Arch. Rational Mech. Anal. 188 (2008) 1–91
Digital Object Identifier (DOI) 10.1007/s00205-007-0085-1

Fully Localised Solitary-Wave Solutions of the Three-Dimensional Gravity–Capillary Water-Wave Problem

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Communicated by A. Mielke

Abstract

A model equation derived by Kadomtsev & Petviashvili (Sov Phys Dokl 15:539–541, 1970) suggests that the hydrodynamic problem for three-dimensional water waves with strong surface-tension effects admits a *fully localised solitary wave* which decays to the undisturbed state of the water in every horizontal spatial direction. This prediction is rigorously confirmed for the full water-wave problem in the present paper. The theory is variational in nature. A simple but mathematically unfavourable variational principle for fully localised solitary waves is reduced to a locally equivalent variational principle with significantly better mathematical properties. The reduced functional is related to the functional associated with the Kadomtsev–Petviashvili equation, and a nontrivial critical point is detected using the direct methods of the calculus of variations.

1. Introduction

1.1. The main result

The classical *three-dimensional gravity–capillary water-wave problem* concerns the irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. The fluid motion is described by the Euler equations in a domain bounded below by a rigid horizontal bottom $\{y = 0\}$ and above by a free surface $\{y = h + \rho(x, z, t)\}$, where h denotes the depth of the water in its undisturbed state and the function ρ depends upon the two horizontal spatial directions x, z and time t. *Steady waves* are water waves which are uniformly translating in a distinguished horizontal direction without change of shape; without loss of generality we assume that the waves propagate in the x-direction with speed c and continue to write x as an abbreviation for $x - ct$. In terms of an Eulerian velocity potential $\phi(x, y, z, t)$ the mathematical problem for steady waves is to solve the equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad 0 < y < 1 + \rho, \tag{1}$$

$$\phi_y = 0 \qquad \text{on } y = 0, \tag{2}$$

$$\phi_y = \rho_x \phi_x + \rho_z \phi_z - \rho_x \qquad \text{on } y = 1 + \rho \tag{3}$$

and

$$-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + \alpha\rho$$
$$-\beta \left[\frac{\rho_x}{\sqrt{1 + \rho_x^2 + \rho_z^2}} \right]_x - \beta \left[\frac{\rho_z}{\sqrt{1 + \rho_x^2 + \rho_z^2}} \right]_z = 0 \quad \text{on } y = 1 + \rho \tag{4}$$

(see STOKER [38]), in which we have introduced dimensionless variables. The equations involve two physical parameters, $\alpha := gh/c^2$ and $\beta := \sigma/hc^2$, where g and σ are the acceleration due to gravity and the coefficient of surface tension, respectively.

The steady water-wave problem (1)–(4) is a free boundary-value problem with nonlinear boundary conditions, and in this respect its solution poses considerable mathematical difficulties. At a formal level these difficulties may be overcome by replacing the above equations by a simpler model equation based upon certain approximations. One of the more widely used model equations is the Kadomtsev–Petviashvili (KP)-I equation

$$\partial_{xx} \left(u_{xx} - u - \frac{3}{2} u^2 \right) - u_{zz} = 0, \tag{5}$$

in which u depends upon two unbounded spatial directions x and z. This equation was derived formally by KADOMTSEV & PETVIASHVILI [23] as a long-wave approximation for solutions of the steady gravity–capillary water-wave problem (1)–(4) in which

$$\beta > 1/3, \quad \alpha = 1 + \varepsilon, \;\; 0 < \varepsilon \ll 1; \tag{6}$$

the variable u is supposed to approximate the free surface of the water via the formula

$$\rho(x, z) = \varepsilon\, u \left(\frac{\varepsilon^{1/2} x}{2(\beta - 1/3)^{1/2}}, \varepsilon z \right) + \mathcal{O}(\varepsilon^2).$$

The KP-I equation (5) admits the the explicit solution

$$u(x, z) = -8 \frac{3 - x^2 + z^2}{(3 + x^2 + z^2)^2}, \tag{7}$$

which defines a *fully localised solitary wave*, that is, a wave which decays to zero at large distances in both spatial directions [1]; this wave is sketched in Fig. 1. In the present paper we confirm the prediction made by the KP-I equation by proving that the steady water-wave problem (1)–(4) has a fully localised solitary-wave solution in the parameter regime (6). Our result contrasts with a recent theorem by CRAIG [13], who showed that in the absence of surface tension there are no fully localised solitary waves with $\rho \geqq 0$.

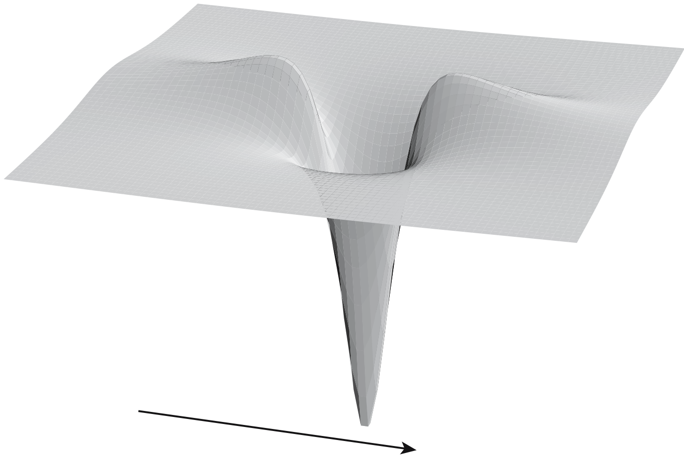

Fig. 1. A fully localised solitary wave; the arrow shows the direction of wave propagation

The present paper is the latest in a series of results justifying the use of the KP-I equation (5) as a model equation for solitary gravity–capillary water waves. This equation actually has several explicit solitary-wave solutions, namely the *line solitary wave*

$$u(x) = -\operatorname{sech}^2\left(\frac{x}{2}\right),$$

which decays exponentially to zero as $x \to \infty$ and does not depend upon the transverse spatial direction z, the family

$$u^\delta(x,z) = -\frac{4(1-\delta^2)}{4-\delta^2}\frac{1-\delta\cosh(a^\delta x)\cos(\omega^\delta z)}{(\cosh(a^\delta x)-\delta\cos(\omega^\delta z))^2},$$

$$a^\delta = \sqrt{\frac{1-\delta^2}{4-\delta^2}}, \qquad \omega^\delta = \frac{\sqrt{3(1-\delta^2)}}{4-\delta^2},$$

where $\delta \in (0,1)$, of *periodically modulated solitary waves*, which decay exponentially to zero as $x \to \pm\infty$ and are periodic with frequency ω^δ in z (see [39]), and of course the fully localised solitary wave (7), which decays algebraically to zero as $|(x,z)| \to \infty$. (In fact the line and fully localised solitary waves correspond to the limiting cases u_0 and u_1 in the above formula.) It was shown, respectively, by Kirchgässner [25] (see also Amick & Kirchgässner [3] and Sachs [36]) and Groves et al. [20] that the steady water-wave problem has a line solitary-wave solution and a family of periodically modulated solitary-wave solutions in the KP-I parameter regime (6).

1.2. Variational methods

The key to our existence theory for fully localised solitary waves is the observation that the hydrodynamic problem (1)–(4) in the parameter regime (6) follows from the formal variational principle

$$\delta \int_{\mathbb{R}^2} \left(\int_0^{1+\rho} \left(-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) \right) \mathrm{d}y \right.$$
$$\left. + \frac{1}{2}(1+\varepsilon)\rho^2 + \beta(\sqrt{1+\rho_x^2+\rho_z^2} - 1) \right) \mathrm{d}x\,\mathrm{d}z = 0, \quad (8)$$

where the variation is taken in (ρ, ϕ) (see LUKE [30]). A more satisfactory version of this variational principle is obtained using the transformation

$$y = \tilde{y}(1+\rho(x,z)), \quad \phi(x,y,z) = \Phi(x,\tilde{y},z),$$

which maps the fluid domain $\mathcal{D}_\rho = \{(x,y,z) : (x,z) \in \mathbb{R}^2, \tilde{y} \in (0, 1+\rho(x,z))\}$ bijectively into the fixed strip $\Sigma = \{(x,\tilde{y},z) : (x,z) \in \mathbb{R}^2, \tilde{y} \in (0,1)\}$, and it is also appropriate to introduce the scaled variables associated with the KP scaling limit, namely

$$(\tilde{\rho}(\tilde{x},\tilde{z}), \tilde{\Phi}(\tilde{x},y,\tilde{z})) = (\varepsilon^{-1}\rho(x,z), \varepsilon^{-\frac{1}{2}}\Phi(x,y,z)), \quad (\tilde{x},\tilde{z}) = (\varepsilon^{\frac{1}{2}}x, \varepsilon z). \quad (9)$$

The hydrodynamic problem (1)–(4) is transformed into the equation

$$(1+\varepsilon)\rho - \beta\varepsilon\rho_{xx} - \beta\varepsilon^2\rho_{zz} = \int_0^1 y\Phi_{xy}\,\mathrm{d}y + \int_0^1 \Phi_x\,\mathrm{d}y + \varepsilon^{-1}N_1(\rho,\Phi), \quad (10)$$

where we have used the identity

$$\Phi|_{y=1} = \int_0^1 y\Phi_y\,\mathrm{d}y + \int_0^1 \Phi\,\mathrm{d}y,$$

and the boundary-value problem

$$-\varepsilon\Phi_{xx} - \varepsilon^2\Phi_{zz} - \Phi_{yy} = \varepsilon^{-\frac{1}{2}}N_2(\rho,\Phi), \quad 0 < y < 1, \quad (11)$$

$$\varepsilon\rho_x + \Phi_y = \varepsilon^{-\frac{1}{2}}N_3(\rho,\Phi) \qquad \text{on } y = 1, \quad (12)$$

$$\Phi_y = 0 \qquad \text{on } y = 0, \quad (13)$$

in which the nonlinearities N_1, N_2, N_3 are given by the formulae

$$N_1(\rho,\Phi) =$$
$$\beta\varepsilon^2 \left[\frac{\rho_x}{\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}} \right]_x - \beta\varepsilon^2\rho_{xx} + \beta\varepsilon^3 \left[\frac{\rho_z}{\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}} \right]_z - \beta\varepsilon^3\rho_{zz}$$
$$- \int_0^1 \left\{ \varepsilon^2 \left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho} \right)^2 + \varepsilon^3 \left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho} \right)^2 \right.$$
$$+ \varepsilon^2 \left(\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho} \right) y\Phi_y \right)_x + \varepsilon^3 \left(\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho} \right) y\Phi_y \right)_z$$
$$+ \varepsilon^{\frac{3}{2}} \left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho} \right) \frac{y\Phi_y}{1+\varepsilon\rho} + \varepsilon^2 \left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho} \right) \frac{y\Phi_y}{1+\varepsilon\rho}$$
$$\left. + \frac{\varepsilon\Phi_y^2}{2(1+\varepsilon\rho)^2} \right\} \mathrm{d}y, \quad (14)$$

$$N_2(\rho,\Phi) = \varepsilon^{\frac{5}{2}}(\rho\Phi_x)_x + \varepsilon^{\frac{7}{2}}(\rho\Phi_z)_z - \varepsilon^{\frac{5}{2}}(y\Phi_y\rho_x)_x - \varepsilon^{\frac{7}{2}}(y\Phi_y\rho_z)_z$$
$$-\varepsilon^{\frac{5}{2}}\left(\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right)y\rho_x\right)_y - \varepsilon^{\frac{7}{2}}\left(\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right)y\rho_z\right)_y$$
$$+\frac{\varepsilon^{\frac{3}{2}}\rho\Phi_{yy}}{1+\varepsilon\rho},$$
$$N_3(\rho,\Phi) = \left[\varepsilon^{\frac{5}{2}}\rho_x\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right) + \varepsilon^{\frac{7}{2}}\rho_z\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right) - \frac{\varepsilon^{\frac{3}{2}}\rho\Phi_y}{1+\varepsilon\rho}\right]_{y=1},$$

while the functional in the variational principle (8) is transformed into

$$\mathcal{V}(\rho,\Phi) = \int_{\mathbb{R}^2}\left\{\frac{1}{2}\varepsilon(1+\varepsilon)\rho^2 + \beta\varepsilon^{-1}[\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}-1]\right.$$
$$+\int_0^1\left(\frac{\varepsilon}{2}\left[\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right]^2 + \frac{\Phi_y^2}{2(1+\varepsilon\rho)^2} + \frac{\varepsilon^2}{2}\left[\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right]^2\right)(1+\varepsilon\rho)\,\mathrm{d}y$$
$$\left.+\varepsilon\int_0^1(\rho_x y\Phi_y - \rho\Phi_x)\,\mathrm{d}y\right\}\,\mathrm{d}x\,\mathrm{d}z,$$

(The tildes have been dropped in the above formulae for notational simplicity.) At a formal level it is readily confirmed that critical points of $\mathcal{V}$ correspond to weak solutions of (10)–(13). Our strategy is therefore to apply the direct methods of the calculus of variations to find critical points of $\mathcal{V}$ (defined upon a suitable function space) and develop a regularity theory which shows that the corresponding weak solutions of (10)–(13) are in fact strong solutions of these equations.

The calculus of variations offers a variety of results for studying functionals of the type

$$\mathcal{F}(u) = \int_{\mathcal{S}} F(u)\,\mathrm{d}x^n$$

which are defined on spatially extended domains $\mathcal{S}$ (that is, subsets of $\mathbb{R}^n$ which are unbounded in one or more spatial directions). A problem of this kind is typically treated in two stages. Firstly one establishes the existence of a *Palais–Smale* sequence $\{u_m\}$ with the property that $\mathcal{F}(u_m) \to a$, $\mathcal{F}'(u_m) \to 0$ as $m \to \infty$ for some nonzero constant a, so that $\{u_m\}$ is a sequence of successively better approximations to a putative critical point $u \neq 0$ with $\mathcal{F}(u) = a$, $\mathcal{F}'(u) = 0$. The second step is to study the convergence properties of $\{u_m\}$ (note that weaker results than the strong convergence of $\{u_m\}$ are sufficient to guarantee the existence of a nonzero critical point). The *concentration-compactness principle* of Lions [28, 29] is frequently helpful in this respect; it has been applied with great success to the following class of problems collectively known as the ‘coercive, semilinear, locally compact case’. Suppose that $\mathcal{F}$ is a smooth functional on $\mathcal{X}(\mathcal{S})$, where $\mathcal{X}(U)$ is a Sobolev space of functions defined upon the spatial domain $U \subseteq \mathbb{R}^n$. Let us write

$$\mathcal{F}(u) = \mathcal{F}_2(u) + \mathcal{F}_{\mathrm{NL}}(u),$$

where $\mathcal{F}_2 : \mathcal{X}(\mathcal{S}) \to \mathbb{R}$ is the quadratic part of $\mathcal{F}$, and suppose that $\mathcal{F}_{\mathrm{NL}}$ extends to a smooth functional $\mathcal{F}_{\mathrm{NL}} : \mathcal{Y}(\mathcal{S}) \to \mathbb{R}$, where

(i) ('coerciveness') $\mathcal{F}_2$ is equivalent to the $\mathcal{X}(\mathcal{S})$-norm;
(ii) ('semilinearity') $\mathcal{Y}(\mathcal{S})$ is continuously embedded in $\mathcal{X}(\mathcal{S})$;
(iii) ('local compactness') $\mathcal{Y}(U)$ is compactly embedded in $\mathcal{X}(U)$ for every compact subset U of $\mathbb{R}^n$.

The use of concentration-compactness methods to find solitary-wave solutions of model equations for two-dimensional water waves was pioneered by WEINSTEIN [42], who considered a variety of third-order equations. The method has been extended to many other equations arising in water-wave theory, including fifth-order models [16, 24, 26], systems of model equations [4] and model equations for three-dimensional water waves [15, 34]; all of these problems satisfy the coerciveness, semilinearity, and local compactness conditions. Let us now examine the variational functional $\mathcal{V}$ associated with the full water-wave problem. A straightforward calculation shows that

$$\mathcal{V}_2(\rho, \Phi) = \int_{\mathbb{R}^2} \left\{ \int_0^1 \left(\frac{\varepsilon}{2}\Phi_x^2 + \frac{\varepsilon^2}{2}\Phi_z^2 + \frac{1}{2}\Phi_y^2 + \varepsilon(\rho_x y \Phi_y - \rho\Phi_x) \right) \mathrm{d}y \right.$$
$$\left. + \frac{1}{2}\varepsilon(1+\varepsilon)\rho^2 + \frac{\beta}{2}\varepsilon^2\rho_x^2 + \frac{\beta}{2}\varepsilon^3\rho_z^2 \right\} \mathrm{d}x\,\mathrm{d}z,$$

$$\mathcal{V}_{\mathrm{NL}}(\rho, \Phi) = \int_{\mathbb{R}^2} \left\{ \int_0^1 \left(\frac{\varepsilon^2}{2}\rho\Phi_x^2 + \frac{1}{2}\varepsilon^3\rho\Phi_z^2 - \frac{\varepsilon\rho\Phi_y^2}{2(1+\varepsilon\rho)} + \frac{\varepsilon^3 y^2 \Phi_y^2 \rho_x^2}{2(1+\varepsilon\rho)} \right. \right.$$
$$\left. + \frac{\varepsilon^4 y^2 \Phi_y^2 \rho_z^2}{2(1+\varepsilon\rho)} - \varepsilon^2 y \Phi_y \Phi_x \rho_x - \varepsilon^3 y \Phi_y \Phi_z \rho_z \right) \mathrm{d}y$$
$$\left. - \frac{\beta\varepsilon^{-1}(\varepsilon^3\rho_x^2 + \varepsilon^4\rho_z^2)^2}{2(\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}+1)^2} \right\} \mathrm{d}x\,\mathrm{d}z,$$

and it is readily confirmed that there are no function spaces $\mathcal{X}(\mathbb{R}^2 \times \Sigma)$, $\mathcal{Y}(\mathbb{R}^2 \times \Sigma)$ that meet the criteria set out above. (In particular, it is not possible to choose a function space for $\mathcal{V}_{\mathrm{NL}}$ which requires less regularity of its elements than that for $\mathcal{V}_2$; the problem is *quasilinear* rather than semilinear in this respect.) We therefore proceed by studying $\mathcal{V}$ in one of the widest possible Sobolev spaces upon which it defines a smooth functional, namely $[W^{1,2}(\mathbb{R}^2) \times U^{0,2}(\Sigma)] \cap [W^{1+\delta,p}(\mathbb{R}^2) \times U^{\delta,p}(\Sigma)]$ for $\delta \in (0,1)$ and $p \in (3/\delta, \infty)$, where

$$U^{s,p}(\Sigma) := \{\Phi : \|\Phi\|_{U^{s,p}(\Sigma)} := \|\Phi_x\|_{W^{s,p}(\Sigma)} + \|\Phi_y\|_{W^{s,p}(\Sigma)} + \|\Phi_z\|_{W^{s,p}(\Sigma)} < \infty\}$$

and using a reduction technique to show that the problem of finding critical points of $\mathcal{V}$ on this function space is locally equivalent to one of finding critical points of a reduced functional J which falls into the coercive, semilinear, locally compact category. Its Euler–Lagrange equation is a sixth-order pseudodifferential equation for a single function Φ_1 of the horizontal coordinates (x, z).

The reduction technique is presented in overview in Section 2, and complete proofs of the reduction theorems are given in Section 3. Section 4 deals with the remaining part of the existence theory, namely the proof that the reduced variational functional J has a nonzero critical point. We show that J is a functional of *mountain-pass type*, that is, it has a strict local minimum at the origin and is negative for some nonzero function. The *mountain-pass lemma* (see, for example, BREZIS & NIRENBERG [5, p. 943]) yields the existence of a Palais–Smale sequence $\{\Phi_{1m}\}$ with $J(\Phi_{1m}) \to a_\varepsilon$, $J'(\Phi_{1m}) \to 0$ as $m \to \infty$, where a_ε is a nonzero constant (which may be interpreted geometrically as the minimum height attained by any path connecting the origin to another point at 'sea level'). The convergence properties of this Palais–Smale sequence are examined with the concentration-compactness principle according to the method given by GROVES [16] in a study of solitary-wave solutions to a fifth-order model equation for water waves.

The reduction of complex variational problems to simpler variational problems is well known in the framework of the Lyapunov–Schmidt reduction procedure (see MIELKE [32, pp. 62–63]). The method is often used to reduce a variational system of partial differential equations to a locally equivalent variational system of ordinary differential equations. In this context it has been applied to several problems involving wave phenomena, in particular by MOSER [33] in his investigation of the resonant case of the Lyapunov centre theorem for periodic solutions of Hamiltonian systems, and by CRAIG & NICHOLLS [14] in their existence theory for doubly periodic three-dimensional water waves. In the present paper we use the method in a more general fashion to reduce our *quasilinear* system of partial differential equations to a locally equivalent *semilinear* partial differential equation which meets the criteria set out above for an application of the concentration-compactness method.

A number of other existence theories for three-dimensional gravity–capillary water waves have recently been published, all of which are based upon variational principles equivalent to (8). A wide variety of three-dimensional water waves has been found using *spatial dynamics* by GROVES & MIELKE [19], GROVES [17] and GROVES & HARAGUS [18]. In these references solutions are found by formulating the hydrodynamic problem as an infinite-dimensional Hamiltonian system in which an unbounded spatial coordinate plays the role of the time-like variable. The Hamiltonian system is derived by performing a Legendre transform on the variational functional $\mathcal{V}$ (in unscaled variables), and its solutions are found using a reduction technique which shows that it is locally equivalent to a Hamiltonian system with finitely many degrees of freedom, whose solution set can be analysed. A different technique was used by CRAIG & NICHOLLS [14] in an existence theory for doubly periodic water waves. The starting point of their analysis is again the variational principle (8), but they overcome the difficulty posed by the variable domain D_ρ by introducing a new variable $\xi = \phi|_{y=1+\rho}$ and expressing the variational functional in terms of ρ and ξ. CRAIG & NICHOLLS apply a version of the variational Lyapunov–Schmidt reduction discussed above to show that their variational principle is locally equivalent to a finite-dimensional variational principle and find critical points of their reduced functional using topological arguments.

The method of CRAIG & NICHOLLS, like the result in the present paper, relies upon a *reduction* method which converts a global variational principle into a more tractable local variational principle. An alternative method is to *extend* a variational principle to a more general problem to which the direct methods of the calculus of variations can be applied. BUFFONI et al. [12] have recently used this approach in a study of two-dimensional periodic steady waves on deep water in the absence of surface tension. Their quasilinear functional is made semilinear by the addition of a regularising term (with higher derivatives), and a priori estimates are used to confirm that the detected critical points of the regularised functional are actually critical points of the original. The method has been extended to gravity–capillary solitary water waves (in finite and infinite depth) by BUFFONI [6, 7].

There are several further variational results in the literature concerning two-dimensional steady water waves. Hamiltonian spatial dynamics methods have been successfully applied to the problem for gravity–capillary waves by BUFFONI et al. [9] and BUFFONI & GROVES [8], who found a plethora of solitary-wave solutions to this problem. BUFFONI et al. [10, 11] and PLOTNIKOV [35] constructed variational global bifurcation theories for respectively Stokes waves (periodic travelling gravity waves) and solitary gravity waves using variational methods, while TURNER [40] found Stokes and solitary gravity waves by applying the direct methods of the calculus of variations. TURNER used semi-Lagrangian coordinates to map the fluid domain into a strip; the resulting quasilinear variational functional is handled by extending it to a tractable semilinear problem and using a priori estimates to return to the original setting.

1.3. The functional-analytic framework

In this section we introduce scaled function spaces in which the subsequent theory is developed. At this stage we merely define the spaces and state their main properties; a detailed commentary on their choice and usefulness is given in the course of the mathematical development at the beginning of Section 3. Here, and in the remainder of this paper, we use the symbol c to denote a general positive constant (which in particular does not depend upon ε).

In the following analysis we use four basic spaces for functions of two real variables, namely

(i) the Hilbert space $X = \{u : |||u||| < \infty\}/\sim$, where $u \sim v$ if and only if $u - v \in \mathbb{R}$,

$$\langle\!\langle u, v\rangle\!\rangle = \int_{\mathbb{R}^2} \Big\{c_0(\varepsilon u_{xxx}v_{xxx} + 3\varepsilon^2 u_{xxz}v_{xxz} + 3\varepsilon^3 u_{xzz}v_{xzz} + \varepsilon^4 u_{zzz}v_{zzz}) + (\beta - \tfrac{1}{3})(u_{xx}v_{xx} + 2\varepsilon u_{xz}v_{xz} + \varepsilon^2 u_{zz}v_{zz}) + u_x v_x + (1+\varepsilon)u_z v_z\Big\}\,\mathrm{d}x\,\mathrm{d}z \qquad (15)$$

and $c_0 = 2\alpha/15 - \beta/3$;

(ii) the Banach space $W_\varepsilon^{\delta,p}(\mathbb{R}^2) = \{u : \|u\|_{\delta,p,\varepsilon} < \infty\}$, where

$$\|u\|_{\delta,p,\varepsilon} = \|\mathcal{F}^{-1}[(1+\mu^2+\varepsilon k^2)^{\frac{\delta}{2}}\mathcal{F}u]\|_p,$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote, respectively, the Fourier and inverse Fourier transforms, (μ, k) is the independent variable associated with the Fourier transform in (x, z) and $\|\cdot\|_p$ is the $L^p(\mathbb{R}^2)$-norm;

(iii) the Banach space $V_\varepsilon^{\delta,p}(\mathbb{R}^2) = \{u : |u|_{\delta,p,\varepsilon} < \infty\}$, where

$$|u|_{\delta,p,\varepsilon} = \|\mathcal{F}^{-1}[(1+\varepsilon^{\frac{1}{2}}(\mu^2+\varepsilon k^2)^{\frac{1}{2}+\frac{\delta}{2}})\mathcal{F}u]\|_p;$$

(iv) the Banach space $U_\varepsilon^{\delta,p}(\mathbb{R}^2) = \{u : \|u\|_{U_\varepsilon^{\delta,p}} < \infty\}/\sim$, where

$$\|u\|_{U_\varepsilon^{\delta,p}} = \|u_x\|_{\delta,p,\varepsilon} + \varepsilon^{\frac{1}{2}}\|u_z\|_{\delta,p,\varepsilon}.$$

The spaces $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ are scaled versions of the standard Sobolev spaces $W^{\delta,p}(\mathbb{R}^2)$ and $W^{1+\delta,p}(\mathbb{R}^2)$ defined using the Bessel potential (see Adams & Fournier [2, Section 7.63]); similarly X and $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ are scaled versions of familiar spaces in which only the derivatives of functions play a role. Both the scaling and the choice of coefficients c_0 and $\beta - 1/3$ used in the definition of X are dictated by the hydrodynamic problem (see Section 2); on the other hand the scalings used in the other spaces are chosen in view of their compatibility with X and usefulness in fixed-point arguments for solving nonlinear equations (see Section 3).

The following proposition states some of the basic properties of the above function spaces. Parts (i)–(iv) are proved by applying straightforward scaling arguments to well-known properties of the standard function spaces from which they are constructed, parts (v) and (vi) follow by scaling the results given by Mazya [31, Section 7.1.2], and part (vii) is obtained using the method described by Wang et al. [41, Lemma 1].

Proposition 1.

(i) *The function spaces $W_\varepsilon^{\delta_2,p}(\mathbb{R}^2)$ and $V_\varepsilon^{\delta_2,p}(\mathbb{R}^2)$ are continuously embedded in, respectively, $W_\varepsilon^{\delta_1,p}(\mathbb{R}^2)$ and $V_\varepsilon^{\delta_1,p}(\mathbb{R}^2)$ whenever $\delta_1 \leqq \delta_2$; in particular we have the embedding inequalities*

$$\|u\|_{\delta_1,p,\varepsilon} \leqq \|u\|_{\delta_2,p,\varepsilon}, \quad |u|_{\delta_1,p,\varepsilon} \leqq |u|_{\delta_2,p,\varepsilon}, \quad \delta_1 \leqq \delta_2.$$

(ii) *The space $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ is a Banach algebra and continuously embedded in $C_\mathrm{b}(\mathbb{R}^2)$ whenever $\delta > 2/p$; in particular we have the inequalities*

$$\|uv\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2p}}\|u\|_{\delta,p,\varepsilon}\|v\|_{\delta,p,\varepsilon}, \quad \|u\|_\infty \leqq c\varepsilon^{-\frac{1}{2p}}\|u\|_{\delta,p,\varepsilon}, \quad \delta > 2/p.$$

(iii) *The inequality*

$$\|u\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{\delta}{2}}|u|_{\delta,p,\varepsilon}$$

holds for each $\delta \geqq 0$.

(iv) *The space X is continuously embedded in $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ for $\delta \in [0,1]$ and we have the embedding inequality*

$$\|u\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{\frac{1}{2p}-\frac{1}{4}-\frac{\delta}{2}}|||u|||, \quad \delta \in [0,1]. \tag{16}$$

(v) *The $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ norm may be replaced by the equivalent norm*

$$\|u\|_{\delta,p,\varepsilon} = \|u\|_p + \|\mathcal{F}^{-1}[(\mu^2+\varepsilon k^2)^{\frac{\delta}{2}}\mathcal{F}u]\|_p.$$

(vi) *The* $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *norm may be replaced by the equivalent norm*

$$|u|_{\delta,p,\varepsilon} = \|u\|_p + \varepsilon^{\frac{1}{2}}\|\mathcal{F}^{-1}[(\mu^2+\varepsilon k^2)^{\frac{1}{2}+\frac{\delta}{2}}\mathcal{F}u]\|_p.$$

(vii) *The sharper embedding inequality*

$$\|u\|_{U_\varepsilon^{1,p}} \leqq c|||u||| \tag{17}$$

holds whenever $p \in (2,6)$.

It is also necessary to consider functions $u = u(x,z)$ defined upon an open subset S of $\mathbb{R}^2$ (with smooth boundary); for this purpose we use the space $X_S = \{u : |||u|||_S < \infty\}/\sim$, where $|||\cdot|||_S$ is defined by formula (15) with the range of integration replaced by S, the space $W_\varepsilon^{\delta,p}(S)$, which is defined by interpolation (see below), and the space $U_\varepsilon^{\delta,p}(S)$, which is obtained from $W_\varepsilon^{\delta,p}(S)$ in the same way that $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ is obtained from $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$. The function space $W_\varepsilon^{\delta,p}(S)$ defined by an interpolation procedure according to the formulae

$$W_\varepsilon^{\delta,p}(S) = \{u : \|u\|_{\delta,p,\varepsilon} < \infty\}, \quad \|u\|_{s,p,\varepsilon} = \sum_{i+k=0}^{s} \varepsilon^{\frac{k}{2}}\|\partial_x^i\partial_z^k u\|_p$$

for $s = 0, 1, 2, \ldots$ and

$$W_\varepsilon^{\delta,p}(S) = [W_\varepsilon^{\lfloor\delta\rfloor,p}(S), W_\varepsilon^{\lceil\delta\rceil,p}(S)]_{\delta-\lfloor\delta\rfloor}$$

for arbitrary $\delta \geqq 0$, in which $\|\cdot\|_p$ is the $L^p(S)$-norm, the symbols $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ refer to the 'floor' and 'ceiling' of a positive real number and the interpolation is carried out in the sense of LIONS & MAGENES [27] (see also ADAMS & FOURNIER [2, Section 7.57]). Of course this procedure can also be used to define the space $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ itself, and in fact leads to a space which coincides with that constructed using the Fourier transform (see also ADAMS & FOURNIER [2, Sections 7.50–7.66]). The following proposition states the key properties of X_S, $W_\varepsilon^{\delta,p}(S)$ and $U_\varepsilon^{\delta,p}(S)$; note that it is the compactness of certain embeddings rather than the size of embedding constants which is of most interest here.

