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Supercritical Geometric Optics for Nonlinear Schrödinger Equations

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Abstract

We consider the small time semi-classical limit for nonlinear Schrödinger equations with defocusing, smooth, nonlinearity. For a super-cubic nonlinearity, the limiting system is not directly hyperbolic, due to the presence of vacuum. To overcome this issue, we introduce new unknown functions, which are defined nonlinearly in terms of the wave function itself. This approach provides a local version of the modulated energy functional introduced by Y. Brenier. The system we obtain is hyperbolic symmetric, and the justification of WKB analysis follows.

1. Introduction

1.1. Presentation

We study the behavior of the solution u^ε to

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \left|u^\varepsilon\right|^{2\sigma} u^\varepsilon; \quad u^\varepsilon_{|t=0} = a_0^\varepsilon e^{i\phi_0/\varepsilon}, \tag{1.1}$$

as the parameter $\varepsilon \in]0,1]$ goes to zero. To fix matters, we work on $\mathbb{R}^n$, yet all the results are valid in the Torus $\mathbb{T}^n$. Throughout all of this paper, we assume that the space dimension is $n \leqq 3$, which corresponds to the physical cases. The unknown u^ε and the initial amplitude a_0^ε are complex valued, the phase ϕ_0 is real-valued. The case of a more general nonlinearity, of the form

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon^\kappa f\left(|u^\varepsilon|^2\right) u^\varepsilon; \quad u^\varepsilon_{|t=0} = a_0^\varepsilon e^{i\phi_0/\varepsilon},$$

was discussed in [11]. In particular, WKB type analysis is justified for $\kappa \geqq 1$ (weak nonlinearity). On the other hand, when $\kappa = 0$, there are only two cases in which the mathematical analysis of the semi-classical limit for nonlinear Schrödinger equations is well developed. First, for analytic initial data. We refer to [23,36] for this approach. Second, for the cubic defocusing nonlinear Schrödinger equation ($\sigma = 1$ in (1.1)). Our goal is to justify geometric optics in Sobolev spaces for (1.1) when $\sigma \geqq 2$ (see also Section 6.2 for the nonhomogeneous case). This question has remained open since the pioneering work of GRENIER [27], where the nonlinearity had to be cubic at the origin.

There are several motivations to study the semi-classical limit for (1.1). Let us mention three. First, (1.1) with $\sigma = 2$ (quintic nonlinearity) is sometimes used as a model for one-dimensional Bose–Einstein condensation [28]. An external potential is usually considered in this framework (most commonly, an harmonic potential); we refer to [11] to show that the results of the present paper can easily be adapted to that case (see also Section 6.3).

Second, the limit $\varepsilon \to 0$ relates classical and quantum wave equations. In particular, the semi-classical limit $\varepsilon \to 0$ for u^ε is expected to be described by the laws of hydrodynamics; see, for example, [21–23,27]. If we assume that $a_0^\varepsilon \to a_0$ as $\varepsilon \to 0$, then formally, u^ε is expected to be well approximated by $a e^{i\phi/\varepsilon}$, where

$$\begin{cases} \partial_t \phi + \dfrac{1}{2}|\nabla\phi|^2 + |a|^{2\sigma} = 0; & \phi_{|t=0} = \phi_0, \\ \partial_t a + \nabla\phi \cdot \nabla a + \dfrac{1}{2} a \Delta\phi = 0; & a_{|t=0} = a_0. \end{cases} \tag{1.2}$$

This system is to be understood as a compressible Euler equation. Indeed, setting $(\rho, v) = (|a|^2, \nabla\phi)$, we see that (ρ, v) solves the following:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla\left(\rho^\sigma\right) = 0; & v_{|t=0} = \nabla\phi_0, \\ \partial_t \rho + \operatorname{div}(\rho v) = 0; & \rho_{|t=0} = |a_0|^2. \end{cases}$$

Note that for $\sigma > 1$, the above system is not directly hyperbolic symmetric, due to the presence of vacuum. We will see that the above system suffices to describe the convergences of the main two quadratic observables for u^ε, that is, position and current densities. We will also see that passing to the limit $\varepsilon \to 0$ in the usual conservation laws for nonlinear Schrödinger equations, we recover important conservation laws for the Euler equation (see Section 6.4). This also serves as a background to note that some blow-up results for the nonlinear Schrödinger equation on the one hand, and the compressible Euler equation on the other hand, follow from very similar identities (see Section 6.5). This remark reinforces the bridge noticed by SERRE [33].

Another motivation lies in the study the Cauchy problem for nonlinear Schrödinger equations with no small parameter ($\varepsilon = 1$ in (1.1), typically). As noticed in [8, Appendix] and [9], one can prove ill-posedness results for energy-supercritical equations by reducing the problem to semi-classical analysis for (1.1). In [9, Appendix C], a result of loss of regularity was proved for the cubic, defocusing nonlinear Schrödinger equation, in the spirit of the pioneering work of LEBEAU [29]. It concerned the flow associated to the nonlinear Schrödinger equation near the origin. This was extended in [10] to the case of data of arbitrary size in Sobolev spaces. When the nonlinearity is defocusing and not necessarily cubic, the result of [9, Appendix C] was extended in [2], by studying the semi-classical limit for (1.1). However, [2] does not use the complete justification of geometric optics, which makes it impossible to extend the results in [10] to the case of super-cubic nonlinearities. The analysis presented in this paper makes it possible.

1.2. Main results

For $s \geqq 0$, we shall denote $H^s(\mathbb{R}^n)$, or simply H^s, the Sobolev space of order s. We equip $H^\infty(\mathbb{R}^n) = H^\infty := \cap_{s>0} H^s(\mathbb{R}^n)$ with the distance

$$d(f,g) = \sum_{s\in\mathbb{N}} 2^{-s} \frac{\|f-g\|_{H^s}}{1+\|f-g\|_{H^s}}.$$

Note that for any $k \in \mathbb{N}$ and any interval I, $C^k(I;H^\infty) = \cap_{s\geqq 0} C^k(I;H^s)$.

Assumption 1.1. *We require $\sigma \in \mathbb{N}\setminus\{0\}$ without recalling this assumption explicitly in the statements. Similarly, it is assumed that $a_0, \phi_0 \in H^\infty$, where recall that ϕ_0 is real-valued. We also suppose that a_0^ε belongs uniformly to H^∞ and converges towards a_0 in H^∞ as $\varepsilon \to 0$. More precisely,*

$$a_0^\varepsilon = a_0 + \mathcal{O}(\varepsilon) \quad \textit{in } H^s(\mathbb{R}^n), \quad \forall s \geqq 0.$$

The first remark, based on a change of unknown due to Makino et al. [31] (see also [15]), is that the limiting system (1.2) is locally well-posed in Sobolev spaces, despite the possible presence of vacuum:

Lemma 1.2. (from [2]) *Let $n \geqq 1$, and let Assumption 1.1 be satisfied. There exists $T^* > 0$ such that (1.2) has a unique maximal solution (ϕ, a) in $C([0,T^*[;H^\infty(\mathbb{R}^n))$.*

The proof is recalled in Section 2. It is based on a change of unknown introduced in [31] (see also [15]), which makes it possible to rewrite the equation under the form of a quasi-linear symmetric hyperbolic system. This transformation of the equations, which consists in introducing $(v,u) := (\nabla\phi, a^\sigma)$, clearly exhibits a key dichotomy between $\sigma = 1$ and $\sigma \geqq 2$. In particular, a stability analysis in the case $\sigma \geqq 2$ is not straightforward, since the above mentioned change of variables does not seem to be well adapted to Schrödinger equations.

Here is the main result of this paper. In the context of Assumption 1.1, we prove that the solutions of (1.1) exist and satisfy uniform estimates on a time interval which is independent of ε.

Theorem 1.3. *Let $n \leqq 3$, and let Assumption 1.1 be satisfied. There exists $T \in]0,T^*[$, where T^* is given by Lemma 1.2, such that the following holds. For all $\varepsilon \in]0,1]$ the Cauchy problem (1.1) has a unique solution $u^\varepsilon \in C([0,T];H^\infty(\mathbb{R}^n))$. Moreover,*

$$\sup_{\varepsilon\in]0,1]} \left\| u^\varepsilon e^{-i\phi/\varepsilon} \right\|_{L^\infty([0,T];H^k(\mathbb{R}^n))} < +\infty, \tag{1.3}$$

where $\phi \in C_b^\infty([0,T]\times\mathbb{R}^n)$ is given by (1.2), and the index k is as follows:

– *If* $\sigma = 1$, *then* $k \in \mathbb{N}$ *is arbitrary.*
– *If* $\sigma = 2$ *and* $n = 1$, *then we can take* $k = 2$.
– *If* $\sigma = 2$ *and* $2 \leqq n \leqq 3$, *then we can take* $k = 1$.
– *If* $\sigma \geqq 3$, *then we can take* $k = \sigma$.

Finally, define $q^\varepsilon = \frac{1}{\varepsilon} B_\sigma \left(|a^\varepsilon|^2, |a|^2\right)$, *where we set, for* $r_1, r_2 \geqq 0$,

$$B_\sigma(r_1, r_2) = (r_1 - r_2) \left(2\sigma \int_0^1 (1-s)\,(r_2 + s(r_1 - r_2))^{\sigma - 1}\,\mathrm{d}s\right)^{1/2}.$$

Then for the same k *as above,*

$$\sup_{\varepsilon \in]0,1]} \left\| q^\varepsilon \right\|_{L^\infty([0,T];H^{k-1}(\mathbb{R}^n))} < +\infty. \tag{1.4}$$

Remark 1. The estimate (1.3) is trivial for $k = 0$, from the conservation of mass, which holds even for weak solutions [25].

Remark 2. For sufficiently large σ, the approach followed in this paper makes it possible to extend Theorem 1.3 to the case of higher dimensions, $n \geqq 4$. We shall not pursue this question.

Remark 3. The assumption $a_0^\varepsilon = a_0 + \mathcal{O}(\varepsilon)$ plays a crucial role in the above result. Indeed, the analysis in [9] shows that in the case $\sigma = 1$, if we assume only $a_0^\varepsilon = a_0 + o(1)$, then the conclusion of Theorem 1.3 fails. For instance, if $a_0^\varepsilon = (1 + \varepsilon^\alpha) a_0$ for some $0 < \alpha \leqq 1$, then for arbitrarily small $t > 0$ independent of ε, $u^\varepsilon \mathrm{e}^{-i\phi/\varepsilon}$ has oscillations of order $\varepsilon^{1-\alpha}$. So if $\alpha < 1$, then $u^\varepsilon \mathrm{e}^{-i\phi/\varepsilon}$ is not bounded in H^1.

For $\sigma = 1$, the above result is a consequence of the analysis due to GRENIER [27], and remains valid in any space dimension $n \geqq 1$. We propose an alternate proof in Section 3.

In the quintic case $\sigma = 2$, for all $\varepsilon > 0$, the Cauchy problem (1.1) has a unique global solution in $C(\mathbb{R}; H^\infty(\mathbb{R}^n))$. Indeed, for $n = 1$ this follows from standard results for semi-linear equations; in the energy-subcritical case $n = 2$, this follows from Strichartz estimate and the conservation laws; for the difficult energy-critical case $n = 3$, this has been proved by COLLIANDER et al. [20]. Therefore, the main point in our result is the uniform bound (1.3). The same is true for the case $n \leqq 2$ and $\sigma \geqq 3$, since the nonlinearity is then H^1-subcritical.

For $\sigma \geqq 3$ and $n = 3$, the equation is H^1-supercritical. Therefore, not only the bound (1.3) is new, but also the fact that we can construct a smooth solution u^ε to (1.1) on a time interval $[0, T]$ independent of $\varepsilon \in]0, 1]$.

Since the estimate (1.4) may seem a little mysterious prior to any analysis, let us state its main consequences for the value $k = 1$. We infer that the quadratic observables converge strongly towards the solution of compressible Euler equations for potential flows in vacuum, hence giving the Wigner measure associated to $(u^\varepsilon)_\varepsilon$ (see, for example, [7,24] for the definition and the main properties). The following result is proved in Section 4.5.

Corollary 1.4. *Let $n \leqq 3$, and let Assumption 1.1 be satisfied. There exists $T \in]0, T^*[$, where T^* is given by Lemma 1.2, such that the position and current densities converge strongly on $[0, T]$ as $\varepsilon \to 0$:*

$$|u^\varepsilon|^2 \underset{\varepsilon\to 0}{\longrightarrow} |a|^2 \qquad \text{in } C\left([0,T]; L^{\sigma+1}(\mathbb{R}^n)\right),$$
$$\operatorname{Im}\left(\varepsilon\overline{u}^\varepsilon \nabla u^\varepsilon\right) \underset{\varepsilon\to 0}{\longrightarrow} |a|^2\nabla\phi \qquad \text{in } C\left([0,T]; L^{\sigma+1}(\mathbb{R}^n) + L^1(\mathbb{R}^n)\right).$$

In particular, there is only one Wigner measure associated to $(u^\varepsilon)_\varepsilon$, and it is given by

$$\mu(t, dx, d\xi) = |a(t,x)|^2 \,\mathrm{d}x \otimes \delta\left(\xi - \nabla\phi(t,x)\right).$$

The analysis proposed to prove Theorem 1.3 allows us to compute the leading order behavior of the wave function u^ε, provided that we know a more precise WKB expansion of the initial amplitude.

