline{S}_\tau)$ (see Remark 2).

As already mentioned, Theorem 4 will be proved by an iteration procedure. To this purpose we need existence, uniqueness and regularity results concerning the problem:

$$\begin{cases} v_t = \alpha\, v_{yy} + \xi' v_y & \text{in } Q_\tau^+ \\ v(0, t) = \kappa & t \in (0, \tau] \\ v(y, 0) = v_0(y) & y \in \mathbb{R}_+, \end{cases} \tag{29}$$

where $\alpha > 0$, $\tau \in (0, T]$, $\kappa \in \mathbb{R}$ and ξ' is the derivative of a given smooth function $\xi = \xi(t)$. For, if u is a solution of the moving boundary problem in S_τ, the function

$$v^{\pm}(y,t) := u(\xi(t) \pm y, t) \quad (y \in \mathbb{R}_+,\ t \in [0,\tau])$$

satisfies the problem (see (26), (27)):

$$(P)_\pm \qquad \begin{cases} v_t = \alpha_\pm v_{yy} \pm \xi' v_y & \text{in } Q_\tau^+ \\ v(0,t) = \kappa_\pm & t \in (0,\tau] \\ v(y,0) = v_0^\pm(y) & y \in \mathbb{R}_+ \end{cases}$$

with $\xi \in C^{\frac{3}{2}}([0,\tau])$, $\xi(0) = 0$ and $v_0^\pm(y) := u_0(\pm y)$.

Remark 8. It is worth pointing out the relationship between the functions $v^\pm$ and the restrictions $v\big|_{Q_\tau^\pm}$ of the function v defined in (24):

$$v\Big|_{Q_\tau^\pm}(y,t) = v^\pm(\pm y, t) \quad \text{for any } y \in \mathbb{R}_+,\ t \in [0,\tau].$$

In particular, there holds

$$(v^\pm)_y(0,t) = \pm v_y(0^\pm, t) \quad (t \in (0,\tau]) \tag{30}$$

whenever the above quantities exist.

Let us recall the definition of some function spaces to be used in the sequel. By $C^{l,\frac{l}{2}}(\overline{Q})$ ($l > 0$ noninteger, $Q \subseteq S_T$), we denote the Banach space of functions u continuous in $\overline{Q}$, together with all derivatives of the form $D_t^r D_x^s u$ for $2r + s < l$, with norm

$$\|u\|_Q^{(l)} := \sum_{j=0}^{[l]} \sum_{2r+s=j} \|D_t^r D_x^s u\|_\infty + \sum_{2r+s=[l]} \langle D_t^r D_x^s u\rangle_{x,Q}^{(l-[l])} + \sum_{0<l-2r-s<2} \langle D_t^r D_x^s u\rangle_{t,Q}^{(\frac{l-2r-s}{2})},$$

where

$$\langle u\rangle_{x,Q}^{(\sigma)} := \sup_{(x,t),(x',t)\in Q, x\neq x'} \frac{|u(x,t) - u(x',t)|}{|x-x'|^\sigma},$$

$$\langle u\rangle_{t,Q}^{(\sigma)} := \sup_{(x,s),(x,t)\in Q, s\neq t} \frac{|u(x,s) - u(x,t)|}{|s-t|^\sigma} \quad (\sigma \in (0,1)).$$

In the above expression and in the following, we denote

$$\|f\|_\infty := \sup_{(x,t)\in Q} |f(x,t)|$$

for any function f defined in a subset $Q \subseteq S_T$.

Concerning problem (29), we can state the following result.

Proposition 2. *Let* $\xi \in C^{1+\frac{\sigma}{2}}([0,\tau])$ $(\sigma \in (0,1))$, $\xi(0) = 0$. *Suppose* $v_0 \in H^{2,\infty}(\overline{\mathbb{R}}_+)$, $v_0(0) = \kappa$, $\lim_{y\to\infty} v_0'(y) = 0$. *Then there exists a unique classical solution* $v \in C^{2,1}(Q_\tau^+) \cap C([0,\tau]; H^{1,\infty}(\overline{\mathbb{R}}_+))$ *of problem* (29). *Moreover,*

(i) *there holds*

$$\lim_{y\to\infty} v_y(y,t) = 0 \quad \text{for any } t \in (0,\tau]\,;$$

(ii) $v_t,\ v_{yy} \in L^\infty(Q_\tau^+)$;

(iii) *there exists a constant* $C_1 > 0$ *(only depending on* $\|v_0'\|_\infty$, $\|v_0''\|_\infty$ *and* $A := \sup_{t\in[0,\tau]} |\xi'(t)|$*) such that*

$$|v_y(y,t_1) - v_y(y,t_2)| \leqq C_1 \sqrt{|t_1 - t_2|} \tag{31}$$

for any $(y,t_1), (y,t_2) \in \overline{Q}_\tau^+$;

(iv) $v \in C^{1+\sigma,\frac{1+\sigma}{2}}(\overline{Q}_\tau^+)$, *and there exists a constant* $C_2 > 0$ (*only depending on the norm* $\|v_0\|_{2,\infty}$ *and* A) *such that*

$$\|v\|_{Q_\tau^+}^{(1+\sigma)} \leqq C_2 \quad (\sigma \in (0,1)). \tag{32}$$

Remark 9. The solution v mentioned in the above proposition does not belong to $C^{2,1}(\overline{Q}_\tau^+)$. In fact, this would imply the first order compatibility condition $\alpha v_0''(0) + \xi'(0)v_0'(0) = 0$, which we do not assume.

The following lemma will be used to prove Proposition 2.

Lemma 2. *Let* $\xi \in C([0,\tau])$, $\xi(0) = 0$. *Suppose* $u_0 \in H^{0,\infty}(\overline{\mathbb{R}}_+)$, $\theta \in C([0,\tau])$, $\alpha u_0(0) = \theta(0)$. *Then there exists a unique classical solution* $u \in C^{2,1}(A_\tau^+) \cap C(\bar{A}_\tau^+) \cap L^\infty(A_\tau^+)$ *of the problem*

$$\begin{cases} u_t = \alpha\, u_{xx} & \text{in } A_\tau^+ \\ \alpha\, u(\xi(t),t) = \theta(t) & t \in (0,\tau] \\ u(x,0) = u_0(x) & x \in \mathbb{R}_+. \end{cases} \tag{33}$$

In addition, if $u_0 \in H^{1,\infty}(\overline{\mathbb{R}}_+)$ *and* $\lim_{x\to\infty} u_0'(x) = 0$, *then* $\lim_{x\to\infty} u_x(x,t) = 0$ *for any* $t \in (0,\tau]$.

Proof. For $(x,t) \in A_\tau^+$, set

$$u(x,t) := -\int_0^t K_x(x - \xi(s), t - s)\, h(s)\, \mathrm{d}s + \int_0^\infty \Lambda(x,y;t)\, u_0(y)\, \mathrm{d}y, \tag{34}$$

where $K(x,t) := (4\pi\,\alpha\,t)^{-1/2}\, \mathrm{e}^{-\frac{x^2}{4\alpha t}}$ denotes the heat kernel,

$$\Lambda(x,y;t) := K(x-y,t) - K(x+y,t) \tag{35}$$

and h solves the integral equation

$$\begin{aligned}\frac{h(t)}{2\alpha} - \int_0^t K_x(\xi(t)-\xi(s), t-s)\, h(s)\,\mathrm{d}s \\ = \frac{\theta(t)}{\alpha} - \int_0^\infty \Lambda(\xi(t), y; t)\, u_0(y)\,\mathrm{d}y.\end{aligned} \tag{36}$$

for $t \in (0, \tau]$. Under the present assumptions there exists a unique solution $h \in C([0,\tau])$ of the above equation (for example see [5]). Then by standard calculations the function u defined in (34) belongs to $C^{2,1}(A_\tau^+) \cap C(\overline{A_\tau^+}) \cap L^\infty(A_\tau^+)$ and solves problem (33) in the classical sense; moreover, it is the unique bounded solution (for example see [31] for details).

In addition, there holds

$$u_x(x,t) = -\int_0^t K_{xx}(x-\xi(s), t-s)\, h(s)\,\mathrm{d}s + \int_0^\infty \Lambda_x(x,y;t)\, u_0(y)\,\mathrm{d}y.$$

Using the Lebesgue convergence theorem, it is easily seen that the first integral on the right-hand side vanishes as $x \to \infty$; moreover,

$$\lim_{x\to\infty} \int_0^\infty \Lambda_x(x,y;t)\, u_0(y)\,\mathrm{d}y = \lim_{x\to\infty} u_0'(x) = 0.$$

Then the conclusion follows. □

Proof (Proposition 2).

(i) Let u be the solution of problem (33) considered in the previous lemma, with $\theta(t) = \alpha\kappa$. Then the function v defined by (24) belongs to $C^{2,1}(Q_\tau^+) \cap C(\overline{Q_\tau^+}) \cap L^\infty(Q_\tau^+)$ and solves problem (29) with $v_0 = u_0$ in the classical sense. By comparison results there holds

$$\|v\|_\infty \leqq \|v_0\|_\infty. \tag{37}$$

Moreover, v is the unique bounded solution of the problem, and its derivative v_y belongs to $L^\infty(Q_\tau^+)$ (for example see [C, Theorem 20.3.1]). A standard calculation shows that

$$\|v_y\|_\infty \leqq M_1, \tag{38}$$

with some constant $M_1 > 0$, which only depends on $\|v_0'\|_\infty$ and A.

(ii) It follows from Lemma 2 that claim (i) of the statement is satisfied. To prove claim (ii), observe that v satisfies the integral equation

$$\begin{aligned}v(y,t) = \kappa + \int_0^\infty \Lambda(y,z;t)(v_0(z)-\kappa)\,\mathrm{d}z \\ + \int_0^t \xi'(s) \int_0^\infty \Lambda(y,z;t-s) v_z(z,s)\,\mathrm{d}z\,\mathrm{d}s.\end{aligned} \tag{39}$$

Differentiating twice with respect to y the above equation and using the equality $\Lambda_{yy} = \Lambda_{zz}$ (see (35)), we obtain

$$v_{yy}(y,t) = \int_0^\infty \Lambda_{zz}(y,z;t)(v_0(z)-\kappa)\,\mathrm{d}z + \int_0^t \xi'(s)\int_0^\infty \Lambda_{zz}(y,z;t-s)v_z(z,s)\,\mathrm{d}z\,\mathrm{d}s.$$

Since $v_0(0)=\kappa$, $\Lambda(y,0;t)=0$ and $\Lambda_z(y,0;t)=-2K_y(y,t)$, integrating by parts gives plainly

$$v_{yy}(y,t) = \int_0^\infty \Lambda(y,z;t)\,v_0''(z)\,\mathrm{d}z + 2\int_0^t \xi'(s)K_y(y,t-s)\,v_z(0,s)\,\mathrm{d}s - \int_0^t \xi'(s)\int_0^\infty \Lambda_z(y,z;t-s)v_{zz}(z,s)\,\mathrm{d}z\,\mathrm{d}s.$$

Then using (38), we obtain easily

$$|v_{yy}(y,t)| \leqq \frac{M_2}{2} + \frac{2A}{\sqrt{\pi\alpha}}\int_0^t \frac{1}{\sqrt{t-s}}\|v_{yy}(\cdot,s)\|_\infty\,\mathrm{d}s \quad ((y,t)\in Q_\tau^+)$$

with a constant $M_2>0$ only depending on $\|v_0'\|_\infty$, $\|v_0''\|_\infty$ and A. It follows that

$$\sup_{s\in[0,t]}\|v_{yy}(\cdot,s)\|_\infty \leqq \frac{M_2\sqrt{\alpha\pi}}{2\left(\sqrt{\alpha\pi}-4A\sqrt{t}\right)},$$

thus

$$\sup_{s\in[0,t]}\|v_{yy}(\cdot,s)\|_\infty \leqq M_2$$

for any $t\in[0,t_1]$ with $t_1 := \frac{\alpha\pi}{64A^2}$. Repeating the argument $\left[\frac{\tau}{t_1}\right]$ times, we obtain

$$\sup_{s\in[0,\tau]}\|v_{yy}(\cdot,s)\|_\infty = \|v_{yy}\|_\infty \leqq M_2. \tag{40}$$

From the first equation in (29) and inequalities (38), (40), we also have

$$\|v_t\|_\infty \leqq AM_1 + \alpha M_2; \tag{41}$$

thus the claim follows.