Proposition 2. *Suppose that S is an open subset of* $\mathbb{R}^2$ *with smooth boundary.*

(i) *The space* $W_\varepsilon^{\delta,p}(S)$ *is a Banach algebra and continuously embedded in* $C_\mathrm{b}(S)$ *whenever* $\delta > 2/p$.
(i) *The space* X_S *is continuously embedded in* $U_\varepsilon^{\delta,p}(S)$ *for* $\delta \in [0,1]$ *and the embedding constant does not depend upon S. The embedding is compact whenever S is bounded.*

We also consider functions of three variables $(x,y,z) \in \Sigma$, where Σ is the strip $\{(x,y,z) : (x,z) \in \mathbb{R}^2, y \in (0,1)\}$, using the function space $W_\varepsilon^{\delta,p}(\Sigma)$ defined by an interpolation procedure according to the formulae

$$W_\varepsilon^{\delta,p}(\Sigma) = \{u : \|u\|_{\delta,p,\varepsilon} < \infty\}, \quad \|u\|_{s,p,\varepsilon} = \sum_{i+j+k=0}^{s} \varepsilon^{\frac{k}{2}}\|\partial_x^i\partial_y^j\partial_z^k u\|_p$$

for $s = 0, 1, 2, \ldots$ and

$$W_\varepsilon^{\delta,p}(\Sigma) = [W_\varepsilon^{\lfloor\delta\rfloor,p}(\Sigma), W_\varepsilon^{\lceil\delta\rceil,p}(\Sigma)]_{\delta-\lfloor\delta\rfloor}$$

for arbitrary $\delta \geqq 0$, where $\|\cdot\|_p$ is the $L^p(\Sigma)$-norm. The space $U_\varepsilon^{\delta,p}(\Sigma) = \{u : \|u\|_{U_\varepsilon^{\delta,p}} < \infty\}/\sim$ is derived from $W^{\delta,p}(\Sigma)$ in the usual fashion, so that

$$\|u\|_{U_\varepsilon^{\delta,p}} = \|u_x\|_{\delta,p,\varepsilon} + \|u_y\|_{\delta,p,\varepsilon} + \varepsilon^{\frac{1}{2}}\|u_z\|_{\delta,p,\varepsilon}.$$

The following properties of $W_\varepsilon^{\delta,p}(\Sigma)$ are readily deduced from the fact that it is a scaled version of the standard interpolation space $W^{\delta,p}(\Sigma)$.

Proposition 3.

(i) *The space $W_\varepsilon^{\delta_2,p}(\Sigma)$ is continuously embedded in $W_\varepsilon^{\delta_1,p}(\Sigma)$ whenever $\delta_1 \leqq \delta_2$; in particular we have the embedding inequality*

$$\|u\|_{\delta_1,p,\varepsilon} \leqq \|u\|_{\delta_2,p,\varepsilon}, \quad \delta_1 \leqq \delta_2.$$

(ii) *The space $W_\varepsilon^{\delta,p}(\Sigma)$ is a Banach algebra and continuously embedded in $C_\mathrm{b}(\Sigma)$ whenever $\delta > 3/p$; in particular we have the inequalities*

$$\|uv\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2p}}\|u\|_{\delta,p,\varepsilon}\|v\|_{\delta,p,\varepsilon}, \quad \|u\|_\infty \leqq c\varepsilon^{-\frac{1}{2p}}\|u\|_{\delta,p,\varepsilon}, \ \ \delta > 3/p.$$

Finally, we state some elementary properties of operators which arise naturally when passing between functions defined on $\mathbb{R}^2$ and functions defined on Σ.

Proposition 4.

(i) *The mapping*

$$u \mapsto \int_0^1 u(\cdot, y)\,\mathrm{d}y$$

defines a bounded linear operator $W_\varepsilon^{\delta,p}(\Sigma) \to W_\varepsilon^{\delta,p}(\mathbb{R}^2)$.

(ii) *The natural extension of $u : \mathbb{R}^2 \to \mathbb{R}$ to $u : \Sigma \to \mathbb{R}$ defines a bounded linear operator $W_\varepsilon^{\delta,p}(\mathbb{R}^2) \to W_\varepsilon^{\delta,p}(\Sigma)$.*

(iii) *The trace mapping $u \mapsto u|_S$ defines bounded linear operators*

$$W_\varepsilon^{1,2}(\Sigma) \to W_\varepsilon^{1/2,2}(\mathbb{R}^2)$$

and

$$W_\varepsilon^{\delta,p}(\Sigma) \to W_\varepsilon^{\delta-1/p,p}(\mathbb{R}^2), \quad p > 2.$$

The norms of the linear operators listed above are all independent of ε.

2. Reduction to a single pseudodifferential equation

In this section we present an overview of the reduction procedure which converts equations (10)–(13) into a locally equivalent semilinear pseudodifferential equation for a single function of the horizontal coordinates (x, z). The method is variational in nature, that is the variational principle (8) associated with (10)–(13) is simultaneously converted into a variational principle for the reduced equation. Full details of the theorems used in the reduction procedure are given in Section 3, and the existence proof is completed in Section 4, which is concerned with finding critical points of the reduced variational functional.

The goal of this paper is to find solutions (ρ, Φ) of equations (10)–(13) which belong to $[W^{1,2}(\mathbb{R}^2) \times U^{0,2}(\Sigma)] \cap [W^{2,p}(\mathbb{R}^2) \times U^{1,p}(\Sigma)]$ for all sufficiently large values of $p > 2$; the trace $\Phi_x|_{y=1}$ and nonlinearities N_1, N_2, N_3 are well defined and smooth (in a neighbourhood of the origin) in these function spaces. We refer to such solutions as *strong solutions* of (10)–(13). Our strategy is to seek *weak solutions* of these equations which lie in the larger function space $[W^{1,2}(\mathbb{R}^2) \times U^{0,2}(\Sigma)] \cap [W^{1+\delta,p}(\mathbb{R}^2) \times U^{\delta,p}(\Sigma)]$ for sufficiently small values of $\delta \in (0, 1)$ and establish a regularity result that weak solutions are in fact strong solutions. We always choose δ and p with $\delta > 3/p$ so that the weak forms of the nonlinearities are well defined and smooth. It is later necessary to work in scaled versions of these function spaces in order to solve certain fixed-point equations (we return to this issue in detail below); we therefore use scaled spaces from the outset, although only their topological properties are relevant for the present discussion of weak and strong solutions. The scaled spaces in question are $[V_\varepsilon^{0,2}(\mathbb{R}^2) \times U_\varepsilon^{0,2}(\Sigma)] \cap [V_\varepsilon^{1,p}(\mathbb{R}^2) \times U_\varepsilon^{1,p}(\Sigma)]$ for strong solutions and $[V_\varepsilon^{0,2}(\mathbb{R}^2) \times U_\varepsilon^{0,2}(\Sigma)] \cap [V_\varepsilon^{\delta,p}(\mathbb{R}^2) \times U_\varepsilon^{\delta,p}(\Sigma)]$ for weak solutions (see Section 1.3 for their definitions).

Definition 1. A *weak solution* of (10)–(13) is a pair (ρ, Φ) of functions which lie in $[V_\varepsilon^{0,2}(\mathbb{R}^2) \times U_\varepsilon^{0,2}(\Sigma)] \cap [V_\varepsilon^{\delta,p}(\mathbb{R}^2) \times U_\varepsilon^{\delta,p}(\Sigma)]$ and satisfy

$$\begin{aligned}&\int_{\mathbb{R}^2} \left\{(1+\varepsilon)\rho\omega + \beta\varepsilon^2 \rho_x \omega_x + \beta\varepsilon^4 \rho_z \omega_z\right\} \mathrm{d}x\,\mathrm{d}z \\ &\quad = -\int_{\mathbb{R}^2}\int_0^1 (y\Phi_y\omega_x - \Phi_x\omega)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z + \int_{\mathbb{R}^2} \varepsilon^{-1} N_1(\rho, \Phi)\omega\,\mathrm{d}x\,\mathrm{d}z, \end{aligned}\tag{18}$$

$$\begin{aligned}&\int_{\mathbb{R}^2} \left\{\int_0^1 (\Phi_y\Psi_y + \varepsilon\Phi_x\Psi_x + \varepsilon^2\Phi_z\Psi_z)\,\mathrm{d}y + \rho_x \Psi|_{y=1}\right\} \mathrm{d}x\,\mathrm{d}z \\ &\quad = \int_{\mathbb{R}^2}\int_0^1 \varepsilon^{-\frac{1}{2}} N_4(\rho, \Phi)\Psi\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z + \int_{\mathbb{R}^2}\int_0^1 \varepsilon^{-\frac{1}{2}} N_5(\rho, \Phi)\hat{\Psi}_y\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z \end{aligned}\tag{19}$$

for all $(\omega, \Psi) \in V_\varepsilon^{0,2}(\mathbb{R}^2) \times W_\varepsilon^{1,2}(\Sigma)$ (or any dense subset thereof). Here

$$N_4(\rho, \Phi) = \varepsilon^{\frac{5}{2}}(\rho\Phi_x)_x + \varepsilon^{\frac{7}{2}}(\rho\Phi_z)_z - \varepsilon^{\frac{5}{2}}(y\Phi_y\rho_x)_x - \varepsilon^{\frac{7}{2}}(y\Phi_y\rho_z)_z,$$

$$N_5(\rho, \Phi) = \varepsilon^{\frac{5}{2}}\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right) y\rho_x + \varepsilon^{\frac{7}{2}}\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right) y\rho_z - \frac{\varepsilon^{\frac{3}{2}}\rho\Phi_y}{1+\varepsilon\rho}$$

and the 'outer' derivatives with respect to x and z in N_4 and N_1 are transferred to, respectively, Ψ and ω by an integration by parts.

The following proposition explains the significance of weak solutions of equations (10)–(13) in our variational existence theory.

Proposition 5. *The weak solutions of equations* (10)–(13) *are precisely the critical points of the smooth functional* $\mathcal{V} : [V_\varepsilon^{0,2}(\mathbb{R}^2) \times U_\varepsilon^{0,2}(\Sigma)] \cap [V_\varepsilon^{\delta,p}(\mathbb{R}^2) \times U_\varepsilon^{\delta,p}(\Sigma)] \to \mathbb{R}$.

Our first step is to formulate the hydrodynamic problem in terms of integral equations which are later solved using fixed-point arguments. We begin by fixing Φ and examining the equation for ρ. Taking the Fourier transform of the strong form (10) of the equation for ρ, one finds that

$$\hat{\rho} = \frac{1}{1+\varepsilon+\beta q^2}\left(\mathrm{i}\mu \int_0^1 y\hat{\Phi}_y \,\mathrm{d}y + \int_0^1 \hat{\Phi}_x \,\mathrm{d}y + \varepsilon^{-1}\hat{N}_1(\rho,\Phi)\right), \tag{20}$$

where $q^2 = \varepsilon\mu^2 + \varepsilon^2 k^2$. It is helpful to write this equation in the form

$$\begin{aligned}\hat{\rho} = {} & \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\left(\int_0^1 y\hat{\Phi}_y \,\mathrm{d}y\right) + \frac{1}{1+\varepsilon+\beta q^2}\left(\int_0^1 \hat{\Phi}_x \,\mathrm{d}y\right) \\ & + \frac{1}{1+\varepsilon+\beta q^2}\left(\varepsilon^{-1}\hat{N}_1^1(\rho,\Phi)\right) + \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\left(\varepsilon^{-1}\hat{N}_1^2(\rho,\Phi)\right) \\ & + \frac{\mathrm{i}\varepsilon^{1/2}k}{1+\varepsilon+\beta q^2}\left(\varepsilon^{-\frac{3}{2}}\hat{N}_1^3(\rho,\Phi)\right),\end{aligned} \tag{21}$$

where

$$\begin{aligned}N_1^1(\rho,\Phi) = -\int_0^1 \Bigg\{ & \varepsilon^2\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right)^2 + \varepsilon^3\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right)^2 \\ & + \frac{\varepsilon\Phi_y^2}{2(1+\varepsilon\rho)^2} + \varepsilon^{\frac{3}{2}}\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right)\frac{y\Phi_y}{1+\varepsilon\rho} \\ & + \varepsilon^2\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right)\frac{y\Phi_y}{1+\varepsilon\rho}\Bigg\}\,\mathrm{d}y,\end{aligned}$$

$$N_1^2(\rho,\Phi) = \frac{\beta\varepsilon^2\rho_x}{\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}} - \beta\varepsilon^2\rho_x - \varepsilon^2\int_0^1\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right)y\Phi_y\,\mathrm{d}y,$$

$$N_1^3(\rho,\Phi) = \frac{\beta\varepsilon^3\rho_z}{\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}} - \beta\varepsilon^3\rho_z - \varepsilon^3\int_0^1\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right)y\Phi_y\,\mathrm{d}y.$$

This equation consists of a series of Fourier-multiplier operators acting upon nonlinear functions of ρ and first derivatives of Φ; in keeping with this structure the x derivative in the second term on the right-hand side of (20) is not absorbed into the corresponding multiplier. Inspecting (21), we find that each of the quantities in brackets defines a mapping from a neighbourhood of the origin in $[V_\varepsilon^{0,2}(\mathbb{R}^2) \times U_\varepsilon^{0,2}(\Sigma)] \cap [V_\varepsilon^{\delta,p}(\mathbb{R}^2) \times U_\varepsilon^{\delta,p}(\Sigma)]$ into $W_\varepsilon^{0,2}(\mathbb{R}^2) \cap W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ (see Propositions 1–4), and in fact each of the Fourier multipliers appearing in (21) defines a mapping from $W_\varepsilon^{0,2}(\mathbb{R}^2) \cap W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ into $V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ (see Lemma 2 below). Equation (20) is therefore well defined for (ρ, Φ) in the larger function class $[V_\varepsilon^{0,2}(\mathbb{R}^2) \times U_\varepsilon^{0,2}(\Sigma)] \cap [V_\varepsilon^{\delta,p}(\mathbb{R}^2) \times U_\varepsilon^{\delta,p}(\Sigma)]$, and in this setting we refer to it as the *integral form of the equation for* ρ.

We can also obtain a weak form of the equation for ρ by multiplying the strong form by a test function and integrating by parts; the weak and integral forms of the equation for ρ are in fact equivalent.

Proposition 6. *Suppose that $\Phi \in U_\varepsilon^{0,2}(\Sigma) \cap U_\varepsilon^{\delta,p}(\Sigma)$. A function $\rho^\star \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ solves the integral form of the equation for ρ if and only if it is a weak solution of the equation for ρ.*

Proof. With slightly more generality we consider the problem posed by the above equations in which N_1 is an arbitrary function of the form

$$\hat{N}_1 = \hat{N}_1^1 + \mathrm{i}\mu\hat{N}_1^2 + \mathrm{i}\varepsilon^{\frac{1}{2}}k\hat{N}_1^3, \quad N_1^1, N_1^2, N_1^3 \in L^2(\mathbb{R}^2).$$

Suppose that $\rho^\star$ solves the integral form of the equation for ρ, so that $\rho^\star$ is related to Φ, N_1^1, N_1^2 and N_1^3 according to equation (21). Clearly

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left\{(1+\varepsilon)\rho^\star\omega + \beta\varepsilon^2\rho_x^\star\omega_x + \beta\varepsilon^4\rho_z^\star\omega_z\right\} dx\,\mathrm{d}z \\
&= \int_{\mathbb{R}^2} \left\{(1+\varepsilon+\beta q^2)\hat{\rho}^\star\bar{\hat{\omega}}\,\mathrm{d}k_1\,\mathrm{d}k_2\right. \\
&= \int_{\mathbb{R}^2}\int_0^1 (\mathrm{i}\mu y\hat{\Phi}_y\bar{\hat{\omega}} + \hat{\Phi}_x\bar{\hat{\omega}})\,\mathrm{d}y\,\mathrm{d}k_1\,\mathrm{d}k_2 \\
&\quad + \varepsilon^{-1}\int_{\mathbb{R}^2} (\hat{N}_1^1 + \mathrm{i}\mu\hat{N}_1^2 + \mathrm{i}\varepsilon^{\frac{1}{2}}k\hat{N}_1^3)\bar{\hat{\omega}}\,\mathrm{d}k_1\,\mathrm{d}k_2 \\
&= -\int_{\mathbb{R}^2}\int_0^1 (y\Phi_y\omega_x - \Phi_x\omega)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z \\
&\quad + \varepsilon^{-1}\int_{\mathbb{R}^2} (N_1^1\omega - N_1^2\omega_x - N_1^3\omega_z)\,\mathrm{d}x\,\mathrm{d}z
\end{aligned}$$

for all $\omega \in V_\varepsilon^{0,2}(\mathbb{R}^2)$, so that $\rho^\star$ is a weak solution of the equation for ρ.

Conversely, suppose that $\rho^\star$ is a weak solution of the equation for ρ. A familiar argument asserts the existence of a function $F \in C(U_\varepsilon^{0,2}(\Sigma) \times (L^2(\Sigma))^3, V_\varepsilon^{0,2}(\mathbb{R}^2))$ with the property that $\rho^\star = F(\Phi, N_1^1, N_1^2, N_1^3)$. In the special case in which $(\Phi, N_1^1, N_1^2, N_1^3)$ belongs to the dense subset $U_\varepsilon^{1,2}(\Sigma) \times (W_0^{1,2}(\mathbb{R}^2))^3$ of $U_\varepsilon^{0,2}(\Sigma) \times (L^2(\Sigma))^3$ the elliptic regularity theory implies that $\rho^\star$ belongs to $V_\varepsilon^{1,2}(\mathbb{R}^2)$ and is a strong solution of the equation for ρ; taking the Fourier transform, one finds that $\rho^\star$ satisfies (21). The right-hand side of (21) defines a function $G \in C(U_\varepsilon^{0,2}(\Sigma) \times (L^2(\Sigma))^3, V_\varepsilon^{0,2}(\mathbb{R}^2))$, and it follows from the above argument that

$$F(\Phi, N_1^1, N_1^2, N_1^3) = G(\Phi, N_1^1, N_1^2, N_1^3)$$

for every $(\Phi, N_1^1, N_1^2, N_1^3) \in U_\varepsilon^{1,2}(\Sigma) \times (W_0^{1,2}(\mathbb{R}^2))^3$. Using a standard density argument, we conclude that this equation holds for every $(\Phi, N_1^1, N_1^2, N_1^3) \in U_\varepsilon^{0,2}(\Sigma) \times (L^2(\Sigma))^3$ and hence that $\rho^\star$ solves the integral form of the equation for ρ. □

The boundary-value problem for Φ yields integral and weak formulations in an analogous fashion. Taking the Fourier transform of the strong form (11)–(13) of the equations for Φ and using (20) to eliminate ρ from the linear part of the equations, we obtain the boundary-value problem

$$-\hat{\Phi}_{yy} + q^2\hat{\Phi} = \varepsilon^{-\frac{1}{2}}\hat{N}_2(\rho, \Phi), \quad 0 < y < 1,$$

$$\hat{\Phi}_y - \frac{\varepsilon\mu^2\hat{\Phi}}{1+\varepsilon+\beta q^2} = \frac{-\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho, \Phi) + \varepsilon^{-\frac{1}{2}}N_3(\rho, \Phi) \qquad \text{on } y = 1,$$

$$\hat{\Phi}_y = 0 \qquad \text{on } y = 0.$$

This boundary-value problem can be recast as the single equation

$$\hat{\Phi} = -\int_0^1 G\varepsilon^{-\frac{1}{2}}\hat{N}_2(\rho, \Phi)\,\mathrm{d}\xi - G|_{\xi=1}\left(\frac{-\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho, \Phi) + \varepsilon^{-\frac{1}{2}}\hat{N}_3(\rho, \Phi)\right),$$

in which the Green's function $G(y, \xi)$ is given by

$$G(y,\xi) = \begin{cases} \dfrac{\cosh qy}{\cosh q}\,\dfrac{(1+\varepsilon+\beta q^2)\cosh q(1-\xi) + (\varepsilon\mu^2/q)\sinh q(\xi-1)}{q^2 - (1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2}, \\ \hfill 0 < y < \xi < 1, \\[2ex] \dfrac{\cosh q\xi}{\cosh q}\,\dfrac{(1+\varepsilon+\beta q^2)\cosh q(1-y) + (\varepsilon\mu^2/q)\sinh q(y-1)}{q^2 - (1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2}, \\ \hfill 0 < \xi < y < 1, \end{cases}$$

and an integration by parts yields the alternative representation

$$\hat{\Phi} = -\int_0^1 G\varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho, \Phi)\,\mathrm{d}\xi - \int_0^1 G_\xi\varepsilon^{-\frac{1}{2}}\hat{N}_5(\rho, \Phi)\,\mathrm{d}\xi + \frac{\mathrm{i}\mu G|_{\xi=1}}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho, \Phi). \tag{22}$$

Examining equation (22) in the same way as equation (20) (see the discussion below (20)), we find that it is well defined for $(\rho, \Phi) \in [V_\varepsilon^{0,2}(\mathbb{R}^2) \times U_\varepsilon^{0,2}(\Sigma)] \cap [V_\varepsilon^{\delta,p}(\mathbb{R}^2) \times U_\varepsilon^{\delta,p}(\Sigma)]$, and in this setting we refer to it as the *integral form of the equation for* Φ. The appropriate weak form of the equation for Φ is found by multiplying the above boundary problem by a test function and integrating by parts; the next proposition, which is proved in the same way as Proposition 6, shows that it is sufficient to consider the integral form of the equation for Φ when seeking weak solutions.

Proposition 7. *Suppose that* $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$. *A function* $\Phi^\star \in U_\varepsilon^{0,2}(\Sigma) \cap U_\varepsilon^{\delta,p}(\Sigma)$ *solves the integral form of the problem for* Φ *if and only if it is a weak solution of the problem for* Φ.

Our study of fully localised solitary waves is motivated by their existence as explicit solutions of the KP-I model equation, which is formally derived in the long-wave limit $|(\mu, k)| \to 0$. This fact suggests that any bifurcation of fully localised solitary wave solutions to the full water-wave problem should be controlled by terms associated with the long-wave limit. We therefore proceed by expanding the dispersion relation, which appears in the denominator of the Green's function, in powers of μ and k; the result is

$$\begin{aligned} & q^2 - (1+\varepsilon+\beta q^2) q \tanh q - \varepsilon^2 k^2 \\ &\quad = -\varepsilon^2 \Big[k^2(1+\varepsilon) + \mu^2 + \left(\beta - \tfrac{1}{3}(1+\varepsilon)\right) \varepsilon^{-2} q^4 \\ &\qquad\qquad + \left(\tfrac{2}{15}(1+\varepsilon) - \tfrac{1}{3}\beta\right) \varepsilon^{-2} q^6 \Big] + \mathcal{O}(q^8) \\ &\quad = -\varepsilon^2 Q - \tfrac{1}{3}\varepsilon q^4 + \mathcal{O}(q^8) \end{aligned} \tag{23}$$

as $|(\mu, k)| \to 0$, where

$$Q = k^2(1+\varepsilon) + \mu^2 + \left(\beta - \tfrac{1}{3}\right) \varepsilon^{-2} q^4 + c_0 \varepsilon^{-2} q^6, \quad c_0 = \tfrac{2}{15}(1+\varepsilon) - \tfrac{1}{3}\beta.$$

Supposing that the coefficients of q^4 and q^6 are positive, one finds that the quantity $\varepsilon^2 Q$ defines both a differential operator (whose first and second terms coincide with the linear operator in the KP-I equation) and the norm of a Hilbert space X (cf. equation (15)). By including the third term in Q we ensure that X is continuously embedded in the Banach spaces $U_\varepsilon^{0,2}(\mathbb{R}^2)$ and $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ for $\delta > p/2$ (see Proposition 1(iv)), and this feature guarantees the semilinearity of the reduced pseudodifferential equation derived below. Moving the term $-\frac{1}{3}\varepsilon q^4$ to the remainder in the above expansion ensures that the coefficient of q^4 is positive whenever $\beta > 1/3$, while the coefficient c_0 of q^6 is clearly positive for $\beta < \frac{2}{5}(1+\varepsilon)$. It is actually possible to apply the method for all values of $\beta > 1/3$ by including higher-order terms in the expansion: one finds that Q defines a norm for $\beta > \frac{26}{63}(1+\varepsilon)$ when the eighth-order terms are included, and an expansion up to tenth order fills the gap $\frac{2}{5}(1+\varepsilon) \leqq \beta \leqq \frac{26}{63}(1+\varepsilon)$. Although we concentrate in this paper upon the case when $c_0 > 0$, it is a straightforward matter to extend our calculations to the remaining cases.

Let us now decompose the Green's function into a singular and a smooth part using the formula

$$G = -\frac{1+\varepsilon}{\varepsilon^2 Q} + \varepsilon^{-2} G_1$$

and define functions $\Phi_1(x, z)$ and $\Phi_2(x, y, z)$ by replacing G with, respectively, its first and second component in the integral form of the equation for Φ, so that

$$\hat{\Phi}_1 = \frac{1+\varepsilon}{\varepsilon^2 Q} \left(\int_0^1 \varepsilon^{-\frac{1}{2}} \hat{N}_4(\rho, \Phi) \, \mathrm{d}\xi - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2} \hat{N}_1(\rho, \Phi) \right), \tag{24}$$

$$\begin{aligned} \hat{\Phi}_2 = & -\int_0^1 \frac{G_1}{\varepsilon^{5/2}} \hat{N}_4(\rho, \Phi) \, \mathrm{d}\xi \\ & - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}} \hat{N}_5(\rho, \Phi) \, \mathrm{d}\xi + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \hat{N}_1(\rho, \Phi). \end{aligned} \tag{25}$$

It is a straightforward matter to confirm that equations (24), (25) are equivalent to equation (22).

Proposition 8.

(i) *Any solution of the integral form* (22) *of the equation for* Φ *can be expressed as the sum* $\Phi = \Phi_1 + \Phi_2$*, where* Φ_1*,* Φ_2 *solve* (24), (25).
(ii) *Suppose conversely that* Φ_1*,* Φ_2 *satisfy equations* (24), (25) *with* $\Phi = \Phi_1 + \Phi_2$*. The function* Φ *satisfies equation* (22).

In keeping with this proposition, we henceforth abandon the integral form of the equation for Φ and work instead with (24), (25) with $\Phi = \Phi_1 + \Phi_2$ on their right-hand sides; these equations are the *integral forms of the equations for* Φ_1 *and* Φ_2. Equation (24) is valid for $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\Phi_1 \in X$, $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$, while equation (25) is valid for $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\Phi_1 \in U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$. Notice the difference in the regularity requirements for Φ_1 between the two equations; upon writing down the weak and strong versions of the integral equations this difference manifests itself in the fact that the equation for Φ_1 is semilinear, while the equation for Φ_2 remains quasilinear (see below). It is actually convenient to place a further requirement upon Φ_1 in relation to the integral form of the problem for Φ_2, namely that it should also lie in $U_\varepsilon^{0,4}(\mathbb{R}^2)$ (into which X is also continuously embedded). This restriction allows one to obtain better estimates for the Φ_2 equation in the subsequent existence theory; we therefore also apply it in the requirements for a weak solution of the equation for Φ_2.

The strong form of the equation for Φ_1 is clearly

$$\frac{\varepsilon^2}{1+\varepsilon}[-c_0\varepsilon(\partial_x^2 + \varepsilon\partial_z^2)^3 + (\beta - \tfrac{1}{3})(\partial_x^2 + \varepsilon\partial_z^2)^2 - (1+\varepsilon)\partial_z^2 - \partial_x^2]\Phi_1$$
$$= \int_0^1 \varepsilon^{-\frac{1}{2}} N_4(\rho, \Phi)\,\mathrm{d}\xi - \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho, \Phi)\right]$$

and is well defined for $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{1,p}(\mathbb{R}^2)$, $\Phi_1 \in U_\varepsilon^{5,p}(\mathbb{R}^2)$, $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{2,p}(\Sigma)$. The equation is semilinear since its right-hand side is well defined for Φ_1 in the larger space $U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{1,p}(\mathbb{R}^2)$, into which X is continuously embedded. The strong form of the equation for Φ_2 is calculated by substituting

$$\Phi = \Phi_2 + \mathcal{F}^{-1}\left[\frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho, \Phi)\,\mathrm{d}\xi - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho, \Phi)\right)\right]$$

into the strong form of the equation for Φ; one finds that

$$-\hat{\Phi}_{2yy} + q^2\hat{\Phi}_2 = \varepsilon^{-\frac{1}{2}}\hat{N}_2(\rho, \Phi)$$
$$-\frac{q^2(1+\varepsilon)}{\varepsilon^2 Q}\left(\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho, \Phi)\,\mathrm{d}y - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho, \Phi)\right),$$
$$0 < y < 1, \qquad (26)$$

$$\hat{\Phi}_{2y} - \frac{\varepsilon\mu^2\hat{\Phi}_2}{1+\varepsilon+\beta q^2} = \varepsilon^{-\frac{1}{2}}\hat{N}_3(\rho,\Phi) - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)$$
$$+\frac{(1+\varepsilon)\varepsilon\mu^2}{\varepsilon^2 Q(1+\varepsilon+\beta q^2)}\left(\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi)\,\mathrm{d}y - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\right)$$
$$\text{on } y=1, \qquad (27)$$

$$\hat{\Phi}_{2y} = 0 \quad \text{on } y = 0, \qquad (28)$$

and this quasilinear boundary-value problem is well defined for $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{2,p}(\mathbb{R}^2)$, $\Phi_1 \in U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{1,p}(\mathbb{R}^2)$ and $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{2,p}(\Sigma)$. The semi- and quasilinearity of the respective equations is also evident in their weak formulations.

Definition 2.

(i) Suppose that $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$. A *weak solution* of the equation for Φ_1 is a function $\Phi_1^\star \in X$ which satisfies

$$\langle\!\langle\Phi_1^\star, \Psi_1\rangle\!\rangle =$$
$$\frac{1+\varepsilon}{\varepsilon^2}\int_{\mathbb{R}^2}\left(\int_0^1 \varepsilon^{-\frac{1}{2}}N_4(\rho,\Phi)\,\mathrm{d}\xi - \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\right]\right)\bar{\hat{\Psi}}_1\,\mathrm{d}x\,\mathrm{d}z$$

for all $\Psi_1 \in X$ (or any dense subset thereof); here $\Phi = \Phi_1^\star + \Phi_2$ and the 'outer' derivatives with respect to x and z in N_4 and N_1 are transferred to Ψ_2 by an integration by parts. Observe that the right-hand side of the equation is well defined for Φ_1 in the larger space $U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, into which X is continuously embedded.