Assumption 1.5. *In addition to Assumption 1.1, we assume that there exists $a_1 \in H^\infty(\mathbb{R}^n)$ such that*

$$a_0^\varepsilon = a_0 + \varepsilon a_1 + \mathcal{O}\left(\varepsilon^2\right) \quad \text{in } H^s(\mathbb{R}^n), \quad \forall s \geqq 0.$$

Theorem 1.6. *Let $n \leqq 3$, and let Assumption 1.5 be satisfied. There exists $\widetilde{a} \in C([0, T^*[; H^\infty)$, and for any $T \in]0, T^*[$, there exists $\varepsilon(T) > 0$, such that $u^\varepsilon \in C([0,T]; H^\infty)$ for $\varepsilon \in]0, \varepsilon(T)]$, and*

$$\begin{aligned} \left\| u^\varepsilon - \widetilde{a}\mathrm{e}^{i\phi/\varepsilon} \right\|_{L^\infty([0,T]; L^2\cap L^p)} &= \mathcal{O}(\varepsilon) \quad \text{when } \sigma = 2 \text{ and } 2 \leqq n \leqq 3, \\ \left\| u^\varepsilon - \widetilde{a}\mathrm{e}^{i\phi/\varepsilon} \right\|_{L^\infty([0,T]; L^2\cap L^\infty)} &= \mathcal{O}(\varepsilon) \quad \text{in the other cases,} \end{aligned} \tag{1.5}$$

where p is such that $H^1(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$.

Remark 4. In general, $\widetilde{a} \neq a$, unless a_0 is real-valued and $a_1 \in i\mathbb{R}$ (see Section 5). Therefore, the system (1.2) does not suffice, in general, to describe the asymptotic behavior of the *wave function* u^ε, even though it suffices to describe the position and current densities (see Corollary 1.4 above).

1.3. Scheme of the proof of Theorem 1.3

To prove that the solutions to the Cauchy problem (1.1) exist for a time independent of ε, it is enough to prove uniform estimates for the L^∞ norm of u^ε (see Lemma 2.1 below). To do so, our approach toward the semi-classical limit is to filter out the oscillations by the following change of unknown, involving the solution (a, ϕ) of the limit system (1.2):

$$a^\varepsilon(t,x) := u^\varepsilon(t,x)\mathrm{e}^{-i\phi(t,x)/\varepsilon}. \tag{1.6}$$

The key point is that, although it is obviously equivalent to prove L^∞ estimates for u^ε and a^ε, it is expected that one can prove uniform estimates in Sobolev spaces for a^ε, thereby obtaining the desired L^∞ estimates from the Sobolev embedding. Obviously, uniform estimates in Sobolev spaces for u^ε are not expected to hold, due to the rapid oscillations described by ϕ.

The amplitude a^ε solves the following evolution equation:

$$\begin{cases} \partial_t a^\varepsilon + \nabla\phi \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta\phi - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{i}{\varepsilon}\left(\left|a^\varepsilon\right|^{2\sigma} - |a|^{2\sigma}\right) a^\varepsilon, \\ a^\varepsilon_{|t=0} = a^\varepsilon_0. \end{cases} \tag{1.7}$$

It is clear that the mass is conserved:

$$\left\|a^\varepsilon(t)\right\|_{L^2} = \left\|u^\varepsilon(t)\right\|_{L^2} = \left\|u^\varepsilon(0)\right\|_{L^2} = \left\|a^\varepsilon_0\right\|_{L^2}.$$

This can be seen by multiplying (1.7) by $\overline{a}^\varepsilon$, taking the real part and integrating over $\mathbb{R}^n$. Note that the large term in ε^{-1} disappears from the energy estimate. Indeed, the large term in ε^{-1} is a nonlinear rotation term. But precisely because this term is nonlinear, it does not disappear from the estimate of the derivatives (the equation is not translation invariant). Indeed, ∇a^ε solves

$$\begin{aligned} &\left(\partial_t + \nabla\phi\cdot\nabla + \frac{1}{2}\Delta\phi - i\frac{\varepsilon}{2}\Delta\right)\nabla a^\varepsilon + \nabla a^\varepsilon \cdot \nabla\nabla\phi + \frac{1}{2}a^\varepsilon \nabla\Delta\phi \\ &\qquad + \frac{i}{\varepsilon}\left(\left|a^\varepsilon\right|^{2\sigma} - |a|^{2\sigma}\right)\nabla a^\varepsilon + \frac{i}{\varepsilon}a^\varepsilon\nabla\left(\left|a^\varepsilon\right|^{2\sigma} - |a|^{2\sigma}\right) = 0. \end{aligned} \tag{1.8}$$

This equation is of the form

$$\left(\partial_t + L(\phi, \partial_x) + \mathcal{L}(\varepsilon, \partial_x)\right)\nabla a^\varepsilon + \frac{i}{\varepsilon}a^\varepsilon\nabla\left(\left|a^\varepsilon\right|^{2\sigma} - |a|^{2\sigma}\right) = 0,$$

where $\mathcal{L}(\varepsilon, \partial_x)$ is skew-symmetric. Again, by multiplying (1.8) by $\nabla\overline{a}^\varepsilon$, taking the real part and integrating over $\mathbb{R}^n$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla a^\varepsilon\right\|^2_{L^2} - \frac{1}{\varepsilon}\int \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon\nabla a^\varepsilon))\left(\left|a^\varepsilon\right|^{2\sigma} - |a|^{2\sigma}\right) \\ &\qquad = -\operatorname{Re}\int_{\mathbb{R}^n}\left(\nabla a^\varepsilon\cdot\nabla\nabla\phi + \frac{1}{2}a^\varepsilon\nabla\Delta\phi\right)\nabla\overline{a}^\varepsilon\,\mathrm{d}x. \end{aligned}$$

Together with the mass conservation, this yields the following identity for the energy $E^\varepsilon := \|a^\varepsilon\|^2_{H^1}$:

$$\frac{1}{2}\frac{\mathrm{d}E^\varepsilon}{\mathrm{d}t} - \frac{1}{\varepsilon}\int \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon\nabla a^\varepsilon))\left(\left|a^\varepsilon\right|^{2\sigma} - |a|^{2\sigma}\right) \leqq C_\phi E^\varepsilon,$$

for some constant C_ϕ depending only on the known solution (a, ϕ) of the limit system. The idea is then to find a second energy functional $\mathcal{E}^\varepsilon$ such that

$$\frac{1}{2}\frac{\mathrm{d}\mathcal{E}^\varepsilon}{\mathrm{d}t} + \frac{1}{\varepsilon}\int \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon\nabla a^\varepsilon))\left(\left|a^\varepsilon\right|^{2\sigma} - |a|^{2\sigma}\right) \leqq C_{a,\phi}(E^\varepsilon + \mathcal{E}^\varepsilon). \tag{1.9}$$

By adding the two inequalities, one obtains a uniform in ε energy estimate

$$E^\varepsilon(t) + \mathcal{E}^\varepsilon(t) \leqq \mathrm{e}^{C_{a,\phi}t}(E^\varepsilon(0) + \mathcal{E}^\varepsilon(0)).$$

The previous strategy has many roots. For the semi-classical limit, this goes back to the work of Brenier [6], Zhang [40], Lin and Zhang [30], and is referred to as a modulated energy estimate. Here, we will get the same result in a different way. Our approach amounts to trying to find a nonlinear change of unknown to symmetrize the equations. We will define g^ε and q^ε such that

$$\partial_t q^\varepsilon + g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + \nabla\phi \cdot \nabla q^\varepsilon + \frac{\sigma+1}{2} q^\varepsilon \Delta\phi = 0,$$

and

$$q^\varepsilon g^\varepsilon = \frac{1}{\varepsilon}\left(\left|a^\varepsilon\right|^{2\sigma} - |a|^{2\sigma}\right).$$

Not only does this allow us to obtain (1.9) with $\mathcal{E}^\varepsilon := \|q^\varepsilon\|^2_{L^2}$, but also to derive uniform estimates in Sobolev spaces. More precisely, we will see that the system of equations satisfied by $(a^\varepsilon, \nabla a^\varepsilon, q^\varepsilon)$ is essentially hyperbolic symmetric (plus some skew-symmetric terms). Therefore, we can derive energy estimates, which in turn imply Theorem 1.3. Note that the idea of introducing new unknown functions to diminish the complexity of the initial problem is a strategy that has proven successful in many occasions: for instance, blow-up for the nonlinear wave equation, [4] (see also [3,5]), low Mach number limit of the full Navier–Stokes equations [1], or geometric optics for the incompressible Euler or Navier–Stokes equations [16–18].

Remark 5. Several months after the submission of this paper, results similar to those presented here were obtained in [19]. The assumptions are slightly different from those considered here, the methods differ much more (the authors do not use a local modulated energy, but a linearization argument, directly on the Schrödinger equation), and the conclusions are similar to ours, but stronger: for instance, there is no restriction on the values of n and k in the analogue of Theorem 1.3.

2. Preliminaries

Since, for $\sigma \in \mathbb{N}$, the nonlinearity in (1.1) is smooth, the usual theorems for semi-linear evolution equations (see, for example, [14]) imply the following result.

Lemma 2.1. *Let $\sigma, n \in \mathbb{N}\backslash\{0\}$. For (fixed) $\varepsilon \in]0,1]$, assume that $u^\varepsilon_{|t=0} \in H^s(\mathbb{R}^n)$ with $s > n/2$. Then there exists T^ε such that (1.1) has a unique maximal solution $u^\varepsilon \in C([0,T^\varepsilon[; H^s(\mathbb{R}^n))$: if $T^\varepsilon < +\infty$, then*

$$\limsup_{t\to T^\varepsilon} \left\|u^\varepsilon(t)\right\|_{L^\infty(\mathbb{R}^n)} = +\infty. \tag{2.1}$$

Consequently, if $u^\varepsilon(0) \in H^\infty(\mathbb{R}^n)$, then $u^\varepsilon \in C^\infty([0,T^\varepsilon[; H^\infty(\mathbb{R}^n))$.

With regards to the limit system (1.2), we recall the proof of Lemma 1.2.

Lemma 2.2. *Let $\sigma \in \mathbb{N}$ and $n \geqq 1$. For all $(\phi_0, a_0) \in H^{s+1}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with $s > n/2 + 1$, there exists $T^* > 0$ such that (1.2) has a unique maximal solution (ϕ, a) in $C([0, T^*[; H^{s+1}(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n))$. In addition, if $\phi_0, a_0 \in H^\infty(\mathbb{R}^n)$, then $\phi, a \in C^\infty([0, T^*[; H^\infty(\mathbb{R}^n))$.*

Remark 6. The lifespan T^* is finite for all compactly support initial data (see Proposition 6.1). If $\sigma = 1$, then a belongs to $C([0, T^*[; H^s(\mathbb{R}^d))$ as soon as $(\phi_0, a_0) \in H^{s+1}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$. What makes the previous result nontrivial is the presence of vacuum when $\sigma \geqq 2$: at the zeroes of a, (1.2) ceases to be hyperbolic, and this may cause a loss of regularity.

Sketch of the proof. One can transform (1.2) into a quasi-linear system by differentiating the equation for ϕ: with $v = \nabla\phi$, one has

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla |a|^{2\sigma} = 0; & v_{|t=0} = \nabla\phi_0, \\ \partial_t a + v \cdot \nabla a + \dfrac{1}{2} a \operatorname{div} v = 0; & a_{|t=0} = a_0. \end{cases} \tag{2.2}$$

For the cubic case where $\sigma = 1$, this system enters the standard framework of quasi-linear symmetric hyperbolic systems, with a constant symmetrizer. Thus, one can solve the Cauchy problem (1.2) in standard fashion: one first solves (2.2) and then checks that $\operatorname{curl} v = 0$, so that $v = \nabla\phi$ for some ϕ. In contrast, for $\sigma > 1$, System (2.2) is no longer symmetric. However, as in [31], one can prove that the Cauchy problem for (2.2) is well-posed, with loss of (at most) one derivative for a, by introducing $A = a^\sigma$. Indeed, (v, A) solves a quasi-linear hyperbolic system with constant symmetrizer:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla |A|^2 = 0; & v_{|t=0} = \nabla\phi_0, \\ \partial_t A + v \cdot \nabla A + \dfrac{\sigma}{2} A \operatorname{div} v = 0; & A_{|t=0} = a_0^\sigma. \end{cases} \tag{2.3}$$

This allows us to determine v, and hence ϕ, by setting

$$\phi(t, x) = \phi_0(x) - \int_0^t \left(\frac{1}{2} |v(\tau, x)|^2 + |A(\tau, x)|^2 \right) \mathrm{d}\tau.$$

Then $\partial_t (\nabla\phi - v) = \nabla\partial_t\phi - \partial_t v = 0$, hence $v = \nabla\phi$. Once this is granted, one can define a as the solution of the second equation in (2.2), where v is now viewed as a given coefficient. Since A and a^σ satisfy the same linear equation, with identical initial data, we obtain $A = a^\sigma$. Therefore, (a, ϕ) solves (1.2). Finally, the local existence time T^* may be chosen independent of $s > n/2 + 1$, thanks to tame estimates (see, for example, [35]). □

For further references, we conclude this paragraph by recalling a standard estimate in Sobolev spaces for systems of the form

$$\partial_t U + \sum_{1 \leqq j \leqq n} A_j(\Phi, U)\partial_j U + \varepsilon \mathcal{L}(\partial_x) U = E(\Phi, U), \tag{2.4}$$

where $U \colon [0, T] \times \mathbb{R}^n \to \mathbb{C}^d$ with $d \geqq 1$, $\varepsilon \in \mathbb{R}$ and:

- $\Phi : [0, T] \times \mathbb{R}^n \to \mathbb{C}^d$ is a given function.
- The A_j's are $d \times d$ Hermitian matrices depending smoothly on their arguments.
- $\mathcal{L}(\partial_x) = \sum L_{jk} \partial_j \partial_k$ is a skew-symmetric second-order differential operator with constant coefficients.
- E a C^∞ function of its arguments, vanishing at the origin.

