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# *Refined Jacobian Estimates and Gross–Pitaevsky Vortex Dynamics*

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## Abstract

We study the dynamics of vortices in solutions of the Gross–Pitaevsky equation $iu_t = \Delta u + \frac{1}{\varepsilon^2} u \left(1 - |u|^2\right)$ in a bounded, simply connected domain $\Omega \subset \mathbb{R}^2$ with natural boundary conditions on $\partial\Omega$. Previous rigorous results have shown that for sequences of solutions $u_\varepsilon$ with suitable well-prepared initial data, one can determine limiting vortex trajectories, and moreover that these trajectories satisfy the classical ODE for point vortices in an ideal incompressible fluid. We prove that the same motion law holds for a small, but fixed $\varepsilon$, and we give estimates of the rate of convergence and the time interval for which the result remains valid. The refined Jacobian estimates mentioned in the title relate the Jacobian $J(u)$ of an arbitrary function $u \in H^1(\Omega; \mathbb{C})$ to its Ginzburg–Landau energy. In the analysis of the Gross–Pitaevsky equation, they allow us to use the Jacobian to locate vortices with great precision, and they also provide a sort of dynamic stability of the set of multi-vortex configurations.


## 1. Introduction

This paper revisits the study of asymptotics of the Gross–Pitaevsky equation

$$iu_t = \Delta u + \frac{1}{\varepsilon^2} u \left(1 - |u|^2\right), \qquad x \in \Omega, \tag{1.1}$$

$$\nu \cdot \nabla u = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega \tag{1.3}$$

for suitable $u_0$, where $\Omega$ is a bounded, simply connected, open domain in $\mathbb{R}^2$ with $C^1$ boundary and $u : \Omega \times [0, T) \to \mathbb{C}$. The equation describes the evolution of the wave function associated with an idealized two-dimensional superfluid, and a solution $u$ encodes various physical attributes of the superfluid. For example $|u|^2$ is interpreted as the density, and $j(u) := \frac{i}{2}(\bar{u}\nabla u - u\nabla\bar{u})$ as the supercurrent. It is natural to interpret $J(u) := \frac{1}{2}\nabla \times j(u)$ as the vorticity. This same quantity is also the Jacobian determinant of $u$; see (2.11). Another relevant quantity is the energy

$$E_\varepsilon(u) = \int_\Omega e_\varepsilon(u)\,\mathrm{d}x, \quad e_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left(1 - |u|^2\right)^2. \tag{1.4}$$

A striking feature of superfluids is the presence of quantized vortices. As early as 1966, it was predicted by FETTER [14], based on a formal analysis of (1.1), that these vortices should evolve to leading order by the same system of ODEs that governs point vortices in an ideal incompressible fluid. The same prediction eventually entered the applied math literature with the work of NEU [27] and E [13], who studied (1.1) using matched asymptotics. A rigorous description of vortex dynamics in solutions of (1.1) in the $\varepsilon \to 0$ limit was established in [9,24] in the late 1990s, for a variety of boundary conditions. These results consider a sequence of solutions $u_\varepsilon$ of (1.1) with initial data $u_{0\varepsilon}$ for which the vorticity converges to a sum of point masses at distinct points $a_{0,j} \in \Omega$, $j = 1, \ldots, n$, and each having a single quantum of vorticity with sign $d_j = \pm 1$. More precisely, it is assumed that

$$\frac{1}{2}\nabla \times j(u_{0\varepsilon}) = J(u_{0\varepsilon}) \to \pi \sum_{j=1}^{n} d_j \delta_{a_{0,j}} \quad \text{as } \varepsilon \to 0 \tag{1.5}$$

in certain negative Sobolev spaces. The initial data are also assumed to be well-prepared, in that the energy $E_\varepsilon(u_{0\varepsilon})$ is asymptotically as small as possible as $\varepsilon \to 0$, given the boundary conditions and the constraint (1.5). The papers alluded to above show that under these assumptions,

$$\frac{1}{2}\nabla \times j(u_\varepsilon(t)) = J(u_\varepsilon(t)) \to \pi \sum_{j=1}^{n} d_j \delta_{a_j(t)} \tag{1.6}$$

for $t > 0$ where $(a_1(t), \ldots, a_n(t))$ solve the point vortex system

$$\dot{a}_j = -\frac{1}{\pi}\nabla_{a_j} \times W(a_1, \ldots, a_n), \quad a_j(0) = a_{0,j}, \quad j = 1, \ldots, n. \tag{1.7}$$

Here $W(a_1, \ldots, a_n) := -\pi \sum d_i d_j \ln|a_i - a_j| +$ *boundary terms* is the renormalized energy introduced by BETHUEL et al. [4], and *also* the conserved Hamiltonian for classical point vortex dynamics. The boundary terms in the definition of $W$ depend on the boundary data for (1.1); see (2.20) for the precise definition in the case of Neumann data (1.2). The conclusion (1.6) states that vortices in solutions of (1.1), understood here as concentration points of the vorticity $\nabla \times j(u_\varepsilon)$, evolve via the ODE (1.7) in the limit $\varepsilon \to 0$.

The same papers also characterize the limits as $\varepsilon \to 0$ of the supercurrents $j(u_\varepsilon(t))$ (in $L^p(\Omega)$, $p < 2$) and of the wave functions $u_\varepsilon(t)$ (in $W^{1,p}(\Omega)$, $p < 2$, modulo a multiplicative phase). If trajectories $a_j(\cdot)$ collide, then these results hold only up to the first collision. Based on results collected in [25], one strongly expects that finite-time collisions can occur, and also that there are no collisions for generic initial data $\{a_{0,j}\}$.

Similar results are established by the second author [34] for a system of equations in which an equation like (1.1) is coupled to an equation for a magnetic potential.

We emphasize that if one fixes $0 < \varepsilon \ll 1$ and initial data $u_0$, these previous results say *nothing* quantitative about the solution of (1.1)–(1.3); they only describe limiting behavior of a sequence of solutions for a suitable sequence of initial data. Moreover, the proofs in [9,24] rely at several points on soft compactness arguments, so that no control of any rate of convergence can be extracted from the proofs.

### *1.1. Goals and motivations*

The main goal of this paper is to study the dynamics of vortices in (1.1) for a small but fixed value of the parameter $\varepsilon$, and to establish quantitative versions of the earlier results. Our main new tools are estimates we establish that can be thought of as refined $\Gamma$-convergence results.

We have several motivations for this project. First, a large number of outstanding issues remain from the rigorous analysis found in [9,24], and many of these open problems require a good quantitative description of vortices even in order to be formulated precisely. For example, an important open problem concerns corrections to the leading-order dynamics and the related question of long-time behavior of vortices. An interesting formal discussion of these issues is given by Ovchinnikov and Sigal [28], who study the radiation generated a pair of rotating vortices and argue that it gives rise to small corrections to the limiting dynamical law (1.7) for $\varepsilon$ small and fixed. Related formal results in the physics literature date back at least as far as work of Klyatskin [21] on vortices in a slightly compressible fluid. A prerequisite for addressing this sort of question is the ability to say something about vortex locations for fixed $\varepsilon > 0$. The results of [9,24] are, thus, too weak even to be a suitable starting-point for this sort of problem, whereas with the results and tools we develop here, one can at least begin to study these issues.

Also, we believe that the overall strategy we employ is new and of broader interest. In particular, we use quantitative forms of $\Gamma$-convergence estimates to obtain the same sort of control more commonly found, in different contexts, from linearized stability estimates. We believe that this basic approach is potentially useful for problems completely unrelated to Ginzburg–Landau equations.

### *1.2. Related results*

The only prior rigorous work we know of that describes effective dynamics of vortices in a nonlinear field theory for finite $\varepsilon$, rather than in the limit $\varepsilon \to 0$, are a paper of Stuart [35] on dynamics of pairs of vortices in the Maxwell–Higgs system near the critical coupling, and later work of Gustafson and Sigal [16] on vortex dynamics in solutions of both the Maxwell–Higgs system and a nonlinear heat flow, for arbitrary (finite) numbers of vortices. In the equations studied in both these papers, a complex scalar wave function $u$ is coupled to a magnetic potential $A$. The analyses in [16,35] ultimately rest on a linear stability analysis of magnetic vortices, carried out most fully in [15]. It turns out that this stability analysis is made easier by the presence of the magnetic terms.

The same general approach, involving estimates obtained via control over the spectrum of some linearized operator, has been used successfully in a variety of settings, including dynamics of bubbles in the Cahn–Hilliard equation [2,3], solitons in nonlinear dispersive equations [7,8,26,33,36], interfaces in the Allen–Cahn equation [11] among many other examples. Roughly speaking, the linearized estimates are used to prove an estimate of the form

$$\begin{gathered}\|U - P(U)\|_X^2 \leqq C[H(U) - H(P(U))] + \text{small error terms} \\ U = (u, A) \in N,\end{gathered} \tag{1.8}$$

where $N$ is an open set in a Hilbert space $X$, and $P : N \to M$ is a nonlinear projection onto some explicitly constructed submanifold $M$ of $X$; and $H$ may be a Hamiltonian or Lyapunov functional associated with the dynamics one wants to study. As far as we know, however, this sort of argument has not been carried out for vortex dynamics in any equation that, like (1.1), does not involve a magnetic field. This is presumably related to the fact that the linear analysis of vortices associated with the energy $E_\varepsilon(\cdot)$ is known to be more delicate than that of the *magnetic* vortices of [16,35], although potentially useful estimates in the nonmagnetic case are given by DEL PINO et al. [12].

Most rigorous work on dynamics of vortices in (1.1) and related nonlinear field theories has employed measure-theoretic methods, whereby vortices are located as concentration points of a measure associated with a solution of the PDE in question. In the work on (1.1) cited above, this measure comes from the Jacobian $J(u)$, and results about Ginzburg–Landau heat equations (see for example [5,19,22,30] for example) typically rely instead on an "energy measure", as do the few rigorous results [17,23] about (nonmagnetic) wave equation analog of (1.1). All of these earlier measure-theoretic results describe only the limiting behavior of a sequence of solutions. Our main result is the first *quantitative* result about vortex dynamics for any shows for problem involving nonmagnetic vortices, and it demonstrates in particular that such results can be established in this measure-theoretic framework.

### *1.3. Main results*

**1.3.1. Dynamics** For a solution $u$ of (1.1) with suitable initial data, we obtain a quantitative description of vortex dynamics, with estimates of

- $\|J(u(t)) - \pi \sum_{j=1}^n d_j \delta_{a_j(t)}\|_{\dot{W}^{-1,1}(\Omega)}$. We show in fact that $J(u(t))$ is very close to a sum of delta functions at points $\xi_i(t)$. The $\dot{W}^{-1,1}$, norm, the definition of which is given in (2.5), controls $\sum |\xi_i(t) - a_i(t)|$, and so this estimate measures the distance between vortices in solutions of (1.1) and ideal vortex trajectories:
- The difference between $j(u(t))$ and the current generated by ideal point vortices at the points $a_i(t)$.
- The interval of time $[0, \tau_\star]$ for which the above estimates are valid.

Precise statements of these results are given in Theorem 1, which is stated in Section 3. These estimates depend on $\varepsilon$, the number $n$ of vortices and minimum inter-vortex distance for the trajectories $\{a_j(t)\}_{j=1}^n$, $t \in [0, \tau_\star]$. In particular, since the minimum vortex separation depends on $\tau_\star$, our formula for the latter is implicit.

Our results improve on earlier work in several ways:

- For $n = O(1)$ and initial vortex configurations $\{a_{0j}\}$ such that the inter-vortex distance is bounded away from zero, our results imply that $\tau_\star \geqq c \ln \frac{1}{\varepsilon}$. This holds in particular if $\{a_{0j}\}$ gives rise to a periodic solution of (1.7).
- For rather large numbers of vortices, say $n \approx |\ln \varepsilon|^\alpha$, $\alpha \leqq 1$, our results are valid for $\tau_\star \gtrsim 1/n$ (this is the natural time scale when there are many vortices, for example if one wants to consider the hydrodynamic limit) if one has initial vortex positions $\{a_{0j}\}_{j=1}^n$ such that trajectories remain separated by distances $|\ln \varepsilon|^{1/2}$ for times of order $1/n$.
- Our results are valid even for large numbers of vortices, say $n \lesssim \varepsilon^{-100}$, albeit for extremely short times $\tau_\star$.

Our estimates, therefore, provide quantitative information about behavior of vortices that remains valid on time scales longer than $O(1)$ or when the number of vortices is greater than $O(1)$, as $\varepsilon \to 0$.

**1.3.2. Refined Jacobian estimates** The refined Jacobian estimates mentioned in the title of this paper are quantitative results in the spirit of $\Gamma$-convergence. We use them to obtain the same sort of control that in other settings is more typically deduced from estimates such as (1.8) that ultimately refer to the spectrum of a linearized operator.

Given a point $a = (a_1, \ldots, a_n) \in \Omega^{n*} = \{a \in \Omega^n : a_i \neq a_j \text{ for } i \neq j\}$, and a vector $d \in \{\pm 1\}^n$, we construct in Lemma 14 in Section 10 a function $u_\star^\varepsilon(\cdot; a, d) \in H^1(\Omega; \mathbb{C})$ that has a vortex of degree $d_i$ near the point $a_i$ for $i = 1, \ldots, n$, and that is very close to energetically optimal among functions with this property. We show that if $u \in H^1(\Omega; C)$ is such that

$$\left\| J(u) - \pi \sum_{i=1}^n d_i \delta_{a_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \leqq C \rho_a n^{-5}, \quad E_\varepsilon(u) - E_\varepsilon(u_\star(a)) \leqq 1, \quad (1.9)$$

where $\rho_a = \frac{1}{4} \min\left(\{|a_i, -a_j|, i \neq j\} \cup \{\mathrm{dist}(a_i, \partial\Omega), i = 1, \ldots, n\}\right)$, then $J(u)$ is very well-localized in the sense that there exists some $\xi \in \Omega^{n*}$ such that

$$\left\| J(u) - \pi \sum_{i=1}^n d_i \delta_{\xi_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \leqq C\varepsilon[E_\varepsilon(u) + n^5 \rho_\xi^{-1}] \leqq C\varepsilon^{9/10}, \quad (1.10)$$

and in addition

$$\int_{\Omega \setminus \cup B_{\varepsilon^{1/3}}(\xi_i)} e_\varepsilon(u/u_\star^\varepsilon(\xi))\, \mathrm{d}x \leqq C[E_\varepsilon(u) - E_\varepsilon(u_\star^\varepsilon(\xi))] + C\varepsilon^{1/3}. \quad (1.11)$$

These estimates are valid for $n \leqq \varepsilon^{-\alpha}$ for some $\alpha > 0$. If we compare these results to the approach of [16], assumption (1.9) is roughly analogous to the condition $u \in N$ appearing there. The map $u \mapsto u_\star^\varepsilon(\xi)$, with $\xi$ as in (1.10), is analogous to the projection $P : N \to M$. And (1.11) is analogous in a general way to (1.8). Like that estimate, it is crucial in proving the dynamic stability of the class of multi-vortex configurations.

Conclusion (1.10) is proved in Theorem 3, which appears in Section 9. The theorem, which we refer to as a "localization" theorem, asserts that if $J(u)$ is close to a sum of point masses, and if the energy is not too large, then $J(u)$ is concentrated on length scales of order at most $\varepsilon[E_\varepsilon(u) + n^5 \rho_\xi^{-1}]$. This theorem is close to sharp when $n$ is $O(1)$. It can be viewed as a quantitative version of the sort of compactness condition that one normally requires in the $\Gamma$-convergence framework.

Stability estimates in the spirit[1] of (1.11) are stated and proved in Theorem 2, in Section 8. We view this theorem as a sort of quantitative analog of $\Gamma$-convergence results [1,10,18,24] which provide information about $\liminf E_\varepsilon(u_\varepsilon)$ when $u_\varepsilon$ is a sequence such that $J(u_\varepsilon) \to \sum \pi d_i \delta_{\xi_i}$; here by contrast we obtain information about $E_\varepsilon(u)$ for a fixed function $u$, assuming quantitative control (1.10) over $J(u) - \pi \sum d_i \delta_{\xi_i}$. Since (1.11) is also analogous to the estimate (1.8), we refer to it as a $\Gamma$-stability estimate.

Theorems 2 and 3 can also be seen as powerful refinements, in different directions, of Jacobian estimates as found for example in [1,18,31]. A typical such estimate has roughly the form

$$\|J(u)\|_{\dot{W}^{-1,1}(\Omega)} \leqq |\ln \varepsilon|^{-1}(E_\varepsilon(u) + o(1)). \tag{1.12}$$

Theorem 2 implies in particular *much* sharper lower bounds for $E_\varepsilon(u)$, once some additional information about the Jacobian $J(u)$ is assumed. And Theorem 3 supplements the basic bounds on $J(u)$ in (1.12) by very precise structural information about the Jacobian, showing that it is extremely close to a sum of point masses.

**1.3.3. Proof of main theorem, and organization of paper** We next sketch the proof of our main theorem. First, we define a time $\tau_1$ such that

$$\left\| J(u(t)) - \sum_{j=1}^{n} \pi d_i \delta_{a_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \leqq \varepsilon^{1/4}$$

and $\rho_{a(t)} \geqq \rho_\star$ for all $0 \leqq t \leqq \tau_1$. Here $\rho_\star$ is a parameter that is fixed at a late stage of the proof. We verify that the hypothesis (1.9) of the $\Gamma$-stability and localization results are satisfied, and that the right-hand side of (1.11) is smaller than $\varepsilon^{1/5}$ for all $t \in [0, \tau_1]$. Our main task is then to show that $\tau_1$ is as large as possible.

To do this we need to control the growth of

$$\left\| J(u(t)) - \sum_{j=1}^{n} \pi d_i \delta_{a_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)}.$$

[1] We do not exactly prove (1.11), but it can easily be deduced by combining (8.5) with results of Theorem 2.

The time derivative of this is difficult to work with directly, so we define a function $\eta(t)$ of the form

$$\eta(t) = \sum_{j=1}^{n} \left| \int J(u)\,\phi(x - a_j(t))\,\mathrm{d}x \right|$$

for a suitable $C^\infty$ function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, linear near the origin and with support in $B_{2\rho_\star}(0)$. Using the localization results we check that

$$\left\| J(u(t)) - \sum_{j=1}^{n} \pi d_i \delta_{a_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} = \eta(t) + O(\varepsilon^{9/10})$$

for $t \in [0, \tau_\star]$, and so the theorem reduces to to controlling the growth of $\eta(t)$.

We estimate $\dot{\eta}$ by directly differentiating and using conservation laws for the Gross–Pitaevsky equation, in particular, an equation (2.15) for the evolution of the Jacobian $J(u)$. (This is akin to the equation for vorticity transport in the 2D Euler equations.) We use this to decompose $\dot{\eta}$ into a number of terms in Section 4. Easy calculations, which however rely on the difficult $\Gamma$-stability and localization results, lead in Section 5 to the conclusion

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \eta(t) \right| \leqq \frac{Cn}{\rho_\star^2} (\eta + \varepsilon^{2/5}) + C \frac{n^2}{\rho_\star^{3/2}} (\eta + \varepsilon^{2/5})^{1/2} \leqq C\varepsilon^{1/50} \tag{1.13}$$

for $t \in [0, \tau_\star]$. This is not strong enough to yield a good estimate for $\tau_\star$ via Grönwall's inequality. There are two bad terms in the decomposition of $\dot{\eta}$ that give rise to the $(\eta + \varepsilon^{2/5})^{1/2}$ in (1.13), and in Section 6 we show that they can be controlled after averaging in time. More precisely, we prove that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \langle \eta \rangle_{\delta_\varepsilon} \right| \leqq C \frac{n}{\rho_\star^2} \langle \eta \rangle_{\delta_\varepsilon} + C\varepsilon^{1/3}, \quad \langle \eta \rangle_{\delta_\varepsilon}(t) := \frac{1}{\delta_\varepsilon} \int_{t-\delta_\varepsilon}^{t} \eta(s)\,\mathrm{d}s. \tag{1.14}$$

We prove this estimate for $\delta_\varepsilon = \varepsilon^{1/2}$. This leads to good estimates of $\langle \eta \rangle_{\delta_\varepsilon}$, and hence (using (1.13)) to good pointwise control of $\eta$.

The crucial point in the time-averaging step is that, via averaging, we are able to convert the equation (2.12) for conservation of mass into estimates of the divergence of $\left\langle j(u(t)) - j(u_\star^\varepsilon(\xi(t)) \right\rangle_{\delta_\varepsilon}$. Thus the time-averaged flow is very nearly incompressible—this is what is gained by the averaging procedure. The curl of $\left\langle j(u(t)) - j(u_\star^\varepsilon(\xi(t)) \right\rangle_{\delta_\varepsilon}$ is controlled using the localization estimate (1.10). These calculations lead to improved estimates of the terms that give rise to the bad scaling in (1.13).

The proof of the theorem is completed in Section 7.

Section 8 is devoted to the proof of the $\Gamma$-stability result, Theorem 2. In Section 9 we establish the localization result, Theorem 3. Some ideas in the proofs of these theorems are explained at the beginning of Section 8. Section 10 contains some results on the renormalized energy and the canonical harmonic map. These are used throughout the paper.

As we noted, the proof of Theorem 1 relies heavily on the compactness estimates of Sections 8 and 9 and estimates on the renormalized energy of Section 10. The results from Sections 8 to 10 are independent of dynamics arguments. Section 8 relies on results from Section 10 and [20], and Section 9 relies on results from Section 8 and Section 10 and [20]. Section 10 uses only notation introduced in Section 2.

### *1.4. Other remarks*

The results proved here are for Neumann boundary conditions, but the basic arguments should work both for Dirichlet boundary conditions for arbitrary vortex configurations and over $\mathbb{R}^2$ when $\sum_{j=1}^n d_j = 0$. In both cases the $\Gamma$-stability argument needs slight modification.