(ii) Suppose that $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and $\Phi_1 \in U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$. A *weak solution* of the problem for Φ_2 is a function $\Phi_2^\star \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$ which satisfies

$$\int_{\mathbb{R}^2}\left\{\int_0^1 (\hat{\Phi}_{2y}^\star\bar{\hat{\Psi}}_y + q^2\hat{\Phi}_2^\star\bar{\hat{\Psi}})\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2^\star|_{y=1}\bar{\hat{\Psi}}|_{y=1}}{1+\varepsilon+\beta q^2}\right.$$
$$+\left(\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi)\,\mathrm{d}y - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\right)$$
$$\left.\times\left(\frac{(1+\varepsilon)q^2}{\varepsilon^2 Q}\int_0^1 \bar{\hat{\Psi}}_2\,\mathrm{d}y - \frac{(1+\varepsilon)\varepsilon\mu^2\bar{\hat{\Psi}}_2}{\varepsilon^2 Q(1+\varepsilon+\beta q^2)}\right)\right\}\mathrm{d}\mu\,\mathrm{d}k$$
$$= \int_{\mathbb{R}^2}\left\{\int_0^1 (\varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi)\bar{\hat{\Psi}}_2 + \varepsilon^{-\frac{1}{2}}\hat{N}_5(\rho,\Phi)\bar{\hat{\Psi}}_{2y})\,\mathrm{d}y\right.$$
$$\left. - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\bar{\hat{\Psi}}_2|_{y=1}\right\}\mathrm{d}\mu\,\mathrm{d}k$$

for all $\Psi_2 \in W_\varepsilon^{1,2}(\Sigma)$ (or any dense subset thereof); here $\Phi = \Phi_1 + \Phi_2^\star$ and the 'outer' derivatives with respect to x and z in N_4 and N_1 are transferred to Ψ_1 by an integration by parts.

Proposition 9.

(i) *Suppose that* $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *and* $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$. *A function* $\Phi_1^\star \in X$ *solves the integral form of the equation for* Φ_1 *if and only if it is a weak solution of the equation for* Φ_1.

(ii) *Suppose that* $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *and* $\Phi_1 \in U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$. *A function* $\Phi_2^\star \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$ *solves the integral form of the problem for* Φ_2 *if and only if it is a weak solution of the problem for* Φ_2.

We now proceed in a fashion reminiscent of the classical Lyapunov–Schmidt reduction. In this method a problem is treated by writing it as a pair of coupled equations for two unknowns X and Y; one of the equations is solved to yield the functional relationship $Y = Y(X)$, and inserting this function into the other equation one obtains a reduced equation for X. We use this two-step approach for our water-wave problem in the following manner. Firstly we apply fixed-point principles to solve the integral forms of the equations for ρ and Φ_2 for ρ, Φ_2 as functions of Φ_1 and secondly we substitute the solutions $\rho = \rho(\Phi_1)$, $\Phi_2 = \Phi_2(\Phi_1)$ into the integral form of the equation for Φ_1 to obtain a reduced equation for Φ_1. It is essential to use the scaled function spaces to solve our integral equations, and we explain this point in detail in Section 3 below, where the following theorem is established.

Theorem 1. *Suppose that*

$$|||\Phi_1||| \leqq M,$$

so that Φ_1 *belongs to* $U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *with* $\|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-1/4-\Delta}$ (*see Proposition* 1(iv)). *For sufficiently small values of* δ *and sufficiently large values of* p (*with* $\delta > 3/p$) *the integral forms of the equations for* ρ *and* Φ_2 *admit unique solutions* $\rho = \rho(\Phi_1)$ *in* $V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *and* $\Phi_2 \in \Phi_2(\Phi_1)$ *in* $W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$ *that satisfy*

$$|\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}(\|\Phi_{1x}\|_{\delta,p,\varepsilon} + P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})),$$

$$\|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}),$$

$$\|\Phi_{2y}\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}),$$

$$|\rho|_{0,2,\varepsilon} \leqq c(\|\Phi_{1x}\|_2 + \varepsilon^{\frac{1}{2}}\|\Phi_1\|^2_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,2}}P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})),$$

$$\|\Phi_2\|_{1,2,\varepsilon} \leqq c(\varepsilon^{\frac{1}{2}}\|\Phi_1\|^2_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,2}}P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})),$$

$$\|\Phi_{2y}\|_2 \leqq c(\varepsilon\|\Phi_1\|^2_{U_\varepsilon^{0,4}} + \varepsilon^{1-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,2}}P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}));$$

the functions ρ *and* Φ_2 *depend smoothly upon* Φ_1 *in the topology defined by these function spaces. The symbols* Δ *and* P_n *denote, respectively, a quantity which is*

$O(\delta+1/p)$ and a polynomial which has unit positive coefficients and no monomials of degree less than n.

Substituting $\rho = \rho(\Phi_1)$ and $\Phi_2 = \Phi_2(\Phi_1)$ into the integral form of the equation for Φ_1, we obtain the (integral form of the) reduced equation

$$\hat{\Phi}_1 = \frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho(\Phi_1), \Phi_1+\Phi_2(\Phi_1))\,\mathrm{d}\xi \right.$$
$$\left. -\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho(\Phi_1), \Phi_1+\Phi_2(\Phi_1))\right) \tag{29}$$

for the single variable $\Phi_1 \in X$; the weak and strong versions of this equation are obtained in the usual fashion, and are clearly semilinear. Our analysis proves that a solution $\Phi_1^\star$ of any formulation (integral, weak or strong) of the reduced equation for Φ_1 generates a weak solution (ρ, Φ) of the original hydrodynamic problem, where $\rho = \rho(\Phi_1^\star)$ and $\Phi = \Phi_1^\star + \Phi_2^\star(\Phi_1^\star)$. We now show that this equation inherits the variational structure of the original hydrodynamic problem, that is the reduction procedure also reduces the functional $\mathcal{V}$ to a variational functional J for the reduced equation for Φ_1. In particular, critical points of J are weak solutions of the reduced equation for Φ_1, and of course the semilinear nature of this equation means that J falls into the 'coercive, semilinear, locally compact case' to which the direct methods of the calculus of variations may be applied.

Let us return to the original hydrodynamic problem (10)–(13) with its associated variational functional $\mathcal{V}$. The Euler–Lagrange equations for $\mathcal{V}$, namely

$$\mathrm{d}_1\mathcal{V}[\rho, \Phi] = 0, \tag{30}$$
$$\mathrm{d}_2\mathcal{V}[\rho, \Phi] = 0, \tag{31}$$

correspond to the weak form of the system (10)–(13) and are given explicitly by equations (18) and (19). Proposition 6 asserts that (30) is equivalent to the integral form of the equation for ρ, and the first step in the reduction procedure is to solve this equation for ρ as a function of Φ and insert $\rho = \rho(\Phi)$ into the equation for Φ, whose weak form is therefore

$$\mathrm{d}_2\mathcal{V}[\rho(\Phi), \Phi] = 0. \tag{32}$$

The following proposition shows that this step in the reduction procedure preserves the variational structure in a natural way.

Proposition 10. *Define a smooth functional $\mathcal{W} : U_\varepsilon^{0,2}(\Sigma) \cap U_\varepsilon^{\delta,p}(\Sigma) \to \mathbb{R}$ by the formula $\mathcal{W}(\Phi) = \mathcal{V}(\rho(\Phi), \Phi)$. The critical points of $\mathcal{W}$ are precisely the solutions of equation* (32).

Proof. Observe that

$$\begin{aligned}\mathrm{d}\mathcal{W}[\Phi] &= \mathrm{d}_1\mathcal{V}[\rho(\Phi), \Phi](\mathrm{d}\rho[\Phi]) + \mathrm{d}_2\mathcal{V}[\rho(\Phi), \Phi] \\ &= \mathrm{d}_2\mathcal{V}[\rho(\Phi), \Phi]\end{aligned}$$

since the defining property of $\rho(\Phi)$ is that it solves equation (30). $\square$

We now proceed formally. Because

$$\int_0^1 \varepsilon^{-\frac{1}{2}} \hat{N}_4(\rho, \Phi)\,\mathrm{d}y - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho, \Phi) = \int_0^1 \hat{H}\,\mathrm{d}y + \hat{h},$$

where

$$H = \varepsilon^{-\frac{1}{2}} N_2(\rho(\Phi), \Phi),$$
$$h = \varepsilon^{-\frac{1}{2}} N_3(\rho(\Phi), \Phi) - \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho(\Phi), \Phi)\right],$$

one can write the strong form (26)–(28) of the problem for Φ_2 as

$$-\hat{\Phi}_{2yy} + q^2\hat{\Phi}_2 = \hat{H} - \frac{q^2(1+\varepsilon)}{\varepsilon^2 Q}\left(\int_0^1 \hat{H}\,\mathrm{d}y + \hat{h}\right), \quad 0 < y < 1, \tag{33}$$

$$\hat{\Phi}_{2y} - \frac{\varepsilon\mu^2\hat{\Phi}_2}{1+\varepsilon+\beta q^2} = \hat{h} + \frac{(1+\varepsilon)\varepsilon\mu^2}{\varepsilon^2 Q(1+\varepsilon+\beta q^2)}\left(\int_0^1 \hat{H}\,\mathrm{d}y + \hat{h}\right) \quad \text{on } y = 1, \tag{34}$$

$$\hat{\Phi}_{2y} = 0 \quad \text{on } y = 0. \tag{35}$$

Integrating (33) with respect to y over (0, 1) and substituting for $\hat{\Phi}_{2y}|_{y=0}$, $\hat{\Phi}_{2y}|_{y=1}$ according to (34), (35), we find that

$$q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2} = S\left(\int_0^1 \hat{H}\,\mathrm{d}y + \hat{h}\right), \tag{36}$$

where

$$S = 1 - \frac{q^2(1+\varepsilon)}{\varepsilon^2 Q} + \frac{(1+\varepsilon)\varepsilon\mu^2}{\varepsilon^2 Q(1+\varepsilon+\beta q^2)}.$$

It follows that each solution $\Phi_2(\Phi_1)$ of (33)–(35) satisfies (36) and hence also solves

$$-\hat{\Phi}_{2yy} + q^2\hat{\Phi}_2 + \frac{q^2(1+\varepsilon)}{\varepsilon^2 QS}\left(q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right) = \hat{H},$$
$$0 < y < 1, \tag{37}$$

$$\hat{\Phi}_{2y} - \frac{\varepsilon\mu^2\hat{\Phi}_2}{1+\varepsilon+\beta q^2}$$
$$+\frac{(1+\varepsilon)\varepsilon\mu^2}{\varepsilon^2 QS(1+\varepsilon+\beta q^2)}\left(q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right) = \hat{h} \quad \text{on } y = 1, \tag{38}$$

$$\hat{\Phi}_{2y} = 0 \quad \text{on } y = 0, \tag{39}$$

This argument is reversible: a similar calculation shows that the identity (36) holds for each solution $\Phi_2(\Phi_1)$ of (37)–(39), which therefore solves (33)–(35), and we conclude that the formulations (33)–(35) and (37)–(39) are equivalent.

The left-hand sides of equations (37)–(39) constitute a formally self-adjoint operator associated with the quadratic form

$$Q_2(\Phi_2) = \frac{1}{2}\int_{\mathbb{R}^2}\left\{\int_0^1 (|\hat{\Phi}_{2y}|^2 + q^2|\hat{\Phi}_2|^2)\,\mathrm{d}y - \frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}|\hat{\Phi}_2|_{y=1}|^2 \right.$$
$$\left. + \frac{1+\varepsilon}{\varepsilon^2 QS}\left|q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right|^2\right\}\mathrm{d}\mu\,\mathrm{d}k.$$

(Notice that the quantity S vanishes for certain values of μ and k; we return to this issue below.) Similarly, the left-hand side of the strong form of the equation for Φ_1, namely

$$\frac{\varepsilon^2}{1+\varepsilon}[-c_0\varepsilon(\partial_x^2+\varepsilon\partial_z^2)^3 + (\beta-\tfrac{1}{3})(\partial_x^2+\varepsilon\partial_z^2)^2 - (1+\varepsilon)\partial_z^2 - \partial_x^2]\Phi_1$$
$$= \int_0^1 H\,\mathrm{d}y + h,$$

constitutes a formally self-adjoint operator associated with the quadratic form $\varepsilon^2 Q_1$, where

$$Q_1(\Phi_1) = \frac{1}{2(1+\varepsilon)}|||\Phi_1|||^2.$$

Let us now write $\mathcal{W}(\Phi) = \mathcal{W}_2(\Phi) + \mathcal{W}_{\mathrm{NL}}(\Phi)$, where $\mathcal{W}_2$ denotes the quadratic part of $\mathcal{W}$, and note that

$$\mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi](\Psi) = -\int_{\mathbb{R}^2}\left\{\int_0^1 (\hat{H}_0\bar{\hat{\Psi}}_y + \hat{H}_1\bar{\hat{\Psi}})\,\mathrm{d}y + \hat{h}_1\bar{\hat{\Psi}}|_{y=1}\right\}\mathrm{d}\mu\,\mathrm{d}k, \qquad (40)$$

where

$$H_0 = \varepsilon^{-\frac{1}{2}}N_5(\rho(\Phi),\Phi), \quad H_1 = \varepsilon^{-\frac{1}{2}}N_4(\rho(\Phi),\Phi),$$
$$h_1 = \mathcal{F}\left[\frac{-\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho(\Phi),\Phi)\right].$$

An inspection of the weak form of the equation for Φ_1 and the weak form of the reformulated problem for Φ_2 shows that they formally correspond to, respectively,

$$\varepsilon^2\mathrm{d}Q_1[\Phi_1](\Psi_1) + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1+\Phi_2](\Psi_1) = 0,$$

$$\mathrm{d}Q_2[\Phi_2](\Psi_2) + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1+\Phi_2](\Psi_2) = 0,$$

so that the weak form of the reduced equation for Φ_1 formally corresponds to

$$\varepsilon^2\mathrm{d}Q_1[\Phi_1](\Psi_1) + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1+\Phi_2(\Phi_1)](\Psi_1) = 0. \qquad (41)$$

Repeating the arguments used in Proposition 10, one finds that the solutions of (41) are precisely the critical points of the functional

$$I(\Phi_1) = \varepsilon^2 Q_1(\Phi_1) + Q_2(\Phi_2(\Phi_1)) + \mathcal{W}_{\mathrm{NL}}(\Phi_1+\Phi_2(\Phi_1)),$$

since

$$\begin{aligned}
\mathrm{d}I[\Phi_1](\Psi_1) &= \varepsilon^2 \mathrm{d}Q_1[\Phi_1](\Psi_1) + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2(\Phi_1)](\Psi_1) \\
&\qquad + (\mathrm{d}Q_2[\Phi_2(\Phi_1)] + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2])(\mathrm{d}\Phi_2[\Phi_1](\Psi_1)) \\
&= \varepsilon^2 \mathrm{d}Q_1[\Phi_1](\Psi_1) + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2(\Phi_1)](\Psi_1),
\end{aligned}$$

where the second line follows from the defining property of $\Phi_2(\Phi_1)$ as a solution of the integral and hence of the weak form of the equation for Φ_2.

It remains to treat the difficulty posed by the vanishing denominator in the formula for Q_2. To this end we use the identity

$$q^2 \int_0^1 \hat{\Phi}_2 \,\mathrm{d}y - \frac{\varepsilon\mu^2 \hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2} = S\left(\int_0^1 \hat{H}_1 \,\mathrm{d}y + \hat{h}_1\right), \tag{42}$$

for $\Phi_2(\Phi_1)$, which is obtained from (36) by noting that

$$\int_0^1 \hat{H} \,\mathrm{d}y + \hat{h} = \int_0^1 \hat{H}_1 \,\mathrm{d}y + \hat{h}_1.$$

Using (42) to eliminate S we obtain the alternative formula

$$\begin{aligned}
&Q_2(\Phi_2) = \\
&\frac{1}{2}\int_{\mathbb{R}^2} \Bigg\{ \int_0^1 (|\hat{\Phi}_{2y}|^2 + q^2|\hat{\Phi}_2|^2)\,\mathrm{d}y - \frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}|\hat{\Phi}_2|_{y=1}|^2 \\
&\qquad + \frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \hat{H}_1 \,\mathrm{d}y + \hat{h}_1\right)\left(q^2\int_0^1 \bar{\hat{\Phi}}_2 \,\mathrm{d}y - \frac{\varepsilon\mu^2 \bar{\hat{\Phi}}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right)\Bigg\}\,\mathrm{d}\mu\,\mathrm{d}k
\end{aligned}$$

for $Q_2(\Phi_2(\Phi_1))$.

The above argument, which is formal in nature, delivers a candidate for the variational functional corresponding to the reduced equation for Φ_1. Rather than making the argument rigorous, we proceed by confirming directly that critical points of I (which, with the new definition of $Q_2(\Phi_2(\Phi_1))$, is a smooth functional on X) correspond to weak solutions of the reduced equation for Φ_1. This result is stated in Lemma 1 below; the following proposition, which asserts that a suitable version of (42) holds for solutions of the integral form of the problem for Φ_2, is required for the proof of the lemma.

Proposition 11. *The solution $\Phi_2(\Phi_1)$ of the integral form of the problem for Φ_2 satisfies the identity*

$$\frac{1}{Q^{1/2}}\left(q^2 \int_0^1 \hat{\Phi}_2 \,\mathrm{d}y - \frac{\varepsilon\mu^2 \hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right) = \frac{S}{Q^{1/2}}\left(\int_0^1 \hat{H}_1 \,\mathrm{d}y + \hat{h}_1\right). \tag{43}$$

Proof. With slightly more generality, we establish the result for the boundary-value problem for Φ_2 obtained by replacing N_5 by an arbitrary function in $L^2(\Sigma)$, N_4 by an arbitrary function of the form

$$\hat{N}_4 = \mathrm{i}\mu \hat{N}_4^1 + \mathrm{i}\varepsilon^{\frac{1}{2}} k \hat{N}_4^2, \quad N_4^1, N_4^2 \in L^2(\Sigma)$$

and N_1 by an arbitrary function of the form

$$\hat{N}_1 = \hat{N}_1^1 + \mathrm{i}\mu\hat{N}_1^2 + \mathrm{i}\varepsilon^{\frac{1}{2}}k\hat{N}_1^3, \quad N_1^1, N_1^2, N_1^3 \in L^2(\mathbb{R}^2).$$

It is a straightforward exercise to show that

$$\begin{aligned} &F(N_1^1, N_1^2, N_1^3, N_4^1, N_4^2, N_5) \\ &\qquad = \frac{1}{Q^{1/2}}\left(q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right) - \frac{S}{Q^{1/2}}\left(\int_0^1 \hat{H}_1\,\mathrm{d}y + \hat{h}_1\right), \end{aligned}$$

where Φ_2 is the solution of the integral form of the problem, is a continuous function $(L^2(\mathbb{R}^2))^3 \times (L^2(\Sigma))^3 \to L^2(\Sigma)$ (the Fourier-multiplier operators appearing in this equation are handled using Parseval's formula). Now suppose that N_1^1, N_1^2, N_1^3 belong to the dense subset $W_0^{1,2}(\mathbb{R}^2)$ of $L^2(\mathbb{R}^2)$ and that N_4^1, N_4^2, N_5 belong to the dense subset $W_0^{1,2}(\Sigma)$ of $L^2(\Sigma)$. Using Lemma 4 below in a 'bootstrap' fashion, we find that Φ_2 belongs to $W^{2,2}(\Sigma)$; because it is a weak solution of the problem for Φ_2 with the required additional regularity it solves the strong form of the problem in $L^2(\Sigma)$ and hence the identity (42) in $L^2(\Sigma)$. It follows that F vanishes for $(N_1^1, N_1^2, N_1^3, N_4^1, N_4^2, N_5) \in (W_0^{1,2}(\mathbb{R}^2))^3 \times (W_0^{1,2}(\Sigma))^3$; a standard density argument asserts that it also vanishes for each $(N_1^1, N_1^2, N_1^3, N_4^1, N_4^2, N_5) \in (L^2(\mathbb{R}^2))^3 \times (L^2(\Sigma))^3 \to L^2(\Sigma)$. □

Lemma 1. *The weak solutions of the reduced equation for Φ_1 are precisely the critical points of $I : X \to \mathbb{R}$.*

Proof. Observe that

$$\begin{aligned} \mathrm{d}I[\Phi_1](\Psi_1) &= \frac{\varepsilon^2}{1+\varepsilon}⟪\Phi_1, \Psi_1⟫ + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2(\Phi_1)](\Psi_1 + \Psi_2) \\ &+ \int_{\mathbb{R}^2}\left\{\int_0^1 (\hat{\Phi}_{2y}\bar{\hat{\Psi}}_{2y} + q^2\hat{\Phi}_2\bar{\hat{\Psi}}_2)\,\mathrm{d}y - \frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}\hat{\Phi}_2\bar{\hat{\Psi}}_2|_{y=1}\right\}\mathrm{d}\mu\,\mathrm{d}k \\ &+ \frac{1}{2}\int_{\mathbb{R}^2}\left\{\frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \mathcal{F}[\partial H_1\Psi]\,\mathrm{d}y + \mathcal{F}[\partial h_1\Psi]\right)\left(q^2\int_0^1 \bar{\hat{\Phi}}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\bar{\hat{\Phi}}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right)\right. \\ &\qquad\left. + \frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \hat{H}_1\,\mathrm{d}y + \hat{h}_1\right)\left(q^2\int_0^1 \bar{\hat{\Psi}}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\bar{\hat{\Psi}}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right)\right\}\mathrm{d}\mu\,\mathrm{d}k, \end{aligned} \tag{44}$$

where $\Psi_2 = \mathrm{d}\Phi_2[\Phi_1](\Psi_1)$ and $\Psi = \Psi_1 + \Psi_2$. Elimination of S between equation (43) and its derivative with respect to Φ_1 yields

$$\frac{1}{Q^{1/2}}\left(\int_0^1 \bar{\hat{H}}_1\,\mathrm{d}y+\bar{\hat{h}}_1\right)\left(\int_0^1 q^2\hat{\Psi}_2\,\mathrm{d}y-\frac{\varepsilon\mu^2\hat{\Psi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right)$$
$$=\frac{1}{Q^{1/2}}\left(\int_0^1 \mathcal{F}[\partial H_1\Psi]\,\mathrm{d}y+\mathcal{F}[\partial\hat{h}_1\Psi]\right)\left(\int_0^1 q^2\bar{\hat{\Phi}}_2\,\mathrm{d}y-\frac{\varepsilon\mu^2\bar{\hat{\Phi}}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right), \tag{45}$$

and it follows from (40) that

$$\begin{aligned}&\mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1+\Phi_2(\Phi_1)](\Psi_1)\\&=-\int_{\mathbb{R}^2}\left\{\left(\int_0^1\hat{H}_1\,\mathrm{d}y+\hat{h}_1\right)\bar{\hat{\Psi}}_1+\int_0^1(\hat{H}_0\bar{\hat{\Psi}}_{2y}+\hat{H}_1\bar{\hat{\Psi}}_2)\,\mathrm{d}y+\hat{h}_1\bar{\hat{\Psi}}_2|_{y=1}\right\}\mathrm{d}\mu\,\mathrm{d}k.\end{aligned} \tag{46}$$

Combining equations (44)–(46) and using the fact that $\Phi_2(\Phi_1)$ is a weak solution of the problem for Φ_2, one finds that

$$\mathrm{d}I[\Phi_1](\Psi_1)=\frac{\varepsilon^2}{1+\varepsilon}⟪\Phi_1,\Psi_1⟫-\int_{\mathbb{R}^2}\left(\int_0^1\hat{H}_1\,\mathrm{d}y+\hat{h}_1\right)\bar{\hat{\Psi}}_1\,\mathrm{d}\mu\,\mathrm{d}k.$$

□

It is convenient to replace I by the equivalent functional

$$J(\Phi_1)=\varepsilon^{-2}I(\Phi_1)=Q_1(\Phi_1)+\varepsilon^{-2}Q_2(\Phi_2(\Phi_1))+\varepsilon^{-2}\mathcal{W}_{\mathrm{NL}}(\Phi_1+\Phi_2(\Phi_1)),$$

which is defined upon a neighbourhood of the origin in its function space X (see Theorem 1). We denote the radius of this neighbourhood by the distinguished symbol M and write $J:\bar{B}_M(0)\subset X\to\mathbb{R}$; note that, although M may be taken arbitrarily large, the greatest permissible magnitude of ε decreases as M is increased. We conclude this section by computing a formula for J which is helpful in our subsequent analysis. We write

$$\rho=\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\hat{\Phi}_{1x}+\mathrm{i}\mu\int_0^1 y\hat{\Phi}_{2y}\,\mathrm{d}y+\int_0^1\hat{\Phi}_{2x}\right)\right]+\rho_{\mathrm{NL}}(\rho,\Phi_1,\Phi_2),$$

and use the fact that $\Phi_2(\Phi_1)$ solves the weak formulation of the problem for Φ_2, whereupon Definition 2(ii) with $\Psi_2=\Phi_2$ implies that

$$Q_2(\Phi_2)=\frac{1}{2}\int_{\mathbb{R}^2}\left\{\int_0^1 H_0\Phi_{2y}+H_1\Phi_2)\,\mathrm{d}y+h_1\Phi_2|_{y=1}\right\}\mathrm{d}\mu\,\mathrm{d}k.$$

Using these formulae to achieve some simplification, one finds that

$$J(\Phi_1)=J_2(\Phi_1)+J_3(\Phi_1)+J_4(\Phi_1),$$

where

$$J_2(\Phi_1)=Q_1(\Phi_1), \tag{47}$$

$$J_3(\Phi_1)=\int_{\mathbb{R}^2}\left\{\frac{1}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]\Phi_{1x}^2+\frac{\varepsilon}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]\Phi_{1z}^2\right\}\mathrm{d}x\,\mathrm{d}z, \tag{48}$$

$$\begin{aligned}J_4(\Phi_1) = \int_{\mathbb{R}^2}\Bigg\{&\int_0^1\bigg(\frac{1}{2}\rho\Phi_{2x}^2+\frac{\varepsilon}{2}\rho\Phi_{2z}^2-\frac{\rho\Phi_{2y}^2}{2\varepsilon(1+\varepsilon\rho)}-y\Phi_{1x}\rho_x\Phi_{2y}\\
&\qquad-\varepsilon y\Phi_{1z}\rho_z\Phi_{2y}-\rho\Phi_x\Phi_{2z}-\varepsilon\rho\Phi_z\Phi_{2z}\bigg)\mathrm{d}y\\
&+\frac{1}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{2x}|_{y=1}}{1+\varepsilon+\beta q^2}\right]\Phi_{1x}^2+\frac{\varepsilon}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{2x}|_{y=1}}{1+\varepsilon+\beta q^2}\right]\Phi_{1z}^2\\
&+\frac{1}{2}\rho_{\mathrm{NL}}\Phi_{1x}^2+\frac{\varepsilon}{2}\rho_{\mathrm{NL}}\Phi_{1z}^2-\frac{\varepsilon^{-1}}{2}\rho_{\mathrm{NL}\,x}\Phi_2|_{y=1}\\
&-\frac{\beta\varepsilon^{-3}(\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2)^2}{2\left(\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}+1\right)^2}\Bigg\}\,\mathrm{d}x\,\mathrm{d}z\\
&+\frac{\varepsilon^{-1}}{2}\|\rho_{\mathrm{NL}}\|_2^2+\frac{1}{2}\|\rho_{\mathrm{NL}}\|_{1,2,\varepsilon}^2\end{aligned} \tag{49}$$

are respectively its quadratic, cubic and higher-order parts (recall that Φ_2 and ρ_{NL} are quadratic functions of Φ_1). This formula shows that J_3 and J_4 define smooth functionals on $U_\varepsilon^{0,2}(\mathbb{R}^2)\cap U_\varepsilon^{0,4}(\mathbb{R}^2)\cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, and since X is continuously embedded in $U_\varepsilon^{0,2}(\mathbb{R}^2)\cap U_\varepsilon^{0,4}(\mathbb{R}^2)\cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ one concludes that $J : X\to\mathbb{R}$ indeed falls into the 'coercive, semilinear, locally compact case'. This structure is exploited in Section 4, where it is confirmed that J has a nonzero critical point.

3. Reduction theorems

In this section we describe in detail the techniques used to solve the integral forms of the equations for ρ and Φ_2 for ρ, Φ_2 as functions of Φ_1 and present a regularity theory which shows that a weak solution of the reduced equation for Φ_1 generates a strong solution of the original hydrodynamic problem. The equations are solved using the following fixed-point theorem, which is a straightforward extension of a standard argument in nonlinear analysis.

Theorem 2. *Let $\mathcal{X}$, $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ be Banach spaces, X, $Y_1, \ldots, Y_n$ be closed subsets of, respectively, $\mathcal{X}$, $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ which contain the origin and $\mathcal{F} : X\times Y_1\times\cdots\times Y_n\to\mathcal{X}$ be a smooth function. Suppose there exists a function $r : Y_1\times\cdots\times Y_n\to[0,\infty)$ such that*

$$\|\mathcal{F}(0,y)\|\leqq r/2,\quad \|\mathrm{d}_1\mathcal{F}[x,y]\|\leqq 1/2$$

for each $x\in\bar{B}_r(0)\subset X$ and each $y\in Y_1\times\cdots\times Y_n$.

Under these hypotheses there exists for each $y \in Y_1 \times \cdots \times Y_n$ *a unique solution* $x = x(y)$ *of the fixed-point equation*

$$x = \mathcal{F}(x, y)$$

satisfying $x(y) \in \bar{B}_r(0)$. *Moreover* $x(y)$ *is a smooth function of* $y \in Y_1 \times \cdots \times Y_n$ *and in particular we have the estimates*

$$\|\mathrm{d}_i x[y_1, \ldots, y_n]\| \leqq 2\|\mathrm{d}_{i+1}\mathcal{F}[x(y), y_1, \ldots, y_n]\|, \quad i = 1, \ldots, n$$

for its first derivatives.

The main step in solving the integral equations lies in showing that their right-hand sides define contractions whose Lipschitz constant is bounded by a positive power of ε, and for this purpose it is essential to work in the scaled function spaces introduced in Section 1.3. The issue here is that the KP scaling (9) acts differently upon x and z (a difference of $\varepsilon^{\frac{1}{2}}$ in their respective scaling) and this difference manifests itself in all subsequent formulae. The spaces $W_\varepsilon^{\delta,p}(\Sigma)$ and $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ take the difference into account in the simplest possible way: they are obtained by changing from z to the scaled variable $\tilde{z} = \varepsilon^{\frac{1}{2}}z$ (or equivalently from k to $\tilde{k} = \varepsilon^{-\frac{1}{2}}k$) in the definitions of the usual spaces $W^{\delta,p}(\Sigma)$ and $U^{\delta,p}(\mathbb{R}^2)$. These spaces are ideal for treating the integral equation for Φ_2, in particular for the proof of Lemma 4, where denominators involving the combination $\mu^2 + \varepsilon k^2$ appear in the formulae defining G_1.