Lemma 2.3. *Let $n \geqq 1$ and $s > n/2 + 1$. There exists a smooth nondecreasing function C from $[0, +\infty[$ to $[0, +\infty[$ such that, for all $T > 0$, all $\varepsilon \in \mathbb{R}$, all coefficient $\Phi \in C([0, T]; H^s(\mathbb{R}^n))$ and all unknown $U \in C([0, T]; H^s(\mathbb{R}^n))$ satisfying (2.4), there holds*

$$\sup_{t \in [0,T]} \|U(t)\|_{H^s} \leqq \|U(0)\|_{H^s} \, \mathrm{e}^{C(M)T},$$

with $M := \|(\Phi, U)\|_{L^\infty([0,T];H^s(\mathbb{R}^n))}$.

Proof. We want to estimate the $L^2(\mathbb{R}^n)$ norm of $\Lambda^s U$, where Λ^s is the Fourier multiplier $(\mathrm{Id} - \Delta)^{s/2}$. To deal with smooth functions, we use the Friedrichs mollifiers: let $\jmath \in C_0^\infty(\mathbb{R}^n)$ be such that $\jmath(\xi) = 1$ for $|\xi| \leqq 1$, then we define $J_\delta = \jmath(\delta D_x)$ as the Fourier multiplier with symbol $\jmath(\delta \xi)$.

With these notations, set $U_\delta := J_\delta \Lambda^s U$. Since $s - 1 > n/2$, $H^{s-1}(\mathbb{R}^n)$ is an algebra which is stable by composition ($F(u) \in H^{s-1}(\mathbb{R}^n)$ whenever $u \in H^{s-1}(\mathbb{R}^n)$ and $F \in C^\infty$ satisfies $F(0) = 0$): $U \in C^1([0, T]; H^{s-2}(\mathbb{R}^n))$. Therefore, U_δ is smooth: $U_\delta \in C^1([0, T]; H^\infty(\mathbb{R}^n))$. Now write

$$\partial_t U_\delta + \sum_{1 \leqq j \leqq n} A_j(\Phi, U) \partial_j U_\delta + \varepsilon \mathcal{L}(\partial_x) U_\delta = f_\delta,$$

with

$$f_\delta = \sum_{1 \leqq j \leqq n} [A_j(\Phi, U), J_\delta \Lambda^s] \partial_j U + J_\delta \Lambda^s E(\Phi, U).$$

Since $\mathcal{L}(\partial_x) = -\mathcal{L}(\partial_x)^*$, and since $U_\delta \in C^1([0, T]; L^2(\mathbb{R}^n))$, by taking the inner product in $L^2(\mathbb{R}^n)$, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|U_\delta\|_{L^2}^2 &= \sum_{1 \leqq j \leqq n} \langle \partial_j A_j(\Phi, U) U_\delta \, , \, U_\delta \rangle + 2 \langle f_\delta \, , \, U_\delta \rangle \\
&\leqq \left(1 + \sum_{1 \leqq j \leqq n} \left\| \partial_j A_j(\Phi, U) \right\|_{L^\infty} \right) \|U_\delta\|_{L^2}^2 + \|f_\delta\|_{L^2}^2 \, ,
\end{aligned}$$

where we have used the symmetry of the matrices A_j. The Sobolev embedding and the usual nonlinear estimates (see [35]) imply

$$\begin{aligned}
\left\| \partial_j A_j(\Phi, U) \right\|_{L^\infty} &\leqq C(\|(\Phi, U)\|_{W^{1,\infty}}) \leqq C(\|(\Phi, U)\|_{H^s}), \\
\left\| [A_j(\Phi, U), J_\delta \Lambda^s] \partial_j U \right\|_{L^2} &\leqq K \left\| \widetilde{A}_j(\Phi, U) \right\|_{H^s} \left\| \partial_j U \right\|_{H^{s-1}} \leqq C(\|(\Phi, U)\|_{H^s}), \\
\left\| J_\delta \Lambda^s E(\Phi, U) \right\|_{L^2} &\leqq K \, \|E(\Phi, U)\|_{H^s} \leqq C(\|(\Phi, U)\|_{H^s}),
\end{aligned}$$

where $\widetilde{A}_j = A_j - A_j(0)$ and C denotes a smooth nondecreasing function independent of δ. To complete the proof, apply Gronwall lemma and let δ go to 0 in the inequality thus obtained. □

3. Proof of Theorem 1.3 in the case $\sigma = 1$

Recall that a^ε is defined as:

$$a^\varepsilon(t,x) := u^\varepsilon(t,x)\mathrm{e}^{-i\phi(t,x)/\varepsilon},$$

where $\phi \in C^\infty([0,T^*[\times\mathbb{R}^n)$ is given by (1.2). Assume in the rest of this paragraph that $\sigma = 1$. Then, (1.7) reads

$$\begin{cases} \partial_t a^\varepsilon + \nabla\phi\cdot\nabla a^\varepsilon + \dfrac{1}{2}a^\varepsilon\Delta\phi - i\dfrac{\varepsilon}{2}\Delta a^\varepsilon = -\dfrac{i}{\varepsilon}\left(\left|a^\varepsilon\right|^2 - |a|^2\right)a^\varepsilon, \\ a^\varepsilon_{|t=0} = a^\varepsilon_0. \end{cases}$$

Let $s > n/2+1$ and set $\tau^\varepsilon := \min(T^*, T^\varepsilon)$, where T^* and T^ε are given by Lemmas 2.1 and 2.2. We prove that there exists a function C from $[0,+\infty[$ to $[0,+\infty[$ such that, for all $\varepsilon\in]0,1]$ and all $t\in[0,\tau^\varepsilon[$,

$$\left\|a^\varepsilon(t)\right\|_{H^s} \leqq \left\|a^\varepsilon(0)\right\|_{H^s}\mathrm{e}^{tC(M^\varepsilon(t))}, \tag{3.1}$$

where

$$M^\varepsilon(t) := \left\|a^\varepsilon\right\|_{L^\infty([0,t];H^s(\mathbb{R}^n))} + \|(a,\phi)\|_{L^\infty([0,t];H^{s+3}(\mathbb{R}^n))}.$$

This suffices to conclude by a standard continuity argument. Indeed, set

$$M_0 := \sup_{\varepsilon\in]0,1]}\left\|a^\varepsilon_0\right\|_{H^s} + \|(a,\phi)\|_{L^\infty([0,T^*/2];H^{s+3}(\mathbb{R}^n))} < +\infty,$$

and choose $T_0\in]0,T^*/2]$ small so that $M_0\exp(T_0C(2M_0)) < 2M_0$. Since $M^\varepsilon(0) < 2M_0$ and since $M^\varepsilon\in C^0([0,\tau^\varepsilon[)$, (3.1) implies

$$M^\varepsilon(t) < 2M_0, \quad \forall t\in[0,\min\{T_0,T^\varepsilon\}[.$$

Sobolev embedding then shows that $\|u^\varepsilon(t)\|_{L^\infty} = \|a^\varepsilon(t)\|_{L^\infty}$ is uniformly bounded for all $\varepsilon\in]0,1]$ and all $t\in[0,\min\{T_0,T^\varepsilon\}[$. Hence, the continuation principle (2.1) implies that $T^\varepsilon \geqq T_0 > 0$ for all $\varepsilon\in]0,1]$. The estimate (1.3) with $\sigma = 1$ then follows from the bound $\sup_{\varepsilon\in]0,1]}\sup_{t\in[0,T_0]} M^\varepsilon(t) \leqq 2M_0$.

Theorem 1.3 for $\sigma = 1$ was first established by GRENIER [27], whose approach is based on a subtle phase/amplitude representation of the solution. Here, we give an alternate proof which consists in symmetrizing the large terms in ε^{-1} in the equation for a^ε by introducing

$$q^\varepsilon := \frac{|a^\varepsilon|^2 - |a|^2}{\varepsilon}.$$

We find directly, in view of Assumption 1.1:

$$\partial_t q^\varepsilon + \operatorname{div}\left(\operatorname{Im}\left(\overline{a}^\varepsilon \nabla a^\varepsilon\right)\right) + \operatorname{div}(q^\varepsilon \nabla \phi) = 0; \quad \|q^\varepsilon_{|t=0}\|_{H^s(\mathbb{R}^n)} = \mathcal{O}(1), \ \forall s \geqq 0.$$

Furthermore, with this notation the equations for a^ε and $\psi^\varepsilon := \nabla a^\varepsilon$ read

$$\begin{cases} \partial_t a^\varepsilon + \nabla\phi \cdot \nabla a^\varepsilon + \dfrac{1}{2} a^\varepsilon \Delta\phi - i\dfrac{\varepsilon}{2}\Delta a^\varepsilon + i q^\varepsilon a^\varepsilon = 0, \\ \partial_t \psi^\varepsilon + \nabla\phi \cdot \nabla\psi^\varepsilon + \dfrac{1}{2}\psi^\varepsilon \Delta\phi + \psi^\varepsilon \cdot \nabla\nabla\phi + \dfrac{1}{2} a^\varepsilon \nabla\Delta\phi \\ \quad + i q^\varepsilon \psi^\varepsilon + i a^\varepsilon \nabla q^\varepsilon = i\dfrac{\varepsilon}{2}\Delta\psi^\varepsilon. \end{cases}$$

It is easily verified that $U^\varepsilon := (2q^\varepsilon, a^\varepsilon, \overline{a}^\varepsilon, \psi^\varepsilon, \overline{\psi^\varepsilon}) \in C^\infty([0, \tau^\varepsilon[; H^\infty(\mathbb{R}^n))$ satisfies a system of the form (2.4), that is

$$\partial_t U^\varepsilon + \sum_{1 \leqq j \leqq n} A_j(\Phi, U^\varepsilon)\partial_j U^\varepsilon + \varepsilon \mathcal{L}(\partial_x) U^\varepsilon = E(\Phi, U^\varepsilon),$$

where $\Phi := (\nabla\phi, \Delta\phi, \nabla\Delta\phi)$. Hence, by Lemma 2.3, we obtain the desired estimate (3.1) and conclude the proof of Theorem 1.3 in the case $\sigma = 1$.

4. The case $\sigma \geqq 2$

We now follow the strategy presented in Section 1.3. We introduce a nonlinear change of unknown functions which, together with (1.7), yields a quasi-linear system of the form (2.4). We conclude the proof of Theorem 1.3 thanks to a general result on the composition by nonsmooth functions in Sobolev spaces.

4.1. A nonlinear change of variable

As already explained, to symmetrize the equations, our idea is to split the term $|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}$ as a product

$$\left|a^\varepsilon\right|^{2\sigma} - |a|^{2\sigma} = g^\varepsilon \beta^\varepsilon = (G\,B)(\left|a^\varepsilon\right|^2, |a|^2) = G(r_1, r_2)B(r_1, r_2)\,|_{(r_1,r_2)=(|a^\varepsilon|^2,|a|^2)},$$

where β^ε satisfies an equation of the form

$$\partial_t \beta^\varepsilon + L(a, \phi, \partial_x)\beta^\varepsilon + g^\varepsilon \operatorname{div}\left(\varepsilon \operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)\right) = 0, \tag{4.1}$$

and L is a first-order differential operator. Proposition 4.3 below shows that it is possible to do so. Before giving this precise statement, we introduce convenient notations, and explain how to formally find β^ε.

Introduce the position densities

$$\rho := |a|^2 \in C^\infty([0, T^*[\times\mathbb{R}^n); \quad \rho^\varepsilon := |a^\varepsilon|^2 = \left|u^\varepsilon\right|^2 \in C^\infty([0, T^\varepsilon[\times\mathbb{R}^n).$$

Let $v = \nabla\phi$. Elementary computations show that:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{4.2}$$

$$\partial_t \rho^\varepsilon + \operatorname{div} \operatorname{Im}\left(\varepsilon \overline{u}^\varepsilon \nabla u^\varepsilon\right) = 0, \tag{4.3}$$

$$\partial_t \rho^\varepsilon + \operatorname{div}\left(\operatorname{Im}(\varepsilon \overline{a}^\varepsilon \nabla a^\varepsilon) + \rho^\varepsilon v\right) = 0. \tag{4.4}$$

Denote

$$J^\varepsilon := \varepsilon \operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon).$$

By writing

$$\partial_t \beta^\varepsilon = (\partial_{r_1} B)(\rho^\varepsilon, \rho)\partial_t \rho^\varepsilon + (\partial_{r_2} B)(\rho^\varepsilon, \rho)\partial_t \rho,$$

we compute, from (4.2) and (4.4):

$$\partial_t \beta^\varepsilon + (\partial_{r_1} B)(\rho^\varepsilon, \rho) \operatorname{div}(J^\varepsilon + \rho^\varepsilon v) + (\partial_{r_2} B)(\rho^\varepsilon, \rho) \operatorname{div}(\rho v) = 0.$$

Hence, in order to have an equation of the desired form (4.1), we impose

$$\partial_{r_1} B(r_1, r_2) = G(r_1, r_2).$$

Since on the other hand,

$$G(r_1, r_2)B(r_1, r_2) = r_1^\sigma - r_2^\sigma,$$

this suggests to choose β^ε such that

$$(\beta^\varepsilon)^2 = \frac{2}{\sigma+1}(\rho^\varepsilon)^{\sigma+1} - 2\rho^\sigma \rho^\varepsilon + f(\rho). \tag{4.5}$$

