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Least Supersolution Approach to Regularizing Free Boundary Problems

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Abstract

In this paper, we study a free boundary problem obtained as a limit as $\varepsilon \to 0$ to the following regularizing family of semilinear equations $\Delta u = \beta_\varepsilon(u)F(\nabla u)$, where β_ε approximates the Dirac delta in the origin and F is a Lipschitz function bounded away from 0 and infinity. The least supersolution approach is used to construct solutions satisfying geometric properties of the level surfaces that are uniform in ε. This allows to prove that the free boundary of a limit has the "right" weak geometry, in the measure theoretical sense. By the construction of some barriers with curvature, the classification of global profiles of the blow-up analysis is carried out and the limit functions are proven to be viscosity and pointwise solution ($\mathcal{H}^{n-1}$ almost everywhere) to a free boundary problem. Finally, the free boundary is proven to be a $C^{1,\alpha}$ surface around $\mathcal{H}^{N-1}$ almost everywhere point.

1. Introduction

Regularizing methods in free boundary problems are models for a wide spectrum of problems in nature. They are of particular interest in the theory of flame propagation to describe laminar flames as an asymptotic limit for high energy activation. These methods go back to Zeldovich and Frank-Kamenetski [30]. However, the rigorous mathematical study was postponed until recently with the pioneering works of Berestycki et al. [3] and of Caffarelli and Vazquez [15].

In the last decade, some attention has been given to the study of the limit as $\varepsilon \to 0$ of solutions to the elliptic equation

$$\Delta u = \beta_\varepsilon(u) \tag{1.1}$$

where $\beta_\varepsilon(s) = 1/\varepsilon\beta(s/\varepsilon)$ and β is a Lipschitz continuous function, with $\beta > 0$ in $(0, 1)$, $\operatorname{supp}(\beta) = [0, 1]$, and $\int \beta = M > 0$. It is known from a series of important papers of Caffarelli et al. [13,14] and Lederman and Wolanski [24] that, under certain geometric conditions about the limit function u_0 and its free boundary, it is a viscosity solution of the following free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \partial\{u > 0\} \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 2M & \text{on } \Omega \cap \partial\{u > 0\} \end{cases} \tag{1.2}$$

and the free boundary is locally a $C^{1,\alpha}$ surface. These assumptions are necessary if one intends to obtain further regularity results since there are limits which do not satisfy the free boundary condition in the classical sense in any portion of the free boundary [13, Remark 5.1].

Recently, CAFFARELLI et al. [12] proved some new monotonicity results so that it applies to inhomogeneous equations in which the right-hand side of the equation does not need to vanish on the free boundary. The new versions of the monotonicity theorem led to some existence and regularity results to the Prandtl–Batchelor equation. In connection with these results, a uniform Lipschitz estimates for solutions to a family of semilinear equations was proven. These regularizing approximations generalize the type of elliptic equations in (1.1) and they are the object of study of this paper. More concretely, we study the limit free boundary problem arising from passing the limit as $\varepsilon \to 0$ of the following family of semilinear equations

$$\Delta u = \beta_\varepsilon(u) F(\nabla u) \tag{1.3}$$

Here, F is a Lipschitz continuous function bounded away from 0 and infinity.

The strategy used in this paper is the following. We use the least supersolution approach to construct solutions u_ε, which are more "stable" from the geometric viewpoint. This is done for equations more general than (1.3) and also allows us to obtain a limit function with some geometric properties and its free boundary having some "weak" geometry. We then move to study the limit problem. The key part here is the classification of global profiles (2-plane functions) of the blow-up analysis. We remark however that, the typical integration by parts method developed in [13] and extensively used in similar problems does not seem to work for this case. Here, the classification depends upon a delicate construction of barriers with some uniform control on the curvature of their free boundaries as well as the asymptotic behavior of their slopes. Finally, limits of the least supersolutions are proven to be a viscosity and pointwise ($\mathcal{H}^{N-1}$) almost everywhere solution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \partial\{u > 0\} \\ H_\nu(u_\nu^+) - H_\nu(u_\nu^-) = M & \text{on } \Omega \cap \partial\{u > 0\} \end{cases} \tag{1.4}$$

with $H_\nu(t) = \int_0^t \frac{s}{F(s\nu)}\, ds$, and the free boundary $\Omega \cap \partial\{u > 0\}$ to be a $C^{1,\alpha}$ surface around $\mathcal{H}^{n-1}$ almost everywhere point.

In this case, the free boundary condition

$$H_\nu(u_\nu^+) - H_\nu(u_\nu^-) = M \text{ on } F(u)$$

also depends on the normal direction to the free boundary. This type of free boundary conditions appear as a limit of homogenization problems in periodic media. For homogenization free boundary problems, we refer to [10,11].

This paper is based on the author's Ph.D. dissertation [26].

Let us first introduce some notation that will be used throughout this paper

- N: dimension of the Euclidean space
- Ω: open, bounded, connected set of $\mathbb{R}^N$
- $|S|$: N-dimensional Lebesgue measure of the set S
- $\mathcal{H}^{N-1}$: $(N-1)$-dimensional Hausdorff measure
- $\mathcal{N}_\delta(E) := \{x \in \mathbb{R}^N : \mathrm{dist}(x, E) < \delta\}$, $E \subset \mathbb{R}^N$
- $B_r(x_0)$ open ball centered at x_0 and radius r
- $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$
- $\displaystyle ⨍_{B_r(x_0)} u(x)\,\mathrm{d}x = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(x)\,\mathrm{d}x$
- $\displaystyle ⨍_{\partial B_r(x_0)} u(x)\,\mathrm{d}\mathcal{H}^{N-1} = \frac{1}{\mathcal{H}^{N-1}(\partial B_r(x_0))} \int_{\partial B_r(x_0)} u(x)\,\mathrm{d}\mathcal{H}^{N-1}.$

2. Existence, continuity, regularity theory of the least supersolution

In this section we will consider the following ε-regularized equations

$$\Delta u = F_\varepsilon(u, \nabla u) \ \text{ in } \Omega, \tag{E_ε}$$

where $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain and $\{F_\varepsilon\}_{\varepsilon>0}$ is under the following structural conditions:

$$F_\varepsilon \in C(\mathbb{R} \times \mathbb{R}^N) \tag{2.1}$$

$$0 \leqq F_\varepsilon(z, p) \leqq \frac{A}{\varepsilon} \chi_{\{0<z<\varepsilon\}} \text{ in } \mathbb{R} \times \mathbb{R}^N, \qquad A > 0. \tag{2.2}$$

Since our goal is the study of the free boundary of a limit configuration as $\varepsilon \to 0$, we will be interested to investigate geometric properties of some level sets of u_ε. For this reason, we should choose in some sense, more "stable" solutions u_ε to deal with. This was the approach in [27], where solutions were chosen to be the minimizers of the corresponding functional associated to the ε-perturbed equations. In this case, due to the lack of variational characterization for solutions of E_ε, we will consider the least viscosity supersolution of the equation above. This will be accomplished by Perron's method.

Let φ be in $C(\partial\Omega)$ and let us define,

$$\mathcal{S}_\varphi^\varepsilon = \mathcal{S}_\varepsilon := \left\{ w \in C(\overline{\Omega}),\, w \text{ viscosity supersolution of } E_\varepsilon;\, w \geqq \varphi \text{ on } \partial\Omega \right\}.$$

Clearly, $\mathcal{S}_\varepsilon \neq \emptyset$ since $h_\varphi \in \mathcal{S}_\varepsilon$, where h_φ is the harmonic function in Ω such that $h = \varphi$ on $\partial\Omega$. Besides, there is also a natural barrier from below for the functions in the set $\mathcal{S}_\varepsilon$. Indeed, if for each $\varepsilon > 0$, we define

$$L_\varepsilon := \sup_{(z,p)\in(0,\varepsilon)\times\mathbb{R}^N} F_\varepsilon(z, p) < +\infty$$

and let Ψ_ε be the unique solution to

$$\begin{cases} \Delta\Psi = L_\varepsilon & \text{in } \Omega \\ \Psi = \varphi & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

by the maximum principle, we have

$$\mathcal{S}_\varepsilon = \left\{ w \in C(\overline{\Omega}),\ w \text{ viscosity supersolution of } E_\varepsilon;\ \ w \geqq \Psi_\varepsilon \text{ in } \overline{\Omega} \right\}.$$

We define the function

$$u_\varepsilon(x) := \inf_{w \in \mathcal{S}_\varepsilon} w(x). \tag{2.4}$$

It will be called the least supersolution of the equation E_ε. From the discussion above, there exist natural barriers for u_ε, namely, $\Psi_\varepsilon \leqq u_\varepsilon \leqq h_\varphi$ in $\overline{\Omega}$.

Remark 2.1. It worth noting that, in general, comparison principle for supersolutions and subsolutions of equation E_ε is not available. In this way, uniqueness of solutions is not expected to hold.

Remark 2.2. We recall some definitions that are going to be used in the next theorem. If $u : \Omega \to \mathbb{R}$ is locally bounded, we define

$$u^*(x) = \inf \left\{ v(x) \mid v \in \mathrm{USC}(\Omega) \text{ and } v \geqq u \text{ in } \Omega \right\},$$
$$u_*(x) = \sup \left\{ v(x) \mid v \in \mathrm{LSC}(\Omega) \text{ and } v \leqq u \text{ in } \Omega \right\}.$$

Clearly, $u^* \in \mathrm{USC}(\Omega)$, $u_* \in \mathrm{LSC}(\Omega)$, and $u_* \leqq u \leqq u^*$. Besides, we have

$$u^*(x) = \lim_{r \searrow 0} \sup \{u(y) \mid y \in \Omega \cap B_r(x)\}$$
$$u_*(x) = \lim_{r \searrow 0} \inf \{u(y) \mid y \in \Omega \cap B_r(x)\}$$

The functions u^*, u_* are called the upper semicontinuous envelope and lower semicontinuous envelope of u, respectively.

Theorem 2.3. *For each $\varepsilon > 0$, the least supersolution to equation E_ε, u_ε, belongs to $C(\overline{\Omega}) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ for any $0 < \alpha < 1$ and any $1 \leqq p < \infty$. It is also a viscosity solution to E_ε. Besides, u_ε is a strong solution to E_ε and it assumes the boundary values φ continuously, that is,*

$$\begin{cases} \Delta u_\varepsilon = F_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) & \textit{almost everywhere in } \ \Omega \\ u_\varepsilon = \varphi & \textit{on } \ \partial\Omega \end{cases} \tag{2.5}$$

In particular, $u_\varepsilon \in \mathcal{S}_\varepsilon$.

Proof. Let us observe first that $u_\varepsilon = (u_\varepsilon)^*$. It follows from Perron's method developed by ISHII [21] that u_ε is a viscosity subsolution and $(u_\varepsilon)_*$ is a viscosity supersolution of E_ε. Since $\Delta u_\varepsilon \geqq 0$ in the viscosity sense and u_ε is upper semicontinuous, from the uniqueness of the subharmonic upper semicontinuous representative [23, Theorem 9.3], we conclude

$$u_\varepsilon(x) = \lim_{r \to 0} \fint_{B_r(x)} u_\varepsilon(y)\,\mathrm{d}y. \tag{2.6}$$

Moreover, for any $w \in \mathcal{S}_\varepsilon$, $\Delta w \leqq L_\varepsilon$ in the viscosity sense. In particular,

$$\Delta(w - \Psi_\varepsilon) \leqq 0 \text{ in } \mathcal{D}^{'}(\Omega)$$

which implies, by the average characterization of superharmonicity,

$$\Delta(u_\varepsilon - \Psi_\varepsilon) \leqq 0 \text{ in } \mathcal{D}^{'}(\Omega).$$

Again, from superharmonicity theory, there exists a unique superharmonic and lower semicontinuous representative ω_ε such that $\omega_\varepsilon = u_\varepsilon - \Psi_\varepsilon$ almost everywhere in Ω and it is given by

$$\omega_\varepsilon(x) = \lim_{r\to 0} \fint_{B_r(x)} [u_\varepsilon(y) - \Psi_\varepsilon(y)]\,\mathrm{d}y = u_\varepsilon(x) - \Psi_\varepsilon(x),$$

where we have used (2.6) in the second inequality. In particular, u_ε is lower semicontinuous, and so, $u_\varepsilon = (u_\varepsilon)_*$ is a continuous viscosity solution to E_ε. From the structural conditions of F_ε and the regularity theory developed in [28], there is a universal $0 < \gamma < 1$ such that, $u_\varepsilon \in C^{1,\gamma}_{\text{loc}}(\Omega)$. It also follows from [29] that u_ε is twice differentiable almost everywhere in Ω, with equation E_ε then holding almost every where. To finish the proof, observe that, if we define $f_\varepsilon(x) = F_\varepsilon(u_\varepsilon(x), \nabla u_\varepsilon(x))$, then $f_\varepsilon \in C(\Omega) \cap L^\infty(\Omega)$ and $\Delta u_\varepsilon = f_\varepsilon$ in the viscosity sense. From $W^{2,p}$ estimates in [9, Theorem 7.1], $u_\varepsilon \in W^{2,p}_{\text{loc}}(\Omega)$ for any $1 \leqq p < \infty$ and thus $u_\varepsilon \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for any $0 < \alpha < 1$. To finish, let $x_0 \in \partial\Omega$, and $x_n \to x_0$. Since, $\Psi_\varepsilon(x_n) \leqq u_\varepsilon(x_n) \leqq h_\varphi(x_n)$, letting $n \to \infty$, we conclude $u(x_0) = \varphi(x_0)$. □

Remark 2.4. It follows from the proof of the Theorem (2.3), that under the continuity assumption of F_ε and structural condition (2.2), any continuous viscosity solution of E_ε belongs to $C^{1,\alpha}_{\text{loc}}(\Omega) \cap W^{2,p}_{\text{loc}}(\Omega)$ for any $0 < \alpha < 1$ and $1 \leqq p < \infty$ and satisfies the equation almost everywhere in Ω and also in the distributional sense.

Remark 2.5. The twice differentiability of u_ε in the theorem above could also be justified by the fact that any function in $W^{2,p}_{\text{loc}}(\Omega)$ with $n < 2p$ is twice differentiable almost everywhere. This fact is a consequence of the Calderon–Zygmund theory. A direct proof can be found in [17, Appendix C].

To finish this section, we state a result about local uniform Lipschitz regularity, due to Caffarelli.

Theorem 2.6. ([8, Corollary 2]) *Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a family of continuous viscosity solutions to E_ε such that $||v_\varepsilon||_{L^\infty(\Omega)} \leqq \mathcal{A}$. Then, if $\Omega^{'} \subset\subset \Omega$ there exists a universal constant $C = C(\Omega^{'}, \mathcal{A})$ such that*

$$||\nabla v_\varepsilon||_{L^\infty(\Omega')} \leqq C.$$

In particular, the family $\{v_\varepsilon\}_{\varepsilon>0}$ is locally uniformly Lipschitz continuous.

3. Geometric properties of the least supersolution

In this section, we prove important geometric properties of the least supersolutions. We will be focused in two of them: linear growth away from certain level sets and strong nondegeneracy. In general, those properties are not expected to hold for general solutions of the equation E_ε. Those properties rely heavily on the special kind of solutions considered, the least supersolutions to E_ε. These features will be crucial for the study of the regularity of the free boundary of the limit functions later on. As we will see, these geometric facts will imply a rather restrictive geometry of the free boundary.

Some notation is now introduced.

$$
\begin{aligned}
&B^\star_\alpha = B_{\delta_\varepsilon}(x_\varepsilon) \text{ where } u_\varepsilon(x_\varepsilon) = \alpha \text{ and } \delta_\varepsilon = \frac{1}{2}\mathrm{dist}(x_\varepsilon, \partial\Omega),\\
&\Omega_\alpha = \left\{x \in \Omega;\ 0 \leqq u_\varepsilon(x) \leqq \alpha\right\} \text{ and } d_\alpha(x) = \mathrm{dist}(x, \Omega_\alpha),\\
&\Omega^+_\alpha = \{x \in \Omega;\ u_\varepsilon(x) > \alpha\},\\
&\Omega' \subset\subset \Omega \text{ and } \Delta = \mathrm{dist}(\Omega', \mathbb{R}^N \setminus \Omega).
\end{aligned}
$$

Theorem 3.1. (Linear growth away from level set ε) *There exists a universal constant $C_3 > 0$ such that if $x_0 \in B^\star_\varepsilon \cap \Omega^+_\varepsilon$*

$$u_\varepsilon(x_0) \geqq C_3 d_\varepsilon(x_0).$$

Proof. Let us prove by contradiction. If this is not the case, for $\varepsilon > 0$ small enough, there exists $y_\varepsilon \in B^\star_\varepsilon \cap \Omega^+_\varepsilon$ such that $u_\varepsilon(y_\varepsilon) \ll d_\varepsilon(y_\varepsilon) = d_\varepsilon$. The idea now is to construct an admissible supersolution (in $\mathcal{S}_\varepsilon$) strictly below u_ε in some point providing a contradiction. Since, $y_\varepsilon \in B^\star_\varepsilon \cap \Omega^+_\varepsilon$, we have $B_{d_\varepsilon}(y_\varepsilon) \subset \Omega^+_\varepsilon$ and thus

$$\Delta u_\varepsilon = 0 \text{ in } B_{d_\varepsilon}(y_\varepsilon).$$

By the Harnack inequality, there exists a universal constant $C > 0$ such that

$$u_\varepsilon \leqq C u_\varepsilon(y_\varepsilon) \text{ in } B_{d_\varepsilon/2}(y_\varepsilon).$$

Now, consider the following function:

$$
\begin{cases}
\Delta v_\varepsilon = 0 & \text{in } \mathcal{R} = B_{d_\varepsilon/2}(y_\varepsilon) \setminus \overline{B_{d_\varepsilon/4}(y_\varepsilon)}\\
v_\varepsilon = 0 & \text{on } \partial B_{d_\varepsilon/4}(y_\varepsilon)\\
v_\varepsilon = 1 & \text{on } \partial B_{d_\varepsilon/2}(y_\varepsilon)
\end{cases}
\tag{3.1}
$$

and define,

$$
\overline{w_\varepsilon} = \begin{cases}
0, & \text{in } \overline{B_{d_{\varepsilon/4}}(y_\varepsilon)}\\
\min\{u_\varepsilon, d_\varepsilon v_\varepsilon\} & \text{in } \mathcal{R} = B_{d_\varepsilon/2}(y_\varepsilon) \setminus \overline{B_{d_\varepsilon/4}(y_\varepsilon)}\\
u_\varepsilon & \text{in } \Omega \setminus \overline{B_{d_\varepsilon/2}(y_\varepsilon)}
\end{cases}
\tag{3.2}
$$

Since $C > 0$ is a universal constant (that appears in the Harnack inequality) and $u_\varepsilon(y_\varepsilon) \ll d_\varepsilon$, we can assume for ε small enough that $C u_\varepsilon(y_\varepsilon) < d_\varepsilon$, and thus, $\overline{w_\varepsilon}$ is continuous along $\partial B_{d_\varepsilon/4}(y_\varepsilon)$. It is easy to check that, $\overline{w_\varepsilon}$ is a supersolution ([9, Proposition 2.8], for instance) and so $\overline{w_\varepsilon} \in \mathcal{S}_\varepsilon$, providing a contradiction since $\overline{w_\varepsilon}(y_\varepsilon) = 0 < u_\varepsilon(y_\varepsilon)$. This finishes the proof of the theorem. □

In what follows, we will assume that the family $\{u_\varepsilon\}_{\varepsilon>0}$ of least supersolutions to the equation E_ε is uniformly bounded, that is,

$$||u_\varepsilon||_{L^\infty(\Omega)} \leqq \mathcal{A}. \tag{3.3}$$

Corollary 3.2. *There exists a universal constant* $C = C(\Omega^{'}, \mathcal{A})$ *such that*

$$x \in \Omega^{'} \cap \Omega_\varepsilon^+, \;\; d_\varepsilon(x) \leqq \frac{\Delta}{4} \Longrightarrow \;\; C_3 d_\varepsilon(x) \leqq u_\varepsilon(x) \leqq C d_\varepsilon(x) + \varepsilon,$$

where Δ *is defined as before,* $\Delta = \mathrm{dist}(\Omega^{'}, \mathbb{R}^N \setminus \Omega)$.