A more sophisticated scaling is used to derive $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ from the standard space $W^{1+\delta,p}(\mathbb{R}^2)$. This space is designed to handle the integral equation for ρ, in which the combination $\varepsilon\mu^2 + \varepsilon^2k^2$ (rather than $\mu^2 + \varepsilon k^2$) appears. At first sight it would appear that the appropriate scaled function space for ρ is obtained by changing from (x, z) to $(\tilde{x}, \tilde{z}) = (\varepsilon^{\frac{1}{2}}x, \varepsilon z)$ in the definition of $W^{1+\delta,p}(\mathbb{R})$. However the variable ρ appears in the equation for Φ_2, which is solved in $W_\varepsilon^{1+\delta,p}(\Sigma)$, and an unfavourable factor of $\varepsilon^{-\frac{1}{2}-\frac{\delta}{2}}$ is acquired when estimating a $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$ norm in terms of the corresponding norm defined using the new scaling. The scaling used for $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ is actually a compromise between the two scaling rules described above; notice that $V_\varepsilon^{0,p}(\mathbb{R}^2)$ coincides with the space obtained from $W^{1,p}(\mathbb{R}^2)$ by the latter rule, while the *additional* scaling performed for $\delta > 0$ is carried out according to the former rule. On the one hand it is still possible to solve the equation for ρ in $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, and on the other hand a less problematic factor of $\varepsilon^{-\frac{\delta}{2}}$ appears when estimating the $W_\varepsilon^{1+\delta,p}(\Sigma)$ norm in terms of the $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ norm (see Proposition 1(iii)).

The rationale for our choices of function spaces may be summarised as follows. Firstly, the unscaled spaces $W^{1,2}(\mathbb{R}^2) \cap W^{1+\delta,p}(\mathbb{R}^2)$ and $U^{0,2}(\Sigma) \times U^{\delta,p}(\Sigma)$ (with $\delta > 3/p$) for ρ and Φ constitute a wide class of functions upon which the variational functional $\mathcal{V}$ and weak solutions of (10)–(13) are well defined. Secondly, the choice of the space X, complete with its scaling, is dictated by the expansion (23) of the dispersion relation used to define the splitting $\Phi = \Phi_1 + \Phi_2$; the expansion is carried out to sixth order in (μ, k) to ensure that X is continuously embedded in the function space $U^{0,2}(\mathbb{R}^2) \times U^{\delta,p}(\mathbb{R}^2)$, which is the obvious candidate function space for the reduced variable Φ_1. Finally, scaled versions of the function spaces identified here are introduced for the purposes of solving the integral equations for ρ and Φ_2.

3.1. Elimination of the variable ρ

Anticipating the later stages of our analysis, we suppose that Φ admits a decomposition of the type

$$\Phi(x, y, z) = \Phi_1(x, z) + \Phi_2(x, y, z)$$

and consider the integral form of the equation for ρ in the form

$$\hat{\rho} = \frac{1}{1+\varepsilon+\beta q^2}\left(\hat{\Phi}_{1x} + \mathrm{i}\mu \int_0^1 y\hat{\Psi}\,\mathrm{d}y + \int_0^1 \hat{\Phi}_{2x}\,\mathrm{d}y + \varepsilon^{-1}\hat{N}_1(\rho, \Phi_1 + \Phi_2)\right).$$

The new variable Ψ is identified with Φ_{2y} later; we introduce it here since it plays a significant role in the solution of the equation for Φ_2 in Section 3.2 below.

Let us therefore write the integral form of the equation for ρ as

$$\rho = \mathcal{F}_1(\rho, \Psi, \Phi_1, \Phi_2) \tag{50}$$

and solve this fixed-point problem for ρ as a function of Φ_1, Φ_2 and Ψ. For this purpose we need precise estimates on the norms of the Fourier-multiplier operators appearing in (50); the requisite information is given in the following lemma. Although its proof is straightforward for $p = 2$ (an application of Parseval's theorem), a more detailed study is necessary to establish the result for $p \neq 2$. We defer this aspect of the analysis to Section 5, where theory for handling Fourier-multiplier operators in L^p-based spaces ($p \neq 2$) is developed and the lemmata in Sections 2–3.3 involving norms of Fourier-multiplier operators are proved.

Lemma 2. *The following statements hold for each $\delta \in [0, 1]$ and $p \in (1, \infty)$.*

(i) *For each $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ the function*

$$\mathcal{G}_1(u) = \mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belongs to $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and satisfies the estimate

$$|\mathcal{G}_1(u)|_{\delta,p,\varepsilon} \leqq c\|u\|_{\delta,p,\varepsilon}.$$

(ii) *For each $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ the functions*

$$\mathcal{G}_2(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right], \quad \mathcal{G}_3(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belong to $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and satisfy the estimates

$$|\mathcal{G}_j(u)|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}}\|u\|_{\delta,p,\varepsilon}, \quad j = 2, 3.$$

We now solve the fixed-point problem (50) by applying our basic fixed-point theorem (Theorem 2); the technique developed for this purpose in the following result involves showing that $\mathcal{F}_1$ is a contraction whose Lipschitz constant is bounded by a positive power of ε. We henceforth adopt the notation introduced in Theorem 1 that Δ is a quantity which is bounded by $c(\delta + 1/p)$; it is always supposed to be as small as required for the result in question by taking δ sufficiently small and p sufficiently large while maintaining the relationship $\delta > 3/p$.

Theorem 3. *Suppose that*

$$\|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}, \quad \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}. \tag{51}$$

(i) *Equation* (50) *has a unique solution* $\rho = \rho(\Psi, \Phi_1, \Phi_2)$ *which satisfies the estimate*

$$|\rho|_{\delta,p,\varepsilon} \leqq c\,(\|\Phi_x\|_{\delta,p,\varepsilon} + \varepsilon^{-\frac{1}{2}}\|\Psi\|_{\delta,p,\varepsilon} + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})^2). \tag{52}$$

Moreover ρ *is a smooth function of* (Ψ, Φ_1, Φ_2) *with respect to the* $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *and* $W_\varepsilon^{\delta,p}(\Sigma) \times U_\varepsilon^{\delta,p}(\mathbb{R}^2) \times W_\varepsilon^{1+\delta,p}(\Sigma)$ *topologies and in particular its first derivatives with respect to* Ψ *and* Φ_2 *satisfy the estimates*

$$|\rho_\Psi \tilde{\Psi}|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}, \quad |\rho_{\Phi_2}\tilde{\Phi}_2|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}.$$

(ii) *The solution* $\rho = \rho(\Psi, \Phi_1, \Phi_2)$ *to* (50) *identified in part* (i) *satisfies the estimate*

$$\begin{aligned}|\rho|_{0,2,\varepsilon} \leqq c\,(&\|\Phi_x\|_2 + \varepsilon^{-\frac{1}{2}}\|\Psi\|_2 \\ &+ \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{0,2}} + \|\Phi_y\|_2)(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})).\end{aligned} \tag{53}$$

Moreover ρ *is a smooth function of* (Ψ, Φ_1, Φ_2) *with respect to the* $V_\varepsilon^{0,2}(\mathbb{R}^2)$ *and* $L^2(\Sigma) \times U_\varepsilon^{0,2}(\mathbb{R}^2) \times W_\varepsilon^{1,2}(\Sigma)$ *topologies and in particular its derivatives with respect to* Ψ *and* Φ_2 *satisfy the estimates*

$$|\rho_\Psi \tilde{\Psi}|_{0,2,\varepsilon} \leqq c\,\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_2, \quad |\rho_{\Phi_2}\tilde{\Phi}_2|_{0,2,\varepsilon} \leqq c\,\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}.$$

Proof. (i) This result is established by applying Theorem 2 with $\mathcal{X} = V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\mathcal{Y}_1 = W_\varepsilon^{\delta,p}(\Sigma)$, $\mathcal{Y}_2 = U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\mathcal{Y}_3 = W_\varepsilon^{1+\delta,p}(\Sigma)$ and X, Y_1, Y_2, Y_3 closed origin-centred balls of radius $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$, $\mathcal{O}(\varepsilon^{\frac{1}{2}-\Delta})$, $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$, $\mathcal{O}(\varepsilon^{-\Delta})$. According to this theorem, we have to verify that

$$\begin{aligned}&|\mathcal{F}_1(0, \Psi, \Phi_1, \Phi_2)|_{\delta,p,\varepsilon} \\ &\quad \leqq c(\|\Phi_x\|_{\delta,p,\varepsilon} + \varepsilon^{-\frac{1}{2}}\|\Psi\|_{\delta,p,\varepsilon} + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})^2)\end{aligned} \tag{54}$$

and that

$$|\mathrm{d}_1\mathcal{F}_1[\rho, \Psi, \Phi_1, \Phi_2]|_{V_\varepsilon^{\delta,p}(\mathbb{R}^2) \to V_\varepsilon^{\delta,p}(\mathbb{R}^2)} \leqq \frac{1}{2} \tag{55}$$

whenever (51) and (52) hold.

To verify (54) note that

$$N_1(0,\Phi) = -\int_0^1 \left\{ \frac{\varepsilon^2}{2}\Phi_x^2 + \frac{\varepsilon^3}{2}\Phi_z^2 + \varepsilon^2(\Phi_x y\Phi_y)_x \right.$$
$$\left. + \varepsilon^3(\Phi_z y\Phi_y)_z + \varepsilon^{\frac{3}{2}}\Phi_x y\Phi_y + \varepsilon^2\Phi_z y\Phi_y + \frac{\varepsilon}{2}\Phi_y^2 \right\} \mathrm{d}y,$$

whence

$$\begin{aligned}
&\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\hat{N}_1(0,\Phi)\right]\right|_{\delta,p,\varepsilon} \\
&\leqq \left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\right.\right. \\
&\qquad \left.\left.\mathcal{F}\left[\int_0^1 \left\{\frac{\varepsilon}{2}\Phi_x^2 + \frac{\varepsilon^2}{2}\Phi_z^2 + \varepsilon^{\frac{1}{2}}\Phi_x y\Phi_y + \varepsilon\Phi_z y\Phi_y + \frac{1}{2}\Phi_y^2\right\}\mathrm{d}y\right]\right]\right|_{\delta,p,\varepsilon} \\
&\quad + \left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1 \varepsilon\Phi_x y\Phi_y\,\mathrm{d}y\right]\right]\right|_{\delta,p,\varepsilon} \\
&\quad + \left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1 \varepsilon^{\frac{3}{2}}\Phi_z y\Phi_z\,\mathrm{d}y\right]\right]\right|_{\delta,p,\varepsilon} \\
&\leqq c\,(\varepsilon\|\Phi_x^2\|_{\delta,p,\varepsilon} + \varepsilon^2\|\Phi_z^2\|_{\delta,p,\varepsilon} + \varepsilon^{\frac{1}{2}}\|\Phi_x y\Phi_x\|_{\delta,p,\varepsilon} \\
&\qquad + \varepsilon\|\Phi_z y\Phi_z\|_{\delta,p,\varepsilon} + \|\Phi_y^2\|_{\delta,p,\varepsilon} + \varepsilon^{\frac{1}{2}}\|\Phi_x y\Phi_x\|_{\delta,p,\varepsilon} + \varepsilon\|\Phi_z y\Phi_z\|_{\delta,p,\varepsilon}) \\
&\leqq c\,(\varepsilon^{1-\Delta}\|\Phi_x\|_{\delta,p,\varepsilon}^2 + \varepsilon^{2-\Delta}\|\Phi_z\|_{\delta,p,\varepsilon}^2 + \varepsilon^{-\Delta}\|\Phi_y\|_{\delta,p,\varepsilon}^2 \\
&\qquad + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_x\|_{\delta,p,\varepsilon}\|\Phi_y\|_{\delta,p,\varepsilon} + \varepsilon^{1-\Delta}\|\Phi_z\|_{\delta,p,\varepsilon}\|\Phi_y\|_{\delta,p,\varepsilon}) \\
&\leqq c\,\varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})^2,
\end{aligned}$$

in which Lemma 2 and the properties of our function spaces have been used. We similarly find that

$$\begin{aligned}
&\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\hat{\Phi}_{1x} + \mathrm{i}\mu\int_0^1 y\hat{\Psi}\,\mathrm{d}y + \int_0^1 \hat{\Phi}_{2x}\,\mathrm{d}y\right)\right]\right|_{\delta,p,\varepsilon} \\
&\qquad \leqq c(\|\Phi_{1x}\|_{\delta,p,\varepsilon} + \|\Phi_{2x}\|_{\delta,p,\varepsilon} + \varepsilon^{-1/2}\|\Psi\|_{\delta,p,\varepsilon}),
\end{aligned}$$

and the estimate (54) follows directly from the above calculations.

The next step is to estimate

$$\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\partial_1\hat{N}_1(\rho,\Phi)\tilde{\rho}\right]\right|_{\delta,p,\varepsilon}$$

under the assumptions (51) and

$$|\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta},$$

which follows from (51) and (52), together with the rules

$$\|\rho\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{\delta}{2}}|\rho|_{\delta,p,\varepsilon}, \quad \|\rho_x\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}}|\rho|_{\delta,p,\varepsilon}, \quad \|\rho_z\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-1}|\rho|_{\delta,p,\varepsilon} \tag{56}$$

(an explicit formula for $\partial_1 \hat{N}_1(\rho, \Phi)\tilde{\rho}$ is readily computed from (14)). We find, for example, that

$$\begin{aligned}
\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1 \frac{\varepsilon^2 y^2 \Phi_y^2 \tilde{\rho}_x}{1+\varepsilon\rho}\right]\right]\right|_{\delta,p,\varepsilon} &\leqq c\varepsilon^{-\frac{3}{2}}\left\|\frac{y^2\Phi_y^2\tilde{\rho}_x}{1+\varepsilon\rho}\right\|_{\delta,p,\varepsilon} \\
&\leqq c\varepsilon^{-\frac{3}{2}}\|y^2\Phi_y^2\tilde{\rho}_x\|_{\delta,p,\varepsilon} \\
&\leqq c\varepsilon^{-\frac{3}{2}-\Delta}\|\Phi_y\|_{\delta,p,\varepsilon}^2\|\tilde{\rho}_x\|_{\delta,p,\varepsilon} \\
&\leqq c\varepsilon^{\frac{3}{2}-\Delta}\|\tilde{\rho}_x\|_{\delta,p,\varepsilon} \\
&\leqq c\varepsilon^{1-\Delta}|\tilde{\rho}|_{\delta,p,\varepsilon},
\end{aligned}$$

where we have used the rule

$$\begin{aligned}
\left\|\frac{u}{1+\varepsilon\rho}\right\|_{\delta,p,\varepsilon} &= \left\|u - \frac{\varepsilon u\rho}{1+\varepsilon\rho}\right\|_{\delta,p,\varepsilon} \\
&\leqq \|u\|_{\delta,p,\varepsilon} + c\varepsilon^{-\Delta}(\varepsilon\|\rho\|_{\delta,p,\varepsilon} + \varepsilon^2\|\rho\|_{\delta,p,\varepsilon}^2 + \cdots) \\
&\leqq c\|u\|_{\delta,p,\varepsilon};
\end{aligned}$$

estimating each term in this fashion we conclude that

$$\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\partial_1\hat{N}_1(\rho,\Phi)\tilde{\rho}\right]\right|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}|\tilde{\rho}|_{\delta,p,\varepsilon},$$

from which (55) follows immediately.

Our fixed-point theorem states that

$$|\rho_\Psi\tilde{\Psi}|_{\delta,p,\varepsilon} \leqq 2\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\int_0^1 y\hat{\tilde{\Psi}}\,\mathrm{d}y\right]\right|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}$$

and

$$\begin{aligned}
&|\rho_{\Phi_2}\tilde{\Phi}_2|_{\delta,p,\varepsilon} \\
&\quad\leqq 2\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\mathrm{i}\mu\int_0^1 y\hat{\tilde{\Phi}}_2\,\mathrm{d}y + \varepsilon^{-1}\partial_2\hat{N}_1(\rho,\Phi)\tilde{\Phi}_2\right)\right]\right|_{\delta,p,\varepsilon},
\end{aligned}$$

in which an explicit formula for $\partial_2\hat{N}_1(\rho,\Phi)\tilde{\Phi}_2$ is computed from (14); arguing as above, we find that

$$|\rho_{\Phi_2}\tilde{\Phi}_2|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}.$$

(ii) Observe that

$$\begin{aligned}
X &= V_\varepsilon^{0,2}(\mathbb{R})^2 \cap \{\rho \in V_\varepsilon^{\delta,p}(\mathbb{R}^2) : |\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}\}, \\
Y_1 &= L^2(\Sigma) \cap \{\Psi \in W_\varepsilon^{\delta,p}(\Sigma) : \|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}\},
\end{aligned}$$

$$Y_2 = W_\varepsilon^{0,2}(\mathbb{R}^2) \cap \{\Phi_1 \in U_\varepsilon^{\delta,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}\},$$
$$Y_3 = W_\varepsilon^{1,2}(\Sigma) \cap \{\Phi_2 \in W_\varepsilon^{1+\delta,p}(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}\}$$

are closed subsets of, respectively, $\mathcal{X} = V_\varepsilon^{0,2}(\mathbb{R}^2)$, $\mathcal{Y}_1 = L^2(\Sigma)$, $\mathcal{Y}_2 = U_\varepsilon^{0,2}(\mathbb{R}^2)$ and $\mathcal{Y}_3 = W_\varepsilon^{1,2}(\Sigma)$. We may therefore apply our fixed-point to equation (50) with these definitions of $\mathcal{X}$, $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and X, Y_1, Y_2, Y_3; the fixed point thus located clearly coincides with that identified in part (i). Estimation techniques similar to those used in the proof of part (i) show that

$$|\mathcal{F}_1(0,\Psi,\Phi_1,\Phi_2)|_{0,2,\varepsilon}$$
$$\leqq c\left(\|\Phi_x\|_2 + \varepsilon^{-\frac{1}{2}}\|\Psi\|_2 + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{0,2}} + \|\Phi_y\|_2)(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})\right),$$

and

$$|\mathrm{d}_1\mathcal{F}_1[\rho,\Psi,\Phi_1,\Phi_2](\tilde{\rho})|_{0,2,\varepsilon} \leqq c\,\varepsilon^{\frac{1}{2}-\Delta}|\tilde{\rho}|_{0,2,\varepsilon} \leqq \frac{1}{2}|\tilde{\rho}|_{0,2,\varepsilon}$$

together with

$$|\rho_\Psi\tilde{\Psi}|_{0,2,\varepsilon} \leqq 2|\mathrm{d}_2\mathcal{F}_1[\rho,\Psi,\Phi_1,\Phi_2](\tilde{\Psi})|_{0,2,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_2,$$
$$|\rho_{\Phi_2}\tilde{\Phi}_2|_{0,2,\varepsilon} \leqq 2|\mathrm{d}_4\mathcal{F}_1[\rho,\Psi,\Phi_1,\Phi_2](\tilde{\Psi})|_{0,2,\varepsilon} \leqq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}$$

whenever

$$|\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta},$$
$$\|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}.$$

□

Finally, we record some further estimates for ρ which are used later; they are proved using the methods developed in Theorem 3.

Lemma 3. *Define*

$$\rho_{\mathrm{NL}}(\rho,\Phi_1,\Phi_2) = \mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\hat{N}_1(\rho,\Phi_1,\Phi_2)\right],$$

so that

$$\rho = \mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\hat{\Phi}_{1x} + \mathrm{i}\mu\int_0^1 y\hat{\Psi}\,\mathrm{d}y + \int_0^1 \hat{\Phi}_{2x}\right)\right] + \rho_{\mathrm{NL}}(\rho,\Phi_1,\Phi_2).$$

The function ρ_{NL} satisfies the estimates

$$|\rho_{\mathrm{NL}}|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}P_2(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}}, \|\Psi\|_{\delta,p,\varepsilon}, \|\Phi_y\|_{\delta,p,\varepsilon}),$$
$$|\rho_{\mathrm{NL}}|_{0,2,\varepsilon} \leqq c(\varepsilon\|\Phi_1\|^2_{U_\varepsilon^{0,4}} + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{0,2}} + \|\Psi\|_2 + \|\Phi_y\|_2)$$
$$\times P_1(\varepsilon^{\frac{3}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}, \varepsilon^{\frac{1}{2}}\|\Phi_2\|_{U_\varepsilon^{\delta,p}}, \|\Psi\|_{\delta,p,\varepsilon}, \|\Phi_y\|_{\delta,p,\varepsilon})).$$

3.2. Elimination of the variable Φ_2

Substituting $\rho = \rho(\Psi,\Phi_1,\Phi_2)$ into the integral form of the equation for Φ_2 and identifying Ψ with Φ_{2y}, one finds that

$$\begin{aligned}\hat{\Phi}_2 = &-\int_0^1 \frac{G_1}{\varepsilon^{5/2}}\hat{N}_4(\rho(\Phi_{2y},\Phi_1,\Phi_2),\Phi_1+\Phi_2)\,\mathrm{d}\xi\\ &-\int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\hat{N}_5(\rho(\Phi_{2y},\Phi_1,\Phi_2),\Phi_1+\Phi_2)\,\mathrm{d}\xi\\ &+\frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_1(\rho(\Phi_{2y},\Phi_1,\Phi_2),\Phi_1+\Phi_2).\end{aligned}\tag{57}$$

In this section we show that the above equation can be solved for Φ_2 as a function of Φ_1. We proceed by replacing it with a pair of equivalent integral equations which have more favourable mapping properties (see below), namely

$$\begin{aligned}\hat{\Phi}_2 = &-\int_0^1 \frac{G_1}{\varepsilon^{5/2}}\hat{N}_6(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi)\,\mathrm{d}\xi\\ &-\int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi)\,\mathrm{d}\xi\\ &+\frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi),\end{aligned}\tag{58}$$

$$\begin{aligned}\hat{\Psi} = &-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\hat{N}_6(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi)\,\mathrm{d}\xi\\ &-\int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi)\,\mathrm{d}\xi\\ &+\frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi).\end{aligned}\tag{59}$$

The first equation is obtained by replacing the nonlinearities N_4, N_5 and N_1 with new nonlinear functions N_6, N_7 and N_8, while the second is obtained by differentiating the first with respect to y and replacing Φ_2 with Ψ on the left-hand side; the functions N_6, N_7 and N_8 are given by the formulae defining N_4, N_5 and N_1 with all occurrences of Φ_{2y} replaced by Ψ.

Proposition 12. *Any solution $\Phi_2^\star$ of* (57) *defines a solution $(\Phi_2^\star, \Phi_{2y}^\star)$ of* (58), (59). *Conversely, any solution $(\Phi_2^\star, \Psi^\star)$ of* (58), (59) *satisfies $\Psi^\star = \Phi_{2y}^\star$ and hence defines a solution of* (57).

The following lemma gives estimates on the norms of the Fourier-multiplier operators that appear in the above equations; its proof is given in Section 5.

Lemma 4. *The following statements hold for each $\delta \in [0, 1]$ and $p \in (1, \infty)$.*

(i) *For each $u \in W_\varepsilon^{\delta,p}(\Sigma)$ the functions*

$$\mathcal{G}_4(u) = \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\mu G_1 \mathcal{F}[u]\,\mathrm{d}\xi\right], \quad \mathcal{G}_5(u) = \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\varepsilon^{\frac{1}{2}} k G_1 \mathcal{F}[u]\,\mathrm{d}\xi\right]$$

belong to $W_\varepsilon^{1+\delta,p}(\Sigma)$ and satisfy the estimate

$$\|\mathcal{G}_j(u)\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon\|u\|_{\delta,p,\varepsilon}, \quad j = 4, 5.$$

(ii) *For each* $u \in W_\varepsilon^{\delta,p}(\Sigma)$ *the function*

$$\mathcal{G}_6(u) = \mathcal{F}^{-1}\left[\int_0^1 G_{1\xi}\mathcal{F}[u]\,\mathrm{d}\xi\right]$$

belongs to $W_\varepsilon^{1+\delta,p}(\Sigma)$ *and satisfies the estimate*

$$\|\mathcal{G}_6(u)\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon\|u\|_{\delta,p,\varepsilon}.$$

(iii) *For each* $u \in W_\varepsilon^{\delta,p}(\Sigma)$ *the function*

$$\mathcal{G}_7(u) = \mathcal{F}^{-1}\left[\int_0^1 G_{1y\xi}\mathcal{F}[u]\,\mathrm{d}\xi\right]$$

belongs to $W_\varepsilon^{\delta,p}(\Sigma)$ *and satisfies the estimate*

$$\|\mathcal{G}_7(u)\|_{\delta,p,\varepsilon} \leqq c\varepsilon^2\|u\|_{\delta,p,\varepsilon}.$$

(iv) *For each* $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *the functions*

$$\mathcal{G}_8(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu G_1|_{\xi=1}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right], \quad \mathcal{G}_9(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}kG_1|_{\xi=1}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belong to $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$ *and satisfy the estimate*

$$\|\mathcal{G}_j(u)\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon\|u\|_{\delta,p,\varepsilon}, \quad j = 8, 9.$$

(v) *For each* $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *the functions*

$$\mathcal{G}_{10}(u)=\mathcal{F}^{-1}\left[\frac{-\mu^2 G_1|_{\xi=1}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right], \quad \mathcal{G}_{11}(u)=\mathcal{F}^{-1}\left[\frac{-\varepsilon^{\frac{1}{2}}\mu kG_1|_{\xi=1}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belong to $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$ *and satisfy the estimate*

$$\|\mathcal{G}_j(u)\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}}\|u\|_{\delta,p,\varepsilon}, \quad j = 10, 11.$$

Our strategy in dealing with the coupled integral equations (58), (59) is to solve (59) for Ψ as a function of Φ_1, Φ_2, substitute $\Psi = \Psi(\Phi_1, \Phi_2)$ into (58) and solve this equation for Φ_2 as a function of Φ_1; the two equations are solved by the method used for the equation for ρ in Section 3.1 above. (Attempting to solve equation (57) directly using this method, one finds that the estimates for certain terms have insufficient powers of ε. This difficulty is overcome by the use of the equivalent equations (58), (59). Part (iii) of Lemma 4 ensures that an additional power of ε appears in the estimate of the problematic term in equation (59), and this additional power is inherited by equation (58) in the form of a good estimate for Ψ.) We carry out the first step in Theorem 4 below by writing equation (59) as

$$\Psi = \mathcal{F}_2(\Psi, \Phi_1, \Phi_2) \tag{60}$$

and applying our fixed-point theorem. Notice that the estimates in Theorem 4(i) for $\|\Psi_{\Phi_2}\|_{\delta,p,\varepsilon}$ are better than those for $\|\Psi\|_{\delta,p,\varepsilon}$ by a factor of $\varepsilon^{1/4}$; one expects this behaviour because Φ_2 always appears polynomially and in the combination $\Phi_1 + \Phi_2$, so that a differentiation with respect to Φ_2 eliminates one power of the $O(\varepsilon^{-\frac{1}{4}-\Delta})$ quantity Φ_1.

Theorem 4. *Suppose that*

$$\|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}, \quad \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}. \tag{61}$$

(i) *Equation* (60) *has a unique solution* $\Psi = \Psi(\Phi_1, \Phi_2)$ *which satisfies the estimate*

$$\|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}, \|\Phi_y\|_{\delta,p,\varepsilon}). \tag{62}$$

Moreover Ψ *is a smooth function of* (Φ_1, Φ_2) *with respect to the* $W_\varepsilon^{\delta,p}(\Sigma)$ *and* $U_\varepsilon^{\delta,p}(\mathbb{R}^2) \times W_\varepsilon^{1+\delta,p}(\Sigma)$ *topologies and in particular its first derivative with respect to* Φ_2 *satisfies the estimate*

$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}.$$

(ii) *The solution* $\Psi = \Psi(\Phi_1, \Phi_2)$ *to* (60) *identified in part* (i) *satisfies the estimate*

$$\begin{aligned}\|\Psi\|_2 \leqq c(&\varepsilon^{1-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta}(\|\Phi_2\|_{1,2,\varepsilon} + \varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{0,2}})\\ &\times P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}, \|\Phi_2\|_{1+\delta,p,\varepsilon})).\end{aligned} \tag{63}$$

Moreover Ψ *is a smooth function of* (Φ_1, Φ_2) *with respect to the* $L^2(\Sigma)$ *and* $[U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)] \times W_\varepsilon^{1,2}(\Sigma)$ *topologies and in particular its derivative with respect to* Φ_2 *satisfies the estimate*

$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_2 \leqq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}.$$

Proof. (i) We obtain this result by applying Theorem 2 with $\mathcal{X} = W_\varepsilon^{\delta,p}(\Sigma)$, $\mathcal{Y}_1 = U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\mathcal{Y}_2 = W_\varepsilon^{1+\delta,p}(\Sigma)$ and X, Y_1, Y_2 closed origin-centred balls of radius $\mathcal{O}(\varepsilon^{\frac{1}{2}-\Delta})$, $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$, $\mathcal{O}(\varepsilon^{-\Delta})$; one has to verify that

$$\|\mathcal{F}_2(0, \Phi_1, \Phi_2)\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}, \|\Phi_y\|_{\delta,p,\varepsilon})$$

and that

$$\|\mathrm{d}_1\mathcal{F}_2[\Psi, \Phi_1, \Phi_2]\|_{W_\varepsilon^{\delta,p}(\Sigma)\to W_\varepsilon^{\delta,p}(\Sigma)} \leqq \frac{1}{2}$$

whenever (61) and (62) hold.

The first estimate is a consequence of the calculation

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left[-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\hat{N}_6(\rho(0,\Phi),\Phi,0)\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho(0,\Phi),\Phi,0)\,\mathrm{d}\xi\right.\right.\\ &\qquad\left.\left. + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho(0,\Phi),\Phi,0)\right]\right\|_{\delta,p,\varepsilon}\\ &\leqq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}, \|\Phi_y\|_{\delta,p,\varepsilon}),\end{aligned}$$

whose ingredients are explicit formulae for $N_j(\rho, \Phi, 0)$, $j = 6, 7, 8$, Lemma 4, the properties of our function spaces and the inequality

$$|\rho(0, \Phi)|_{\delta,p,\varepsilon} \leqq c(\|\Phi_x\|_{\delta,p,\varepsilon} + \varepsilon^{-\Delta}(\|\Phi_y\|_{\delta,p,\varepsilon} + \varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}})^2). \tag{64}$$

Similarly, the second estimate follows from the calculation

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_1\hat{N}_6(\rho, \Phi, \Psi)\bar{\rho}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_3\hat{N}_6(\rho, \Phi, \Psi)\tilde{\Psi}\,\mathrm{d}\xi\right.\right.\\
&\qquad - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_1\hat{N}_7(\rho, \Phi, \Psi)\bar{\rho}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_3\hat{N}_7(\rho, \Phi, \Psi)\tilde{\Psi}\,\mathrm{d}\xi\\
&\qquad + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_1\hat{N}_8(\rho, \Phi, \Psi)\bar{\rho}\\
&\qquad \left.\left. + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_3\hat{N}_8(\rho, \Phi, \Psi)\tilde{\Psi}\right]\right\|_{\delta,p,\varepsilon}\\
&\leqq c(\varepsilon^{\frac{3}{4}-\Delta}|\bar{\rho}|_{\delta,p,\varepsilon} + \varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_{\delta,p,\varepsilon})\\
&\leqq c\varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_{\delta,p,\varepsilon},
\end{aligned}$$

where $\bar{\rho} = \rho_\Psi\tilde{\Psi}$; this result is obtained using explicit formulae for the derivatives of $N_j(\rho, \Phi, \Psi)$, $j = 6, 7, 8$ together with the estimates (61) and

$$|\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}, \quad |\rho_\Psi\tilde{\Psi}|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}}|\Psi\|_{\delta,p,\varepsilon}$$

(which follow from (61), (62), (64) and Theorem 3).