To obtain an operator L which is linear with respect to β^ε we choose

$$(\beta^\varepsilon)^2 = \frac{2}{\sigma+1}(\rho^\varepsilon)^{\sigma+1} - \frac{2}{\sigma+1}\rho^{\sigma+1} - 2\rho^\sigma(\rho^\varepsilon - \rho). \tag{4.6}$$

With this choice, we formally compute:

$$\partial_t \beta^\varepsilon + \varepsilon g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + v \cdot \nabla \beta^\varepsilon + \frac{\sigma+1}{2}\beta^\varepsilon \operatorname{div} v = 0.$$

Before deriving this equation rigorously, examine the right-hand side of (4.6). Taylor's formula yields

$$\frac{2}{\sigma+1}(\rho^\varepsilon)^{\sigma+1} - \frac{2}{\sigma+1}\rho^{\sigma+1} - 2\rho^\sigma(\rho^\varepsilon - \rho) = (\rho^\varepsilon - \rho)^2 Q_\sigma(\rho^\varepsilon, \rho),$$

where Q_σ is given by:

$$Q_\sigma(r_1, r_2) := 2\sigma \int_0^1 (1-s)\,(r_2 + s(r_1 - r_2))^{\sigma-1}\, \mathrm{d}s. \tag{4.7}$$

Note that there exists C_σ such that:

$$Q_\sigma(r_1, r_2) \geqq C_\sigma \left(r_1^{\sigma-1} + r_2^{\sigma-1}\right). \tag{4.8}$$

Notation 4.1. *Let $\sigma \in \mathbb{N}$. Introduce*

$$G_\sigma(r_1, r_2) = \frac{P_\sigma(r_1, r_2)}{\sqrt{Q_\sigma(r_1, r_2)}}; \quad B_\sigma(r_1, r_2) := (r_1 - r_2)\sqrt{Q_\sigma(r_1, r_2)},$$

where Q_σ is given by (4.7) and

$$P_\sigma(r_1, r_2) = \frac{r_1^\sigma - r_2^\sigma}{r_1 - r_2} = \sum_{\ell=0}^{\sigma-1} r_1^{\sigma-1-\ell} r_2^\ell.$$

Note that the definition of B_σ is the definition given in Theorem 1.3.

Example 4.2. For $\sigma = 1, 2, 3$, we compute

$$G_1 = 1, \qquad B_1 = r_1 - r_2.$$
$$G_2 = \sqrt{\frac{3}{2}} \frac{r_1 + r_2}{\sqrt{r_1 + 2r_2}}, \qquad B_2 = \sqrt{\frac{2}{3}}(r_1 - r_2)\sqrt{r_1 + 2r_2}.$$
$$G_3 = \sqrt{2}\frac{r_1^2 + r_1 r_2 + r_2^2}{\sqrt{(r_1 - r_2)^2 + 2r_2^2}}, \quad B_3 = \frac{1}{\sqrt{2}}(r_1 - r_2)\sqrt{(r_1 - r_2)^2 + 2r_2^2}.$$

A remarkable fact is that, although the functions G_σ and B_σ are not smooth for $\sigma \geqq 2$, one can compute an evolution equation for the unknown $\beta^\varepsilon := B_\sigma\left(|a^\varepsilon|^2, |a|^2\right)$. We have the following key proposition.

Proposition 4.3. *With G_σ and B_σ as above, define*

$$\beta^\varepsilon := B_\sigma\left(\left|a^\varepsilon\right|^2, |a|^2\right), \quad g^\varepsilon := G_\sigma\left(\left|a^\varepsilon\right|^2, |a|^2\right).$$

Then $\beta^\varepsilon \in C^1([0, \tau^\varepsilon[\times\mathbb{R}^n)$ and $g^\varepsilon \in C^0([0, \tau^\varepsilon[\times\mathbb{R}^n)$, where $\tau^\varepsilon = \min(T^*, T^\varepsilon)$. Moreover,

$$\partial_t \beta^\varepsilon + \varepsilon g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + v \cdot \nabla\beta^\varepsilon + \frac{\sigma+1}{2}\beta^\varepsilon \operatorname{div} v = 0. \tag{4.9}$$

Remark 7. Again, note the dichotomy between $\sigma = 1$ and $\sigma \geqq 2$. If $\sigma = 1$ then, by definition, $g^\varepsilon = 1$ and $\beta^\varepsilon = \rho^\varepsilon - \rho$ are C^∞ functions. Moreover (4.9) simply reads

$$\partial_t \beta^\varepsilon + \varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + \operatorname{div}(v\beta^\varepsilon) = 0,$$

corresponding to the equation for $q^\varepsilon = \varepsilon^{-1}\beta^\varepsilon$ in Section 3, and which follows directly by subtracting (4.2) from (4.4).

Proof. The regularity properties of β^ε and g^ε follow from Lemmas 2.1 and 2.2, along with the definition of β^ε and g^ε (see Notation 4.1, and (4.8)).

Since by definition

$$\beta^\varepsilon(\partial_{r_1} B_\sigma)(\rho^\varepsilon, \rho) = (\rho^\varepsilon)^\sigma - \rho^\sigma,$$
$$\beta^\varepsilon(\partial_{r_2} B_\sigma)(\rho^\varepsilon, \rho) = \sigma(\rho^\sigma - \rho^{\sigma-1}\rho^\varepsilon),$$

we have

$$\begin{aligned}
&\beta^\varepsilon \partial_t \beta^\varepsilon \\
&= \beta^\varepsilon (\partial_{r_1} B_\sigma)(\rho^\varepsilon, \rho)\partial_t \rho^\varepsilon + \beta^\varepsilon (\partial_{r_2} B_\sigma)(\rho^\varepsilon, \rho)\partial_t \rho \\
&= -\beta^\varepsilon (\partial_{r_1} B_\sigma)(\rho^\varepsilon, \rho) \operatorname{div}(J_\varepsilon + \rho^\varepsilon v) - \beta^\varepsilon (\partial_{r_2} B_\sigma)(\rho^\varepsilon, \rho) \operatorname{div}(\rho v) \\
&= -((\rho^\varepsilon)^\sigma - \rho^\sigma) \operatorname{div}(J_\varepsilon + \rho^\varepsilon v) - \sigma(\rho^\sigma - \rho^{\sigma-1}\rho^\varepsilon) \operatorname{div}(\rho v).
\end{aligned}$$

From this we compute

$$\beta^\varepsilon \left(\partial_t \beta^\varepsilon + \varepsilon g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + v \cdot \nabla \beta^\varepsilon + \frac{\sigma+1}{2}\beta^\varepsilon \operatorname{div} v \right) = 0.$$

Introduce

$$\begin{aligned}
\omega^\varepsilon := \{\rho^\varepsilon = \rho\} &= \left\{(t,x) \in [0,\tau^\varepsilon[\times \mathbb{R}^n \mid \rho^\varepsilon(t,x) = \rho(t,x)\right\} \\
&= \left([0,\tau^\varepsilon[\times\mathbb{R}^n\right)\backslash\{\beta^\varepsilon \neq 0\} \quad \text{(by (4.8))}.
\end{aligned}$$

Then (4.9) holds on $([0,\tau^\varepsilon[\times\mathbb{R}^n)\backslash\omega^\varepsilon$; hence on $\overline{([0,\tau^\varepsilon[\times\mathbb{R}^n)\backslash\omega^\varepsilon}$ by continuity. To prove the proposition, it thus suffices to show

$$\partial_t \beta^\varepsilon + \varepsilon g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + v \cdot \nabla \beta^\varepsilon + \frac{\sigma+1}{2}\beta^\varepsilon \operatorname{div} v = 0 \quad \text{on} \quad \mathring{\omega}^\varepsilon,$$

where $\mathring{A}$ denotes the interior of the set A. Since $\beta^\varepsilon = 0$ on $\mathring{\omega}^\varepsilon$, it is enough to prove that $\operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) = 0$ on $\mathring{\omega}^\varepsilon$. This in turn follows from (4.2) and (4.4), which yield the following:

$$\operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) = -\varepsilon^{-1}\left(\partial_t(\rho^\varepsilon - \rho) + \operatorname{div}((\rho^\varepsilon - \rho)v)\right).$$

This completes the proof. □

We will see that $|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}$ is of order $\mathcal{O}(\varepsilon)$, so we naturally set

$$\psi^\varepsilon := \nabla a^\varepsilon; \quad q^\varepsilon := \varepsilon^{-1}\beta^\varepsilon.$$

We infer from the previous computations that $(a^\varepsilon, \psi^\varepsilon, q^\varepsilon)$ solves the following:

$$\begin{cases}
\partial_t a^\varepsilon + v\cdot\nabla a^\varepsilon + \dfrac{1}{2}a^\varepsilon \operatorname{div} v - i\dfrac{\varepsilon}{2}\Delta a^\varepsilon = -ig^\varepsilon q^\varepsilon a^\varepsilon. \\
\partial_t \psi^\varepsilon + v\cdot\nabla\psi^\varepsilon + \dfrac{1}{2}\psi^\varepsilon \operatorname{div} v + \psi^\varepsilon\cdot\nabla v + \dfrac{1}{2}a^\varepsilon \nabla \operatorname{div} v - i\dfrac{\varepsilon}{2}\Delta\psi^\varepsilon \\
\quad = -iq^\varepsilon\nabla\left(a^\varepsilon g^\varepsilon\right) - ia^\varepsilon g^\varepsilon \nabla q^\varepsilon, \\
\partial_t q^\varepsilon + v\cdot\nabla q^\varepsilon + g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon\psi^\varepsilon)) + \dfrac{\sigma+1}{2}q^\varepsilon \operatorname{div} v = 0.
\end{cases} \tag{4.10}$$

Simply by writing

$$g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon\psi^\varepsilon)) = \operatorname{Im}(g^\varepsilon\overline{a}^\varepsilon \operatorname{div}\psi^\varepsilon),$$

we can rewrite the previous system as

$$\begin{cases} \partial_t a^\varepsilon + v\cdot\nabla a^\varepsilon - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{1}{2}a^\varepsilon \operatorname{div} v - ig^\varepsilon q^\varepsilon a^\varepsilon. \\ \partial_t \psi^\varepsilon + v\cdot\nabla\psi^\varepsilon + ia^\varepsilon g^\varepsilon \nabla q^\varepsilon - i\frac{\varepsilon}{2}\Delta\psi^\varepsilon \\ \quad = -\frac{1}{2}\psi^\varepsilon \operatorname{div} v - \psi^\varepsilon\cdot\nabla v - \frac{1}{2}a^\varepsilon\nabla \operatorname{div} v - iq^\varepsilon\nabla\left(a^\varepsilon g^\varepsilon\right), \\ \partial_t q^\varepsilon + v\cdot\nabla q^\varepsilon + \operatorname{Im}(g^\varepsilon \overline{a}^\varepsilon \operatorname{div}\psi^\varepsilon) = -\frac{\sigma+1}{2}q^\varepsilon \operatorname{div} v. \end{cases} \tag{4.11}$$

Note that in view of Assumption 1.1,

$$\left\| a^\varepsilon_{|t=0}\right\|_{H^s(\mathbb{R}^n)} + \left\|\psi^\varepsilon{}_{|t=0}\right\|_{H^s(\mathbb{R}^n)} = \mathcal{O}(1),\ \forall s \geqq 0. \tag{4.12}$$

A similar estimate for the initial data of q^ε is a more delicate issue, since B_σ is not a smooth function. We postpone this estimate to Section 4.2.

The left-hand side of (4.11) is a first-order quasi-linear symmetric hyperbolic system, plus a second-order skew-symmetric term. The right-hand side can be viewed as a semi-linear source term. We deduce from Proposition 4.3:

Corollary 4.4. *On* $[0,\tau^\varepsilon[\times\mathbb{R}^n$*, the function* $U^\varepsilon := (2q^\varepsilon, a^\varepsilon, \overline{a}^\varepsilon, \psi^\varepsilon, \overline{\psi}^\varepsilon)$ *satisfies an equation of the form*

$$\partial_t U^\varepsilon + \sum_{1\leqq j\leqq n} A_j(v, a^\varepsilon g^\varepsilon, \overline{a}^\varepsilon g^\varepsilon)\partial_j U^\varepsilon + \varepsilon\mathcal{L}(\partial_x)U^\varepsilon = E(\Phi, U^\varepsilon, a^\varepsilon g^\varepsilon, \nabla(a^\varepsilon g^\varepsilon)), \tag{4.13}$$

where $\Phi = (\nabla\phi, \nabla^2\phi, \nabla^3\phi)$*, the* A_j*'s are Hermitian matrices linear in their arguments,* $\mathcal{L}(\partial_x) = \sum L_{jk}\partial_j\partial_k$ *is a skew-symmetric second-order differential operator with constant coefficients, and* E *is a* C^∞ *function of its arguments, vanishing at the origin.*

We can restate Theorem 1.3:

Theorem 4.5. *Let* $n \leqq 3$*, and let Assumption 1.1 be satisfied. There exists* $T\in]0,T^*[$*, where* T^* *is given by Lemma 1.2, such that the following holds. For all* $\varepsilon\in]0,1]$*, the Cauchy problem (1.1) has a unique solution* $u^\varepsilon \in C([0,T];H^\infty(\mathbb{R}^n))$*. Moreover,*

$$\sup_{\varepsilon\in]0,1]}\left(\|a^\varepsilon\|_{L^\infty([0,T];H^k(\mathbb{R}^n))} + \|q^\varepsilon\|_{L^\infty([0,T];H^{k-1}(\mathbb{R}^n))}\right) < +\infty, \tag{4.14}$$

where the index k *is as follows:*

- *If* $\sigma = 1$*, then* $k\in\mathbb{N}$ *is arbitrary.*
- *If* $\sigma = 2$ *and* $n = 1$*, then we can take* $k = 2$*.*
- *If* $\sigma = 2$ *and* $2 \leqq n \leqq 3$*, then we can take* $k = 1$*.*
- *If* $\sigma \geqq 3$*, then we can take* $k = \sigma$*.*

4.2. Quasi-linear analysis

We now have to estimate $(a^\varepsilon, q^\varepsilon, \psi^\varepsilon)$ in Sobolev spaces. Let us briefly explain the difficulty. To clarify matters, suppose that $g^\varepsilon = G(|a^\varepsilon|^2, |a|^2)$ for some smooth function $G \in C^\infty(\mathbb{R}^2)$. In particular this is so in the cubic case $\sigma = 1$. Then, in view of Corollary 4.4, $U^\varepsilon := (2q^\varepsilon, a^\varepsilon, \overline{a}^\varepsilon, \psi^\varepsilon, \overline{\psi^\varepsilon}) \in C([0, \tau^\varepsilon[; H^\infty(\mathbb{R}^n))^{3+2n}$ satisfies a system of the form (2.4),

$$\partial_t U^\varepsilon + \sum_{1 \leqq j \leqq n} \mathcal{A}_j(\Phi, U^\varepsilon)\partial_j U^\varepsilon + \varepsilon \mathcal{L}(\partial_x) U^\varepsilon = \mathcal{E}(\Phi, U^\varepsilon),$$

where $\Phi := (|a|^2, \nabla |a|^2, \nabla\phi, \nabla^2\phi, \nabla^3\phi)$. The key difference with the system in Corollary 4.4 is the absence of dependence upon the extra unknown g^ε. Then, Lemma 2.3 yields estimates in Sobolev spaces (of arbitrary order).