(iii) The above estimate of v_{yy} implies that $v_y(\cdot,t)$ is Lipschitz continuous for any fixed $t\in(0,\tau]$. In fact, by (40) there holds

$$|v_y(y_1,t)-v_y(y_2,t)| \leqq M_2\,|y_1-y_2| \tag{42}$$

for any $(y_1,t),(y_2,t)\in Q_\tau^+$. This implies

$$|v_y(y,t_1)-v_y(y,t_2)| \leqq C'\sqrt{|t_1-t_2|} \tag{43}$$

for any $(y, t_1), (y, t_2) \in Q^+_{\tau,\delta} := \{(y, t) \in Q^+_\tau \mid y \geqq \delta\}$, with $C' > 0$ only depending on $\|v_0'\|_\infty$, $\|v_0''\|_\infty$, A and δ (see [12]). Observe that inequality (42) implies the continuity of v_y in $\overline{\mathbb{R}}_+ \times (0, \tau]$. Then by the results in [12] claim (iii) will follow, if we prove that

$$|v_y(0, t_1) - v_y(0, t_2)| \leqq C'' \sqrt{|t_1 - t_2|} \tag{44}$$

for any $t_1, t_2 \in [0, \tau]$, with $C'' > 0$ only depending on $\|v_0'\|_\infty$, $\|v_0''\|_\infty$ and A. It also follows that $v_y \in C(\overline{Q}^+_\tau)$; thus $v \in C([0, \tau]; H^{1,\infty}(\overline{\mathbb{R}}_+))$ (see (38)).

To prove (44) we proceed as above to prove claim (ii). Differentiating with respect to y (39), integrating by parts and using again the equalities $v_0(0) = \kappa$, $\Lambda_z(y, 0; t) = -2\, K_y(y, t)$ we obtain

$$v_y(0, t) = 2 \int_0^\infty K(z, t) v_0'(z)\, dz - 2 \int_0^t \xi'(s) \int_0^\infty K_z(z, t - s) v_z(z, s)\, dz\, ds.$$

For any $t_1, t_2 \in [0, \tau]$, there holds

$$\begin{aligned} &\left| \int_0^\infty (K(z, t_1) - K(z, t_2))\, v_0'(z)\, dz \right| \\ &\quad \leqq \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-z^2) \left| v_0'(\sqrt{4\alpha t_1}\, z) - v_0'(\sqrt{4\alpha t_2}\, z) \right| dz \\ &\quad \leqq \sqrt{\frac{\alpha}{\pi}}\, \|v_0''\|_\infty \sqrt{|t_1 - t_2|}. \end{aligned}$$

Similarly, using inequality (40) we obtain by a lengthy calculation: (with $v_z = v_z(z, s)$)

$$\begin{aligned} &\left| \int_0^{t_1} \xi'(s) \int_0^\infty K_z(z, t_1 - s) v_z\, dz\, ds - \int_0^{t_2} \xi'(s) \int_0^\infty K_z(z, t_2 - s) v_z\, dz\, ds \right| \\ &\quad \leqq C''' \sqrt{|t_1 - t_2|} \end{aligned}$$

with $C''' > 0$ only depending on $\|v_0'\|_\infty$, $\|v_0''\|_\infty$ and A. Hence inequality (44) follows.

(iv) Observe that

$$\begin{aligned} \|v\|^{(1+\sigma)}_{Q^+_\tau} = \|v\|_\infty + \|v_y\|_\infty &+ \sup_{(y,s),(y,t) \in Q^+_\tau, s \neq t} \frac{|v(y, s) - v(y, t)|}{|s - t|^{(1+\sigma)/2}} \\ &+ \sup_{(y,t),(y',t) \in Q^+_\tau, y \neq y'} \frac{|v_y(y, t) - v_y(y', t)|}{|y - y'|^\sigma} \\ &+ \sup_{(y,s),(y,t) \in Q^+_\tau, s \neq t} \frac{|v_y(y, s) - v_y(y, t)|}{|s - t|^{\sigma/2}}. \end{aligned}$$

Then Claim (iv) of the statement follows from the above results (in particular, see (31), (37), (38), (41) and (42)). This completes the proof. □

For any $k \in \mathbb{N}$, consider the problem

$$(P_k)_\pm \qquad \begin{cases} v_{kt} = \alpha_\pm v_{kyy} \pm \xi'_{k-1}(t) v_{ky} & \text{in } Q_\tau^+ \\ v_k(0,t) = \kappa_\pm & t \in (0,\tau] \\ v_k(y,0) = v_0^\pm(y) & y \in \mathbb{R}_+, \end{cases}$$

where the sequence $\{\xi_k\}$ is defined as follows:

$$\xi_0(t) := Mt \quad (M \in \mathbb{R}),$$

$$\xi'_k(t) := -\frac{\alpha_+ v^+_{ky}(0,t) + \alpha_- v^-_{ky}(0,t)}{\kappa_+ - \kappa_-}, \quad \xi_k(0) = 0 \quad (k \in \mathbb{N}) \tag{45}$$

for any $t \in (0,\tau]$.

In view of Proposition 2, a recursive argument shows that for any $k \in \mathbb{N}$ there exists a unique classical solution $v_k^\pm$ of problem $(P_k)_\pm$. Both v_k^- and v_k^+ have the regularity asserted in Proposition 2 for the solution v of problem (29); thus in particular $v_k^\pm \in C^{1+\sigma,\frac{1+\sigma}{2}}(\overline{Q}_\tau^+)$, $v_{ky}^\pm(0,\cdot) \in C^{\frac{\sigma}{2}}([0,\tau])$ and $\xi_k \in C^{1+\frac{\sigma}{2}}([0,\tau])$ (see (45); $\sigma \in (0,1)$).

Theorem 4 will be proved by letting $k \to \infty$ both in $(P_k)_\pm$ and in (45). To this purpose uniform estimates of the sequences $\{\|v_k^\pm\|^{(1+\sigma)}_{Q_\tau^+}\}$ and $\{A_k\}$, where $A_k := \sup_{t\in[0,\tau]} |\xi'_k(t)|$, are needed. Actually, since $\|v_k^\pm\|^{(1+\sigma)}_{Q_\tau^+}$ can be estimated in terms of $\|v_0^\pm\|_{2,\infty}$ and A_{k-1} (see Proposition 2(iv)), a uniform estimate of $\{A_k\}$ will do. Such estimate is the content of Proposition 3, whose proof requires two preliminary lemmata.

Lemma 3. *Let w be the solution of the problem*

$$\begin{cases} w_t = \alpha w_{yy} + M w_y & \text{in } Q_\tau^+ \\ w(0,t) = 0 & t \in (0,\tau] \\ w(y,0) = w_0(y) & y \in \mathbb{R}_+, \end{cases} \tag{46}$$

where $w_0 \in H^{0,\infty}(\overline{\mathbb{R}}_+)$, $w_0(0) = 0$ with derivative $w'_0 \in L^\infty(\mathbb{R}_+)$. Then

$$|w_y(0,t)| \leqq \|w'_0\|_\infty \left\{ M\sqrt{\frac{t}{\alpha\pi}} \exp\left(-\frac{M^2 t}{4\alpha}\right) + \left(2 + \frac{M^2 t}{\alpha}\right) \operatorname{Erf}\left(\frac{M}{2}\sqrt{\frac{t}{\alpha}}\right)\right\} \tag{47}$$

for any $t \in (0,\tau]$, where

$$\operatorname{Erf}(y) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{y} \exp(-z^2)\, \mathrm{d}z \quad (y \in \mathbb{R}).$$

Proof. The solution of problem (46) reads

$$w(y,t) = \exp\left(-\frac{My}{2\alpha} - \frac{M^2 t}{4\alpha}\right)\int_0^\infty \Lambda(y,z;t)\, w_0(z)\, \exp\left(\frac{Mz}{2\alpha}\right)\, \mathrm{d}z,$$

with the kernel Λ defined in (35) (for example see [31]). Plainly, we have

$$\begin{aligned} w_y(0,t) &= \exp\left(-\frac{M^2 t}{4\alpha}\right)\int_0^\infty \Lambda_y(0,z;t)\, w_0(z)\, \exp\left(\frac{Mz}{2\alpha}\right)\, \mathrm{d}z \\ &= \frac{1}{2\sqrt{\pi}\,(\alpha\, t)^{3/2}}\int_0^\infty z\, w_0(z)\, \exp\left[-\frac{1}{4\alpha}\left(\frac{z}{\sqrt{t}} - M\sqrt{t}\right)^2\right] \mathrm{d}z \\ &= \frac{1}{\alpha\sqrt{\pi t}}\int_{-\frac{M}{2}\sqrt{\frac{t}{\alpha}}}^\infty \left(2\sqrt{\alpha}\, p + M\sqrt{t}\right)\, w_0\left(2\sqrt{\alpha t}\, s + Mt\right) \mathrm{e}^{-p^2}\, \mathrm{d}p, \end{aligned}$$

whence

$$|w_y(0,t)| \leqq \frac{\|w_0'\|_\infty}{\alpha\sqrt{\pi}}\int_{-\frac{M}{2}\sqrt{\frac{t}{\alpha}}}^\infty \left(2\sqrt{\alpha}\, p + M\sqrt{t}\right)^2\, \mathrm{e}^{-p^2}\, \mathrm{d}p.$$

From the above inequality the estimate (47) follows by elementary calculations. □

Lemma 4. *Let v be the solution of problem (29) considered in Proposition 2. Then for any $t \in (0,\tau]$*

$$|v_y(0,t)| \leqq \|v_0'\|_\infty\, P\left(\frac{A}{2}\sqrt{\frac{t}{\alpha}}\right), \tag{48}$$

where $A := \sup_{t\in[0,\tau]} |\xi'(t)|$ and

$$P(p) := \frac{2p^3}{\sqrt{\pi}} + p^2 + \frac{4\,p}{\sqrt{\pi}} + 1. \tag{49}$$

Proof. The function $w := v - \kappa$ solves the problem

$$\begin{cases} w_t = \alpha\, w_{yy} + \xi' w_y & \text{in } Q_\tau^+ \\ w(0,t) = 0 & t \in (0,\tau] \\ w(y,0) = w_0(y) & y \in \mathbb{R}_+ \end{cases} \tag{50}$$

with initial data $w_0 := v_0 - \kappa$. Set

$$\begin{aligned} [w_0]_+(y) &:= \max\{w_0(y), 0\} \quad (y \geqq 0), \\ \tilde{w}_0(y) &:= \begin{cases} 0 & \text{if } y = 0 \\ \max_{z\in[0,y]} [w_0]_+(z) & \text{if } y > 0. \end{cases} \end{aligned}$$

It is easily seen that $\tilde{w}_0 \in H^{0,\infty}(\overline{\mathbb{R}}_+)$, $\tilde{w}_0(0) = 0$ with derivative $\tilde{w}_0' \in L^\infty(\mathbb{R}_+)$; moreover, $\tilde{w}_0$ is nondecreasing and $\tilde{w}_0 \geqq 0$ in $\overline{\mathbb{R}}_+$. Hence by the maximum principle there holds $\tilde{w}_y \geqq 0$ in Q_τ^+, $\tilde{w}$ being the solution of the problem

$$\begin{cases} w_t = \alpha\, w_{yy} + A\, w_y & \text{in } Q_\tau^+ \\ w(0,t) = 0 & t \in (0,\tau] \\ w(y,0) = \tilde{w}_0(y) & y \in \mathbb{R}_+. \end{cases} \tag{51}$$

Plainly, this implies that $\tilde{w}$ is a supersolution of problem (50) (recall that by definition $\tilde{w}_0 \geqq [w_0]_+ \geqq w_0$). Thus by classical comparison results

$$w(y,t) \leqq \tilde{w}(y,t) \quad \text{in } \; Q_\tau^+.$$

Since $w(0,t) = \tilde{w}(0,t) = 0$, we also have

$$v_y(0,t) = w_y(0,t) \leqq \tilde{w}_y(0,t) \quad \text{for any } t \in (0,\tau]. \tag{52}$$

In addition, there holds

$$\|\tilde{w}_0'\|_\infty \leqq \|v_0'\|_\infty. \tag{53}$$

Then by inequalities (47), (52) and (53) we have

$$\begin{aligned} v_y(0,t) \leqq \|v_0'\|_\infty \Bigg\{ & A\sqrt{\frac{t}{\alpha\pi}}\, \exp\left(-\frac{A^2 t}{4\alpha}\right) \\ & + \left(2 + \frac{A^2 t}{2\alpha}\right) \operatorname{Erf}\left(\frac{A}{2}\sqrt{\frac{t}{\alpha}}\right)\Bigg\} \end{aligned} \tag{54}$$

for any $t \in (0,\tau]$.