Our fixed-point theorem states that

$$\begin{aligned}
&\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_{\delta,p,\varepsilon}\\
&\leqq 2\left\|\mathcal{F}^{-1}\left[-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_1\hat{N}_6(\rho, \Phi, \Psi)\tilde{\rho}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_2\hat{N}_6(\rho, \Phi, \Psi)\tilde{\Phi}_2\,\mathrm{d}\xi\right.\right.\\
&\qquad - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_1\hat{N}_7(\rho, \Phi, \Psi)\tilde{\rho}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_2\hat{N}_7(\rho, \Phi, \Psi)\tilde{\Phi}_2\,\mathrm{d}\xi\\
&\qquad + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_1\hat{N}_8(\rho, \Phi, \Psi)\tilde{\rho}\\
&\qquad \left.\left. + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_2\hat{N}_8(\rho, \Phi, \Psi)\tilde{\Phi}_2\right]\right\|_{1+\delta,p,\varepsilon},
\end{aligned}$$

where $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$; arguing as above, we find that

$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_{\delta,p,\varepsilon} \leqq c(\varepsilon^{\frac{3}{4}-\Delta}|\tilde{\rho}|_{\delta,p,\varepsilon} + \varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}) \leqq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon},$$

where we have used the estimate $|\rho_{\Phi_2}\tilde{\Phi}_2| \leqq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}$ (see Theorem 3).

(ii) We apply our fixed-point theorem to (60), working in the closed subsets

$$X = L^2(\Sigma) \cap \{\Psi \in W_\varepsilon^{\delta,p}(\Sigma) : \|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}\},$$
$$Y_1 = [U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)] \cap \{\Phi_1 \in U_\varepsilon^{\delta,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}\},$$
$$Y_2 = W_\varepsilon^{1,2}(\Sigma) \cap \{\Phi_2 \in W_\varepsilon^{1+\delta,p}(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}\}$$

of, respectively, $\mathcal{X} = L^2(\Sigma)$, $\mathcal{Y}_1 = U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)$, $\mathcal{Y}_2 = W_\varepsilon^{1,2}(\Sigma)$. We therefore verify that

$$\|\mathcal{F}_2(0, \Phi_1, \Phi_2)\|_2 \leqq c(\varepsilon^{1-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta}(\|\Phi_2\|_{1,2,\varepsilon} + \varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{0,2}}) \times P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}, \|\Phi_2\|_{1+\delta,p,\varepsilon})) \tag{65}$$

and that

$$\|\mathrm{d}_1\mathcal{F}_2[\Psi, \Phi_1, \Phi_2]\|_{L^2(\Sigma)\to L^2(\Sigma)} \leqq \frac{1}{2} \tag{66}$$

whenever

$$\|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}, \quad \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}$$

and hence $|\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}$.

To obtain (65) we first substitute

$$\rho(0, \Phi) = \mathcal{F}^{-1}\left[\frac{\hat{\phi}_{1x}}{1+\varepsilon+\beta q^2}\right] + \mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\int_0^1 \hat{\Phi}_{2x}\,dz\right] + \rho_{NL}(0, \phi)$$

in the terms $\rho_x(0, \Phi)\Phi_{1x}$ and $\rho_z(0, \Phi)\Phi_{1z}$ in the formulae for $N_j(\rho(0, \Phi), \Phi, 0)$, $j = 6, 7, 8$; the required estimate is then derived using the methods in the proof of part (i) together with the inequalities (64) and

$$|\rho(0, \Phi)|_{0,2,\varepsilon} \leqq c(\|\Phi_x\|_2 + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{0,2}} + \|\Phi_y\|_2)(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})),$$

$$|\rho_{\mathrm{NL}}(0, \Phi)|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}P_2(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}}, \|\Phi_y\|_{\delta,p,\varepsilon}).$$

The estimates (66) and

$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_2 \leqq c(\varepsilon^{\frac{3}{4}-\Delta}|\tilde{\rho}|_{0,2,\varepsilon} + \varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}) \leqq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon},$$

in which $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$, are obtained using the methods in the proof of part (i) together with the estimates

$$|\rho_{\Phi_2}\tilde{\Phi}_2|_{0,2,\varepsilon} \leqq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}, \quad |\rho_\Psi\tilde{\Psi}|_{0,2,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_2$$

(see Theorem 3). □

We now substitute $\Psi = \Psi(\Phi_1, \Phi_2)$ into (58), write the resulting equation as

$$\Phi_2 = \mathcal{F}_3(\Phi_1, \Phi_2) \tag{67}$$

and solve this equation for Φ_2 as a function of Φ_1 using our fixed-point theorem.

Theorem 5. *Suppose that*

$$\|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}. \tag{68}$$

(i) *Equation* (67) *has a unique solution* $\Phi_2 = \Phi_2(\Phi_1)$ *which satisfies the estimate*

$$\|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta} P_2(\varepsilon^{\frac{1}{4}} \|\Phi_1\|_{U_\varepsilon^{\delta,p}}). \tag{69}$$

Moreover Φ_2 *depends smoothly upon* Φ_1 *with respect to the* $W_\varepsilon^{1+\delta,p}(\Sigma)$ *and* $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *topologies.*

(ii) *The solution* $\Phi_2 = \Phi_2(\Phi_1)$ *to* (67) *identified in part* (i) *satisfies the estimate*

$$\|\Phi_2\|_{1,2,\varepsilon} \leqq c(\varepsilon^{\frac{1}{2}} \|\Phi_1\|_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta} \|\Phi_1\|_{U_\varepsilon^{0,2}} P_1(\varepsilon^{\frac{1}{4}} \|\Phi_1\|_{U_\varepsilon^{\delta,p}})). \tag{70}$$

Moreover Φ_2 *depends smoothly upon* Φ_1 *with respect to the* $W_\varepsilon^{1,2}(\Sigma)$ *and* $U_\varepsilon^{0,2}(\mathbb{R}^2) \times U_\varepsilon^{0,4}(\mathbb{R}^2)$ *topologies.*

Proof. (i) This result is established by applying Theorem 2 with $\mathcal{X} = W_\varepsilon^{1+\delta,p}(\Sigma)$, $\mathcal{Y} = U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and X, Y closed origin-centred balls of radius $\mathcal{O}(\varepsilon^{-\Delta})$, $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$; we show that

$$\|\mathcal{F}_3(\Phi_1, 0)\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta} P_2(\varepsilon^{\frac{1}{4}} \|\Phi\|_{U_\varepsilon^{\delta,p}}) \tag{71}$$

and that

$$\|\mathrm{d}_2\mathcal{F}_3[\Phi_1, \Phi_2]\|_{W_\varepsilon^{1+\delta,p}(\Sigma)\to W_\varepsilon^{1+\delta,p}(\Sigma)} \leqq \frac{1}{2}$$

whenever (68) and (69) hold.

Inequality (71) follows from the calculation

$$\begin{aligned}
&\Bigg\| \mathcal{F}^{-1}\Bigg[-\int_0^1 \frac{G_1}{\varepsilon^{5/2}} \hat{N}_6(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1))\,\mathrm{d}\xi \\
&\qquad - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}} \hat{N}_7(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1))\,\mathrm{d}\xi \\
&\qquad + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \hat{N}_8(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1)) \Bigg] \Bigg\|_{1+\delta,p,\varepsilon} \\
&\leqq c\varepsilon^{-\Delta} P_2(\varepsilon^{\frac{1}{4}} \|\Phi\|_{U_\varepsilon^{\delta,p}}),
\end{aligned}$$

where we have used the estimates

$$\|\Psi(\Phi_1)\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}} \|\Phi_1\|_{U_\varepsilon^{\delta,p}}), \tag{72}$$

$$|\rho(\Psi(\Phi_1), \Phi_1)|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}(\|\Phi_{1x}\|_{\delta,p,\varepsilon} + P_2(\varepsilon^{\frac{1}{4}} \|\Phi_1\|_{U_\varepsilon^{\delta,p}})). \tag{73}$$

Furthermore, writing $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$, $\bar{\rho} = \rho_\Psi \tilde{\Psi}$, $\tilde{\Psi} = \Psi_{\Phi_2}\tilde{\Phi}_2$ and using the estimates (68) and

$$|\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}, \quad \|\Phi_2\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}$$

(which follow from (68), (69), (72), (73)), we find that

$$\begin{aligned}
&\|\mathrm{d}\mathcal{F}_3[\Phi_1, \Phi_2](\tilde{\Phi}_2)\|_{1+\delta,p,\varepsilon} \\
&\leqq \Bigg\| \mathcal{F}^{-1}\Bigg[-\int_0^1 \frac{G_1}{\varepsilon^{5/2}} \partial_1 \hat{N}_6(\rho, \Phi, \Psi)(\tilde{\rho} + \bar{\rho})\,\mathrm{d}\xi - \int_0^1 \frac{G_1}{\varepsilon^{5/2}} \partial_2 \hat{N}_6(\rho, \Phi, \Psi)\tilde{\Phi}_2\,\mathrm{d}\xi \\
&\qquad - \int_0^1 \frac{G_1}{\varepsilon^{5/2}} \partial_3 \hat{N}_6(\rho, \Phi, \Psi)\tilde{\Psi}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}} \partial_1 \hat{N}_7(\rho, \Phi, \Psi)(\tilde{\rho} + \bar{\rho})\,\mathrm{d}\xi \\
&\qquad - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}} \partial_2 \hat{N}_7(\rho, \Phi, \Psi)\tilde{\Phi}_2\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}} \partial_3 \hat{N}_7(\rho, \Phi, \Psi)\tilde{\Psi}\,\mathrm{d}\xi \\
&\qquad + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \partial_1 \hat{N}_8(\rho, \Phi, \Psi)(\tilde{\rho} + \bar{\rho}) \\
&\qquad + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \partial_2 \hat{N}_8(\rho, \Phi, \Psi)\tilde{\Phi}_2 \\
&\qquad + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \partial_3 \hat{N}_8(\rho, \Phi, \Psi)\tilde{\Psi} \Bigg] \Bigg\|_{1+\delta,p,\varepsilon} \\
&\leqq c(\varepsilon^{\frac{1}{4}-\Delta}(|\tilde{\rho}|_{\delta,p,\varepsilon} + \bar{\rho}|_{\delta,p,\varepsilon}) + \varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon} + \varepsilon^{-\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}) \\
&\leqq c(\varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon} + \varepsilon^{-\frac{1}{4}}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}) \\
&\leqq \varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon} \\
&\leqq \frac{1}{2}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon},
\end{aligned}$$

in which the further inequalities

$$|\rho_{\Phi_2}\tilde{\Phi}_2|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}, \quad |\rho_\Psi \tilde{\Psi}|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}-\Delta}\|\tilde{\Psi}\|_{\delta,p,\varepsilon},$$
$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{\delta,p,\varepsilon}$$

have been used (see Theorems 3 and 4).

(ii) We again note that

$$X = W_\varepsilon^{1,2}(\Sigma) \cap \{\Phi_2 \in W_\varepsilon^{1+\delta,p}(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}\},$$

$$Y = [U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)] \cap \{\Phi_1 \in U_\varepsilon^{\delta,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}\}$$

are closed subsets of, respectively, $\mathcal{X} = W^{1,2}(\Sigma)$, $\mathcal{Y} = U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)$ and apply our fixed-point equation to (60) with these definitions of $\mathcal{X}$, $\mathcal{Y}$ and X, Y.

Estimation techniques similar to those used in the proof of Theorem 4(ii) show that

$$\|\mathcal{F}_3(\Phi_1, 0)\|_2 \leqq c(\varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,2}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})),$$

where the inequalities (72), (73) and

$$\|\Psi(\Phi_1)\|_2 \leqq c(\varepsilon\|\Phi_1\|^2_{U^{0,4}_\varepsilon} + \varepsilon^{1-\Delta}\|\Phi_1\|_{U^{0,2}_\varepsilon} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon})),$$
$$|\rho_{\mathrm{NL}}(\Psi_1(\Phi_1), \Phi_1)|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon})$$

have also been used. Estimating

$$\|\Phi_1\|_{U^{\delta,p}_\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}$$

and hence

$$|\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta},$$

we similarly find that

$$\begin{aligned}
&\|\mathrm{d}_2\mathcal{F}_3[\Phi_1, \Phi_2](\tilde{\Phi}_2)\|_{1,2,\varepsilon} \\
&\quad \leqq c(\varepsilon^{\frac{1}{4}-\Delta}(|\tilde{\rho}|_{0,2,\varepsilon} + |\bar{\rho}|_{0,2,\varepsilon}) + \varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon} + \varepsilon^{-\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_2) \\
&\quad \leqq c(\varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon} + \varepsilon^{-\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}) \\
&\quad \leqq c\varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_2,
\end{aligned}$$

where $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$, $\bar{\rho} = \rho_\Psi\tilde{\Psi}$, $\tilde{\Psi} = \Psi_{\Phi_2}\tilde{\Phi}_2$ and we have made use of the further inequalities

$$|\rho_{\Phi_2}\tilde{\Phi}_2|_{0,2,\varepsilon} \leqq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}, \quad |\rho_\Psi\tilde{\Psi}|_{0,2,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}-\Delta}\|\tilde{\Psi}\|_2,$$
$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_2 \leqq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}$$

(see Theorems 3 and 4). □

3.3. Regularity theory

We now return to the integral form (29) of the reduced equation for Φ_1. According to the material presented in Sections 3.1 and 3.2 above, the quantity in brackets on the right-hand side of this equation is well defined provided that

$$\|\Phi_1\|_{U^{\delta,p}_\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \tag{74}$$

whence

$$|\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}. \tag{75}$$

The corresponding weak formulation of the reduced equation for Φ_1 (see Definition 2(i)) requires that $\Phi_1 \in X$; in view of the embedding (16) we therefore study the integral and weak formulations of this equation in the closed origin-centred ball $\{\Phi_1 \in X : |||\Phi_1||| \leqq M\}$ of X.

Any solution Φ_1 of (29) defines a weak solution $(\rho(\Phi_1), \Phi_1 + \Phi_2(\Phi_1))$ of the scaled water-wave problem (10)–(13), and in Section 4 this aspect of the existence theory is completed with the confirmation that (29) indeed has a nonzero solution. In this section we complete the analysis of the reduction procedure by presenting regularity theory which asserts that Φ_1, Φ_2 and ρ actually belong to the smaller function spaces $U_\varepsilon^{5,p}(\mathbb{R}^2)$, $W_\varepsilon^{2,p}(\Sigma)$ and $V_\varepsilon^{1,p}(\mathbb{R}^2)$ and solve the strong forms of their equations; it follows that $(\rho(\Phi_1), \Phi_1 + \Phi_2(\Phi_1))$ is a strong solution of the equations (10)–(13).

Our first regularity result (Proposition 13 below) shows that Φ_1 belongs to $U_\varepsilon^{2,p}(\mathbb{R}^2)$. In order to establish this result we need the following lemma, which deals with Fourier-multiplier operators appearing in the integral form of the equation for Φ_1; its proof is given in Section 5.

Lemma 5.

(i) *For each $u \in L^p(\mathbb{R}^2)$ the functions*

$$\mathcal{G}_{12}(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{Q}\mathcal{F}[u]\right], \quad \mathcal{G}_{13}(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{Q}\mathcal{F}[u]\right]$$

belong to $U_\varepsilon^{2,p}(\mathbb{R}^2)$ and satisfy the estimate

$$\|\mathcal{G}_j(u)\|_{U_\varepsilon^{2,p}} \leqq c\|u\|_p, \quad j = 12, 13.$$

(ii) *For each $u \in L^p(\mathbb{R}^2)$ the function*

$$\mathcal{G}_{14}(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{(1+\varepsilon+\beta q^2)Q}\mathcal{F}[u]\right]$$

belongs to $U_\varepsilon^{2,p}(\mathbb{R}^2)$ and satisfies the estimate

$$\|\mathcal{G}_{14}(u)\|_{U_\varepsilon^{2,p}} \leqq c\|u\|_p.$$

(iii) *For each $u \in L^p(\mathbb{R}^2)$ the functions*

$$\mathcal{G}_{15}(u) = \mathcal{F}^{-1}\left[\frac{-\mu^2}{(1+\varepsilon+\beta q^2)Q}\mathcal{F}[u]\right],$$
$$\mathcal{G}_{16}(u) = \mathcal{F}^{-1}\left[\frac{-\varepsilon^{\frac{1}{2}}\mu k}{(1+\varepsilon+\beta q^2)Q}\mathcal{F}[u]\right]$$

belong to $U_\varepsilon^{2,p}(\mathbb{R}^2)$ and satisfy the estimate

$$\|\mathcal{G}_j(u)\|_{U_\varepsilon^{2,p}} \leqq c\varepsilon^{-\frac{1}{2}}\|u\|_p, \quad j = 15, 16.$$

Proposition 13. *A solution of the integral form of the equation for Φ_1 which satisfies $|||\Phi_1||| \leqq M$ belongs to $U_\varepsilon^{2,p}(\mathbb{R}^2)$ and satisfies the estimates*

$$\|\Phi_1\|_{U_\varepsilon^{1,p}} \leqq c\varepsilon^{-\frac{3}{8}-\Delta}, \quad \|\Phi_1\|_{U_\varepsilon^{2,p}} \leqq c\varepsilon^{-\frac{1}{2}-\Delta}.$$

Proof. Using Lemma 5(i)–(ii) and the estimates (75) we find that

$$\left\|\mathcal{F}^{-1}\left[\frac{1}{Q}\int_0^1 \varepsilon^{-\frac{5}{2}}N_4(\rho, \Phi)\,\mathrm{d}y\right]\right\|_{U_\varepsilon^{2,p}} \leqq c\varepsilon^{-\frac{1}{2}-\Delta},$$

and a similar calculation using Lemma 5(iii)–(v) and (75) shows that

$$\left\| \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{Q(1+\varepsilon+\beta q^2)}\varepsilon^{-2}\hat{N}_1(\rho,\Phi)\right]\right\|_{U_\varepsilon^{2,p}} \leqq c\varepsilon^{-\frac{1}{2}-\Delta}.$$

An inspection of the reduced equation (29) shows that

$$\|\Phi_1\|_{U_\varepsilon^{2,p}} \leqq c\varepsilon^{-\frac{1}{2}-\Delta}, \tag{76}$$

and the remaining estimate

$$\|\Phi_1\|_{U_\varepsilon^{1,p}} \leqq c\varepsilon^{-\frac{3}{8}-\Delta}$$

follows by interpolation between (76) and

$$\|\Phi_1\|_{U_\varepsilon^{0,p}} \leqq \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}$$

(see equation (74)). □

The next step is to reappraise the integral equations for ρ, Ψ and Φ_2 in the light of the improved regularity of Φ_1. We proceed in the spirit of Theorems 3(ii), 4(ii) and 5(ii), which show how these integral equations, which were originally solved in $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $W_\varepsilon^{\delta,p}(\Sigma)$ and $W_\varepsilon^{1+\delta,p}(\Sigma)$, are also solvable in $V_\varepsilon^{0,2}(\mathbb{R}^2)$, $L^2(\Sigma)$ and $W_\varepsilon^{0,2}(\Sigma)$; here we give three lemmata which show that they are solvable in $V_\varepsilon^{1,p}(\mathbb{R}^2)$, $W_\varepsilon^{1,p}(\Sigma)$ and $W_\varepsilon^{2,p}(\Sigma)$.

Lemma 6. *Suppose that*

$$\|\Phi_1\|_{U_\varepsilon^{1,p}} \leqq c\varepsilon^{-\frac{3}{8}-\Delta}, \quad \|\Phi_2\|_{2,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{8}-\Delta}, \quad \|\Psi\|_{1,p,\varepsilon} \leqq c\varepsilon^{\frac{3}{8}-\Delta}. \tag{77}$$

The solution $\rho = \rho(\Psi, \Phi_1, \Phi_2)$ to (50) *identified in Theorem* 3 *satisfies the estimate*

$$\begin{aligned}|\rho|_{1,p,\varepsilon} \leqq c(&\|\Phi_x\|_{1,p,\varepsilon} + \varepsilon^{-\frac{1}{2}}\|\Psi\|_{1,p,\varepsilon} \\ &+\varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{1,p}} + \|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})).\end{aligned} \tag{78}$$

Moreover ρ depends smoothly upon (Ψ_1, Φ_1, Φ_2) with respect to the $V_\varepsilon^{1,p}(\mathbb{R}^2)$ and $W_\varepsilon^{1,p}(\Sigma) \times U_\varepsilon^{1,p}(\mathbb{R}^2) \times W_\varepsilon^{2,p}(\Sigma)$ toplogies and in particular its derivatives with respect to Ψ and Φ_2 satisfy the estimates

$$|\rho_\Psi \tilde{\Psi}|_{1,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{1,p,\varepsilon}, \quad |\rho_{\Phi_2}\tilde{\Phi}_2|_{1,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{8}-\Delta}\|\tilde{\Phi}_2\|_{2,p,\varepsilon}.$$

Proof. We apply our fixed-point theorem to (50), working in the closed subsets

$$\begin{aligned}X = \{&\rho \in V_\varepsilon^{1,p}(\mathbb{R}^2) : |\rho|_{1,p,\varepsilon} \leqq c\varepsilon^{-\frac{3}{8}-\Delta}\} \\ &\cap\{\rho \in V_\varepsilon^{\delta,p}(\mathbb{R}^2) : |\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}\},\end{aligned}$$

$$\begin{aligned}
Y_1 &= \{\Psi_2 \in W_\varepsilon^{1,p}(\Sigma) : \|\Psi\|_{1,p,\varepsilon} \leqq c\varepsilon^{\frac{3}{8}-\Delta}\} \\
&\qquad \cap \{\Psi \in W_\varepsilon^{\delta,p}(\Sigma) : \|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}\}, \\
Y_2 &= \{\Phi_1 \in U_\varepsilon^{1,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{1,p}} \leqq c\varepsilon^{-\frac{3}{8}-\Delta}\} \\
&\qquad \cap \{\Phi_1 \in U_\varepsilon^{\delta,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}\}, \\
Y_3 &= \{\Phi_2 \in W_\varepsilon^{2,p}(\Sigma) : \|\Phi_2\|_{2,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{8}-\Delta}\} \\
&\qquad \cap \{\Phi_2 \in W_\varepsilon^{1+\delta,p}(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}\}
\end{aligned}$$

of, respectively, $\mathcal{X} = V_\varepsilon^{1,p}(\mathbb{R}^2)$, $\mathcal{Y}_1 = W_\varepsilon^{1,p}(\Sigma)$. $\mathcal{Y}_2 = U_\varepsilon^{1,p}(\mathbb{R}^2)$, $\mathcal{Y}_3 = W_\varepsilon^{2,p}(\Sigma)$. Employing our usual estimation methods, we find that

$$\begin{aligned}
&|\mathcal{F}_1(0, \Psi, \Phi_1, \Phi_2)|_{1,p,\varepsilon} \\
&\quad \leqq c(\|\Phi_x\|_{1,p,\varepsilon} + \varepsilon^{-\frac{1}{2}}\|\Psi\|_{1,p,\varepsilon} \\
&\qquad + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{1,p}} + \|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon}))
\end{aligned}$$

and that

$$|\mathrm{d}_1\mathcal{F}_1[\rho, \Psi, \Phi_1, \Phi_2](\tilde{\rho})|_{1,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}|\tilde{\rho}|_{1,p,\varepsilon} \leqq \frac{1}{2}|\tilde{\rho}|_{1,p,\varepsilon} \tag{79}$$

whenever

$$\begin{aligned}
&|\rho|_{\delta,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}, \\
&\|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}, \quad \|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}
\end{aligned} \tag{80}$$

and (77), (78) hold, so that in particular $|\rho|_{1,p,\varepsilon} \leqq c\varepsilon^{-\frac{3}{8}-\Delta}$.

According to our fixed-point theorem, the estimates for $\rho_\Psi\tilde{\Psi}$ and $\rho_{\Phi_2}\tilde{\Phi}_2$ are given by the formulae

$$\begin{aligned}
|\rho_\Psi\tilde{\Psi}|_{1,p,\varepsilon} &\leqq 2\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\int_0^1 y\hat{\tilde{\Psi}}\,\mathrm{d}y\right]\right|_{1,p,\varepsilon} \\
&\leqq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{1,p,\varepsilon}, \\
|\rho_{\Phi_2}\tilde{\Phi}_2|_{1,p,\varepsilon} &\leqq 2\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\mathrm{i}\mu\int_0^1 y\hat{\tilde{\Phi}}_2\,\mathrm{d}y + \varepsilon^{-1}\partial_2\hat{N}_1(\rho,\Phi)\tilde{\Phi}_2\right)\right]\right|_{1,p,\varepsilon} \\
&\leqq c\varepsilon^{-\frac{1}{8}-\Delta}\|\tilde{\Phi}_2\|_{2,p,\varepsilon},
\end{aligned}$$

where the final inequality is obtained in the same fashion as (79). □

Before proceeding to the equations for Ψ and Φ_2, let us record some further estimates which are useful in the analysis of these equations; they are proved using the estimation techniques developed above.

Proposition 14. *The function $\rho = \rho(\Psi, \Phi_1\Phi_2)$ discussed in the previous lemma satisfies the further inequalities*

$$\|\rho\|_{U_\varepsilon^{0,p}} \leqq c(\|\Phi_x\|_{1,p,\varepsilon} + \varepsilon^{-\frac{1}{2}}\|\Psi\|_{1,p,\varepsilon} + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{1,p}} + \|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})) \qquad (81)$$

and

$$\|\rho_{\Phi_2}\tilde{\Phi}_2\|_{U_\varepsilon^{0,p}} \leqq c\varepsilon^{-\frac{1}{8}-\Delta}\|\tilde{\Phi}_2\|_{2,p,\varepsilon}, \quad \|\rho_\Psi\tilde{\Psi}\|_{U_\varepsilon^{0,p}} \leqq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Phi}_2\|_{1,p,\varepsilon}.$$

Lemma 7. *Suppose that*

$$\|\Phi_1\|_{U_\varepsilon^{1,p}} \leqq c\varepsilon^{-\frac{3}{8}-\Delta}, \quad \|\Phi_2\|_{2,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{8}-\Delta}. \qquad (82)$$

The solution $\Psi = \Psi(\Phi_1, \Phi_2)$ to (60) *identified in Theorem* 4 *satisfies the estimate*

$$\|\Psi\|_{1,p,\varepsilon} \leqq c(\varepsilon^{\frac{3}{4}-\Delta}\|\Phi\|_{U_\varepsilon^{1,p}} + \varepsilon^{\frac{3}{4}-\Delta}\|\Phi_y\|_{1,p,\varepsilon})P_1(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}, \|\Phi_y\|_{\delta,p,\varepsilon}). \qquad (83)$$

Moreover Ψ depends smoothly upon (Φ_1, Φ_2) with respect to the $W_\varepsilon^{1,p}(\Sigma)$ and $U_\varepsilon^{1,p}(\mathbb{R}^2) \times W_\varepsilon^{2,p}(\Sigma)$ topologies and in particular its derivative with respect to Φ_2 satisfies the estimate

$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_{1,p,\varepsilon} \leqq c\varepsilon^{\frac{5}{8}-\Delta}\|\tilde{\Phi}_2\|_{2,p,\varepsilon}.$$

Proof. We obtain this result by applying our fixed-point theorem to (60) with $\mathcal{X} = W_\varepsilon^{1,p}(\Sigma)$, $\mathcal{Y}_1 = U_\varepsilon^{1,p}(\mathbb{R}^2)$, $\mathcal{Y}_2 = W_\varepsilon^{2,p}(\Sigma)$ and

$$\begin{aligned} X &= \{\Psi \in W_\varepsilon^{1,p}(\Sigma) : \|\Psi\|_{1,p,\varepsilon} \leqq c\varepsilon^{\frac{3}{8}-\Delta}\} \\ &\quad \cap \{\Psi \in W_\varepsilon^{\delta,p}(\Sigma) : \|\Psi\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}\}, \\ Y_1 &= \{\Phi_1 \in U_\varepsilon^{1,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{1,p}} \leqq c\varepsilon^{-\frac{3}{8}-\Delta}\} \\ &\quad \cap \{\Phi_1 \in U_\varepsilon^{\delta,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leqq c\varepsilon^{-\frac{1}{4}-\Delta}\}, \\ Y_2 &= \{\Phi_2 \in W_\varepsilon^{2,p}(\Sigma) : \|\Phi_2\|_{2,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{8}-\Delta}\} \\ &\quad \cap \{\Phi_2 \in W_\varepsilon^{1+\delta,p}(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}\}. \end{aligned}$$

Employing the methods developed in the proofs of Theorem 4 and Lemma 6 together with the estimates

$$\begin{aligned} |\rho(0,\Phi)|_{\delta,p,\varepsilon} &\leqq c\varepsilon^{-\Delta}(\|\Phi_x\|_{\delta,p,\varepsilon} + (\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})^2), \\ |\rho(0,\Phi)|_{1,p,\varepsilon} &\leqq c\varepsilon^{-\Delta}(\|\Phi_x\|_{1,p,\varepsilon} \\ &\qquad + (\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{1,p}} + \|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})), \\ \|\rho(0,\Phi)\|_{U_\varepsilon^{0,p}} &\leqq c\varepsilon^{-\Delta}(\|\Phi_x\|_{1,p,\varepsilon} \\ &\qquad + (\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{1,p}} + \|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})), \end{aligned}$$

one finds that

$$\begin{aligned}&\|\mathcal{F}_2(0,\Phi_1,\Phi_2)\|_{1,p,\varepsilon}\\&\quad\leqq c(\varepsilon^{\frac{3}{4}-\Delta}\|\Phi\|_{U_\varepsilon^{1,p}}+\varepsilon^{\frac{3}{4}-\Delta}\|\Phi_y\|_{1,p,\varepsilon})P_1(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}},\|\Phi_y\|_{\delta,p,\varepsilon}).\end{aligned}$$

Similarly, using inequalities (80), (82) and

$$|\rho|_{1,p,\varepsilon}\leqq c\varepsilon^{-\frac{3}{8}-\Delta},\quad \|\rho\|_{U_\varepsilon^{0,p}}\leqq c\varepsilon^{-\frac{3}{8}-\Delta},\quad \|\Psi\|_{1,p,\varepsilon}\leqq c\varepsilon^{\frac{3}{8}-\Delta}$$