Assume now $\sigma \geqq 2$. One can check that the previous symmetrization provides us with uniform a priori estimates in L^2. However, the estimates of the derivatives require a careful analysis. Indeed, recall that

$$g^\varepsilon = G_\sigma(\left|a^\varepsilon\right|^2, |a|^2) \quad \text{with} \quad G_\sigma(r_1, r_2) = \frac{P_\sigma(r_1, r_2)}{\sqrt{Q_\sigma(r_1, r_2)}},$$

where P_σ and Q_σ are defined in Notation 4.1 and (4.7), respectively. Therefore, G_σ need not be smooth at the origin. The classical approach, which consists in differentiating the equations, thus certainly fails here. Yet, as we will see, we need only estimate $a^\varepsilon g^\varepsilon$ in H^σ. Introduce

$$F_\sigma(z, z') = zG_\sigma\left(|z|^2, \left|z'\right|^2\right) : \quad a^\varepsilon g^\varepsilon = F_\sigma\left(a^\varepsilon, a\right). \tag{4.15}$$

One can check that $F_\sigma \in C^{\sigma-1}$ but $F_\sigma \notin C^\sigma$. Hence, to estimate $a^\varepsilon g^\varepsilon$ in H^σ, one cannot use the usual nonlinear estimates. Instead, we will use that F_σ is homogeneous of degree σ and the following lemma.

Lemma 4.6. *Let $p \geqq 1$ and $m \geqq 2$ be integers and consider $F : \mathbb{R}^p \to \mathbb{C}$. Assume that $F \in C^\infty(\mathbb{R}^p \setminus \{0\})$ is homogeneous of degree m, that is:*

$$F(\lambda y) = \lambda^m F(y), \quad \forall \lambda \geqq 0, \quad \forall y \in \mathbb{R}^p.$$

Then, for $n \leqq 3$, there exists $K > 0$ such that, for all $u \in H^m(\mathbb{R}^n)$ with values in $\mathbb{R}^p$, $F(u) \in H^m(\mathbb{R}^n)$ and

$$\|F(u)\|_{H^m} \leqq K \|u\|_{H^m}^m .$$

The same is true when $m = 1$ and $n \in \mathbb{N}$.

Remark 8. Note that the result is false for $n \geqq 4$ and $m = 2$. Also, one must not expect $F(u) \in H^{m+1}(\mathbb{R}^n)$, even for $u \in H^\infty(\mathbb{R}^n)$. For instance, if

$$n = 1 = p, \quad m = 2, \quad F(y) = y\,|y|, \quad u(x) = xe^{-x^2},$$

then $F(u) \in H^2(\mathbb{R})$ and $F(u) \notin H^3(\mathbb{R})$. Similarly, in general, one must not expect $F_\sigma(u, v) \in H^{\sigma+1}(\mathbb{R}^n)$, even for $(u, v) \in H^\infty(\mathbb{R}^n)^2$.

Proof. We prove the result by induction on m. Consider first the case $m = 2$. Observe that, by assumption, $F \in C^{m-1}(\mathbb{R}^p)$. To regularize F, let $\chi \in C_0^\infty(\mathbb{R}^p)$ be such that $0 \leqq \chi \leqq 1$, $\chi(y) = 1$ for $|y| \leqq 1$ and $\chi(y) = 0$ for $|y| \geqq 3$, with $|\nabla\chi(y)| \leqq 1$. For $\ell \in \mathbb{N}$, define $F_\ell \in C^\infty(\mathbb{R}^p)$ by

$$F_\ell(y) = (1 - \chi\,(\ell y))\, F(y).$$

We claim that, for all $y \in \mathbb{R}^p$ and all $\ell \in \mathbb{N}$,

$$|F_\ell(y)| \leqq C_F|y|^2, \quad \left|\partial_j F_\ell(y)\right| \leqq 4C_F\,|y|\,, \quad \left|\partial_j\partial_k F_\ell(y)\right| \leqq 4C_F,$$

where $\partial_j = \partial_{y_j}$ and

$$C_F := \sup_{|z|\leqq 3} |F(z)| + \sup_{1\leqq j\leqq p}\ \sup_{|z|\leqq 3} \left|\partial_j F(z)\right| + \sup_{1\leqq j,k\leqq p}\ \sup_{|z|=1} \left|\partial_j\partial_k F(z)\right|.$$

Since F_ℓ vanishes in a neighborhood of the origin, it suffices to establish these bounds for $y \neq 0$. The first bound follows from the homogeneity: $|F_\ell(y)| \leqq |F(y)| = |y|^2|F(y/|y|)|$. For the second one, compute

$$\partial_j F_\ell(y) = (1 - \chi(\ell y))\partial_j F(y) - \ell^{-1}(\partial_j\chi)(\ell y)F(\ell y),$$

where we used $\ell F(y) = \ell^{-1}F(\ell y)$. Since $1 \leqq |\ell y| \leqq 3$ on the support of $(\partial_j\chi)(\ell y)$, and since $\partial_j F\colon \mathbb{R}^p \to \mathbb{C}$ is homogeneous of degree 1, we infer

$$|\partial_j F_\ell(y)| \leqq |y|\left(\sup_{|z|\leqq 3}\left|\partial_j F\,(z)\right| + 3\sup_{z\in\mathbb{R}^p}\left|(\partial_j\chi)(z)F(z)\right|\right) \leqq 4C_F|y|.$$

The same reasoning yields

$$\begin{aligned}\partial_j\partial_k F_\ell(y) &= (1 - \chi(\ell y))\partial_j\partial_k F(y) - (\partial_j\chi)(\ell y)(\partial_k F)(\ell y)\\ &\quad -(\partial_k\chi)(\ell y)(\partial_j F)(\ell y) - (\partial_j\partial_k\chi)(\ell y)F(\ell y).\end{aligned}$$

The last three terms are clearly bounded by C_F since $|\ell y| \leqq 3$ on the support of $\chi(\ell y)$. Also, the first term is bounded by C_F since $\partial_j\partial_k F\colon \mathbb{R}^p\setminus\{0\} \to \mathbb{C}$ is homogeneous of degree 0. This completes the proof of the claim.

With these preliminary established, we easily obtain that there exists K such that for all $\ell \in \mathbb{N}$ and all $u \in H^2(\mathbb{R}^n)$ with values in $\mathbb{R}^p$,

$$\begin{aligned}&\|F_\ell(u)\|_{L^2} \leqq K\,\|u\|_{L^\infty}\,\|u\|_{L^2}\,,\\ &\|\nabla F_\ell(u)\|_{L^2} \leqq K\,\|u\|_{L^\infty}\,\|\nabla u\|_{L^2}\,,\\ &\left\|\nabla^2 F_\ell(u)\right\|_{L^2} \leqq K\,\|u\|_{L^\infty}\left\|\nabla^2 u\right\|_{L^2} + K\,\|\nabla u\|_{L^4}^2\,.\end{aligned}$$

The Sobolev embeddings $H^1(\mathbb{R}^n) \subset L^6(\mathbb{R}^n)$ and $H^2(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ for $n \in \{1, 2, 3\}$ then imply that there exists a constant K such that, for all $\ell \in \mathbb{N}$ and all $u \in H^2(\mathbb{R}^n)$,

$$\|F_\ell(u)\|_{H^2} \leqq K\,\|u\|_{H^2}^2\,.$$

This in turn implies the desired result for $F(u)$ by using the dominated convergence theorem and a duality argument. Indeed, for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\left|\int F(u)\varphi \,\mathrm{d}x\right| = \left|\lim_{\ell\to+\infty}\int F_\ell(u)\varphi\,\mathrm{d}x\right| \leqq \limsup_{\ell\to+\infty} \|F_\ell(u)\|_{H^2}\, \|\varphi\|_{H^{-2}}$$
$$\leqq K\, \|u\|_{H^2}^2\, \|\varphi\|_{H^{-2}}\,,$$

which implies $F(u) \in H^2(\mathbb{R}^n)$ together with $\|F(u)\|_{H^2} \leqq K\,\|u\|_{H^2}^2$.

Assume now the result at order $m \geqq 2$, and prove the result at order $m+1$. Let $F \in C^\infty(\mathbb{R}^p\setminus\{0\})$ be homogeneous of degree $m+1$. We have

$$\|F(u)\|_{L^2} \leqq K\, \|u\|_{L^\infty}^m\, \|u\|_{L^2} \lesssim \|u\|_{H^{m+1}}^{m+1}\,.$$

Since $m > 3/2 \geqq n/2$, $H^m(\mathbb{R}^n)$ is an algebra and

$$\|\nabla F(u)\|_{H^m} \leqq K\, \|\nabla u\|_{H^m}\, \left\|F'(u)\right\|_{H^m}\,.$$

By assumption, $F' \in C^\infty(\mathbb{R}^p\setminus\{0\})$ is homogeneous of degree m; hence the induction assumption yields the following:

$$\left\|F'(u)\right\|_{H^m} \leqq K\,\|u\|_{H^m}^m\,.$$

Therefore,

$$\|\nabla F(u)\|_{H^m} \leqq K\,\|u\|_{H^{m+1}}^{m+1}\,.$$

The case $m = 1$ can be treated in a similar fashion. □

The lemma turns out to be useful to estimate the source term in (4.11), but also to estimate the initial data for q^ε. By definition, we have

$$q^\varepsilon = \frac{|z|^2 - |z'|^2}{\varepsilon}\,\mathcal{Q}_\sigma(z,z')\,|_{(z,z')=(a^\varepsilon,a)}\,,$$

where

$$\mathcal{Q}_\sigma(z,z') = \sqrt{Q_\sigma\left(|z|^2,|z'|^2\right)}$$
$$= \left(2\sigma\int_0^1 (1-s)\left(|z'|^2 + s(|z|^2 - |z'|^2)\right)^{\sigma-1}\mathrm{d}s\right)^{1/2}.$$

The function $\mathcal{Q}_\sigma$ is not smooth, but homogeneous of degree $\sigma-1$. So when $\sigma \geqq 3$, we can estimate q^ε in $H^{\sigma-1}$ at time $t=0$ thanks to this lemma. See Section 4.3.

To complete the proof of Theorem 4.5, in view of Lemma 2.1, we seek an H^2 estimate of a^ε, since

$$H^2(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n), \quad n \leqq 3.$$

This boils down to an H^1 estimate of U^ε defined in Corollary 4.4. Moreover, Lemma 2.3 requires to control U^ε in H^s with $s > n/2+1$, so we would demand $s = 3$ for $n = 3$ and $s \in \mathbb{N}$. In view of Lemma 2.3 and Corollary 4.4, we have to estimate $a^\varepsilon g^\varepsilon$ in H^4. Because of the lack of smoothness of G_σ, such an estimate seems hopeless in general. We, therefore, proceed in two steps. First, using the particular structure exhibited in Corollary 4.4, we relax the assumption $s > n/2 + 1$ in Lemma 2.3, to $s > n/2$. Next, we use Lemma 4.6 to overcome the lack of smoothness of G_σ, and obtain the desired a priori estimates.

Proposition 4.7. *Assume $\sigma \geqq 2$. Let U^ε be the vector-valued function given by Corollary 4.4, and $m > n/2$. Then for all $t \in [0, \tau^\varepsilon[$, it satisfies the following a priori estimate:*

$$\sup_{s\in[0,t]} \left\|U^\varepsilon(s)\right\|_{H^m} \leqq \left\|U^\varepsilon(0)\right\|_{H^m} \mathrm{e}^{tC(N^\varepsilon(t))},$$

with $N^\varepsilon(t) := \|\Phi\|_{L^\infty([0,t];H^m)} + \|U^\varepsilon\|_{L^\infty([0,t];H^m)} + \|a^\varepsilon g^\varepsilon\|_{L^\infty([0,t];H^{m+1})}$.