## 2. Notation and background

In this section we first fix notation and define some weak norms that are used throughout the paper. We then recall the system of conservation laws, often referred to as the Madelung transformation, that a solution to (1.1) satisfies. These show that conserved quantities for GP equations satisfy a set of nearly incompressible 2D Euler equations. At the end of the section we recall the definitions of the canonical harmonic map and renormalized energy $W(a; d)$ of BETHUEL et al. [4], and we introduce the notion of *surplus energy*. We need a number of specific lemmas concerning these functions; most of these facts are established in Section 10.

### *2.1. General notation*

We first define some notation.

Throughout this paper we implicitly sum over repeated indices, except where explicitly noted otherwise.

We always assume that $\Omega$ is a bounded, connected, simply connected domain with $C^1$ boundary. We believe that it would not be terribly difficult to extend our results to non-simply connected domains.

For $u, w \in \mathbb{R}^2$ let $v \times w = v_1 w_2 - v_2 w_1$. If $w : \mathbb{R}^2 \to \mathbb{R}^2$ we let $\nabla \times w = \partial_{x_1} w_2 - \partial_{x_2} w_1$ whereas if $\phi : \mathbb{R}^2 \to \mathbb{R}$, we let $\nabla \times \phi = (\partial_{x_2}\phi, -\partial_{x_1}\phi)$. Furthermore, for $v, w \in \mathbb{C}$ we use the *real* inner product $(v, w) = \frac{1}{2}(v\overline{w} + \overline{v}w)$. For $v, w \in \mathbb{C}^2$, we define the tensor product $v \otimes w$ to be the $2 \times 2$ matrix with $i, j$ entry $(v_i, w_j)$. We use the notation

$$U_s(x) = \{y \in \mathbb{R}^2 : |x - y| < s\}, \quad U_s = U_s(0),$$
$$B_s(x) = \{y \in \mathbb{R}^2 : |x - y| \leqq s\}, \quad B_s = B_s(0)$$

for open and closed balls, respectively. Let $\Omega$ be a bounded, simply connected, open subset of $\mathbb{R}^2$ with a $C^1$ boundary. For $a = (a_1, \ldots, a_n) \in \Omega^n$, we define

$$\rho_a = \frac{1}{4} \min \left\{ \min_{j \neq k} |a_j - a_k|, \operatorname{dist}(a_j, \partial\Omega) \right\} \tag{2.1}$$

and

$$\Omega_s(a) = \Omega \setminus \cup_{j=1}^n B_s(a_j). \tag{2.2}$$

We often work with functions $W : \Omega^n \subset \mathbb{R}^{2n} \to \mathbb{R}$. For such a function we write $\nabla_i W(x)$ to denote $(\frac{\partial}{\partial x_{2i-1}} W, \frac{\partial}{\partial x_{2i}} W)$, so that if we think of $W$ as a function of arguments $a_1, \dots, a_n \in \Omega$, then $\nabla_i W$ is the gradient of $W$ with respect to $a_i$.

We will write $\mathbb{J}$ to denote the $2 \times 2$ matrix

$$\mathbb{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

### 2.2. *Some weak norms*

For an open set $U \subset \mathbb{R}^n$ and a closed set $\Gamma$ (typically a subset of $\partial U$), we use the notation

$$W_\Gamma^{1,p}(U) := \{\phi \in W^{1,p}(U) \ : \ \phi = 0 \text{ on } \Gamma\}, \tag{2.3}$$

or more precisely the closure in $W^{1,p}(U)$ of the set of smooth functions that vanish on $\Gamma$. For $\Gamma = \emptyset$, we use the convention that $W_\emptyset^{1,p} = W^{1,p}(\Omega)$. We also define the dual norms

$$\begin{aligned} \|\mu\|_{\dot W_\Gamma^{-1,q}(U)} &:= \sup \left\{ \int \phi \, \mathrm{d}\mu \ : \ \|\nabla \phi\|_{L^p} \leqq 1, \ \phi \in W_\Gamma^{1,p}(U) \right\}, \\ 1 &= \frac{1}{p} + \frac{1}{q}. \end{aligned} \tag{2.4}$$

In this paper we will only consider $\|\mu\|_{\dot W_\Gamma^{-1,q}(U)}$ for $\frac{1}{q} > 1 - \frac{1}{n}$ and $\mu$ a (finite signed) measure; in this situation $\|\mu\|_{\dot W_\Gamma^{-1,q}(U)}$ is always finite, by the Sobolev embedding theorem and the Riesz representation theorem. Note that these norms scale nicely if $\mu$, $\Gamma$, and $U$ are all dilated. We use special notation for certain norms that are employed frequently throughout the paper:

$$\|\mu\|_{\dot W^{-1,q}(U)} := \|\mu\|_{\dot W_{\partial U}^{-1,q}(U)}, \quad \|\mu\|_{Lip^*(U)} := \|\mu\|_{\dot W_\emptyset^{-1,1}(U)}. \tag{2.5}$$

Note that $\|\mu\|_{Lip^*(U)} = +\infty$ unless $\int_U \mu = 0$. Clearly $\|\mu\|_{\dot W^{-1,1}(U)} \leqq \|\mu\|_{Lip^*(U)}$ for every measure $\mu$ on every open set $U$.

The $\dot W^{-1,1}(\Omega)$ and $Lip^*(\Omega)$ norms of measures of the form $\sum(\delta_{p_i} - \delta_{n_i})$ have interpretations as the "length of a minimal connection", see Brezis et al. [6], and from this it follows that if $a, \xi \in \Omega^{n*}$ and $|a_i - \xi_i| \leqq \rho_a$ for all $i$, then

$$\begin{aligned} \left\| \pi \sum d_i (\delta_{a_i} - \delta_{\xi_i}) \right\|_{\dot W^{-1,1}(\Omega)} &= \left\| \pi \sum d_i (\delta_{a_i} - \delta_{\xi_i}) \right\|_{Lip^*(\Omega)} \\ &= \pi \sum |d_i| \, |a_i - \xi_i|. \end{aligned} \tag{2.6}$$

### 2.3. Conserved quantities

For the reader's convenience, we collect here some definitions given in the Introduction, where the physical relevance of these quantities is discussed.

$$j(u) = (iu, \nabla u)\,, \tag{2.7}$$

$$J(u) = \frac{1}{2}\nabla \times j(u), \tag{2.8}$$

$$e_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2. \tag{2.9}$$

In view of the importance for our analysis of $j(u)$, $J(u)$, we note that they can be written in several different ways. If we write $u$ locally in the form $u = \rho e^{i\phi}$, with $\rho$, $\phi$ real-valued, then

$$j(u) = \rho^2 \nabla\phi, \quad J(u) = \rho\nabla\rho \times \nabla\phi. \tag{2.10}$$

And if we identify $u$ with the $\mathbb{R}^2$-valued function $(Re\,u, Im\,u) = (u_1, u_2)$, then

$$J(u) = \det \nabla u = \det \begin{pmatrix} u_{1,x_1} & u_{1,x_2} \\ u_{2,x_1} & u_{2,x_2} \end{pmatrix}. \tag{2.11}$$

A solution $u$ of (1.1)–(1.2) satisfies the following set of conservation laws:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 = \operatorname{div} j(u), \tag{2.12}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} j(u) = \operatorname{div}(\nabla u \otimes \nabla u) + \nabla P, \tag{2.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\varepsilon(u) = \operatorname{div}(u_t, \nabla u)\,, \tag{2.14}$$

where in (2.14) we use the notation

$$P = -\frac{1}{2}|\nabla u|^2 - \frac{1}{2}(u, \Delta u) + \frac{|u|^4 - 1}{4\varepsilon^2}.$$

In (2.12)–(2.14), our boundary conditions (1.2) are such that the corresponding integrals are conserved, so that for example $t \mapsto E_\varepsilon(u(t))$ is constant. By taking the curl of (2.13), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} J(u) = \operatorname{curl}\operatorname{div}(\nabla u \otimes \nabla u) = \mathbb{J}_{kl}\partial_{x_k x_m}\left(u_{x_m}, u_{x_l}\right). \tag{2.15}$$

*2.4. Canonical harmonic map, renormalized energy, and surplus energy*

A number of lemmas concerning the renormalized energy are stated and proved in the final section. Here we just give the definitions of these quantities, mostly following Bethuel et al. [4].

Given $a \in \Omega^{n*}$ and $d \in \mathbb{Z}^n$, the canonical harmonic map $u_\star \in W^{1,1}(\Omega; S^1)$ with singularities at points $a = (a_1, \dots, a_n)$ of degree $d = (d_1, \dots d_n)$ and natural boundary condition (corresponding to (1.2)) satisfies

$$\nabla \cdot j(u_\star) = 0, \quad \nabla \times j(u_\star) = 2\pi \sum d_i \delta_{a_i}, \tag{2.16}$$

and $\nu \cdot j(u_\star) = 0$ on $\partial\Omega$. The first equation in (2.16) states that $u_\star$ is a harmonic map into $S^1$, and the second equation specifies the positions and degrees of the singularities. These conditions uniquely determine $j(u_\star)$. In addition, $j(u_\star)$ determines $u_\star$ up to a constant phase; see [4] Chapter 1.

We will sometimes write $u_\star(\cdot; a, d)$, but more often we do not explicitly indicate the dependence of $u_\star$ on $a, d$, and we *never* indicate in our notation the dependence of $u_\star$ on the domain $\Omega$.

It is easy to check that $j(u_\star) = -\nabla \times G$, where $G$ satisfies

$$\Delta G = 2\pi \sum_{i=1}^n d_i \delta_{a_i} \quad \text{in } \Omega, \quad G = 0 \text{ on } \partial\Omega. \tag{2.17}$$

Note also that if we define $H(\cdot; y)$ for $y \in \Omega$ as the solution of

$$\Delta_x H(\cdot, y) = 0 \text{ in } \Omega, \quad H(x, y) = -\ln|x-y| \quad \text{for } x \in \partial\Omega, \quad y \in \Omega \tag{2.18}$$

then

$$G(x; a) = \sum_{i=1}^n d_i[\ln|x - a_i| + H(x, a_i)]. \tag{2.19}$$

Following Bethuel et al. [4], we define the renormalized energy $W_\Omega(a, d)$ by

$$W_\Omega(a, d) = \lim_{r\to 0} \left( \int_{\Omega_r(a)} |\nabla u_\star|^2 \, dx - n\pi \ln \frac{1}{r} \right), \tag{2.20}$$

and we recall from [4] that

$$W_\Omega(a, d) = -\pi \left( \sum_{i\neq j} d_i d_j \log|a_i - a_j| + \sum_{i,j} d_i d_j H(a_i, a_j) \right). \tag{2.21}$$

We give a proof of the equivalence of (2.21) and (2.20), with estimates of the rate of convergence of the right-hand side of (2.20), in Lemma 12, in Section 10.

Next, we recall from [4] the notation

$$I(r, \varepsilon) := \inf \left\{ \int_{U_r} e_\varepsilon(u) \; ; \; u \in H^1(B_r; \mathbb{C}), \; u = e^{i\theta} \text{ on } \partial B_r \right\}, \tag{2.22}$$

and we define

$$\gamma = \lim_{r\to\infty} \Big( I(r,\varepsilon) - \pi \ln \frac{r}{\varepsilon}\Big). \tag{2.23}$$

It is known that $\gamma$ exists, is finite and is independent of $\varepsilon$. Moreover, in Lemma 9 we prove that $\gamma - (I(r,\varepsilon) - \pi \ln \frac{r}{\varepsilon}) = O((\varepsilon/r)^2)$. For $a \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$ we will write

$$W^\varepsilon_\Omega(a,d) := n\left(\gamma + \pi \ln \frac{1}{\varepsilon}\right) + W_\Omega(a,d). \tag{2.24}$$

Like the renormalized energy, $W^\varepsilon_\Omega$ depends on the domain and the prescribed boundary conditions (here Neumann) in a way that is not explicitly indicated in the notation. Informally, $W^\varepsilon_\Omega(a,d)$ provides an approximate lower bound for the energy $E_\varepsilon$ of a function with vortices of degree $d_i$ near $a_i$, $i = 1,\dots,n$. This is made precise in Theorem 2, see Section 8. This lower bound is very close to sharp; this follows from Lemma 14 (see Section 10), in which we construct, for given $a \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$, a function $u^\varepsilon_\star$ with a vortex of degree $d_i$ at the point $a_i$, $i = 1,\dots,n$, and with energy extremely close to $W^\varepsilon_\Omega(a,d)$. As remarked in the Introduction, our results can be seen as, among other things, establishing the dynamic stability of the manifold $\{u^\varepsilon_\star(a,d) : a \in \Omega^{*n}\} \subset H^1(\Omega)$.

We will also use the notation

$$\begin{aligned}\Sigma^\varepsilon_\Omega(u;a,d) &:= \int_\Omega e_\varepsilon(u)\,\mathrm{d}x - W^\varepsilon_\Omega(a,d)\\ &\approx \int_\Omega [e_\varepsilon(u) - e_\varepsilon(u^\varepsilon_\star(a,d))]\,\mathrm{d}x.\end{aligned} \tag{2.25}$$

We refer to this quantity as the *surplus energy*; the terminology is justified again by Theorem 2. This quantity is only meaningful when $\|J(u) - \pi \sum d_i \delta_{a_i}\|_{\dot W^{-1,1}(\Omega)}$ is small.

We remark that one can check that for a single vortex of degree $\pm 1$ at the center of a ball of radius $r$, the associated renormalized energy is $W_{U_r(a)}(a,\pm 1) = \pi \ln r$, and so the associated surplus energy is

$$\Sigma^\varepsilon_{U_r(a)}(u,a,\pm 1) = \int_{U_r(a)} e_\varepsilon(u)\,\mathrm{d}x - \left(\pi \ln \frac{r}{\varepsilon} + \gamma\right). \tag{2.26}$$

This quantity appears in the statement of a number of results.

## 3. Vortex dynamics: main result

We will study solutions of (1.1)–(1.2) for initial data $u_0$ with vortices of degree[2] $d_i = \pm 1$ near points $a^0_i$, $i = 1,\dots,n$. We always write $a(t) = (a_1(t),\dots a_n(t))$

[2] Throughout the next few sections $d = (d_1,\dots,d_n) \in \{\pm 1\}^n$ is fixed.

to denote the solution of the ODE

$$\begin{aligned} \dot{a}_j(t) &= \frac{1}{\pi}\mathbb{J}\nabla_j W_\Omega(a(t), d) \\ a_j(0) &= a_j^0, \end{aligned} \tag{3.1}$$

Our main theorem about solutions of (1.1)–(1.3) is the following

**Theorem 1.** *Let $u$ solve the Schrödinger equation with initial $u_0$ satisfying*

$$\left\| J(u_0) - \sum_{j=1}^{n} \pi d_j \delta_{a_j^0} \right\|_{\dot{W}^{-1,1}(\Omega)} \leqq C\varepsilon^{9/10} \tag{3.2}$$

*for some $n \leqq \varepsilon^{-1/100}$, with $d \in \{\pm 1\}^n$ and $a^0 = (a_1^0, \ldots, a_n^0) \in \Omega^{n*}$ such that*

$$\rho_{a_0} \geqq C\varepsilon^{1/25} \tag{3.3}$$

*and assume also that*

$$\Sigma_\Omega^\varepsilon(u; a^0, d) = E_\varepsilon(u_0) - W_\Omega^\varepsilon(a^0, d) \leqq C\varepsilon^{1/2} \tag{3.4}$$

*for some $C > 0$.*

*Then there exist $\varepsilon_0 > 0$ and $C > 0$, depending only on $\Omega$ and the constants in (3.2), (3.4) above, with the following properties:*

*Given any $\varepsilon < \varepsilon_0$, let $\tau_\star$ be implicitly defined by*

$$\tau_\star = \frac{C}{n}\rho_{\min}(\tau_\star)^2 \ln\frac{1}{\varepsilon}, \quad \rho_{\min}(t) = \inf\{\rho_{\alpha(s)} \ : \ 0 \leqq s \leqq t\}. \tag{3.5}$$

*Then*

$$\left\| J(u(t)) - \sum_{j=1}^{n} \pi d_j \delta_{a_j(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \leqq C\varepsilon^{1/4}. \tag{3.6}$$

*Moreover,*

$$\int_{\Omega_{\rho_{\min}(\tau_\star)}(a(t))} e_\varepsilon(|u(t)|) + \frac{1}{4}\left| \frac{j(u(t))}{|u(t)|} - j(u_\star(a(t), d)) \right|^2 \leqq C\varepsilon^{1/5}, \tag{3.7}$$

*and*

$$\| j(u(t)) - j(u_\star(a(t), d)) \|_{L^{4/3}(\Omega)} \leqq C\varepsilon^{1/9} \tag{3.8}$$

*for all $0 \leqq t \leqq \tau_\star$.*

The theorem asserts that the vortices in the solution of (1.1) remain close to the point vortex trajectories from the ODE (3.1), and moreover that the associated supercurrents $j(u(t))$, $j(u_\star(a(t)))$ are close[3] for relatively long time intervals, unless two vortices nearly collide or one vortex comes very close to $\partial\Omega$.

For example, for $a^0 \in \Omega^{n*}$ such that the solution of (3.1) is periodic, $\rho_\star(t)$ is bounded away from 0, and so (3.6), (3.8) remain valid at least for times of order $\ln \frac{1}{\varepsilon}$.

All the powers of $\varepsilon$ appearing in the hypotheses are a bit arbitrary, and they could be jiggled a little, with corresponding small changes in the conclusions. We have no reason to believe that the conclusions are sharp.

In view of Lemma 13 and the conservation of energy, the assumptions on $n$, $\rho_{a_0}$ imply that

$$E_\varepsilon(u(t)) \ll \varepsilon^{-1/5} \quad \text{for all} \quad t \in \mathbb{R}. \tag{3.9}$$

In the remainder of this section we carry out the first part of the proof of the above theorem, in which we reduce the theorem to controlling the rate of growth of a scalar quantity that we call $\eta(t)$, defined in (3.15). In the subsequent three sections, we compute and bound $\frac{\mathrm{d}}{\mathrm{d}t}\eta$ and $\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta\rangle_\delta$ for a suitable $\delta$, where

$$\langle\eta\rangle_\delta(t) = \frac{1}{\delta}\int_{t-\delta}^{t} \eta(s)\,\mathrm{d}s.$$

The proof of the theorem is finally completed in Section 7 by applying Grönwall's inequality to $\langle\eta\rangle_\delta$ and using a preliminary, weaker estimate of $\frac{\mathrm{d}}{\mathrm{d}t}\eta$ to deduce pointwise bounds on $\eta$.

The reduction in this section to the problem of controlling $\eta(t)$, and the subsequent estimates in the proof of Theorem 1 rely crucially on Theorems 2 and 3, which are proved in Sections 8 and 9, respectively.

Throughout the proof of the theorem, one can check that, whenever we require $\varepsilon$ to be small, we actually require $\varepsilon^p \leqq C$, where $p$ is some fixed positive number and $C$ depends only on the domain $\Omega$ and on the constants in assumptions (3.2), (3.4). This occurs several times, so that in the end we require $\varepsilon^{p_i} \leqq C_i$ for some positive constants $p_1, \ldots, p_K$ and $C_1, \ldots, C_K$, which are not identified explicitly. The number $\varepsilon_0$ in the statement of the theorem can be taken to be $\varepsilon_0 = \min\{(C_i)^{1/p_i}\}$,

**Step 1: finding good points $\xi_j$.** We first define

$$\tau_0 = \inf\left\{t > 0 : \rho_{a(t)} \leqq \rho_\star\right\}, \tag{3.10}$$

where $\rho_\star \geqq \varepsilon^{1/20}$ is a parameter that will be fixed at the end the proof; and

$$\tau_1 = \sup\left\{0 \leqq t \leqq \tau_0 : \|J(u) - \sum_{i=1}^{n} \pi d_i \delta_{a_i(s)}\|_{\dot W^{-1,1}(\Omega)} \leqq \varepsilon^{1/4}\ \forall s \in [0,t]\right\}. \tag{3.11}$$

[3] in $L^{4/3}$. It would not be difficult to obtain estimates of $\|j(u) - j(u_\star)\|_{L^p(\Omega)}$ for $4/3 < p < 2$, by interpolating between (3.8) and easy bounds on $j(u)$, $j(u_\star)$ in $L^2$ and $L^q$, $p < q < 2$, respectively.

We will see that every choice of $\rho_\star$ leads to a lower bound for $\tau_1$. We will eventually choose $\rho_\star$ to optimize this lower bound.