Similarly, set

$$\begin{aligned} &[w_0]_-(y) := -\min\{w_0(y), 0\} \quad (y \geqq 0), \\ &\hat{w}_0(y) := \begin{cases} 0 & \text{if } y = 0 \\ -\max\limits_{z \in [0,y]} [w_0]_-(z) & \text{if } y > 0\,; \end{cases} \end{aligned}$$

then $\hat{w}_0 \in H^{0,\infty}(\overline{\mathbb{R}}_+)$, $\hat{w}_0(0) = 0$ with derivative $\hat{w}_0' \in L^\infty(\mathbb{R}_+)$.; moreover, $\hat{w}_0 \leqq 0$ and nonincreasing in $\overline{\mathbb{R}}_+$. Arguing as before, it is easily checked that the solution $\hat{v}$ of problem (51) with $\tilde{w}_0$ replaced by $\hat{w}_0$ is a subsolution of problem (50), and

$$v_y(0,t) = w_y(0,t) \geqq \hat{w}_y(0,t) \quad \text{for any } t \in (0,\tau]. \tag{55}$$

Since

$$\|\hat{w}_0'\|_\infty \leqq \|v_0'\|_\infty, \tag{56}$$

by inequalities (47), (55) and (56) we have

$$v_y(0,t) \geqq -\|v_0'\|_\infty \left\{ A\sqrt{\frac{t}{\alpha\pi}}\,\exp\left(-\frac{A^2 t}{4\alpha}\right) + \left(2+\frac{A^2 t}{2\alpha}\right)\mathrm{Erf}\left(\frac{A}{2}\sqrt{\frac{t}{\alpha}}\right)\right\} \tag{57}$$

for any $t \in (0,\tau]$. Then from the inequality

$$\mathrm{Erf}\,(p) \leqq \mathrm{Erf}\,(0) + \mathrm{Erf}\,'(0)\,p = \frac{1}{2} + \frac{p}{\sqrt{\pi}} \quad (p \in \mathbb{R}_+)$$

and inequalities (54), (57) we easily obtain (48). Hence the conclusion follows. □

Proposition 3. *Let* $v_k^\pm$ *solve problem* $(P_k)_\pm$ *and* ξ_k *be defined by* (45) ($k \in \mathbb{N}$). *Then there exist a constant* $K > 0$ *and* $\tau = \tau(K) \in (0,T]$, *only depending on* $\|v_0'\|_\infty$, *such that*

$$A_k := \sup_{t\in[0,\tau]} |\xi_k'(t)| \leqq K \quad \textit{for any}\ \ k \in \mathbb{N}. \tag{58}$$

Proof. Set

$$K := \max\left\{A_0,\, 2\max\left\{\|(v_0^+)'\|_\infty, \|(v_0^-)'\|_\infty\right\}\frac{\alpha_- + \alpha_+}{\kappa_+ - \kappa_-}\right\},$$

$$Q(t) := C\,\left\{\alpha_+ P\left(\frac{K}{2}\sqrt{\frac{t}{\alpha_+}}\right) + \alpha_- P\left(\frac{K}{2}\sqrt{\frac{t}{\alpha_-}}\right)\right\} \quad (t \geqq 0),$$

where

$$C := \frac{\max\left\{\|(v_0^+)'\|_\infty, \|(v_0^-)'\|_\infty\right\}}{\kappa_+ - \kappa_-}$$

and the function P is defined by (49) (recall that by definition $\xi_0'(t) \equiv M$; thus $A_0 = |M|$). Set $\tau := \min\left\{Q^{-1}(K), T\right\}$; the definition is well posed and $\tau \in (0,T]$, since Q is increasing and

$$Q(0) = \max\left\{\|(v_0^+)'\|_\infty, \|(v_0^-)'\|_\infty\right\}\frac{\alpha_- + \alpha_+}{\kappa_+ - \kappa_-}.$$

Consider problem $(P_k)_\pm$ with the above choice of τ. By inequality (48), we have

$$|v_{ky}^\pm(0,t)| \leqq \|(v_0^\pm)'\|_\infty\, P\left(\frac{A_{k-1}}{2}\sqrt{\frac{t}{\alpha_\pm}}\right) \quad (k \in \mathbb{N})$$

for any $t \in (0,\tau]$. Then from the definition (45), we obtain immediately

$$|\xi_k'(t)| \leqq C\,\left\{\alpha_+ P\left(\frac{A_{k-1}}{2}\sqrt{\frac{t}{\alpha_+}}\right) + \alpha_- P\left(\frac{A_{k-1}}{2}\sqrt{\frac{t}{\alpha_-}}\right)\right\}$$

for any $t \in (0, \tau]$ and $k \in \mathbb{N}$. Since P is increasing, this gives

$$A_k \leqq C\left\{\alpha_+ P\left(\frac{A_{k-1}}{2}\sqrt{\frac{\tau}{\alpha_+}}\right) + \alpha_- P\left(\frac{A_{k-1}}{2}\sqrt{\frac{\tau}{\alpha_-}}\right)\right\} \tag{59}$$

for any $k \in \mathbb{N}$. Since $A_0 \leqq K$, from (59), we obtain

$$A_1 \leqq C\left\{\alpha_+ P\left(\frac{K}{2}\sqrt{\frac{\tau}{\alpha_+}}\right) + \alpha_- P\left(\frac{K}{2}\sqrt{\frac{\tau}{\alpha_-}}\right)\right\} = Q(\tau) \leqq K.$$

By the same token, the inequalities $A_{k-1} \leqq K$ and (59) imply $A_k \leqq K$ for any $k \in \mathbb{N}$. Hence the conclusion follows. □

Now we can prove Theorem 4.

Proof. Consider problem $(P_k)_\pm$ with initial data $v_0^\pm(y) := u_0(\pm y)$. Let $v_k^\pm$ be its solution, and ξ_k be defined by (45) ($k \in \mathbb{N}$). Choose $K > 0$ and $\tau \in (0, T]$ as in Proposition 3; observe that now

$$K = \max\left\{A_0, 2\|u_0'\|_\infty \frac{\alpha_- + \alpha_+}{\kappa_+ - \kappa_-}\right\}.$$

Then inequality (58) holds; moreover, by Proposition 2 there exists $K_1 \geqq K$ (only depending on the norm $\|u_0\|_{2,\infty}$) such that

$$\|v_k^\pm\|_{Q_\tau^+}^{(1+\sigma)} \leqq K_1 \quad \text{for any } k \in \mathbb{N} \quad (\sigma \in (0,1)). \tag{60}$$

In particular, there holds

$$\|v_{ky}^\pm\|_{Q_\tau^+}^{(\sigma)} \leqq K_1 \quad \text{for any } k \in \mathbb{N}; \tag{61}$$

thus

$$\begin{aligned} \|\xi_k\|_{(0,\tau)}^{(1+\frac{\sigma}{2})} &:= \sup_{t\in(0,\tau)}\left\{|\xi_k(t)| + |\xi_k'(t)|\right\} + \sup_{s,t\in(0,\tau), s\neq t}\frac{|\xi_k'(s) - \xi_k'(t)|}{|s-t|^{\sigma/2}} \\ &\leqq \frac{\alpha_- + \alpha_+}{\kappa_+ - \kappa_-}(1+\tau)K_1 \quad \text{for any } k \in \mathbb{N} \quad (\sigma \in (0,1)). \end{aligned} \tag{62}$$

Now recall that for any $\sigma, \sigma' \in (0, 1)$, $\sigma' > \sigma$ the embedding operators from $C^{1+\sigma', \frac{1+\sigma'}{2}}(\overline{B}_\tau^+)$ to $C^{1+\sigma, \frac{1+\sigma}{2}}(\overline{B}_\tau^+)$, B_τ^+ denoting any bounded subset of Q_τ^+, and from $C^{1+\frac{\sigma'}{2}}([0, \tau])$ to $C^{1+\frac{\sigma}{2}}([0, \tau])$ are compact. Hence for any $\sigma \in (0, 1)$:

– in view of the uniform estimate (62), there exist a subsequence $\{\xi_k\} \subseteq C^{1+\frac{\sigma}{2}}([0, \tau])$ and $\xi \in C^{1+\frac{\sigma}{2}}([0, \tau])$ such that

$$\xi_k \to \xi \quad \text{in } C^{1+\frac{\sigma}{2}}([0, \tau]); \tag{63}$$

– in view of the uniform estimate (60), by a diagonal argument there exist a subsequence $\{v_k^\pm\} \subseteq C^{1+\sigma,\frac{1+\sigma}{2}}(\overline{Q}_\tau^+)$ and a function $v^\pm \in C^{1+\sigma,\frac{1+\sigma}{2}}(\overline{B}_\tau^+)$ such that

$$v_k^\pm \to v^\pm \quad \text{in } C^{1+\sigma,\frac{1+\sigma}{2}}(\overline{B}_\tau^+)$$

for any bounded subset $B_\tau^+ \subseteq Q_\tau^+$. In particular, there holds

$$v_{ky}^\pm(0,\cdot) \to v_y^\pm(0,\cdot) \quad \text{in } C^{\frac{\sigma}{2}}([0,\tau]). \tag{64}$$

Consider the weak formulation of the differential equation in problem $(P_k)_\pm$, namely:

$$\int_{\mathbb{R}_+} v_k^\pm(y,t_1)\,\eta(y,t_1)\,\mathrm{d}y - \int_0^{t_1}\int_{\mathbb{R}_+} v_k^\pm\,\eta_t\,\mathrm{d}y\,\mathrm{d}t = \int_0^{t_1}\int_{\mathbb{R}_+} v_{ky}^\pm\left\{-\alpha_\pm\,\eta_y \pm \xi_{k-1}'(t)\eta\right\}\mathrm{d}y\,\mathrm{d}t$$

for any $t_1 \in (0,\tau]$, $\eta \in C_0^\infty(Q_\tau^+)$. In view of the above remarks, taking the limit as $k \to \infty$ in the above equality gives:

$$\int_{\mathbb{R}_+} v^\pm(y,t_1)\,\eta(y,t_1)\,\mathrm{d}y - \int_0^{t_1}\int_{\mathbb{R}_+} v^\pm\,\eta_t\,\mathrm{d}y\,\mathrm{d}t = \int_0^{t_1}\int_{\mathbb{R}_+} v_y^\pm\left\{-\alpha_\pm\,\eta_y \pm \xi'(t)\eta\right\}\mathrm{d}y\,\mathrm{d}t$$

for any t_1 and η as above. Moreover, $v^\pm$ solves problem $(P)_\pm$ in Q_τ^+ with ξ given by (63), for it belongs to $C(\overline{Q}_\tau^+) \cap L^\infty(Q_\tau^+)$ and satisfies the initial and boundary condition of problem $(P)_\pm$. By uniqueness results in the class of bounded solutions, $v^\pm$ coincides with the classical solution of the same problem, whose existence is ensured by Proposition 2; in particular, $v_y^\pm \in L^\infty(Q_\tau^+)$, and there holds

$$\lim_{y\to\infty} v_y^\pm(y,t) = 0 \quad \text{for any } t \in (0,\tau].$$

On the other hand, taking the limit as $k \to \infty$ in equality (45) gives by (63), (64):

$$\xi'(t) = -\frac{\alpha_+ v_y^+(0,t) + \alpha_- v_y^-(0,t)}{\kappa_+ - \kappa_-} \quad \text{for any } t \in [0,\tau]. \tag{65}$$

Since the trace $v_y(0,\cdot)$ belongs to $C^{\frac{1}{2}}([0,\tau])$ (see Proposition 2(iii)), we obtain that $\xi \in C^{\frac{3}{2}}([0,\tau])$. Observe also that for $t = 0$ the above equation reads

$$\xi'(0) = -\frac{\alpha_+\,u_0'(0^+) - \alpha_-\,u_0'(0^-)}{\kappa_+ - \kappa_-}. \tag{66}$$

Define $v = v(y,t)$ in S_τ as follows:

$$v(y,t) := \begin{cases} v^+(y,t) & \text{if } (y,t) \in \overline{Q}_\tau^+ \\ v^-(-y,t) & \text{if } (y,t) \in \overline{Q}_\tau^- \setminus \{y=0\}. \end{cases}$$

It is easily seen that the above function has all the properties mentioned in Remark 6. In particular,

(α) The invariance property (25) follows from assumption (A)(i) and the inequalities $\kappa_- \leqq b$, $\kappa_+ \geqq c$. In fact, b is a supersolution of problem $(P_k)_-$, c a subsolution of problem $(P_k)_+$ for any $k \in \mathbb{N}$, and the property follows by comparison results.