(which follow from (78), (81) and (83)), we find that

$$\|\mathrm{d}_2\mathcal{F}_2[\Psi,\Phi_1,\Phi_2](\tilde{\Psi})\|_{1,p,\varepsilon}\;\leqq\; c\varepsilon^{\frac{1}{8}-\Delta}\|\tilde{\Psi}\|_{1,p,\varepsilon}\;\leqq\;\frac{1}{2}\|\tilde{\Psi}\|_{1,p,\varepsilon}$$

and

$$\|\mathrm{d}_3\mathcal{F}_2[\Psi,\Phi_1,\Phi_2](\tilde{\Phi}_2)\|_{1,p,\varepsilon}\leqq\varepsilon^{\frac{5}{8}-\Delta}\|\tilde{\Phi}_2\|_{2,p,\varepsilon};$$

here $\bar{\rho}=\rho_\Psi\tilde{\Psi}$ and $\tilde{\rho}=\rho_{\Phi_2}\tilde{\Phi}_2$ are estimated by

$$|\bar{\rho}|_{\delta,p,\varepsilon}\leqq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{\delta,p,\varepsilon},\quad |\bar{\rho}|_{1,p,\varepsilon}\leqq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{1,p,\varepsilon},\quad \|\bar{\rho}\|_{U_\varepsilon^{0,p}}\leqq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{1,p,\varepsilon},$$

$$|\tilde{\rho}|_{\delta,p,\varepsilon}\leqq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{\delta,p,\varepsilon},\quad |\tilde{\rho}|_{1,p,\varepsilon}\leqq c\varepsilon^{-\frac{1}{8}}\|\tilde{\Phi}_2\|_{1,p,\varepsilon},\quad \|\tilde{\rho}\|_{U_\varepsilon^{0,p}}\leqq c\varepsilon^{-\frac{1}{8}}\|\tilde{\Phi}_2\|_{1,p,\varepsilon}.$$

□

Lemma 8. *Suppose that*

$$\|\Phi_1\|_{U_\varepsilon^{1,p}}\leqq c\varepsilon^{-\frac{3}{8}-\Delta}. \tag{84}$$

The solution $\Phi_2=\Phi_2(\Phi_1)$ *to* (67) *identified in Theorem* 5 *satisfies the estimate*

$$\|\Phi_2\|_{2,p,\varepsilon}\leqq c\varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}}P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}). \tag{85}$$

Moreover Φ_2 *depends smoothly upon* Φ_1 *with respect to the* $W_\varepsilon^{2,p}(\Sigma)$ *and* $U_\varepsilon^{1,p}(\mathbb{R}^2)$ *topologies.*

Proof. This result is established by applying our fixed-point theorem to (67) with $\mathcal{X}=W_\varepsilon^{2,p}(\Sigma)$, $\mathcal{Y}=U_\varepsilon^{1,p}(\mathbb{R}^2)$ and

$$\begin{aligned}X&=\{\Phi_2\in W_\varepsilon^{2,p}(\Sigma):\|\Phi_2\|_{2,p,\varepsilon}\leqq c\varepsilon^{-\frac{1}{8}-\Delta}\}\\&\qquad\cap\{\Phi_2\in W_\varepsilon^{1+\delta,p}(\Sigma):\|\Phi_2\|_{1+\delta,p,\varepsilon}\leqq c\varepsilon^{-\Delta}\},\\Y&=\{\Phi_1\in U_\varepsilon^{1,p}(\mathbb{R}^2):\|\Phi_1\|_{U_\varepsilon^{1,p}}\leqq c\varepsilon^{-\frac{3}{8}-\Delta}\}\\&\qquad\cap\{\Phi_1\in U_\varepsilon^{\delta,p}(\mathbb{R}^2):\|\Phi_1\|_{U_\varepsilon^{\delta,p}}\leqq c\varepsilon^{-\frac{1}{4}-\Delta}\}.\end{aligned}$$

The methods developed in the proofs of Theorem 5 and Corollary 6 together with the estimates (72), (73) and

$$|\rho(\Psi(\Phi_1),\Phi_1)|_{1,p,\varepsilon}\leqq c(\|\Phi_{1x}\|_{1,p,\varepsilon}+\varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}}P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})),$$

$$\|\rho(\Psi(\Phi_1),\Phi_1)\|_{U_\varepsilon^{0,p}}\leqq c(\|\Phi_{1x}\|_{1,p,\varepsilon}+\varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}}P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}))$$

yield

$$\|\mathcal{F}_3(\Phi_1, 0)\|_{2,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}).$$

Furthermore, writing $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$, $\bar{\rho} = \rho_\Psi\tilde{\Psi}$, $\tilde{\Psi} = \Psi_{\Phi_2}\tilde{\Phi}_2$ and using the estimates (80), (84) and

$$|\rho|_{1,p,\varepsilon} \leqq c\varepsilon^{-\frac{3}{8}-\Delta}, \quad \|\rho\|_{U_\varepsilon^{0,p}} \leqq c\varepsilon^{-\frac{3}{8}-\Delta},$$

$$\|\Phi_2\|_{1,p,\varepsilon} \leqq c\varepsilon^{-\frac{1}{2}-\Delta}, \quad \|\Psi\|_{1,p,\varepsilon} \leqq c\varepsilon^{\frac{3}{8}-\Delta}$$

(which follow from (78), (81), (83) and (85)) we find that

$$\|\mathrm{d}_2\mathcal{F}_3[\Phi_1, \Phi_2](\tilde{\Phi}_2)\|_{2,p,\varepsilon} \;\leqq\; c\varepsilon^{\frac{1}{8}-\Delta}\|\tilde{\Phi}_2\|_{2,p,\varepsilon} \;\leqq\; \frac{1}{2}\|\tilde{\Phi}_2\|_{2,p,\varepsilon},$$

in which the estimates for $\rho_{\Phi_2}\tilde{\Phi}_2$, $\rho_\Psi\tilde{\Psi}$ and $\Psi_{\Phi_2}\tilde{\Phi}_2$ stated in Theorems 3 and 4 and Lemmata 6 and 7 have also been used. □

Altogether, the above results show that

$$\|\Phi_2\|_{2,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}),$$

$$\|\Phi_{2y}\|_{2,p,\varepsilon} \leqq c\varepsilon^{\frac{3}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})$$

and

$$|\rho(\Phi_1)|_{1,p,\varepsilon} \leqq c(\|\Phi_{1x}\|_{1,p,\varepsilon} + \varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})),$$

$$\|\rho(\Phi_1)\|_{U_\varepsilon^{0,p}} \leqq c(\|\Phi_{1x}\|_{1,p,\varepsilon} + \varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}} + P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})),$$

where $\rho(\Phi_1)$ is an abbreviation for $\rho(\Phi_{2y}(\Phi_1), \Phi_2(\Phi_1), \Phi_1)$ and Ψ has been identified with Φ_{2y}. Observe that $\rho(\Phi_1)$ is a weak solution of the equation for ρ (with $\Phi_2 = \Phi_2(\Phi_1)$) which meets the additional regularity requirements of a strong solution; a familiar argument asserts that it is a strong solution. One similarly finds that $\Phi_2(\Phi_1)$ is a strong solution of the equation for Φ_2 (with $\rho = \rho(\Phi_1)$), and that $(\rho(\Phi_1), \Phi_1 + \Phi_2(\Phi_1))$ is a strong solution of the original equations (10)–(13). Finally, it is possible to repeat the proof of Proposition 13 in a 'bootstrap' fashion to conclude that Φ_1 belongs to $U_\varepsilon^{5,p}(\mathbb{R}^2)$ and is therefore a strong solution of equation (29); this step is however only of academic interest since it does not play a role in the regularity theory for (10)–(13).

4. Critical-point theory

In this section we complete our existence theory by showing that the functional $J : X \to \mathbb{R}^2$ has at least one nontrivial critical point. We employ a well-established strategy from the calculus of variations, namely an application of the mountain-pass lemma (to find a Palais–Smale sequence) and the concentration-compactness principle (to deduce the existence of a nonzero critical point). The present situation is however complicated by the presence of nonlocal terms in J and the fact that it is defined only upon a neighbourhood $\bar{B}_M(0)$ of the origin in its function space X; recall that although M may be taken arbitrarily large, the greatest permissible magnitude of ε decreases as M is increased.

We begin by collecting together several auxiliary results necessary for the subsequent application of the calculus of variations. Let us first note two topological facts concerning J. Examining the formulae (47), (48), we find that J_2 and J_3 admit natural extensions from $\bar{B}_M(0)$ to the whole of X, and we henceforth consider them as functions $X \to \mathbb{R}$. Recall also that the cubic and higher-order parts J_3 and J_4 of J define smooth functionals on (a neighbourhood of the origin in) $U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$. Turning to an algebraic property of J, we may eliminate $J_3(\Phi_1)$ between

$$J(\Phi_1) = J_2(\Phi_1) + J_3(\Phi_1) + J_4(\Phi_1)$$

and

$$\begin{aligned}\langle\!\langle\!\langle J'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle &= \langle\!\langle\!\langle J_2'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_3'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle \\ &= 2J_2(\Phi_1) + 3J_3(\Phi_1) + \langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle, \end{aligned} \tag{86}$$

to obtain the identities

$$J(\Phi_1) = \frac{1}{3}J_2(\Phi_1) + J_4(\Phi_1) - \frac{1}{3}\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle + \frac{1}{3}\langle\!\langle\!\langle J'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle,$$

$$J_2(\Phi_1) = 3J(\Phi_1) - 3J_4(\Phi_1) + \langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle - \langle\!\langle\!\langle J'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle,$$

which are exploited repeatedly below.

Observe that

$$|J_2(\Phi_1)| = \frac{1}{2(1+\varepsilon)}|||\Phi_1|||^2, \tag{87}$$

$$|J_3(\Phi_1)| \leqq c\|\Phi_1\|_{U_\varepsilon^{0,3}}^3 \leqq c|||\Phi_1|||^3; \tag{88}$$

the following proposition presents corresponding estimates for the higher-order terms in J.

Proposition 15. *The inequalities*

$$|J_4(\Phi_1)| \leqq c\varepsilon^{\frac{1}{4}-\Delta}P_4(|||\Phi_1|||), \tag{89}$$

$$|\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle| \leqq c\varepsilon^{\frac{1}{4}-\Delta}P_4(|||\Phi_1|||) \tag{90}$$

hold for each $\Phi_1 \in \bar{B}_M(0) \subset X$.

Proof. We proceed by estimating each term in the explicit formula (49) for J_4 using the inequalities

$$\|\Phi_2\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon^{-\Delta}P_2(|||\Phi_1|||),$$

$$\|\Phi_{2y}\|_{\delta,p,\varepsilon} \leqq c\varepsilon^{\frac{1}{2}-\Delta}P_2(|||\Phi_1|||),$$

$$
\begin{aligned}
|\rho|_{\delta,p,\varepsilon} &\leqq c(\varepsilon^{-\frac{1}{4}-\Delta}|||\Phi_1||| + \varepsilon^{-\Delta}P_2(|||\Phi_1|||)),\\
|\rho_{\mathrm{NL}}|_{\delta,p,\varepsilon} &\leqq c\varepsilon^{-\Delta}P_2(|||\Phi_1|||),\\
\|\Phi_2\|_{1,2,\varepsilon} &\leqq c\varepsilon^{\frac{1}{2}-\Delta}P_2(|||\Phi_1|||),\\
\|\Phi_{2y}\|_2 &\leqq c\varepsilon^{1-\Delta}P_2(|||\Phi_1|||),\\
|\rho|_{0,2,\varepsilon} &\leqq c(|||\Phi_1||| + \varepsilon^{\frac{1}{2}-\Delta}P_2(|||\Phi_1|||)),\\
|\rho_{\mathrm{NL}}|_{0,2,\varepsilon} &\leqq c\varepsilon^{1-\Delta}P_2(|||\Phi_1|||),
\end{aligned}
$$

which are obtained by combining the estimates presented in Theorem 1 and Proposition 3 with the embeddings (16), (17); the result is

$$|J_4(\Phi_1)| \leqq c\varepsilon^{\frac{1}{4}-\Delta}P_4(|||\Phi_1|||).$$

The second estimate is obtained by noting that

$$
\begin{aligned}
\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle &= \langle\!\langle\!\langle J'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle - 2J_2(\Phi_1) - 3J_3(\Phi_1)\\
&= -\int_{\mathbb{R}^2}\varepsilon^{-2}\left(\int_0^1 H_1\,\mathrm{d}y + h_1\right)\Phi_1\,\mathrm{d}x\,\mathrm{d}z - 3J_3(\Phi_1), \quad (91)
\end{aligned}
$$

where we have used the fact that

$$\langle\!\langle\!\langle J'(\Phi_1), \Psi_1\rangle\!\rangle\!\rangle = \frac{1}{1+\varepsilon}\langle\!\langle\!\langle \Phi_1, \Psi_1\rangle\!\rangle\!\rangle - \int_{\mathbb{R}^2}\varepsilon^{-2}\left(\int_0^1 H_1\,\mathrm{d}y + h_1\right)\Psi_1\,\mathrm{d}x\,\mathrm{d}z.$$

An expression for $\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle$ is therefore obtained by substituting the explicit formulae for J_3, H_1 and h_1 into the right-hand side of (91). Estimating each term in this expression using the rules explained above, we arrive at the requisite inequality

$$|\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle| \leqq c\varepsilon^{\frac{1}{4}-\Delta}P_4(|||\Phi_1|||).$$

□

Let us now recall the mountain-pass lemma as stated by BREZIS & NIRENBERG [5, p. 943].

Lemma 9. *Consider a Banach space $\mathcal{X}$ and a functional $\mathcal{J} \in C^1(\mathcal{X}, \mathbb{R})$ with the properties that $\mathcal{J}(0) = 0$, that 0 is a strict local minimum of $\mathcal{J}$ and that there is an element $x \in \mathcal{X}$ with $\mathcal{J}(x) < 0$. There exists a Palais–Smale sequence $\{x_m\} \subset \mathcal{X}$ such that $\mathcal{J}(x_m) \to a$, $\mathcal{J}'(x_m) \to 0$ as $m \to \infty$, where*

$$a = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]}\mathcal{J}(\gamma(s)), \qquad \Gamma = \{\gamma \in C([0,1], \mathcal{X}) : \gamma(0) = 0, \mathcal{J}(\gamma(1)) < 0\}.$$

A functional that satisfies the hypotheses of Lemma 9 is said to have a *mountain-pass structure.*

It is not possible to apply Lemma 9 directly to $J : \bar{B}_M(0) \subset X \to \mathbb{R}$ since it is not defined upon the whole of X. Notice however that it does meet the geometric requirements of a mountain-pass functional: it follows from (87) to (89) that 0 is a strict local minimum of J, and choosing $\Phi_1^\star$ such that $J_3(\Phi_1^\star) \neq 0$, we find that there exists a real number $\lambda^\star$ which has the property that $J(\lambda^\star \Phi_1^\star) < 0$. We proceed by extending J to a smooth functional $\tilde{J} : X \to \mathbb{R}$ in such a way that J and $\tilde{J}$ coincide on a sufficiently large neighbourhood of the origin; the new functional therefore inherits the geometric structure of J and can be treated using Lemma 9.

Define

$$M_1 = \sup\{J(\Phi_1) : |||\Phi_1||| \leqq 2|||\lambda^\star \Phi_1^\star|||\},$$

choose $M_2 \geqq \max(2|||\lambda^\star \Phi_1^\star|||, (24(1+\varepsilon)M_1)^{\frac{1}{2}})$ and let $\psi : X \to \mathbb{R}$ be a smooth 'cut-off' function with the properties that

$$\psi(x) = 1, \quad |||x||| \leqq M_2,$$
$$\psi(x) = 0, \quad |||x||| \geqq M_2 + 1.$$

The new functional $\tilde{J} : X \to \mathbb{R}$ is defined by the formula

$$\tilde{J}(\Phi_1) = \tilde{J}_2(\Phi_1) + \tilde{J}_3(\Phi_1) + \tilde{J}_4(\Phi_1),$$

where

$$\tilde{J}_2(\Phi_1) = J_2(\Phi_1), \quad \tilde{J}_3(\Phi_1) = J_3(\Phi_1), \quad \tilde{J}_4(\Phi_1) = \psi(\Phi_1) J_4(\Phi_1).$$

Because $\tilde{J}$ coincides with J on $\bar{B}_{M_2}(0) \subset X$, one concludes that 0 is a strict local minimum of $\tilde{J}$ and that $\tilde{J}(\lambda^\star \Phi_1^\star) < 0$. The functional $\tilde{J}$ therefore has a mountain-pass structure, and Lemma 9 implies the existence of a Palais–Smale sequence $\{\Phi_{1m}\} \subset X$ such that $\tilde{J}(\Phi_{1m}) \to a_\varepsilon$, $\tilde{J}'(\Phi_{1m}) \to 0$ as $m \to \infty$, where

$$a_\varepsilon = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \tilde{J}(\gamma(s)), \quad \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \tilde{J}(\gamma(1)) < 0\}.$$

(Here, and in the remainder of this section, we attach the subscript ε to certain quantities as a reminder of their ε-dependence.)

The functional $\tilde{J}$ clearly satisfies the same identities as J, namely

$$\tilde{J}(\Phi_1) = \frac{1}{3}\tilde{J}_2(\Phi_1) + \tilde{J}_4(\Phi_1) - \frac{1}{3}\langle\!\langle\!\langle \tilde{J}_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle + \frac{1}{3}\langle\!\langle\!\langle \tilde{J}'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle, \tag{92}$$
$$\tilde{J}_2(\Phi_1) = 3\tilde{J}(\Phi_1) - 3\tilde{J}_4(\Phi_1) + \langle\!\langle\!\langle \tilde{J}_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle \tilde{J}'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle; \tag{93}$$

we now use these identities to establish some bounds for a and the Palais–Smale sequence $\{\Phi_{1m}\}$ which are needed later.

Proposition 16.

(i) *The constant a_ε satisfies $0 < a_\varepsilon \leqq M_1$.*
(ii) *There exists a positive constant C_ε such that $|||\Phi_{1m}||| \geqq C_\varepsilon$ for all $m \in \mathbb{N}$.*
(iii) *The Palais–Smale sequence $\{\Phi_{1m}\}$ satisfies $|||\Phi_{1m}||| \leqq M_2$ for all sufficiently large values of m.*

Proof. (i) The positivity of a_ε follows from the fact that 0 is a strict local minimum of $\tilde{J}$, while the upper bound for a_ε follows from the calculation

$$\begin{aligned} a_\varepsilon &= \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} \tilde{J}(\gamma(s)) \\ &\leqq \max_{s\in[0,1]} \tilde{J}(s\lambda^\star\Phi_1^\star) \\ &\leqq \sup\{\tilde{J}(\Phi_1) : |||\Phi_1||| \leqq 2|||\lambda^\star\Phi_1^\star|||\} \\ &= M_1. \end{aligned}$$

(ii) Suppose that there were no positive lower bound for $|||\Phi_{1m}|||$. It would be possible to extract a subsequence (still denoted by $\{\Phi_{1m}\}$) such that $|||\Phi_{1m}||| \to 0$ and hence $J(\Phi_{1m}) \to 0$ as $m \to \infty$, which would imply that $a_\varepsilon = 0$ and contradict part (i).

(iii) The first step is to show that $|||\Phi_{1m}|||$ is bounded above (without loss of generality one may assume that any upper bound is independent of ε). Suppose that there were no upper bound for $|||\Phi_{1m}|||$. It would be possible to extract a subsequence (still denoted by $\{\Phi_{1m}\}$) such that $|||\Phi_{1m}||| \to \infty$ as $m \to \infty$; in particular $|||\Phi_{1m}||| \geqq M_2+1$ for all sufficiently large values of m, so that $\tilde{J}_4(\Phi_{1m}) = 0$ and

$$\tilde{J}_2(\Phi_{1m}) = 3\tilde{J}(\Phi_{1m}) - \langle\!\langle\!\langle \tilde{J}'(\Phi_{1m}), \Phi_{1m}\rangle\!\rangle\!\rangle$$

(see equation (93)). It would follow that

$$\frac{1}{2(1+\varepsilon)}|||\Phi_{1m}|||^2 \leqq 3|\tilde{J}(\Phi_{1m})| + |||\tilde{J}'(\Phi_{1m})|||\,|||\Phi_{1m}|||$$

and hence that

$$\frac{1}{2(1+\varepsilon)} \leqq \frac{3|\tilde{J}(\Phi_{1m})|}{|||\Phi_{1m}|||^2} + \frac{|||\tilde{J}'(\Phi_{1m})|||}{|||\Phi_{1m}|||};$$

this inequality is a contradiction since its right-hand side tends to zero as $m \to \infty$.

The specific upper bound stated in the proposition is obtained using the fact that $|||\Phi_{1m}|||$ is bounded above. Observe that

$$\begin{aligned} &|\tilde{J}(\Phi_{1m})| \\ &\quad\geqq \frac{1}{3}|\tilde{J}_2(\Phi_{1m})| - |\tilde{J}_4(\Phi_{1m})| - \frac{1}{3}|\langle\!\langle\!\langle \tilde{J}_4'(\Phi_{1m}), \Phi_{1m}\rangle\!\rangle\!\rangle| - \frac{1}{3}|\langle\!\langle\!\langle \tilde{J}'(\Phi_{1m}), \Phi_{1m}\rangle\!\rangle\!\rangle| \end{aligned}$$

(see equation (92)) and

$$\begin{aligned} \langle\!\langle\!\langle \tilde{J}_4'(\Phi_{1m}), \Phi_{1m}\rangle\!\rangle\!\rangle &= \psi'(\Phi_{1m})\langle\!\langle\!\langle J_4(\Phi_{1m}), \Phi_{1m}\rangle\!\rangle\!\rangle + \psi(\Phi_{1m})\langle\!\langle\!\langle J_4'(\Phi_{1m}), \Phi_{1m}\rangle\!\rangle\!\rangle \\ &\leqq c(\psi(\Phi_{1m}) + \psi'(\Phi_{1m}))\varepsilon^{\frac{1}{4}-\Delta}P_4(|||\Phi_{1m}|||) \\ &\leqq c\varepsilon^{\frac{1}{4}-\Delta}P_4(|||\Phi_{1m}|||). \end{aligned}$$

Substituting the second inequality into the first, we find that

$$\begin{aligned} |\tilde{J}(\Phi_{1m})| &\geqq \frac{1}{6(1+\varepsilon)}(1 - c\varepsilon^{\frac{1}{4}-\Delta}P_4(|||\Phi_{1m}|||))|||\Phi_{1m}|||^2 - \frac{1}{3}|||\tilde{J}'(\Phi_{1m})|||\,|||\Phi_{1m}||| \\ &\geqq \frac{1}{12(1+\varepsilon)}|||\Phi_{1m}|||^2 - \frac{1}{3}|||\tilde{J}'(\Phi_{1m})|||\,|||\Phi_{1m}||| \end{aligned}$$

(because $|||\Phi_{1m}|||$ is bounded). The left-hand side of this expression approaches a_ε as $m \to \infty$ while the second term on its right-hand side vanishes as $m \to \infty$ (because $\tilde{J}'(\Phi_{1m}) \to 0$ and $|||\Phi_{1m}|||$ is bounded); we conclude that

$$|||\Phi_{1m}|||^2 \leqq 24(1+\varepsilon)a_\varepsilon \leqq 24(1+\varepsilon)M_1 \leqq M_2^2$$

for sufficiently large values of m. □

Proposition 16(iii) implies that $\tilde{J}(\Phi_{1m}) = J(\Phi_{1m})$ for sufficiently large values of m; hence, by extracting a subsequence if necessary, one may assume that $\{\Phi_{1m}\}$ is a Palais–Smale sequence for the original functional J, so that $J(\Phi_{1m}) \to a_\varepsilon$ and $J'(\Phi_{1m}) \to 0$ as $m \to \infty$. In the following discussion we therefore return to the original functional $J : \bar{B}_M(0) \subset X \to \mathbb{R}$.

Let us now turn to the second element of the variational theory, namely the concentration-compactness principle [28, 29].

Theorem 6. *Any sequence $\{u_m\} \subset L^1(\mathbb{R}^2)$ of non-negative functions with the property that*

$$\lim_{m\to\infty} \int_{\mathbb{R}^2} u_m(x,z)\,\mathrm{d}x\,\mathrm{d}z = \ell > 0$$

contains a subsequence for which one of the following phenomena occurs.

Vanishing: *For each $R > 0$ one has that*

$$\lim_{m\to\infty}\left(\sup_{(\tilde{x},\tilde{z})\in\mathbb{R}^2} \int_{B_R(\tilde{x},\tilde{z})} u_m(x,z)\,\mathrm{d}x\,\mathrm{d}z\right) = 0.$$

Concentration: *There is a sequence $\{(x_m, z_m)\} \subset \mathbb{R}^2$ with the property that for each $\tilde{\varepsilon} > 0$ there exists a positive real number R with*

$$\int_{B_R(0,0)} u_m(x+x_m, z+z_m)\,\mathrm{d}x\,\mathrm{d}z \geqq \ell - \tilde{\varepsilon}$$

for each $m \in \mathbb{N}$.

Dichotomy: *There are sequences $\{(x_m, z_m)\} \subset \mathbb{R}^2$, $\{R_m\}, \{S_m\} \subset \mathbb{R}$ and a real number $\lambda \in (0, \ell)$ with the properties that $R_m, S_m \to \infty$, $R_m/S_m \to 0$,*

$$\int_{B_{R_m}(0,0)} u_m(x+x_m, z+z_m)\,\mathrm{d}x\,\mathrm{d}z \to \lambda,$$

$$\int_{B_{S_m}(0,0)} u_m(x+x_m, z+z_m)\,\mathrm{d}x\,\mathrm{d}z \to \lambda,$$

as $m \to \infty$. Furthermore, for each $\tilde{\varepsilon} > 0$ there is a positive, real number R such that

$$\int_{B_R(0,0)} u_m(x+x_m, z+z_m)\,\mathrm{d}x\,\mathrm{d}z \geqq \lambda - \tilde{\varepsilon}$$

for each $m \in \mathbb{N}$.

It follows from Proposition 16(ii), (iii) that a subsequence of our Palais–Smale sequence (still denoted by $\{\Phi_{1m}\}$) satisfies $|||\Phi_{1m}|||^2 \to \ell_\varepsilon$ as $m \to \infty$, where $\ell_\varepsilon \neq 0$. This observation suggests exploring the convergence properties of $\{\Phi_{1m}\}$ by applying Theorem 6 to the sequence $\{u_m\}$ defined by

$$\begin{aligned} u_m = {} & c_0(\varepsilon\Phi_{1mxxx}^2 + 3\varepsilon^2\Phi_{1mxxz}^2 + 3\varepsilon^3\Phi_{1mxzz}^2 + \varepsilon^4\Phi_{1mzzz}^2) \\ & +(\beta - \tfrac{1}{3})(\Phi_{1mxx}^2 + 2\varepsilon\Phi_{1mxz}^2 + \varepsilon^2\Phi_{1mzz}^2) + \Phi_{1mx}^2 + (1+\varepsilon)\Phi_{1mz}^2. \end{aligned}$$

The consequences of 'vanishing', 'concentration' and 'dichotomy' are investigated in turn below, where $\{u_m\}$ is replaced by the subsequence identified by the relevant clause in Theorem 6 and we use the notation given there, writing $\ell_\varepsilon, \lambda_\varepsilon$ as a reminder of the ε-dependence of these quantities. Lemma 10 states that 'vanishing' does not occur, while Lemma 11 asserts that 'concentration' leads to the weak convergence of $\{\Phi_{1m}\}$ to a nonzero critical point of J. The discussion of 'dichotomy' is more involved and requires several steps, the conclusion of which is again the existence of a nonzero critical point of J.

Lemma 10. *The sequence $\{u_m\}$ does not have the 'vanishing' property.*

Proof. This result is proved by contradiction. Suppose that $\{u_m\}$ has the vanishing property. Proposition 2(ii) implies that

$$\|\Phi_{1m}\|^3_{U_0^3(B_1(\tilde{x},\tilde{z}))} \leqq c_\varepsilon |||\Phi_{1m}|||^3_{B_1(\tilde{x},\tilde{z})},$$

so that

$$\int_{B_1(\tilde{x},\tilde{z})} (|\Phi_{1x}|^3 + \varepsilon^{\frac{1}{2}}|\Phi_{1z}|^3)\,\mathrm{d}x\,\mathrm{d}z \leqq c_\varepsilon \left(\sup_{(\tilde{x},\tilde{z})\in\mathbb{R}^2} \int_{B_1(\tilde{x},\tilde{z})} u_m \,\mathrm{d}x\,\mathrm{d}z \right)^{\frac{1}{2}} \int_{B_1(\tilde{x},\tilde{z})} u_m \,\mathrm{d}x\,\mathrm{d}z$$

for each $(\tilde{x},\tilde{z}) \in \mathbb{R}^2$. Cover $\mathbb{R}^2$ by unit balls in such a fashion that each point of $\mathbb{R}^2$ is contained in at most three balls. Summing over all the balls, we find that

$$\begin{aligned} \|\Phi_{1m}\|^3_{U_\varepsilon^{0,3}} & \leqq c_\varepsilon \left(\sup_{(\tilde{x},\tilde{z})\in\mathbb{R}^2} \int_{B_1(\tilde{x},\tilde{z})} u_m \,\mathrm{d}x\,\mathrm{d}z \right)^{\frac{1}{2}} |||\Phi_{1m}|||^2 \\ & \leqq c_\varepsilon \left(\sup_{(\tilde{x},\tilde{z})\in\mathbb{R}^2} \int_{B_1(\tilde{x},\tilde{z})} u_m \,\mathrm{d}x\,\mathrm{d}z \right)^{\frac{1}{2}} \\ & \to 0 \end{aligned}$$

and hence $J_3(\Phi_{1m}) \to 0$ as $m \to \infty$ (see equation (88)).

On the other hand it follows from (86) that

$$\begin{aligned} |J_3(\Phi_{1m})| & \geqq \frac{2}{3}|J_2(\Phi_{1m})| - \frac{1}{3}|\langle\!\langle J'(\Phi_{1m}), \Phi_{1m}\rangle\!\rangle| - \frac{1}{3}|\langle\!\langle J_4'(\Phi_{1m}), \Phi_{1m}\rangle\!\rangle| \\ & \geqq \frac{1}{6}|||\Phi_{1m}|||^2(1 - c\varepsilon^{\frac{1}{4}-\Delta}P_2(M_2)) + o(1) \\ & \geqq \frac{1}{12}|||\Phi_{1m}|||^2 + o(1) \\ & \geqq \frac{1}{12}C_\varepsilon^2 + o(1) \end{aligned}$$

as $m \to \infty$, and this inequality clearly contradicts our previous conclusion that $J_3(\Phi_{1m}) \to 0$ as $m \to \infty$. $\square$

Lemma 11. *Suppose that* $\{u_m\}$ *has the 'concentration' property. The sequence* $\{\Phi_{1m}(x_m + \cdot, z_m + \cdot)\}$ *converges weakly to a nonzero critical point of* J.