By conservation of the Hamiltonian for the ODE (3.1) and the PDE (1.1) with Neumann conditions (1.2), we deduce from (3.4) that

$$\Sigma^\varepsilon_\Omega(u(t); a(t), d) = \Sigma^\varepsilon_\Omega(u(0); a(0), d) \leqq C\varepsilon^{1/2} \quad \text{for all } t.$$

The definition of $\tau_1$ implies that

$$\rho_{a(t)} \geqq \rho_\star \geqq \varepsilon^{1/20}, \quad \left\| J(u) - \sum_{i=1}^n \pi d_i \delta_{a_i(s)} \right\|_{\dot{W}^{-1,1}(\Omega)} \leqq \varepsilon^{1/4}, \qquad t \in [0, \tau_1] \tag{3.12}$$

and hence that the hypotheses of Theorem 3 (see Section 9) are satisfied by $u(t), a(t), d$ for all $t \in [0, \tau_1]$, when $\varepsilon$ is sufficiently small. Therefore when this holds, there exist $\xi(t) = (\xi_1(t), \ldots, \xi_n(t)) \in \Omega^{n*}$ such that $|\xi_i - a_i| \leqq \frac{\rho_{a(t)}}{4}$ for all $i$, and

$$\left\| J(u)(s) - \sum_{i=1}^n \pi d_i \delta_{\xi_i(s)} \right\|_{\dot{W}^{-1,1}(\Omega)} \leqq s_\varepsilon, \tag{3.13}$$

where here and throughout this proof,

$$s_\varepsilon := C\varepsilon \left[ \frac{n^5}{\rho_\star} + E_\varepsilon(u) \right] \leqq C\varepsilon^{9/10}. \tag{3.14}$$

**Step 2: definition, basic properties of $\eta(t)$.** In some sense the main point of the theorem is to estimate $|\xi(t) - a(t)|$. It is difficult to work directly with this quantity, however, and so we define

$$\eta(t) := \sum_{j=1}^n \left| \int J(u)\, \Phi_j(x,t)\, \mathrm{d}x \right| := \sum_{j=1}^n |\eta_j(t)|, \tag{3.15}$$

where

$$\Phi_j(x,t) = \varphi(x - a_j(t)), \quad \varphi(x) = x\chi_{\rho_\star}(x)$$

and $\chi_{\rho_\star}(x) = \chi(\frac{x}{\rho_\star})$ for a fixed $\chi \in C_0^\infty(\mathbb{R}^2)$ satisfying $\chi(x) = \begin{cases} 1 \text{ for } |x| \leqq 1 \\ 0 \text{ for } |x| \geqq 2 \end{cases}$. Note that $\Phi_j(x,t)$ is supported on $B_{2\rho_\star}(a_j(t))$, so that $\{\operatorname{supp} \Phi_j(x,t)\}$ are pairwise disjoint when $\rho_{a(t)} \geqq \rho_\star$ and in particular for all $0 \leqq t \leqq \tau_1$. As we shall see, it is easy to compute $d\eta/\mathrm{d}t$, using the equation (2.15) for $\frac{\mathrm{d}}{\mathrm{d}t} J(u)$.

We now argue that

$$\eta(t) = \pi \sum_i |\xi_i(t) - a_i(t)| + O(s_\varepsilon) = \left\| J(u(t)) - \sum_{i=1}^n \pi d_i \delta_{a_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} + O(s_\varepsilon)$$

as long as all these quantities remain small, and in particular for $0 \leqq t \leqq \tau_1$.

First note that in view of (2.6), the definition of $\tau_1$, and (3.13),

$$\pi \sum_j |\xi_j(t) - a_j(t)| = \left\| \sum_{i=1}^n \pi d_i (\delta_{\xi_i(t)} - \delta_{a_i(t)}) \right\|_{\dot{W}^{-1,1}(\Omega)}$$
$$\leqq \varepsilon^{1/4} + s_\varepsilon \leqq \rho_\star \tag{3.16}$$

when $\varepsilon$ is sufficiently small, for all $t \in [0, \tau_1]$. As a result, the definition of $\Phi_j$ implies that $\xi_j(t) - a_j(t) = \Phi_j(\xi_j(t), t)$ for all such $t$. It follows that there exists a unit vector $v_j(t)$ such that $|\xi_j(t) - a_j(t)| = d_j v_j \cdot \Phi_j(\xi_j(t), t)$, and hence (using the support properties of $\Phi_j$) that

$$\begin{aligned}
&\pi \sum_j |\xi_j(t) - a_j(t)| \\
&= \int \left(\pi \sum d_i \delta_{\xi_i(t)}\right) \left(\sum v_j \cdot \Phi_j(t)\right) \mathrm{d}x \\
&\leqq \int \left(\pi \sum d_i \delta_{\xi_i(t)} - J(u(t))\right) \left(\sum v_j \cdot \Phi_j(t)\right) \mathrm{d}x + \eta(t) \\
&\leqq \left\| J(u(t)) - \pi \sum d_i \delta_{\xi_i(t)} \right\|_{\dot{W}^{-1,1}} \left\| \sum_j v_j \cdot \Phi_j(t) \right\|_{W^{1,\infty}} + \eta(t) \\
&\leqq C s_\varepsilon + \eta(t) \quad \text{for all } t \in [0, \tau_1].
\end{aligned} \tag{3.17}$$

A similar argument shows that for such $t$,

$$\eta(t) \leqq \pi \sum |\xi_i(t) - a_i(t)| + C s_\varepsilon. \tag{3.18}$$

We also note that, in view of the triangle inequality and the interpretation (2.6) of the $\dot{W}^{-1,1}$ norm as the length of a minimal connection,

$$\begin{aligned}
&\left\| J(u(t)) - \sum_{i=1}^n \pi d_i \delta_{a_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \\
&\quad \leqq \left\| J(u(t)) - \sum_{i=1}^n \pi d_i \delta_{\xi_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \\
&\qquad + \left\| \sum_{i=1}^n \pi d_i (\delta_{\xi_i(t)} - \delta_{a_i(t)}) \right\|_{\dot{W}^{-1,1}(\Omega)} \\
&\quad \leqq s_\varepsilon + \pi \sum |\xi_i(t) - a_i(t)| \\
&\quad \leqq C s_\varepsilon + \eta(t) \quad \text{for all } t \in [0, \tau_1].
\end{aligned} \tag{3.19}$$

And one can similarly check that

$$\eta(t) \leqq C s_\varepsilon + \left\| J(u(t)) - \sum_{i=1}^n \pi d_i \delta_{a_i(t)} \right\|_{\dot{W}^{-1,1}(\Omega)} \quad \text{for all } t \in [0, \tau_1]. \tag{3.20}$$

In particular this implies that when $\varepsilon$ is sufficiently small,

$$\eta(t) \leqq 2\varepsilon^{1/4} \quad \text{for all } \ t \in [0, \tau_1]. \tag{3.21}$$

**Step 3: approximation by canonical harmonic maps.** We next will use Theorem 2 to show that $u(t)$ is well approximated in certain ways by the canonical harmonic map $u_\star(t) := u_\star(\cdot; \xi(t), d)$ for $t \leqq \tau_1$. To do this, we need to estimate the surplus energy $\Sigma^\varepsilon_\Omega(u(t); \xi(t), d)$ with respect to the points $\xi(t)$ found in Step 1 above. In fact, we will show that this too is controlled by $\eta(t)$ when $\varepsilon$ is small.

Fix $t \in [0, \tau_1]$, and observe from the definition (2.25) of $\Sigma^\varepsilon_\Omega$ that

$$\begin{aligned} &\Sigma^\varepsilon_\Omega(u(t); \xi(t), d) \\ &\quad = \Sigma^\varepsilon_\Omega(u(t); a(t), d) + W_\Omega(a(t), d) - W_\Omega(\xi(t), d) \\ &\quad \leqq C\varepsilon^{1/2} + \left(\sum_{j=1}^n |\xi_j(t) - a_j(t)|\right)\left(\sup_j \sup_{|y-a(t)| \leqq |\xi(t)-a(t)|} |\nabla_{y_j} W(y)|\right). \end{aligned} \tag{3.22}$$

From (3.16) it follows that if $y \in \Omega^n$ is such that $|y - a(t)| \leqq |\xi(t) - a(t)|$, then $\rho_y \geqq \frac{1}{2}\rho_{a(t)}$ for all sufficiently small $\varepsilon$, and so we can use (10.3) to find that $|\nabla_{y_j} W(y)| \leqq \frac{Cn}{\rho_\star}$. Hence (3.22) and (3.17) yield

$$\Sigma^\varepsilon_\Omega(u(t); \xi(t), d) \leqq C\varepsilon^{1/2} + (Cs_\varepsilon + \eta(t))\frac{Cn}{\rho_\star}. \tag{3.23}$$

Thus the first conclusion (8.3) of Theorem 2 implies that

$$\begin{aligned} &\int_{\Omega_{\rho_\star}(\xi(t))} e_\varepsilon(|u(t)|) + \frac{1}{4}\left|\frac{j(u(t))}{|u(t)|} - j(u_\star(t))\right|^2 \mathrm{d}x \\ &\quad \leqq \frac{Cn}{\rho_\star}(\eta(t) + Cs_\varepsilon) + C\varepsilon^{1/2} + C\left(\frac{n^5}{\rho_\star}(s_\varepsilon + \varepsilon E_\varepsilon(u))\right)^{1/2} \\ &\quad \leqq \frac{Cn}{\rho_\star}(\eta(t) + \varpi_\varepsilon) \end{aligned} \tag{3.24}$$

for all $t \in [0, \tau_1]$, where (since $s_\varepsilon \geqq C\varepsilon E_\varepsilon(u)$, see the definition (3.14))

$$\varpi_\varepsilon := C\left(\rho_\star n^3 s_\varepsilon\right)^{1/2} + C\varepsilon^{1/2} n^{-1}\rho_\star \ \leqq \ C\varepsilon^{2/5}.$$

(Note that a condition $\sigma^* \leqq \rho_a$ appearing as a hypothesis for conclusion (8.3) of Theorem 3 is satisfied as a result of the definitions of $\tau_1, s_\varepsilon$, etc.)

From the other conclusion (8.4) of Theorem 2, we deduce that

$$\|j(u)(t) - j(u_\star)(t)\|_{L^{4/3}(\Omega)} \leqq C\left(\frac{n\eta}{\rho_\star}\right)^{1/2} + \lambda_\varepsilon \tag{3.25}$$

where

$$\lambda_\varepsilon = C\varepsilon^{1/2}E_\varepsilon(u)^{3/4} + Cns_\varepsilon^{1/4}\left[\left(\frac{n}{\rho_{a(t)}}\right)^{1/4} + \rho_{a(t)}^{1/4}\left(1+\sqrt{\frac{E_\varepsilon(u)}{n^3}}\right)\right]$$
$$\leqq C\varepsilon^{1/5}. \tag{3.26}$$

In the sections that follow, we use (3.24), (3.25) and conservation laws for the Gross–Pitaevsky equation to control the growth in time of $\eta$. The conclusion of the proof appears in Section 7, where we use these estimates to show that $\tau_1$ cannot be too small.

## 4. Decomposition of $\dot\eta$

We use the notation $\eta_j(t) := \int J(u(t))\,\Phi_j(x,t)\,\mathrm{d}x \in \mathbb{R}^2$ as introduced in (3.15).

**Lemma 1.** *Let $u$ be a solution to the Schrödinger equation. Then for $0 \leqq t \leqq \tau_1$ and $j = 1, \ldots, n$*

$$\dot\eta_j = T_{j,1} + T_{j,2} + T_{j,3} + T_{j,4} + T_{j,5} + T_{j,6}, \tag{4.1}$$

*where*

$$T_{j,1} = d_j \nabla\varphi(\xi_j - a_j)\cdot \mathbb{J}\left(\nabla_j W_\Omega(\xi) - \nabla_j W_\Omega(a)\right),$$
$$T_{j,2} = -\int\left(J(u) - \sum_{i=1}^n \pi d_i \delta_{\xi_i}\right)\mathbb{J}\nabla_j W_\Omega(a)\cdot\nabla\Phi_j\,\mathrm{d}x,$$
$$T_{j,3} = \int \mathbb{J}_{kl}\partial_{x_l x_m}\Phi_j\partial_{x_k}|u|\,\partial_{x_m}|u|\,\mathrm{d}x,$$
$$T_{j,4} = \int \mathbb{J}_{kl}\partial_{x_l x_m}\Phi_j\left(\frac{j(u)}{|u|} - j(u_\star)\right)_k\left(\frac{j(u)}{|u|} - j(u_\star)\right)_m \mathrm{d}x,$$
$$T_{j,5} = \int \mathbb{J}_{kl}\partial_{x_l x_m}\Phi_j\left(\frac{j(u)}{|u|} - j(u_\star)\right)_k (j(u_\star))_m \mathrm{d}x,$$
$$T_{j,6} = \int \mathbb{J}_{kl}\partial_{x_l x_m}\Phi_j\left(\frac{j(u)}{|u|} - j(u_\star)\right)_m (j(u_\star))_k \mathrm{d}x,$$

*where $(j(u))_m$ is the $m$th component of the vector $j(u(t))$ and $u_\star = u_\star(\cdot;\xi(t),d)$.*

In the statement and proof of the lemma, we do *not* implicitly sum over the index $j$ when it is repeated.

**Proof.** Differentiating in time yields

$$\dot\eta_j = \int J(u)\frac{\mathrm{d}}{\mathrm{d}t}\Phi_j\mathrm{d}x + \int \Phi_j\frac{\mathrm{d}}{\mathrm{d}t}J(u)\mathrm{d}x.$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_j(x) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(x-a_j) = \left(-\dot{a_j}\right)\cdot\nabla\varphi(x-a_j)$, we can use the ODE (3.1) and the fact that $\Phi_j(\xi_i(t)) = 0$ for $i \neq j$ to write

$$\begin{aligned}
&\int J(u)\frac{\mathrm{d}}{\mathrm{d}t}\Phi_j \mathrm{d}x\\
&= \int J(u)\left(-\dot{a_j}\right)\cdot\nabla\varphi(x-a_j)\mathrm{d}x\\
&= -d_j\mathbb{J}\nabla_j W_\Omega(a)\cdot\nabla\varphi(\xi_j - a_j) \qquad (4.2)\\
&\quad+\int\left(J(u)-\sum_{i=1}^n \pi d_i\delta_{\xi_i}\right)\left(-\frac{1}{\pi}\mathbb{J}\nabla_j W_\Omega(a)\right)\cdot\nabla\varphi(x-a_j)\mathrm{d}x. \qquad (4.3)
\end{aligned}$$

For the second term $\int \Phi_j \frac{\mathrm{d}}{\mathrm{d}t}J(u)\mathrm{d}x$ we use the conservation law for the Jacobian (2.15). In particular for each $j$

$$\begin{aligned}
\int \Phi_j\frac{\mathrm{d}}{\mathrm{d}t}J(u)\mathrm{d}x &= \int \Phi_j \mathbb{J}_{kl}\partial_{x_k x_m}\left(u_{x_m}, u_{x_l}\right)\\
&= \int \mathbb{J}_{kl}\partial_{x_k x_m}\Phi_j\left(u_{x_m}, u_{x_l}\right).
\end{aligned}$$

Noting that

$$\nabla u = \nabla|u|\frac{u}{|u|} + i\frac{j(u)}{|u|}\frac{u}{|u|},$$

and that $(\nabla|u|\frac{u}{|u|}, i\frac{j(u)}{|u|}\frac{u}{|u|}) = (\nabla|u|\cdot\frac{j(u)}{|u|})(\frac{u}{|u|}, i\frac{u}{|u|}) = 0$, we continue by writing

$$\begin{aligned}
&\int \Phi_j\frac{\mathrm{d}}{\mathrm{d}t}J(u)\mathrm{d}x\\
&= \int \mathbb{J}_{kl}\partial_{x_k x_m}\Phi_j\partial_{x_m}|u|\partial_{x_l}|u|\\
&\quad+\int \mathbb{J}_{kl}\partial_{x_k x_m}\Phi_j\left(\frac{j(u)}{|u|}\right)_m\left(\frac{j(u)}{|u|}\right)_l\mathrm{d}x\\
&= \int \mathbb{J}_{kl}\partial_{x_k x_m}\Phi_j\partial_{x_m}|u|\partial_{x_l}|u|\mathrm{d}x \qquad (4.4)\\
&\quad+\int \mathbb{J}_{kl}\partial_{x_k x_m}\Phi_j\left(\frac{j(u)}{|u|}-j(u_\star)\right)_m\left(\frac{j(u)}{|u|}-j(u_\star)\right)_l\mathrm{d}x \qquad (4.5)\\
&\quad+\int \mathbb{J}_{kl}\partial_{x_k x_m}\Phi_j\left(\frac{j(u)}{|u|}-j(u_\star)\right)_m(j(u_\star))_l\,\mathrm{d}x \qquad (4.6)\\
&\quad+\int \mathbb{J}_{kl}\partial_{x_k x_m}\Phi_j\left(\frac{j(u)}{|u|}-j(u_\star)\right)_l(j(u_\star))_m\,\mathrm{d}x \qquad (4.7)\\
&\quad+\int \mathbb{J}_{kl}\partial_{x_k x_m}\Phi_j\,(j(u_\star))_m\,(j(u_\star))_l\,\mathrm{d}x. \qquad (4.8)
\end{aligned}$$

It is known (see Lemma 8 in Section 10) that (4.8) satisfies

$$\int \mathbb{J}_{kl}\partial_{x_k x_m}\Phi_j\,(j(u_\star))_m\,(j(u_\star))_l = d_j\nabla\varphi(\xi_j - a_j)\cdot\nabla_j W_\Omega(\xi). \qquad (4.9)$$

Now combining (4.2) and (4.9) gives $T_{j,1}$, and the remaining terms $T_{j,2}, \dots, T_{j,6}$ are exactly (4.3)–(4.7).

## 5. Estimate of $\dot{\eta}$

In this section we obtain an estimate of $\dot{\eta}$ by separately considering contributions from the different terms isolated in Lemma 1. We will prove

**Lemma 2.** *For $t \in [0, \tau_1]$ and $\varepsilon < \varepsilon_0$,*

$$|\dot{\eta}(t)| \leqq \frac{Cn}{\rho_\star^2}(\eta + \varpi_\varepsilon) + C\frac{n^2}{\rho_\star^{3/2}}(\eta + \varpi_\varepsilon)^{1/2}.$$

This is not good enough to get any very strong result from Gronwall's inequality, but in view of the assumptions about $n$, $\rho_\star$, the definition of $\varpi_\varepsilon$ and the bounds (3.21) on $\eta$, it implies the useful estimate

$$|\dot{\eta}| \leqq C\varepsilon^{1/50} \quad \text{for } t \in [0, \tau_1]. \tag{5.1}$$

The proof of Lemma 2 relies on the powerful $\Gamma$-stability and Localization Theorems 2 and 3.

We now present the

*Proof of Lemma 2.* The condition $\varepsilon < \varepsilon_0$ is needed to guarantee the validity of estimates (3.17), (3.20), (3.24) from Section 3.

Note from Lemma 1 and the definition (3.15) of $\eta$ that

$$\dot{\eta} = T_1 + \cdots T_6, \quad \text{where } T_k = \sum_{j=1}^{n} \frac{\eta_j}{|\eta_j|} \cdot T_{j,k}. \tag{5.2}$$

We estimate these terms in turn. We suppress the argument $t$ throughout the proof.

First, note that $\nabla\phi(\xi_j - a_j) = \xi_j - a_j$ for $0 \leqq t \leqq \tau_1$, by the definition of $\phi$ and (3.16). Thus, in view of (3.17),

$$|T_1| \leqq \sum_j |T_{j,1}| \leqq C(\eta + s_\varepsilon)\sum_j |\nabla_j W_\Omega(\xi) - \nabla_j W_\Omega(a)|.$$

And arguing as in the proof of (3.23) we see that

$$\begin{aligned} |\nabla_j W_\Omega(\xi) - \nabla_j W_\Omega(a)| &\leqq \sum_{k=1}^{n} |\xi_k(t) - a_k(t)| \left( \sup_k \sup_{|y-a(t)| \leqq |\xi(t)-a(t)|} |\nabla_k \nabla_j W(y)| \right) \\ &\leqq (\eta(t) + Cs_\varepsilon)C\frac{n}{\rho_\star^2}, \end{aligned}$$

using (3.17) again, as well as bounds on $\nabla^2 W_\Omega$ from (10.3). Thus

$$|T_1| \leqq C\frac{n^2}{\rho_\star^2}(\eta(t) + Cs_\varepsilon)^2. \tag{5.3}$$

Next,

$$|T_2| = \left| \int \left( J(u) - \sum_{i=1}^{n} \pi d_i \delta_{\xi_i} \right) \left( \sum_j \mathbb{J} \nabla_j W_\Omega(a) \cdot \nabla \left( \Phi_j \cdot \frac{\eta_j}{|\eta_j|} \right) \right) \mathrm{d}x \right|$$
$$\leqq \left\| J(u) - \sum_i \pi d_i \delta_{\xi_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \left\| \nabla \sum_j \nabla_j W_\Omega(a) \cdot \nabla \left( \Phi_j \cdot \frac{\eta_j}{|\eta_j|} \right) \right\|_{L^\infty}.$$

Since the $\Phi_j$'s have disjoint support

$$\left\| \nabla \sum_j \nabla_j W_\Omega(a) \cdot \nabla \left( \Phi_j \cdot \frac{\eta_j}{|\eta_j|} \right) \right\|_{L^\infty} \leqq \sup_j |\nabla_j W_\Omega(a)| \, \|\nabla^2 \Phi_j\|_\infty \leqq C \frac{n}{\rho_\star^2}.$$

We conclude from (3.13) and the above that

$$|T_2| \leqq C s_\varepsilon \frac{n}{\rho_\star^2}. \tag{5.4}$$

Continuing, we use the fact that $\nabla^2 \Phi_j$ vanishes in $B_{\rho_\star}(a_j)$, together with (3.24), to find that

$$|T_3| \leqq \left\| \sum_j \frac{\eta_j}{|\eta_j|} \cdot \nabla^2 \Phi_j \right\|_{L^\infty} \int_{\Omega_{\rho_\star}(a)} |\nabla |u||^2 \, \mathrm{d}x \leqq \frac{Cn}{\rho_\star^2} (\eta + \varpi_\varepsilon). \tag{5.5}$$

Exactly the same considerations show that

$$|T_4| \leqq \frac{Cn}{\rho_\star^2} (\eta(t) + \varpi_\varepsilon). \tag{5.6}$$

Next,

$$|T_5| \leqq \left\| \sum_j \frac{\eta_j}{|\eta_j|} \cdot \nabla^2 \Phi_j \right\|_{L^\infty} \left\| \frac{j(u)}{|u|} - j(u_\star) \right\|_{L^2(\Omega_{\rho_\star})} \|j(u_\star)\|_{L^2(\cup_j \operatorname{supp} \nabla^2 \Phi_j)}.$$

Using (10.2), one can easily check that $\|j(u_\star)\|_{L^2(\cup_j \operatorname{supp} \nabla^2 \Phi_j)} \leqq \frac{Cn}{\rho_\star} (Cn\rho_\star^2)^{1/2}$, hence we conclude that

$$|T_5| \leqq \frac{Cn^2}{\rho_\star^{3/2}} (\eta + \varpi_\varepsilon)^{1/2}. \tag{5.7}$$

Exactly the same argument shows that $|T_6| \leqq \frac{Cn^2}{\rho_\star^{3/2}} (\eta + \varpi_\varepsilon)^{1/2}$.