(β) The Rankine–Hugoniot condition (28) follows from (65) and the definition of v (see (30)).

(γ) by assumption (ii) and (66)) there holds $\xi'(0) \neq 0$; thus $\xi'(t) \neq 0$ for any $t \in (0, \tau]$ by continuity, with a possibly smaller τ.

Then the couple (u, ξ) with

$$u(x,t) := v(x - \xi(t), t) \quad ((x,t) \in S_\tau)$$

is a solution in S_τ of the moving boundary problem at the value C given by (21). This completes the proof. □

4.2. Steady boundary problem

Let us make the following definition, which is the counterpart of Definition 4 in the present case.

Definition 5. *By a solution in S_τ ($\tau \in (0, T]$) of the steady boundary problem we mean any couple $(0, u)$ such that the function $u = u(x,t)$ satisfies the following:*

(i) $u \in L^\infty(S_\tau) \cap C\left(\overline{Q_\tau^+} \setminus \{x = 0\}\right) \cap C\left(\overline{Q_\tau^-} \setminus \{x = 0\}\right)$ *and for any $t \in (0, \tau]$ there exist finite the limits*

$$\lim_{\eta \to 0^+} u(\pm\eta, t) =: u(0^\pm, t)\,;$$

(ii) $u \in C^{2,1}\left(Q_\tau^+\right) \cap C^{2,1}\left(Q_\tau^-\right)$, $u_x \in L^\infty(S_\tau)$ *and for any $t \in (0, \tau]$ there exist finite the limits*

$$\lim_{\eta \to 0^+} u_x(\pm\eta, t) =: u_x(0^\pm, t)\,;$$

(iii) *u is a classical solution of the problem*

$$\begin{cases} u_t = \alpha_\pm u_{xx} & \text{in } Q_\tau^\pm \\ u(x,0) = u_0(x) & x \in \mathbb{R}_\pm \setminus \{0\}, \end{cases}$$

such that

$$\lim_{x \to \pm\infty} u_x(x,t) = 0\,;$$

(iv) *for any $t \in (0, \tau]$*

$$\alpha_- u(0^-, t) + \beta_- = \alpha_+ u(0^+, t) + \beta_+, \tag{67}$$

$$\alpha_- u_x(0^-, t) = \alpha_+ u_x(0^+, t). \tag{68}$$

Remark 10. If a solution u of the steady boundary problem satisfies the invariance condition:

$$u\left(Q_\tau^-\right) \subseteq (-\infty, b], \quad u\left(Q_\tau^+\right) \subseteq [c, \infty), \tag{69}$$

the couple $(0, u)$ satisfies the requirements (i)–(vi) of Definition 1. Moreover, since $\xi' = 0$ and (68) holds, both Rankine–Hugoniot and entropy conditions are satisfied (see (8)–(9)); then $(0, u)$ is a two-phase entropy solution of problem (1).

Concerning existence of solutions to the steady boundary problem, the following holds.

Theorem 5. *Let assumption (A_1) be satisfied. Then there exists a solution in S_T of the steady boundary problem.*

To prove the above result, consider preliminarily the problems

$$(N)_\pm \quad \begin{cases} u_t = \alpha_\pm u_{xx} & \text{in } Q_T^+ \\ u_x(0,t) = \pm\frac{\theta(t)}{\alpha_\pm} & t \in (0,T] \\ u(x,0) = u_0^\pm(x) := u_0(\pm x) & x \in \mathbb{R}_+ \end{cases}$$

with $\theta \in C^{\frac{1}{2}}([0,T])$. For any $(x,t) \in Q_T^+$ define:

$$u_\theta^\pm(x,t) := \mp 2\int_0^t K_\pm(x, t-s)\,\theta(s)\,\mathrm{d}s + \int_0^\infty \Gamma_\pm(x, y; t)\, u_0^\pm(y)\,\mathrm{d}y, \tag{70}$$

where

$$\Gamma_\pm(x, y; t) := K_\pm(x-y, t) + K_\pm(x+y, t)$$

and

$$K_\pm(x,t) := \frac{1}{\sqrt{4\pi\,\alpha_\pm\, t}}\,\mathrm{e}^{-\frac{x^2}{4\alpha_\pm t}}.$$

It is easily seen that $u_\theta^\pm$ belongs to $C^{2,1}\left(Q_T^+\right) \cap C\left(\overline{Q_T^+}\right) \cap L^\infty(Q_T^+)$. From (70) we obtain (with $\theta = \theta(s)$ and $u_0^\pm = u_0^\pm(y)$, $(u_0^\pm)' = (u_0^\pm)'(y)$)

$$\begin{aligned}(u_\theta^\pm)_x(x,t) &= \mp 2\int_0^t (K_\pm)_x(x, t-s)\,\theta\,\mathrm{d}s + \int_0^\infty (\Gamma_\pm)_x(x, y; t)\, u_0^\pm\,\mathrm{d}y\,\mathrm{d}s \\ &= \mp 2\int_0^t (K_\pm)_x(x, t-s)\,\theta\,\mathrm{d}s + \int_0^\infty \Lambda_\pm(x, y; t)\,(u_0^\pm)'\,\mathrm{d}y\,\mathrm{d}s, \end{aligned} \tag{71}$$

where $\Lambda_\pm$ denotes the function (35) with K replaced by $K_\pm$. Arguing as in the proof of Lemma 2, one easily sees that

$$(u_\theta^\pm)_x(0,t) := \lim_{\eta\to 0^+} (u_\theta^\pm)_x(\eta, t) = \pm\frac{\theta(t)}{\alpha_\pm} \tag{72}$$

for any $t \in (0,T]$; thus $u_\theta^\pm$ is a classical solution of problem $(N)_\pm$. Moreover, $(u_\theta^\pm)_x \in L^\infty(Q_T^+)$ and there holds:

$$\lim_{x\to\infty} (u_\theta^\pm)_x(x,t) = 0.$$

Remark 11. The derivative $(u_\theta^\pm)_x$ satisfies the problem

$$\begin{cases} w_t = \alpha\, w_{xx} & \text{in } Q_\tau^+ \\ w(0,t) = \pm\frac{\theta(t)}{\alpha} & t \in (0,\tau] \\ w(x,0) = (u_0^\pm)'(x) & x \in \mathbb{R}_+. \end{cases} \tag{73}$$

Observe that equality (73) corresponds to (34), with h replaced by $\pm 2\theta$ and u_0 by $(u_0^\pm)'$ (in fact, the integral equation (36) reduces to the equality $h = 2\theta$ if $\xi \equiv 0$).

Since by assumption $\theta \in C^{\frac{1}{2}}([0,T])$, there holds:

$$(u_\theta^\pm)_x(0,0) := \lim_{t\to 0}(u_\theta^\pm)_x(0,t) = \pm\frac{\theta(0)}{\alpha_\pm}.$$

However, $(u_\theta^\pm)_x$ is not continuous at $(0,0)$, since we do not assume the compatibility condition of order zero for problem (73), namely $\theta(0) = \pm\alpha_\pm(u_0^\pm)'(0)$. Observe that the latter would imply the equality $\alpha_+\, u_0'(0^+) = \alpha_-\, u_0'(0^-)$ (in this connection, see Remark 12).

Now we can prove Theorem 5.

Proof. Fix $\theta \in C^{\frac{1}{2}}([0,T])$. Define u_θ in S_T as follows:

$$u_\theta(x,t) := \begin{cases} u_\theta^+(x,t) & \text{if } (x,t) \in \overline{Q_T^+} \\ u_\theta^-(-x,t) & \text{if } (x,t) \in \overline{Q_T^-} \setminus \{x = 0\}, \end{cases} \tag{74}$$

with $u_\theta^\pm$ given by (70). In view of the properties of $u_\theta^\pm$, the above function has the properties (i)–(iii) of Definition 5; moreover, equality (68) follows from (72) and the definition (74) of u. Then the conclusion follows, if we prove the following *Claim:* There exists $\bar\theta \in C^{\frac{1}{2}}([0,T])$ such that $u_{\bar\theta}$ satisfies equality (67).

To this purpose, observe that the definition (70) for $x = 0$ gives easily:

$$\begin{aligned}\left[\phi_+(u_\theta^+) - \phi_-(u_\theta^-)\right](0,t) \;=\; &-\frac{\sqrt{\alpha_+}+\sqrt{\alpha_-}}{\sqrt{\pi}}\int_0^t \frac{\theta(s)}{\sqrt{t-s}}\,\mathrm{d}s \\ &+2\int_0^\infty K_+(y,t)\,[\phi_+(u_0^+)](y)\,\mathrm{d}y \\ &-2\int_0^\infty K_-(y,t)\,[\phi_-(u_0^-)](y)\,\mathrm{d}y.\end{aligned}$$

Hence equality (67) holds, if the function θ satisfies the Abel integral equation:

$$\int_0^t \frac{\theta(s)}{\sqrt{t-s}}\,\mathrm{d}s = 2\sqrt{\pi}\,F(t), \tag{75}$$

where

$$\begin{aligned}F(t) := \frac{1}{\sqrt{\alpha_+}+\sqrt{\alpha_-}}\int_0^\infty &\big\{K_+(y,t)\,[\phi_+(u_0^+)](y) \\ &- K_-(y,t)\,[\phi_-(u_0^-)](y)\big\}\,\mathrm{d}y.\end{aligned} \tag{76}$$

Clearly, $F \in C^1((0,T])$ and $\lim_{t\to 0} F(t) = 0$; thus $F \in C([0,T])$ (here use of assumption (A_1)(iii) is made). Moreover, observe that

$$\begin{aligned}\int_0^\infty & (K_\pm)_t(y,t)[\phi_\pm(u_0^\pm)](y)\,\mathrm{d}y \\ &= \alpha_\pm \int_0^\infty (K_\pm)_{yy}(y,t)\,[\phi_\pm(u_0^\pm)](y)\,\mathrm{d}y \\ &= \frac{\alpha_\pm[\phi_\pm(u_0^\pm)]'(0)}{\sqrt{4\alpha_\pm t}} + \alpha_\pm \int_0^\infty K_\pm(y,t)\,[\phi_\pm(u_0^\pm)]''(y)\,\mathrm{d}y. \end{aligned} \tag{77}$$

Then it is easily seen that

$$F'(t) = \frac{C}{\sqrt{t}} + g(t) \quad \text{for any} \ \ t \in (0,T], \tag{78}$$

where

$$\begin{aligned} C &:= \frac{1}{2\sqrt{\pi}}\,\frac{\sqrt{\alpha_+^3}\,u_0'(0^+) + \sqrt{\alpha_-^3}\,u_0'(0^-)}{\sqrt{\alpha_+}+\sqrt{\alpha_-}}, \\ g(t) &:= \frac{1}{\sqrt{\alpha_+}+\sqrt{\alpha_-}} \int_0^\infty \big\{\alpha_+ K_+(y,t)\,[\phi_+(u_0^+)]''(y) \\ &\qquad\qquad - \alpha_- K_-(y,t)\,[\phi_-(u_0^-)]''(y)\big\}\ \mathrm{d}y. \end{aligned}$$

In view of (78), there holds $F \in W^{1,1}((0,T))$. Observe that

$$\lim_{t\to 0} g(t) = \frac{\alpha_+^2 u_0''(0^+) - \alpha_-^2 u_0''(0^-)}{2\left(\sqrt{\alpha_+}+\sqrt{\alpha_-}\right)},$$

thus $g \in C([0,T])$.