Proof. With a slight abuse of notation, let us abbreviate $\{\Phi_{1m}(x_m + \cdot, z_m + \cdot)\}$ to $\{\Phi_{1m}\}$. Clearly $|||\Phi_{1m}|||^2 \to \ell_\varepsilon$ as $m \to \infty$, so that $\{\Phi_{1m}\}$ admits a subsequence (still denoted by $\{\Phi_{1m}\}$) which is weakly convergent in X; here we denote its weak limit by Φ_1 and confirm that $\Phi_1 \neq 0$, $J'(\Phi_1) = 0$.

The first step is to show that $\Phi_{1m} \to \Phi_1$ in $U_\varepsilon^{\delta,p}(\mathbb{R}^2) \cap U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)$. Choose $\tilde{\varepsilon} > 0$. The 'concentration' property asserts the existence of $R > 0$ such that

$$|||\Phi_{1m}|||_{\{|(x,z)| \geqq R\}} < \tilde{\varepsilon}$$

for each $m \in \mathbb{N}$. By replacing R with a larger number if necessary we also have that

$$|||\Phi_1|||_{\{|(x,z)| \geqq R\}} < \tilde{\varepsilon}$$

because Φ_1 is an element of X. It follows from the continuity of the embedding $X_{\{|(x,z)| \geqq R\}} \subset U_\varepsilon^{\delta,p}(\{|(x,z)| \geqq R\})$ that

$$\begin{aligned}\|\Phi_{1m} - \Phi_1\|_{U_\varepsilon^{\delta,p}(\{|(x,z)| \geqq R\})} &\leqq c_\varepsilon |||\Phi_{1m} - \Phi_1|||_{\{|(x,z)| \geqq R\}} \\ &\leqq c_\varepsilon |||\Phi_{1m}|||_{\{|(x,z)| \geqq R\}} + c_\varepsilon |||\Phi_1|||_{\{|(x,z)| \geqq R\}} \\ &\leqq c_\varepsilon \tilde{\varepsilon}\end{aligned}$$

for each $m \in \mathbb{N}$. (Here, and in the remainder of this paper, the symbol c_ε is used to denote a general positive constant which may depend upon ε.) Furthermore, since $X_{B_R(0,0)}$ is compactly embedded in $U_\varepsilon^{\delta,p}(B_R(0,0))$ and $\Phi_{1m} \rightharpoonup \Phi_1$ in $X_{B_R(0,0)}$, one has that $\Phi_{1m} \to \Phi_1$ in $U_\varepsilon^{\delta,p}(B_R(0,0))$; the inequality

$$\|\Phi_{1m} - \Phi_1\|_{U_\varepsilon^{\delta,p}(B_R(0,0))} \leqq \tilde{\varepsilon}$$

therefore holds for all sufficiently large values of m. The previous two inequalities assert that

$$\|\Phi_{1m} - \Phi_1\|_{\delta,p,\varepsilon} \leqq c_\varepsilon \tilde{\varepsilon}$$

for all sufficiently large values of m, so that $\Phi_{1m} \to \Phi_1$ in $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, and a similar argument yields the strong convergence in $U_\varepsilon^{0,2}(\mathbb{R}^2)$ and $U_\varepsilon^{0,4}(\mathbb{R}^2)$ (and in fact in any Sobolev space which is locally compactly embedded in X).

It follows from the strong convergence of $\{\Phi_{1m}\}$ in $U_\varepsilon^{\delta,p}(\mathbb{R}^2) \cap U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)$ and the fact that J_3, J_4 are continuous functionals on (a sufficiently large neighbourhood of the origin in) this space that

$$\begin{aligned}J_3(\Phi_{1m}) &\to J_3(\Phi_1), \quad J_4(\Phi_{1m}) \to J_4(\Phi_1), \\ J_3'(\Phi_{1m}) &\to J_3'(\Phi_1), \quad J_4'(\Phi_{1m}) \to J_4'(\Phi_1)\end{aligned}$$

as $m \to \infty$, and noting that

$$\langle\!\langle\!\langle \Phi_{1m}, \Psi_1 \rangle\!\rangle\!\rangle \to \langle\!\langle\!\langle \Phi_1, \Psi_1 \rangle\!\rangle\!\rangle$$

as $m \to \infty$ for each fixed $\Psi_1 \in X$ (by the definition of weak convergence), we find that

$$\begin{aligned} \langle\!\langle\!\langle J'(\Phi_{1m}), \Psi_1 \rangle\!\rangle\!\rangle &= \frac{1}{1+\varepsilon} \langle\!\langle\!\langle \Phi_{1m}, \Psi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_3'(\Phi_{1m}), \Psi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_4'(\Phi_{1m}), \Psi_1 \rangle\!\rangle\!\rangle \\ &\to \frac{1}{1+\varepsilon} \langle\!\langle\!\langle \Phi_1, \Psi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_3'(\Phi_1), \Psi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_4'(\Phi_1), \Psi_1 \rangle\!\rangle\!\rangle \\ &= \langle\!\langle\!\langle J'(\Phi_1), \Psi_1 \rangle\!\rangle\!\rangle \end{aligned}$$

as $m \to \infty$. On the other hand one has that

$$\langle\!\langle\!\langle J'(\Phi_{1m}), \Psi_1 \rangle\!\rangle\!\rangle \to 0$$

as $m \to \infty$, and it follows from the uniqueness of limits that

$$\langle\!\langle\!\langle J'(\Phi_1), \Psi_1 \rangle\!\rangle\!\rangle = 0.$$

We conclude that $J'(\Phi_1) = 0$ since this equation holds for every $\Psi_1 \in X$.

It remains to confirm that $\Phi_1 \neq 0$. Notice that

$$\langle\!\langle\!\langle J_2'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle = -\langle\!\langle\!\langle J_3'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle,$$

from which it follows that

$$\begin{aligned} \frac{1}{1+\varepsilon} |||\Phi_1|||^2 &= -3J_3(\Phi_1) - \langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle \\ &\leqq 3|J_3(\Phi_1)| + |\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle| \end{aligned}$$

and hence that

$$\frac{1}{1+\varepsilon} \leqq c \left(|||\Phi_1||| + c\varepsilon^{\frac{1}{4}-\Delta} \frac{P_4(|||\Phi_1|||)}{|||\Phi_1|||^2} \right)$$

(see equations (88), (90)); the right-hand side of this equation would vanish for $\Phi_1 = 0$ and contradict the positivity of its left-hand side. □

We now examine the remaining case ('dichotomy'), again abbreviating $\{u_m(x_m + \cdot, z_m + \cdot)\}$ and $\{\Phi_{1m}(x_m + \cdot, z_m + \cdot)\}$ to, respectively, $\{u_m\}$ and $\{\Phi_{1m}\}$. We begin by recalling an argument due to GROVES [16, Section 3.3] which shows that this scenario also leads to the existence of a nonzero critical point of J; it relies upon a convergence result (equation (94) below) whose proof in the current situation is complicated by the presence of nonlocal terms in J.

Let $\{\chi_m\} \subset C_0^\infty(\mathbb{R}^2, \mathbb{R})$ be a sequence of 'cut-off' functions with the properties that

$$\begin{array}{ll} \chi_m(x, z) = 1, & |(x, z)| \leqq R_m, \\ 0 < \chi_m(x, z) < 1, & R_m < |(x, z)| < S_m, \\ \chi_m(x, z) = 0, & |(x, z)| \geqq S_m \end{array}$$

and $|\chi_m'(x,z)|$, $|\chi_m''(x,z)| \leqq c$ for each $m \in \mathbb{N}$ and each $(x,z) \in \mathbb{R}^2$. (The existence of a sequence $\{\chi_m\}$ with these properties is assured by the facts that R_m, S_m, $S_m - R_m \to \infty$ as $m \to \infty$.) Define sequences $\{\Phi_{1m}^{(1)}\}$, $\{\Phi_{1m}^{(2)}\}$ and $\{u_m^{(1)}\}$ by the formulae

$$\Phi_{1m}^{(1)} = \Phi_{1m}\chi_m, \quad \Phi_{1m}^{(2)} = \Phi_{1m}(1-\chi_m)$$

and

$$\begin{aligned} u_m^{(1)} = c_0 \Big(&\varepsilon(\Phi_{1mxxx}^{(1)})^2 + 3\varepsilon^2(\Phi_{1mxxz}^{(1)})^2 + 3\varepsilon^3(\Phi_{1mxzz}^{(1)})^2 + \varepsilon^4(\Phi_{1mzzz}^{(1)})^2\Big) \\ &+(\beta - \tfrac{1}{3})\Big((\Phi_{1mxx}^{(1)})^2 + 2\varepsilon(\Phi_{1mxz}^{(1)})^2 + \varepsilon^2(\Phi_{1mzz}^{(1)})^2\Big) \\ &+(\Phi_{1mx}^{(1)})^2 + (1+\varepsilon)(\Phi_{1mz}^{(1)})^2. \end{aligned}$$

The method described by Groves [16, Proposition 12 and Lemma 14] shows that

$$|||\Phi_{1m}^{(1)}|||^2 \to \lambda_\varepsilon, \quad |||\Phi_{1m}^{(2)}|||^2 \to \ell_\varepsilon - \lambda_\varepsilon$$

as $m \to \infty$, that there are positive constants $C_\varepsilon^{(1)}$, $C_\varepsilon^{(2)}$ such that

$$|||\Phi_{1m}^{(1)}||| \geqq C_\varepsilon^{(1)}, \quad |||\Phi_{1m}^{(2)}||| \geqq C_\varepsilon^{(2)}$$

for all $m \in \mathbb{N}$, that $|||\Phi_{1m}^{(1)}|||$ and $|||\Phi_{1m}^{(2)}|||$ are bounded above (by replacing M_2 with a larger number if necessary we may assume that the upper bounds do not exceed M_2) and that $\{u_m^{(1)}\}$ has the 'concentration' property: for each $\tilde{\varepsilon} > 0$ there exists a positive number R such that

$$\int_{B_R(0,0)} u_m(x,z)\,\mathrm{d}x\,\mathrm{d}z \geqq \lambda_\varepsilon - \tilde{\varepsilon}$$

for each $m \in \mathbb{N}$. Suppose that

$$\langle\langle\langle J'(\Phi_{1m}^{(1)}), \Psi_1 \rangle\rangle\rangle \to 0 \tag{94}$$

as $m \to \infty$ for each $\Psi_1 \in X$; repeating the argument used in the proof of Lemma 11, we find that the weak limit $\Phi_1^{(1)}$ of $\{\Phi_{1m}^{(1)}\}$ in X is a nonzero critical point of J.

It therefore remains to establish the limit (94). This task is accomplished by showing that

$$\langle\langle\langle J'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1 \rangle\rangle\rangle - \langle\langle\langle J'(\Phi_{1m}^{(1)}), \Psi_1 \rangle\rangle\rangle \to 0 \tag{95}$$

as $m \to \infty$ for each $\Psi_1 \in C_0^\infty(\mathbb{R}^2)$ (and hence, by a density argument, for each $\Psi_1 \in X$); the desired result follows from this limit together with the fact that

$$\langle\langle\langle J'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1 \rangle\rangle\rangle = \langle\langle\langle J'(\Phi_{1m}), \Psi_1 \rangle\rangle\rangle \to 0$$

as $m \to \infty$. It is a straightforward matter to show that $⟪J_2'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1⟫ - ⟪J_2'(\Phi_{1m}^{(1)}), \Psi_1⟫$ vanishes as $m \to \infty$. Observe that

$$\begin{aligned} &⟪J_2'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1⟫ - ⟪J_2'(\Phi_{1m}^{(1)}), \Psi_1⟫ \\ &= \frac{1}{1+\varepsilon}⟪\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}, \Psi_1⟫ - \frac{1}{1+\varepsilon}⟪\Phi_{1m}^{(1)}, \Psi_1⟫ \\ &= \frac{1}{1+\varepsilon}⟪\Phi_{1m}^{(2)}, \Psi_1⟫, \end{aligned}$$

and since the integrand in the formula for $⟪\Phi_{1m}^{(2)}, \Psi_1⟫$ is calculated by pointwise multiplication of derivatives of Φ_{1m} by derivatives of Ψ_1, we find that $⟪\Phi_{1m}^{(2)}, \Psi_1⟫$ vanishes whenever R_m exceeds the radius of support of Ψ_1, so that in particular the above expression vanishes as $m \to \infty$. The same argument shows that $⟪(J_3 + J_4)'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1⟫ - ⟪(J_3 + J_4)'(\Phi_{1m}^{(1)}), \Psi_1⟫ \to 0$ as $m \to \infty$ provided that the integrand defining $⟪J'(\Phi_1), \Psi_1⟫$ contains only *local* operations with respect to (x, z), that is differentiation, integration with respect to y, pointwise addition and pointwise multiplication. The presence of the functional relationships $\rho = \rho(\Phi_1)$, $\Phi_2 = \Phi_2(\Phi_1)$ however means that *nonlocal* effects also have to be taken into account and the simple argument given above no longer suffices.

The functional relationships $\rho = \rho(\Phi_1)$, $\Phi_2 = \Phi_2(\Phi_1)$ are constructed using the basic Fourier-multiplier operators $\mathcal{G}_i$ described in Lemmata 2 and 4. The next result asserts that these operators, although nonlocal, enjoy a particular property of local operators, namely that $\|\Psi_1\mathcal{G}_i(\Phi_{1m}^{(2)})\|_{1+\delta,p,\varepsilon} \to 0$ as $m \to \infty$ for each $\Psi_1 \in C_0^\infty(\mathbb{R}^2)$; its proof is deferred to Section 5. (The corresponding estimate

$$\|\mathcal{G}_7^{N,m}(u)\|_{\delta,p,\varepsilon} \leqq c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}$$

for $\mathcal{G}_7^{N,m}(u) = \chi_N\mathcal{G}_7((1 - \chi_{R_m})u)$ is an immediate consequence of Lemma 12 because $\mathcal{G}_7^{N,m} = \partial_y\mathcal{G}_6^{N,m}$; since the size of the operator norm in terms of ε is not at issue it is however unnecessary to consider $\mathcal{G}^7 = \partial_7\mathcal{G}_6$ as a separate entity.)

Lemma 12. *Choose $N > 0$, suppose that $\{R_m\}$ is a sequence of positive, real numbers such that $R_m \to \infty$ as $m \to \infty$ and let $\chi_N : \mathbb{R}^2 \to \mathbb{R}$, $\chi_{R_m} : \mathbb{R}^2 \to \mathbb{R}$ be smooth 'cut-off' functions whose support is contained in respectively $\bar{B}_N(0)$ and $\bar{B}_{R_m}(0)$. The functions*

$$\mathcal{G}_i^{N,m}(u) = \chi_N\mathcal{G}_i((1 - \chi_{R_m})u), \quad i = 1, \dots, 6, 8, \dots, 11$$

satisfy

$$\|\mathcal{G}_i^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leqq c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}$$

for each $\delta \in [0, 1]$ and each sufficiently large value of p, in which the symbol $c_\varepsilon^{N,m}$ denotes a quantity that, for each fixed value of N and ε, tends to zero as $m \to \infty$.

Our final result shows that the 'local' property of the basic Fourier-multiplier operators described in Lemma 12 is sufficient to guarantee the asymptotic behaviour (95) required of J.

Theorem 7. *One has that*

$$\langle\!\langle\!\langle (J_3+J_4)'(\Phi_{1m}^{(1)}+\Phi_{1m}^{(2)}),\Psi_1\rangle\!\rangle\!\rangle - \langle\!\langle\!\langle (J_3+J_4)'(\Phi_{1m}^{(1)}),\Psi_1\rangle\!\rangle\!\rangle \to 0$$

as $m\to\infty$ *for each* $\Psi_1 \in C_0^\infty(\mathbb{R}^2)$.

Proof. Recall that $\rho(\Phi_1)$ and $\Phi_2(\Phi_1)$ are constructed by solving fixed-point problems using the contraction-mapping principle, in other words using an iteration scheme. The key to proving this theorem is to approximate $\rho(\Phi_1)$ and $\Phi_2(\Phi_1)$ by the result of a finite number of iterations in the scheme. Let us therefore begin by reviewing the four main steps in the construction of $\rho(\Phi_1)$ and $\Phi_2(\Phi_1)$. In the entirety of the discussion ρ, Ψ, Φ_1 and Φ_2 are supposed to lie in origin-centred balls of respective radii $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$ in $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\mathcal{O}(\varepsilon^{\frac{1}{2}-\Delta})$ in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$ in $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and $\mathcal{O}(\varepsilon^{-\Delta})$ in $W_\varepsilon^{1+\delta,p}(\Sigma)$; all estimates hold uniformly in and suprema are taken over these sets.

(i) One solves a fixed-point problem of the form

$$\rho = \sum_i \mathcal{G}_i\mathcal{N}_i(\Phi_1,\Phi_2,\Psi,\rho)$$

in $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, in which $\mathcal{G}_i : W_\varepsilon^{\delta,p}(\mathbb{R}^2)\to V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ is a Fourier-multiplier operator and $\mathcal{N}_i : U_\varepsilon^{\delta,p}(\mathbb{R}^2)\times W_\varepsilon^{1+\delta,p}(\Sigma)\times W_\varepsilon^{\delta,p}(\Sigma)\times V_\varepsilon^{\delta,p}(\mathbb{R}^2)\to W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ is a 'local' nonlinear function (that is, a function of its arguments that involves only differentiation, integration with respect to y, pointwise addition and pointwise multiplication). This fixed-point problem is solved using the iteration scheme

$$\rho_0 = \sum_i \mathcal{G}_i\mathcal{N}_i(\Phi_1,\Phi_2,\Psi,0),$$

$$\rho_{n+1} = \sum_i \mathcal{G}_i\mathcal{N}_i(\Phi_1,\Phi_2,\Psi,\rho_n), \quad n=1,2,\ldots,$$

which converges uniformly in Φ_1, Φ_2, Ψ to the unique solution $\rho_\infty(\Phi_1,\Phi_2,\Psi)$. There are estimates for ρ_∞ and its derivatives in terms of Φ_1, Φ_2 and Ψ (see Theorem 3).

(ii) One solves a fixed-point problem of the form

$$\Psi = \sum_i \mathcal{G}_i\mathcal{N}_i(\Phi_1,\Phi_2,\Psi,\rho_\infty(\Phi_1,\Phi_2,\Psi))$$

in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$, in which $\mathcal{G}_i : W_\varepsilon^{\delta,p}(\Sigma)\to W_\varepsilon^{\delta,p}(\Sigma)$ is a Fourier-multiplier operator and $\mathcal{N}_i : U_\varepsilon^{\delta,p}(\mathbb{R}^2)\times W_\varepsilon^{1+\delta,p}(\Sigma)\times W_\varepsilon^{\delta,p}(\Sigma)\times V_\varepsilon^{\delta,p}(\mathbb{R}^2)\to W_\varepsilon^{\delta,p}(\Sigma)$ is a 'local' nonlinear function. This fixed-point problem is solved using the iteration scheme

$$\Psi_0 = \sum_i \mathcal{G}_i\mathcal{N}_i(\Phi_1,\Phi_2,0,\rho_\infty(\Phi_1,\Phi_2,0)),$$

$$\Psi_{n+1} = \sum_i \mathcal{G}_i\mathcal{N}_i(\Phi_1,\Phi_2,\Psi_n,\rho_\infty(\Phi_1,\Phi_2,\Psi_n)), \quad n=1,2,\ldots,$$

which converges uniformly in Φ_1, Φ_2 to the unique solution $\Psi_\infty(\Phi_1, \Phi_2)$. There are estimates for Ψ_∞ and its derivatives in terms of Φ_1 and Φ_2 (see Theorem 4).

(iii) One solves a fixed-point problem of the form

$$\Phi_2 = \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, \Psi_\infty(\Phi_1, \Phi_2), \rho_\infty(\Phi_1, \Phi_2, \Psi_\infty(\Phi_1, \Phi_2)))$$

in $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$, in which $\mathcal{G}_i : W_\varepsilon^{\delta,p}(\Sigma) \to W_\varepsilon^{1+\delta,p}(\Sigma)$ is a Fourier-multiplier operator and $\mathcal{N}_i : U_\varepsilon^{\delta,p}(\mathbb{R}^2) \times W_\varepsilon^{1+\delta,p}(\Sigma) \times W_\varepsilon^{\delta,p}(\Sigma) \times V_\varepsilon^{\delta,p}(\mathbb{R}^2) \to W_\varepsilon^{\delta,p}(\Sigma)$ is a 'local' nonlinear function. This fixed-point problem is solved using the iteration scheme

$$\Phi_{2,0} = \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, 0, \Psi_\infty(\Phi_1, 0), \rho_\infty(\Phi_1, 0, \Psi_\infty(\Phi_1, 0)))$$

$$\Phi_{2,n+1} = \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_{2,n}, \Psi_\infty(\Phi_1, \Phi_{2,n}), \rho_\infty(\Phi_1, \Phi_{2,n}, \Psi_\infty(\Phi_1, \Phi_{2,n}))), \qquad n = 1, 2, \ldots,$$

which converges uniformly in Φ_1 to the unique solution $\Phi_{2,\infty}(\Phi_1)$. There are estimates for $\Phi_{2,\infty}$ and its derivatives in terms of Φ_1 (see Theorem 5).

(iv) A supplementary argument shows that $\Psi_n = \partial_y \Phi_{2,n}$ for each $n \in \mathbb{N}$ and $\Psi_\infty = \partial_y \Phi_{2,\infty}$.

Choose $\tilde{\varepsilon} > 0$. It follows from the uniform convergence described in step (i) that

$$|\rho_\infty(\Phi_1, \Phi_2, \Psi) - \rho_n(\Phi_1, \Phi_2, \Psi)|_{\delta,p,\varepsilon} \leqq \tilde{\varepsilon} \tag{96}$$

for all sufficiently large values of n, where ρ_n satisfies the same estimates as ρ_∞. Next consider the fixed-point problem

$$\Psi = \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, \Psi, \rho_n(\Phi_1, \Phi_2, \Psi))$$

obtained by replacing ρ_∞ with ρ_n in step (ii). Applying the iteration scheme described there to this modified fixed-point problem, we obtain a solution $\tilde{\Psi}_\infty$ which satisfies the same estimates as Ψ_∞, and the argument used above for ρ shows that

$$\|\tilde{\Psi}_\infty(\Phi_1, \Phi_2) - \Psi_n(\Phi_1, \Phi_2)\|_{\delta,p,\varepsilon} \leqq \tilde{\varepsilon}$$

for all sufficiently large values of n, where Ψ_n satisfies the same estimates as Ψ_∞. Moreover, we find that

$$\begin{aligned}
&\|\Psi_\infty(\Phi_1, \Phi_2) - \tilde{\Psi}_\infty(\Phi_1, \Phi_2)\|_{\delta,p,\varepsilon} \\
&\quad \leqq \sum_i \sup \|\mathcal{G}_i \partial_4 \mathcal{N}_i(\Phi_1, \Phi_2, \Psi, \rho)\|_{\delta,p,\varepsilon} \|\rho_\infty(\Phi_1, \Phi_2, \Psi) - \rho_n(\Phi_1, \Phi_2, \Psi)\|_{\delta,p,\varepsilon} \\
&\quad \leqq c_\varepsilon \tilde{\varepsilon},
\end{aligned}$$

and it follows from the previous two estimates that

$$\|\Psi_\infty(\Phi_1, \Phi_2) - \Psi_n(\Phi_1, \Phi_2)\|_{\delta,p,\varepsilon} \leqq c_\varepsilon \tilde{\varepsilon}. \tag{97}$$

Similarly, examining the fixed-point problem

$$\Phi = \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, \Psi_n(\Phi_1, \Phi_2), \rho_n(\Phi_1, \Phi_2, \Psi_n(\Phi_1, \Phi_2)))$$

obtained by replacing ρ_∞, Ψ_∞ with ρ_n, Ψ_n in step (iii), we find that

$$\|\Phi_{2,\infty}(\Phi_1) - \Phi_{2,n}(\Phi_1)\|_{1+\delta,p,\varepsilon} \leqq c_\varepsilon \tilde{\varepsilon} \tag{98}$$

for sufficiently large values of n, where $\Phi_{2,n}$ satisfies the same estimates as $\Phi_{2,\infty}$; by construction we have that $\Psi_n = \partial_y \Phi_{2,n}$.

Let $K_n(\Phi_1, \Psi_1)$ be the functional obtained by replacing each occurrence of ρ_∞ and $\Phi_{2,\infty}$ in the integrand defining $⟪ J_{\mathrm{NL}}'(\Phi_1), \Psi_1 ⟫$ with, respectively, ρ_n and $\Phi_{2,n}$. (The $W_\varepsilon^{\delta,p}(\Sigma)$-norm of the integrand defining $K_n(\Phi_1, \Psi_1)$ is finite, and, since Ψ_1 has compact support, the same is true of its $W_\varepsilon^{\delta,p}(B_N(0))$-norm, where N denotes the radius of support of Ψ_1; its integrability follows from the embedding $W_\varepsilon^{\delta,p}(B_N(0)) \subset L^1(B_N(0))$.) It follows from (96)–(98) that the difference between the two integrands is bounded in the $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$-norm and hence in the $W_\varepsilon^{\delta,p}(B_N(0))$-norm by $c_\varepsilon \tilde{\varepsilon}$, and using the continuity of the embedding $W_\varepsilon^{\delta,p}(B_N(0)) \subset L^1(B_N(0))$, we find that

$$|K_n(\Phi_1, \Psi_1) - ⟪ J_{\mathrm{NL}}'(\Phi_1), \Psi_1 ⟫| \leqq c_\varepsilon \tilde{\varepsilon}.$$

In order to establish that

$$⟪ (J_3 + J_4)'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1 ⟫ - ⟪ (J_3 + J_4)'(\Phi_{1m}^{(1)}), \Psi_1 ⟫ \to 0$$

as $m \to \infty$ it therefore suffices to prove that

$$K_n(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}, \Psi_1) - K_n(\Phi_{1m}^{(1)}, \Psi_1) \to 0$$

as $m \to \infty$.

The integrand defining $K_n(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}, \Psi_1) - K_n(\Phi_{1m}^{(1)}, \Psi_1)$ is a finite sum, each term of which is constructed recursively as follows. A *level 0 formula* has the form

$$\mathcal{G}_i \mathcal{N}_i(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}),$$

while a *level s formula*, $s = 1, 2, \ldots$ has the form

$$\mathcal{G}_i \mathcal{N}_i(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \text{level 0 formulae, level 1 formulae, …, level } s-1 \text{ formulae});$$

here

$$\mathcal{G}_i : \left\{ \begin{array}{c} W_\varepsilon^{\delta,p}(\mathbb{R}^2) \\ W_\varepsilon^{\delta,p}(\Sigma) \end{array} \right\} \to \left\{ \begin{array}{c} W_\varepsilon^{1+\delta,p}(\mathbb{R}^2) \\ W_\varepsilon^{1+\delta,p}(\Sigma) \end{array} \right\}$$

is a Fourier-multiplier operator and

$$\mathcal{N}_i : \left\{ \begin{array}{c} W_\varepsilon^{1+\delta,p}(\mathbb{R}^2) \\ W_\varepsilon^{1+\delta,p}(\Sigma) \end{array} \right\} \times \cdots \times \left\{ \begin{array}{c} W_\varepsilon^{1+\delta,p}(\mathbb{R}^2) \\ W_\varepsilon^{1+\delta,p}(\Sigma) \end{array} \right\} \to \left\{ \begin{array}{c} W_\varepsilon^{\delta,p}(\mathbb{R}^2) \\ W_\varepsilon^{\delta,p}(\Sigma) \end{array} \right\}$$

is a 'local' nonlinear function. Each term in our integrand is the product of Ψ_1 and a level s formula for some $s \geqq 0$; the target space of its nonlinearity at level s is $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$, the Fourier-multiplier operator at level s may be replaced by the identity and $\Phi_{1m}^{(2)}$ appears in at least one nonlinearity, that is at least one nonlinearity in the recursion scheme satisfies $\mathcal{N}_i(\cdot, 0, \ldots) = 0$. We now show that each term in our integrand tends to zero in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ as $m \to \infty$ for sufficiently large values of p; by replacing $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ by $W_\varepsilon^{\delta,p}(B_N(0))$ (see above) and using the continuity of the embedding $W_\varepsilon^{\delta,p}(B_N(0)) \subset L^1(B_N(0))$, one concludes that $K_n(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}, \Psi_1) - K_n(\Phi_{1m}^{(1)}, \Psi_1) \to 0$ as $m \to \infty$.

Consider the expression

$$\Psi_1 \left\{ \begin{array}{c} \mathcal{G}_s \\ I \end{array} \right\} \mathcal{N}_s(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \text{ level 0 formulae, level 1 formulae, } \ldots, \text{ level } s-1 \text{ formulae}). \tag{99}$$

Suppose first that $\Phi_{1m}^{(2)}$ appears in the nonlinearity at level s, which therefore satisfies

$$\mathcal{N}_s(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots) = \chi_m \mathcal{N}_s(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots),$$

so that

$$\Psi_1 \mathcal{G}_s \mathcal{N}_s(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots) = \Psi_1 \mathcal{G}_s^{N,m} \mathcal{N}_s(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots).$$

It follows that

$$\begin{aligned} \|\Psi_1 \mathcal{G}_s \mathcal{N}_s(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots)\|_{\delta,p,\varepsilon} &\leqq \|\Psi_1 \mathcal{G}_s^{N,m} \mathcal{N}_s(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots)\|_{1+\delta,p,\varepsilon} \\ &\leqq c_\varepsilon^{N,m} \|\mathcal{N}_s(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots)\|_{\delta,p,\varepsilon} \\ &\leqq c_\varepsilon^{N,m} \end{aligned}$$

for sufficiently large values of p, in which Lemma 12 and the fact that all arguments of $\mathcal{N}_s$ are bounded in $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$ or $W_\varepsilon^{1+\delta,p}(\Sigma)$ have been used. (Recall that the symbol $c_\varepsilon^{N,m}$ denotes a quantity that, for each fixed value of N and ε, tends to zero as $m \to \infty$.) The same result clearly holds when $\mathcal{G}_s$ is replaced by the identity since $\Psi_1 \mathcal{N}_s(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots)$ is identically zero for sufficiently large values of m.