## 6. Time averages of $\eta$

In this section we get improved estimates of $T_5$ and $T_6$, after eventually averaging in time.

**Definition 1.** *Define the time average*

$$\langle g\rangle_{\delta_\varepsilon}(t) = \frac{1}{\delta_\varepsilon}\int_{t-\delta_\varepsilon}^{t} g(s)\mathrm{d}s.$$

Note that in view of (5.1),

$$|\eta(s) - \langle\eta(t)\rangle_{\delta_\varepsilon}| \leqq C\varepsilon^{1/50}\delta_\varepsilon \quad \text{if } 0 \leqq t-\delta_\varepsilon \leqq s \leqq t \leqq \tau_1. \tag{6.1}$$

We state our result

**Proposition 1.** *For* $\delta_\varepsilon = \varepsilon^{1/2}$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta\rangle_{\delta_\varepsilon}(t)\right| \leqq C\frac{n}{\rho_\star^2}\langle\eta(t)\rangle_{\delta_\varepsilon} + C\varepsilon^{1/3}. \tag{6.2}$$

*for all* $t \in [\delta_\varepsilon, \tau_1]$.

We will verify later that $\tau_1 \geqq \delta_\varepsilon$ for the initial data that we consider.

**Proof.** Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\eta\rangle_{\delta_\varepsilon} = \langle T_1\rangle_{\delta_\varepsilon} + \cdots + \langle T_6\rangle_{\delta_\varepsilon}$$

using the notation of (5.2). In view of (5.3)—(5.6),

$$\sum_{i=1}^4 |\langle T_i\rangle_{\delta_\varepsilon}| \leqq \sum_{i=1}^4 \langle |T_i|\rangle_{\delta_\varepsilon} \leqq \frac{Cn}{\rho_\star^2}\langle\eta+\varpi_\varepsilon\rangle_{\delta_\varepsilon} \leqq \frac{Cn}{\rho_\star^2}\langle\eta\rangle_{\delta_\varepsilon} + C\varepsilon^{1/3}.$$

Thus it is only necessary to show that

$$\sum_{i=5}^6 |\langle T_i\rangle_{\delta_\varepsilon}| \leqq \frac{Cn}{\rho_\star^2}\langle\eta\rangle_{\delta_\varepsilon} + C\varepsilon^{1/3}. \tag{6.3}$$

Since the proof for $T_6$ is identical to the proof for $T_5$ we only consider the latter.

Because $|\langle g\rangle| \leqq \langle |g|\rangle$, estimates valid for every $t$ automatically imply estimates for time-averaged quantities. Thus it is not necessary to average in $t$ until rather late in the proof.

We generally write $\delta_\varepsilon$ instead of $\varepsilon^{1/2}$ when we want to make it clear how our estimates depend on the interval over which we are averaging.

Throughout the proof we frequently use the facts that $n \leqq \varepsilon^{-1/100}$, $\rho_\star \geqq \varepsilon^{1/20}$.

**Step 1**: For simplicity let

$$\zeta_k := \sum_j \mathbb{J}_{kl}\partial_{x_l x_m}\left(\frac{\eta_j}{|\eta_j|}\cdot\Phi_j\right) j_m(u_\star), \quad k = 1, 2, \tag{6.4}$$

where $j_m$ denotes the $m$ component of $j(u_\star)$, $m=1,2$, and $u_\star(x,t)=u_\star(x;\xi(t),d)$ as usual. From the definitions and (10.2) it is easy to see that $|\zeta| \leqq C\frac{n}{\rho_\star^2}$, and in addition $\zeta$ is supported on a set of measure $cn\rho_\star^2$. It follows that

$$\|\zeta\|_{L^q(\Omega)} \leqq Cn^{1+\frac{1}{q}}\rho_\star^{\frac{2}{q}-2}. \tag{6.5}$$

Now we write

$$\begin{aligned} T_5 &= \int \zeta \cdot \frac{j(u)}{|u|}(1-|u|)\mathrm{d}x \; + \int \zeta \cdot (j(u) - j(u_\star))\,\mathrm{d}x \\ &= T_{5,1} + T_{5,2}. \end{aligned}$$

The first term is easily estimated. Indeed, by Cauchy–Schwarz,

$$|T_{5,1}| \leqq \|\zeta\|_{L^\infty} \left\| \frac{j(u)}{|u|}\right\|_{L^2} \|(1-|u|^2)\|_{L^2} \leqq C\frac{n}{\rho_\star^2}\varepsilon E_\varepsilon(u) \leqq C\varepsilon^{1/2}. \tag{6.6}$$

**Step 2**. The boundary conditions for (1.1) and the definition of the canonical harmonic map imply that $\nu\cdot(j(u)-j(u_\star))=0$ on $\partial\Omega$. As a result, we can write

$$j(u)-j(u_\star) = \nabla f + \nabla\times g$$

for some $f\in W^{1,p}(\Omega)$ and $g\in W^{1,p}(\Omega;\mathbb{R}^2)$, for $p<2$, such that

$$\nu\cdot\nabla f = 0, \quad g=0 \quad \text{on } \partial\Omega. \tag{6.7}$$

In fact $f$ and $g$ can be found by solving

$$\Delta f = \nabla\cdot(j(u)-j(u_\star)) = \nabla\cdot j(u), \tag{6.8}$$

$$-\Delta g = \nabla\times(j(u)-j(u_\star)) = 2J(u) - 2\pi\sum d_i\delta_{\xi_i} \tag{6.9}$$

in $\Omega$ with boundary conditions (6.7). We write $T_{5,2}=T_{5,2a}+T_{5,2b}$, where

$$T_{5,2a} := \int \zeta\cdot\nabla\times g\,\mathrm{d}x, \quad T_{5,2b} := \int \zeta\cdot\nabla f\,\mathrm{d}x.$$

**Step 2a**. We claim that

$$|T_{5,2a}| \leqq C{s_\varepsilon}^{3/5}\left[n^{6/5}\,\rho_\star^{-8/5}(E_\varepsilon(u)+n\pi)^{2/5}\right] \leqq C\varepsilon^{1/3}. \tag{6.10}$$

To prove this, we first use the equation (6.9), (6.7) satisfied by $g$, together with standard elliptic estimates (see for example [29], Chapter 5)

$$\|g\|_{W^{1,p}(\Omega)} \leqq C\|J(u)-\pi\sum d_i\delta_{\xi_i}\|_{\dot W^{-1,p}(\Omega)}$$

for $p>1$, where the constant depends on $p$ and $\Omega$. By duality and the Sobolev embedding theorem, the dual space $C^{0,\alpha}(\Omega)^*$ embeds into $\dot W^{-1,p}(\Omega)$ for $1\leqq p < 2$ and $\frac{\alpha}{1}+\frac{1-\alpha}{2}=\frac{1}{p}$, and

$$\|v\|_{\dot W^{-1,p}(\Omega)} \leqq C\|v\|_{C_0^{0,\alpha}(\Omega)^*}$$

for all $v \in C_0^{0,\alpha}(\Omega)^*$. In addition, an interpolation inequality [18] states that

$$\|v\|_{C_0^{0,\alpha}(\Omega)^*} \leqq C(\|v\|_{C_0^{0,1}(\Omega)^*})^{\alpha}(\|v\|_{C_0^{0,0}(\Omega)^*})^{1-\alpha},$$

where $C_0^{0,0}(\Omega)^*$ denotes the space of finite signed Radon measures on $\Omega$, and the norm is just the total mass of the measure. Note that the $C_0^{0,1}(\Omega)^*$ norm is equivalent to the $\dot{W}^{-1,1}(\Omega)$ norm, and so $\|J(u) - \pi \sum d_i \delta_{\xi_i}\|_{C_0^{0,1}(\Omega)^*}$ is estimated by (3.13). Also,

$$\left\| J(u) - \pi \sum d_i \delta_{\xi_i} \right\|_{C_0^{0,0}(\Omega)^*} \leqq \left\| |\nabla u|^2 \right\|_{L^1(\Omega)} + n\pi \leqq C E_\varepsilon(u) + n\pi.$$

Combining these, we find that

$$\|\nabla \times g\|_{L^p(\Omega)} \leqq \|g\|_{W^{1,p}(\Omega)} \leqq C s_\varepsilon^{\frac{2}{p}-1}(E_\varepsilon(u) + n)^{2-\frac{2}{p}}$$

for $1 \leqq p < 2$, with a constant depending on $p$. Taking $\frac{1}{q} = 1 - \frac{1}{p}$ in (6.5) for $p \in [1, 2)$ to be selected, we conclude that

$$|T_{5,2a}| \leqq \|\zeta\|_{L^q} \|\nabla \times g\|_{L^p} \leqq C n^{2-\frac{1}{p}} \rho_\star^{-\frac{2}{p}} s_\varepsilon^{\frac{2}{p}-1} (E_\varepsilon(u) + n\pi)^{2-\frac{2}{p}}.$$

Choosing $p = \frac{5}{4}$, we arrive at (6.10).

**Step 2b**.

The time-averaging that appears in the statement of the lemma is needed to deal with the final term $T_{5,2b}$. We first note that

$$\begin{aligned}
&\left\langle T_{5,2b}\right\rangle_{\delta_\varepsilon}(t) \\
&\quad = \frac{1}{\delta_\varepsilon} \int_{t-\delta_\varepsilon}^{t} \int \zeta \cdot \nabla f \,\mathrm{d}x\mathrm{d}s \\
&\quad = \int \langle \zeta \rangle_{\delta_\varepsilon} \cdot \nabla \langle f \rangle_{\delta_\varepsilon} \,\mathrm{d}x + \frac{1}{\delta_\varepsilon} \int_{t-\delta_\varepsilon}^{t} \int \left(\zeta - \langle \zeta \rangle_{\delta_\varepsilon}\right) \cdot \nabla \left(f - \langle f \rangle_{\delta_\varepsilon}\right) \mathrm{d}x\mathrm{d}s \\
&\quad =: T_{5,2b(i)} + T_{5,2b(ii)},
\end{aligned}$$

where the cross terms disappear since $\int g - \langle g \rangle \,\mathrm{d}s = 0$.

**Step 2b(i)**. Recalling the equation that defines $f$ and the equation for conservation of mass (2.12), we compute

$$\begin{aligned}
\left\| \Delta \langle f \rangle_{\delta_\varepsilon} \right\|_{L^2(\Omega)} &= \left\| \langle \nabla \cdot j(u) \rangle_{\delta_\varepsilon} \right\|_{L^2(\Omega)} \\
&= \left\| \left\langle \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(|u|^2 - 1\right) \right\rangle_{\delta_\varepsilon} \right\|_{L^2(\Omega)} \\
&= \frac{1}{2\delta_\varepsilon} \left\| \left(|u|^2 - 1\right) \Big|_{t-\delta_\varepsilon}^{t} \right\|_{L^2(\Omega)} \\
&\leqq \frac{\varepsilon}{\delta_\varepsilon} \sqrt{E_\varepsilon(u)}.
\end{aligned}$$

Hence $\|\nabla\cdot\langle f\rangle_{\delta_\varepsilon}\|_{L^2} \leqq \|\langle f\rangle_{\delta_\varepsilon}\|_{W^{2,2}} \leqq C\|\Delta\langle f\rangle_{\delta_\varepsilon}\|_{L^2} \leqq C\frac{\varepsilon}{\delta_\varepsilon}\sqrt{E_\varepsilon(u)}$, and so recalling (6.5), we conclude that

$$|T_{5,2b(i)}| \leqq Cn^{3/2}\rho_\star^{-1}\frac{\varepsilon}{\delta_\varepsilon}\sqrt{E_\varepsilon(u)} \leqq C\varepsilon^{1/3}. \tag{6.11}$$

**Step 2b(ii)**. The arguments in this final step are rather involved, but the overall point is to take advantage of the fact that $\delta_\varepsilon$ is small to show that $\zeta$ is close to $\langle\zeta\rangle_{\delta_\varepsilon}$, and similarly $\nabla f$ and $\langle\nabla f\rangle_{\delta_\varepsilon}$. First observe that

$$\begin{aligned}&|T_{5,2b(ii)}|\\&\leqq \sup_{s\in[t-\delta_\varepsilon,t]}\|\zeta(s)-\langle\zeta\rangle_{\delta_\varepsilon}\|_{L^4(\Omega)}\sup_{s\in[t-\delta_\varepsilon,t]}\|\nabla(f(s)-\langle f\rangle_{\delta_\varepsilon})\|_{L^{4/3}(\Omega)}\\&\leqq \sup_{s,s'\in[t-\delta_\varepsilon,t]}\|\zeta(s)-\zeta(s')\|_{L^4(\Omega)}\sup_{s,s'\in[t-\delta_\varepsilon,t]}\|\nabla(f(s)-f(s'))\|_{L^{4/3}(\Omega)}.\end{aligned} \tag{6.12}$$

We choose $L^4$ and $L^{4/3}$ because the estimates of $\|\nabla(f(s)-f(s'))\|_{L^{4/3}(\Omega)}$ are slightly easier for $p=4/3$ than for other choices $1\leqq p<2$. In estimating the quantities in (6.12), we will repeatedly use the fact that for $s,s'\in[t-\delta_\varepsilon,t]$ with $t\in[\delta_\varepsilon,\tau_1]$,

$$|a_j(s)-a_j(s')|\leqq C\frac{n}{\rho_\star}\delta_\varepsilon\leqq C\varepsilon^{2/5}. \tag{6.13}$$

This follows from the ordinary differential equation (3.1) satisfied by $a(\cdot)$, which together with (10.3) implies that $|\dot a_j|\leqq\frac{Cn}{\rho_\star}$. From (6.13) and (3.17), (3.21), it follows that for $s,s'$ as above,

$$\sum_{j=1}^n\left|\xi_j(s)-\xi_j(s')\right| \leqq C\frac{n^2}{\rho_\star}\delta_\varepsilon+\eta(s)+\eta(s')+Cs_\varepsilon\leqq C\varepsilon^{1/4}. \tag{6.14}$$

**Estimate of** $\|\zeta(s)-\zeta(s')\|_{L^4(\Omega)}$ .

Throughout this discussion we assume that $0\leqq t-\delta_\varepsilon\leqq s,s'\leqq t\leqq\tau_1$. To find a time-Lipschitz bound on $\zeta$, we note from the definition (6.4) that

$$\begin{aligned}\zeta_k(s)-\zeta_k(s') &= \sum_j \mathbb{J}_{kl}\partial_{x_lx_m}\left(\frac{\eta_j}{|\eta_j|}\cdot\Phi_j\right)(s)\ j_m(u_\star)(s)\\&\quad-\sum_j \mathbb{J}_{kl}\partial_{x_lx_m}\left(\frac{\eta_j}{|\eta_j|}\cdot\Phi_j\right)(s')\ j_m(u_\star)(s')\\&=\sum_j \mathbb{J}_{kl}\partial_{x_lx_m}\left[\frac{\eta_j}{|\eta_j|}\cdot(\Phi_j(s)-\Phi_j(s'))\right]\ j_m(u_\star)(s)\\&\quad+\sum_j \mathbb{J}_{kl}\partial_{x_lx_m}\left(\frac{\eta_j}{|\eta_j|}\cdot\Phi_j\right)(s')\ \left[j_m(u_\star)(s)-j_m(u_\star)(s')\right]\\&=Z_1+Z_2.\end{aligned}$$

We first consider $Z_1$. From the definitions,

$$\begin{aligned}\left\| \partial_{x_l x_m} \left[ \frac{\eta_j}{|\eta_j|} \cdot (\Phi_j(s) - \Phi_j(s')) \right] \right\|_{L^\infty(\Omega)} &\leqq \left\| \partial_{x_l x_m} [\varphi(x - a_j(s)) - \varphi(x - a_j(s'))] \right\|_{L^\infty(\Omega)} \\ &\leqq C \left\| \partial_{x_l x_m x_n} \varphi \right\|_{L^\infty} \left| a_j(s) - a_j(s') \right| \\ &\leqq C \frac{n}{\rho_\star^3} \delta_\varepsilon\end{aligned}$$

using (6.13).

We next assert that

$$\operatorname{supp} \nabla^2 \Phi_j(s) \cup \operatorname{supp} \nabla^2 \Phi_j(s') \subset B_{3\rho_\star}(\xi_j(s)) \setminus B_{\frac{1}{2}\rho_\star}(\xi_j(s)) \tag{6.15}$$

for all $\varepsilon$ sufficiently small. This follows from (6.13), (6.14), and (3.17), which imply that the distances separating $a_i(s), a_i(s'), \xi_i(s), \xi_i(s')$ are *much* smaller than $\varepsilon^{1/20} \leqq \rho_\star$.

From (6.15) we infer that $|j(u_\star)(\xi(s))| \leqq \frac{Cn}{\rho_\star}$ on the support of $Z_1$, and since the support of $Z_1$ has measure bounded by $Cn\rho_\star^2$, we conclude that

$$\|Z_1\|_{L^4} \leqq \frac{Cn^2}{\rho_\star^4} \left( Cn\rho_\star^2 \right)^{1/4} \delta_\varepsilon = C \frac{n^{9/4}}{\rho_\star^{7/2}} \delta_\varepsilon \leqq C\varepsilon^{3/10}. \tag{6.16}$$

Next we consider $Z_2$. Since $\left\| \sum_j \partial_{x_l x_m} \Phi_j \right\|_{L^\infty} \leqq \frac{C}{\rho_\star}$, and noting that supp $Z_2$ has measure at most $Cn\rho_\star^2$, we use Hölder's inequality to estimate

$$\|Z_2\|_{L^4} \leqq \frac{C}{\rho_\star} \left\| j(u_\star)(s) - j(u_\star)(s') \right\|_{L^\infty(\cup_j \operatorname{supp} \nabla^2 \Phi_j(s'))} (Cn\rho_\star^2)^{1/4}.$$

Note that (6.15) is still true if we reverse the roles of $s$ and $s'$. It follows that $\operatorname{supp} \cup_j \nabla^2 \Phi_j(s') \subset \Omega_{\rho_\star/2}(\xi(s)) \cap \Omega_{\rho_\star/2}(\xi(s'))$. We can thus use (10.6) to find that

$$\left\| j(u_\star)(s) - j(u_\star)(s') \right\|_{L^\infty(\cup_j \operatorname{supp} \nabla^2 \Phi_j(s'))} \leqq \frac{C}{\rho_\star^2} \sum_{j=1}^n \left| \xi_j(s) - \xi_j(s') \right|.$$

Consequently, using the left-hand inequality in (6.14) together with (6.1), we deduce that

$$\begin{aligned}\|Z_2\|_{L^4} &\leqq C \left( \frac{n}{\rho_\star^2} \right)^{5/4} \left( \frac{n^2}{\rho_\star} \delta_\varepsilon + \eta(s) + \eta(s') + C s_\varepsilon \right) &(6.17)\\ &\leqq C\varepsilon^{-1/30} \frac{n}{\rho_\star^2} \langle \eta(t) \rangle + C\varepsilon^{1/4}. &(6.18)\end{aligned}$$

Combining (6.16) and (6.18) yields

$$\left\| \zeta(s) - \zeta(s') \right\|_{L^4(\Omega)} \leqq C\varepsilon^{-1/30} \frac{n}{\rho_\star^2} \langle \eta(t) \rangle + C\varepsilon^{1/4}. \tag{6.19}$$

**Estimate of** $\|\nabla(f(s)-f(s'))\|_{L^{4/3}(\Omega)}$**.**
We continue to assume that $s, s' \in [t-\delta_\varepsilon, t]$ for $t \in [\delta_\varepsilon, \tau_1]$.

First note that, by elliptic regularity, and using the equation and boundary conditions (6.7), (6.8) that define $f$,

$$\begin{aligned}\|\nabla(f(s)-f(s'))\|_{L^{4/3}(\Omega)} &\leqq C\|\Delta(f(s)-f(s'))\|_{W^{-1,4/3}(\Omega)}\\ &= \|\nabla\cdot[j(u)(s)-j(u)(s')]\|_{W^{-1,4/3}(\Omega)}\\ &\leqq \|j(u)(s)-j(u)(s')\|_{L^{4/3}(\Omega)}. \end{aligned}\tag{6.20}$$

Using the triangle inequality and (3.25), we see that it follows that

$$\begin{aligned}&\|j(u)(s)-j(u)(s')\|_{L^{4/3}(\Omega)}\\ &\quad\leqq \frac{Cn}{\rho_\star}(\eta(s)+\eta(s'))+2\lambda_\varepsilon+\|j(u_\star)(s)-j(u_\star)(s')\|_{L^{4/3}(\Omega)}.\end{aligned}$$

The last term on the right-hand side is estimated in by combining (10.7) and (6.14). This leads to

$$\begin{aligned}\|j(u_\star)(s)-j(u_\star)(s')\|_{L^{4/3}(\Omega)} &\leqq Cn^{1/2}\left(\frac{n^2}{\rho_\star}\delta_\varepsilon+\eta(s)+\eta(s')+Cs_\varepsilon\right)^{1/2}\\ &\leqq C\varepsilon^{1/9}.\end{aligned}$$

The other terms on the right-hand side of (6.20) are smaller, in view of the estimate (3.26) and the constraints on $n, \rho_\star$, so we conclude that

$$\|\nabla(f(s)-f(s'))\|_{L^{4/3}(\Omega)} \leqq C\varepsilon^{1/9}. \tag{6.21}$$

Finally we combine the above with (6.12) and (6.19) to deduce that

$$|T_{5,2b(ii)}| \leqq C\frac{n}{\rho_\star^2}\langle\eta(t)\rangle + C\varepsilon^{1/3}.$$

Together with (6.11), (6.10), and (6.6), this implies (6.3), which is what we needed to show.