By a standard calculation from (75) we obtain

$$\int_0^t \theta(s)\,\mathrm{d}s = \frac{2}{\sqrt{\pi}} \int_0^t \frac{F(s)}{\sqrt{t-s}}\,\mathrm{d}s.$$

In view of the regularity of F, we can integrate by parts the right-hand side of the above equality, obtaining

$$\int_0^t \theta(s)\,\mathrm{d}s = \frac{4}{\sqrt{\pi}} \int_0^t F'(s)\sqrt{t-s}\,\mathrm{d}s,$$

which in turn implies:

$$\theta(t) = \frac{2}{\sqrt{\pi}} \int_0^t \frac{F'(s)}{\sqrt{t-s}}\,\mathrm{d}s \quad \text{for any} \ \ t \in (0,T].$$

To complete the proof of the claim, let us show that the function

$$\bar\theta(t) := \frac{2}{\sqrt{\pi}} \int_0^t \frac{F'(s)}{\sqrt{t-s}}\,\mathrm{d}s$$

belongs to $C^{\frac{1}{2}}([0, T])$. In fact, using (78), we obtain:

$$\bar{\theta}(t) = \frac{2C}{\sqrt{\pi}} \int_0^t \frac{ds}{\sqrt{s(t-s)}} + \frac{2}{\sqrt{\pi}} \int_0^t \frac{g(s)}{\sqrt{t-s}} ds$$
$$= \frac{\sqrt{\alpha_+^3}\, u_0'(0^+) + \sqrt{\alpha_-^3}\, u_0'(0^-)}{\sqrt{\alpha_+} + \sqrt{\alpha_-}} + G(t), \quad (79)$$

where

$$G(t) := \frac{2}{\sqrt{\pi}} \int_0^t \frac{g(s)}{\sqrt{t-s}} ds.$$

Since $g \in C([0, T])$, for any $t \in (0, T]$ there holds:

$$|G(t)| \leqq \frac{4\|g\|_\infty}{\sqrt{\pi}} \sqrt{t}. \quad (80)$$

In addition, for any $t_1, t_2 \in (0, T]$ with $t_1 < t_2$, we have plainly:

$$|G(t_1) - G(t_2)| \leqq \frac{2}{\sqrt{\pi}} \left\{ \int_{t_1}^{t_2} \frac{|g(s)|}{\sqrt{t_2 - s}} ds + \int_0^{t_1} |g(s)| \left[\frac{1}{\sqrt{t_1 - s}} - \frac{1}{\sqrt{t_2 - s}} \right] ds \right\}$$
$$\leqq 6\|g\|_\infty \sqrt{t_2 - t_1}.$$

This shows that $\bar{\theta} \in C^{\frac{1}{2}}([0, T])$; thus completing the proof of the claim; hence the conclusion follows. □

Remark 12. As already observed, the space derivative $(u_{\bar{\theta}})_x$ is discontinuous at the origin whenever $\alpha_+ u_0'(0^+) \neq \alpha_- u_0'(0^-)$. In fact, by (79), (80) there holds

$$\lim_{t\to 0} \bar{\theta}(t) = \alpha_+ \lim_{t\to 0} (u_{\bar{\theta}})_x(0^+, t) = \frac{\sqrt{\alpha_+^3}\, u_0'(0^+) + \sqrt{\alpha_-^3}\, u_0'(0^-)}{\sqrt{\alpha_+} + \sqrt{\alpha_-}},$$

hence

$$\lim_{t\to 0} (u_{\bar{\theta}})_x(0^+, t) = u_0'(0^+) - \frac{\sqrt{\alpha_-}}{\alpha_+} \frac{\alpha_+ u_0'(0^+) - \alpha_- u_0'(0^-)}{\sqrt{\alpha_+} + \sqrt{\alpha_-}}.$$

This is a remarkable difference with respect to the moving boundary problem, where $u_x \in C(\bar{A}_\tau^+)$ (see Proposition 2 and the proof of Theorem 4).

4.3. Proof of existence results

Now we can prove Theorem 1.

Proof. (i) Let $\phi(u_0)(0) \in (A, B)$—namely, $u_0(0^-) < b$ and $u_0(0^+) > c$. We claim that in this case the solution $(0, u_{\bar\theta})$ of the steady boundary problem given by Theorem 5 is for small times a two-phase entropy solution of the problem.

To prove the claim, it suffices to show that the invariance conditions (69) are satisfied (see Remark 10). Since $u^{\pm}_{\bar\theta}$ is continuous in $\overline{Q^{\pm}_T}$, by the definition of $u_{\bar\theta}$ (see (74)) there exists $\tau > 0$ such that $u_{\bar\theta}(0^-, t) < b$, $u_{\bar\theta}(0^+, t) > c$ for any $t \in (0, \tau]$. Hence by assumption (A_1)(i) $v = c$ is a subsolution of the problem:

$$\begin{cases} v_t = \alpha_+ v_{xx} & \text{in } Q^+_\tau \\ v(0,t) = u_{\bar\theta}(0^+, t) & t \in (0, \tau] \\ v(x, 0) = u_0(x) & x \in \mathbb{R}_+. \end{cases}$$

This implies $u\left(Q^+_\tau\right) \subseteq [c, \infty)$ by comparison results. The proof of $u\left(Q^-_\tau\right) \subseteq (-\infty, b]$ is similar; thus the conclusion follows in this case.

(ii) Let $\phi(u_0)(0) = B$, $\alpha_+ u_0'(0^+) < \alpha_- u_0'(0^-)$. We claim that the couple $(0, u_{\bar\theta})$ is a local two-phase entropy solution in this case, too.

Consider the function

$$\phi(u_{\bar\theta})(0,t) = \alpha_- u_{\bar\theta}(0^-, t) + \beta_- = \alpha_+ u_{\bar\theta}(0^+, t) + \beta_+ \quad (t \in (0, T])$$

(see (67)). From (70) and (75), (76) we obtain:

$$\begin{aligned} \phi(u_{\bar\theta})(0,t) &= 2\int_0^\infty K_+(y,t)\,[\phi_+(u_0^+)](y)\,\mathrm{d}y - \sqrt{\frac{\alpha_+}{\pi}}\int_0^t \frac{\theta(s)}{\sqrt{t-s}}\,\mathrm{d}s \\ &= \frac{2}{\sqrt{\alpha_+}+\sqrt{\alpha_-}}\left\{\sqrt{\alpha_-}\int_0^\infty K_+(y,t)\,[\phi_+(u_0^+)](y)\,\mathrm{d}y \right. \\ &\qquad \left. + \sqrt{\alpha_+}\int_0^\infty K_-(y,t)\,[\phi_-(u_0^-)](y)\,\mathrm{d}y\right\}. \end{aligned} \tag{81}$$

Deriving the above expression and using (77) plainly gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\phi(u_{\bar\theta})(0,t)\right] = \frac{D}{\sqrt{t}} + h(t) \quad \text{for any } t \in (0, T], \tag{82}$$

where

$$D := \sqrt{\frac{\alpha_+\alpha_-}{\pi}}\,\frac{\alpha_+ u_0'(0^+) - \alpha_- u_0'(0^-)}{\sqrt{\alpha_+}+\sqrt{\alpha_-}}, \tag{83}$$

$$\begin{aligned} h(t) := \frac{\sqrt{\alpha_+\alpha_-}}{\sqrt{\alpha_+}+\sqrt{\alpha_-}}&\left\{\sqrt{\alpha_+}\int_0^\infty K_+(y,t)\,[\phi_+(u_0^+)]''(y)\,\mathrm{d}y\right. \\ &\left. + \sqrt{\alpha_-}\int_0^\infty K_-(y,t)\,[\phi_-(u_0^-)]''(y)\,\mathrm{d}y\right\}. \end{aligned}$$

Observe that $h \in C([0,T])$, since

$$\lim_{t\to 0} h(t) = \frac{\sqrt{\alpha_+\alpha_-}}{\sqrt{\alpha_+}+\sqrt{\alpha_-}} \frac{\sqrt{\alpha_+^3}\, u_0''(0^+) + \sqrt{\alpha_-^3}\, u_0''(0^-)}{2}.$$

Then from (82), (83), we obtain

$$\lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\phi(u_{\bar\theta})(0,t) = -\infty.$$

This entails $\phi(u_{\bar\theta})(0,t) \in (A,B)$ for any t in some neighbourhood $(0,\tau]$. The invariance property (69) is satisfied with this choice of τ; hence the claim follows.

The case $\phi(u_0)(0) = A,\ \alpha_+\, u_0'(0^+) > \alpha_-\, u_0'(0^-)$ can be dealt with similarly, since in this case

$$\lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t}\phi(u_{\bar\theta})(0,t) = \infty,$$

thus again $\phi(u_{\bar\theta})(0,t) \in (A,B)$ for any t sufficiently small.

(iii) Let $\phi(u_0)(0) = B,\ \alpha_+\, u_0'(0^+) > \alpha_-\, u_0'(0^-)$. We claim that in such case the solution (ξ, u) of the moving boundary problem given by Theorem 4 is a two-phase entropy solution. In fact, by (66) there holds $\xi'(0) < 0$; thus by continuity $\xi'(t) < 0$ for any t in some neighbourhood $(0,\tau]$. Since the entropy condition (9)(b) is satisfied, the claim follows.

The case $\phi(u_0)(0) = A,\ \alpha_+\, u_0'(0^+) < \alpha_-\, u_0'(0^-)$ is similar. In fact, by (66) there holds $\xi'(0) > 0$; thus by the same argument the entropy condition (9)(a) is satisfied. This completes the proof. □

Remark 13. It is informative to add some comments to the above proof. Let $\phi(u_0)(0) = B,\ \alpha_+\, u_0'(0^+) < \alpha_-\, u_0'(0^-)$. If use of the solution (ξ, u) of the moving boundary problem were made in this case, we would have $\xi'(0) > 0$ (see (66)); thus $\xi'(t) > 0$ for any t in some neighbourhood $(0,\tau]$, which contradicts the entropy condition (9)(b). On the other hand, using the solution $u_{\bar\theta}$ of the steady boundary problem was expedient, for

$$\operatorname{sgn}\left\{\lim_{t\to 0}\sqrt{t}\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\phi(u_{\bar\theta})(0,t)\right]\right\} = \operatorname{sgn}\left\{\alpha_+\, u_0'(0^+) - \alpha_-\, u_0'(0^-)\right\} \tag{84}$$

(see (82), (83)). Similar remarks hold for the remaining three cases, namely:

(α) $\phi(u_0)(0) = B, \alpha_+\, u_0'(0^+) > \alpha_-\, u_0'(0^-)$;
(β) $\phi(u_0)(0) = A, \alpha_+\, u_0'(0^+) > \alpha_-\, u_0'(0^-)$;
(γ) $\phi(u_0)(0) = A, \alpha_+\, u_0'(0^+) < \alpha_-\, u_0'(0^-)$.

In view of the entropy conditions (9)(a), (b) and of equalities (66), (84), using the solution (ξ, u) of the moving boundary problem is right in cases (α) and (γ), while it is wrong in case (β). Conversely, using the solution of the steady boundary problem is wrong in cases (α) and (γ), and right in case (β).

Finally, let us prove Theorem 2.

Proof. (Theorem 2) Let us show that the couple $(0, u_{\bar{\theta}})$, $u_{\bar{\theta}}$ being the solution of the steady boundary problem in S_T given by Theorem 5, is a global two-phase entropy solution. In fact, by assumption there holds $\phi_{\pm}(u_0^{\pm})(x) \in [A, B]$ for any $x \in \mathbb{R}_+$. By (81), this implies $\phi(u_{\bar{\theta}})(0, t) \in [A, B]$, namely $u_{\bar{\theta}}(0^-, t) \leqq b$, $u_{\bar{\theta}}(0^+, t) \geqq c$ for any $t \in [0, T]$. Arguing as in part (i) of the proof of Theorem 1, we see that the invariance conditions (69) are satisfied; thus $(0, u_{\bar{\theta}})$, is a two-phase solution of the problem for any $t \in [0, T]$. This proves the result. □

Remark 14. In connection with the above proof, observe that $\phi(u_0)(0) = B$ and $\phi_{\pm}(u_0^{\pm})(x) \in [A, B]$ for any $x \in \mathbb{R}_+$ imply $u_0'(0^+) \leqq 0 \leqq u_0'(0^-)$. Similarly, if $\phi(u_0)(0) = A$ there holds $u_0'(0^+) \geqq 0 \geqq u_0'(0^-)$. Hence cases (α) and (γ) above cannot arise. In the other cases using the solution of the moving boundary problem contradicts the entropy condition, as already discussed.