Proof. The first inequality follows from the Theorem (3.1) just by observing that, if $d_\varepsilon(x) < \frac{\Delta}{3}$, then $x \in B_\varepsilon^\star$. Indeed, let $x_\varepsilon \in \partial\Omega_\varepsilon^+$ with $d_\varepsilon(x) = |x - x_\varepsilon|$, then

$$2|x - x_\varepsilon| = 2d_\varepsilon(x) < \mathrm{dist}(x, \partial\Omega) - d_\varepsilon(x) \leqq \mathrm{dist}(x_\varepsilon, \partial\Omega) = 2\delta_\varepsilon(x)$$

The other inequality follows from uniform Lipschitz continuity, Theorem (2.6).
□

We turn our attention to a strong nondegeneracy result for the least supersolutions. Below, we state the strong nondegeneracy Lemma. The proof can be found in [16, Theorem 1.19] for the Laplacian or in [27, Lemma 3.3] for a general divergence operator with Holder coefficients.

Lemma 3.3. (*Strong nondegeneracy lemma*) *Assume that* $v \geqq 0$ *is Lipschitz and harmonic in* $\Omega \cap B_R(\xi)$*, such that*

(1) $v \equiv \delta$ *on* $\partial\Omega \cap B_R(\xi), \xi \in \partial\Omega,$
(2) $v(x_0) \geqq C\delta > 0, \;\; C \gg 1$ *with* $x_0 \in B_{R/2}(\xi),$
(3) $v(x) \geqq D \cdot \mathrm{dist}(x, \partial\Omega)$ *in* $\left\{v \geqq C\delta\right\} \cap B_{R/2}(\xi).$

Then, there exists a universal constant $M = M(C, D, \mathrm{Lip}(v))$ *such that:*

$$\sup_{B_r(x_0)} v \geqq Mr \quad \textit{for} \;\; 0 < r \leqq \frac{R}{4}.$$

As a consequence of this Lemma, the strong nondegeneracy follows.

Theorem 3.4. (Strong nondegeneracy [27, Theorem 3.4]) *Given* $C_4 \gg 1$ *there exists* $C = C(\Omega^{'}, C_3, C_4, \mathcal{A})$ *such that*

$$\sup_{B_\rho(x_0)} u_\varepsilon \geqq C\rho \qquad \textit{for } \rho \leqq \frac{\Delta}{12}$$

for

$$x_0 \in \Omega^{'} \cap \left\{u_\varepsilon \geqq C_4\varepsilon\right\}, \;\; d_\varepsilon(x_0) \leqq \frac{\Delta}{6}.$$

4. Limits of the least supersolutions

This section will be devoted to establish the first results about the limit functions and the weak geometry of their free boundary. Before, we introduce the following notation for a continuous function $v : \Omega \to \mathbb{R}$

$$\Omega^+(v) = \{x \in \Omega \mid v(x) > 0\} \quad ; \quad \Omega^-(v) = (\Omega \setminus \Omega^+(v))^\circ$$
$$F(v) = \partial \{x \in \Omega \mid v(x) > 0\} \cap \Omega = \partial\Omega^+ \cap \Omega.$$

The set $F(v)$ is called the free boundary of v. Again, in what follows, we assume $\Omega^{'} \subset\subset \Omega$.

Theorem 4.1. (Properties of a limit of the least supersolutions) *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be the family of least supersolutions of E_ε. Assume,*

$$||u_\varepsilon||_{L^\infty(\Omega)} \leqq \mathcal{A}.$$

Then for every sequence $\varepsilon_k \to 0$ there exists a subsequence $\varepsilon_k^{'} \to 0$ such that

(a) $u_{\varepsilon_k^{'}} \to u_0 \in C^{0,1}_{\text{loc}}(\Omega)$ *uniformly in compact subsets of* Ω,

(b) (*Regularity*) $u_0 \in C^{0,1}_{\text{loc}}(\Omega)$, $\Delta u_0 \geqq 0$ *in* $\mathcal{D}^{'}(\Omega)$, *and* $\Delta u_0 = 0$ *in* $\Omega^+(u_0)$ *and in* $\Omega^-(u_0)$.

(c) (*Linear growth away from the free boundary*) *Let $C_3 > 0$ be the constant given by Theorem* (3.1), *then*

$$u_0^+(x_0) \geqq C_3 dist\left(x_0, \left\{u_0 \leqq 0\right\}\right)$$
$$if \quad x_0 \in \Omega^{'}, dist\left(x_0, \left\{u_0 \leqq 0\right\}\right) \leqq \frac{\Delta}{4}.$$

(d) (*Strong nondegeneracy*) *There exists a constant $C = C(\Omega^{'}, \mathcal{A})$ such that:*

$$\sup_{B_\rho(x_0)} u_0 \geqq C\rho \quad for \quad \rho \leqq \frac{\Delta}{12}$$

provided

$$x_0 \in \Omega^{'} \cap (\Omega_0 \cup F(u_0)) \; with \; dist\left(x_0, \left\{u_0 \leqq 0\right\}\right) \leqq \frac{\Delta}{6}.$$

(e) (*Nondegeneracy*) *There exists a constant $\underline{C} = \underline{C}(\Omega^{'}, \mathcal{A})$ and $\overline{C} = \overline{C}(\Omega^{'}, \mathcal{A})$ such that:*

$$\underline{C} \leqq \frac{1}{\rho} \fint_{\partial B_\rho(x_0)} u_0^+(y)\, \mathrm{d}\mathcal{H}^{N-1}(y) \leqq \overline{C} \quad for \; \rho \leqq \frac{\Delta}{12}$$

whenever

$$x_0 \in \Omega^{'} \cap F(u_0) \; with \; dist\left(x_0, \left\{u_0 \leqq 0\right\}\right) \leqq \frac{\Delta}{6}.$$

Proof. The properties (a), (b), (c) and (d) were already proven in [27, Theorem 4.1]. So, let us concentrate on proving (e), which will follow from (b) and (d). Indeed, if we define $K = \mathcal{N}_{\frac{\Delta}{2}}(\Omega^{'})$, then $\overline{B_\rho(x_0)} \subset K$ and, by Lipschitz continuity,

$$u_0^+ \leqq \left(\frac{\mathrm{Lip}(u_0 \mid K)}{12}\right)\rho \text{ in } B_\rho(x_0),$$

yielding

$$\frac{1}{\rho}\fint_{\partial B_\rho(x_0)} u_0^+ \, \mathrm{d}\mathcal{H}^{N-1} \leqq \overline{C}.$$

To prove the other inequality, let us consider x_0 in the conditions described in (e). So, by (d), there exists $x_1 \in \overline{B_{\rho/2}(x_0)}$ such that $u_0(x_1) \geqq \frac{C\rho}{4}$. By, Lipschitz continuity, if $\tau \leqq \frac{1}{3}$, since $B_{\rho\tau}(x_1) \subset\subset B_\rho(x_0) \subset K$,

$$u_0 \geqq \left(\frac{C}{4} - \mathrm{Lip}(u_0 \mid K)\tau\right)\rho \text{ in } B_{\rho\tau}(x_1).$$

Taking τ small enough, $u_0 \geqq \frac{C\rho}{8} > 0$ in $B_{\rho\tau}(x_1)$, and thus

$$\fint_{B_\rho(x_0)} u_0^+ \, \mathrm{d}x \geqq \tau^{N-1}\fint_{B_{\rho\tau}(x_1)} u_0^+ \, \mathrm{d}x \geqq \frac{\tau^N C}{8}\rho.$$

By now, we have proven (e) for the volume average, that is, there exist a constant $C_1 = C_1(\Omega^{'}, \mathcal{A}) > 0$ such that, whenever x_0 fulfils the conditions of (5), we have:

$$\frac{1}{\rho}\fint_{B_\rho(x_0)} u_0^+ \, \mathrm{d}x \geqq C_1 \tag{4.1}$$

From the fact that $u_0^+ \geqq 0$ is locally Lipschitz continuous and harmonic in $\{u_0^+ > 0\}$, the same conclusion holds for the area average as in the statement of (e). Indeed, suppose by contradiction, that this is not the case. Then, we can find a sequence $\{x_n\}_{n\geqq 1} \subset F(u_0) \cap \Omega^{'}$ with $\mathrm{dist}(x_n, \{u_0 \leqq 0\}) \leqq \frac{\Delta}{6}$, such that

$$\fint_{\partial B_{\rho_n}(x_n)} u_0^+ \, \mathrm{d}\mathcal{H}^{N-1} \leqq \frac{1}{n}\rho_n \quad \text{with } \rho_n \to 0. \tag{4.2}$$

Considering the rescaling functions, $v_n(x) := \frac{1}{\rho_n} u_0^+(x_n + \rho_n x)$, it follows that there exists a subsequence, which we still denote by v_n, such that $v_n \to V$ uniformly in compact sets of $\mathbb{R}^N$, $V \geqq 0$, V Lipschitz continuous and harmonic in $\{V > 0\}$. Now, rewriting (4.2) in terms of v_n, we find

$$\fint_{\partial B_1(0)} v_n \, \mathrm{d}\mathcal{H}^{N-1} \leqq \frac{1}{n}\rho_n.$$

Since, u_0^+ is globally subharmonic, we have

$$0 \leqq \fint_{B_1(0)} u_0^+ \, \mathrm{d}x \leqq \fint_{\partial B_1(0)} u_0^+ \, \mathrm{d}\mathcal{H}^{N-1} = 0,$$

which implies that $u_0^+ \equiv 0$ in $B_1(0)$. On the other hand, we have proven that

$$\fint_{B_1(0)} v_n \, \mathrm{d}x \geqq C_1 > 0.$$

Letting $n \to \infty$, we obtain $\fint_{B_1(0)} u_0^+ \, \mathrm{d}x \geqq C_1 > 0$, a contradiction. This finishes the proof of the Theorem. □

Now, we establish some properties of the free boundary of u_0, $F(u_0)$. First, we need the following definition:

Definition 4.2. Let $v : \Omega \to \mathbb{R}$ be a continuous function. A unit vector $\nu \in \mathbb{R}^N$ is said to be the inward unit normal in the measure theoretic sense to the free boundary $F(v)$ at a point $x_0 \in F(v)$ if

$$\lim_{\rho \to 0} \frac{1}{\rho^N} \int_{B_\rho(x_0)} \left| \chi_{\{v>0\}} - \chi_{H_\nu^+(x_0)} \right| \, \mathrm{d}x = 0, \tag{4.3}$$

where $H_\nu^+(x_0) = \left\{ x \in \mathbb{R}^N \mid \langle x - x_0, \nu \rangle > 0 \right\}$. If A is a set of locally finite perimeter, then for every point in the reduced boundary, $\partial_{\mathrm{red}} A$, the inward unit normal is defined. The details can be found in [19, Section 5.7].

Theorem 4.3. (Properties of the free boundary $F(u_0)$) *Let u_0 be a function given by Theorem (4.1). Then,*

(a) $\mathcal{H}^{N-1}(\Omega' \cap \partial \{u_0 > 0\}) < \infty$.

(b) *There exist Borelian functions $q_{u_0}^+$ and $q_{u_0}^-$ defined on $F(u_0)$ such that*

$$\Delta u_0^+ = q_{u_0}^+ \mathcal{H}^{N-1} \lfloor \partial \{u_0 > 0\},$$
$$\Delta u_0^- = q_{u_0}^- \mathcal{H}^{N-1} \lfloor \partial \{u_0 > 0\}.$$

(c) *There exists universal constants $\underline{C} > 0$, $\overline{C} > 0$ and $\rho_0 > 0$ depending on Ω', $\mathcal{A}$ such that*

$$\underline{C} \rho^{N-1} \leqq \mathcal{H}^{N-1}(B_\rho(x_0) \cap \partial \{u_0 > 0\}) \leqq \overline{C} \rho^{N-1}$$

for every $x_0 \in \Omega' \cap \partial \{u_0 > 0\}$, $0 < \rho < \rho_0$.

(d) $0 < \underline{C} \leqq q_{u_0}^+ \leqq \overline{C}$ *and* $0 \leqq q_{u_0}^- \leqq \overline{C}$ *in* $\Omega' \{u_0 > 0\}$. *In addition,* $q_{u_0}^- = 0$ *in* $\partial \{u_0 > 0\} \setminus \partial \{u_0 < 0\}$.

(e) *u_0 has the following asymptotic development at $\mathcal{H}^{N-1}$-almost every point x_0 in $F(u_0)_{\mathrm{red}}$*

$$u_0(x) = q_{u_0}^+(x_0) \langle x - x_0, \nu \rangle^+ - q_{u_0}^-(x_0) \langle x - x_0, \nu \rangle^- + o(|x - x_0|).$$

(f) *There exists a constant $\tau = \tau(\Omega', \mathcal{A}) > 0$ such that*

$$\mathcal{H}^{N-1}(F(u_0)_{\mathrm{red}} \cap B_\rho(x_0)) \geqq \tau \rho^{N-1}$$

for any $x_0 \in F(u_0) \cap \Omega'$. In particular, we have

$$\mathcal{H}^{N-1}(F(u_0) \setminus F(u_0)_{\mathrm{red}}) = 0. \tag{4.4}$$

Proof. It follows from Theorem (4.1) that all the assumptions of the ALT-CAFFARELLI theory developed in [2, Section 4] are satisfied, thus proving there that (a), (b), (c), (d), and (e) hold; a brief overview can be found in [24, Theorem 3.2]. We observe however, that because of the lack of variational characterization for solutions u_ε (and therefore for u_0), we are unable to obtain a positive uniform density from above of the positive phase, like in [2, Lemma 3.7]. So, the $\mathcal{H}^{N-1}$ measure totality in (4.4) of the reduced free boundary, $F(u_0)_{\text{red}}$, does not follow from the ALT-CAFFARELLI theory in [2]. Instead, a subtle construction like that developed in [7] is necessary. So, let us concentrate on proving (f). By rescaling, that is, considering the function

$$(u_0)_\rho(x) = \frac{1}{\rho} u_0(\rho(x - x_0))$$

it is enough to prove the case where $\rho = 1$ and $x_0 = 0$. For $0 < \sigma < 1/4$, let us define the following auxiliary function v_σ

$$\begin{cases} \Delta v_\sigma = -\dfrac{1}{|B_\sigma(0)|}\chi_{B_\sigma(0)} & \text{in } B_1(0) \\ v_\sigma = 0 & \text{on } \partial B_1(0). \end{cases} \tag{4.5}$$

In fact, if $G(x, y)$ denotes the positive Green function of the unit ball, we have

$$v_\sigma(x) = \fint_{B_\sigma(0)} G(x, y)\,\mathrm{d}y.$$

By the maximum principle, $v_\sigma \geqq 0$. It follows from the LITMANN-STAMPACCHIA-WEINBERGER theorem [25, Theorem 7.1] that $v_\sigma \leqq \overline{C}\sigma^{2-N}$ outside $B_{2\sigma}(0)$ ($\overline{C} > 0$ universal constant) and $\partial_\nu v_\sigma \sim C > 0$ (here, C is also a universal constant) along $\partial B_1(0)$, where ν is the unit outwards normal vector to $\partial B_1(0)$. Now, by the Harnack inequality [20, Theorems 8.17 and 8.18], for any $q > \frac{N}{2}$,

$$\sup_{B_{2\sigma}(0)} v_\sigma \leqq C^\star \left\{ \inf_{B_{2\sigma}(0)} v_\sigma + \sigma^{2-\frac{2N}{q}} \left|\left| \frac{1}{|B_\sigma(0)|}\chi_{B_\sigma(0)} \right|\right|_{L^q(B_{2\sigma}(0))} \right\},$$

where $C^\star = C^\star(N, q)$. Since $\inf\limits_{B_\sigma(0)} v_\sigma \leqq \overline{C}\sigma^{2-N}$, we finally obtain that

$$v_\sigma \leqq C\sigma^{2-N} \text{ in } B_1(0), \quad \text{where } C = C(N, \sigma). \tag{4.6}$$

Since, $u_{\varepsilon_k}, v_\sigma \in C^{1,\alpha}(\overline{B_1(0)})$ for any $0 < \alpha < 1$, we can apply the second Green's formula, obtaining

$$\begin{aligned} &\int_{\Omega^+(u_0)\cap B_1(0)} (v_\sigma \Delta u_{\varepsilon_k} - u_{\varepsilon_k}\Delta v_\sigma)\,\mathrm{d}x \\ &= \int_{B_1(0)\cap F(u_0)_{\text{red}}} (v_\sigma \partial_\nu u_{\varepsilon_k} - u_{\varepsilon_k}\partial_\nu v_\sigma)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial B_1(0)\cap\Omega^+(u_0)} u_{\varepsilon_k}\partial_\nu v_\sigma\,\mathrm{d}\mathcal{H}^{N-1}. \end{aligned} \tag{4.7}$$

From the uniform Lipschitz continuity of u_{ε_k} in $\overline{B_1(0)}$ and (4.6),

$$\left|\int_{B_1(0)\cap F(u_0)_{\mathrm{red}}} v_\sigma\, \partial_\nu u_{\varepsilon_k}\, \mathrm{d}\mathcal{H}^{N-1}\right| \leqq C\sigma^{2-N}\mathcal{H}^{N-1}(F(u_0)_{\mathrm{red}} \cap B_1(0)).$$

Moreover, as $\varepsilon_k \to 0$,

$$\int_{B_1(0)\cap F(u_0)_{\mathrm{red}}} u_{\varepsilon_k}\partial_\nu v_\sigma\, \mathrm{d}\mathcal{H}^{N-1} \to 0$$

$$\int_{\partial B_1(0)\cap\Omega^+(u_0)} u_{\varepsilon_k}\partial_\nu v_\sigma\, \mathrm{d}\mathcal{H}^{N-1} \to \int_{\partial B_1(0)} u_0^+\partial_\nu v_\sigma\, \mathrm{d}\mathcal{H}^{N-1}$$

$$-\int_{\Omega^+(u_0)\cap B_1(0)} u_{\varepsilon_k}\Delta v_\sigma\, \mathrm{d}x = \frac{1}{|B_\sigma(0)|}\int_{\Omega^+(u_0)\cap B_\sigma(0)} u_{\varepsilon_k}\, \mathrm{d}x \to ⨍_{B_\sigma(0)} u_0^+\, \mathrm{d}x$$