## 7. Conclusion of the proof of Theorem 1

*Proof* (*conclusion of the proof of Theorem 1*). **Step 1**. Note from (3.2) and (3.20) that $\eta(0) \leqq C\varepsilon^{9/10}$. It also follows from (3.20) that

$$\tau_1 \geqq \tau_2 := \sup\left\{0 \leqq t \leqq \tau_0 : \eta(s) \leqq \frac{1}{2}\varepsilon^{1/4}\ \forall s \in [0,t]\right\}.$$

Thus for $t \in [0, \tau_2]$, all the conclusions of the previous sections hold. In particular, from the estimate (5.1) of $|\dot\eta|$, it follows that

$$\eta(t) \leqq C\varepsilon^{9/10} + Ct\varepsilon^{1/50} \quad \text{for } 0 \leqq t \leqq \tau_2. \tag{7.1}$$

This implies that $\tau_2 \geqq \varepsilon^{1/4}$ and that

$$\langle\eta\rangle_{\delta_\varepsilon}(\delta_\varepsilon) = \frac{1}{\delta_\varepsilon}\int_0^{\delta_\varepsilon} \eta(s)\mathrm{d}s \leqq \delta_\varepsilon = \sqrt{\varepsilon}$$

for $\varepsilon$ sufficiently small. Next observe from (6.1) that

$$\tau_2 \geqq \tau_3 := \sup\left\{\delta_\varepsilon \leqq t \leqq \tau_0 : \langle\eta(s)\rangle \leqq \frac{1}{4}\varepsilon^{1/4}\ \forall s \in [\delta_\varepsilon, t]\right\} \geqq \delta_\varepsilon$$

and that (5.1), (6.2) hold for all $t \in [\delta_\varepsilon, \tau_3]$. Hence from Grönwall's inequality,

$$\langle\eta\rangle_{\delta_\varepsilon}(t) \leqq C\exp\left(\frac{Cn(t-\delta_\varepsilon)}{\rho_\star^2}\right)\varepsilon^{1/3} \leqq C\exp\left(\frac{Cnt}{\rho_\star^2}\right)\varepsilon^{1/3} \tag{7.2}$$

for $t \in [\delta_\varepsilon, \tau_3]$. It follows that

$$\tau_1 \geqq \tau_3 \geqq \min\left\{C\frac{\rho_\star^2}{n}\ln\frac{1}{\varepsilon}, \tau_0\right\}.$$

**Step 2**. We now fix $\rho_\star \geqq \varepsilon^{1/20}$ by requiring that

$$\tau_0 = \tau(\rho_\star) = C\frac{\rho_\star^2}{n}\ln\frac{1}{\varepsilon}, \tag{7.3}$$

where we write

$$\tau(\rho) = \inf\{t \geqq 0 \ :\ \rho_{a(t)} \leqq \rho\}.$$

To prove this, we first note that $\tau(\varepsilon^{1/20}) \geqq \varepsilon^{1/12}$. This follows from the ODE (3.1), and estimates (10.3) on the renormalized energy, which imply that $\frac{\mathrm{d}}{\mathrm{d}t}\rho_{a(t)} \geqq -\frac{C}{n\rho_{a(t)}}$, and consequently that

$$\rho_{a(t)} \geqq \left[\rho_{a^0}^2 - Cnt\right]^{1/2}.$$

In (3.3) we assumed that $\rho_{a^0} \geqq \varepsilon^{1/25}$, and it follows that $\rho_{a(t)} \geqq \varepsilon^{1/20}$ whenever $t \leqq \varepsilon^{1/12}$. As a result,

$$g(\rho) := \tau(\rho) - C\frac{\rho^2}{n}\ln\frac{1}{\varepsilon} \geqq 0$$

when $\rho = \varepsilon^{1/20}$. It is clear that $\rho \mapsto g(\rho)$ is a continuous, strictly decreasing function and that it is negative for large values of $\rho$, so the existence of $\rho_\star$ as in (7.3) follows.

We remark that $\tau_\star$ as defined in (3.5) is equal to the common value $\tau(\rho_\star) = C\frac{\rho_\star^2}{n}\ln\frac{1}{\varepsilon}$; this is just a rewriting of (7.3).

**Step 3**. It follows from (7.2) and (6.1) that $\eta(t) \leqq C\exp\left(\frac{Cnt}{\rho_\star^2}\right)\varepsilon^{1/3} \leqq \frac{1}{2}\varepsilon^{1/4}$ for $0 \leqq t \leqq \tau_\star$. The conclusion (3.6) of the theorem follows from this and (3.19).

To prove (3.7), note first from Theorem 2 that (3.24) remains valid if the integral $\int_{\Omega_{\rho_\star}(\xi(t))} \dots \mathrm{d}x$ on the left-hand side is replaced by an integral over the larger set $\Omega_{\varepsilon^{2/5}}(\xi(t))$. Since $\Omega_{\rho_\star}(a(t)) \subset \Omega_{\varepsilon^{2/5}}(\xi(t))$, it follows that

$$\int_{\Omega_{\rho_\star}(a(t))} e_\varepsilon(|u(t)|) + \frac{1}{4}\left|\frac{j(u(t))}{|u(t)|} - j(u_\star(\xi(t)))\right|^2 \leqq C\varepsilon^{1/5}.$$

In addition, since $\rho_\star \geqq \varepsilon^{1/20}$ and $\sum |\xi_i(t) - a_i(t)| \leqq C\varepsilon^{1/4}$ by (3.16), we readily estimate from (10.6) that $\| j(u_\star(\xi(t))) - j(u_\star(a(t)))\|_{L^2(\Omega_{\rho_\star}(a))} \leqq C\varepsilon^{1/10}$. With the triangle inequality and the above estimate, this yields (3.7).

Finally, again using the fact that $\sum |\xi_i(t) - a_i(t)| \leqq C\varepsilon^{1/4}$, we deduce (3.8) from (3.25) and (10.7).

## 8. $\Gamma$-stability

This section proves the $\Gamma$-stability theorem discussed in the introduction and used extensively in the previous arguments.

The proof rests on the identity

$$\begin{aligned}\int_{\Omega_\sigma(\alpha)} e_\varepsilon(|u|) + \frac{1}{2}\left|\frac{j(u)}{|u|} - j(u_\star(\alpha))\right|^2 \mathrm{d}x \\ = \int_{\Omega_\sigma(\alpha)} [e_\varepsilon(u) - e_\varepsilon(u_\star(\alpha))]\, \mathrm{d}x \\ + \int_{\Omega_\sigma(\alpha)} |j(u_\star(\alpha))|^2 - \frac{j(u)}{|u|} \cdot j(u_\star(\alpha))\, \mathrm{d}x. \end{aligned} \tag{8.1}$$

In fact the integrands on the left and right-hand sides are pointwise equal. The above follows from a short calculation, using the fact that $|\nabla u_\star| = |j(u_\star)|$. The main hypothesis of the theorem is that $\|J(u) - \pi \sum d_i \delta_{\alpha_i}\|_{\dot{W}^{-1,1}}$ is small, and under this hypothesis we wish to bound the left-hand side of (8.1) by the surplus energy $\Sigma_\varepsilon(u; \alpha, d) \approx \int_\Omega [e_\varepsilon(u) - u_\star^\varepsilon(\alpha, d)]$, see (2.25) for the definition. We rewrite the first term on the right-hand side as a sum of the surplus energy and contributions from balls $U_\sigma(\alpha_i)$:

$$\begin{aligned}\int_{\Omega_\sigma(\alpha)} [e_\varepsilon(u) - e_\varepsilon(u_\star(\alpha))]\, \mathrm{d}x \\ \approx \Sigma_\varepsilon(u; \alpha, d) + \sum_i \int_{U_\sigma(\alpha_i)} [e_\varepsilon(u_\star(\alpha)) - e_\varepsilon(u)]\, \mathrm{d}x.\end{aligned}$$

The integrals over the balls $U_\sigma(\alpha_i)$ are shown to be small using results from [20], which require the hypothesis $\|J(u) - \pi \sum d_i \delta_{\alpha_i}\|_{\dot{W}^{-1,1}}$. The second term on the right-hand side of (8.1) is approximately (suppressing the dependence on $\alpha$)

$$\begin{aligned}\int_{\Omega_\sigma} j(u_\star) \cdot (j(u_\star) - j(u)) = \int_{\Omega_\sigma} \nabla \times G \cdot (j(u_\star) - j(u)) \\ \approx \int_\Omega \tilde{G}\, \nabla \times (j(u_\star) - j(u)).\end{aligned}$$

Here $G$ is defined in (2.17), and $\tilde{G}$ is a modification of $G$ obtained by modifying $\Omega_\sigma$ sightly (so that $G$ is constant on each component of $\partial\Omega_\sigma$) and then setting $\tilde{G}$ equal to the suitable constant on each component of $\Omega \setminus \Omega_\sigma$. The right-hand side is then controlled using the assumed bounds on $\|J(u) - \pi \sum d_i \delta_{\alpha_i}\|_{\dot{W}^{-1,1}} = \frac{1}{2}\|\nabla \times (j(u) - j(u_\star))\|_{\dot{W}^{-1,1}}$; note that this hypothesis turns out to be extremely natural at this point. These arguments yield an estimate of the left-hand side of (8.1) in terms of the surplus energy and small error terms.

Thus, the main result of this section is

**Theorem 2.** *Let $\Omega$ be a bounded, open simply connected subset of $\mathbb{R}^2$ with $C^1$ boundary. Then there exists absolute constants $C$ and $K_1$ such that for any $u \in H^1(\Omega;\mathbb{C})$, if there exist $n \geqq 0$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$ such that*

$$\left\| J(u) - \sum_{j=1}^{n} \pi d_j \delta_{\alpha_j} \right\|_{\dot{W}^{-1,1}(\Omega)} \leqq s_\varepsilon \quad \text{for some } s_\varepsilon \in [\varepsilon\sqrt{\ln(\rho_\alpha/\varepsilon)}, \tfrac{\rho_\alpha}{4nK_1}\,], \tag{8.2}$$

*and if $4s_\varepsilon \leqq \sigma^* := \left[\frac{\rho_\alpha}{n^3}(s_\varepsilon + \varepsilon E_\varepsilon(u))\right]^{1/2} \leqq \frac{\rho_\alpha}{nK_1}$, then*

$$\begin{aligned} &\int_{\Omega_{\sigma^*}(\alpha)} e_\varepsilon(|u|) + \frac{1}{4}\left|\frac{j(u)}{|u|} - j(u_\star)\right|^2 \mathrm{d}x \\ &\leqq \Sigma_\Omega^\varepsilon(u;\alpha,d) + C\left[\frac{n^5}{\rho_\alpha}(s_\varepsilon + \varepsilon E_\varepsilon(u))\right]^{1/2} \end{aligned} \tag{8.3}$$

*for $u_\star = u_\star(\cdot;\alpha,d)$ as defined in (2.16). Finally,*

$$\|j(u) - j(u_\star)\|_{L^{4/3}(\Omega)} \leqq C\sqrt{\Sigma(u;\alpha.d)} + \textit{error terms} \tag{8.4}$$

*with*

$$\begin{aligned} \textit{error terms} &\leqq C\varepsilon^{1/2}E_\varepsilon(u)^{3/4} \\ &\quad + Cn(s_\varepsilon + \varepsilon E_\varepsilon(u))^{1/4}\left[\left(\frac{n}{\rho_\alpha}\right)^{1/4} + \rho_\alpha^{1/4}\left(1 + \sqrt{\frac{E_\varepsilon(u)}{n^3}}\right)\right]. \end{aligned}$$

The conclusion (8.3) is deduced from a more general estimate, which is displayed in (8.8).

Note that the left-hand side of (8.3) approximately equals $\int_{\Omega_{\sigma^*}(\alpha)} e_\varepsilon(u/u_\star)$. Indeed, a short calculation shows that $e_\varepsilon(u/u_\star) = e_\varepsilon(|u|) + \frac{1}{2}|\frac{j(u)}{|u|} - |u|j(u_\star))|^2$. As a result,

$$\begin{aligned} &e_\varepsilon(|u|) + \frac{1}{2}\left|\frac{j(u)}{|u|} - j(u_\star)\right|^2 - e_\varepsilon(u/u_\star) \\ &\quad = \frac{j(u)}{|u|}\cdot j(u_\star)(|u|-1) + \frac{1}{2}|j(u_\star)|^2(1-|u|^2) \end{aligned} \tag{8.5}$$

and the terms on the right-hand side are small in $L^1(\Omega_{\sigma^*})$ if the energy is not too large.

In many situations, such as in the study of vortex dynamics in the first part of this paper, the error terms in (8.4) are of order $\varepsilon^{1/5}$ or smaller. We remark that it would be possible to establish estimates of $\|j(u) - j(u_\star)\|_{L^p(\Omega)}$ for $1 \leqq p < 2$, in the spirit of (8.4).

The proof of the theorem uses the following lemma from [20]. Note that it is essentially the $n = 1$ case of Theorem 2 on a ball with a single vortex at the center, except that the positive terms on the left-hand side of (8.3) are missing.

**Lemma 3.** *There exists an absolute constant $C > 0$ such that if $u \in H^1(U_\sigma; \mathbb{C})$ satisfies*

$$\|J(u) \pm \pi\delta_0\|_{\dot{W}^{-1,1}(U_\sigma)} \leqq \frac{\sigma}{4},$$

*then*

$$0 \leqq \Sigma^\varepsilon_{U_\sigma}(u; 0, \pm 1) + C\frac{\varepsilon}{\sigma}\sqrt{\ln\frac{\sigma}{\varepsilon}} + \frac{C}{\sigma}\|J(u) - \pi\delta_0\|_{\dot{W}^{-1,1}(U_\sigma)}. \tag{8.6}$$

This is Theorem 5 in [20]; the statement there appears slightly different, but the two versions are easily seen to be equivalent using Lemma 9.

For the time being we assume one additional lemma, the proof of which is given below, and we use it to complete the

*Proof of Theorem 2.* **Step 1**. We first rewrite $W^\varepsilon_\Omega(\alpha, d)$ using facts about the renormalized energy that are collected in Section 10. Recall from (2.24) that $W^\varepsilon_\Omega$ is defined by $W^\varepsilon_\Omega(\alpha, d) = W_\Omega(\alpha, d) + n(\gamma + \pi \ln \frac{1}{\varepsilon})$, where $\gamma$ is defined in (2.23). Hence by Lemma 12,

$$W^\varepsilon_\Omega(\alpha, d) = \int_{\Omega_\sigma} e_\varepsilon(u_\star)\,\mathrm{d}x + O\left(\left(\frac{n\sigma}{\rho_\alpha}\right)^2\right) + n\left(\gamma + \pi\ln\frac{\sigma}{\varepsilon}\right)$$

for any $0 \leqq \sigma \leqq \rho_\alpha$. Thus, recalling the formula (2.26) for the surplus energy $\Sigma^\varepsilon_{U_r(a)}(u, a, \pm 1)$ on a ball,

$$\begin{aligned}
&\Sigma^\varepsilon_\Omega(u; \alpha, d) \\
&\quad = \int_\Omega e_\varepsilon(u)\,\mathrm{d}x \; - \; W^\varepsilon_\Omega(\alpha, d) \\
&\quad = \int_{\Omega_\sigma(\alpha)} [e_\varepsilon(u) - e_\varepsilon(u_\star)]\mathrm{d}x \; + \; \sum_{i=1}^n \Sigma^\varepsilon_{U_\sigma(\alpha_i)}(u, \alpha_i, d_i) + O\left(\left(\frac{n\sigma}{\rho_\alpha}\right)^2\right).
\end{aligned} \tag{8.7}$$

**Step 2**. We give a lower bound for $\int_{\Omega_\sigma(\alpha)}[e_\varepsilon(u) - e_\varepsilon(u_\star)]\mathrm{d}x$ in Lemma 4 below; valid for all $\sigma \leqq \frac{\rho_\alpha}{nK_1}$, with $K_1$ being fixed in the course of the proof of this lemma. The contributions from $B_\sigma(\alpha_i)$, $i = 1, \ldots, n$ are estimated using Lemma 3. Note that the definitions of the norms imply that $\|J(u)-\pi\delta_{\alpha_i}\|_{\dot W^{-1,1}(U_\sigma(\alpha_i))} \leqq s_\varepsilon$, so the hypotheses of this lemma are satisfied whenever $4s_\varepsilon \leqq \sigma$. We may, thus, apply that lemma to deduce that $\Sigma^\varepsilon_{U_\sigma(\alpha_i)}(u,\alpha_i,d_i) \geqq -C\frac{\varepsilon}{\sigma}\sqrt{\ln\frac{\sigma}{\varepsilon}} - \frac{C}{\sigma}s_\varepsilon$ for each $i$. We assemble these estimates and simplify, using our assumption that $s_\varepsilon \geqq \varepsilon\sqrt{\ln \rho_\alpha/\varepsilon}$ and the fact that $(\frac{n\sigma}{\rho_\alpha})^2 \leqq n^4\frac{\sigma}{\rho_\alpha}$, to find that

$$\begin{aligned}\int_{\Omega_\sigma(\alpha)} e_\varepsilon(|u|) + \frac14\left|\frac{j(u)}{|u|} - j(u_\star)\right|^2 \mathrm{d}x \\ \leqq \Sigma^\varepsilon_\Omega(u,\alpha,d) + C\left(n^4\frac{\sigma}{\rho_\alpha} + \frac{n}{\sigma}(s_\varepsilon + \varepsilon E_\varepsilon(u))\right).\end{aligned} \tag{8.8}$$

for any $\sigma$ such that $4s_\varepsilon \leqq \sigma \leqq \frac{\rho_\alpha}{nK_1}$. By taking $\sigma=\sigma^*$ in (8.8), we arrive at (8.3).

**Step 3**. The remaining conclusion (8.4) is essentially a corollary of (8.3) and is proved as follows. First note that

$$\begin{aligned}&\|j(u)-j(u_\star)\|_{L^{4/3}(\Omega)} \\ &\leqq \left\|j(u) - \frac{j(u)}{|u|}\right\|_{L^{4/3}(\Omega)} + \left\|\frac{j(u)}{|u|} - j(u_\star)\right\|_{L^{4/3}(\Omega)} \\ &= A_1 + A_2.\end{aligned} \tag{8.9}$$

The first term is easily estimated:

$$\begin{aligned}A_1 &\leqq \|\,|\nabla u|\,|1-|u|\,|\|_{L^{4/3}(\Omega)} \leqq \|\nabla u\|_{L^2(\Omega)}\,\|1-|u|\|_{L^4(\Omega)} \\ &\leqq E_\varepsilon(u)^{1/2}(\varepsilon^2 E_\varepsilon(u))^{1/4}.\end{aligned}$$

As for the second term, note that

$$A_2 \leqq \left\|\frac{j(u)}{|u|} - j(u_\star)\right\|_{L^{4/3}(\Omega_{\sigma^*}(\alpha))} + \left\|\frac{j(u)}{|u|} - j(u_\star)\right\|_{L^{4/3}(\cup_i B_{\sigma^*}(\alpha_i))},$$

for $\sigma^*$ as in Theorem 2. By Hölder's inequality,

$$\left\|\frac{j(u)}{|u|} - j(u_\star)\right\|_{L^{4/3}(\Omega_{\sigma^*}(\alpha))} \leqq C\left\|\frac{j(u)}{|u|} - j(u_\star)\right\|_{L^2(\Omega_{\sigma^*}(\alpha))}, \tag{8.10}$$

and the right-hand side is estimated in (8.3), so we move on by observing that

$$\left\|\frac{j(u)}{|u|} - j(u_\star)\right\|_{L^{4/3}(\cup_i B_{\sigma^*}(\alpha_i))} \leqq \left\|\frac{j(u)}{|u|}\right\|_{L^{4/3}(\cup_i B_{\sigma^*}(\alpha_i))} + \|j(u_\star)\|_{L^{4/3}(\cup_i B_{\sigma^*}(\alpha_i))}.$$

Both terms are easily handled. First, by Hölder's inequality,

$$\begin{aligned}\left\|\frac{j(u)}{|u|}\right\|_{L^{4/3}(\cup_i B_{\sigma^*}(\alpha_i))} &\leqq (\pi n\sigma^{*2})^{1/4}\left\|\frac{j(u)}{|u|}\right\|_{L^2(\cup_i B_{\sigma^*}(\alpha_i))} \\ &\leqq Cn^{1/4}\sqrt{\sigma^*}(E_\varepsilon(u))^{1/2}.\end{aligned}$$

Second, using (10.2) and the co-area formula, we compute

$$\|j(u_\star)\|_{L^{4/3}(\cup_i B_{\sigma^*}(\alpha_i))} \leqq C\left(\sum_i \int_0^{\sigma^*} \left(\frac{n}{r}\right)^{4/3} r\,\mathrm{d}r\right)^{3/4} \leqq Cn^{7/4}\sqrt{\sigma^*}.$$

We obtain (8.4) by combining (8.9), (8.10), (8.3) and the other estimates above, and then recalling the definition of $\sigma^*$.

We finish the proof by giving the lower bound for $e_\varepsilon(u)$ used above.