Proof (Sketch of the proof). Resume the proof of Lemma 2.3. The quantities that appear in N^ε are those on the last three lines of the proof of Lemma 2.3. First, we have the following:

$$\begin{aligned}\left\|\nabla A_j(v, a^\varepsilon g^\varepsilon, \overline{a}^\varepsilon g^\varepsilon)\right\|_{L^\infty} &\leqq C\left(\|v\|_{W^{1,\infty}} + \left\|a^\varepsilon g^\varepsilon\right\|_{W^{1,\infty}}\right)\\ &\leqq C\left(\|v\|_{H^{m+1}} + \left\|a^\varepsilon g^\varepsilon\right\|_{H^{m+1}}\right).\end{aligned}$$

Since A_j is linear in its arguments one has $\widetilde{A}_j = A_j$. In addition, since $m+1 > n/2+1$, a standard commutator estimate implies that

$$\begin{aligned}\left\|[A_j, \Lambda^m]\partial_j U^\varepsilon\right\|_{L^2} &\leqq K\left\|A_j\right\|_{H^{m+1}}\left\|U^\varepsilon\right\|_{H^m}\\ &\leqq C\left(\|v\|_{H^{m+1}} + \left\|a^\varepsilon g^\varepsilon\right\|_{H^{m+1}}\right)\left\|U^\varepsilon\right\|_{H^m}.\end{aligned}$$

Finally,

$$\left\|E(\Phi, U^\varepsilon, a^\varepsilon g^\varepsilon, \nabla(a^\varepsilon g^\varepsilon))\right\|_{H^m} \leqq C\left(\|\Phi\|_{H^m}, \left\|U^\varepsilon\right\|_{H^m}, \left\|a^\varepsilon g^\varepsilon\right\|_{H^{m+1}}\right).$$

We conclude the proof thanks to Gronwall lemma. □

4.3. The case $\sigma \geqq 3$

Recall that from (4.15),

$$a^\varepsilon g^\varepsilon = F_\sigma\left(a^\varepsilon, a\right),$$

where F_σ is homogeneous of degree σ. For $\sigma \geqq 3$ and $n \leqq 3$, Lemma 4.6 yields

$$\|a^\varepsilon g^\varepsilon\|_{H^\sigma} \leqq K\left(\|a^\varepsilon\|_{H^\sigma} + \|a\|_{H^\sigma}\right)^\sigma.$$

Hence Proposition 4.7 with $m = \sigma - 1 \geqq 2 > n/2$ shows that there exists a function C from $[0, +\infty[$ to $[0, +\infty[$ such that, for all $\varepsilon \in]0, 1]$ and all $t \in [0, \tau^\varepsilon[$,

$$\left\|U^\varepsilon(t)\right\|_{H^{\sigma-1}} \leqq \left\|U^\varepsilon(0)\right\|_{H^{\sigma-1}} \exp(tC(M^\varepsilon(t))),$$

where

$$M^\varepsilon(t) := \left\| U^\varepsilon \right\|_{L^\infty([0,t];\,H^{\sigma-1}(\mathbb{R}^n))} + \|(a,\phi)\|_{L^\infty([0,t];\,H^{\sigma+2}(\mathbb{R}^n))} \, .$$

It remains to estimate the initial data. By definition, we have

$$\left\| U^\varepsilon(0) \right\|_{H^{\sigma-1}} \lesssim \left\| q^\varepsilon(0) \right\|_{H^{\sigma-1}} + \left\| a^\varepsilon(0) \right\|_{H^\sigma} \, .$$

The second term is uniformly bounded by assumption. To estimate the first term, recall that

$$q^\varepsilon = \frac{|z|^2 - |z'|^2}{\varepsilon} \mathcal{Q}_\sigma(z, z') \mid_{(z,z')=(a^\varepsilon, a)} ,$$

where

$$\begin{aligned} \mathcal{Q}_\sigma(z, z') &= \sqrt{Q_\sigma\left(|z|^2, |z'|^2\right)} \\ &= \left(2\sigma \int_0^1 (1-s)\left(|z'|^2 + s(|z|^2 - |z'|^2)\right)^{\sigma-1} \mathrm{d}s\right)^{1/2} . \end{aligned}$$

The function $\mathcal{Q}_\sigma$ is not smooth, but homogeneous of degree $\sigma - 1$. To estimate q^ε at time $t = 0$, we use the usual product rule in Sobolev space and Lemma 4.6 (applied with $F(y_1, \ldots, y_4) = \mathcal{Q}_\sigma(y_1 + iy_2, y_3 + iy_4)$): if $\sigma \geqq 3$, with $m = \sigma - 1 \geqq 2$, we obtain

$$\begin{aligned} \left\| q^\varepsilon(0) \right\|_{H^{\sigma-1}} &\lesssim \left\| \varepsilon^{-1}\left(\left|a^\varepsilon(0)\right|^2 - |a(0)|^2 \right) \right\|_{H^{\sigma-1}} \left\| \mathcal{Q}_\sigma\left(a^\varepsilon(0), a(0)\right) \right\|_{H^{\sigma-1}} \\ &\lesssim \left\| \varepsilon^{-1}\left(\left|a^\varepsilon(0)\right|^2 - |a(0)|^2 \right) \right\|_{H^{\sigma-1}} \left\| \left(a^\varepsilon(0), a(0)\right) \right\|_{H^{\sigma-1}}^{\sigma-1} . \end{aligned}$$

The assumption $a_0^\varepsilon - a_0 = \mathcal{O}(\varepsilon)$ in H^s for all $s > 0$ then implies

$$\sup_{\varepsilon \in]0,1]} \left\| q^\varepsilon(0) \right\|_{H^{\sigma-1}} < +\infty, \tag{4.16}$$

hence

$$\sup_{\varepsilon \in]0,1]} \left\| U^\varepsilon(0) \right\|_{H^{\sigma-1}} < +\infty.$$

Consequently, since $\left\| u^\varepsilon \mathrm{e}^{-i\phi/\varepsilon} \right\|_{H^\sigma} = \|a^\varepsilon\|_{H^\sigma} \leqq \|U^\varepsilon\|_{H^{\sigma-1}}$, the same continuity argument as in Section 3 completes the proof of Theorem 4.5 in the case $\sigma \geqq 3$.

4.4. The case $\sigma = 2$

For $\sigma = 2$, we have $m = \sigma - 1 > n/2$ only when $n = 1$. The last point in Lemma 4.6 shows that

$$\sup_{\varepsilon \in]0,1]} \left\| q^\varepsilon(0) \right\|_{H^1(\mathbb{R})} < +\infty.$$

We can then proceed as in the case $\sigma \geqq 3$, to prove the second case in Theorem 4.5.

Finally, when $\sigma = 2$ and $2 \leqq n \leqq 3$, recall that we already know that for fixed $\varepsilon \in]0,1]$, u^ε is global in time, $u^\varepsilon \in C(\mathbb{R}, H^1)$. For $n = 2$, this is so since every defocusing, homogeneous nonlinearity is H^1-subcritical. For $n = 3$, the nonlinearity is H^1 critical, and this property follows from [20]. The proof of the estimate is based on an interesting feature of the equation for β^ε (see Proposition 4.3), which does not appear in Corollary 4.4. In the introduction, we claimed that the previous nonlinear symmetrization of the equations implies a local version of the modulated energy estimate. To see this, introduce

$$\mathrm{e}^\varepsilon := \left|a^\varepsilon\right|^2 + \left|\psi^\varepsilon\right|^2 + \left|q^\varepsilon\right|^2 \in C^1([0, \tau^\varepsilon[\times \mathbb{R}^n).$$

It satisfies an equation of the form $\partial_t \mathrm{e}^\varepsilon + \operatorname{div}(\eta^\varepsilon) + \flat^\varepsilon = \mathcal{O}(\mathrm{e}^\varepsilon)$, where $\int \flat^\varepsilon = 0$. Indeed, directly from (4.10), we compute

$$\begin{aligned}
&\partial_t \mathrm{e}^\varepsilon + \operatorname{div}(v \mathrm{e}^\varepsilon) + 2 \operatorname{div}\left(\operatorname{Im}(g^\varepsilon q^\varepsilon \overline{a}^\varepsilon \psi^\varepsilon)\right) + \varepsilon \operatorname{Im}\left(\overline{a}^\varepsilon \Delta a^\varepsilon + \overline{\psi^\varepsilon} \Delta \psi^\varepsilon\right) \\
&\quad = -\sigma \left|q^\varepsilon\right|^2 \operatorname{div} v - \operatorname{Re}\left((2\psi^\varepsilon \cdot \nabla v + a^\varepsilon \nabla \operatorname{div} v)\overline{\psi^\varepsilon}\right).
\end{aligned}$$

Hence, we have obtained an evolution equation for a modulated energy, which yields the desired modulated energy estimate. Gronwall lemma yields

$$\|\mathrm{e}^\varepsilon(t)\|_{L^1(\mathbb{R}^n)} \leqq \|\mathrm{e}^\varepsilon(0)\|_{L^1(\mathbb{R}^n)} \exp(Ct).$$

Finally, $(\mathrm{e}^\varepsilon(0))_\varepsilon$ is bounded in $L^1(\mathbb{R}^n)$. This is obvious for the first two terms of e^ε. For q^ε, a rough estimate yields:

$$\left\|q^\varepsilon(0)\right\|_{L^2} \leqq \left\|\varepsilon^{-1}\left(|a^\varepsilon(0)|^2 - |a(0)|^2\right)\right\|_{L^2} \|\mathcal{Q}_2(a^\varepsilon(0), a(0))\|_{L^\infty},$$

and the assumption $a_0^\varepsilon - a_0 = \mathcal{O}(\varepsilon)$ in H^∞ shows that

$$\sup_{0<\varepsilon\leqq 1} \|\mathrm{e}^\varepsilon(0)\|_{L^1(\mathbb{R}^n)} < \infty.$$

This completes the proof of Theorem 4.5.

4.5. Convergence of position and current densities

Corollary 1.4 follows from both information in (4.14). Indeed, for $k = 1$, (4.14) implies the "usual" modulated energy estimate, as in [6,30,40] (see also [2]). The boundedness of q^ε in $C([0,T];L^2)$, and the convexity argument (4.8), yield

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^n}\left(|a^\varepsilon(t,x)|^2-|a(t,x)|^2\right)^2\left(|a^\varepsilon(t,x)|^{2\sigma-2}+|a(t,x)|^{2\sigma-2}\right)^2 \mathrm{d}x \lesssim \varepsilon^2.$$

Therefore,

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^n}\left||a^\varepsilon(t,x)|^2-|a(t,x)|^2\right|^{\sigma+1} \mathrm{d}x \lesssim \varepsilon^2. \tag{4.17}$$

This yields the first part of Corollary 1.4, along with a bound on the rate of convergence as $\varepsilon \to 0$. For the current density, write

$$\operatorname{Im}\left(\varepsilon\overline{u}^\varepsilon\nabla u^\varepsilon\right) = |a^\varepsilon|^2\nabla\phi + \operatorname{Im}\left(\varepsilon\overline{a}^\varepsilon\nabla a^\varepsilon\right).$$

Since $\nabla\phi \in L^\infty([0,T]\times\mathbb{R}^n)$, (4.17) yields

$$|a^\varepsilon|^2\nabla\phi \underset{\varepsilon\to 0}{\longrightarrow} |a|^2\nabla\phi \quad \text{in } C([0,T];L^{\sigma+1}).$$

On the other hand, since a^ε is bounded in $C([0,T];H^1)$, we have the following:

$$\operatorname{Im}\left(\varepsilon\overline{a}^\varepsilon\nabla a^\varepsilon\right) = \mathcal{O}(\varepsilon) \quad \text{in } C([0,T];L^1).$$

This completes the proof of Corollary 1.4.

Remark 9. Since we have used (4.14) with $k = 1$ only, we could also refine the statements of Corollary 1.4 when $k \geqq 2$ is allowed in (4.14).