Since $v_\sigma\Delta u_{\varepsilon_k} \geqq 0$, from (4.7), we deduce

$$⨍_{B_\sigma(0)} u_0^+\, \mathrm{d}x + \int_{\partial B_1(0)} u_0^+\partial_\nu v_\sigma\, \mathrm{d}\mathcal{H}^{N-1} \leqq C\sigma^{2-N}\mathcal{H}^{N-1}(F(u_0)_{\mathrm{red}} \cap B_1(0)) \quad (4.8)$$

By Theorem (4.1)(e),

$$\int_{\partial B_1(0)\cap\Omega^+(u_0)} u_0^+\partial_\nu v_\sigma\, \mathrm{d}\mathcal{H}^{N-1} \geqq C_1 > 0.$$

In particular, again by nondegeneracy (4.1), the relation (4.8) implies

$$C^\star\sigma \leqq ⨍_{B_\sigma(0)} u_0^+\, \mathrm{d}x \leqq C\sigma^{2-N}\mathcal{H}^{N-1}(F(u_0)_{\mathrm{red}} \cap B_1(0)).$$

Since there exist $\underline{C}$, $\overline{C}$ depending on Ω' and $\mathcal{A}$, such that

$$\underline{C} \leqq \frac{1}{\sigma}⨍_{B_\sigma(0)} u_0^+\, \mathrm{d}x \leqq \overline{C}$$

We can then choose, $\sigma = \underline{C}/8\overline{C}$, a universal constant. The last conclusion follows from the density Theorem for a lower dimensional Hausdorff measure [19, Theorem 1, p. 72], just by observing that $\mathcal{H}^{N-1}\lfloor F(u_0)$ is a Radon measure. □

5. Special form for the perturbation and blow-up preliminaries

In the previous sections, we described the "weak" geometry of the free boundary $F(u_0)$ for a limit of the least supersolutions u_ε to the equation E_ε. In order to study in more depth the limit free boundary problem, we will restrict ourselves to deal with the special case where equation E_ε assumes the following form:

$$\Delta u = \beta_\varepsilon(u)F(\nabla u) \ \text{ in } \Omega, \qquad (SE_\varepsilon)$$

where F satisfies

$F-1)$ $F \in C^{0,1}(\mathbb{R}^N)$;
$F-2)$ $0 < F_{\min} \leqq F(p) \leqq F_{\max} < \infty \quad \forall p \in \mathbb{R}^N$;

and β satisfies the conditions in specified in [15], that is,

$\beta-1)$ $\beta \in C^{0,1}(\mathbb{R})$;
$\beta-2)$ $\beta > 0$ in $(0, 1)$ and support of β is $[0, 1]$;
$\beta-3)$ β is increasing in $[0, 1/2)$ and decreasing in $(1/2, 1]$;
$\beta-4)$ $\displaystyle\int_0^1 \beta(s)\,ds := M > 0$;

and additionally,

$\beta-5)$ $\beta(t) \geqq B_0 t^+$ for all $t \leqq 3/4$, where $B_0 > 0$.

Observe that, from the condition $(\beta-2)$, we conclude that there exists $\tau_0 > 0$ such that

$$\beta(t) \geqq \frac{\tau_0}{F_{\min}} \quad \text{for } t \in [1/4, 3/4] \tag{5.1}$$

and we define the following universal constant

$$A_0 := \frac{\tau_0}{3N} > 0. \tag{5.2}$$

As we mentioned in the introduction, the semilinear equations SE_ε have connections with the Prandtl–Batchelor free boundary problems as they were pointed out by Caffarelli et al. [12].

Remark 5.1. From the assumption $F-1$, we can improve the regularity obtained in Theorem (2.3). Indeed, it follows from [9, Theorem 8.1] or [22, Theorem 5.20] that, if v_ε is a continuous viscosity solution to SE_ε, then v_ε is actually a classical solution of SE_ε.

The presence of the gradient in the equations SE_ε does not affect the rescaling properties (see Remark (5.5) below). In this way, the convergence of blow-ups and their compatibility condition proven in [13,24] are preserved. Since the proofs are a small variant of the original ones, they will be omitted.

Proposition 5.2. (Blow-up convergence—[13, Lemma 3.2]) *Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a family of viscosity solutions to SE_ε. Assume for a subsequence $\varepsilon_j \to 0$, $v_{\varepsilon_j} \to v$ uniformly in compact subsets of Ω. Let $x_0, x_n \in \Omega \cap \partial\{v > 0\}$ be such that $x_n \to x_0$ as $n \to \infty$. Let $\lambda_n \to 0$, $v_{\lambda_n}(x) = (1/\lambda_n)v(x_n + \lambda_n x)$ and $(v_{\varepsilon_j})_{\lambda_n}(x) = (1/\lambda_n)v_{\varepsilon_j}(x_n + \lambda_n x)$. Suppose that $v_{\lambda_n} \to V$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}^N$. Then, there exists $j(n) \to \infty$ such that for every $j_n \geqq j(n)$ there holds that $\varepsilon_{j_n}/\lambda_n \to 0$, and*

(i) $(v_{\varepsilon_{j_n}})_{\lambda_n} \to V$ *uniformly in compact subsets of* $\mathbb{R}^N$;
(ii) $\nabla(v_{\varepsilon_{j_n}})_{\lambda_n} \to \nabla V$ *in* $L^2_{\mathrm{loc}}(\mathbb{R}^N)$;
(iii) $\nabla v_{\lambda_n} \to \nabla V$ *in* $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

Proposition 5.3. (Blow-up compatibility condition—[24, Lemma 3.1]) *Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a family of viscosity solutions to SE_ε. Assume for a subsequence $\varepsilon_j \to 0$, $v_{\varepsilon_j} \to v$ uniformly in compact subsets of Ω. Let $x_0 \in F(v)$ and, for $\lambda > 0$, let $v_\lambda(x) = \frac{1}{\lambda} v(x_0 + \lambda x)$. Let $\lambda_n \to 0$ and $\widetilde{\lambda}_n \to 0$ be such that*

$$v_{\lambda_n} \to V = \alpha x_1^+ - \gamma x_1^- + o(|x|),$$
$$v_{\widetilde{\lambda}_n} \to \widetilde{V} = \widetilde{\alpha} x_1^+ - \widetilde{\gamma} x_1^- + o(|x|)$$

uniformly in compact sets of $\mathbb{R}^N$, with $\alpha, \widetilde{\alpha}, \gamma, \widetilde{\gamma} \geqq 0$. Then $\alpha\gamma = \widetilde{\alpha}\widetilde{\gamma}$.

Definition 5.4. A continuous family $\{v_\varepsilon\}_{\varepsilon>0}$ of viscosity solutions to SE_ε is said to be a family of least viscosity supersolutions to SE_ε in Ω if, for every open set $V \subset\subset \Omega$, we have for every $\varepsilon > 0$

$$v_\varepsilon \mid V = \omega_\varepsilon^V,$$

where

$$w_\varepsilon^V(x) := \inf_{w \in \mathcal{S}_\varepsilon(V)} w(x).$$

$$\mathcal{S}_\varepsilon(V) := \Big\{ w \in C^0(\overline{V}),\ w \text{ viscosity supersolution of } SE_\varepsilon;\ w \geqq v_\varepsilon \text{ on } \partial V \Big\}.$$

Clearly, proceeding by Perron's method, as in Theorem (2.3), ω_ε^V is a continuous viscosity solution of SE_ε in V. It follows directly from the theory developed in Theorem (2.3) that $\{u_\varepsilon\}_{\varepsilon>0}$ is a family of least viscosity supersolutions of SE_ε.

Remark 5.5. (Transformations that preserves SE_ε)

(i) (Rescaling) Assume that v is a solution to SE_ε in Ω. If $x_0 \in \Omega$ and $\lambda > 0$, let $T_{x_0}^\lambda(x) := x_0 + \lambda x$. We define the open set $\Omega_{x_0}^\lambda = (T_{x_0}^\lambda)^{-1}(\Omega) = \{x \in \mathbb{R}^N \mid x_0 + \lambda x \in \Omega\}$ and the function $(v_{x_0})_\lambda(x) := \frac{1}{\lambda} v(T_{x_0}^\lambda(x)) = \frac{1}{\lambda} v(x_0 + \lambda x)$. It is immediate that, $(v_{x_0})_\lambda$ is a solution in $\Omega_{x_0}^\lambda$ to

$$\Delta u = \beta_{\frac{\varepsilon}{\lambda}}(u) F(\nabla u).$$

Conversely, if w is a solution to $(SE_{\frac{\varepsilon}{\lambda}})$ in $\Omega_{x_0}^\lambda$, we define in Ω the function $(w_{x_0})^\lambda(y) := \lambda w((T_{x_0}^\lambda)^{-1}(y)) = \lambda w(\frac{y - x_0}{\lambda})$. Again, it is clear that $(w_{x_0})^\lambda$ is a solution to SE_ε.

So, the correspondences $v \mapsto (v_{x_0})_\lambda$ and $w \mapsto (w_{x_0})^\lambda$ establish a bijection among solutions of SE_ε and $(SE)_{\frac{\varepsilon}{\lambda}}$. Since those maps preserve order, that is, $v^1 \leqq v^2 \Longrightarrow (v_{x_0}^1)_\lambda \leqq (v_{x_0}^2)_\lambda$ and $w^1 \leqq w^2 \Longrightarrow (w_{x_0}^1)^\lambda \leqq (w_{x_0}^2)^\lambda$, we conclude: $\{v_\varepsilon\}_{\varepsilon>0}$ is a family of least viscosity supersolution to SE_ε in Ω if and only if $\big\{((v_\varepsilon)_{x_0})_\lambda\big\}_{\varepsilon>0}$ is a family of least viscosity supersolutions to $(SE)_{\frac{\varepsilon}{\lambda}}$ in $\Omega_{x_0}^\lambda$.

(ii) (Invariance under translations) Since the equation SE_ε does not depend on x, the equation is translation invariant, that is, translations of solutions (subsolutions, supersolutions) u, $v = u(\cdot + h)$, $h \in \mathbb{R}^N$ are still solutions (subsolutions, supersolutions), respectively.

6. Some qualitative results

In this section, we will prove some results that will be used in a decisive way to obtain the classifications of global profiles later on. We will start with some definitions.

Definition 6.1. We set for $\sigma > 0$ the scaled function

$$\mathcal{E}_\sigma(\beta)(x) := \sigma\beta\left(\frac{x}{\sigma} - \frac{1}{2\sigma} + \frac{1}{2}\right). \tag{6.1}$$

Geometrically, the graph of $\mathcal{E}_\sigma(\beta)$ corresponds to a σ-rescaling of the graph of β with respect to $x = \frac{1}{2}$. So, supp $(\mathcal{E}_\sigma(\beta)) = [\kappa_\sigma^-, \kappa_\sigma^+]$, where $\kappa_\sigma^- := \frac{1}{2} - \frac{\sigma}{2}$ and $\kappa_\sigma^+ := \frac{1}{2} + \frac{\sigma}{2}$. Also, for any $\sigma > 0$, $\mathcal{E}_\sigma(\beta) \in C^{0,1}(\mathbb{R})$ with $\mathrm{Lip}(\mathcal{E}_\sigma(\beta)) = \mathrm{Lip}(\beta)$. By $(\beta - 3)$, it is easy to verify that

$$0 < \sigma < 1 \Longrightarrow \mathcal{E}_\sigma(\beta)(t) < \beta(t) \quad \text{for } t \in \mathrm{supp}(\mathcal{E}_\sigma(\beta)),$$
$$\sigma > 1 \Longrightarrow \mathcal{E}_\sigma(\beta)(t) > \beta(t) \quad \text{for } t \in [0,1] = \mathrm{supp}(\beta).$$

Moreover, from the relation

$$\sigma_1, \sigma_2 > 0 \Longrightarrow \mathcal{E}_{\frac{\sigma_2}{\sigma_1}}(\mathcal{E}_{\sigma_1}(\beta)) = \mathcal{E}_{\sigma_2}(\beta)$$

it follows that

$$0 < \sigma_1 < \sigma_2 \Longrightarrow \mathcal{E}_{\sigma_1}(\beta)(t) < \mathcal{E}_{\sigma_2}(\beta)(t) \quad \text{for } t \in \mathrm{supp}(\mathcal{E}_{\sigma_1}(\beta)). \tag{6.2}$$

We set,

$$M_\sigma := \int_{\kappa_\sigma^-}^{\kappa_\sigma^+} \mathcal{E}_\sigma(\beta)(t)\,\mathrm{d}t = \sigma^2 M. \tag{6.3}$$

As usual, we use the same notation for the ε-rescaling, that is,

$$(\mathcal{E}_\sigma(\beta))_\varepsilon(t) = \frac{1}{\varepsilon}\mathcal{E}_\sigma(\beta)\left(\frac{t}{\varepsilon}\right).$$

Let us also define, for $|\mu| < F_{\min}/2$ and $|\delta| < \frac{1}{2}$,

$$F_{\delta,\mu}(p) = (1+\delta)(F(p)+\mu) > \frac{\delta F_{\min}}{4} > 0, \tag{6.4}$$

and finally let $e_1 = (1, 0, \ldots, 0)$ be the first canonical vector in $\mathbb{R}^N$,

$$H_{\delta,\mu}(t) = \int_0^t \frac{s}{(F_{\delta,\mu}(se_1))}\,\mathrm{d}s = \int_0^t \frac{s}{(1+\delta)(F(se_1)+\mu)}\,\mathrm{d}s. \tag{6.5}$$

We also denote

$$H(t) = H_{0,0}(t) = \int_0^t \frac{s}{F(se_1)}\,\mathrm{d}s. \tag{6.6}$$

In the next lemma, we show that the monotonicity relation in (6.2) still holds if we perturb $\mathcal{E}_\sigma(\beta)$ by a scaling factor close enough to 1.

Lemma 6.2. *Assume* $0 < \sigma_1 < \sigma_2$. *If* θ *is close enough to* 1, *then for every* $\varepsilon > 0$ *we have the following inequalities*

$$(\mathcal{E}_{\sigma_2}(\beta))_\varepsilon(t) \geqq (\mathcal{E}_{\sigma_1}(\beta))_\varepsilon(\theta t) \quad \textit{for all } t \in \mathbb{R}, \tag{6.7}$$

$$(\mathcal{E}_{\sigma_2}(\beta))_\varepsilon(\theta t) \geqq (\mathcal{E}_{\sigma_1}(\beta))_\varepsilon(t) \quad \textit{for all } t \in \mathbb{R}. \tag{6.8}$$

Proof. Clearly, by rescaling, it is enough to prove the lemma for $\varepsilon = 1$. So, let us define the following functions in $\mathbb{R}$,

$$G_\theta(t) = \mathcal{E}_{\sigma_2}(\beta)(\theta t) - \mathcal{E}_{\sigma_1}(\beta)(t),$$
$$J_\theta(t) = \mathcal{E}_{\sigma_2}(\beta)(t) - \mathcal{E}_{\sigma_1}(\beta)(\theta t).$$

We will prove that $G_\theta, J_\theta \geqq 0 \ \forall t \in \mathbb{R}$. Indeed, let $K \subset \mathbb{R}$ be a compact interval such that $\operatorname{supp}\mathcal{E}_{\sigma_1}(\beta) \subsetneq K \subsetneq \operatorname{supp}\mathcal{E}_{\sigma_2}(\beta)$. Setting $G(t) = \mathcal{E}_{\sigma_2}(\beta)(t) - \mathcal{E}_{\sigma_1}(\beta)$, since $\mathcal{E}_\sigma(\beta)(t)$ is Lipschitz continuous for $\sigma > 0$, we have $G_\theta \to G$ and $J_\theta \to G$ locally uniformly in compact subsets of $\mathbb{R}$, as $\theta \to 1$. By (6.2), $G > 0$ in K. In particular, by the uniform convergence, $G_\theta, J_\theta > 0$ in K for θ close enough to 1. On the other hand, clearly $G_\theta(t) \geqq 0$ for $t \notin K$. If $g_\theta(t) = \mathcal{E}_{\sigma_1}(\beta)(\theta t)$ then, for θ close enough to 1, $\operatorname{supp} g_\theta = \dfrac{1}{\theta}(\operatorname{supp}\mathcal{E}_{\sigma_1}(\beta)) \subsetneq K$, and thus $J_\theta(t) \geqq 0$ for $t \notin K$. This finishes the Lemma. □

Now, we prove a Lemma that says essentially that if an "almost" strict subsolution to SE_ε is below a supersolution to SE_ε, then they cannot touch inside the domain. This Lemma will be used later on with the help of some barriers to prevent the slopes of the blow-up limits to have "too closed" an aperture.

Lemma 6.3. (No interior contact) *Let* $u_1, u_2 \in C^2(B_1) \cap C^0(\overline{B}_1)$ *and* $\sigma > 1$ *such that*

$$\Delta u_1 \leqq \beta_\varepsilon(u_1) F(\nabla u_1) \textit{ in } B_1,$$
$$\Delta u_2 \geqq (\mathcal{E}_\sigma(\beta))_\varepsilon(u_2) F(\nabla u_2) \textit{ in } B_1,$$
$$u_1 \geqq u_2 \textit{ in } B_1.$$

Then, u_2 *cannot touch* u_1 *in an interior point.*

Proof. Let us prove the renormalized case $\varepsilon = 1$. The general case will follow analogously. So, let us assume, by contradiction, that u_2 touches u_1 from below at $x_0 \in B_1$. In this way $\Delta u_2(x_0) \leqq \Delta u_1(x_0)$. Moreover, since $\nabla u_1(x_0) = \nabla u_2(x_0)$ and $\mathcal{E}_\sigma(\beta) \geqq \beta$ $(\sigma > 1)$, we have the opposite inequality and thus

$$\Delta u_2(x_0) = \Delta u_1(x_0).$$

If we choose $1 < \overline{\sigma} < \sigma$, then $\beta \leqq \mathcal{E}_{\overline{\sigma}} < \mathcal{E}_\sigma$ in supp $\mathcal{E}_{\overline{\sigma}} = [a, b]$, where $a < 0$ and $b > 1$. Thus, $c = u_1(x_0) = u_2(x_0) \notin [a, b]$. Let us suppose $c > b$. Consider $r = \operatorname{dist}(x_0, \{u_1 \leqq \frac{1+b}{2}\})$ and consider the convex set $A = \overline{B}_r(x_0) \cap \overline{B}_1$. Since u_1 is harmonic in A°, u_2 is subharmonic in A°, and A° is connected, the strong maximum principle implies $u_1 \equiv u_2$ in A. In particular, $\nabla u_1 \equiv \nabla u_2$ and $\Delta u_1 \equiv \Delta u_2$ in A°. If x_1 is such that $r = |x_1 - x_0|$, then $u_1(x_1) = \frac{1+b}{2}$. So, the segment $(x_1, x_0) \subset A^\circ$. In particular, by the intermediate value theorem, we can find x_2 in the open segment, for which $\frac{1+b}{2} < u_1(x_2) = \frac{1+b}{2} + \frac{b-1}{8} = \overline{b} < b$. In this way, since $x_2 \in A^\circ$, we have $u_1(x_2) = \overline{b} = u_2(x_2)$, $\nabla u_1(x_2) = p = \nabla u_2(x_2)$ and $\Delta u_1(x_2) = \Delta u_2(x_2)$. Thus,

$$\beta(\overline{b})F(p) = \Delta u_1(x_2) = \Delta u_2(x_2) = \mathcal{E}_\sigma(\beta)(\overline{b})F(p),$$

which implies, since $F > 0$, $\beta(\overline{b}) = \mathcal{E}_\sigma(\overline{b})$, a contradiction since $\overline{b} \in (a, b)$. If $c \leqq a$ we proceed similarly. So, u_2 never touches u_1 and the Lemma is proven. □

In the next proposition, we construct a radially symmetric supersolution to SE_ε where its value in a inner disk is much smaller that its value on the boundary. This will be used to prove that the least supersolutions u_ε have some type of exponential decay inside the domain.