**Lemma 4.** *Let $\Omega \subset \mathbb{R}^2$ be bounded, open, and simply connected, with $\partial\Omega$ of class $C^1$, and let $u \in H^1(\Omega;\mathbb{C})$, $\alpha \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$ satisfy (8.2). Then there exist constants $C$ and $K_1$, depending only on $\Omega$, such that for any $\sigma \in (0, \frac{\rho_\alpha}{nK_1})$,*

$$\begin{aligned}\int_{\Omega_\sigma(\alpha)} e_\varepsilon(|u|) + \frac{1}{4}\left|\frac{j(u)}{|u|} - j(u_\star)\right|^2 \mathrm{d}x \\ \leqq \int_{\Omega_\sigma(\alpha)} [e_\varepsilon(u) - e_\varepsilon(u_\star)] + C\left(\frac{n}{\sigma}(s_\varepsilon + \varepsilon E_\varepsilon(u)) + n^4\frac{\sigma}{\rho_\alpha}\right)\end{aligned} \tag{8.11}$$

*for $u_\star = u_\star(\cdot;\alpha,d)$ as defined in (2.16).*

This lemma is used again in the next section, in the proof of Theorem 3.

**Proof. Step 1**. Assume that $u, \alpha, d$ satisfy (8.2), and let $\sigma > 0$ be such that

$$\sigma \leqq \frac{\rho_\alpha}{K_1 n} \tag{8.12}$$

for $K_1$ to be fixed below. Throughout the proof of this lemma, $C$ will denote a constant that may depend on $\Omega$ but is independent of all other parameters. We write $\Omega_\sigma$ for $\Omega_\sigma(\alpha)$ and $G$ for $G(\cdot;\alpha,d)$ as defined in (2.17).

**Step 2**. In Step 3 below we will verify that when (8.12) holds, there exists a set $\tilde{\Omega}_\sigma \subset \Omega_\sigma$ such that

$$|\Omega_\sigma \backslash \tilde{\Omega}_\sigma| \leqq C\frac{n^2\sigma^3}{\rho_\alpha}, \tag{8.13}$$

and a function $\tilde{G}_\sigma$ of the form

$$\tilde{G}_\sigma = \begin{cases} G & \text{in } \tilde{\Omega}_\sigma, \\ \text{constant} & \text{on each connected component of } \Omega\backslash\tilde{\Omega}_\sigma, \end{cases} \tag{8.14}$$

such that $\tilde{G}_\sigma \in W^{1,\infty}(\Omega)$ (in particular $\tilde{G}_\sigma$ is continuous across $\Omega \cap \partial\tilde{\Omega}_\sigma$), with

$$\tilde{\chi}\; j(u_\star) = \nabla \times \tilde{G}_\sigma. \tag{8.15}$$

Here $\tilde{\chi}$ is the characteristic function of $\tilde{\Omega}_\sigma$.

For now we assume the existence of $\tilde{\Omega}_\sigma(\alpha)$, $\tilde{G}_\sigma$ as described above, and we use them to prove that

$$\left|\int_{\Omega_\sigma} |j(u_\star)|^2 - \frac{j(u)}{|u|}\cdot j(u_\star)\,\mathrm{d}x\right| \leqq C\left[\frac{n}{\sigma}(s_\varepsilon + \varepsilon E_\varepsilon(u)) + n^4\frac{\sigma}{\rho_\alpha}\right] + \frac{1}{4}\int_{\Omega_\sigma}\left|\frac{j(u)}{|u|} - j(u_\star)\right|^2\mathrm{d}x. \tag{8.16}$$

Note that in view of (8.1), this immediately implies the conclusion of the lemma.

To prove (8.16), we write $\int_{\Omega_\sigma} |j(u_\star)|^2 - \frac{j(u)}{|u|}\cdot j(u_\star)\,\mathrm{d}x = A_1 + A_2 + A_3$, where

$$A_1 = \int_{\tilde{\Omega}_\sigma} j(u_\star)\cdot(j(u_\star) - j(u))\,\mathrm{d}x,$$

$$A_2 = \int_{\tilde{\Omega}_\sigma} j(u_\star)\cdot\frac{j(u)}{|u|}(|u|-1)\,\mathrm{d}x,$$

and

$$A_3 = \int_{\Omega_\sigma\setminus\tilde{\Omega}_\sigma} j(u_\star)\cdot\left(j(u_\star) - \frac{j(u)}{|u|}\right)\mathrm{d}x.$$

We analyze these terms in turn. First, using (8.15),

$$A_1 = \int_\Omega \nabla\times\tilde{G}_\sigma\cdot(j(u_\star) - j(u)).$$

Since $\tilde{G}_\sigma = G = 0$ on $\partial\Omega$, we can integrate by parts and use (2.16) and (2.7) to find that

$$A_1 = 2\int_\Omega \tilde{G}_\sigma\left(\pi\sum_{i=1}^n d_i\delta_{\alpha_i} - J(u)\right).$$

So

$$A_1 \leqq C\left\|\nabla\tilde{G}_\sigma\right\|_{L^\infty(\Omega)}\left\|J(u) - \pi\sum d_i\delta_{\alpha_i}\right\|_{\dot{W}^{-1,1}(\Omega)} \leqq Cs_\varepsilon\frac{n}{\sigma}$$

by (10.2), since $\|\nabla\tilde{G}_\sigma\|_{L^\infty(\Omega)} = \|j(u_\star)\|_{L^\infty(\tilde{\Omega}_\sigma)}$.

Next, since $(1-|u|) \leqq |1-|u|^2|$, Cauchy–Schwarz implies

$$A_2 \leqq \|j(u_\star)\|_{L^\infty(\tilde{\Omega}_\sigma)}\left\|\frac{j(u)}{|u|}\right\|_2 \|1-|u|^2\|_2.$$

and using (10.2) again we get an estimate of the first term, leading to

$$A_2 \leqq C\frac{n}{\sigma}\varepsilon E_\varepsilon(u). \tag{8.17}$$

Finally, (8.13) and (10.2) imply that

$$A_3 \leqq \int_{\Omega_\sigma \setminus \tilde{\Omega}_\sigma} |j(u_\star)|^2 + \frac{1}{4}\int_{\Omega_\sigma \setminus \tilde{\Omega}_\sigma} \left| j(u_\star) - \frac{j(u)}{u}\right|^2$$
$$\leqq Cn^4 \frac{\sigma}{\rho_\alpha} + \frac{1}{4}\int_{\Omega_\sigma \setminus \tilde{\Omega}_\sigma} \left| j(u_\star) - \frac{j(u)}{u}\right|^2 .$$

**Step 3**. To complete the proof, we construct the set $\tilde{\Omega}_\sigma$ used in Step 2 above. We introduce some notation: First, for $i = 1, \ldots n$ and $\sigma \leqq \rho_\alpha/2$, we define

$$\ell_i(\sigma) = \min_{|x-\alpha_i|=\sigma} G(x) \text{ if } d_i < 0, \quad \ell_i(\sigma) = \max_{|x-\alpha_i|=\sigma} G(x) \text{ if } d_i > 0. \tag{8.18}$$

We write $R := \sigma(1 + K_1 \frac{n\sigma}{\rho_\alpha})$, where $K_1$ is the constant in (8.12), and we define

$$\begin{aligned} \tilde{B}_{i,\sigma} &= \{x \in B_R(\alpha_i) : G(x) \geqq \ell_i(r)\} \quad \text{if } d_i < 0, \\ \tilde{B}_{i,\sigma} &= \{x \in B_R(\alpha_i) : G(x) \leqq \ell_i(r)\} \quad \text{if } d_i > 0. \end{aligned} \tag{8.19}$$

We fix $K_1$ to be large enough that

$$R \leqq 2\sigma \leqq \frac{\rho_\alpha}{2n} \leqq \rho_\alpha \quad \text{and} \quad \ln\left(1 + K_1 \frac{n\sigma}{\rho_\alpha}\right) > 8\frac{n\sigma}{\rho_\alpha} \quad \text{whenever (8.12) holds.} \tag{8.20}$$

In fact it is enough to take $K_1 = 40$ say. Finally we define

$$\tilde{\Omega}_\sigma(\alpha) = \Omega \setminus \left(\cup_i \tilde{B}_{i,\sigma}\right) \tag{8.21}$$

and

$$\tilde{G}_\sigma(x) := \begin{cases} G(x) & \text{if } x \in \tilde{\Omega}_\sigma \\ \ell_i(\sigma) & \text{if } x \in \tilde{B}_{i,\sigma}. \end{cases} \tag{8.22}$$

We now verify that the required properties hold. First, as a consequence of Remark 1 (which appears immediately after Lemma 10) we infer that $B_\sigma(\alpha_i) \subset \tilde{B}_{i,\sigma}$ for all $i$, and hence that $\tilde{\Omega}_\sigma \subset \Omega_\sigma$ as claimed.

Second, it is clear that

$$|\Omega_\sigma \setminus \tilde{\Omega}_\sigma| \leqq \sum_{i=1}^n |B_R(\alpha_i) \setminus B_\sigma(\alpha_i)| \leqq Cn^2 \frac{\sigma^3}{\rho_\alpha}.$$

Finally, it is obvious that (8.15) holds almost everywhere, so we only need to verify that $\tilde{G}_\sigma$ is continuous across $\Omega \cap \partial\tilde{\Omega}_\sigma$ and consequently globally Lipschitz. For concreteness, consider the case $d_i = -1$. Then it suffices to verify that $G(x) \leqq \ell_i(\sigma)$ for $x \in \partial B_R(\alpha_i)$.

We use the notation $H_i(x) = G(x) - d_i \ln|x - \alpha_i|$. Recall from (8.20) that $R \leqq \rho_\alpha$, and so (10.4) implies that $|\nabla H_i| \leqq \frac{2n}{\rho_\alpha}$ in $B_R(\alpha_i)$. Fix a point $x_0 \in \partial B_\sigma(\alpha)$ at which $G(x_0) = \ell_i(\sigma)$. For any $y \in \partial B_R(\alpha_i)$

$$G(y) = \ell_i(\sigma) + G(y) - G(x_0) = \ell_i(\sigma) + H_i(y) - H_i(x_0) + \ln\frac{\sigma}{R}.$$

Also, $|x_0 - y| \leqq R + \sigma < 4\sigma$, so

$$|H_i(y) - H_i(x_0)| \leqq 4\sigma \|\nabla H_i\|_{L^\infty(B_R(\alpha_i))} \leqq 8\sigma \frac{n}{\rho_\alpha}$$

and $\ln \frac{\sigma}{R} = -\ln(1 + K_1 \frac{n\sigma}{\rho_\alpha})$, so it follows from (8.20) that $H_i(y) < \ell_i(\sigma)$ as required. This completes the proof.

## 9. Localization

In this section we prove the localization theorem discussed in the introduction and used throughout the dynamics proof.

The analysis is a continuation of estimates of the authors in [20], and the proof relies crucially on two results from that paper. We know from Lemma 6 how to resolve a single vortex in a ball, and with error depending on the surplus energy in the ball. We also have a global estimate from Lemma 7 that controls $\|J(u)\|_{\dot W^{-1,1}(\Omega_r)}$ by the total energy over $\Omega_r$. In order to use these results, we use some techniques from Section 8 to compute bounds the energy about each vortex and bounds on the energy in $\Omega_r$.

Our main result of this section is the following:

**Theorem 3.** *Let $\Omega$ be a bounded, open, simply connected subset of $\mathbb{R}^2$ with $C^1$ boundary. Then there exists constants $C$ and $K_2$, depending on* $\operatorname{diam}(\Omega)$, *with the following property:*

*For any $u \in H^1(\Omega;\mathbb{C})$, if there exist $n \geqq 0$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$ such that*

$$\left\| J(u) - \sum_{j=1}^n \pi d_j \delta_{\alpha_j} \right\|_{\dot W^{-1,1}(\Omega)} \leqq \frac{\rho_\alpha}{8K_2 n^5}, \tag{9.1}$$

*and if in addition $E_\varepsilon(u) \geqq 1$ and*

$$\left[ \frac{n^5}{\rho_\alpha} E_\varepsilon(u) + \frac{n^{10}}{\rho_\alpha^2} \sqrt{E_\varepsilon(u)} \right] \leqq \frac{1}{\varepsilon}, \tag{9.2}$$

*then there exist $(\xi_1, \ldots, \xi_d) \in \Omega^{n*}$ such that $|\xi_i - \alpha_i| \leqq \frac{\rho_\alpha}{2K_2 n^4}$ for all $i$, and*

$$\left\| J(u) - \pi \sum_{i=1}^n d_i \delta_{\xi_i} \right\|_{\dot W^{-1,1}} \tag{9.3}$$
$$\leqq C\,\varepsilon \left[ n(C + \Sigma_\Omega^\varepsilon)^2 e^{\frac{1}{\pi}\Sigma_\Omega^\varepsilon} + (C + \Sigma_\Omega^\varepsilon)\frac{n^5}{\rho_\alpha} + E_\varepsilon(u), \right]$$

*where $\Sigma_\Omega^\varepsilon = \Sigma_\Omega^\varepsilon(u, \alpha, d) = E_\varepsilon(u) - W_\Omega^\varepsilon(\alpha, d)$.*

The first assumption (9.1) says that vortices are well-localized compared to the length-scale determined by the vortex separation $\rho_\alpha$. The second assumption (9.2) is a weak assumption that allows $n \leqq \varepsilon^{-\alpha}$, $\rho_\alpha \geqq \varepsilon^{\beta}$ for certain $\alpha, \beta > 0$.

Some of the results from [20] that we will need are stated in terms of the modified Jacobian $J'(u)$, a useful technical device that we learned about from [1]. They define

$$J'(u) = \zeta(|u|)J(u), \tag{9.4}$$

where $\zeta : [0,\infty) \to [0,\infty)$ is a smooth function with support in $[0, 1/2)$, and such that $\int_{\mathbb{R}^2} \zeta(|y|)\,\mathrm{d}y = \pi$. In other words, the two-form $J'(u)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ is the pullback by $u$ of $\zeta(|y|)\mathrm{d}y^1 \wedge \mathrm{d}y^1$. The choice of $\zeta$ implies that

$$\operatorname{supp} J'(u) \subset \left\{ x : |u(x)| < \frac{1}{2} \right\} \tag{9.5}$$

so that $J'(u)$ is more concentrated than $J(u)$. In addition, the following lemma implies that $J'(u)$ is close to $J(u)$ if $\int e_\varepsilon(u)$ is not too large.

**Lemma 5.** ([1, Lemma 3.6]) *If $\Omega$ is a bounded open subset of $\mathbb{R}^2$ and $u \in H^1(\Omega;\mathbb{C})$ then*

$$\|J'(u) - J(u)\|_{\dot{W}^{-1,1}(\Omega)} \leqq C\|\nabla u\|_{L^2(\Omega)}\,\|1-|u|^2\|_{L^2(\Omega)} \leqq C\varepsilon E_\varepsilon(u)$$

*for a constant $C$ depending only on the choice of the auxiliary function $\zeta$ appearing in the definition of $J'(u)$.*

It follows from calculations in Section 3.5 of [1] that

$$J'(u) = J(u'), \quad \text{where } u' = g(|u|)u, \\ \text{and } g(s) = \frac{1}{s}\left(\int_0^s 2\zeta(t)\,\mathrm{d}t\right)^{1/2}. \tag{9.6}$$

The first result from [20] that we will use in this section is essentially the $n = 1$ case of Theorem 3 when $\Omega$ is a ball:

**Lemma 6.** ([20, Theorem 3]) *There exists an absolute constant $C$ such that for any $0 < \varepsilon \leqq \tau$ and any $u \in H^1(U_\tau;\mathbb{C})$ satisfying*

$$\|J(u) - \pi d\delta_0\|_{\dot{W}^{-1,1}(U_\tau)} < \frac{\tau}{4} \quad \textit{with } d = \pm 1, \tag{9.7}$$

*if we write $\Sigma^\varepsilon_{U_\tau} := \Sigma^\varepsilon_{U_\tau}(u;0,1) = \int_{U_\tau} e_\varepsilon(u)\,\mathrm{d}x - (\pi \ln\frac{\tau}{\varepsilon} + \gamma)$ and*

$$\ell_\varepsilon := \varepsilon\, C(C + \Sigma^\varepsilon_{U_\tau})e^{\Sigma^\varepsilon_{U_\tau}/\pi}, \tag{9.8}$$

*then there exists a point $\xi \in U_{\tau/2}$ such that for any $\sigma < \tau - \ell_\varepsilon$,*

$$|\{s \in [\sigma,\tau] : \ u \textit{ satisfies (9.10) on } U_s\}| \geqq \tau - \sigma - \ell_\varepsilon. \tag{9.9}$$

*where the estimate referred to is*

$$\|J'(u) - \pi d\delta_\xi\|_{Lip^*(U_s)} \leqq \varepsilon C(C + \Sigma^\varepsilon_{U_\tau})^2 e^{\Sigma^\varepsilon_{U_\tau}/\pi} \ = \ \ell_\varepsilon(C + \Sigma^\varepsilon_{U_\tau}). \tag{9.10}$$

The result clearly remains true, with appropriate modifications, if $U_\tau = U_\tau(0)$ is replaced by an open ball $U_\tau(y)$ centered at an arbitrary point $y \in \mathbb{R}^2$.

The second result from [20] can be thought of as the $n = 0$ case of Theorem 3. For present purposes we only need this result when the domain is a set of the form $\Omega_\sigma(\alpha) = \Omega \setminus \cup_{i=1}^n B_\sigma(\alpha_i)$.

**Lemma 7.** ([20, Theorem 4]) *There exists an absolute constant C such that for any bounded open $\Omega \subset \mathbb{R}^2$, any $\alpha \in \Omega^n$. any $w \in H^1(\Omega; \mathbb{C})$, any $0 < \varepsilon \leqq 1/2$, and any $\tau > 0$, if we write*

$$\ell_\varepsilon := C\varepsilon \exp\left(\frac{1}{\pi}\int_{\Omega_\tau(a)} e_\varepsilon(w)\mathrm{d}x\right), \tag{9.11}$$

*then for any $\sigma > s_\varepsilon$,*

$$|\{s \in [\tau, \tau+\sigma] : \ w \text{ satisfies (9.14), (9.13) below on } \Omega_s(\alpha)\}| \geqq \sigma - s_\varepsilon. \tag{9.12}$$

*The conditions appearing in (9.12) are*

$$|u| > \frac{1}{2} \quad \text{on } \cup_i \partial B_s(\alpha_i) \tag{9.13}$$

*and*

$$\|J'(w)\|_{\dot{W}_\Gamma^{-1,1}(\Omega_s(\alpha))} \leqq \ell_\varepsilon \int_{\Omega_\tau(\alpha)} e_\varepsilon(w)\mathrm{d}x, \tag{9.14}$$

*where $W_\Gamma^{-1,1}$ is defined in (2.4) and $\Gamma = \partial\Omega$.*

*Proof of Theorem 3.* **Step 1**. We will take $K_2 = \max\{K_1, \frac{1}{4}\operatorname{diam}(\Omega)\}$. In particular this implies that $\frac{\rho_\alpha}{K_2} \leqq \frac{1}{2}$, which is used below.

We will start by using inequalities (8.7) and (8.11) from the previous section, for various choices of the parameter $\sigma$, and with $\rho_\alpha/8K_2n^5$ from (9.1) playing the role of $s_\varepsilon$ in the hypothesis (8.2) for these estimates. We will always select

$$\sigma \in \left[\frac{3}{4}\sigma_1, \sigma_1\right], \quad \sigma_1 := \frac{\rho_\alpha}{n^4K_2}.$$

(Actually, we will only need the two endpoints).

Throughout the proof we will write $\Sigma_\Omega^\varepsilon$ for the surplus energy $\Sigma_\Omega^\varepsilon(u; \alpha, d)$ on the whole domain $\Omega$, and $\Sigma_i^\varepsilon(\sigma)$ for the surplus energy $\Sigma_{U_\sigma(\alpha_i)}^\varepsilon(u; \alpha_i, d_i)$ of $u$ on a ball $U_\sigma(\alpha_i)$ about the $i$th point.