5. Proof of Theorem 1.6

To prove Theorem 1.6, resume the approach of GRENIER [27]. His idea was to seek

$$u^\varepsilon(t,x) = a^\varepsilon(t,x)\mathrm{e}^{i\phi^\varepsilon(t,x)/\varepsilon},$$

where the pair $U^\varepsilon = (a^\varepsilon, \nabla\phi^\varepsilon)$ is given by a system of the form (2.4) (with $E \equiv 0$). The point is that the form (2.4) for this U^ε meets all the requirements that we have listed, if and only if the nonlinearity is defocusing, and cubic at the origin. In the case of the homogeneous nonlinearity of (1.1), the only admissible case is then $\sigma = 1$. The second step of the analysis in [27] consists in showing that under suitable assumptions, a^ε and ϕ^ε have an asymptotic expansion of the form

$$a^\varepsilon \underset{\varepsilon\to 0}{\sim} a + \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \cdots; \quad \phi^\varepsilon \underset{\varepsilon\to 0}{\sim} \phi + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \cdots$$

The pair (a, ϕ) solves (the analogue of) (1.2). Note that because the phase ϕ^ε is divided by ε, we need to take $\phi^{(1)}$ into account in order to have a pointwise description of u^ε:

$$u^\varepsilon \underset{\varepsilon\to 0}{\sim} a e^{i\phi^{(1)}} e^{i\phi/\varepsilon}.$$

Therefore, the rapidly oscillatory phase for u^ε is given by ϕ, and its amplitude at leading order is given by $a e^{i\phi^{(1)}}$ (which does not depend on ε). If u^ε solves

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = f\left(|u^\varepsilon|^2\right)u^\varepsilon; \quad u^\varepsilon_{|t=0} = a_0^\varepsilon e^{i\phi_0/\varepsilon},$$

where a_0^ε satisfies Assumption 1.5, then $\phi^{(1)}$ is given by the system

$$\begin{cases} \partial_t\phi^{(1)} + \nabla\phi\cdot\nabla\phi^{(1)} + 2\,\mathrm{Re}\left(\overline{a}a^{(1)}\right) f'\left(|a|^2\right) = 0, \\ \partial_t a^{(1)} + \nabla\phi\cdot\nabla a^{(1)} + \nabla\phi^{(1)}\cdot\nabla a + \frac{1}{2}a^{(1)}\Delta\phi + \frac{1}{2}a\Delta\phi^{(1)} = \frac{i}{2}\Delta a, \\ \phi^{(1)}\,|_{t=0} = 0; \quad a^{(1)}\,|_{t=0} = a_1. \end{cases}$$

This coupling shows that $\phi^{(1)}$ is a (nonlinear) function of a, ϕ, and a_1, the term of order ε in the expansion of the initial data a_0^ε. In our case, $f(y) = y^\sigma$, we introduce the system

$$\begin{cases} \partial_t\phi^{(1)} + \nabla\phi\cdot\nabla\phi^{(1)} + 2\sigma\,\mathrm{Re}\left(\overline{a}a^{(1)}\right)|a|^{2\sigma-2} = 0, \\ \partial_t a^{(1)} + \nabla\phi\cdot\nabla a^{(1)} + \nabla\phi^{(1)}\cdot\nabla a + \frac{1}{2}a^{(1)}\Delta\phi + \frac{1}{2}a\Delta\phi^{(1)} = \frac{i}{2}\Delta a, \\ \phi^{(1)}\,|_{t=0} = 0; \quad a^{(1)}\,|_{t=0} = a_1. \end{cases} \tag{5.1}$$

Lemma 5.1. *Let $n \geqq 1$, and let Assumption 1.5 be satisfied. Then (5.1) has a unique solution $(\phi^{(1)}, a^{(1)})$ in $C([0, T^*[; H^\infty(\mathbb{R}^n))$, where T^* is given by Lemma 1.2.*

Proof. Again, at the zeroes of a, (5.1) ceases to be hyperbolic, and we cannot solve the Cauchy problem by a standard argument. The strategy of the proof is to transform the equations so as to obtain an auxiliary hyperbolic system for $(\nabla\phi^{(1)}, A_1)$ for some good unknown A_1, depending linearly upon $a^{(1)}$. The definition of A_1 depends on the parity of σ. This allows one to determine a function $\phi^{(1)}$ and next to define a function $a^{(1)}$ by solving the second equation in (5.1). We conclude the proof by checking that $(\phi^{(1)}, a^{(1)})$ does solve (5.1). The first change of unknown consists in considering $v_1 := \nabla\phi^{(1)}$. The first equation in (5.1) yields:

$$\partial_t v_1 + v\cdot\nabla v_1 + 2\sigma\nabla\,\mathrm{Re}\left(|a|^{2\sigma-2}\overline{a}a^{(1)}\right) = -v_1\cdot\nabla v,$$

where we have denoted $v = \nabla\phi$.
First case: $\sigma \geqq 2$ is even. Consider the new unknown

$$A_1 := |a|^{\sigma-2}\,\mathrm{Re}\left(\overline{a}a^{(1)}\right).$$

We check that, if $(\phi^{(1)}, a^{(1)})$ solves (5.1), then

$$\begin{cases} \partial_t v_1 + v \cdot \nabla v_1 + 2\sigma |a|^\sigma \nabla A_1 = -v_1 \cdot \nabla v - 2\sigma A_1 \nabla \left(|a|^\sigma\right), \\ \partial_t A_1 + v \cdot \nabla A_1 + \frac{1}{2}|a|^\sigma \operatorname{div} v_1 = -\frac{1}{\sigma}\nabla \left(|a|^\sigma\right) \cdot v_1 - \frac{\sigma}{2} A_1 \operatorname{div} v \\ \qquad\qquad + \frac{i}{2} \operatorname{Re}\left(|a|^{\sigma-2}\overline{a}\Delta a\right). \end{cases} \tag{5.2}$$

This linear system is hyperbolic symmetric, and its coefficients are smooth since $\sigma \in 2\mathbb{N}$ and $a, v \in C^\infty([0, T^*[; H^\infty(\mathbb{R}^n))$, from Lemma 2.2. In particular, uniqueness for (5.1) follows from the uniqueness for (5.2). Note that, since $\sigma - 2 \in 2\mathbb{N}$,

$$(v_1, A_1)\,|_{t=0} = \left(0, |a_0|^{\sigma-2} \operatorname{Re}\left(\overline{a}_0 a_1\right)\right) \in H^\infty \left(\mathbb{R}^n\right)^2.$$

Therefore, (5.2) possesses a unique solution in $C^\infty([0, T^*[; H^\infty(\mathbb{R}^n))$. We next define $\phi^{(1)} \in C^\infty([0, T^*[: H^\infty(\mathbb{R}^n))$ by

$$\phi^{(1)}(t, x) = -\int_0^t \left(v(\tau, x) \cdot v_1(\tau, x) + 2\sigma |a(\tau, x)|^\sigma A_1(\tau, x)\right) \mathrm{d}\tau.$$

Then $\partial_t \left(\nabla\phi^{(1)} - v_1\right) = 0$; therefore $v_1 = \nabla\phi^{(1)}$ and hence $\phi^{(1)}$ satisfies

$$\partial_t\phi^{(1)} + v \cdot \nabla\phi^{(1)} + 2\sigma |a|^\sigma A_1 = 0, \quad \phi^{(1)}\,|_{t=0} = 0.$$

Once this is granted, we can define $a^{(1)} \in C^\infty([0, T^*[: H^\infty(\mathbb{R}^n))$ as the unique solution of the linear equation

$$\begin{cases} \partial_t a^{(1)} + v \cdot \nabla a^{(1)} + \nabla\phi^{(1)} \cdot \nabla a + \frac{1}{2}a^{(1)} \operatorname{div} v + \frac{1}{2}a\Delta\phi^{(1)} = \frac{i}{2}\Delta a, \\ a^{(1)}\,|_{t=0} = a_1. \end{cases}$$

By construction, A_1 and $|a|^{\sigma-2} \operatorname{Re}\left(\overline{a}a^{(1)}\right)$ solve the same linear equation, where $\phi^{(1)}$ is viewed as a smooth coefficient. Therefore, these two functions coincide, and $(\phi^{(1)}, a^{(1)})$ solves (5.1).

Second case: σ is odd. In this case, $\sigma = 2m + 1$, for some $m \in \mathbb{N}$. We consider the new unknown

$$A_1 := |a|^{\sigma-1}a^{(1)} = |a|^{2m}a^{(1)}.$$

We check that (v_1, A_1) must solve

$$\begin{cases} \partial_t v_1 + v \cdot \nabla v_1 + 2\sigma \operatorname{Re}\left(|a|^{2m}\overline{a}\nabla A_1\right) = -\, v_1 \cdot \nabla v - 2\sigma \operatorname{Re}\left(A_1 \nabla\left(|a|^{2m}\overline{a}\right)\right), \\ \partial_t A_1 + v \cdot \nabla A_1 + \frac{1}{2}|a|^{2m} a \operatorname{div} v_1 = -\,\frac{\sigma}{2}A_1 \operatorname{div} v - |a|^{2m}\nabla a \cdot v_1 \\ \qquad\qquad + \frac{i}{2}|a|^{2m}\Delta a. \end{cases}$$

We can then conclude as in the first case, by considering $(v_1, A_1, \overline{A_1})$. □

Theorem 1.6 follows from:

Proposition 5.2. *Let $n \leqq 3$, and let Assumption 1.5 be satisfied. Set $\widetilde{a} := a e^{i\phi^{(1)}}$. Then for any $T \in]0, T^*[$, there exists $\varepsilon(T) > 0$ such that $a^\varepsilon \in C([0,T]; H^\infty)$ for $\varepsilon \in]0, \varepsilon(T)]$, and*

$$\left\| a^\varepsilon - \widetilde{a} \right\|_{L^\infty([0,T];H^k)} = \mathcal{O}(\varepsilon),$$

where k is as in Theorem 4.5.

Proof. Since the proof follows the same lines as the proof of Theorem 1.3, we shall indicate its main steps only. Denote

$$r^\varepsilon = a^\varepsilon - \widetilde{a}; \quad \widetilde{a}^{(1)} = a^{(1)} e^{i\phi^{(1)}}.$$

From (1.2), (1.7) and (5.1), we see that r^ε solves

$$\begin{cases} \partial_t r^\varepsilon + v \cdot \nabla r^\varepsilon + \frac{1}{2} r^\varepsilon \operatorname{div} v - i\frac{\varepsilon}{2}\Delta r^\varepsilon = i\frac{\varepsilon}{2}\Delta \widetilde{a} - iS^\varepsilon, \\ r^\varepsilon_{|t=0} = a_0^\varepsilon - a_0 = \varepsilon a_1 + \mathcal{O}\left(\varepsilon^2\right), \end{cases}$$

where the term S^ε is given by the following:

$$S^\varepsilon = \frac{1}{\varepsilon}\left(\left|a^\varepsilon\right|^{2\sigma} - |\widetilde{a}|^{2\sigma} \right) a^\varepsilon - 2\sigma \widetilde{a} |\widetilde{a}|^{2\sigma-2} \operatorname{Re}\left(\overline{\widetilde{a}} \widetilde{a}^{(1)} \right).$$

We check that for all $s \geqq 0$, we have, in $H^s(\mathbb{R}^n)$:

$$S^\varepsilon = \frac{1}{\varepsilon}\left(\left|a^\varepsilon\right|^{2\sigma} - \left| \widetilde{a} + \varepsilon \widetilde{a}^{(1)} \right|^{2\sigma} \right) a^\varepsilon + 2\sigma r^\varepsilon |\widetilde{a}|^{2\sigma-2} \operatorname{Re}\left(\overline{\widetilde{a}} \widetilde{a}^{(1)} \right) + \mathcal{O}(\varepsilon).$$

The last term should be viewed as a small source term. The second one is linear in r^ε, and is suitable in view of an application of the Gronwall Lemma. There remains to handle the first term. At this stage, we can mimic the approach detailed in Section 4. Introduce the nonlinear change of unknown:

$$\widetilde{q}^\varepsilon = \frac{1}{\varepsilon} B_\sigma \left(\left|a^\varepsilon\right|^2, \left| \widetilde{a} + \varepsilon \widetilde{a}^{(1)} \right|^2 \right); \quad \widetilde{g}^\varepsilon = G_\sigma \left(\left|a^\varepsilon\right|^2, \left| \widetilde{a} + \varepsilon \widetilde{a}^{(1)} \right|^2 \right),$$

where B_σ and G_σ are defined in Notation 4.1. We check that $(r^\varepsilon, \nabla r^\varepsilon, \widetilde{q}^\varepsilon)$ solves a system of the form (4.11), plus some extra source terms of order $\mathcal{O}(\varepsilon)$ in $H^s(\mathbb{R}^n)$. We also note that the initial data are of order $\mathcal{O}(\varepsilon)$, from Assumption 1.5:

$$\left\| (r^\varepsilon, \nabla r^\varepsilon) \,|_{t=0} \right\|_{H^s} = \mathcal{O}(\varepsilon), \quad \forall s \geqq 0.$$

We also have

$$\left\| \widetilde{q}^\varepsilon \,|_{t=0} \right\|_{H^{k-1}} = \mathcal{O}(\varepsilon),$$

where k is as Theorem 4.5.

Following the approach of Section 4, the proposition stems from the Gronwall lemma. Note also that the time T can be taken arbitrarily close to T^*, by the usual continuity argument, since we now have an error estimate that goes to zero with ε. □

To conclude this paragraph, we note that unless a_0 is real valued and $a_1 \in i\mathbb{R}$, one must not expect $\widetilde{a} = a$. Indeed, we see that

$$\phi^{(1)}_{|t=0} = 0; \quad \partial_t \phi^{(1)}_{|t=0} = -2\sigma \operatorname{Re}(\overline{a}_0 a_1) |a_0|^{2\sigma-2}.$$

So in general, $\phi^{(1)} \not\equiv 0$, and $\widetilde{a} \neq a$. On the other hand if a_0 is real-valued, then so is a. In this case,

$$\operatorname{Im}(\overline{a}\Delta a) \equiv 0,$$

and $(\phi^{(1)}, \operatorname{Re}(\overline{a}a^{(1)}))$ solves an homogeneous linear system. Therefore, if $\operatorname{Re}(\overline{a}a^{(1)}) = 0$ at time $t = 0$, then $\phi^{(1)} \equiv 0$.

6. Further remarks

The following remarks clarify some features of the systems we produced.

6.1. Regularity of the initial data

It is a matter of routine to extend the previous analysis to the case where the initial data belong to $H^s(\mathbb{R}^n)$ with $s < +\infty$ large enough.