Proposition 6.4. (Radially symmetric supersolution) *Given $\eta > 0$, there exist radially symmetric functions $\Theta_\varepsilon \in C^1(\mathbb{R}^N) \cap W^{2,\infty}_{\text{loc}}(\mathbb{R}^N)$ and universal constants $\kappa_2 > 0$ and $0 < \kappa_1 < 1$ such that*

(i) $\Theta_\varepsilon \equiv \dfrac{\varepsilon}{4}$ *in* $B_{\kappa_1 \eta}$,
(ii) $\Theta_\varepsilon \geqq \kappa_2 \eta$ *in* $\mathbb{R}^N \setminus B_\eta$,
(iii) Θ_ε *is a viscosity supersolution to* SE_ε *for* ε *small enough.*

Proof. We will work assuming first that $\varepsilon = 1$. After that, we will rescale the construction to obtain Θ_ε. Let $L \geqq \dfrac{10}{\sqrt{2A_0}}$, where A_0 is the universal constant defined in (5.2). Then, we define,

$$\overline{\Theta}(r) = \begin{cases} 1/4, & \text{for } 0 \leqq r \leqq L \\ G(r) = A_0(r-L)^2 + 1/4 & \text{for } L \leqq r \leqq L + 1/\sqrt{2A_0} \\ \Gamma(r) & \text{for } r \geqq L + 1/\sqrt{2A_0} \end{cases} \tag{6.9}$$

where Γ solves

$$\Gamma_{rr} + \frac{N-1}{r}\Gamma_r = 0 \text{ for } r \geqq L + 1/\sqrt{2A_0}, \tag{6.10}$$

$$\Gamma(L + 1/\sqrt{2A_0}) = 3/4, \qquad \Gamma_r(L + 1/\sqrt{2A_0}) = \sqrt{2A_0}.$$

Let us assume $N \geqq 3$. Then, setting

$$K_L := 3/4 + \frac{\sqrt{2A_0}}{N-2}(L + 1/\sqrt{2A_0})$$

and

$$f(r) := \frac{\sqrt{2A_0}}{N-2}(L + 1/\sqrt{2A_0})^{N-1} r^{2-N}$$

we obtain,

$$\begin{aligned}\Gamma(r) &= 3/4 + \frac{\sqrt{2A_0}}{N-2}(L + 1/\sqrt{2A_0}) - \frac{\sqrt{2A_0}}{N-2}(L + 1/\sqrt{2A_0})^{N-1} r^{2-N} \\ &= K_L - f(r)\end{aligned}$$

We would like to show that

$$f(\kappa_3 L) \leqq \frac{1}{2} K_L \qquad \text{for } \kappa_3^{2-N} = \frac{1}{4}\left(\frac{10}{11}\right)^{N-1} < 1.$$

Indeed, this a consequence of the following sequence of estimates,

$$\begin{aligned} \kappa^{2-N} &< \frac{1}{2}\left(\frac{10}{11}\right)^{N-1} \Rightarrow \kappa^{2-N} < \left(\frac{10}{11}\right)^{N-1} \frac{1}{2L}\left(L + 1/\sqrt{2A_0}\right) \\ &\Longrightarrow \frac{\sqrt{2A_0}}{N-2}\left(\frac{11}{10}\right)^{N-1} \kappa^{2-N} L \leqq \frac{1}{2}\frac{\sqrt{2A_0}}{N-2}\left(L + 1/\sqrt{2A_0}\right) \end{aligned}$$

once we translate the inequality above in terms of K_L, $f(r)$ and recall that $L > \frac{10}{\sqrt{2A_0}}$.

In particular, since Γ is increasing, $\Gamma(r) > \frac{1}{2}K_L \geqq \kappa_4 L$ for $r \geqq \kappa_3 L$, where $\kappa_4 = \sqrt{2A_0}/2(N-2)L$. Finally, let us observe that, for $r \in (L, L + 1/\sqrt{2A_0})$, $1/4 \leqq \Theta \leqq 3/4$. In this way, recalling the universal constant τ_0 defined earlier in (5.1), we have

$$\Theta_{rr} + \frac{N-1}{r}\Theta_r = G_{rr} + \frac{N-1}{r}G_r \leqq 2A_0 N \leqq \tau_0 \leqq \beta(\Theta(r))F\left(\Theta_r(r)\frac{x}{|x|}\right).$$

Thus, setting $\Theta(x) := \overline{\Theta}(|x|)$, by construction, $\Theta \in C^1(\mathbb{R}^N) \cap W^{2,\infty}_{\text{loc}}(\mathbb{R}^N)$ is a L^∞_{loc}-strong solution to the equation (that is, it belongs to $W^{2,\infty}_{\text{loc}}(\mathbb{R}^N)$ and solves the equation almost everywhere)

$$\Delta u = \beta(u)F(\nabla u).$$

If $\varepsilon < \varepsilon_0 := \dfrac{\eta\sqrt{2A_0}}{10\kappa_3}$, we can find $L > \frac{10}{\sqrt{2A_0}}$ such that $\varepsilon = \frac{\eta}{\kappa_3 L}$ and set

$$\Theta_\varepsilon(x) := \varepsilon\Theta\left(\frac{x}{\varepsilon}\right)$$

We see that $\Theta_\varepsilon \in C^1(\mathbb{R}^N) \cap W^{2,\infty}_{\text{loc}}(\mathbb{R}^N)$ and (i) and (ii) are satisfied with $k_1 = 1/\kappa_3$ and $\kappa_2 = \kappa_4/\kappa_3$. The fact the Θ_ε are viscosity solutions of SE_ε follows from [18, Theorem 2.1] or more generally by the results in [17]. The case $N = 2$, where $\Gamma(r) = 3/4 + \sqrt{2A_0}(L + 1/\sqrt{2A_0})\log(\dfrac{r}{L + \sqrt{2A_0}})$, is proven similarly. $\square$

We will prove an interesting geometric property of the least supersolution to SE_ε. Essentially, it says that if they are small in a certain region of the domain, as soon as we enter a bit inside this region, they become much smaller. In some sense, this decay is exponentially fast in ε as further inside we enter into that region. For our purposes, it is enough to show that the decay is cubic in ε. This is the content of the next proposition, where we use the notation

$$Q_r = \left\{(x_1, x') \in \mathbb{R}^N; |x_1| \leqq r, |x'| \leqq r\right\}.$$

Proposition 6.5. (Cubic decay inside) *Suppose* $\{v_\varepsilon\}_{\varepsilon>0}$ *is a family of least supersolutions to* SE_ε *and that, for some* $\eta > 0$ *(small),* $||v_\varepsilon^+||_{L^\infty(Q_1)} < \kappa_2\eta$*. Then, there exist a constant* $C_\eta > 0$ *depending on* η *such that*

$$v_\varepsilon^+(x) \leqq C_\eta \varepsilon^3 \text{ for all } x \in Q_{1-2\eta} \text{ and } \varepsilon \text{ small enough.}$$

Proof. Indeed, if $x_0 \in Q_{1-\eta}$, $B_\eta(x_0) \subset Q_1$. We can now place the radially symmetric barrier constructed in the previous Proposition (6.4) in this ball, and since v_ε is the least supersolution to SE_ε, we conclude, $v_\varepsilon(x_0) \leqq \frac{\varepsilon}{4}$. So,

$$v_\varepsilon(x) \leqq \frac{\varepsilon}{4} \text{ for all } x \in Q_{1-\eta}.$$

Let us denote by G_x the positive Green's function of the ball $B_\eta(x)$. If $x_1 \in Q_{1-2\eta}$, $\overline{B}_\eta(x_1) \subset Q_{1-\eta}$. Using the Green's representation formula:

$$v_\varepsilon(x_1) = \fint_{\partial B_\eta(x_1)} v_\varepsilon \, d\mathcal{H}^{N-1} - \int_{B_\eta(x_1)} G_{x_1}(y)\Delta v_\varepsilon(y)\, dy.$$

We have by property $(\beta - 5)$,

$$\frac{F_{\min} B_0}{\varepsilon^2}\left(\inf_{B_{\frac{\eta}{2}}(x_1)} G_{x_1}\right)\int_{B_{\eta/2}(x_1)} v_\varepsilon^+(y)\, dy \leqq F_{\min}\int_{B_{\eta/2}(x_1)} G_{x_1}(y)\beta_\varepsilon(v_\varepsilon(y)) \leqq \frac{\varepsilon}{2}. \tag{6.11}$$

Since v_ε^+ is subharmonic,

$$v_\varepsilon^+(x_1) \leqq \fint_{B_{\eta/2}(x_1)} v_\varepsilon^+(y)\, dy. \tag{6.12}$$

Recalling that $\inf_{B_{\eta/2}(x_1)} G_{x_1} = A_\eta$, where A_η is a universal constant depending on η and combining (6.11) and (6.12), we have

$$v_\varepsilon^+(x_1) \leqq \frac{\varepsilon^3}{2F_{\min} B_0 A_\eta |B_{\eta/2}(x_1)|} = C_\eta \varepsilon^3.$$

□

Finally, to end this section, we prove a qualitative lemma concerning the behavior of solutions to some nonlinear ordinary differential equations (ODEs). The heuristic idea here is the following: the key point to understand the free boundary condition of the limit problem is the classification of global profiles. By the linear behavior of harmonic functions around regular free boundary points [16, Lemma 11.17], it will be enough to study profiles $P(x)$ that are 2-plane functions, that is, profiles of the form $P(x) = Ax_1^+ - Bx_1^-$, with $A, B \geqq 0$. Since, these profiles are essentially one dimensional, their geometry "should be" very much related to the one dimensional version of our equation SE_ε, which is

$$u_{ss} = \beta_\varepsilon(u) F(u_s e_1). \tag{6.13}$$

So, in the next Lemma we study the equation (6.13) in its perturbed version. These technicalities are needed in the next section to "create the adequate room" to bend uniformly the free boundaries of these profiles and to do a careful perturbation of their slopes.

Lemma 6.6. (One dimensional profiles) *Assume that $P \in C^2(\mathbb{R})$ is the unique solution of*

$$u_{ss} = \mathcal{E}_\sigma(\beta)(u)F_{\delta,\mu}(u_s e_1) = (1+\delta)(\mathcal{E}_\sigma(\beta)(u))(F(u_s e_1)+\mu) \tag{6.14}$$
$$u(0) = \kappa_\sigma^+ \text{ and } u_s(0) = \alpha > 0.$$

Then,

(a) *If $\gamma \geqq 0$ and $H_{\delta,\mu}(\alpha) - H_{\delta,\mu}(\gamma) > M_\sigma$, there exist $\overline{\gamma} > \gamma$ and $\overline{s} < 0$ depending on $\alpha, \gamma, \delta, \sigma, \mu$ such that*

$$P(s) = \begin{cases} \kappa_\sigma^+ + \alpha s, & s \geqq 0 \\ \overline{\gamma}(s-\overline{s}) + \kappa_\sigma^-, & s \leqq \overline{s}, \end{cases} \tag{6.15}$$

(b) *If $\gamma \geqq 0$ with $H_{\delta,\mu}(\alpha) - H_{\delta,\mu}(\gamma) < M_\sigma$ we have two cases:*
 (b.1) *If $H_{\delta,\mu}(\alpha) > M_\sigma$, there exist $\overline{\gamma} < \gamma$ and $\overline{s} < 0$ depending on $\alpha, \gamma, \delta, \sigma, \mu$ such that*

$$P(s) = \begin{cases} \kappa_\sigma^+ + \alpha s, & s \geqq 0 \\ \overline{\gamma}(s-\overline{s}) + \kappa_\sigma, & s \leqq \overline{s}, \end{cases} \tag{6.16}$$

 or
 (b.2) *If $H_{\delta,\mu}(\alpha) < M_\sigma$, there exist $\overline{\gamma} > 0$ and $\overline{s} < 0$ depending on $\alpha, \gamma, \delta, \sigma, \mu$ such that*

$$P(s) = \begin{cases} \kappa_\sigma^+ + \alpha s, & s \geqq 0 \\ \kappa_\sigma^+ - \overline{\gamma}(s-\overline{s}) & s \leqq \overline{s}, \end{cases} \tag{6.17}$$

Moreover, in this case, there exists $\overline{\kappa}_\sigma$ such that $\kappa_\sigma^- < \overline{\kappa}_\sigma < P(s) < \kappa_\sigma^+$ for $\overline{s} < s < 0$. Furthermore, setting $P_\varepsilon(s) = \varepsilon P(\frac{s}{\varepsilon})$, it solves:

$$u_{ss} = (\mathcal{E}_\sigma(\beta))_\varepsilon(u)F_{\delta,\mu}(u_s e_1), \tag{$\mathcal{E}^\varepsilon_{\alpha,\delta,\mu,\sigma}$}$$

$$u(0) = \varepsilon\kappa_\sigma^+ \text{ and } u_s(0) = \alpha > 0.$$

Proof. We start by observing that $H_{\delta,\mu}$ is a bijection from $[0,+\infty)$ over itself. This follows since $H_{\delta,\mu}(s) \geqq \dfrac{s^2}{3F_{\max}}$, and $(H_{\delta,\mu})_s > 0$ for $s > 0$. Multiplying the equation (6.14) by P_s we find,

$$(H_{\delta,\mu}(P_s))_s = B^\sigma(P)_s,$$

where $B^\sigma(\zeta) = \int_{\kappa_\sigma^-}^{\zeta} \mathcal{E}_\sigma(\beta)(t)\,dt$. Integrating this equation, we obtain, in cases (a) and (b.1), for some $\overline{\gamma} > 0$,

$$H_{\delta,\mu}(P_s(s)) - B^\sigma(P(s)) = H_{\delta,\mu}(\alpha) - M_\sigma = H_{\delta,\mu}(\overline{\gamma}) > 0. \tag{6.18}$$

This way, since from the expression above, $P_s \geqq 0$

$$0 < \overline{\gamma} \leqq P_s(s) \leqq \alpha, \quad \text{for } t \in \mathbb{R}$$

In case (a), we have $H_{\delta,\mu}(\overline{\gamma}) > H_{\delta,\mu}(\gamma) \geqq 0$ and so $\overline{\gamma} > \gamma$. In case (b), $H_{\delta,\mu}(\overline{\gamma}) < H_{\delta,\mu}(\gamma)$, and thus $\overline{\gamma} < \gamma$. From the inequality (6.18) above, the conclusion of (a) and (b.1) is straightforward. Using the ODE above and the expression (for case (b.2)),

$$H_{\delta,\mu}(P_s(s)) - B^{\sigma}(P(s)) = H_{\delta,\mu}(\alpha) - M_{\sigma} < 0. \tag{6.19}$$

We can see that $P(s) \to +\infty$ as $s \to -\infty$, and therefore (b.2) follows. □

7. Slope barriers with curved free boundary

In this section, we will construct some barriers with uniformly curved free boundaries. They are essentially obtained by a uniform bending of the one dimensional profiles given by Lemma (6.6). The key tool used to accomplish this is a sequence of Kelvin transforms with respect to large spheres, that is, spheres having centers and radii approaching infinity. These barriers will be the fundamental ingredient to classify global profiles (2-plane functions) in the next section.

Remark 7.1. For later reference, we will recall some facts about inversions and Kelvin transforms that will be used in the sequel. For $L > 0$, we denote

$$\mathbb{S}_L = \Big\{x \in \mathbb{R}^N;\, |x + Le_1| = L\Big\},$$
$$\mathbb{S}^\star_L = \Big\{x \in \mathbb{R}^N;\, |x - Le_1| = L\Big\}.$$

The Kelvin transforms of a continuous function u with respect to $\mathbb{S}_L$ and $\mathbb{S}^\star_L$ are given, respectively, by K_L and T_L:

$$K_L[u](x) = (\rho_L(x))^{N-2}\, u(I_L(x)) \tag{7.1}$$
$$T_L[u](x) = (\varrho_L(x))^{N-2}\, u(J_L(x)), \tag{7.2}$$

where I_L, J_L are the inversions with respect to $\mathbb{S}_L$ and $\mathbb{S}^\star_L$, respectively, given by

$$I_L(x) = -Le_1 + \frac{L^2}{|x + Le_1|^2}(x + Le_1)$$
$$J_L(x) = Le_1 + \frac{L^2}{|x - Le_1|^2}(x - Le_1)$$
$$\rho_L(x) = \frac{L}{|x + Le_1|} \qquad \text{and} \qquad \varrho_L(x) = \frac{L}{|x - Le_1|}.$$

It follows also that,

$$\Delta K_L[u](x) = (\rho_L(x))^{N+2}\, \Delta u(I_L(x)) \tag{7.3}$$
$$\Delta T_L[u](x) = (\varrho_L(x))^{N+2}\, \Delta u(J_L(x)). \tag{7.4}$$

Furthermore, if $\mathcal{R}_1$ is the orthogonal reflection with respect to the hyperplane $\{x_1 = 0\}$, then for any $L_0 > 0$, and $L > L_0$ we have

$$\rho_L \to 1 \text{ in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{le_1; l \leqq -10L_0\}), \tag{7.5}$$

$$\varrho_L \to 1 \text{ in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{le_1; l \geqq 10L_0\}), \tag{7.6}$$

$$I_L \to \mathcal{R}_1 \text{ in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{le_1; l \leqq -10L_0\}), \tag{7.7}$$

$$J_L \to \mathcal{R}_1 \text{ in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{le_1; l \geqq 10L_0\}). \tag{7.8}$$

For more details about inversions and Kelvin transforms, see [1,4].

Now, we discuss heuristically the construction of the barriers with curved free boundaries. For that, we use the notation introduced in the beginning of Section 6. Let us suppose that $H(\alpha) - H(\gamma) > M$. Then, by continuity, we can find $\delta, \mu > 0$, $\overline{\alpha} < \alpha$ and $\overline{\sigma} > \sigma > 1$ such that

$$H_{\delta,\mu}(\overline{\alpha}) - H_{\delta,\mu}(\gamma) > M_{\overline{\sigma}} > M.$$

So, Lemma (6.6)(a) provides $\overline{P}_\varepsilon$, the solution to $(\mathcal{E}^\varepsilon_{\alpha,\delta,\mu,\sigma})$, that is,

$$u_{ss} = (\mathcal{E}_\sigma(\beta))_\varepsilon(u)F_{\delta,\mu}(u_s e_1), \qquad u_s(0) = \alpha > 0. \tag{7.9}$$

Since, the equation (7.9) is translation invariant, we can assume $\overline{P}_\varepsilon(0) = 0$. Let us observe that $P_\varepsilon(x) := \overline{P}_\varepsilon(x_1)$ is a subsolution to SE_ε, and its free boundary is flat (actually the hyperplane $\{x_1 = 0\}$). Furthermore, also by Lemma (6.6)(a), P_ε is a "regularization" of the 2-plane function $P(x) := \overline{\alpha}x_1^+ - \overline{\gamma}x_1^-$, where $\overline{\gamma} > \gamma$. For technical reasons (see discussion after Theorem (8.1)), the free boundary of P_ε needs to be uniformly curved in ε. We then modify P_ε to obtain the barriers ϑ^ε_L by considering the composition of a sequence of Kelvin transforms (with respect to large spheres $\mathbb{S}_L$) with the reflection across the hyperplane $\{x_1 = 0\}$, $\vartheta^\varepsilon_L = K_L[P_\varepsilon] \circ \mathcal{R}_1$.