For $\sigma \in \left[\frac{3}{4}\sigma_1, \sigma_1\right]$ in this range and $s_\varepsilon$ as specified above:

$$4s_\varepsilon = \frac{\rho_\alpha}{2K_2n^5} \leqq \frac{\sigma}{n} \leqq \sigma \leqq \frac{\rho_\alpha}{nK_1}. \tag{9.15}$$

In particular the hypotheses for (8.7) and (8.11) are always satisfied. In addition, for these choices of $\sigma$, $s_\varepsilon$, the error terms in these two inequalities are always bounded by constants $C$, independent of $\varepsilon$, $n$, $\rho_\alpha$. Thus

$$\begin{aligned}
&\Sigma_\Omega^\varepsilon(u;\alpha,d)\\
&\geqq \int_{\Omega_\sigma(\alpha)} [e_\varepsilon(u) - e_\varepsilon(u_\star)]\mathrm{d}x + \sum \Sigma_i^\varepsilon(\sigma) - C \quad \text{by (8.7)}\\
&\geqq \int_{\Omega_\sigma(\alpha)} e_\varepsilon(|u|) + \frac{1}{4}\left|\frac{j(u)}{|u|} - j(u_\star)\right|^2 + \sum \Sigma_i^\varepsilon(\sigma) - C \text{ by (8.11)}.
\end{aligned} \tag{9.16}$$

**Step 2**. We will next apply Lemma 6 on each $U_{\sigma_1}(\alpha_i)$. First note that by the definition of the norms, and since $\{\alpha_1,\ldots,\alpha_n\} \cap U_{\sigma_1}(\alpha_i) = \{\alpha_i\}$,

$$\begin{aligned}
\|J(u) - \pi d_i \delta_{\alpha_i}\|_{\dot{W}^{-1,1}(U_{\sigma_1}(\alpha_i))} &\leqq \left\| J(u) - \pi \sum_{i=1}^n d_i \delta_{\alpha_i} \right\|_{\dot{W}^{-1,1}(\Omega)}\\
&\leqq \frac{\rho_\alpha}{8K_2 n^5} \leqq \frac{1}{4}\frac{\sigma_1}{n}
\end{aligned} \tag{9.17}$$

for each $i$, by (9.1) and (9.15). Thus the hypotheses of Lemma 6 are satisfied on each ball. In addition, we see from (9.16) that for each such ball

$$\Sigma_i^\varepsilon(\sigma) \leqq \Sigma_\Omega^\varepsilon + C, \quad \text{for all } i = 1,\ldots,n \quad \text{and} \quad \sigma \in \left[\frac{3}{4}\sigma_1, \sigma_1\right]$$

and so for each $i$

$$\ell_\varepsilon^i := \varepsilon C(C + \Sigma_i^\varepsilon(\sigma_1))e^{\Sigma_i^\varepsilon(\sigma_1)/\pi} \leqq \varepsilon C(C + \Sigma_\Omega^\varepsilon)e^{\Sigma_\Omega^\varepsilon/\pi} =: \ell_\Omega^\varepsilon \tag{9.18}$$

after increasing the constant $C$ as necessary. Then Lemma 6 implies that for each $i$ there exists a point $\xi_i \in U_{\sigma_1/2}(\alpha_i)$ such that

$$\left|\left\{ s \in \left[\frac{3}{4}\sigma_1, \sigma_1\right] : u \text{ does not satisfy (9.20) on } U_s(\alpha_i) \right\}\right| \leqq \ell_\Omega^\varepsilon, \tag{9.19}$$

where the estimate referred to is

$$\|J'(u) - \pi d \delta_{\xi_i}\|_{Lip^*(U_s(\alpha_i))} \leqq \varepsilon C(C + \Sigma_\Omega^\varepsilon)^2 e^{\Sigma_\Omega^\varepsilon/\pi} = \ell_\Omega^\varepsilon(C + \Sigma_\Omega^\varepsilon). \tag{9.20}$$

**Step 3**: In this step we will prove that, after taking $C$ still larger if necessary, the estimate

$$\|J'u\|_{\dot{W}_\Gamma^{-1,1}(\Omega_s(\alpha))} \leqq C(C + \Sigma_\Omega^\varepsilon)\left[\ell_\Omega^\varepsilon + \varepsilon\frac{n^5}{\rho_\alpha}\right] \tag{9.21}$$

holds for many choices of the parameter $s$. More precisely, we show that

$$\left|\left\{ s \in \left[\frac{3}{4}\sigma_1, \sigma_1\right] : (9.21) \text{ does not hold} \right\}\right| \leqq \ell_\Omega^\varepsilon. \tag{9.22}$$

The argument has three parts.

**Step 3a**. lower bound $\Sigma_i^\varepsilon \geqq -C/n$ for all $i$, using Lemma 3.

First note that from Lemma 3 and (9.17), if $\sigma \in [\frac{3}{4}\sigma_1, \sigma_1]$, then

$$\Sigma_i^\varepsilon(\sigma) \geqq -C\frac{\varepsilon}{\sigma_1}\sqrt{\ln\frac{\sigma_1}{\varepsilon}} - \frac{C}{\sigma_1}\,\|J(u) - \pi d_i \delta_{\alpha_i}\|_{\dot W^{-1,1}(U_{\sigma_1(\alpha_i)})} \geqq -\frac{C}{n}.$$

In view of (9.2), it follows that $\Sigma_i^\varepsilon(\sigma) \leqq \frac{C}{n}$ for all such $\sigma$. It then follows from (9.16) that for $\sigma$ as above,

$$\int_{\Omega_\sigma} e_\varepsilon(|u|) + \frac{1}{4}\left|\frac{j(u)}{|u|} - j(u_\star)\right|^2 \leqq C + \Sigma_\Omega^\varepsilon. \tag{9.23}$$

**Step 3b**: Let $w := u/u_\star$, where $u_\star = u_\star(\cdot;\alpha, d)$ denotes the canonical harmonic map. By (8.5) and (9.23) for $\sigma \in \left[\frac{3}{4}\sigma_1, \sigma_1\right]$,

$$\begin{aligned}\int_{\Omega_\sigma} e_\varepsilon(w) \leqq{}& 2\Sigma_\Omega^\varepsilon + C\\ &+ \int_{\Omega_\sigma} \frac{j(u)}{|u|}\cdot j(u_\star)(1-|u|) - \frac{1}{2}(1-|u|^2)|j(u_\star)|^2\,\mathrm{d}x.\end{aligned} \tag{9.24}$$

One can easily check from the definitions in (9.6) that $\left|\frac{j(u)}{|u|}\right|^2 \leqq |\nabla u|^2$, and it is clear that $|1-|u|\,| \leqq |1-|u|^2|$, so that (very much as in the proof of (8.17))

$$\int_{\Omega_\sigma} \frac{j(u)}{|u|}\cdot j(u_\star)(1-|u|) \leqq \|j(u_\star)\|_{L^\infty(\Omega_\sigma)}\varepsilon\, E_\varepsilon(u) \leqq \varepsilon C\frac{n}{\sigma}E_\varepsilon(u) \leqq C$$

when $\sigma \geqq \frac{3}{4}\sigma_1 = C\frac{\rho_\alpha}{n^4}$, in view of (9.2) and (10.2). Using the same two inequalities and the choice of $\sigma_1$, we similarly estimate

$$\begin{aligned}-\int_{\Omega_\sigma}\frac{1}{2}(1-|u|^2)|j(u_\star)|^2\,\mathrm{d}x &\leqq \varepsilon\sqrt{E_\varepsilon(u)}\||j(u_\star)|^2\|_{L^2(\Omega_\sigma)}\\ &\leqq \varepsilon\sqrt{E_\varepsilon(u)}\|j(u_\star)\|^2_{L^\infty(\Omega_\sigma)}|\Omega|^{1/2}\\ &\leqq C\varepsilon\sqrt{E_\varepsilon(u)}\frac{n^2}{\sigma^2} \leqq C.\end{aligned} \tag{9.25}$$

It follows that

$$\int_{\Omega_\sigma} e_\varepsilon(w) \leqq C + \Sigma_\Omega^\varepsilon. \tag{9.26}$$

Taking $\sigma = \frac{3}{4}\sigma_1$, we conclude from Lemma 7 that

$$\left|\left\{s \in \left[\frac{3}{4}\sigma_1, \sigma_1\right] :\ w \text{ satisfies (9.28), (9.29) below on } \Omega_\sigma(\alpha)\right\}\right| \geqq \frac{1}{4}\sigma - \ell_\Omega^\varepsilon \tag{9.27}$$

with

$$\|J'(w)\|_{\dot W_\Gamma^{-1,1}(\Omega_s(\alpha))} \leqq \varepsilon C(C + \Sigma_\Omega^\varepsilon)^2 e^{\Sigma_\Omega^\varepsilon/\pi} = C\ell_\Omega^\varepsilon(C + \Sigma_\Omega^\varepsilon) \tag{9.28}$$

and

$$|w| > \frac{1}{2} \text{ on } \cup \partial B_s(\alpha_i). \tag{9.29}$$

(Lemma 7 actually gives somewhat better estimates, but this is all that we need.)

**Step 3c**: Next we check that $J'(u)$ is close to $J'(w)$ on $\Omega_s(\alpha)$ for $s$ such that (9.29) holds. To do this we use the notation of (9.6) to write $J'(u) = J(u')$ and $J'(w) = J(w')$, with $u' = g(|u|)u$ and similarly $w' = g(|w|)w = g(|u|)w$ (since $|u| = |w|$). Then $u' = w'u_\star$, and so one can check that

$$j(u') = j(w') + |w'|^2 j(u_\star)$$

and hence that

$$J'(u) - J'(w) = \frac{1}{2}\nabla \times \left[(|w'|^2 - 1)j(u_\star)\right].$$

If $\phi \in W^{1,\infty}_\Gamma(\Omega_s)$ (and hence vanishes on $\Gamma = \partial\Omega$), then for $s$ such that $|u| = |w| > 1/2$ on $\cup\partial B_s(\alpha_i)$, the definitions imply that $|w'| = 1$ in a neighborhood of $\cup\partial B_s(\alpha_i)$, so we can integrate by parts without any contributions coming from boundary terms, to conclude that

$$\begin{aligned}\int_{\Omega_s} \phi(J'(u) - J'(w)) &= \int_{\Omega_s} \nabla \times \phi \cdot \left[(|w'|^2 - 1)j(u_\star)\right] \\ &\leqq \|\nabla\phi\|_{L^\infty(\Omega_s)} \, \|j(u_\star)\|_{L^2(\Omega_s)} \, \||w'|^2 - 1\|_{L^2(\Omega_s)}.\end{aligned}$$

From the definitions, $\left||w'|^2 - 1\right| \leqq C\left||w|^2 - 1\right|$ and so $\||w'|^2 - 1\|_{L^2(\Omega_s)}$ is controlled by (9.26). Also, $\|j(u_\star)\|_{L^2(\Omega_s)} \leqq \|j(u_\star)\|_{L^\infty(\Omega_s)} \, |\Omega|^{1/2} \leqq Cn/s$. Thus

$$\|J'(u) - J'(w)\|_{\dot{W}^{-1,1}_\Gamma(\Omega_s)} \leqq C\varepsilon(C + \Sigma^\varepsilon_\Omega)\frac{n^5}{\rho_\alpha}$$

for $s \in \left[\frac{3}{4}\sigma_1, \sigma_1\right]$ such that $|w| > \frac{1}{2}$ on $\partial\Omega_s \backslash \Gamma$. By combining this with the conclusions of Step 3a, we find that (9.21), (9.22) hold once $C$ is taken to be large enough.

**Step 4**. Now define the set

$$S := \left\{ s \in \left[\frac{3}{4}\sigma_1, \sigma_1\right] : (9.20) \text{ holds } \forall i, \text{ and } (9.21) \text{ also holds} \right\}.$$

In this step we show that if $S$ is nonempty then the points $\xi_1, \ldots, \xi_n$ found above satisfy

$$\begin{aligned}&\left\| J(u) - \pi \sum_{i=1}^n d_i \delta_{\xi_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \\ &\quad \leqq C(n+1)(C + \Sigma^\varepsilon_\Omega)\left[\ell^\varepsilon_\Omega + \varepsilon\frac{n^4}{\rho_\alpha}\right] + C\varepsilon E_\varepsilon(u)\end{aligned} \tag{9.30}$$

for a suitable constant $C$. For future use, we assume that $C \geqq 1$.

Assuming $S \neq \emptyset$, we fix some $s \in S$. Then if $\phi \in W_0^{1,\infty}(\Omega)$, the restriction of $\phi$ to $U_s(\alpha_i)$ belongs to $Lip(U_s)$, and similarly the restriction of $u$ to $\Omega_s(\alpha)$ belongs to $W_\Gamma^{1,\infty}(\Omega_s)$. If in addition $\|\nabla\phi\|_\infty \leqq 1$, then

$$\begin{aligned}
\int_\Omega \phi\left(J'(u) - \pi\sum_{i=1}^n d_i\delta_{\xi_i}\right)
&= \int_{\Omega_s} \phi J'(u) + \sum_{i=1}^n \int_{U_s(\alpha_i)} \phi(J'(u) - \pi d_i\delta_{\xi_i}) \\
&\leqq \|J'(u)\|_{\dot{W}_\Gamma^{-1,1}(\Omega_s)} + \sum_{i=1}^n \|J'(u) - \pi d_i\delta_{\xi_i}\|_{Lip^*(U_s(\alpha_i))} \\
&\leqq C(n+1)(C+\Sigma_\Omega^\varepsilon)\left[\ell_\Omega^\varepsilon + \varepsilon\frac{n^4}{\rho_\alpha}\right].
\end{aligned}$$

By taking the supremum over all such $\phi$, we find that

$$\left\| J'(u) - \pi\sum_{i=1}^n d_i\delta_{\xi_i}\right\|_{\dot{W}^{-1,1}(\Omega)}$$

is bounded by the right-hand side of the above inequality. Now (9.30) follows from the above estimate and Lemma 5.

**Step 5**. We now prove that (9.3) holds, with the constant $C$ appearing in (9.30) and the points $\xi_1, \ldots, \xi_n$ found above. To do this, note that we may assume that

$$\frac{\sigma_1}{4} = \frac{\rho_\alpha}{4K_2 n^4} \geqq C(n+1)(C+\Sigma_\Omega^\varepsilon)\left[\ell_\Omega^\varepsilon + \varepsilon\left(n\ \ln\frac{Cn}{\rho_\alpha}\right)^{1/2}\right] + C\varepsilon E_\varepsilon(u) \qquad (9.31)$$

since otherwise (9.3) (with $\xi_i = \alpha_i$ for all $i$) follows from (9.1). So in view of Step 4, it suffices to verify that the $S$ is nonempty whenever (9.31) is satisfied.

To see that $S \neq \emptyset$, note that by (9.19) and (9.22), $\left[\frac{3}{4}\sigma_1, \sigma_1\right] \setminus S$ has measure at most $(n+1)\ell_\Omega^\varepsilon$. So we must check that

$$(n+1)\ell_\Omega^\varepsilon \leqq \frac{1}{4}\sigma_1.$$

Since we have assumed that $C \geqq 1$, this follows directly from (9.31) and (9.15), and so the proof is finished.

## 10. More about the canonical harmonic map and the renormalized energy

In this section, we first give a series of lemmas concerning the canonical harmonic map and the renormalized energy. At the end of the section, we construct maps that are close to energetically optimal, for a fixed $\varepsilon$ and prescribed configuration of vortices. This construction proves in particular that one can find initial data satisfying the hypotheses of Theorem 1.

We start with a characterization of the gradient of the renormalized energy. We include the proof for the sake of completeness.

**Lemma 8.** *Let $\xi \in \Omega^{n*}$ and $d \in \{\pm 1\}$ then the canonical harmonic map $u_\star = u_\star(\cdot;\xi, d)$ and the renormalized energy $W_\Omega(\xi, d)$ satisfy*

$$\int \mathbb{J}_{kl}\partial_{x_k x_m}\eta\,(j(u_\star))_m\,(j(u_\star))_l = \sum_{j=1}^{n} d_j\partial_k\eta(\xi_j)\left(\nabla_{\xi_j}W_\Omega(\xi, d)\right)_k, \quad (10.1)$$

*where $\eta \in C^2(\Omega)$ and $\nabla^2\eta$ has support in a neighborhood of the $\xi_j$'s.*

**Proof.** This statement has been proved in [9] for periodic boundary conditions and [24] for Dirichlet boundary conditions. The proof in the case of Neumann boundary conditions follows along exactly the same lines.

Recall from (2.23) the constant $\gamma := \lim_{r\to\infty}[I(r,\varepsilon)-\pi\ln r/\varepsilon]$. The following lemma establishes the rate at which the right-hand side converges.

**Lemma 9.** ([20, Lemma 16]) $\left|\gamma - (I(r,\varepsilon) - \pi\ln\frac{r}{\varepsilon})\right| \leqq C(\frac{\varepsilon}{r})^2$.

Next we estimate the derivatives of the canonical harmonic map and renormalized energy.

**Lemma 10.** *There exists absolute constants $C$ such that for every bounded, open $\Omega \subset \mathbb{R}^2$, $a \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$, the renormalized energy $W_\Omega(a, d)$, canonical harmonic map $u_\star(\cdot, ; a, d)$ and its potential $G(\cdot; a, d)$ as defined in (2.17) satisfy*

$$\|j(u_\star)\|_{L^\infty(\Omega_r(a))} = \|\nabla G\|_{L^\infty(\Omega_r(a))} \leqq \frac{2n}{r} \quad (10.2)$$

*for all $r \leqq \rho_a$, and*

$$|\nabla_i W_\Omega(a, d)| \leqq \frac{Cn}{\rho_a}, \quad |\nabla_i\nabla_j W_\Omega(a, d)| \leqq \frac{Cn}{\rho_a^2} \quad (10.3)$$

*for every $i, j \in \{1, \ldots, n\}$. Finally, for $H_i(x) := G(x) - d_i\ln|x - a_i|$,*

$$\|\nabla H_i\|_{L^\infty(B_{\rho_\alpha}(\alpha_i))} \leqq \frac{2n}{\rho_\alpha}. \quad (10.4)$$

**Remark 1.** For every $i$, it follows from (10.4) that $G$ cannot have any critical points in $\{x : 0 < |x - \alpha_i| < \rho_\alpha/2n\}$.

**Proof.** In view of the definitions (2.19), (2.20), the conclusions all follow from the estimates

$$|\nabla_x H(x, y)| \leqq \frac{1}{\operatorname{dist}(y, \partial\Omega)}, \quad |\nabla_x^2 H(x, y)| \leqq \frac{C}{\operatorname{dist}(y, \partial\Omega)^2}.$$

By differentiating the definition (2.18) of the auxiliary function $H$, we find that

$$\begin{aligned} -\Delta_x H_{x_i}(\cdot, y) &= 0 \quad \text{in } \Omega, \\ H_{x_i}(x, y) &= -\frac{x_i - y_i}{|x - y|^2} \quad \text{for } x \in \partial\Omega,\ y \in \Omega \end{aligned} \quad (10.5)$$

for $i = 1, 2$. It follows that $-\Delta|\nabla H| \leqq 0$ in $\Omega$, and so the maximum principle implies that

$$|\nabla H(x; y)| \leqq \operatorname{dist}(y, \partial\Omega)^{-1}$$

for all $x \in \Omega$, $y \in \Omega$. A similar argument shows that $|\nabla_x^2 H(x, y)| \leqq \frac{C}{\operatorname{dist}(y,\partial\Omega)^2}$.

In order to determine rate of change of the canonical harmonic map, we have

**Lemma 11.** *Let* $\xi = (\xi_1, \dots, \xi_n)$ *and* $\xi' = (\xi'_1, \dots, \xi'_n)$ *with* $\xi, \xi' \in \Omega^{n*}$. *Let* $\Omega_r(\xi, \xi') = \Omega \setminus \left(\cup_{j=1}^n B_r(\xi_j) \cup B_r(\xi'_j)\right)$, *then for every* $d \in \{\pm 1\}^n$,

$$\left\| j(u_\star)(\xi, d) - j(u_\star)(\xi', d) \right\|_{L^\infty(\Omega_r(\xi,\xi'))} \leqq \frac{1}{r^2} \sum_{j=1}^n \left|\xi_j - \xi'_j\right| \tag{10.6}$$

*for all* $r \leqq \min\{\rho_\xi, \rho_{\xi'}\}$. *In addition, for* $1 < p < 2$,

$$\|j(u(\xi)) - j(u(\xi'))\|_{L^p(\Omega)} \leqq \left(\pi \sum |\xi_i - \xi'_i|\right)^{\frac{2}{p}-1} (2n\pi)^{2-\frac{2}{p}}. \tag{10.7}$$

**Proof.** The local Lipschitz bound (10.6) follows from the vector identity

$$\left| \frac{x-a}{|x-a|^2} - \frac{x-b}{|x-b|^2} \right|^2 = \frac{|a-b|^2}{|x-a|^2\,|x-b|^2} \tag{10.8}$$

and the maximum principle. In particular

$$j(u_\star)(\xi) = -\nabla \times G(x, \xi) = -\sum_{j=1}^n d_j \nabla \times \left(\log\left|x - \xi_j\right| + H(x, \xi_j)\right)$$

so

$$j(u_\star)(\xi) - j(u_\star)(\xi') = -\sum_{j=1}^n d_j \nabla \times \left( \log \left| \frac{x-\xi_j}{x-\xi'_j} \right| + H(x, \xi_j) - H(x, \xi'_j) \right).$$

We argue as in the proof of Lemma 10. Letting $Q^j = H(x, \xi_j) - H(x, \xi'_j)$ then $-\Delta Q^j_{x_m} = 0$ in $\Omega$ and $Q^j_{x_m} = -d_j \left[ \frac{(x-\xi_j)_m}{|x-\xi_j|^2} - \frac{(x-\xi'_j)_m}{|x-\xi'_j|^2} \right]$ on $\partial\Omega$, and since $-\Delta\left|\nabla Q^j\right| \leqq 0$, by (10.8) and the maximum principle

$$\left|\nabla Q^j\right| \leqq \frac{\left|\xi_j - \xi'_j\right|}{\operatorname{dist}(\xi_j, \partial\Omega)\operatorname{dist}(\xi'_j, \partial\Omega)}.$$

Summing over indices we find

$$\left| \sum_{j=1}^n \nabla \left( H(x, \xi_j) - H(x, \xi'_j) \right) \right| \leqq \sum_{j=1}^n \frac{\left|\xi_j - \xi'_j\right|}{\operatorname{dist}(\xi_j, \partial\Omega)\operatorname{dist}(\xi'_j, \partial\Omega)}.$$

On the other hand,

$$\left|\sum_{j=1}^{n} \nabla \log \left|\frac{x-\xi_j}{x-\xi_j'}\right|\right| \leqq \sum_{j=1}^{n} \left|\frac{x-\xi_j}{\left|x-\xi_j\right|^2} - \frac{x-\xi_j'}{\left|x-\xi_j'\right|^2}\right| \leqq \sum_{j=1}^{n} \frac{\left|\xi_j-\xi_j'\right|}{\left|x-\xi_j\right|\left|x-\xi_j'\right|}$$

also follows from (10.8). Combining both bounds yields (10.6).