5. Uniqueness

In this section we prove Theorem 3 (the same proof holds for a general "cubic-like" function ϕ). The following result will be needed.

Proposition 4. *Let assumptions (A_1)–(A_2) be satisfied. Let (ξ, u) be a solution of the moving boundary problem, or $(0, u)$ a solution of the steady boundary problem in S_τ. Then for any $t \in (0, \tau]$ the functions $\phi(u)(\cdot, t) - A$, $\phi(u)(\cdot, t) - B$ change sign at most a finite number of times in any compact subset of $\mathbb{R}$.*

Proof. In view of the invariance property (5), there holds

$$\phi(u) - B = \begin{cases} \alpha_- u + \beta_- - B & \text{in } A_\tau^- \\ \alpha_+ u + \beta_+ - B & \text{in } A_\tau^+ \end{cases}$$

(recall that $A_\tau^{\pm} = Q_\tau^{\pm}$ if $\xi \equiv 0$). Then the function

$$w^{\pm}(y, t) := \phi(u)(\xi(t) \pm y, t) - B \quad (y \in \mathbb{R}_+,\ t \in [0, \tau])$$

satisfies the problem

$$\begin{cases} w_t = \alpha_{\pm} w_{yy} \pm \xi' w_y & \text{in } Q_\tau^+ \\ w(0, t) = \phi(u)(\xi(t), t) - B & t \in (0, \tau] \\ w(y, 0) = w_0^{\pm}(y) := \phi(u_0)(\pm y) - B & y \in \mathbb{R}_+. \end{cases}$$

By assumption (A_2) $w_0^{\pm}$ changes sign at most a finite number of times in any compact subset of $\mathbb{R}_+$. Moreover, $\phi(u)(\xi(t), t) \in [A, B]$; thus $\phi(u)(\xi(t), t) - B \leqq 0$ for any $t \in (0, \tau]$. In view of [18] (Lemma 2.6), the function $w^{\pm}(\cdot, t)$ changes sign at most a finite number of times in any compact subset of $\mathbb{R}_+$; then the result follows in this case. The proof is the same for $\phi(u) - A$; thus the conclusion follows. □

Let (ξ_1, u_1), (ξ_2, u_2) be two two-phase entropy solutions of problem (1), (2) in S_{τ_1} and S_{τ_2}, respectively ($\tau_1, \tau_2 \in (0, T]$). Define $\tau := \min\{\tau_1, \tau_2\}$. Recall that $v_1 := \phi(u_1)$ and $v_2 := \phi(u_2)$ are continuous in $\overline{S}_\tau$ (see Remark 2).

Consider a family of functions $\{\eta_\varepsilon\} \subseteq C^2(\mathbb{R})$ ($\varepsilon > 0$), such that

(i) η_ε converges to the absolute value $|\cdot|$ in $C(\mathbb{R})$ as $\varepsilon \to 0$;
(ii) $\eta'_\varepsilon(s) \to \operatorname{sgn}(s)$ as $\varepsilon \to 0$ for any $s \neq 0$, and $|\eta'_\varepsilon(s)| \leqq 1$ for any $s \in \mathbb{R}$ and $\varepsilon > 0$;
(iii) there holds for some $C > 0$

$$0 \leqq \eta''_\varepsilon(s) \leqq \frac{C}{\varepsilon} \quad \text{for any } \; s \in \mathbb{R}, \quad \eta''_\varepsilon(s) = 0 \quad \text{for any } \; s \notin (-\varepsilon, \varepsilon).$$

Multiplying by $\eta'_\varepsilon(u_1 - u_2)$ the difference of the equations satisfied by u_1 and u_2, one easily sees that the following equality holds in S_τ:

$$\begin{aligned} 0 &= [\eta_\varepsilon(u_1 - u_2)]_t - \left[\eta'_\varepsilon(u_1 - u_2)(v_{1x} - v_{2x})\right]_x \\ &\quad + \eta''_\varepsilon(u_1 - u_2)\,(u_{1x} - u_{2x})\,(v_{1x} - v_{2x}) \\ &= [\eta_\varepsilon(u_1 - u_2)]_t - \left[\eta'_\varepsilon(u_1 - u_2)(v_{1x} - v_{2x})\right]_x \\ &\quad + \eta''_\varepsilon(u_1 - u_2)\,(\phi'(u_1) - \phi'(u_2))\,u_{2x}(u_{1x} - u_{2x}) \\ &\quad + \eta''_\varepsilon(u_1 - u_2)\,\phi'(u_1)\,(u_{1x} - u_{2x})^2. \end{aligned}$$

Since any two-phase entropy solution only takes values in the stable branches of ϕ', there holds $\phi'(u_1) \geqq 0$. Hence the last term on the right-hand side of the above equality is nonnegative, and we obtain

$$\begin{aligned} &[\eta_\varepsilon(u_1 - u_2)]_t - \left[\eta'_\varepsilon(u_1 - u_2)(v_{1x} - v_{2x})\right]_x \\ &\qquad + \eta''_\varepsilon(u_1 - u_2)\,(\phi'(u_1) - \phi'(u_2))\,u_{2x}(u_{1x} - u_{2x}) \leqq 0. \end{aligned}$$

Set $I_\varepsilon := \{(x, t) \mid |u_1 - u_2| \leqq \varepsilon\}$. Then for any subset $\Omega \subseteq S_\tau$

$$\begin{aligned} &\iint_\Omega \eta''_\varepsilon(u_1 - u_2)\,|(\phi'(u_1) - \phi'(u_2))\,u_{2x}(u_{1x} - u_{2x})|\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqq \max\{\alpha_-, \alpha_+\}\,\frac{M}{\varepsilon}\,\|u_{2x}\|_\infty \iint_{\Omega \cap I_\varepsilon} |u_1 - u_2|\,|(u_1 - u_2)_x|\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqq \max\{\alpha_-, \alpha_+\}\,M\,\|u_{2x}\|_\infty \iint_{\Omega \cap I_\varepsilon} |(u_1 - u_2)_x|\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

for some $M > 0$. By Saks' Lemma (see [27]), the last term in the above inequalities converges to zero as $\varepsilon \to 0$, thus giving

$$\lim_{\varepsilon \to 0} \iint_\Omega \left\{[\eta_\varepsilon(u_1 - u_2)]_t - \left[\eta'_\varepsilon(u_1 - u_2)(v_{1x} - v_{2x})\right]_x\right\}\,\mathrm{d}x\,\mathrm{d}t \leqq 0 \qquad (85)$$

for any $\Omega \subset S_\tau$. Set

$$\zeta_1 := \min\{\xi_1, \xi_2\}, \quad \zeta_2 := \max\{\xi_1, \xi_2\}.$$

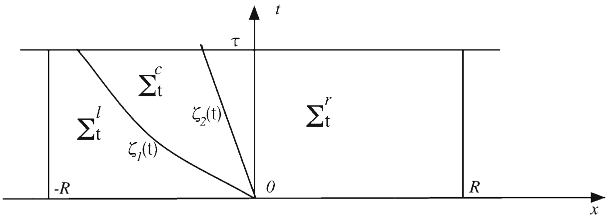

Fig. 2. The subsets Σ_t^l, Σ_t^c and Σ_t^r

Both ζ_1 and ζ_2 are Lipschitz continuous, thus differentiable almost everywhere in $[0,\tau]$. Set also

$$
\begin{aligned}
\gamma_i &:= \{(\xi_i(t),t) \mid t \in [0,\tau]\} \quad (i=1,2),\\
H_t &:= \{s \in (0,t] \mid \zeta_1(s) < \zeta_2(s)\} \quad t \in (0,\tau],\\
K_t &:= \{s \in (0,t] \mid \zeta_1(s) = \zeta_2(s)\} \quad t \in (0,\tau].
\end{aligned}
$$

Observe that H_τ and K_τ are a countable union of intervals.

We shall use the following notations:

$$
\begin{aligned}
&h^{i,\pm}(t) := \lim_{\eta\to 0^+} h(\zeta_i(t) \pm \eta, t) \equiv h(\zeta_i(t)^{\pm}, t),\\
&[h]_i \equiv [h]_i(t) := h^{i,+}(t) - h^{i,-}(t) \quad (i=1,2;\ t \in [0,\tau])
\end{aligned}
$$

for any piecewise continuous function h defined in S_τ. On the subset K_τ we use the simpler notation $\xi \equiv \zeta \equiv \zeta_1 \equiv \zeta_2$, $[h] \equiv [h]_1 \equiv [h]_2$.

Let $R > \max\{\|\xi_1\|_\infty, \|\xi_2\|_\infty\}$, $Q_{R,t} := [-R,R] \times [0,t]$ ($t \in (0,\tau]$). For any $t \in (0,\tau]$, there holds

$$
Q_{R,t} = \Sigma_t^l \cup \Sigma_t^c \cup \Sigma_t^r,
$$

where (see Fig. 2):

$$
\begin{aligned}
\Sigma_t^l &:= \{(x,s) \in S_t \mid -R \leqq x \leqq \zeta_1(s),\ s \in (0,t]\},\\
\Sigma_t^c &= \{(x,s) \in S_t \mid \zeta_1(s) < x \leqq \zeta_2(s),\ s \in (0,t]\},\\
\Sigma_t^r &= \{(x,s) \in S_t \mid \zeta_2(s) < x \leqq R,\ s \in (0,t]\}.
\end{aligned}
$$

Plainly, there holds:

$$
\begin{aligned}
&\iint_{\Sigma_t^l} \left\{[\eta_\varepsilon(u_1-u_2)]_t - \left[\eta'_\varepsilon(u_1-u_2)(v_{1x}-v_{2x})\right]_x\right\} \,\mathrm{d}x\,\mathrm{d}s\\
&= \int_{-R}^{\zeta_1(t)} \eta_\varepsilon(u_1-u_2)(x,t)\,\mathrm{d}x - \int_0^t \left\{\eta'_\varepsilon(u_1-u_2)\,(-v_{1x}+v_{2x})\right\}(-R,s)\,\mathrm{d}s\\
&\quad + \int_0^t \left\{-\eta_\varepsilon(u_1^{1,-}-u_2^{1,-})\,\zeta_1' + \eta'_\varepsilon(u_1^{1,-}-u_2^{1,-})\left(-v_{1x}^{1,-}+v_{2x}^{1,-}\right)\right\}\,\mathrm{d}s\,;
\end{aligned}
$$

$$\iint_{\Sigma_t^c} \left\{[\eta_\varepsilon(u_1-u_2)]_t - \left[\eta_\varepsilon'(u_1-u_2)(v_{1x}-v_{2x})\right]_x\right\} \mathrm{d}x\,\mathrm{d}s$$
$$= \int_{\zeta_1(t)}^{\zeta_2(t)} \eta_\varepsilon(u_1-u_2)(x,t)\,\mathrm{d}x$$
$$+ \int_{H_t} \left\{\eta_\varepsilon(u_1^{1,+}-u_2^{1,+})\,\zeta_1' - \eta_\varepsilon'(u_1^{1,+}-u_2^{1,+})\left(-v_{1x}^{1,+}+v_{2x}^{1,+}\right)\right\}\mathrm{d}s$$
$$+ \int_{H_t} \left\{-\eta_\varepsilon(u_1^{2,-}-u_2^{2,-})\,\zeta_2' + \eta_\varepsilon'(u_1^{2,-}-u_2^{2,-})\left(-v_{1x}^{2,-}+v_{2x}^{2,-}\right)\right\}\mathrm{d}s\,;$$

$$\iint_{\Sigma_t^r} \left\{[\eta_\varepsilon(u_1-u_2)]_t - \left[\eta_\varepsilon'(u_1-u_2)(v_{1x}-v_{2x})\right]_x\right\} \mathrm{d}x\,\mathrm{d}s$$
$$= \int_{\zeta_2(t)}^{R} \eta_\varepsilon(u_1-u_2)(x,t)\,\mathrm{d}x + \int_0^t \left\{\eta_\varepsilon'(u_1-u_2)\,(-v_{1x}+v_{2x})\right\}(R,s)\mathrm{d}s$$
$$+ \int_0^t \left\{\eta_\varepsilon(u_1^{2,+}-u_2^{2,+})\,\zeta_2' - \eta_\varepsilon'(u_1^{2,+}-u_2^{2,+})\left(-v_{1x}^{2,+}+v_{2x}^{2,+}\right)\right\}\mathrm{d}s.$$