It is intuitive to see that, the larger the L, the closer ϑ^ε_L will be to P_ε. We now use Lemma (6.2) to make a large and uniform in ε choice for a radius and pole of inversion, so far away that, ϑ^ε_L has not changed "that much" from P_ε. In this way, ϑ^ε_L will still be a subsolution to SE_ε, Proposition (7.2)(a), and its free boundary is now curved, Proposition (7.2)(b). Finally it is desirable to keep track of how much the geometry of ϑ^ε_L has changed from P. This is the content of Proposition (7.2)(c), where ϑ^ε_L is compared with some suitable, close 2-plane function in the interior and along the boundary of the domain. This will later provide an estimate of how much the barriers should be moved.

The details of the proof follows below. We point out that Propositions (7.3) and (7.4) have similar heuristic interpretations based on the other accounts of Lemma (6.6). In what follows, for $L_0 > 0$, we use the cylinder,

$$Q_{L_0} := \left\{x = (x_1, x') \in \mathbb{R}^N |\ \ |x|_\infty = \max\left\{|x_1|, |x'|\right\} \leqq 4L_0\right\}.$$

Proposition 7.2. (Above condition barrier) *Suppose $\overline{\sigma} > \sigma > 1$, $\delta, \mu > 0$, and $\overline{\alpha} > 0$, $\gamma \geqq 0$ are such that $H_{\delta,\mu}(\overline{\alpha}) - H_{\delta,\mu}(\gamma) > M_{\overline{\sigma}}$. There exists $\vartheta_\varepsilon \in C^2(Q_{L_0})$ such that*

(a) $\Delta\vartheta_\varepsilon(x) \geqq (\mathcal{E}_\sigma(\beta))_\varepsilon(\vartheta_\varepsilon(x))F(\nabla\vartheta_\varepsilon(x))$ *for* $x \in Q_{L_0}$;
(b) *one has*

$$\begin{gathered}\vartheta_\varepsilon < 0 \textit{ in } Q_{L_0} \cap \mathbb{B}^C \\ \vartheta_\varepsilon > 0 \textit{ in } Q_{L_0} \cap \mathbb{B}^\circ \\ \vartheta_\varepsilon = 0 \textit{ on } Q_{L_0} \cap \mathbb{S} \\ \mathbb{S} \cap \partial Q_{L_0} \subset \{x_1 = d\}\ , \qquad d = d(\textit{radius of } \mathbb{S}, L_0) > 0,\end{gathered}$$

where $\mathbb{S} = \partial\mathbb{B}$, *and* $\mathbb{B}$ *is a closed ball completely contained in the half space* $\left\{x_1 \geqq 0\right\}$, *centered in the positive semiaxis generated by* e_1 *and tangent to the hyperplane* $\{x_1 = 0\}$*;*

(c) *There exists* $\widetilde{\alpha} > \overline{\alpha} > \widetilde{\gamma} > \gamma$ *such that for* $\mathcal{W}(x) = \widetilde{\alpha}x_1^+ - \widetilde{\gamma}x_1^-$, *we have*

$$\mathcal{W}(x) \geqq \vartheta_\varepsilon(x) \textit{ in } Q_{L_0} \quad \textit{and } \mathcal{W}(0) = \vartheta_\varepsilon(0) \tag{7.10}$$

$$\mathcal{W}(x - de_1) \geqq \vartheta_\varepsilon(x) \quad \textit{for } x \in Q_{L_0} \cap \left\{x = (x_1, x^{'}) \in \mathbb{R}^N;\, |x^{'}| = L_0\right\} \tag{7.11}$$

$$Q_\varepsilon \leqq \vartheta_\varepsilon \textit{ along span}\,\{e_1\}\,, \tag{7.12}$$

where $Q_\varepsilon(x) := \overline{Q}_\varepsilon(x_1)$ *and* $\overline{Q}_\varepsilon(s) := P_\varepsilon(s + a_\varepsilon)$, P_ε *is the solution to* $(\mathcal{E}^\varepsilon_{\overline{\alpha},\delta,\mu,\sigma})$ *in Lemma* (6.6) *and* a_ε *is chosen such that* $\overline{Q}_\varepsilon(0) = 0$. *Moreover,* $\widetilde{\alpha}$ *can be taken as close as we wish from* α.

Proof. As suggested in (c), let us define

$$Q_\varepsilon(x) := \overline{Q}_\varepsilon(x_1)$$

and recall that $\mathcal{R}_1$ denotes the reflection with respect to the hyperplane $\{x_1 = 0\}$. Taking $L > 20L_0$, we set

$$\vartheta_\varepsilon^L(x) := (K_L[Q_\varepsilon] \circ \mathcal{R}_1)(x) = K_L[Q_\varepsilon](\mathcal{R}_1(x)) = (\overline{\rho}_L(x))^{N-2}Q_\varepsilon(\overline{I}_L(x)), \tag{7.13}$$

where

$$\overline{I}_L = I_L \circ \mathcal{R}_1, \quad \overline{\rho}_L = \rho \circ \mathcal{R}_1.$$

By Remark (7.1),

$$\begin{aligned}\Delta\vartheta_\varepsilon^L(x) &= (\Delta K_L[Q_\varepsilon] \circ \mathcal{R}_1)(x) = \Delta K_L[Q_\varepsilon](\mathcal{R}_1(x)) \\ &= (\overline{\rho}_L(x))^{N+2}(\mathcal{E}_{\overline{\sigma}}(\beta))_\varepsilon((1/\rho_L(x))^{N-2}\vartheta_\varepsilon^L(x))F_{\delta,\mu}(\nabla\vartheta_\varepsilon^L(x) + A_L^\varepsilon(x)),\end{aligned} \tag{7.14}$$

where

$$A_L^\varepsilon(x) = \nabla Q_\varepsilon(\overline{I}_L(x)) - \nabla\vartheta_\varepsilon^L(x).$$

Since $Q_\varepsilon(\overline{I}_L(x))$ and $\nabla Q_\varepsilon(\overline{I}_L(x))$ are uniformly bounded in Q_{L_0} (recall that Q_ε are translations of rescalings of P given in Lemma (6.6)) by (7.5) and (7.7), we obtain that

$$A_L^\varepsilon \to 0 \text{ uniformly in } Q_{L_0} \text{ as } L \to \infty \text{ uniformly in } \varepsilon.$$

Since F is Lipschitz continuous, we have for $x \in Q_{L_0}$ and L large enough

$$F(\nabla\vartheta_\varepsilon^L(x) + A_L^\varepsilon) + \mu \geqq F(\nabla\vartheta_\varepsilon^L(x) + A_L^\varepsilon) + \mathrm{Lip}(F)|A_L^\varepsilon| \geqq F(\nabla\vartheta_\varepsilon^L(x)), \tag{7.15}$$

$$(1+\delta)(\overline{\rho}_L(x))^{N-2} \geqq 1 + \frac{\delta}{2}. \tag{7.16}$$

Also, by Lemma (6.2), since $\overline{\sigma} > \sigma > 1$

$$(\mathcal{E}_{\overline{\sigma}}(\beta))_\varepsilon((1/\rho_L(x))^{N-2}\vartheta_\varepsilon^L(x)) \geqq (\mathcal{E}_\sigma(\beta))_\varepsilon(\vartheta_\varepsilon^L(x)).$$

Combining the estimates above, we conclude that choosing L large enough, for $x \in Q_{L_0}$ uniformly in ε

$$\Delta\vartheta_\varepsilon^L(x) \geqq (\mathcal{E}_\sigma(\beta))_\varepsilon(\vartheta_\varepsilon^L(x))F(\nabla\vartheta_\varepsilon^L(x)). \tag{7.17}$$

It follows from the proof of Lemma (6.6)(a) that there exists $\overline{\gamma} > \gamma$ such that

$$\overline{\gamma} < (\overline{Q}_\varepsilon)_s < \overline{\alpha} \text{ with } \overline{Q}_\varepsilon(0) = 0.$$

We can easily check that the following properties hold

1. $\overline{Q}_\varepsilon(s) \leqq \overline{\gamma}s$ for $s \in (-\infty, 0]$ and $\overline{Q}_\varepsilon(s) \leqq \overline{\alpha}s$ for $s \in [0,\infty)$;
2. $x \in Q_{L_0} \Rightarrow (\overline{I}_L(x))_1 \leqq -L + \frac{L^2}{L-x_1} =: \tau_L(x_1)$ with
$$\tau_L \geqq 0 \text{ in } \{x_1 \geqq 0\} \text{ and } \tau_L \leqq 0 \text{ in } \{x_1 \leqq 0\};$$
3. If $\tau > 0$ is a small number, for L large enough, we have
$$1-\tau \leqq \overline{\rho}_L \leqq 1+\tau \text{ in } Q_{L_0}$$
$$1-\tau \leqq \frac{\mathrm{d}}{\mathrm{d}x_1}\tau_L \leqq 1+\tau \text{ in } [-4L_0, 4L_0].$$

From these, it is easy to observe the following estimates.
For $x \in \{x_1 \leqq 0\} \cap Q_{L_0}$,

$$\begin{aligned}\vartheta_\varepsilon^L(x) &= (\overline{\rho}_L(x))^{N-2}Q_\varepsilon(\overline{I}_L(x)) \leqq (1-\tau)^{N-2}\overline{Q}_\varepsilon((\overline{I}_L(x))_1)\\ &\leqq (1-\tau)^{N-2}\overline{Q}_\varepsilon(\tau_L(x_1))\\ &\leqq (1-\tau)^{N-1}\overline{\gamma}x_1 = -\widetilde{\gamma}x_1^-.\end{aligned}$$

Similarly, for $x \in \{x_1 \geqq 0\} \cap Q_{L_0}$

$$\begin{aligned}\vartheta_\varepsilon^L(x) &= (\overline{\rho}_L(x))^{N-2}\overline{Q}_\varepsilon((\overline{I}_L(x))_1) \leqq (1+\tau)^{N-2}\overline{Q}_\varepsilon(\tau_L(x_1))\\ &\leqq (1+\tau)^{N-1}\overline{\alpha}x_1 = \widetilde{\alpha}x_1^+.\end{aligned}$$

We can also use similar ideas, to obtain estimates along the boundary. In this case, the estimates will be one dimensional. Indeed, if $x \in Q_{L_0} \cap \Big\{ x = (x_1, x') \in \mathbb{R}^N; |x'| = L_0 \Big\}$ then

$$\vartheta_\varepsilon(x) = (\widetilde{\rho}_L(x_1))^{N-2} Q_\varepsilon(\overline{I}_L(x)) = (\widetilde{\rho}_L(x_1))^{N-2} \overline{Q}_\varepsilon(\varphi_L(x_1)),$$

where

$$\widetilde{\rho}_L(x_1) = \frac{L}{\sqrt{(L - x_1)^2 + L_0^2}}$$

and

$$\varphi_L(x_1) := (\widetilde{I}_L(x))_1 = -L + \frac{L^2}{(L - x_1)^2 + L_0^2}(L - x_1).$$

Now, let us observe that φ_L has the following properties,

$$\varphi_L(x_1) = 0 \Longleftrightarrow x_1 = d := \frac{L - \sqrt{L^2 - 4L_0^2}}{2} > 0,$$
$$\varphi_L \geqq 0 \text{ in } [-4L_0, d], \qquad \varphi_L \leqq \text{ in } [d, 4L_0].$$

Also,

$$\Big\{ \vartheta_\varepsilon^L(x) = 0 \Big\} \cap Q_{L_0} \cap \Big\{ x \in Q_{L_0}; |x'| = L_0 \Big\} \Longleftrightarrow x_1 = d.$$

If $\tau > 0$ is a small enough, again for L large enough,

$$1 - \tau \leqq \frac{\mathrm{d}\varphi_L}{\mathrm{d}x_1}(x_1) \leqq 1 + \tau \text{ for } x_1 \in [-4L_0, 4L_0]$$

and

$$1 - \tau \leqq \widetilde{\rho}_L(x_1) \leqq 1 + \tau \text{ for } x_1 \in [-4L_0, 4L_0].$$

In this way, we have for $x \in Q_{L_0} \cap \Big\{ x = (x_1, x') \in \mathbb{R}^N; |x'| = L_0 \Big\} \cap \{x_1 \geqq d\}$

$$\begin{aligned} \vartheta_\varepsilon^L(x) &\leqq (1 + \tau)^{N-2} \overline{Q}_\varepsilon(\varphi_L(x_1)) \leqq (1 + \tau)^{N-2} \overline{\alpha} \varphi_L(x_1) \\ &\leqq (1 + \tau)^{N-1} \overline{\alpha}(x_1 - d) = \widetilde{\alpha}(x_1 - d)^+. \end{aligned}$$

Similarly,

$$\begin{aligned} &x \in Q_{L_0} \cap \Big\{ x = (x_1, x') \in \mathbb{R}^N; |x'| = L_0 \Big\} \cap \{x_1 \leqq d\} \\ &\Rightarrow \vartheta_\varepsilon^L(x) \leqq -(1 - \tau)^{N-1} \overline{\gamma}(d - x_1) = -\widetilde{\gamma}(x_1 - d)^-. \end{aligned}$$

The fact that $Q_\varepsilon \leqq \vartheta_\varepsilon^L$ along *span* $\{e_1\}$ is straightforward. If we now choose $\overline{L}$ large enough in such a way that all the estimates above holds, we define for every ε

$$\vartheta_\varepsilon := \vartheta_\varepsilon^{\overline{L}}.$$

Thus, (a) and (c) are proven. (b) follows from the geometric properties of inversions. □

Proposition 7.3. (Below condition barrier—I) *Let $0 < \overline{\sigma} < \sigma < 1$, $\delta, \mu < 0$ and $\overline{\alpha}, \gamma > 0$ be such that*

$$0 < H_{\delta,\mu}(\overline{\alpha}) - M_{\overline{\sigma}} < H_{\delta,\mu}(\gamma).$$

Let $0 < \alpha^\star < \overline{\alpha}$ be close to $\overline{\alpha}$. There exists a function $\chi_\varepsilon \in C^2(Q_{L_0})$ such that for every $\varepsilon > 0$

(a) $\Delta\chi_\varepsilon(x) \leqq \beta_\varepsilon(\chi_\varepsilon(x))F(\nabla\chi_\varepsilon(x))$ *for x in Q_{L_0};*
(b) *one has*

$$\begin{gathered}\chi_\varepsilon > 0 \textit{ in } Q_{L_0} \cap \mathbb{B}_\star{}^C\\ \chi_\varepsilon < 0 \textit{ in } Q_{L_0} \cap \mathbb{B}_\star^\circ\\ \chi_\varepsilon = 0 \textit{ on } Q_{L_0} \cap \mathbb{S}_\star\\ \mathbb{S}_\star \cap \partial Q_{L_0} \subset \{x_1 = d_\star\}\ , \qquad d_\star = d_\star(\textit{radius of } \mathbb{S}_\star, L_0) < 0,\end{gathered}$$

where $\mathbb{S}_\star = \partial\mathbb{B}_\star$, and $\mathbb{B}_\star$ is a closed ball completely contained in the half space $\left\{x_1 \leqq 0\right\}$, centered in the negative semiaxis generated by e_1 and tangent to the hyperplane $\{x_1 = 0\}$;

(c) *There exist $0 < \widetilde{\alpha} < \alpha^\star$ and $0 < \widetilde{\gamma} < \gamma$ and constants $C, D > 0$ not depending on ε such that if $\mathcal{W}^\star(x) = \widetilde{\alpha}x_1^+ - \widetilde{\gamma}x_1^-$, then*

$$\mathcal{W}^\star_\varepsilon(x) := \mathcal{W}^\star(x - \varepsilon De_1) + C\varepsilon \leqq \chi_\varepsilon(x) \textit{ for all } x \in Q_{L_0} \tag{7.18}$$

$$\begin{aligned}&\mathcal{W}^\star(x + (d_\star - \varepsilon D)e_1)\\ &\quad \leqq \chi_\varepsilon(x) \textit{ for } x \in Q_{L_0} \cap \left\{x = (x_1, x') \in \mathbb{R}^N;\ |x'| = L_0\right\}\end{aligned} \tag{7.19}$$

$$Q_\varepsilon \geqq \chi_\varepsilon \textit{ along } \ \textit{span}\,\{e_1\}, \tag{7.20}$$

where $Q_\varepsilon(x) := \overline{Q}_\varepsilon(x_1)$, and $\overline{Q}_\varepsilon(s) = P_\varepsilon(s + a_\varepsilon)$, P_ε is the solution to $(\mathcal{E}^\varepsilon_{\overline{\alpha},\delta,\mu,\sigma})$ in Lemma (6.6), where a_ε is chosen such that $\overline{Q}_\varepsilon(0) = 0$. Moreover, $\widetilde{\alpha}$ can be taken as close as we wish to $\alpha^\star$.