To prove (10.7), note that for $1 < p < 2$

$$\begin{aligned}
\left\| j(u(\xi)) - j(u(\xi')) \right\|_{L^p(\Omega)} &= \left\| \nabla \times (G(\xi) - G(\xi')) \right\|_{L^p(\Omega)} \\
&\leqq C \left\| G(\xi) - G(\xi') \right\|_{W^{1,p}(\Omega)} \\
&\leqq C \left\| \Delta(G(\xi) - G(\xi')) \right\|_{W^{-1,p}(\Omega)} \\
&= C \left\| \pi \sum d_i (\delta_{\xi_i} - \delta_{\xi_i'}) \right\|_{W^{-1,p}(\Omega)} \\
&\leqq C \left\| \pi \sum d_i (\delta_{\xi_i} - \delta_{\xi_i'}) \right\|_{C_0^{0,\frac{2}{p}-1}(\Omega)^*}.
\end{aligned}$$

The last line is the dual of the standard Sobolev embedding theorem. We use the interpolation inequality

$$\|\mu\|_{C_0^{0,\alpha}(\Omega)^*} \leqq (\|\mu\|_{C_0^{0,1}(\Omega)^*})^{\alpha} (\|\mu\|_{C_0(\Omega)^*})^{1-\alpha}$$

(see [18, Lemma 3.3]) together with (2.6), to deduce (10.7).

We next estimate the rate of convergence of the limit used in (2.20) to define the renormalized energy.

**Lemma 12.** *There exists an absolute constant $C$ such that*

$$W_\Omega(\alpha, d) + n\pi \ln \frac{1}{r} - \int_{\Omega_r(\alpha)} \frac{1}{2} |\nabla u_\star|^2 \, \mathrm{d}x \;\leqq\; C n^3 \left(\frac{r}{\rho_\alpha}\right)^2 \tag{10.9}$$

*for all bounded, open $\Omega \subset \mathbb{R}^2$, all $n \geqq 1$, $\alpha \in \Omega^{n*}$, $d \in \{\pm 1\}^n$, and $r < \rho_\alpha$.*

**Proof.** We will write $\Omega_r$ as shorthand for $\Omega_r(\alpha)$. We define $H_i$ as in Lemma 10 and compute

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_r} |\nabla u_\star|^2 \, \mathrm{d}x &= \frac{1}{2}\int_{\Omega_r} |\nabla G|^2 \, \mathrm{d}x \\
&= \frac{1}{2}\int_{\partial\Omega_r} \frac{1}{2} G \nabla_\nu G \, \mathrm{d}\mathcal{H}^1 \\
&= -\frac{1}{2}\sum_i \left[ \int_{\partial B_r(\alpha_i)} (H_i + d_i \ln r) \left(\nabla_\nu H_i + \frac{d_i}{r}\right) \mathrm{d}\mathcal{H}^1 \right].
\end{aligned}$$

The sign changes because the outward normal to $\partial\Omega_r$ is the inward normal to $\partial B_r$ and vice versa. Using the mean value property of harmonic functions and integrating by parts again in the terms involving $H \nabla_\nu H$, we get

$$\frac{1}{2}\int_{\Omega_r} |\nabla u_\star|^2 \, \mathrm{d}x = \sum_i \pi \left( d_i^2 \ln \frac{1}{r} - d_i \, H_i(\alpha_i) \right) - \frac{1}{2} \int_{B_r(\alpha_i)} |\nabla H_i|^2 \, \mathrm{d}x.$$

When $|d_i| = 1$ for all $i$, (2.20) implies that $W_\Omega(\alpha, d) = -\pi \sum d_i H_i(\alpha_i)$, and so

$$W_\Omega(\alpha, d) + n\pi \ln \frac{1}{r} - \int_{\Omega_r} \frac{1}{2}|\nabla u_\star|^2 \, dx = \frac{1}{2}\sum_i \int_{B_r(\alpha_i)} |\nabla H_i|^2 \, dx \tag{10.10}$$
$$\leqq Cn^3 \left(\frac{r}{\rho_\alpha}\right)^2$$

by (10.4).

The next lemma gives a very crude estimate of the how the renormalized energy scales with the number $n$ of vortices.

**Lemma 13.** *For a smooth, bounded domain $\Omega \subset \mathbb{R}^n$, if $a \in \Omega^{n^*}$ and $d \in \{\pm 1\}^n$, then*

$$W_\Omega(a, d) \leqq C\left(n^3 + \frac{n^2}{\rho_a}\right). \tag{10.11}$$

**Proof.** Let $u_\star = u_\star(\cdot, a, d)$. For $r < \rho_a$ we compute

$$\int_{\Omega_r} |\nabla u_\star|^2 = \int_{\Omega_{\rho_a}} |\nabla u_\star|^2 + \sum_{i=1}^n \int_{B_{\rho_a} \setminus B_r(a_i)} \left|\nabla H_i + d_i \frac{x - a_i}{|x - a_i|^2}\right|^2 dx$$

using the notation of Lemma 10. From (10.2), we estimate

$$\int_{\Omega_{\rho_a}} |\nabla u_\star|^2 \, dx \leqq \frac{Cn^2}{\rho_a^2}.$$

Next, since $H_i$ is harmonic in $B_{\rho_a}(a_i)$,

$$\int_{B_{\rho_a} \setminus B_r(a_i)} \nabla H_i \cdot \frac{x - a_i}{|x - a_i|^2} dx = \int_r^{\rho_a} \frac{1}{s} \int_{\partial B_s(a_i)} \nu \cdot \nabla H_i \, \mathcal{H}^1(dx) \, ds = 0,$$

so using (10.4), we check that for $i = 1, \dots, n$,

$$\int_{B_{\rho_a} \setminus B_r(a_i)} \left|\nabla H_i + d_i \frac{x - a_i}{|x - a_i|^2}\right|^2 dx = \int_{B_{\rho_a} \setminus B_r(a_i)} |\nabla H_i|^2 + \left|\frac{x - a_i}{|x - a_i|^2}\right|^2 dx$$
$$\leqq Cn^2 + 2\pi \ln \frac{\rho_a}{r}.$$

Combining these estimates and recalling the characterization of $W_\Omega(a, d)$ in (2.20), we deduce (10.11).

We conclude this section by constructing maps that are close to energetically optimal for fixed $\varepsilon$ and configuration of vortices $a, d$. Recall the definition

$$I(r, \varepsilon) = \inf \left\{ \int_{U_r} e_\varepsilon(u) \; ; \; u \in H^1(B_r; \mathbb{C}), \; u = e^{i\theta} \text{ on } \partial B_r \right\}.$$

It is known that the infimum on the right-hand side of the above definition is attained, and moreover the minimizer $u_{\varepsilon,r}$ has the form

$$u_{\varepsilon,r}(x) = f_{\varepsilon,r}(|x|)\frac{x}{r} \tag{10.12}$$

for an increasing function $f_{\varepsilon,r} : [0,\infty) \to [0,1]$ such that $f_{\varepsilon,r}(0) = 0$ and $f_{\varepsilon,r}(r) = 1$. One can easily check that $f_{\varepsilon,r}(s) = f_{\lambda\varepsilon,\lambda r}(\lambda s)$ for all $\lambda > 0$, and hence $I(r,\varepsilon) = I(r/\varepsilon, 1)$ for all $r, \varepsilon$.

We will use the notation

$$u_\star^{r,\varepsilon}(x; a, d) = u_\star(x; a, d) \prod_{i=1}^{d} f_{\varepsilon,r}(|x - a_i|). \tag{10.13}$$

For $r \lesssim \rho_a$, this yields a map with vortex configuration $a, d$ and with nearly optimal energy. We will usually write simply $u_\star^{r,\varepsilon}$ when no confusion can result.

**Lemma 14.** *For any $a \in \Omega^{n*}$ and $d \in \{\pm 1\}^n$ and for $r \leqq \rho_a$, the map $u_\star^{r,\varepsilon}(\cdot; a, d)$ constructed above satisfies*

$$\int_\Omega e_\varepsilon(u_\star^{r,\varepsilon})\,\mathrm{d}x \leqq W_\Omega^\varepsilon(a, d) + Cn\left(\frac{\varepsilon}{r}\right)^2 \tag{10.14}$$

*and*

$$\left\| J(u_\star^{r,\varepsilon}) - \pi \sum_{i=1}^{n} d_i \delta_{a_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \leqq Cn\varepsilon \left(1 + \varepsilon \frac{n^3}{\rho_a^2}\right). \tag{10.15}$$

The proof will show that

$$\left\| J(u_\star^{r,\varepsilon}) - \pi \sum_{i=1}^{n} d_i \delta_{a_i} \right\|_{\dot{W}^{-1,1}(\Omega)} \leqq Cn\varepsilon \tag{10.16}$$

if $r \leqq \rho_a / cn$ for a suitable constant $c$. Throughout the body of this paper we refer to a function $u_\star^\varepsilon(a, d)$. We define

$$u_\star^\varepsilon(a, d) := u_\star^{r_\star,\varepsilon}(a, d) \quad \text{for } r_\star := \frac{\rho_a}{cn},$$

so that in particular (10.16) holds for $u_\star^\varepsilon(a, d)$.

**Proof.** 1. To prove (10.14), note that

$$\int_{\Omega} e_\varepsilon(u_\star^{r,\varepsilon})\,dx \;=\; \int_{\Omega_r(a)} \frac{1}{2}|\nabla u_\star|^2\,dx \;+\; \sum_i \int_{B_r(a_i)} e_\varepsilon(u_\star^{r,\varepsilon})\,dx. \quad (10.17)$$

In $B_r(a_i) = B_r$, $u_\star^{r,\varepsilon} = f_{\varepsilon,r}(|x-a_i|)u_\star(x)$ and so we compute that

$$e_\varepsilon(u_\star^{r,\varepsilon}) \;=\; \frac{1}{2}(|\nabla f_{\varepsilon,r}|^2 + f_{\varepsilon,r}^2|\nabla u_\star|^2) + \frac{1}{4\varepsilon^2}(f_{\varepsilon,r}^2 - 1)^2.$$

Writing $f_{\varepsilon,r}^2|\nabla u_\star|^2 = f_{\varepsilon,r}^2|\nabla H_i + d_i \frac{x-a_i}{|x-a_i|^2}|^2$ as in the proof of Lemma 13, we find as before that the cross-terms integrate to 0, using the radial symmetry of $f_{\varepsilon,r}$. Thus

$$\begin{aligned}\int_{B_r(a_i)} e_\varepsilon(u_\star^{r,\varepsilon}) = \int_{B_r(a_i)} \left[ e_\varepsilon(f_{\varepsilon,r}) + \frac{2f_{\varepsilon,r}^2}{|x-a_i|^2}\right] dx \\ + \int_{B_r(a_i)} \frac{1}{2} f_{\varepsilon,r}^2 |\nabla H_i|^2\,dx.\end{aligned} \quad (10.18)$$

The first integral on the right-hand side is exactly $I(r,\varepsilon)$, by the definition of $f_{\varepsilon,r}$. So combining (10.17) with (10.18) and recalling (10.10), we deduce that

$$\int_{\Omega} e_\varepsilon(u_\star^{r,\varepsilon})\,dx = W_\Omega(a,d) + n\left(\pi \ln\frac{1}{r} + I(r,\varepsilon)\right) + \frac{1}{2}\sum_i \int_{B_r(a_i)} (f_{\varepsilon,r}^2 - 1)|\nabla H_i|^2.$$

The integrals on the right-hand side are all negative, and by using (9) we find that

$$\int_{\Omega} e_\varepsilon(u_\star^{r,\varepsilon})\,dx \;\leqq\; W_\Omega^\varepsilon(a,d) + O\left(n\left(\frac{\varepsilon}{r}\right)^2\right).$$

2. Because $J(u_\star^{r,\varepsilon}) = 0$ in $\Omega_r(a)$, (10.15) will follow once we check that

$$\|J(u_\star^{r,\varepsilon}) - \pi d_i \delta_{a_i}\|_{Lip^*(B_r(a_i))} \leqq Cn\varepsilon\left(1 + \varepsilon\frac{n^3}{\rho_a^2}\right), \quad i = 1,\dots,n. \quad (10.19)$$

We assume for convenience that $a_i = 0$ and that $d_i = 1$. We also write $f$ instead of $f_{\varepsilon,r}$. Using (2.10) and the definition of the potential $G$ associated with $u_\star$, we see that

$$J(u_\star^{r,\varepsilon})(x) = f(|x|)f'(|x|)\frac{x}{|x|} \times j(u_\star) = \left(\frac{f^2}{2}\right)' \frac{x}{|x|}\cdot\nabla G.$$

Writing $G = \ln|x| + H_i(x)$ in $B_r(a_i) = B_i(0)$, as in Lemma 10, the above becomes

$$J(u_\star^{r,\varepsilon})(x) = \left(\frac{f^2}{2}\right)'\left(\frac{1}{|x|} + \frac{x}{|x|}\cdot\nabla H_i\right). \quad (10.20)$$

It then follows from (10.4) that $J(u_\star^{r,\varepsilon}) > 0$ in $B_{\rho/Cn}(0)$. Various arguments show that

$$\int_{B_r} J(u_\star^{r,\varepsilon})(x) \;=\; \pi.$$

For example, this follows from (10.20) and integration by parts. Thus if $Lip(\phi) \leqq 1$ then

$$\begin{aligned}
&\int_{B_r} \phi[J(u_\star^{r,\varepsilon}) - \pi\delta_0] \\
&\quad = \int_{B_r} (\phi(x) - \phi(0)) J(u_\star^{r,\varepsilon})(x)\, \mathrm{d}x \\
&\quad \leqq \int_{B_r} |x|\, |J(u_\star^{r,\varepsilon})(x)|\, \mathrm{d}x \\
&\quad = \int_{B_{\rho_a/Cn}} |x| J(u_\star^{r,\varepsilon})(x)\, \mathrm{d}x + \int_{B_r \setminus B_{\rho_a/Cn}} |x|\, |J(u_\star^{r,\varepsilon})(x)|\, \mathrm{d}x.
\end{aligned}$$

Again using (10.20) and arguing as in the proof of Lemma 13, one can check that

$$\int_{B_{\rho_a/Cn}} |x| J(u_\star^{r,\varepsilon})(x)\, \mathrm{d}x = \int_{B_{\rho_a/Cn}} \left(\frac{f^2}{2}\right)' \mathrm{d}x = \pi \int_0^{\rho_a/Cn} s(f^2)'(s)\mathrm{d}s,$$

where $f = f_{\varepsilon,r}$. After integrating by parts and using the fact that $f_{\varepsilon,r} \geqq f_{\varepsilon,\infty} \geqq \max\{0, 1 - (C\varepsilon/s)^2\}$, a short calculation shows that

$$\pi \int_0^{\rho_a/Cn} s(f^2)'(s)\mathrm{d}s \leqq C\varepsilon.$$

Finally, from (10.20) and (10.4) it is easy to see that $|J(u_\star^{r,\varepsilon})| \leqq \frac{1}{2}(f^2)' \frac{Cn}{\rho_a}$ in $B_r \setminus B_{\rho_a}$, and from this one can check (using again $f \geqq \max\{0, 1 - (C\varepsilon/s)^2\}$) that

$$\int_{B_r \setminus B_{\rho_a/Cn}} |x|\, |J(u_\star^{r,\varepsilon})(x)|\, \mathrm{d}x \ \leqq \left(\frac{Cn}{\rho_a}\right)^3 \varepsilon^2 r \ \leqq \frac{Cn^3}{\rho_a^2}\varepsilon^2.$$

Combining the above inequalities, we arrive at (10.19)

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## References

1. Alberti, G., Baldo, C., Orlandi, G.: Functions with prescribed singularities. *J. Eur. Math. Soc*. **5**, 275–311 (2003)
2. Alikakos, N.D., Bronsard, L., Fusco, G.: Slow motion in the gradient theory of phase transitions via energy and spectrum. *Calc. Var. Partial Differ. Equ.* **6**, 39–66 (1998)
3. Alikakos, N., Fusco, G.: Slow dynamics for the Cahn–Hilliard equation in higher space dimensions: the motion of bubbles. *Arch. Rational Mech. Anal.* **141**, 1–61 (1998)
4. Bethuel, F., Brezis, H., Helein, F.: *Ginzburg–Landau Vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser, Boston, 1994
5. Bethuel, F., Orlandi, G., Smets, D.: Collisions and phase-vortex interactions in dissipative Ginzburg–Landau dynamics. *Duke Math. J.* **130**, 523–614 (2005)

6. BREZIS, H., CORON, J.-M., LIEB, E.: Harmonic maps with defects. *Comm. Math. Phys.* **107**, 649–705 (1986)
7. BUSLAEV, V.S., PERELMAN, G.S.: Scattering for the nonlinear Schrödinger equation: states close to a soliton. *St. Petersburg Math. J.* **4**, 1111–1142 (2003)
8. BUSLAEV, V.S., SULEM, C.: On asymptotic stability of solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20**, 419–475 (2003)
9. COLLIANDER, J.E., JERRARD, R.L.: Vortex dynamics for the Ginzburg–Landau–Schrödinger equation. *Internat. Math. Res. Notices* **1998**, 333–358 (1998)
10. COLLIANDER, J.E., JERRARD, R.L.: Ginzburg–Landau vortices: weak stability and Schrödinger equation dynamics. *J. Anal. Math.* **77**, 129–205 (1999)
11. DE MOTTONI, P., SCHATZMAN, M.: Geometrical evolution of developed interfaces. *Trans. Am. Math. Soc.* **347**, 1533–1589 (1995)
12. DEL PINO, M., FELMER, P., KOWALCZYK, M.: Minimality and nondegeneracy of degree-one Ginzburg–Landau vortex as a Hardy's type inequality. *Int. Math. Res. Not.* **2004**, 1511–1527 (2004)
13. E, W.: Dynamics of vortices in Ginzburg–Landau theories with applications to superconductivity. *Phys. D* **77**, 383–404 (1994)
14. FETTER, A.L.: Vortices in an imperfect Bose Gas. IV. Translational Velocity. *Phys. Rev.* **151**, 100–104 (1966)
15. GUSTAFSON, S., SIGAL, I.M.: The stability of magnetic vortices. *Comm. Math. Phys.* **212**, 257–275 (2000)
16. GUSTAFSON, S., SIGAL, I.M.: Effective dynamics of magnetic vortices. *Adv. Math.* **199**, 448–498 (2006)
17. JERRARD, R.L.: Vortex dynamics for the Ginzburg–Landau wave equation. *Calc. Var. Partial Differ. Equ.* **9**, 683–688 (1999)
18. JERRARD, R.L., SONER, H.M.: The Jacobian and the Ginzburg–Landau energy. *Calc. Var. Partial Differ. Equ.* **14**, 151–191 (2002)
19. JERRARD, R.L., SONER, H.M.: Dynamics of Ginzburg–Landau vortices. *Arch. Rational Mech. Anal.* **142**, 99–125 (1998)
20. JERRARD, R.L., SPIRN, D.: Refined Jacobian estimates for Ginzburg–Landau functionals. *Indiana Univ. Math. J.* **56**, 135–186 (2007)
21. KLYATSKIN, V.I.: Sound radiation by a system of vortices. *Izv. AN SSSR Mekhanika Zhidkosti i Gza* **1**, 87–92 (1966)
22. LIN, F.-H.: Some dynamical properties of Ginzburg–Landau vortices. *Comm. Pure Appl. Math.* **49**, 323–359 (1996)
23. LIN, F.-H.: Vortex dynamics for the nonlinear wave equation. *Comm. Pure Appl. Math.***52**, 737–761 (1999)
24. LIN, F.-H., XIN, J.: On the Incompressible Fluid Limit and the Vortex Motion Law of the Nonlinear Schrödinger Equation. *Comm. Math. Phys.* **200**, 249–274 (1999)
25. MARCHIORO, C., PULVIRENTI, M.: *Mathematical Theory of Incompressible Nonviscous Fluids*. Appl. Math. Sci., 96. Springer, New York, 1994
26. MARTEL, Y., MERLE, F., TSAI, T.-P.: Stability and asymptotic stability in the energy space of the sum of $N$ solitons for subcritical gKdV equations. *Comm. Math. Phys.* **231**, 347–373 (2002)
27. NEU, J.: Vortices in the complex scalar fields. *Phys. D* **43**, 385–406 (1990)
28. OVCHINNIKOV, YU.N., SIGAL, I.M.: Long-time behaviour of Ginzburg–Landau vortices. *Nonlinearity* **11**, 1295–1309 (1998)
29. ROITBERG, Y.: *Elliptic Boundary Value Problems in the Space of Distributions*. Mathematics and its Applications, vol. 498. Kluwer, Dordrecht, 1999
30. SANDIER, E., SERFATY, S.: Gamma-convergence of gradient flows with applications to Ginzburg–Landau. *Comm. Pure Appl. Math.* **57**, 1627–1672 (2004)
31. SANDIER, E., SERFATY, S.: *Vortices in the magnetic Ginzburg–Landau model.* Progress in Nonlinear Differential Equations and their Applications, vol. 70. Birkhäuser, Boston, 2007

32. Shafrir, I.: Remarks on solutions of $-\Delta u = (1 - |u|^2)u$ in $R^2$. *C. R. Acad. Sci. Paris S^r. I Math.* **318**, 327–331 (1994)
33. Soffer, A., Weinstein, M.I.: Multichannel nonlinear scattering for non-integrable equations. *Comm. Math. Phys.* **133**, 119–146 (1990)
34. Spirn, D.: Vortex motion law for the Schrödinger–Ginzburg–Landau equations. *SIAM J. Math. Anal.* **34**, 1435–1476 (2003)
35. Stuart, D.: Dynamics of Abelian Higgs vortices in the near Bogomolny regime. *Comm. Math. Phys.* **159**, 51–91 (1994)
36. Tsai, T.-P., Yau, H.-T.: Relaxation of excited states in nonlinear Schrödinger equations. *Int. Math. Res. Notices* **2002**, 1629–1673 (2002)

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