In view of inequality (85), summing up the above equalities and passing to the limit as $\varepsilon \to 0$, we obtain for any $t \in (0, \tau]$

$$\int_{-R}^{R} |u_1(x,t) - u_2(x,t)|\,\mathrm{d}x$$
$$\leqq - \int_0^t \{\mathrm{sgn}\,(u_1-u_2)\,(-v_{1x}+v_{2x})\}\,(R,s)\,\mathrm{d}s$$
$$+ \int_0^t \{\mathrm{sgn}\,(u_1-u_2)\,(-v_{1x}+v_{2x})\}\,(-R,s)\,\mathrm{d}s$$
$$+ \int_{H_t} \left\{-\,[|u_1-u_2|]_1\;\zeta_1' + \left[\mathrm{sgn}\,(u_1-u_2)(-v_{1x}+v_{2x})\right]_1\right\}\mathrm{d}s$$
$$+ \int_{H_t} \left\{-\,[|u_1-u_2|]_2\;\zeta_2' + \left[\mathrm{sgn}\,(u_1-u_2)(-v_{1x}+v_{2x})\right]_2\right\}\mathrm{d}s$$
$$+ \int_{K_t} \left\{-\,[|u_1-u_2|]\;\zeta' + \left[\mathrm{sgn}\,(u_1-u_2)(-v_{1x}+v_{2x})\right]\right\}\mathrm{d}s. \qquad (86)$$

Now we can prove Theorem 3.

Proof. Let (ξ_1, u_1), (ξ_2, u_2), S_{τ_1}, S_{τ_2} as above and $\tau := \min\{\tau_1, \tau_2\}$. The conclusion will follow by an iterative argument, if we prove the following
Claim: There exists a time $\vartheta \in (0, \tau]$ such that $\xi_1 = \xi_2$ in $[0, \vartheta]$ and $u_1 = u_2$ in S_ϑ.

Two cases are possible: (i) $\phi(u_0)(0) \in (A, B)$; (ii) either $\phi(u_0)(0) = A$, or $\phi(u_0)(0) = B$.

(i) Let $\phi(u_0)(0) \in (A, B)$. In view of the continuity of $\phi(u_1)$, $\phi(u_2)$, ξ_1 and ξ_2, there exists $\vartheta \in (0, \tau]$ such that $\phi(u_1)(\xi_1(t), t) \in (A, B)$, $\phi(u_2)(\xi_2(t), t) \in$

(A, B) for any $t \in (0, \vartheta]$. Then by the entropy condition (9)(c), we have $\xi_1 = \xi_2 = \zeta_1 = \zeta_2 = 0$ in $[0, \vartheta]$. Hence $H_t = \emptyset$ for any $t \in (0, \vartheta]$ and equality (86) reads:

$$\int_{-R}^{R} |u_1(x,t) - u_2(x,t)|\,\mathrm{d}x \leqq \int_0^t [\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})]\mathrm{d}s + \\ - \int_0^t \{\operatorname{sgn}\,(u_1 - u_2)\,(-v_{1x} + v_{2x})\}\,(R, s)\,\mathrm{d}s \\ + \int_0^t \{\operatorname{sgn}\,(u_1 - u_2)\,(-v_{1x} + v_{2x})\}\,(-R, s)\,\mathrm{d}s \quad (87)$$

for any $t \in (0, \vartheta]$.

By the Rankine–Hugoniot condition, we have $[v_{1x}] = [v_{2x}] = 0$ in $(0, t)$. On the other hand, the continuity of $\phi(u_1)$ and $\phi(u_2)$ across the line $x = 0$ gives

$$\alpha_- u_1(0^-, t) + \beta_- = \alpha_+ u_1(0^+, t) + \beta_+,$$
$$\alpha_- u_2(0^-, t) + \beta_- = \alpha_+ u_2(0^+, t) + \beta_+$$

(see (10)). Since $\alpha_\pm > 0$, from the above equalities we obtain

$$\operatorname{sgn}(u_1 - u_2)(0^-, t) = \operatorname{sgn}(u_1 - u_2)(0^+, t) \quad (88)$$

for any $t \in (0, \vartheta]$. It follows that the first integral in the right-hand side of (87) is equal to zero; thus

$$\int_{-R}^{R} |u_1(x,t) - u_2(x,t)|\,\mathrm{d}x \\ \leqq - \int_0^t \{\operatorname{sgn}\,(u_1 - u_2)\,(-v_{1x} + v_{2x})\}\,(R, s)\,\mathrm{d}s \\ + \int_0^t \{\operatorname{sgn}\,(u_1 - u_2)\,(-v_{1x} + v_{2x})\}\,(-R, s)\,\mathrm{d}s \quad (89)$$

for any $R > 0$ sufficiently large and $t \in (0, \vartheta]$. Now observe that by (6)

$$\lim_{x\to-\infty} v_{ix} = \alpha^- \lim_{x\to-\infty} u_{ix} = \lim_{x\to\infty} v_{ix} = \alpha_+ \lim_{x\to\infty} u_{ix} = 0 \quad (i = 1, 2).$$

Then as $R \to \infty$ from (89), we obtain

$$\int_{-\infty}^{\infty} |u_1(x,t) - u_2(x,t)|\,\mathrm{d}x \leqq 0 \quad (90)$$

for any $(t \in (0, \vartheta])$; thus $u_1 = u_2$ in S_ϑ. This proves the claim in this case.

(ii) Let $\phi(u_0)(0) = B$ (the case $\phi(u_0(0)) = A$ is analogous; thus we leave it to the reader). By the continuity of $\phi(u_1)$, $\phi(u_2)$, ξ_1 and ξ_2, there exists a time $\overline{\tau} \in (0, \tau]$ such that $\phi(u_1)(\xi_1(t), t) > A$, $\phi(u_2)(\xi_2(t), t) > A$ for any $t \in (0, \overline{\tau}]$. Then by the entropy conditions (9)(b), (c) we have $\xi_1'(t) \leqq 0$, $\xi_2'(t) \leqq 0$ for any $t \in [0, \overline{\tau}]$.

In view of the assumption on ξ' made in Definition 1(i), there exists $\tau^* \in (0, \overline{\tau}]$ such that either $\xi_i'(t) = 0$, or $\xi_i'(t) < 0$ for any $t \in (0, \tau^*]\,(i = 1, 2)$.

Hence three cases are possible: (a) $\xi_1'(t) = \xi_2'(t) = 0$; (b) $\xi_1'(t) < 0$, $\xi_2'(t) < 0$; (c) $\xi_1'(t) < 0$, $\xi_2'(t) = 0$ $(t \in (0, \tau^*])$. Let us prove the claim in cases (a), (b) and show that the case (c) is excluded, since it would lead to a contradiction.

(a) If $\xi_1'(t) = \xi_2'(t) = 0$ for any $t \in (0, \tau^*]$, there holds $\xi_1 = \xi_2 = 0$ in $[0, \tau^*]$ and the claim is proved arguing as in (i) with $\vartheta = \tau^*$.

(b) Let $\xi_1'(t) < 0$, $\xi_2'(t) < 0$ (thus $\xi_1(t) < 0$, $\xi_2(t) < 0$) for any $t \in (0, \tau^*]$. Then u_1 and u_2 are solutions of the moving boundary problem at the level B; in particular, there holds $u_i(\xi_i(t)^+, t) = d$ $(i = 1, 2)$. Moreover, for any $l > 0$ and $\vartheta \in (0, \tau^*]$ u_i satisfies the problem

$$(L)_i \quad \begin{cases} w_t = (\phi(w))_{xx} & \text{in } B_{\vartheta}^{(i)} \\ w(\xi_i(t)^+, t) = d, \quad w(l, t) = u_i(l, t) & t \in (0, \vartheta] \\ w(x, 0) = u_0(x) & x \in (0, l), \end{cases}$$

where

$$B_{l,\vartheta}^{(i)} := \left\{(x, t) \in S_\tau \;\middle|\; x \in (\xi_i(t), l),\ t \in (0, \vartheta)\right\} \quad (i = 1, 2).$$

By assumption (A_2) there exist $0 \leqq \delta_1 < \delta_2$ such that $\phi(u_0)(x) = B$ for any $x \in [0, \delta_1]$ and either $\phi(u_0)(x) < B$, or $\phi(u_0)(x) > B$ for any $x \in (\delta_1, \delta_2)$. Let us show that the former possibility is excluded.

In fact, assume $\phi(u_0)(x) < B$ for any $x \in (\delta_1, \delta_2)$. Fix $\bar{x} \in (\delta_1, \delta_2)$; thus $\phi(u_0)(\bar{x}) < B$, that is $u_0(\bar{x}) < d$. Since $\phi(u_1)$ and $\phi(u_2)$ are continuous in $\overline{S}_\tau$, there exists $\vartheta \in (0, \tau^*]$ such that $\phi(u_1)(\bar{x}, t) < B$, $\phi(u_2)(\bar{x}, t) < B$—namely, $u_1(\bar{x}, t) < d$, $u_2(\bar{x}, t) < d$ for any $t \in (0, \tau^*]$. This implies that $w \equiv d$ is a supersolution of problem $(L)_i$ with $l = \bar{x}$; thus by comparison $u_i \leqq d$ in $B_{\bar{x},\vartheta}^{(i)}$ $(i = 1, 2)$.

The above remarks imply $u_{i\,x}(\xi_i(t)^+, t) \leqq 0$. On the other hand, by the same condition (9)(b) there holds $u_i(\xi_i(t)^-, t) = b$; thus $u_{i\,x}(\xi_i(t)^-, t) \geqq 0$ (observe that $u_i(x, t) \leqq b$ for any $x \leqq \xi_i(t)$, $t \in (0, \tau^*]$; $(i = 1, 2)$). Then by the Rankine–Hugoniot condition we obtain $\xi_1'(t) \geqq 0$, $\xi_2'(t) \geqq 0$ for any $t \in (0, \vartheta]$, a contradiction. This rules out the possibility that $\phi(u_0) < B$ in (δ_1, δ_2).

Hence there holds $u_0(x) \geqq d$ for any $x \in (0, \delta_2)$. Arguing as above, we find that there exists $\vartheta \in (0, \tau^*]$ such that $w = d$ is a subsolution of problem $(L)_i$ with $l = \bar{x}$; thus by comparison $u_i \geqq d$ in $B_{\bar{x},\vartheta}^{(i)}$ $(i = 1, 2)$. Since $\xi_1(t) < 0$, $\xi_2(t) < 0$ for any $t \in (0, \tau^*]$, we conclude that for any $t \in (0, \vartheta]$

$$\begin{aligned} u_1(\xi_2(t), t) \geqq d \quad &\text{if } \xi_1(t) \leqq \xi_2(t), \\ u_2(\xi_1(t), t) \geqq d \quad &\text{if } \xi_2(t) \leqq \xi_1(t). \end{aligned} \tag{91}$$

Now consider the third and fourth integral in the right-hand side of equality (86). If $H_t \neq \emptyset$, there holds $\xi_1(s) \neq \xi_2(s)$ for any $s \in H_t$ $(t \in (0, \vartheta])$. Let $I := \{s \in H_t \,|\, \xi_1(s) < \xi_2(s)\}$; thus $\zeta_1 \equiv \xi_1$, $\zeta_2 \equiv \xi_2$ on I. Since u_2 is regular along γ_1, we have

$$u_2^{1,-}(t) = u_2^{1,+}(t) = u_2(\xi_1(t), t) \leqq b < d\,; \tag{92}$$

moreover,

$$u_1^{1,-}(t) = b, \quad u_1^{1,+}(t) = d \quad (t \in (0, \vartheta]).$$

From (92) and the above equality we obtain:

$$\begin{aligned}
&[|u_1 - u_2|]_1 = |u_1^{1,+} - u_2^{1,+}| - |u_1^{1,-} - u_2^{1,-}| = d - b = [u_1]_1,\\
&\operatorname{sgn}\left(u_1^{1,-} - u_2^{1,-}\right) = \operatorname{sgn}\left(u_1^{1,+} - u_2^{1,+}\right) = 1,\\
&\left[\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})\right]_1 = -[v_{1x}]_1 + [v_{2x}]_1 = -[v_{1x}]_1.
\end{aligned}$$