Proof. The proof is very similar to the proof of Proposition (7.2). As suggested in (c), if we define

$$Q_\varepsilon(x) := \overline{Q}_\varepsilon(x_1)$$

and for $\overline{J}_L = J_L \circ \mathcal{R}_1$ and $\overline{\varrho}_L = \varrho \circ \mathcal{R}_1$ we set

$$\chi^L_\varepsilon(x) := (T_L[Q_\varepsilon] \circ \mathcal{R}_1)(x) = T_L[Q_\varepsilon](\mathcal{R}_1(x)) = (\overline{\varrho}_L(x))^{N-2}Q_\varepsilon(\overline{J}_L(x)). \tag{7.21}$$

Thus, by proceeding analogously to Proposition (7.2), we obtain for L large enough and for $x \in Q_{L_0}$,

$$\Delta\chi^L_\varepsilon(x) \leqq (\mathcal{E}_\sigma(\beta))_\varepsilon(\chi^L_\varepsilon(x))F(\nabla\chi^L_\varepsilon(x)) \leqq \beta_\varepsilon(\chi^L_\varepsilon(x))F(\nabla\chi^L_\varepsilon(x)).$$

Also, similarly to the proof of Proposition (7.2), by Lemma (6.6)(b.1), there exists $0 < \overline{\gamma} < \gamma$ such that

$$\overline{\gamma} \leqq (\overline{Q}_\varepsilon)_s \leqq \overline{\alpha}.$$

It is easy to check the properties below

1. There exists a constant $\overline{D}$ such that

$$\begin{cases} \overline{Q}_\varepsilon(s) \geqq \alpha^\star s, & \text{for } s \geqq \overline{D}\varepsilon \\ \overline{Q}_\varepsilon(s) = \overline{\gamma} s, & \text{for } s \leqq 0 \\ \overline{Q}_\varepsilon(s) \geqq \overline{\gamma} s, & \text{for every } s. \end{cases} \tag{7.22}$$

2. $x \in Q_{L_0} \Rightarrow (J_L(x))_1 \geqq L - \frac{L^2}{L+x_1} := \tau_L^\star(x_1)$, with

$$\tau_L^\star \geqq 0 \text{ in } \{x_1 \geqq 0\} \qquad \text{and} \qquad \tau_L^\star \leqq 0 \text{ in } \{x_1 \leqq 0\}.$$

3. If $\tau > 0$ is a small number, for L large enough we have

$$1 - \tau \leqq \varrho_L \leqq 1 + \tau \text{ in } Q_{L_0}$$
$$1 - \tau \leqq \frac{\mathrm{d}}{\mathrm{d}x_1}\tau_L^\star \leqq 1 + \tau \text{ in } [-4L_0, 4L_0].$$

From (3), there exists D such that $x_1 \geqq D\varepsilon \Rightarrow \tau_L^\star(x_1) \geqq \overline{D}\varepsilon$, and thus,

$$\begin{aligned} \{x_1 \geqq \overline{D}\varepsilon\} \cap Q_{L_0} \Rightarrow \chi_\varepsilon^L(x) &= (\varrho_L(x))^{N-2}\overline{Q}_\varepsilon((\overline{J}_L(x))_1) \\ &\geqq (1-\tau)^{N-2}\overline{Q}_\varepsilon(\tau_L^\star(x_1)) \geqq (1-\tau)^{N-1}\alpha^\star x_1. \end{aligned}$$

Proceeding similarly, we find

$$\begin{aligned} x_1 \leqq 0 &\Rightarrow \chi_\varepsilon^L(x) \geqq -(1+\tau)^{N-1}\overline{\gamma}x_1^- = -\widetilde{\gamma}x_1^- \\ 0 \leqq x_1 \leqq \overline{D}\varepsilon &\Rightarrow \chi_\varepsilon^L(x) \geqq (1-\tau)^{N-1}\overline{\gamma}x_1 = \gamma^\star x_1^+. \end{aligned}$$

Setting $C = \gamma^\star D$, it follows that

$$\mathcal{W}_\varepsilon^\star(x) = \mathcal{W}^\star(x - D\varepsilon) + C\varepsilon \leqq \chi_\varepsilon(x) \quad \text{for all } x \in Q_{L_0}.$$

Following the ideas above and proceeding as in the proof of Proposition (7.2), we finish this proof. □

Now, to finish this section, we construct the last barrier.

Proposition 7.4. (Below condition barrier—II) *Let* $0 < \sigma < \overline{\sigma} < 1$ *and* $\delta, \mu < 0$ *with* $\overline{\alpha} > 0$ *such that*

$$H_{\delta,\mu}(\overline{\alpha}) < M_{\overline{\sigma}}.$$

Then, there exist a function $\chi_\varepsilon \in C^2(Q_{L_0})$ *and constants* $C, D > 0$ *(independent of* ε*) satisfying for every* $\varepsilon > 0$

(a) $\Delta\chi_\varepsilon(x) \leqq \beta_\varepsilon(\chi_\varepsilon(x))F(\nabla\chi_\varepsilon(x))$ *for* x *in* Q_{L_0};
(b) $\chi_\varepsilon \geqq C\varepsilon$ *in* Q_{L_0} *and* $\chi_\varepsilon \leqq Q_\varepsilon$ *for* $\{x_1 \geqq 0\}$; *where* $Q_\varepsilon(x) := P_\varepsilon(x_1)$, P_ε *solution to* $(\mathcal{E}^\varepsilon_{\overline{\alpha},\delta,\mu,\sigma})$ *in Lemma* (6.6)*;*
(c) *There exist* $0 < \widetilde{\alpha} < \overline{\alpha}$ *and a constant* $C > 0$ *independent of* ε *such that*

$$\chi_\varepsilon \geqq \widetilde{\alpha}x_1^+ + D\varepsilon \quad \text{for } x \in Q_{L_0} \cap \{x_1 \geqq 0\}. \tag{7.23}$$

Moreover, $\widetilde{\alpha}$ *can be taken as close as we wish to* $\overline{\alpha}$.

(d) *There exists a negative number $d_\star$ independent of ε such that*

$$\chi_\varepsilon(x) \to g(x_1) \text{ uniformly on } \left\{x = (x_1, x') \in Q_{L_0}; |x'| = L_0\right\}$$

and

$$g(x_1) \geqq \widetilde{\alpha}(x_1 - d_\star) \qquad \text{for } x_1 \geqq d_\star.$$

Proof. Defining $\chi_\varepsilon^L(x) = (T_L[Q_\varepsilon] \circ \mathcal{R}_1)(x)$ as in Proposition (7.3), where $Q_\varepsilon(x)$ is specified above, then, similarly to the proof of Proposition (7.2), for L large enough,

$$\Delta \chi_\varepsilon^L(x) \leqq \beta_\varepsilon(\chi_\varepsilon(x)) F(\nabla \chi_\varepsilon(x)) \text{ for } x \text{ in } Q_{L_0}.$$

But now, by Lemma (6.6)(b.2), we have for $C = (1-\tau)^{N-2}\overline{\kappa_{\overline{\sigma}}}$

$$\chi_\varepsilon^L(x) = (\varrho_L(x))^{N-2}\overline{Q}_\varepsilon((\overline{J}_L(x))_1) \geqq C\varepsilon \qquad \forall x \in Q_{L_0}$$

and also, for $D = (1-\tau)^{N-2}\kappa_{\overline{\sigma}}^+$ and $x \in Q_{L_0} \cap \left\{x_1 \geqq 0\right\}$,

$$\chi_\varepsilon^L(x) \geqq (1-\tau)^{N-2}\overline{Q}_\varepsilon(\tau_L^\star(x_1)) \geqq (1-\tau)^{N-1}\alpha x_1^+ + D\varepsilon = \widetilde{\alpha} x_1^+ + D\varepsilon.$$

Now, as before, we choose a universal L for which the estimates above hold uniformly in ε. From Lemma (6.6)(b.2) we conclude that, for some $\overline{\gamma} > 0$,

$$Q_\varepsilon \to P^\star(x) := \overline{\alpha} x_1^+ + \overline{\gamma} x_1^- \text{ uniformly in } \mathbb{R}^N.$$

Since, Kelvin transforms preserve uniform convergence, we have

$$\chi_\varepsilon \to T_L[P^\star] \circ \mathcal{R}_1 \text{ uniformly in } Q_{L_0} \text{ as } \varepsilon \to 0.$$

In particular, for $x \in Q_{L_0} \cap \left\{x = (x_1, x') \in \mathbb{R}^N; |x'| = L_0\right\}$,

$$\chi_\varepsilon \to g(x_1) := (\widetilde{\varrho}_L(x_1))^{N-2} P^\star(\varphi_L^\star(x_1)) \text{ uniformly as } \varepsilon \to 0, \tag{7.24}$$

where

$$\widetilde{\varrho}_L(x_1) = \frac{L}{\sqrt{(L+x_1)^2 + L_0^2}}$$

and

$$\varphi_L^\star(x_1) = L - \frac{L^2}{(L+x_1)^2 + L_0^2}(L + x_1).$$

Clearly,

$$x_1 \in [-4L_0, 4L_0] \text{ with } g(x_1) = 0 \Longleftrightarrow x_1 = d_\star := \frac{1}{2}(-L + \sqrt{L^2 - 4L_0^2}) < 0.$$

L can be taken large enough such that, if τ is a small number,

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\varphi_L^\star(x_1) \geqq 1-\tau \qquad \forall x \in Q_{L_0},$$

and thus

$$\varphi_L^\star(x_1) \geqq (1-\tau)(x_1-d_\star) \qquad \forall x \in Q_{L_0}.$$

So,

$$x_1 \geqq d_\star \Rightarrow g(x_1) \geqq (1-\tau)^{N-1}\overline{\alpha}(x_1-d_\star) = \widetilde{\alpha}(x_1-d_\star).$$

From the convergence, (7.24), (d) follows, finishing the proof. □

8. Classification of global profiles

The purpose of this section is to classify the global profiles (2-plane functions) that will appear in the blow-up analysis of our free boundary problem in the next section. The precise statement of the result is the following:

Theorem 8.1. (Classification of global profiles) *Let v_{ε_j} be a family of least viscosity supersolutions to $(SE)_{\varepsilon_j}$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j=1}^{\infty}\Omega_j = \mathbb{R}^N$. Suppose v_{ε_j} converge to $v(x)$ uniformly in compact subsets of $\mathbb{R}^N$, then*

$$\begin{aligned} v(x) &= \alpha x_1^+ - \gamma x_1^- \quad \textit{with } \alpha > 0, \gamma \geqq 0 \Longrightarrow H(\alpha) - H(\gamma) = M,\\ v(x) &= \alpha x_1^+ + \gamma x_1^+ \quad \textit{with } \alpha > 0, \gamma \geqq 0 \Longrightarrow H(\alpha) \leqq M. \end{aligned}$$

Heuristically, the idea of the proof is the following. If the slopes of a limit would satisfy $H(\alpha) - H(\gamma) > M$, then we could use the above condition barriers (see heuristic discussion before Proposition (7.2)) to construct ε-regularized 2-plane functions which are "almost" strict subsolution to SE_ε with uniformly curved free boundary and bigger opening v. If we bring them from the right starting at "infinity", this geometry would force (for ε small enough) a interior contact with v_ε violating Lemma (6.3). Analogously, if $H(\alpha) - H(\gamma) < M$, then we could use the below condition barriers to construct ε-regularized 2-plane functions also with uniformly curved free boundary and smaller opening than v. If we bring them from the left, starting at "infinity", this geometry would force (for ε small enough) a interior crossing of their graph with the graph of v_ε violating the least supersolution condition. The proof will be divided in several Propositions, analyzing different scenarios.

Proposition 8.2. *Let v_{ε_j} be viscosity solutions to $(SE)_{\varepsilon_j}$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j=1}^{\infty}\Omega_j = \mathbb{R}^N$. Suppose v_{ε_j} converge to $v = \alpha x_1^+ - \gamma x_1^-$ uniformly in compact subsets of $\mathbb{R}^N$, with $\alpha > 0$, $\gamma \geqq 0$ as $\varepsilon_j \to 0$. Then,*

$$H(\alpha) - H(\gamma) \leqq M. \tag{8.1}$$

Proof. Let us suppose by contradiction that

$$H_{0,0}(\alpha) - H_{0,0}(\gamma) = H(\alpha) - H(\gamma) > M.$$

In this way, we find $0 < \overline{\alpha} < \alpha$ and $\overline{\sigma} > 1$ such that $H_{0,0}(\overline{\alpha}) - H_{0,0}(\gamma) > M_\sigma = \overline{\sigma}^2 M > M$. So, by continuity, there exist $\delta > 0$, $\mu > 0$ such that

$$H_{\delta,\mu}(\overline{\alpha}) - H_{\delta,\mu}(\gamma) > M_{\overline{\sigma}}. \tag{8.2}$$

Thus, we are now in a position to use the above condition barriers constructed in Proposition (7.2) with $\alpha > \widetilde{\alpha}$. In what follows, we will freely use them as well as the notation employed there. Let small $\eta > 0$ be given. By assumption, we can find $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that

$$\varepsilon < \varepsilon_0 \Longrightarrow ||v_\varepsilon - v||_{L^\infty(Q_{L_0})} < \eta$$

Setting $c_1 = \dfrac{1}{\overline{\gamma}}$ and $c_2 = c_1 + \dfrac{1}{\overline{\gamma}}$, we may assume that η is so small that

$$(c_1 + c_2)\,\eta < \frac{d}{4} < \frac{L_0}{4}. \tag{8.3}$$

Setting $Q_0 = \frac{1}{2} Q_{L_0}$ and $Q_{00} = \frac{1}{4} Q_{L_0}$, we have that, for every $\xi \in \mathbb{R}^N$ with $|\xi| \leqq L_0$, the functions $(\vartheta_\varepsilon)_\xi : Q_0 \to \mathbb{R}$ given by $(\vartheta_\varepsilon)_\xi(x) = \vartheta_\varepsilon(x + \xi)$ are well defined. In particular, we can define $\vartheta^\star_\varepsilon : Q_0 \to \mathbb{R}$, given by

$$\vartheta^\star_\varepsilon(x) = \vartheta_\varepsilon(x - c_1 \eta e_1).$$

It is easy to see that

$$\vartheta^\star_\varepsilon(x) \leqq \mathcal{W}(x - c_1 \eta e_1) < v(x) - \eta < v_\varepsilon(x) \quad \text{for } x \in Q_{00}.$$

If $|T| \leqq \frac{L_0}{4}$ then we can define $(\vartheta^\star_\varepsilon)_T : Q_{00} \to \mathbb{R}$ by

$$(\vartheta^\star_\varepsilon)_T(x) := \vartheta^\star_\varepsilon(x + T e_1).$$

So, let us consider the set of translations

$$\Gamma_\varepsilon = \left\{ 0 \leqq T \leqq \frac{L_0}{4}; \quad (\vartheta^\star_\varepsilon)_t \leqq v_\varepsilon \quad \text{in } Q_{00}, \quad 0 \leqq t \leqq T \right\} \quad \text{and} \quad \mathcal{I}_\varepsilon = \sup \Gamma_\varepsilon.$$

Let us recall that $Q_\varepsilon(x) = \overline{Q}_\varepsilon(x_1) \geqq \overline{\gamma} x_1$ for $x_1 \geqq 0$ and (7.12). In particular, considering $x = l e_1$, with $|l| \leqq \frac{L_0}{4}$ and $l \geqq -\dfrac{\eta}{\overline{\gamma}}$

$$\vartheta_\varepsilon\left((l + \frac{\eta}{\overline{\gamma}}) e_1 \right) \geqq \overline{\gamma} l + \eta,$$

but

$$\vartheta_\varepsilon\left(\left(l + \frac{\eta}{\overline{\gamma}} \right) e_1 \right) = \vartheta_\varepsilon((l - c_1\eta + c_2\eta) e_1)) = (\vartheta^\star_\varepsilon)_{c_2\eta}(l e_1).$$

Taking now $l = 0$, we find

$$(\vartheta_\varepsilon^\star)_{c_2\eta}(0) \geqq \eta > v_\varepsilon(0).$$

In other words, if we translate $\vartheta_\varepsilon^\star$ by $c_2\eta$, we have gone too far in terms of touching v_ε from below. This clearly implies that

$$\mathcal{I}_\varepsilon \leqq c_2\eta. \tag{8.4}$$

Moreover, for each $n \geqq 1$, we can find $0 \leqq \mathcal{T}_n^\varepsilon \leqq \mathcal{I}_\varepsilon + \frac{1}{n}$ and $x_n^\varepsilon \in Q_{00}$ such that

$$(\vartheta_\varepsilon^\star)_{\mathcal{T}_n^\varepsilon}(x_n^\varepsilon) > v_\varepsilon(x_n).$$

Passing to a subsequence if necessary, we can assume $x_n^\varepsilon \to x_0^\varepsilon$ and $\mathcal{T}_n^\varepsilon \to \mathcal{T}_\varepsilon$ as $n \to \infty$, where $x_0^\varepsilon \in Q_{00}$ and $0 \leqq \mathcal{T}_\varepsilon \leqq \mathcal{I}_\varepsilon$. Thus, we have

$$\begin{aligned}(\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon}(x_0^\varepsilon) &= v_\varepsilon(x_0),\\ (\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon} &\leqq v_\varepsilon \text{ in } Q_{00}.\end{aligned}$$

Now, since for $x \in Q_{00}$, by (7.10),

$$v_\varepsilon(x) - (\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon}(x) \geqq v(x) - \eta - (\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon}(x) \geqq v(x) - \eta - \mathcal{W}(x - c_1\eta e_1 + \mathcal{T}_\varepsilon e_1).$$

We have

$$\begin{aligned}& x \in \partial Q_{00} \cap \{x_1 = \pm L_0\} \Rightarrow v_\varepsilon - (\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon}(x)\\ &\quad \geqq \min\{(\alpha - \widetilde{\alpha})L_0 + A(\varepsilon,\eta), (\widetilde{\gamma} - \gamma)L_0 + B(\varepsilon,\eta)\} \geqq c_3 > 0,\end{aligned}$$

if η is chosen small enough, since

$$\begin{aligned}A(\varepsilon,\eta) &= (\widetilde{\alpha}c_1\eta - \widetilde{\alpha}\mathcal{T}_\varepsilon - \eta) \to 0 \text{ as } \eta \to 0,\\ B(\varepsilon,\eta) &= (\widetilde{\gamma}c_1\eta - \widetilde{\gamma}\mathcal{T}_\varepsilon - \eta) \to 0 \text{ as } \eta \to 0.\end{aligned}$$

Now, once

$$\rho > 0 \Longrightarrow v(x) - \mathcal{W}(x - \rho e_1) \geqq \min\{\alpha, \widetilde{\gamma}\}\,\rho \quad \forall x \in \mathbb{R}^N$$

we can estimate, using (7.11),

$$\begin{aligned}& x \in \partial Q_{00} \cap \left\{x = (x_1, x') \in \mathbb{R}^N; |x'| = L_0\right\}\\ &\quad \Rightarrow v_\varepsilon(x) - (\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon}(x) \geqq v(x) - \eta - \vartheta_\varepsilon(x - c_1\eta e_1 + \mathcal{T}_\varepsilon e_1)\\ &\quad \geqq v(x) - \eta - \mathcal{W}(x - c_1\eta e_1 + \mathcal{T}_\varepsilon e_1 - d e_1)\\ &\quad \geqq \min\{\alpha, \widetilde{\gamma}\}\,(c_1\eta - \mathcal{T}_\varepsilon + d) - \eta \geqq \frac{\min\{\alpha, \widetilde{\gamma}\}}{4} d,\end{aligned}$$

for η small enough, since $c_1\eta - \mathcal{T}_\varepsilon \to 0$ as $\eta \to 0$, by (8.4). In particular, we conclude that, if $\eta > 0$ is chosen small enough, on the boundary of Q_{00}, $(\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon}$ is strictly below v_ε for ε small enough. This forces the contact point $x_0^\varepsilon \in \operatorname{int}(Q_{00})$.

Now, from the translation invariance, Remark (5.5), $\overline{\vartheta}_\varepsilon = (\vartheta^\star_\varepsilon)_{\mathcal{T}_\varepsilon}$ satisfies, for some $\sigma > 1$,

$$\Delta\overline{\vartheta}_\varepsilon(x) \geqq (\mathcal{E}_\sigma(\beta))_\varepsilon(\overline{\vartheta}_\varepsilon(x))F(\nabla\overline{\vartheta}_\varepsilon(x)) \text{ in } Q_{00}.$$

Since v_ε are solutions to SE_ε, this contradicts Lemma (6.3). In this way,

$$H(\alpha) - H(\gamma) \leqq M,$$

and the Theorem is proven. □

Using the same ideas of Theorem (8.2), we can state the following corollary. The proof will follow *mutatis mutandis*. Details are left to the reader.