6.2. Nonhomogeneous nonlinearity

It is important to note that one can consider nonlinearity which are not homogeneous. More precisely, consider the equation

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = f\left(|u^\varepsilon|^2\right) u^\varepsilon; \quad u^\varepsilon_{|t=0} = a_0^\varepsilon e^{i\phi_0/\varepsilon},$$

where f is a finite sum of smooth defocusing homogeneous nonlinearities:

$$f(r) = \sum_{m=1}^{M} c_m r^m \quad (c_m \geqq 0).$$

Let us set $\sigma := \min\{m : c_m > 0\}$. If $\sigma = 1$, then $f' > 0$, and we are in the situation already studied in [27]. We explain how to symmetrize the equations in the case where $\sigma \geqq 2$, provided we are given a smooth classical solution (a, ϕ) of the limit system

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla f(|a|^2) = 0; & v_{|t=0} = \nabla\phi_0, \\ \partial_t a + v \cdot \nabla a + \dfrac{1}{2} a \operatorname{div} v = 0; & a_{|t=0} = a_0. \end{cases}$$

Again, set $a^\varepsilon(t,x) := u^\varepsilon(t,x) e^{-i\phi(t,x)/\varepsilon}$ which satisfies

$$\partial_t a^\varepsilon + \nabla\phi \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta\phi - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{i}{\varepsilon}\left(f(|a^\varepsilon|^2) - f(|a|^2)\right) a^\varepsilon.$$

For $m = 1, \dots, M$, introduce

$$q_m^\varepsilon = \frac{1}{\varepsilon}(\rho^\varepsilon - \rho)\sqrt{Q_m(\rho^\varepsilon, \rho)}, \quad g_m^\varepsilon = \frac{(\rho^\varepsilon)^m - \rho^m}{(\rho^\varepsilon - \rho)\sqrt{Q_m(\rho^\varepsilon, \rho)}},$$

where Q_m is as defined in (4.7). Then, one has

$$\begin{cases} \partial_t a^\varepsilon + v \cdot \nabla a^\varepsilon - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{1}{2} a^\varepsilon \operatorname{div} v - i \sum_m c_m g_m^\varepsilon q_m^\varepsilon a^\varepsilon. \\ \partial_t \psi^\varepsilon + v \cdot \nabla \psi^\varepsilon + i a^\varepsilon \sum_m c_m g_m^\varepsilon \nabla q_m^\varepsilon - i\frac{\varepsilon}{2}\Delta\psi^\varepsilon \\ \quad = -\frac{1}{2}\psi^\varepsilon \operatorname{div} v - \psi^\varepsilon \cdot \nabla v - \frac{1}{2} a^\varepsilon \nabla \operatorname{div} v - i \sum_m c_m q_m^\varepsilon \nabla\left(a^\varepsilon g_m^\varepsilon\right), \\ \partial_t q_m^\varepsilon + v \cdot \nabla q_m^\varepsilon + \operatorname{Im}(g_m^\varepsilon \overline{a}^\varepsilon \operatorname{div} \psi^\varepsilon) = -\frac{m+1}{2} q_m^\varepsilon \operatorname{div} v, \quad m = 1, \dots, M. \end{cases}$$

We thus find that

$$U^\varepsilon := \left(2\sqrt{c_1} q_1^\varepsilon, \dots, 2\sqrt{c_M} q_M^\varepsilon, a^\varepsilon, \overline{a}^\varepsilon, \psi^\varepsilon, \overline{\psi}^\varepsilon\right)$$

satisfies a equation of the form (4.13) with g^ε replaced with $(g_1^\varepsilon, \dots, g_M^\varepsilon)$. We are now in position to establish the uniform estimates (4.14) (recall that we have set $\sigma := \inf\{m, c_m > 0\}$).

To conclude, let us note that one can solve the Cauchy problem for the limit system if $f(r) = F(r^\sigma)$ for some smooth function satisfying $F' > 0$, by following the same approach as in the proof of Lemma 2.2. This includes for instance the case

$$f\left(|u^\varepsilon|^2\right) = \alpha|u^\varepsilon|^4 + \beta|u^\varepsilon|^8, \quad \alpha, \beta > 0.$$

In the case

$$f\left(|u^\varepsilon|^2\right) = \alpha|u^\varepsilon|^4 + \beta|u^\varepsilon|^6, \quad \alpha, \beta > 0,$$

one may use the same trick as in [31]: the main difference with the proof of Lemma 2.2 is that a non constant symmetrizer must be introduced, which is bounded from above, and from below away from zero. See [19].

6.3. Introducing an external potential

To treat a possibly more physically relevant case, one might want to consider (1.1) with an extra external potential:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = V u^\varepsilon + \left|u^\varepsilon\right|^{2\sigma} u^\varepsilon; \quad u^\varepsilon_{|t=0} = a_0^\varepsilon e^{i\phi_0/\varepsilon},$$

where $V = V(t, x)$ is real-valued, and possibly time-dependent. As noticed in [11], it is sensible to consider an external potential V and an initial phase ϕ_0 which are smooth and sub-quadratic:

$$\partial_x^\alpha V \in C(\mathbb{R}; L^\infty(\mathbb{R}^n)), \quad \partial^\alpha \phi_0 \in L^\infty(\mathbb{R}^n), \quad \forall \alpha \in \mathbb{N}^n, \quad |\alpha| \geqq 2.$$

This includes the case of the harmonic oscillator, commonly used in the theory of Bose–Einstein condensation [28]. The main remark in [11] is that the introduction of this assumption does not deeply change the analysis. Indeed, we can resume the analysis of (1.1): introduce the solution to the standard eikonal equation

$$\partial_t \phi_{\text{eik}} + \frac{1}{2}|\nabla \phi_{\text{eik}}|^2 + V = 0; \quad \phi_{\text{eik}}\,|_{t=0} = \phi_0\,.$$

Decomposing the phase ϕ of the above quasi-linear analysis as

$$\phi = \phi_{\text{eik}} + \underline{\phi},$$

and seeking $\underline{\phi}$ in Sobolev spaces, we see that the extra terms appearing after this sort of linearization can be treated like semi-linear terms. Therefore, mimicking the above computations, and using only extra perturbative arguments, it is easy to adapt Theorems 1.3 and 1.6 to this case.

6.4. *About conservation laws*

Recall some important evolution laws for (1.1):

$$\text{Mass: } \frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon(t)\|_{L^2} = 0\,.$$

$$\text{Energy: } \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\varepsilon\nabla u^\varepsilon\|_{L^2}^2 + \frac{1}{\sigma+1}\|u^\varepsilon\|_{L^{2\sigma+2}}^{2\sigma+2}\right) = 0\,.$$

$$\text{Momentum: } \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Im}\int \overline{u}^\varepsilon(t,x)\varepsilon\nabla u^\varepsilon(t,x)\,\mathrm{d}x = 0\,.$$

$$\text{Pseudo-conformal law: } \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|J^\varepsilon(t)u^\varepsilon\|_{L^2}^2 + \frac{t^2}{\sigma+1}\|u^\varepsilon\|_{L^{2\sigma+2}}^{2\sigma+2}\right) = \frac{t}{\sigma+1}(2-n\sigma)\|u^\varepsilon\|_{L^{2\sigma+2}}^{2\sigma+2},$$

where $J^\varepsilon(t) = x + i\varepsilon t\nabla$. These evolutions are deduced from the usual ones ($\varepsilon = 1$, see, for example, [13,34]) via the scaling $\psi(t,x) = u(\varepsilon t, \varepsilon x)$.

Writing $u^\varepsilon = a^\varepsilon \mathrm{e}^{i\phi/\varepsilon}$ and passing to the limit formally in the above formulae yield:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|_{L^2} = 0\,.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2}|a(t,x)|^2|\nabla\phi(t,x)|^2 + \frac{1}{\sigma+1}|a(t,x)|^{2\sigma+2}\right)\mathrm{d}x = 0\,.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |a(t,x)|^2\nabla\phi(t,x)\,\mathrm{d}x = 0\,.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2}\,|(x - t\nabla\phi(t,x))\,a(t,x)|^2 + \frac{t^2}{\sigma+1}|a(t,x)|^{2\sigma+2}\right)\mathrm{d}x = \frac{t}{\sigma+1}(2-n\sigma)\int |a(t,x)|^{2\sigma+2}\,\mathrm{d}x.$$

Note that we also have the conservation [12]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Re}\int \overline{u}^{\varepsilon}(t,x)J^{\varepsilon}(t)u^{\varepsilon}(t,x)\,\mathrm{d}x = 0,$$

which yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (x - t\nabla\phi(t,x))\,|a(t,x)|^2\,\mathrm{d}x = 0\,.$$

All these expressions involve only $(|a|^2, \nabla\phi) = (|\widetilde{a}|^2, \nabla\phi)$. Recall that if we set $(\rho, v) = (|a|^2, \nabla\phi)$, then (1.2) implies

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla\left(\rho^{\sigma}\right) = 0; & v_{|t=0} = \nabla\phi_0,\\ \partial_t\rho + \mathrm{div}\,(\rho v) = 0; & \rho_{|t=0} = |a_0|^2. \end{cases} \tag{6.1}$$

Rewriting the above evolution laws, we get the following:

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}t}\int \rho(t,x)\,\mathrm{d}x = 0\,.\\
\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2}\rho(t,x)|v(t,x)|^2 + \frac{1}{\sigma+1}\rho(t,x)^{\sigma+1}\right)\mathrm{d}x = 0 :\ \text{energy.}\\
\frac{\mathrm{d}}{\mathrm{d}t}\int \rho(t,x)v(t,x)\,\mathrm{d}x = 0\,.\\
\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2}\left|(x - tv(t,x))\right|^2\rho(t,x) + \frac{t^2}{\sigma+1}\rho(t,x)^{\sigma+1}\right)\mathrm{d}x\\
= \frac{t}{\sigma+1}(2 - n\sigma)\int \rho(t,x)^{\sigma+1}\,\mathrm{d}x\,.\\
\frac{\mathrm{d}}{\mathrm{d}t}\int (x - tv(t,x))\,\rho(t,x)\,\mathrm{d}x = 0\,.
\end{gathered} \tag{6.2}$$

We thus retrieve formally some evolution laws for the compressible Euler equation (6.1) (see, for example, [33,38]), with the pressure law $p(\rho) = c\rho^{\sigma+1}$.

6.5. About global in time results

We point out that the solution to (1.2) must not be expected to be smooth for all time: the time T^* in Lemma 2.2 is finite in general. Recall that $(\rho, v) = (|a|^2, \nabla\phi)$ solves (6.1). Theorem 3 in [31] (see also [38]) implies that, if $\nabla\phi_0$ and $|a_0|^2$ are compactly supported, then the life span T^* in Lemma 2.2 is necessarily finite. Note that these initial data can be chosen arbitrarily small: the phenomenon remains.

Proposition 6.1. *Let* $n \geqq 1$ *and* $\sigma \geqq 1$. *For all initial data* $(a_0, \phi_0) \in C^2(\mathbb{R}^n)$ *with compact support, there does not exist* $(a, \phi) \in C^2([0, +\infty[\times\mathbb{R}^n)$ *satisfying the Cauchy problem* (1.2).

A word of caution: because of one technical assumption in the definition of regular solution in [31], Theorem 3 in [31] does not apply directly. Yet, one can prove our claim by combining the proof of Lemma 2.2 with the approach in [31]. Indeed, recall that $U := (a^\sigma, \nabla\phi)$ satisfies $\partial_t U + \sum A_j(U)\partial_j U = 0$ where the A_j's are $n \times n$ matrices linear in their argument. Therefore, the proof of Theorem 2 in [31] shows that U is compactly supported, and so is $(\rho, v) := (|a|^2, \nabla\phi)$, with support included in the support of $(|a_0|^2, \nabla\phi_0)$. And this is the only point which requires the above mentioned technical assumption.

Note also that the proof of this result in [38] relies on the evolution law for the total pressure

$$\int_{\mathbb{R}^n} p(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \rho(t,x)^{\sigma+1}\,\mathrm{d}x. \tag{6.3}$$

This approach is very similar to the Zakharov–Glassey method [26,39], which yields a sufficient condition for the finite time blow-up of solutions to the *focusing* nonlinear Schrödinger equation. As noticed by WEINSTEIN [37], the identity used by Zakharov, and generalized by Glassey, follows from the pseudo-conformal law, along with the conservation of energy. For $\sigma \geqq 2/n$ and a defocusing nonlinearity, this approach yields an upper bound for the L^2-norm of xu, the momentum of u. When this upper bound may become negative, finite time blow-up occurs.

In the present context, the nonlinearity is defocusing, but the idea is similar. Note that (the generalized version of) (6.2) is the key ingredient in the proof of XIN [38] (Xin considers Navier–Stokes equations). Expanding (6.2), and using the conservation of energy, we recover an upper bound for (6.3) which goes to zero as $t \to \infty$. But so long as v remains bounded, (6.1) is an ordinary differential equations for ρ, thus contradicting the upper bound for (6.3), unless v ceases to be smooth in finite time (see [38] for the details).

6.6. About focusing nonlinearities

The main feature of the limit system we used is that it enters, up to a change of unknowns, into the framework of quasi-linear hyperbolic systems. This comes from the fact that we consider the defocusing case. Had we worked instead with the focusing case, where $+|u|^{2\sigma}u$ is replaced with $-|u|^{2\sigma}u$, the corresponding limit system would have been ill-posed. We refer to [32], in which G. Métivier establishes Hadamard's instabilities for nonhyperbolic nonlinear equations.

As an example, consider the Cauchy problem

$$\begin{cases} \partial_t\phi + \dfrac{1}{2}|\partial_x\phi|^2 - |a|^{2\sigma} = 0; & \phi_{|t=0} = \phi_0, \\ \partial_t a + \partial_x\phi\partial_x a + \dfrac{1}{2}a\partial_x^2\phi = 0; & a_{|t=0} = a_0. \end{cases} \tag{6.4}$$

The following result follows from Hadamard's argument (see [32]).

Proposition 6.2. *Suppose that* (ϕ, a) *in* $C^2([0,T]\times\mathbb{R})$ *solves* (6.4). *If* $\phi_0(x)$ *is real analytic near* $\underline{x}$ *and if* $a_0(\underline{x}) > 0$, *then* $a_0(x)$ *is real analytic near* $\underline{x}$. *Consequently, there are smooth initial data for which the Cauchy problem has no solution.*

This shows that to study the semi-classical limit for the focusing analogue of (1.1), working with analytic data, as in [23,36], is not only convenient: it is necessary.

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