By the Rankine–Hugoniot condition, this gives

$$-[|u_1 - u_2|]_1\ \xi_1' + \left[\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})\right]_1 = -[u_1]_1\,\xi_1' - [v_{1x}]_1 = 0$$

at any point of I. Hence

$$\int_I \left\{-[|u_1 - u_2|]_1\ \zeta_1' + \left[\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})\right]_1\right\} \mathrm{d}s = 0. \tag{93}$$

Similarly, observe that

$$u_2^{2,-}(t) = b, \quad u_2^{2,+}(t) = d.$$

From the first inequality in (91) and the above equality we get:

$$\begin{aligned}
&[|u_1 - u_2|]_2 = |u_1^{2,+} - u_2^{2,+}| - |u_1^{2,-} - u_2^{2,-}| = -d + b = -[u_2]_2,\\
&\operatorname{sgn}\left(u_1^{2,-} - u_2^{2,-}\right) = \operatorname{sgn}\left(u_1^{2,+} - u_2^{2,+}\right) = 1,\\
&[\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})]_2 = -[v_{1x}]_2 + [v_{2x}]_2 = [v_{2x}]_2.
\end{aligned}$$

(recall that u_1 is regular along γ_2). Then by the Rankine–Hugoniot condition

$$-[|u_1 - u_2|]_2\,\xi_2' + [\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})]_2 = [u_2]_2\,\xi_2' + [v_{2x}]_2 = 0$$

at any point of I; whence

$$\int_I \left\{-[|u_1 - u_2|]_2\ \zeta_2' + \left[\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})\right]_2\right\}\mathrm{d}s = 0. \tag{94}$$

It is similarly seen that

$$\begin{aligned}
&\int_J \left\{-[|u_1 - u_2|]_1\ \zeta_1' + \left[\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})\right]_1\right\}\mathrm{d}s\\
&\quad = \int_J \left\{-[|u_1 - u_2|]_2\ \zeta_2' + \left[\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})\right]_2\right\}\mathrm{d}s = 0,
\end{aligned} \tag{95}$$

where $J := \{s \in H_t \mid \xi_1(s) > \xi_2(s)\}$ (here use of the second inequality in (91) is made).

In view of (93)–(95), the third and the fourth integral in the right-hand side of (86) vanish. Concerning the last integral, since $\zeta_1 \equiv \zeta_2$ we can argue as in (i) above to prove that

$$\operatorname{sgn}(u_1 - u_2)(\zeta(s)^-, s) = \operatorname{sgn}(u_1 - u_2)(\zeta(s)^+, s) \quad (s \in K_t, t \in (0, \vartheta]).$$

Hence by the Rankine–Hugoniot condition

$$\begin{aligned} &-[|u_1 - u_2|]\,\xi' + \left[\operatorname{sgn}(u_1 - u_2)(-v_{1x} + v_{2x})\right] \\ &\quad = \operatorname{sgn}(u_1 - u_2)\left\{-[u_1]\,\xi' - [v_{1x}] + [u_2]\,\xi' + [v_{2x}]\right\} = 0 \end{aligned}$$

at any point of K_t. Then the last integral in the right-hand side of (86) is also equal to zero.

Arguing as in (i), we conclude that inequality (90) holds. Therefore $u_1(x,t) = u_2(x,t)$ for any $x \in \mathbb{R}\backslash\{\xi_1(t) \cup \xi_2(t)) \,|\, t \in (0,\vartheta]\}$. In view of the Rankine–Hugoniot condition, this implies $\xi_1'(t) = \xi_2'(t)$; hence $\xi_1(t) = \xi_2(t)$ for any $t \in (0,\vartheta]$. Then the claim follows in the present case, too.

(*c*) Let us prove that the case $\xi_1'(t) < 0$, $\xi_2'(t) = 0$ (thus $\xi_1(t) < 0$, $\xi_2(t) = 0$) for any $t \in (0,\tau^*]$ is excluded under the present assumptions.

By contradiction, let $\xi_1'(t) < 0$, $\xi_2'(t) = 0$ for any $t \in (0,\tau^*]$. Since u_1 is a solution of the moving boundary problem at the level B, we can argue as in the previous case (b) and prove that there exists $\vartheta \in (0,\tau^*]$ such that

$$u_1(0,t) \geqq d \tag{96}$$

for any $t \in (0,\vartheta]$.

In this case $\xi_1(s) < \xi_2(s)$ for any $s \in H_t$; thus $\zeta_1 \equiv \xi_1$, $\zeta_2 \equiv \xi_2$ on H_t. Using inequality (96) instead of the first inequality in (91), we can argue as in (b) and prove that the third and the fourth integral in the right-hand side of (86) vanish in this case, too. The last integral is obviously equal to zero, since $K_t = \emptyset$ for any $t \in \vartheta$; hence equality (90) follows.

As before, this implies $u_1(x,t) = u_2(x,t)$ for any $x \in \mathbb{R}\backslash\{\xi_1(t) \cup \xi_2(t)) \,|\, t \in (0,\vartheta]\}$ and $\xi_1(t) = \xi_2(t)$ for any $t \in (0,\vartheta]$. However, this is impossible, since $\xi_1(t) < \xi_2(t)$ for any $t \in (0,\tau^*]$. The contradiction proves that the case (*c*) is excluded. This completes the proof of the claim.

To complete the proof, define

$$E := \{\vartheta \in (0,\tau] \,|\, \xi_1 = \xi_2 \text{ in } [0,\vartheta] \text{ and } u_1 = u_2 \text{ in } S_\vartheta\}.$$

In view of the claim, the set E is nonempty; thus $\vartheta^* := \sup E \in (0,\tau]$ is well defined. Were $\vartheta^* < \tau$, two cases again would be possible: (i) $\phi(u_i)(0,\vartheta^*) \in (A,B)$ for any $i = 1,2$; (ii) either $\phi(u_i)(0,\vartheta^*) = A$ or $\phi(u_i)(0,\vartheta^*) = B$ for some $i = 1,2$. In both cases we could argue as before to show that ϑ^* is not an upper bound of E, in contrast with its definition; here use of Proposition 4 is made, to repeat the above arguments in case (ii) (namely, when $\phi(u_0)(0) = A$ or $\phi(u_0)(0) = B$). The contradiction shows that $\vartheta^* = \tau$; then the conclusion follows. □

References

1. Barenblatt, G.I., Bertsch, M., Dal Passo, R., Ughi, M.: A degenerate pseudoparabolic regularization of a nonlinear forward–backward heat equation arising in the theory of heat and mass exchange in stably stratified turbulent shear flow. *SIAM J. Math. Anal.* **24**, 1414–1439 (1993)

2. BELLETTINI, G., FUSCO, G., GUGLIELMI, N.: A concept of solution and numerical experiments for forward–backward diffusion equations. *Discr. Contin. Dyn. Syst.* **16**, 783–842 (2006)
3. BINDER, K., FRISCH, H.L., JÄCKLE, J.: Kinetics of phase separation in the presence of slowly relaxing structural variables. *J. Chem. Phys.* **85**, 1505–1512 (1986)
4. BROKATE, M., SPREKELS, J.: *Hysteresis and Phase Transitions*. Applied Mathematical Sciences, vol. **121**. Springer, Berlin, 1996
5. CANNON, J.R.: *The One-Dimensional Heat Equation*. Encyclopedia of Mathematics and Its Applications, vol. **23**. Addison-Wesley, Reading, 1984
6. EVANS, L.C.: *Weak convergence methods for nonlinear partial differential equations*, CBMS Reg. Conf. Ser. Math., vol. **74**. American Mathematical Society, Providence, 1990
7. EVANS, L.C.: A survey of entropy methods for partial differential equations. *Bull. Am. Math. Soc.* **41**, 409–438 (2004)
8. EVANS, L.C., PORTILHEIRO, M.: Irreversibility and hysteresis for a forward–backward diffusion equation. *Math. Models Methods Appl. Sci.* **14**, 1599–1620 (2004).
9. FRIEDMAN, A.: Free boundary problems for parabolic equations I: Melting of solids. *J. Math. Mech.* **8**, 499–517 (1959)
10. FRIEDMAN, A.: *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs, 1991
11. GHISI, M., GOBBINO, M.: Gradient estimates for the Perona–Malik equation. *Math. Ann.* **337**, 557–590 (2007)
12. GILDING, B.H.: Hölder continuity of solutions of parabolic equations. *J. Lond. Math. Soc.* **13**, 103–106 (1976)
13. GILDING, B.H., TESEI, A.: *The Riemann problem for a forward–backward parabolic equation*, preprint, 2008
14. HÖLLIG, K.: Existence of infinitely many solutions for a forward backward heat equation. *Trans. Am. Math. Soc.* **278**, 299–316 (1983)
15. LADYZENSKAJA, O.A., SOLONNIKOV, V.A., URALCEVA, N.N.: *Linear and Quasi-linear Equations of Parabolic Type*. American Mathematical Society, Providence, 1991
16. LAFITTE, P., MASCIA, C.: *Numerical exploration of a forward–backward diffusion equation*, in preparation
17. MASCIA, C., TERRACINA, A., TESEI, A.: Evolution of stable phases in forward–backward parabolic equations. *Asymptotic Analysis and Singularities* (Eds. Kozono, H., Ogawa, T., Tanaka, K., Tsutsumi, Y., Yanagida E.), Advanced Studies in Pure Mathematics **47-2**. Mathematical Socity, Japan, 451–478, 2007
18. MATANO, H.: Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **29**, 401–441 (1982)
19. NOVICK-COHEN, A., PEGO, R.L.: Stable patterns in a viscous diffusion equation. *Trans. Am. Math. Soc.* **324**, 331–351 (1991)
20. PADRÓN, V.: Sobolev regularization of a nonlinear ill-posed parabolic problem as a model for aggregating populations. *Comm. Partial Differ. Equ.* **23**, 457–486 (1998)
21. PADRÓN, V.: Effect of aggregation on population recovery modeled by a forward–backward pseudoparabolic equation. *Trans. Am. Math. Soc.* **356**, 2739–2756 (2003)
22. PERONA, P., MALIK, J.: Scale space and edge detection using anisotropic diffusion. *IEEE Trans. Pattern Anal. Mach. Intell.* **12**, 629–639 (1990)
23. PLOTNIKOV, P.I.: Equations with alternating direction of parabolicity and the hysteresis effect. *Russian Acad. Sci. Dokl. Math.* **47**, 604–608 (1993)
24. PLOTNIKOV, P.I.: Passing to the limit with respect to viscosity in an equation with variable parabolicity direction. *Differ. Equ.* **30**, 614–622 (1994)
25. PLOTNIKOV, P.I.: Forward–backward parabolic equations and hysteresis. *J. Math. Sci.* **93**, 747–766 (1999)
26. ROUBÍČEK,T., HOFFMANN, K.-H.: About the concept of measure-valued solutions to distributed parameter systems. *Math. Meth. Appl. Sci.* **18**, 671–685 (1995)
27. SAKS, S.: *Theory of the Integral*. Dover, New York, 1964

28. SERRE, D.: *Systems of Conservation Laws,* vol. **1**: *Hyperbolicity, Entropies, Shock Waves*. Cambridge University Press, Cambridge, 1999
29. SLEMROD, M.: Dynamics of measure valued solutions to a backward-forward heat equation. *J. Dyn. Differ. Equ.* **3**, 1–28 (1991)
30. SMARRAZZO, F.: On a class of equations with variable parabolicity direction. *Discr. Contin. Dyn. Syst.* (to appear)
31. TYCHONOV, A.N., SAMARSKI, A.A.: *Partial Differential Equations of Mathematical Physics*. Holden-Day, San Francisco, 1964
32. VISINTIN, A.: Forward–backward parabolic equations and hysteresis. *Calc. Var.* **15**, 115–132 (2002)
33. ZHANG, K.: Existence of infinitely many solutions for the one-dimensional Perona–Malik model. *Calc. Var.* **26**, 171–199 (2006)

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