Corollary 8.3. *Let v_{ε_j} be viscosity solutions to $(SE)_{\varepsilon_j}$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j=1}^{\infty}\Omega_j = \mathbb{R}^N$. Suppose v_{ε_j} converge to $v = \alpha x_1^+ + \gamma x_1^+$ uniformly on compact subsets of $\mathbb{R}^N$, with $\alpha > 0, \gamma \geqq 0$ as $\varepsilon_j \to 0$. Then,*

$$H(\alpha) \leqq M.$$

Now, we study the situation where the limit is a strict 2-phase case. The idea is very similar to the Proposition (8.2) but approaching the curved barriers from the other side.

Proposition 8.4. *Let v_{ε_j} be a family of least viscosity supersolutions to $(SE)_{\varepsilon_j}$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j=1}^{\infty}\Omega_j = \mathbb{R}^N$. Suppose v_{ε_j} converge to $v = \alpha x_1^+ - \gamma x_1^-$ uniformly in compact subsets of $\mathbb{R}^N$, with $\alpha > 0, \gamma > 0$ as $\varepsilon_j \to 0$. Then,*

$$H(\alpha) - H(\gamma) \geqq M. \tag{8.5}$$

Proof. Let us suppose by contradiction that,

$$H_{0,0}(\alpha) - H_{0,0}(\gamma) = H(\alpha) - H(\gamma) < M.$$

In this way, we can find $0 < \overline{\sigma} < 1$ such that

$$H_{0,0}(\alpha) - M_{\overline{\sigma}} < H_{0,0}(\gamma).$$

Since $\gamma > 0$, we can find $\overline{\alpha} > \alpha$ such that

$$0 < H_{0,0}(\overline{\alpha}) - M_{\overline{\sigma}} < H_{0,0}(\gamma).$$

By continuity, there exist $\delta, \mu < 0$ such that

$$0 < H_{\delta,\mu}(\overline{\alpha}) - M_{\overline{\sigma}} < H_{\delta,\mu}(\gamma).$$

Now, let $\alpha < \alpha^\star < \overline{\alpha}$. In this way, we are in a position to use the below condition barriers χ_ε constructed in Proposition (7.3). Let small $\eta > 0$ be given. We can find $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that

$$\varepsilon < \varepsilon_0 \Rightarrow ||v_\varepsilon - v||_{L^\infty(Q_{L_0})} < \eta.$$

Let us set $c_1 := \max\left\{\frac{1}{\alpha}, \frac{1}{\gamma}\right\}$, $c_2 = c_1 + \frac{1}{\gamma}$ and assume η is so small that

$$(c_1 + c_2)\,\eta < \frac{d_\star}{4} < \frac{L_0}{4}.$$

If $Q_0 = \frac{1}{2} Q_{L_0}$ and $Q_{00} = \frac{1}{4} Q_{L_0}$ are as before, we can define

$$\chi_\varepsilon^\star(x) = \chi_\varepsilon(x + c_1 \eta e_1), \quad x \in Q_0.$$

It is easy to check that

$$\chi_\varepsilon^\star(x) \geqq \mathcal{W}^\star(x + c_1\eta e_1) > v(x) + \eta > v_\varepsilon(x) \text{ for } x \in Q_{00}. \tag{8.6}$$

As before, for $|T| \leqq \dfrac{L_0}{4}$, we can define $(\chi_\varepsilon^\star)_T : Q_{00} \to \mathbb{R}$ by

$$(\chi_\varepsilon^\star)_T(x) := \chi_\varepsilon^\star(x - Te_1)$$

and consider the set

$$\Gamma_\varepsilon = \left\{0 \leqq T \leqq \frac{L_0}{4};\ (\chi_\varepsilon^\star)_t \geqq v_\varepsilon \text{ in } Q_{00},\quad 0 \leqq t \leqq T\right\} \text{ and } \mathcal{I}_\varepsilon = \sup \Gamma_\varepsilon.$$

Let us recall that $Q_\varepsilon(x) = \overline{Q}_\varepsilon(x_1) \leqq \overline{\gamma} x_1$ for $x_1 \leqq 0$ and (7.20). In particular, considering $x = le_1$, with $|l| \leqq \frac{L_0}{4}$ and $l \leqq \dfrac{\eta}{\gamma}$

$$\chi_\varepsilon\left(\left(l - \frac{\eta}{\overline{\gamma}}\right)e_1\right) \leqq \overline{\gamma} l - \eta,$$

but

$$\chi_\varepsilon\left(\left(l - \frac{\eta}{\overline{\gamma}}\right)e_1\right) = \chi_\varepsilon((l + \eta c_1 - \eta c_2)e_1) = (\chi_\varepsilon^\star)_{c_2\eta}(le_1).$$

Taking now, $l = 0$, we find

$$(\chi_\varepsilon^\star)_{c_2\eta}(0) \leqq -\eta < v_\varepsilon(0).$$

This means that if we translate $\chi_\varepsilon^\star$ by $c_2\eta$ we have gone too far in terms of touching v_ε from above. This clearly implies,

$$\mathcal{I}_\varepsilon \leqq c_2\eta. \tag{8.7}$$

For $\tau > 0$ small, we can find $0 \leqq \mathcal{T}_\varepsilon^\tau \leqq \mathcal{I}_\varepsilon + \tau$ and $x_\varepsilon^\tau \in Q_{00}$ such that, setting

$$Z_\varepsilon^\tau(x) := (\chi_\varepsilon^\star)_{\mathcal{T}_\varepsilon^\tau}(x), \qquad \tau > 0,$$

we have

$$Z_\varepsilon^\tau(x_\varepsilon^\tau) < v_\varepsilon(x_\varepsilon^\tau).$$

Let us observe that for $x \in Q_{00}$, by (7.18)

$$\begin{aligned} Z_\varepsilon^\tau(x) - v_\varepsilon(x) &\geqq \mathcal{W}_\varepsilon^\star(x + (c_1\eta - \mathcal{T}_\varepsilon - \tau)e_1) - v(x) - \eta \\ &\geqq \mathcal{W}^\star((x + (c_1\eta - \mathcal{T}_\varepsilon - \tau)e_1) - \varepsilon D e_1) + C\varepsilon - v(x) - \eta. \end{aligned}$$

In particular,

$$\begin{aligned} &x \in Q_{00} \cap \{x_1 = \pm L_0\} \\ &\Rightarrow Z_\varepsilon^\tau(x) - v_\varepsilon(x) \\ &\geqq \min\left\{(\widetilde{\alpha} - \alpha)L_0 + \overline{A}(\eta, \varepsilon, \tau), (\gamma - \widetilde{\gamma})L_0 + \overline{B}(\eta, \varepsilon, \tau)\right\} \geqq c_4 > 0 \end{aligned}$$

if η and τ are chosen small enough, since by (8.7),

$$\begin{aligned} \overline{A}(\eta, \varepsilon, \tau) &= \widetilde{\alpha}(c_1\eta - \mathcal{T}_\varepsilon^\tau - \varepsilon D) + C\varepsilon - \eta \to 0 \text{ as } \varepsilon, \eta, \tau \to 0 \\ \overline{B}(\eta, \varepsilon, \tau) &= \widetilde{\gamma}(c_1\eta - \mathcal{T}_\varepsilon^\tau - \varepsilon D) + C\varepsilon - \eta \to 0 \text{ as } \varepsilon, \eta, \tau \to 0. \end{aligned}$$

Furthermore, by (7.19), if $x \in Q_{00} \cap \left\{x = (x_1, x') \in \mathbb{R}^N; |x'| = L_0\right\}$

$$\begin{aligned} Z_\varepsilon^\tau(x) - v_\varepsilon(x) &\geqq \mathcal{W}^\star(x + (c_1\eta - \mathcal{T}_\varepsilon^\tau)e_1 + (d_\star - \varepsilon D)e_1) - v(x) - \eta \\ &\geqq \min\{\overline{\alpha}, \gamma\}(c_1\eta - \mathcal{T}_\varepsilon^\tau + d_\star - \varepsilon D) > c_5 > 0 \end{aligned}$$

for ε, η, τ small enough, since $c_1\eta - \mathcal{T}_\varepsilon^\tau - \varepsilon D \to 0$ as $\varepsilon, \eta, \tau \to 0$. In this way, by the translation invariance of SE_ε, Remark (5.5), Z_ε^τ is a supersolution of SE_ε in Q_{00}. In conclusion, for η, τ, ε small enough, we have

$$\begin{aligned} &Z_\varepsilon^\tau > v_\varepsilon \text{ in } \partial Q_{00} \\ &Z_\varepsilon^\tau(x_\varepsilon^\tau) < v_\varepsilon(x_\varepsilon^\tau) \text{ with } x_\varepsilon^\tau \in \operatorname{int}(Q_{00}), \end{aligned}$$

which contradicts the fact that v_ε is the least supersolution of SE_ε. So, $H(\alpha) - H(\gamma) \geqq M$ and the Theorem is proven. □

Finally, we treat the case where the profile is of one-phase type. The idea is the same as the previous Theorem, taking into account, the cubic decay of the least supersolutions to prevent an "early" touching.

Proposition 8.5. *Let v_{ε_j} be a family of least viscosity supersolutions to $(SE)_{\varepsilon_j}$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j=1}^\infty \Omega_j = \mathbb{R}^N$. Suppose v_{ε_j} converge to $v = \alpha x_1^+$ uniformly on compact subsets of $\mathbb{R}^N$, with $\alpha > 0$ and $\varepsilon_j \to 0$. Then,*

$$H(\alpha) \geqq M. \tag{8.8}$$

Proof. The proof is similar to the proof of Theorem (8.4). Once more, let us assume by contradiction that $H(\alpha) < M$. As before, we can find $\overline{\alpha} > \alpha$, $\delta, \mu < 0$ and $\overline{\sigma} < 1$ such that

$$H_{\delta,\mu}(\overline{\alpha}) < M_{\overline{\sigma}}.$$

Let us now choose $\alpha < \widetilde{\alpha} < \overline{\alpha}$. We are now in a position to use the barriers constructed in Proposition (7.4). By assumption, there exists $\varepsilon_0 = \varepsilon_0(\eta)$ such that

$$\varepsilon \leqq \varepsilon_0 \Rightarrow ||v_\varepsilon - v||_{L^\infty(Q_{L_0})} < \kappa_2 \eta.$$

Let $Q_0 = \frac{1}{2} Q_{L_0}$ and $Q_{00} = \frac{1}{4} Q_{L_0}$ as before and set

$$Q_{L_0}^\eta = \left\{ x = (x_1, x') \in \mathbb{R}^N |\ \ |x|_\infty = \max \left\{ |x_1|, |x'| \right\} \leqq 4L_0 - 2\eta \right\}.$$

Let us define $c_1 := 2\kappa_2/\widetilde{\alpha} + 3$ and $c_2 := c_1 + 2/\overline{\alpha}$, and consider η so small that

$$(c_1 + c_2)\eta < \frac{d_\star}{4} < \frac{L_0}{4}.$$

Observe that, by the cubic decay in the interior, Lemma (6.5), there exists a constant C_η such that

$$x \in Q_{00} \cap \left\{ x_1 \leqq -2\eta \right\} \Rightarrow v_\varepsilon^+ \leqq C_\eta \varepsilon^3.$$

Now, taking the barrier constructed in Proposition (7.4), by (b), $\chi_\varepsilon \geqq C\varepsilon$ in Q_{L_0}. Let us define

$$\chi_\varepsilon^\star(x) = \chi_\varepsilon(x + c_1 \eta e_1).$$

Then, if for $x_1 \geqq -c_1\eta$ we have

$$\chi_\varepsilon^\star(x) \geqq \widetilde{\alpha}(x_1 + c_1\eta e_1) + D\varepsilon \geqq \widetilde{\alpha} x_1 + 2\kappa_2\eta + 3\widetilde{\alpha}\eta + D\varepsilon.$$

In particular, $x_1 \geqq -3\eta \Rightarrow \chi_\varepsilon^\star(x) > 2\kappa_2\eta$. In this way, there exists $\varepsilon_1 = \varepsilon_1(\eta) < \varepsilon_0$, such that, for $\varepsilon < \varepsilon_1$, we have $\chi_\varepsilon - v_\varepsilon > 0$ in Q_0 since

$$\begin{aligned} \chi_\varepsilon^\star - v_\varepsilon &\geqq C\varepsilon - C_\eta \varepsilon^3 > 0 \qquad in \qquad Q_0 \cap \left\{ x_1 \leqq -2\eta \right\} \\ \chi_\varepsilon^\star - v_\varepsilon &\geqq \kappa_2 \eta \qquad in \qquad Q_0 \cap \left\{ x_1 \geqq -3\eta \right\}. \end{aligned}$$

For $|T| \leqq \dfrac{L_0}{4}$, we can define $(\chi_\varepsilon^\star)_T : Q_{00} \to \mathbb{R}$ by

$$(\chi_\varepsilon^\star)_T(x) := \chi_\varepsilon^\star(x - Te_1).$$

and consider the set of translations

$$\Gamma_\varepsilon = \left\{ 0 \leqq T \leqq \frac{L_0}{4}; (\chi_\varepsilon^\star)_t \geqq v_\varepsilon \text{ in } Q_{00},\ \ 0 \leqq t \leqq T \right\} \text{ and } \mathcal{I}_\varepsilon = \sup \Gamma_\varepsilon.$$

Now, let us recall that

$$\chi_\varepsilon(le_1) \leqq Q_\varepsilon(le_1) = P_\varepsilon(l) = \overline{\alpha} l + \varepsilon\kappa_\sigma^+ \qquad \text{for} \qquad l \geqq 0.$$

In particular, if $l \geqq (c_2 - c_1)\eta$ and $l \leqq \frac{L_0}{4}$, then

$$(\chi_\varepsilon^\star)_{c_2\eta}(le_1) = \chi_\varepsilon(le_1 + (c_1 - c_2)\eta e_1) \leqq \overline{\alpha} l + \overline{\alpha}(c_1 - c_2)\eta + \varepsilon\kappa_\sigma^+.$$

Taking $l = 2\eta/\overline{\alpha}$, for ε small enough,

$$(\chi_\varepsilon^\star)_{c_2\eta}\left(\frac{2\eta}{\overline{\alpha}}e_1\right) = \chi_\varepsilon(0) = \varepsilon\kappa_{\sigma}^{+} < \frac{2\alpha\eta}{\overline{\alpha}} = v\left(\frac{2\eta}{\overline{\alpha}}e_1\right).$$

In other words, if we translate $\chi_\varepsilon^\star$ by $c_2\eta$ we have gone too far in terms of touching v_ε from above. This implies, that

$$0 \leqq \mathcal{I}_\varepsilon \leqq c_2\eta. \tag{8.9}$$

For $\tau > 0$ small, we can find $0 \leqq \mathcal{T}_\varepsilon^\tau \leqq \mathcal{I}_\varepsilon + \tau$ and $x_\varepsilon^\tau \in Q_{00}$ such that, setting

$$\begin{aligned} &Z_\varepsilon^\tau(x) := (\chi_\varepsilon^\star)_{\mathcal{T}_\varepsilon^\tau}(x), \qquad \tau > 0 \\ &Z_\varepsilon^\tau(x_\varepsilon^\tau) < v_\varepsilon(x_\varepsilon^\tau). \end{aligned}$$

Now, we estimate, by (7.23)

$$\begin{aligned} x \in \partial Q_{00} \cap \{x_1 = L_0\} &\Rightarrow Z_\varepsilon^\tau(x) - v_\varepsilon(x) \geqq Z_\varepsilon^\tau(x) - v(x) - \eta \\ &\geqq \chi_\varepsilon(x + c_1\eta e_1 - \mathcal{T}_\varepsilon^\tau e_1) - v(x) - \eta \\ &\geqq (\widetilde{\alpha} - \alpha)L_0 + \overline{A}(\varepsilon, \eta, \tau) \\ &\geqq \frac{1}{4}(\widetilde{\alpha} - \alpha)L_0, \end{aligned}$$

since $\overline{A}(\varepsilon, \eta, \tau) = \widetilde{\alpha}(c_1\eta - \mathcal{T}_\varepsilon^\tau) + D\varepsilon - \eta \to 0$ as $\varepsilon, \eta, \tau \to 0$, by (8.9). Clearly, for η, τ, ε small enough,

$$x \in \partial Q_{00} \cap \{x_1 = -L_0\} \Rightarrow Z_\varepsilon^\tau - v_\varepsilon > C\varepsilon - C_\eta\varepsilon^3 > 0.$$

Finally, let us note that

$$x \in \partial Q_{00} \cap \left\{x = (x_1, x') \in Q_{L_0};\, |x'| = L_0\right\} \Rightarrow Z_\varepsilon^\tau(x) > v_\varepsilon(x).$$

Indeed, choosing η, τ small enough, $d_\star - c_1\eta + \mathcal{T}_\varepsilon^\tau < \frac{3}{4}d_\star$. We can assume, passing to a subsequence if necessary, that $\mathcal{T}_\varepsilon \to \mathcal{T}^\tau$ as $\varepsilon \to 0$. By the convergence given in Proposition (7.4)(d),

$$Z_\varepsilon^\tau \to G \qquad \text{uniformly in} \qquad \partial Q_{00} \cap \left\{x = (x_1, x') \in Q_{L_0};\, |x'| = L_0\right\},$$

where

$$G(x_1) = g(x_1 + c_1\eta - \mathcal{T}^\tau).$$

Additionally, if $x_1 \geqq d_\star - c_1\eta + \mathcal{T}^\tau$, then

$$G(x_1) \geqq \widetilde{\alpha}(x_1 - d_\star + c_1\eta - \mathcal{T}^\tau).$$

So, for ε small enough and $x \in \partial Q_{00} \cap \left\{x = (x_1, x') \in Q_{L_0};\, |x'| = L_0\right\} \cap \left\{x_1 \geqq \frac{3}{4}d_\star\right\}$

$$Z_\varepsilon^\tau \geqq G - \eta.$$

In this way,

$$x_1 \geqq d_\star/2 \Rightarrow Z_\varepsilon^\tau > \mathcal{L}(x_1) := \widetilde{\alpha}(x_1 - d_\star) + \overline{B}(\eta, \mathcal{T}, \tau),$$

where $\overline{B}(\eta, \mathcal{T}, \tau) = \widetilde{\alpha}(c_1\eta - \mathcal{T}^\tau) - \eta \to 0$ as $\eta, \tau \to 0$.

Since $\mathcal{L}(d_\star/2) > -\widetilde{\alpha}/4 > \kappa_2\eta$ for η, τ small enough, and $\frac{\mathrm{d}}{\mathrm{d}x_1}\mathcal{L}(x_1) = \widetilde{\alpha} > \alpha$, we conclude that

$$x_1 \geqq d_\star/2 \Rightarrow Z_\varepsilon^\tau \geqq \mathcal{L} > v + \eta > v_\varepsilon,$$

and clearly

$$x_1 \leqq d_\star/2 \Rightarrow Z_\varepsilon^\tau - v_\varepsilon > C\varepsilon - C_\eta\varepsilon^3 > 0.$$

In this way, again by the translation invariance of SE_ε, Remark (5.5), Z_ε^τ is a supersolution of SE_ε in Q_{00}. In conclusion, for η, τ, ε small enough,

$$\begin{aligned} &Z_\varepsilon^\tau > v_\varepsilon \text{ in } \partial Q_{00} \\ &Z_\varepsilon^\tau(x_\varepsilon^\tau) < v_\varepsilon(x_\varepsilon^\tau) \text{ with } x_\varepsilon^\tau \in \operatorname{int}(Q_{00}), \end{aligned}$$

which contradicts the fact that v_ε is the least supersolution of SE_ε. So, $H(\alpha) \geqq M$ and the Theorem is proven. □

9. Limit free boundary problem

In this section, we prove that a limit of the least viscosity supersolutions, u_0, given by Theorem (4.1) is a solution in the Caffarelli's viscosity sense as well as in the pointwise sense ($\mathcal{H}^{N-1}$ almost everywhere) to the free boundary problem

$$\Delta u = 0 \text{ in } \Omega \setminus \partial\{u > 0\} \qquad (FBP)$$

$$H_\nu(u_\nu^+) - H_\nu(u_\nu^-) = M \text{ on } \Omega \cap \partial\{u > 0\},$$

where $u^+ = \max(u, 0)$, $u^- = max(-u, 0)$, ν is the inward unit normal to the free boundary $F(u) = \Omega \cap \partial\{u > 0\}$ and

$$H_\nu(t) = \int_0^t \frac{s}{F(s\nu)}\,\mathrm{d}s.$$

This notion of viscosity solutions to free boundary problems was introduced by Caffarelli in the classical papers [5–7]. Now, we provide these definitions for our problem.

Definition 9.1. Let Ω be a domain in $\mathbb{R}^N$ and $u \in C^0(\Omega)$. Then, u is called a viscosity supersolution to (FBP) if

(i) $\Delta u \leqq 0$ in $\Omega^+ = \Omega \cap \{u > 0\}$;
(ii) $\Delta u \leqq 0$ in $\Omega^- = (\Omega \setminus \Omega^+)^\circ$;

(iii) Along $F(u)$, u satisfies $H_\nu(u_\nu^+) - H_\nu(u_\nu^-) \leqq M$ in the following sense: If $x_0 \in F(u)$ is a regular point from the nonnegative side (that is, there exists $B_r(y) \subset \Omega^+$ with $x_0 \in \partial B_r(y)$) and

$$u^+(x) \geqq \alpha \langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \text{ in } B_r(x_0), \qquad (\alpha > 0)$$

and

$$u^-(x) \geqq \beta \langle x - x_0, \nu\rangle^- + o(|x - x_0|) \text{ in } B_r(x_0)^C, \qquad (\beta \geqq 0)$$

with equality along every nontangential domain in both cases, then $H_\nu(\alpha) - H_\nu(\beta) \leqq M$.

Analogously, we have:

Definition 9.2. Let Ω be a domain in $\mathbb{R}^N$ and $u \in C^0(\Omega)$. Then, u is called a viscosity subsolution to (FBP) if

(i) $\Delta u \geqq 0$ in $\Omega^+ = \Omega \cap \{u > 0\}$;
(ii) $\Delta u \geqq 0$ in $\Omega^- = (\Omega \setminus \Omega^+)^\circ$;
(iii) Along $F(u)$, u satisfies $H_\nu(u_\nu^+) - H_\nu(u_\nu^-) \geqq M$ in the following sense: If $x_0 \in F(u)$ is a regular point from the nonpositive side (that is, there exists $B_r(y) \subset \Omega^-$ with $x_0 \in \partial B_r(y)$) and

$$u^-(x) \geqq \beta \langle x - x_0, \nu\rangle^- + o(|x - x_0|) \text{ in } B_r(x_0), \qquad (\beta \geqq 0)$$

and

$$u^+(x) \geqq \alpha \langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \text{ in } B_r(x_0)^C, \qquad (\alpha \geqq 0)$$

with equality along every nontangential domain in both cases, then $H_\nu(\alpha) - H_\nu(\beta) \geqq M$.

Remark 9.3. There are equivalent definitions for supersolutions and subsolutions to (FBP) above. We mention an equivalent one for supersolutions that will be used in the next results. For this and further details, see [16, Chapter 2].

Equivalently, $u \in C^0(\Omega)$ is a supersolution of (FBP) if conditions (i), (ii) of definition (9.1) are satisfied and if x_0 is a regular point from the nonnegative side with tangent ball B

$$u^+(x) \geqq \alpha \langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \text{ in } B, \qquad (\alpha \geqq 0)$$

then,

$$u^-(x) \geqq \beta \langle x - x_0, \nu\rangle^- + o(|x - x_0|) \text{ in } B^C, \qquad (\beta \geqq 0),$$

for any β such that $H_\nu(\alpha) - H_\nu(\beta) > M$.

Now, we move towards the proof of the major results in this section.

Proposition 9.4. *Let u_0 be a limit of the least supersolutions given by Theorem* (4.1)*. Then, u_0 is a viscosity subsolution to* (*FBP*)*.*

Proof. Clearly, conditions (i), (ii) of definition (9.1) are satisfied. Now, let us suppose that $x_0 \in F(u_0)$ is a regular point from the nonpositive side with tangent ball B. We can assume without lost of generality that $x_0 = 0$ and $\nu = e_1$. In this way, by linear behavior at regular boundary points [16, Lemma 11.17] there exist $\alpha \geqq 0$ and $\beta > 0$

$$u_0^+(x) = \alpha x_1^+ + o(|x|) \text{ in } B^C$$

and

$$u_0^-(x) = \beta x_1^- + o(|x|) \text{ in } B$$

Since u_0^+ is nondegenerate, by Theorem (4.1)(e), or more specifically since (4.1) holds, we conclude that $\alpha > 0$ and thus [16, Lemma 11.17b] B is tangent to $F(u_0)$. In this way, u_0 admits full asymptotic development, that is,

$$u_0(x) = \alpha x_1^+ - \beta x_1^- + o(|x|).$$

Taking now any sequence $\lambda_n \to 0$ and using the blow-up sequence $(u_{\varepsilon_k^{\prime}})_{\lambda_n}$ given in Proposition (5.2), we conclude that there exists a subsequence that we still denote by $\varepsilon_k^{\prime}$ such that $(u_{\varepsilon_k^{\prime}})_{\lambda_k} \to \alpha x_1^+ - \beta x_1^-$ uniformly in compact subsets of $\mathbb{R}^N$. Since by Remark (5.5) the equation SE_ε and the least supersolution property are preserved under the blow-up process, by Theorem (8.1), we conclude that

$$H(\alpha) - H(\beta) = M,$$

where $H = H_{e_1}$. □

Proposition 9.5. *Let u_0 be a limit of the least supersolutions given by Theorem* (4.1)*. Then, u_0 is a supersolution to* (*FBP*)*.*

Proof. As we already observed, u_0 satisfies conditions (i), (ii) of definition (9.1). We will show that the condition in the Remark (9.3) holds. In this way, let us assume that $B = B_r(y)$ is a touching ball from the nonnegative side at x_0 and let us assume that, for some $\alpha \geqq 0$,

$$u_0^+(x) \geqq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \quad \text{in } B, \tag{9.1}$$

where ν is given by the inward unit radial direction of the ball at x_0. If $H_\nu(\alpha) \leqq M$ there is nothing to prove. Otherwise, if

$$H_\nu(\alpha) > M \tag{9.2}$$

let $\gamma \geqq 0$ such that $H_\nu(\alpha) - H_\nu(\gamma) > M$ (we can find such γ, since H_ν is a bijection from $[0, +\infty]$ into itself). We will show that

$$u_0^-(x) \geqq \gamma \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \text{ in } B^C. \tag{9.3}$$

As usual, we assume without loss of generality that $\nu = e_1$ and $x_0 = 0$. We will prove the following:

Claim. There exist $\overline{\alpha}, \overline{\gamma} > 0$ such that

$$u_0(x) = \overline{\alpha} x_1^+ - \overline{\gamma} x_1^- + o(|x|)$$

Proof of the Claim. Indeed, by Lemma 4.1 in [24],

$$u_0^-(x) = \overline{\gamma} x_1^- + o(|x|) \quad \text{in } \{x_1 < 0\} \tag{9.4}$$

for some $\overline{\gamma} \geqq 0$. Let us consider the blow-up sequence, for $\lambda > 0$, given by

$$(u_0)_\lambda(x) = \frac{1}{\lambda} u_0(\lambda x).$$

Since u_0 is locally Lipschitz continuous and $u_0(0) = 0$ then, for every sequence, $\lambda_n \to 0$, there exists a subsequence, which we still denote by λ_n, such that $(u_0)_{\lambda_n} \to U_0$ uniformly in compact sets of $\mathbb{R}^N$, where U_0 is Lipschitz in $\mathbb{R}^N$. By (9.1) and (9.4) we know that

$$U_0^- = \overline{\gamma} x_1^- \text{ in } \mathbb{R}^N$$

and

$$U_0 > 0 \quad \text{and harmonic in } \{x_1 > 0\}.$$

We have to analyze two cases:

Case I: $\overline{\gamma} > 0$.

In this case, $U_0 < 0$ in $\{x_1 < 0\}$. Therefore $U_0 = 0$ on the hyperplane $\{x_1 = 0\}$ and since it is Lipschitz continuous, we have

$$U_0^+(x) = \overline{\alpha} x_1^+ \quad \text{in } \mathbb{R}^N$$

for some $\overline{\alpha} > 0$. In this way, we conclude that

$$U_0(x) = \overline{\alpha} x_1^+ - \overline{\gamma} x_1^-, \qquad \overline{\alpha}, \overline{\gamma} > 0. \tag{9.5}$$

Case II: $\overline{\gamma} = 0$.

In this case, $U_0 \geqq 0$ in $\mathbb{R}^N$. Since $U_0 > 0$ and harmonic in $\{x_1 > 0\}$, then by Lemma A1 in [6], there exist $\overline{\alpha} > 0$ such that

$$U_0(x) = \overline{\alpha} x_1^+ + o(|x|) \text{ in } \{x_1 > 0\}. \tag{9.6}$$

Since $\overline{\alpha} > \alpha$, then (recall $H = H_{e_1}$)

$$H(\overline{\alpha}) \geqq H(\alpha) > M. \tag{9.7}$$

Let us consider, for $\lambda > 0$, the blow-up sequence

$$(U_0)_\lambda(x) = \frac{1}{\lambda} U_0(\lambda x).$$

Since U_0 is Lipschitz continuous and $U_0(0) = 0$, there exists a subsequence $\overline{\lambda}_n \to 0$, such that $(U_0)_{\overline{\lambda}_n} \to U_{00}$ uniformly in compact sets of $\mathbb{R}^N$, where $U_{00} \in \mathrm{Lip}(\mathbb{R}^N)$. By (9.6),

$$U_{00}(x) = \overline{\alpha} x_1^+ \text{ in } \{x_1 > 0\}.$$

We observe that $U_{00} \geqq 0$ in $\mathbb{R}^N$ is harmonic in its positivity set $\{U_{00} > 0\}$ and that $U_{00} = 0$ on the hyperplane $\{x_1 = 0\}$. Then, again by Lemma A1 in [6], we have

$$U_{00}(x) = \widetilde{\alpha} x_1^- + o(|x|) \text{ in } \{x_1 < 0\}$$

for some $\widetilde{\alpha} \geqq 0$. Finally, we consider once more for $\lambda > 0$ the blow-up sequence

$$(U_{00})_\lambda = \frac{1}{\lambda} u_{00}(\lambda x).$$

As before, there is still a subsequence $\widetilde{\lambda}_n \to 0$ and $U_{000} \in \mathrm{Lip}(\mathbb{R}^N)$ such that $(U_{00})_{\widetilde{\lambda}_n} \to U_{000}$ uniformly in compact subsets of $\mathbb{R}^N$. From the computations above, we conclude

$$U_{000}(x) = \overline{\alpha} x_1^+ + \widetilde{\alpha} x_1^-, \qquad \overline{\alpha} > 0,\ \widetilde{\alpha} \geqq 0.$$

Applying Proposition (5.2) and recalling that least supersolutions are preserved under blow-ups, we can see that there exists a sequence $\delta_n \to 0$ and least supersolutions u_{δ_n} to (SE_{δ_n}) such that

$$u_{\delta_n} \to U_0 \tag{9.8}$$

uniformly in compact sets of $\mathbb{R}^N$. Applying the same Proposition twice, we see that there exist a sequence $\widetilde{\delta_n} \to 0$ and solutions $u_{\widetilde{\delta}_n}$ to $(SE_{\widetilde{\delta}_n})$ such that $u_{\widetilde{\delta}_n} \to U_{000}$ uniformly on compact sets of $\mathbb{R}^N$. By Theorem (8.1) and by (9.2)

$$H(\overline{\alpha}) \leqq M < H(\alpha),$$

which contradicts (9.7). Then, Case II does not occur and (9.5) holds. In this way, by (9.8), we can apply again Theorem (8.1) to U_0 to conclude

$$H_{e_1}(\overline{\alpha}) - H_{e_1}(\overline{\gamma}) = M. \tag{9.9}$$

By Proposition (5.3), the blow-up compatibility condition, there exists $\delta > 0$ independent of the sequence λ_n such that

$$\overline{\alpha\gamma} = \delta. \tag{9.10}$$

So, $\overline{\alpha}$ and $\overline{\gamma}$ are determined univocally, and therefore U_0 does not depend on the sequence λ_n. In particular,

$$(u_0)_\lambda \to U_0$$

uniformly in compact subsets of $\mathbb{R}^N$ (as $\lambda \to 0$). Thus, (see Remark (9.6) below)

$$u_0(x) = \overline{\alpha} x_1^+ - \overline{\gamma} x_1^- + o(|x|),$$

proving the claim. In particular,

$$u_0^-(x) = \overline{\gamma} x_1^- + o(|x|) \quad \text{ in } B^C. \tag{9.11}$$

By (9.9), we obtain since $\overline{\alpha} \geqq \alpha$

$$H_{e_1}(\overline{\gamma}) = H_{e_1}(\overline{\alpha}) - M \geqq H_{e_1}(\alpha) - M > H_{e_1}(\gamma), \tag{9.12}$$

from which we conclude $\overline{\gamma} > \gamma$ and therefore by (9.11),

$$u_0^-(x) \geqq \gamma x_1^- + o(|x|) \quad \text{in } B^C.$$

This finishes the proof. □

Remark 9.6. We observe that, if $u \in C^0(\mathbb{R}^N)$ and the sequence of blow-ups defined by $u_\lambda(x) := \dfrac{1}{\lambda}u(\lambda x)$ converges as $\lambda \to 0$ locally and uniformly in $\mathbb{R}^N$ to H, a homogeneous function of degree 1 (that is, $H(\alpha x) = \alpha H(x)$, $\forall \alpha > 0$, $\forall x \in \mathbb{R}^N$), then

$$u(x) = H(x) + o(|x|) \text{ near the origin.}$$

We now establish the pointwise result.

Theorem 9.7. *Let u_0 be a limit of the least supersolutions given by Theorem (4.1). For $\mathcal{H}^{N-1}$ almost everywhere $x_0 \in F(u_0)$, u_0 has the following asymptotic development*

$$u_0(x) = \alpha \langle x - x_0, \nu\rangle^+ - \gamma \langle x - x_0, \nu\rangle^- + o(|x - x_0|),$$

where

$$H_\nu(\alpha) - H_\nu(\gamma) = M.$$

In particular, around such points, the free boundary $F(u_0)$ is flat in the sense of Theorem $2'$ in [6].

Proof. Indeed, by Theorem (4.3), for $\mathcal{H}^{N-1}$ almost everywhere in $F(u_0)$, we have

$$u_0(x) = q_{u_0}^+(x_0) \langle x - x_0, \nu\rangle^+ - q_{u_0}^-(x_0) \langle x - x_0, \nu\rangle^- + o(|x - x_0|).$$

Considering now the blow-up sequence $(u_0)_\lambda(x) = \frac{1}{\lambda}u(x_0 + \lambda x)$, $\lambda > 0$, we have

$$(u_0)_\lambda \to q_{u_0}^+(x_0) \langle x - x_0, \nu\rangle^+ - q_{u_0}^-(x_0) \langle x - x_0, \nu\rangle^- .$$

Since least supersolutions are preserved under the blow-up process, as in the previous Theorem, by Proposition (5.2) and Theorem (8.1), we conclude that

$$H_\nu(q_{u_0}^+(x_0)) - H_\nu(q_{u_0}^-(x_0)) = M.$$

Flatness follows now by the arguments in [6,7]. This finishes the proof. □

Finally, we prove our last theorem concerning the regularity of the free boundary $F(u_0)$.

Theorem 9.8. (Free boundary regularity) *Let u_0 be a limit of the least supersolutions given by Theorem (4.1). Then, the free boundary $F(u_0) = \partial\{u_0 > 0\} \cap \Omega$ is a $C^{1,\gamma}$ surface in a neighborhood of $\mathcal{H}^{N-1}$ almost everywhere point $x_0 \in F(u_0)_{\text{red}}$. In particular, $F(u_0)$ is a $C^{1,\gamma}$ surface in a neighborhood of $\mathcal{H}^{N-1}$ almost everywhere point in $F(u_0)$.*

Proof. We already know that u_0 is a viscosity solution of (FBP). In this case, u_0 satisfies

$$u_\nu^+ = G(u_\nu^-, \nu) \text{ on } F(u)$$

in the viscosity sense, where

$$G(z, \nu) = H_\nu^{-1}(M + H_\nu(z)). \tag{9.13}$$

Let us observe that G depends on ν in a Lipschitz continuous fashion. Indeed, there is a constant $C > 0$ such that $G(z, \nu) \geqq C$. Since $t^2/2F_{\max} \leqq H_\nu(t) \leqq t^2/2F_{\min}$, for $t \geqq 0$, we obtain

$$\frac{G(z, \nu)^2}{2F_{\min}} \geqq H_\nu(G(z, \nu)) \geqq M + H_\nu(z) \geqq M.$$

Furthermore,

$$|H_\nu(x) - H_\nu(y)| \geqq \frac{\sigma}{F_{\min}}|x - y| \text{ for } x, y \in [\sigma, +\infty).$$

In this way, for $\nu_1, \nu_2 \in \mathbb{S}^{N-1}$, by (9.13),

$$|H_{\nu_1}(G(z, \nu_1)) - H_{\nu_2}(G(z, \nu_2))| = |H_{\nu_1}(z) - H_{\nu_2}(z)|.$$

Therefore for $|z| \leqq C_0$, there exists $\overline{C_0} = \overline{C_0}(C_0, \operatorname{Lip}(F))$ such that

$$\frac{C}{F_{\min}}|G(\nu_1, z) - G(\nu_2, z)| \leqq |H_{\nu_1}(G(z, \nu_1)) - H_{\nu_2}(G(z, \nu_2))|$$
$$= |H_{\nu_1}(z) - H_{\nu_2}(z)| \leqq \overline{C_0}|\nu_1 - \nu_2|.$$

Moreover, by Theorem (4.1), u_0 is locally Lipschitz continuous and it has linear growth away from its free boundary $F(u_0)$. Also, since $F(u_0)_{\text{red}}$ has full $\mathcal{H}^{N-1}$ measure in $F(u_0)$, u_0 is for $\mathcal{H}^{N-1}$ almost everywhere point on $F(u_0)$ a 2-plane function. In particular, for any such point x_0, a suitable dilation

$$(u_0)_\tau = \frac{u(\tau(x - x_0))}{\tau}, \qquad \tau \text{ small enough}$$

falls under conditions of Theorem 3 in [6], concluding the proof. □

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