Next suppose that $\Phi_{1m}^{(2)}$ appears in a nonlinearity at level $s-1$, so that (99) takes the form

$$\Psi_1 \left\{ \begin{array}{c} \mathcal{G}_s \\ I \end{array} \right\} \mathcal{N}_s(\Phi_{1m}^{(1)}, \mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots), \ldots),$$

where $\mathcal{N}_{s-1}(\cdot, 0, \ldots) = 0$. The above expression is clearly equal to

$$\Psi_1 \left\{ \begin{array}{c} \mathcal{G}_s \\ I \end{array} \right\} \mathcal{N}_s(\Phi_{1m}^{(1)}, (1-\chi_{N_1})\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots), \ldots)$$
$$+\Psi_1 \left\{ \begin{array}{c} \mathcal{G}_s \\ I \end{array} \right\} \tilde{\mathcal{N}}_s(\Phi_{1m}^{(1)}, \chi_{N_1}\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots),$$
$$(1-\chi_{N_1})\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots), \ldots), \qquad (100)$$

in which

$$\begin{aligned} &\tilde{\mathcal{N}}_s(\Phi_{1m}^{(1)}, \chi_{N_1}\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots), (1-\chi_{N_1})\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots) \\ &\quad = \mathcal{N}_s(\Phi_{1m}^{(1)}, \mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots), \ldots) \\ &\qquad -\mathcal{N}_s(\Phi_{1m}^{(1)}, (1-\chi_{N_1})\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots), \ldots) \end{aligned}$$

and N_1 is any positive number greater than N; note that $\tilde{\mathcal{N}}_s(\cdot, 0, \ldots) = 0$. The previous argument shows that

$$\|\chi_{N_1}\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots)\|_{1+\delta,p,\varepsilon} \leqq c_\varepsilon^{N_1,m}$$

for sufficiently large values of p, and by continuity the second term in (100) tends to zero in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ as $m \to \infty$ for each fixed value of N_1. The previous argument also shows that

$$\left\| \Psi_1 \left\{ \begin{array}{c} \mathcal{G}_s \\ I \end{array} \right\} \mathcal{N}_s(\Phi_{1m}^{(1)}, (1-\chi_{N_1})u_m, \ldots) \right\|_{1+\delta,p,\varepsilon} \leqq c_\varepsilon^{N,N_1}$$

uniformly in m for any bounded sequence $\{u_m\}$ in $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$ or $W_\varepsilon^{1+\delta,p}(\Sigma)$, and in particular for $u_m = \mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots)$. Taking the limit $m \to \infty$ followed by the limit $N_1 \to \infty$ in (100), one concludes that this expression tends to zero in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ as $m \to \infty$.

An appearance of $\Phi_{1m}^{(2)}$ in a level $s-2$ nonlinearity is similarly handled using two new 'cut-off' functions χ_{N_1}, χ_{N_2} with $N_2 > N_1 > N$, and proceeding recursively in this fashion we find that each term in the integrand defining $K_n(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}, \Psi_1) - K_n(\Phi_{1m}^{(1)}, \Psi_1)$ tends to zero in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ as $m \to \infty$ for sufficiently large values of p. □

5. Fourier-multiplier operators

It remains to establish the results stated in Sections 3 and 4 concerning Fourier-multiplier operators, namely their mapping properties (in particular the estimates on their norms given in Lemmata 2, 4 and 5) and the convergence properties given in Lemma 12.

5.1. Basic tools

Here we present our basic tools for studying Fourier-multiplier operators in L^p-based spaces, $p \neq 2$, beginning with well-known results known as 'Marcinkiewicz's theorem' (Lemma 13) and 'Mikhlin's theorem' (Lemma 14); detailed proofs are given by STEIN [37, Chapter IV].

Lemma 13. *Consider the operator T defined by*

$$(Tu)(x,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-x_1, z-z_1)u(x_1,z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1.$$

Suppose that the kernel K satisfies $|\hat{K}| \leqq c\varepsilon^\alpha$ and

$$\sup_{j\in\mathbb{Z}} \int_{I_j} |\partial_\mu \hat{K}|\,\mathrm{d}\mu \leqq c\varepsilon^\alpha,$$
$$\sup_{j\in\mathbb{Z}} \int_{I_j} |\partial_k \hat{K}|\,\mathrm{d}k \leqq c\varepsilon^\alpha,$$
$$\sup_{j_1,j_2\in\mathbb{Z}} \int_{I_{j_1}}\int_{I_{j_2}} |\partial_\mu\partial_k \hat{K}|\,\mathrm{d}\mu\,\mathrm{d}k \leqq c\varepsilon^\alpha,$$

where I_j is the dyadic interval $(2^j, 2^{j+1})$ or $(-2^{j+1}, -2^j)$. The operator T maps $L^p(\mathbb{R}^2)$ continuously into itself and

$$\|Tu\|_p \leqq c\varepsilon^\alpha \|u\|_p.$$

Lemma 14. *Consider the operator T defined by*

$$(Tu)(x,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-x_1, z-z_1)u(x_1,z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1.$$

Suppose that the kernel K satisfies

$$|\hat{K}| \leqq c\varepsilon^\alpha,$$
$$|\partial_\mu \hat{K}| + |\partial_k \hat{K}| \leqq \frac{c\varepsilon^\alpha}{(\mu^2+k^2)^{1/2}},$$
$$|\partial_\mu^2 \hat{K}| + |\partial_\mu\partial_k \hat{K}| + |\partial_k^2 \hat{K}| \leqq \frac{c\varepsilon^\alpha}{\mu^2+k^2}$$

for each $(\mu,k) \neq (0,0)$. The operator T maps $L^p(\mathbb{R}^2)$ continuously into itself and

$$\|Tu\|_p \leqq c\varepsilon^\alpha \|u\|_p.$$

The next result is a scaled version of Lemma 14 which is useful in dealing with scaled function spaces such as $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$.

Lemma 15. *Consider the operator T defined by*

$$(Tu)(x,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-x_1, z-z_1)u(x_1,z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1.$$

Suppose that the kernel K satisfies

$$\begin{aligned}|\hat{K}| &\leqq c\varepsilon^{\alpha},\\ |\partial_\mu \hat{K}| + \varepsilon^{-\frac{1}{2}}|\partial_k \hat{K}| &\leqq \frac{c\varepsilon^{\alpha}}{(\mu^2+\varepsilon k^2)^{1/2}},\\ |\partial_\mu^2 \hat{K}| + \varepsilon^{-\frac{1}{2}}|\partial_\mu\partial_k \hat{K}| + \varepsilon^{-1}|\partial_k^2 \hat{K}| &\leqq \frac{c\varepsilon^{\alpha}}{\mu^2+\varepsilon k^2}\end{aligned}$$

for each $(\mu,k) \neq (0,0)$. The operator T maps $L^p(\mathbb{R}^2)$ continuously into itself and

$$\|Tu\|_p \leqq c\varepsilon^{\alpha}\|u\|_p.$$

We now turn to Fourier-multiplier operators in $L^p(\Sigma)$-based function spaces. Our first result in this direction is obtained by a straightforward application of Hölder's inequality.

Theorem 8. *Consider the operator T defined by*

$$(Tu)(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1 K(x-x_1, z-z_1; y, \xi)u(x_1,\xi,z_1)\,\mathrm{d}\xi\,\mathrm{d}x_1\,\mathrm{d}z_1.$$

Suppose the kernel $K(x,z;y,\xi)$ satisfies the hypotheses of Lemma 13, *Lemma* 14 *or Lemma* 15 *uniformly for $y, \xi \in [0,1]$. The operator T maps $L^p(\Sigma)$ continuously into itself and*

$$\|Tu\|_p \leqq c\varepsilon^{\alpha}\|u\|_p.$$

A natural tactic in dealing with more general Fourier-multiplier operators on $L^p(\Sigma)$ is to consider them as operators on $L^p(\mathbb{R}^2, L^p(0,1) \to L^p(0,1))$. Unfortunately the multiplier theorems of Marcinkiewicz and Mikhlin do not generalise to this operator-valued setting in a straightforward manner. An operator-valued generalisation of a theorem by Stein [37, p. 29], in which the hypotheses upon derivatives of $\hat{K}$ are replaced by hypotheses upon the derivatives of K itself, is however available (see the discussion in Section II5.1 of this reference); the following result is a scaled version of the appropriate theorem.

Theorem 9. *Consider the operator T defined by*

$$(Tu)(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1 K(x-x_1, z-z_1; y, \xi)u(x_1,\xi,z_1)\,\mathrm{d}\xi\,\mathrm{d}x_1\,\mathrm{d}z_1.$$

Suppose the kernel $K(x,z;y,\xi)$ satisfies

$$\left\|\int_0^1 \hat{K}w\,\mathrm{d}\xi\right\|_{L^p(0,1)} \leqq c\varepsilon^{\alpha}\|w\|_{L^p(0,1)} \tag{101}$$

and

$$\left\| \left\{ \begin{matrix} \partial_x \\ \varepsilon^{1/2}\partial_z \end{matrix} \right\} \int_0^1 K w \, \mathrm{d}\xi \right\|_{L^p(0,1)} \leqq \frac{c\varepsilon^{\alpha-1/2}}{(x^2+\varepsilon^{-1}z^2)^{3/2}} \|w\|_{L^p(0,1)}, \quad (x,z) \neq (0,0) \tag{102}$$

for each $w \in L^p(0,1)$. *The operator* T *maps* $L^p(\Sigma)$ *continuously into itself and*

$$\|Tu\|_p \leqq c\varepsilon^\alpha \|u\|_p.$$

5.2. Mapping properties

The next step is to use the results stated above to analyse the mapping properties of the operators $\mathcal{G}_1, \ldots, \mathcal{G}_{16}$ defined in Lemmata 2, 4 and 5. Our first result is the proof of Lemma 2(i); part (ii) is proved in a similar fashion.

Lemma 16. *Choose* $\delta \in [0,1]$ *and* $p \in (1,\infty)$. *For each* $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *the function*

$$\mathcal{G}_1(u) = \mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belongs to $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *and satisfies the estimate*

$$|\mathcal{G}_1(u)|_{\delta,p,\varepsilon} \leqq c\|u\|_{\delta,p,\varepsilon}.$$

Proof. A straightforward calculation shows that the multipliers $(1+\varepsilon+\beta q^2)^{-1}$ and $\varepsilon^{\frac{1}{2}}(\mu^2+\varepsilon k^2)^{\frac{1}{2}}(1+\varepsilon+\beta q^2)^{-1}$ satisfy the hypotheses of Lemma 15 (with $\alpha = 0$), which therefore implies that

$$\left\|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]\right\|_p \leqq c\|u\|_p,$$

$$\left\|\mathcal{F}^{-1}\left[\frac{\varepsilon^{\frac{1}{2}}(\mu^2+\varepsilon k^2)^{\frac{1}{2}}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]\right\|_p \leqq c\|u\|_p.$$

It follows that

$$\begin{aligned}
&|\mathcal{G}_1(u)|_{\delta,p,\varepsilon} \\
&\quad = \left\|\mathcal{F}^{-1}\left[\frac{1+\varepsilon^{\frac{1}{2}}(\mu^2+\varepsilon k^2)^{\frac{1}{2}+\frac{\delta}{2}}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]\right\|_p \\
&\quad \leqq \left\|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]\right\|_p + \left\|\mathcal{F}^{-1}\left[\frac{\varepsilon^{\frac{1}{2}}(\mu^2+\varepsilon k^2)^{\frac{1}{2}}}{1+\varepsilon+\beta q^2}(\mu^2+\varepsilon k^2)^{\frac{\delta}{2}}\mathcal{F}[u]\right]\right\|_p \\
&\quad \leqq c(\|u\|_p + \|\mathcal{F}^{-1}[(\mu^2+\varepsilon k^2)^{\frac{\delta}{2}}\mathcal{F}[u]]\|_p) \\
&\quad \leqq c\|u\|_{\delta,p,\varepsilon}.
\end{aligned}$$

□

The next result gives the proof of Lemma 5(i); parts (ii) and (iii) are deduced from part (i) and Lemma 2. Observe that

$$\begin{aligned}
\|\partial_x \mathcal{G}_{14}(u)\|_{2,p,\varepsilon} &= \|\mathcal{G}_1(\partial_x \mathcal{G}_{12}(u))\|_{2,p,\varepsilon} \\
&\leqq \|\mathcal{G}_1\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \|\mathcal{G}_{12}(u)\|_{2,p,\varepsilon} \\
&\leqq c\|u\|_p, \\
\varepsilon^{\frac{1}{2}}\|\partial_z \mathcal{G}_{14}(u)\|_{2,p,\varepsilon} &= \|\mathcal{G}_1(\varepsilon^{\frac{1}{2}}\partial_z \mathcal{G}_{12}(u))\|_{2,p,\varepsilon} \\
&\leqq \|\mathcal{G}_1\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}\, \varepsilon^{\frac{1}{2}} \|\mathcal{G}_{12}(u)\|_{2,p,\varepsilon} \\
&\leqq c\|u\|_p,
\end{aligned}$$

where the final inequalities in each line follow from Lemma 2(i) and Lemma 5(i); the estimates for $\mathcal{G}_{15}$ and $\mathcal{G}_{16}$ are obtained by the same method.

Lemma 17. *Choose $p \in (1, \infty)$. For each $u \in L^p(\mathbb{R}^2)$ the functions*

$$\mathcal{G}_{12}(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{Q}\mathcal{F}[u]\right], \quad \mathcal{G}_{13}(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{Q}\mathcal{F}[u]\right]$$

belong to $U^{2,p}_\varepsilon(\mathbb{R}^2)$ and satisfy the estimate

$$\|\mathcal{G}_j(u)\|_{U^{2,p}_\varepsilon} \leqq c\|u\|_p, \quad j = 12, 13.$$

Proof. A straightforward calculation shows that the multiplier μ^2/Q satisfies the hypotheses of Lemma 14 (with $\alpha = 0$). It follows that

$$\left\|\partial_x \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{Q}\mathcal{F}[u]\right]\right\|_p = \left\|\mathcal{F}^{-1}\left[\frac{-\mu^2}{Q}\mathcal{F}[u]\right]\right\|_p \leqq c\|u\|_p,$$

and repeating this argument with the multiplier $(\mu^2 + \varepsilon k^2)\mu^2/Q$, one finds that

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[(\mu^2+\varepsilon k^2)\mathcal{F}\left[\partial_x \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{Q}\mathcal{F}[u]\right]\right]\right]\right\|_p \\
&= \left\|\mathcal{F}^{-1}\left[\frac{-\mu^2(\mu^2+\varepsilon k^2)}{Q}\mathcal{F}[u]\right]\right\|_p \\
&\leqq c\|u\|_p.
\end{aligned}$$

The previous two inequalities imply that

$$\|\partial_x \mathcal{G}_{12}(u)\|_{2,p,\varepsilon} \leqq c\|u\|_p,$$

and a similar argument yields the complementary estimate

$$\varepsilon^{\frac{1}{2}}\|\partial_z \mathcal{G}_{12}(u)\|_{2,p,\varepsilon} \leqq c\|u\|_p.$$

The result for $\mathcal{G}_{13}$ is derived by the same method. $\square$

It is instructive to compare the proofs of Lemmata 16 and 17. The former uses the scaled version of Mikhlin's theorem (Lemma 15), while the latter relies upon the standard version (Lemma 14). In general, the scaled version of Mikhlin's theorem is appropriate for multipliers which depend upon μ and k only through the combination q and for multipliers whose support is bounded away from the origin (see, for example, Lemmata 22 and 23 below); in all other circumstances one requires the standard version of Mikhlin's theorem.

We now turn to the more involved analysis necessary for Lemma 4. The first step in the proof of part (i) of this lemma is to establish the basic estimate that for each $u \in L^p(\Sigma)$ the function

$$\mathcal{G}(u) = \mathcal{F}^{-1}\left[\int_0^1 G_1 \mathcal{F}[u]\,\mathrm{d}\xi\right]$$

belongs to $W_\varepsilon^{2,p}(\Sigma)$ and satisfies

$$\|\mathcal{G}(u)\|_{2,p,\varepsilon} \leqq c\varepsilon\|u\|_p;$$

to this end we show that

$$\|\mathcal{G}(u)\|_p \leqq c\varepsilon\|u\|_p, \quad \|\tilde{\mathcal{G}}(u)\|_p \leqq c\varepsilon\|u\|_p, \tag{103}$$

where

$$\tilde{\mathcal{G}}(u) = \mathcal{F}^{-1}\left[\int_0^1 (\mu^2 + \varepsilon k^2) G_1 \mathcal{F}[u]\,\mathrm{d}\xi\right],$$

and

$$\|\partial_y^2 \mathcal{G}(u)\|_p \leqq c\varepsilon\|u\|_p. \tag{104}$$

The expansion

$$\begin{aligned} q^2 - (1 + \varepsilon + \beta q^2) q \tanh q - \varepsilon^2 k^2 = {} & -\varepsilon^2\mu^2 - (1+\varepsilon)\varepsilon^2 k^2 \\ & - \left(\beta - \frac{\alpha}{3}\right) q^4 - c_0 q^6 + \mathcal{O}(q^8) \end{aligned}$$

as $q \to 0$ implies the existence of a constant q_0 such that

$$|q^2 - (1 + \varepsilon + \beta q^2) q \tanh q - \varepsilon^2 k^2| \geqq c\varepsilon^2 Q \tag{105}$$

whenever $q \leqq q_0$. Let $\chi \in C_0^\infty([0,\infty), \mathbb{R})$ be a smooth 'cut-off' function with the properties that

$$\begin{aligned} \chi(q) &= 1, \quad q \leqq q_0/2, \\ \chi(q) &= 0, \quad q \geqq q_0 \end{aligned}$$

and consider the decompositions $\mathcal{G} = \mathcal{G}_\mathrm{a} + \mathcal{G}_\mathrm{b}$, $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_\mathrm{a} + \tilde{\mathcal{G}}_\mathrm{b}$, where

$$\mathcal{G}_\mathrm{a}(u) = \mathcal{F}^{-1}\left[\int_0^1 \chi(q) G_1 \mathcal{F}[u]\,\mathrm{d}\xi\right], \quad \mathcal{G}_\mathrm{b}(u) = \mathcal{F}^{-1}\left[\int_0^1 (1-\chi(q)) G_1 \mathcal{F}[u]\,\mathrm{d}\xi\right]$$

and $\tilde{\mathcal{G}}_\mathrm{a}$, $\tilde{\mathcal{G}}_\mathrm{b}$ are defined in the same way. We establish (103) by proving that it holds for $\mathcal{G}_\mathrm{a}$, $\tilde{\mathcal{G}}_\mathrm{a}$ and $\mathcal{G}_\mathrm{b}$, $\tilde{\mathcal{G}}_\mathrm{b}$ separately and use an auxiliary argument to deduce (104). The first step in this programme is accomplished by Lemmata 18 and 19 below, which present the required estimates for $\mathcal{G}_\mathrm{a}$ and $\tilde{\mathcal{G}}_\mathrm{a}$.

Lemma 18. *Choose $p \in (1, \infty)$. For each $u \in L^p(\Sigma)$ the function $\mathcal{G}_{\mathrm{a}}(u)$ belongs to $L^p(\Sigma)$ and satisfies the estimate*

$$\|\mathcal{G}_{\mathrm{a}}(u)\|_p \leqq c\varepsilon \|u\|_p.$$

Proof. We show that the hypotheses of Lemma 14, namely

$$|\chi G_1| \leqq c\varepsilon,$$
$$(\mu^2+k^2)^{\frac{1}{2}} \left| \left\{ \begin{array}{c} \partial_\mu \\ \partial_k \end{array} \right\} (\chi G_1) \right| \leqq c\varepsilon,$$
$$(\mu^2+k^2) \left| \left\{ \begin{array}{c} \partial_\mu^2 \\ \partial_k^2 \\ \partial_\mu \partial_k \end{array} \right\} (\chi G_1) \right| \leqq c\varepsilon,$$

are satisfied uniformly for $y, \xi \in [0, 1]$, so that the result follows by an application of Theorem 8. To this end we write

$$\frac{G_1}{\varepsilon^2} = \tilde{G}_1 + \tilde{G}_2 + \tilde{G}_3, \tag{106}$$

where

$$\begin{aligned}
\tilde{G}_1 &= \frac{(1+\varepsilon)\tilde{G}}{q^2 - (1+\varepsilon+\beta q^2) q \tanh q - \varepsilon^2 k^2}, \\
\tilde{G}_2 &= \frac{1+\varepsilon}{q^2 - (1+\varepsilon+\beta q^2) q \tanh q - \varepsilon^2 k^2} \\
&\quad + \frac{1+\varepsilon}{\varepsilon^2\mu^2 + (1+\varepsilon)\varepsilon^2 k^2 + (\beta - \frac{1}{3}(1+\varepsilon)) q^4 + c_0 q^6}, \\
\tilde{G}_3 &= \frac{1+\varepsilon}{\varepsilon^2\mu^2 + (1+\varepsilon)\varepsilon^2 k^2 + (\beta - \frac{1}{3}) q^4 + c_0 q^6} \\
&\quad - \frac{1+\varepsilon}{\varepsilon^2\mu^2 + (1+\varepsilon)\varepsilon^2 k^2 + (\beta - \frac{1}{3}(1+\varepsilon)) q^4 + c_0 q^6}
\end{aligned}$$

and

$$\tilde{G} = \begin{cases}
\dfrac{\cosh qy}{\cosh q} \left(\dfrac{\beta q^2}{1+\varepsilon} \cosh q(1-\xi) + \dfrac{\varepsilon\mu^2}{(1+\varepsilon)q} \sinh q(\xi - 1) \right) & \\
\quad + \dfrac{\cosh qy}{\cosh q} \cosh q(1-\xi) - 1, & 0 < y < \xi < 1, \\
\dfrac{\cosh q\xi}{\cosh q} \left(\dfrac{\beta q^2}{1+\varepsilon} \cosh q(1-y) + \dfrac{\varepsilon\mu^2}{(1+\varepsilon)q} \sinh q(y - 1) \right) & \\
\quad + \dfrac{\cosh q\xi}{\cosh q} \cosh q(1-y) - 1, & 0 < \xi < y < 1.
\end{cases}$$

Using (105) and the fact that $\tilde{G} = \mathcal{O}(q^2)$ as $q \to 0$ uniformly for $y, \xi \in [0, 1]$, we find that

$$|\tilde{G}_1| \leqq \frac{cq^2}{\varepsilon^2 Q} \leqq c\varepsilon^{-2}\frac{q^2}{Q} \leqq c\varepsilon^{-1}$$

for $q \leqq q_0$ and hence that $|\chi G_1| \leqq c\varepsilon$. Direct calculations yield the estimates

$$\left|\frac{1}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}\right| \leqq \frac{c}{\varepsilon^2 Q},$$

$$\left|\partial_\mu\left(\frac{1}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}\right)\right| \leqq \frac{c}{\varepsilon^4 Q^2}(\varepsilon q+q^3)|\partial_\mu q|, \quad (107)$$

$$\left|\partial_k\left(\frac{1}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}\right)\right| \leqq c\left(\frac{1}{\varepsilon^4 Q^2}(\varepsilon q+q^3)|\partial_k q| + \frac{\varepsilon^2|k|}{\varepsilon^4 Q^2}\right) \quad (108)$$

and

$$|\partial_\mu \tilde{G}| \leqq c\varepsilon^{\frac{1}{2}}q, \quad |\partial_k \tilde{G}| \leqq c\varepsilon q$$

for $q \leqq q_0$. Combining the above estimates, we find that

$$(\mu^2+k^2)^{\frac{1}{2}}|\partial_\mu \tilde{G}_1| \leqq c(\mu^2+k^2)^{\frac{1}{2}}\left(\frac{\varepsilon^{\frac{1}{2}}q}{\varepsilon^2 Q}+\frac{\varepsilon^{\frac{3}{2}}q^3}{\varepsilon^4 Q^2}+\frac{\varepsilon^{\frac{1}{2}}q^5}{\varepsilon^4 Q^2}\right) \leqq c\varepsilon^{-1},$$

$$(\mu^2+k^2)^{\frac{1}{2}}|\partial_k \tilde{G}_1| \leqq c(\mu^2+k^2)^{\frac{1}{2}}\left(\frac{\varepsilon q}{\varepsilon^2 Q}+\frac{\varepsilon^2 q^3}{\varepsilon^4 Q^2}+\frac{\varepsilon q^5}{\varepsilon^4 Q^2}+\frac{\varepsilon^2 q^2|k|}{\varepsilon^4 Q^2}\right) \leqq c\varepsilon^{-1}$$

for $q \leqq q_0$. Observe that

$$|\partial_\mu \chi| = |\chi'(q)\partial_\mu q| \leqq c\varepsilon^{\frac{1}{2}}, \quad |\partial_k \chi| = |\chi'(q)\partial_k q| \leqq c\varepsilon,$$

whence

$$\begin{aligned}
(\mu^2+k^2)^{\frac{1}{2}}|\tilde{G}_1\partial_\mu\chi| &\leqq (\varepsilon\mu^2+\varepsilon k^2)^{\frac{1}{2}}|\tilde{G}_1| \\
&\leqq |\tilde{G}_1| + \varepsilon^{-1}|q^2\tilde{G}_1| \\
&\leqq c\left(\varepsilon^{-1}+\frac{\varepsilon^{-1}q^4}{\varepsilon^2 Q}\right) \\
&\leqq c\varepsilon^{-1}, \\
(\mu^2+k^2)^{\frac{1}{2}}|\tilde{G}_1\partial_k\chi| &\leqq (\varepsilon^2\mu^2+\varepsilon^2 k^2)^{\frac{1}{2}}|\tilde{G}_1| \\
&\leqq |\tilde{G}_1| + |q^2\tilde{G}_1| \\
&\leqq c\varepsilon^{-1},
\end{aligned}$$

in which $q \leqq q_0$ on the right-hand side, and altogether we have that

$$(\mu^2+k^2)^{\frac{1}{2}}\left|\left\{\begin{matrix}\partial_\mu\\ \partial_k\end{matrix}\right\}(\chi\tilde{G}_1)\right| = (\mu^2+k^2)^{\frac{1}{2}}\left|\chi\left\{\begin{matrix}\partial_\mu\\ \partial_k\end{matrix}\right\}\tilde{G}_1+\tilde{G}_1\left\{\begin{matrix}\partial_\mu\\ \partial_k\end{matrix}\right\}\chi\right| \leqq c\varepsilon^{-1}.$$

A similar analysis yields the estimates

$$(\mu^2+k^2)^{\frac{1}{2}}\left|\left\{\begin{array}{c}\partial_\mu\\ \partial_k\end{array}\right\}(\chi\tilde{G}_2)\right| \leqq c\varepsilon^{-1}, \quad (\mu^2+k^2)^{\frac{1}{2}}\left|\left\{\begin{array}{c}\partial_\mu\\ \partial_k\end{array}\right\}(\chi\tilde{G}_3)\right| \leqq c\varepsilon^{-1}$$

and

$$(\mu^2+k^2)\left|\left\{\begin{array}{c}\partial_\mu^2\\ \partial_k^2\\ \partial_\mu\partial_k\end{array}\right\}(\chi\tilde{G}_i)\right| \leqq c\varepsilon^{-1}, \quad i=1,2,3,$$

and the required estimates for G_1 are obtained from equation (106). □

Lemma 19. *Choose $p \in (1,\infty)$. For each $u \in L^p(\Sigma)$ the function $\tilde{\mathcal{G}}_{\mathrm{a}}(u)$ belongs to $L^p(\Sigma)$ and satisfies the estimate*

$$\|\tilde{\mathcal{G}}_{\mathrm{a}}(u)\|_p \leqq c\varepsilon\|u\|_p.$$

Proof. Here we use the formula

$$\tilde{\mathcal{G}}_{\mathrm{a}}(u) = \varepsilon^{-1}\mathcal{F}^{-1}\left[\int_0^1 q^2\chi G_1\mathcal{F}[u]\,\mathrm{d}\xi\right]$$

and show that the hypotheses of Lemma 13, namely that

$$|q^2\chi G_1| \leqq c\varepsilon^2, \tag{109}$$

$$\int_{I_j}|\partial_\mu(q^2\chi G_1)|\,\mathrm{d}\mu \leqq c\varepsilon^2, \tag{110}$$

$$\int_{I_j}|\partial_k(q^2\chi G_1)|\,\mathrm{d}k \leqq c\varepsilon^2, \tag{111}$$

$$\int_{I_{j_1}}\int_{I_{j_2}}|\partial_\mu\partial_k(q^2\chi G_1)|\,\mathrm{d}\mu\,\mathrm{d}k \leqq c\varepsilon^2 \tag{112}$$

uniformly over all dyadic intervals I_j, I_{j_1}, I_{j_2}, hold uniformly for $y, \xi \in [0,1]$, so that the result follows by Theorem 8. To this end we again use the decomposition (106), and recall the estimates

$$|q^2\tilde{G}_i| \leqq c, \quad i=1,2,3$$

for $q \leqq q_0$ established in the proof of Lemma 18, from which (109) is an immediate consequence.

Using the fact that $|\partial_q(q^2\tilde{G})| \leqq cq^3$ for $q \leqq q_0$ together with estimates (107), (108), we find that

$$|\partial_\mu(q^2\tilde{G}_1)| \leqq \frac{c}{|\mu|}, \quad |\partial_k(q^2\tilde{G}_1)| \leqq \frac{c}{|k|}$$

for $q \leqq q_0$. It follows that

$$\begin{aligned}\int_{2^j}^{2^{j+1}} |\partial_\mu(q^2\chi\tilde{G}_1)|\,\mathrm{d}\mu &\leqq \int_{2^j}^{2^{j+1}} (|\partial_\mu(q^2\tilde{G}_1)| + \varepsilon^{\frac{1}{2}}|\chi'(q)||q^2\tilde{G}_1|)\,\mathrm{d}\mu \\ &\leqq \int_{2^j}^{2^{j+1}} \frac{1}{\mu}\,\mathrm{d}\mu + c\varepsilon^{\frac{1}{2}} \int_0^{q_0\varepsilon^{-\frac{1}{2}}} \mathrm{d}\mu \\ &\leqq \log 2 + c \\ &\leqq c\end{aligned}$$

(because χ is identically zero for $q \geqq q_0$, and in particular for $\mu \geqq q_0\varepsilon^{-1/2}$) and similarly

$$\int_{2^j}^{2^{j+1}} |\partial_k(q^2\chi\tilde{G}_1)|\,\mathrm{d}k \leqq c$$

for $j \in \mathbb{N}_0$; the same estimates clearly hold for the dyadic intervals $(2^{-j-1}, 2^{-j})$, $j \in \mathbb{N}_0$ and those in the negative half-line.

We similarly find that

$$|\partial_\mu(q^2\tilde{G}_j)| \leqq \frac{c}{|\mu|}, \quad |\partial_k(q^2\tilde{G}_j)| \leqq \frac{c}{|k|}, \quad j = 2, 3$$

for $q \leqq q_0$, whence

$$\int_{I_j} |\partial_\mu(q^2\chi\tilde{G}_j)|\,\mathrm{d}\mu \leqq c, \quad \int_{I_j} |\partial_k(q^2\chi\tilde{G}_j)|\,\mathrm{d}k \leqq c, \quad j = 2, 3$$

for every dyadic interval I_j. An analogous argument shows that

$$\int_{I_{j_1}}\int_{I_{j_2}} |\partial_\mu\partial_k(q^2\chi\tilde{G}_i)|\,\mathrm{d}\mu\,\mathrm{d}k \leqq c, \quad i = 1, 2, 3$$

for every pair (I_{j_1}, I_{j_2}) of dyadic intervals, and the estimates (110)–(112) for G_1 follow from equation (106). $\square$

To obtain the estimates for $\mathcal{G}_\mathrm{b}$ and $\tilde{\mathcal{G}}_\mathrm{b}$ we write

$$G_1 = \varepsilon^2 G + \frac{1+\varepsilon}{Q} \tag{113}$$

and introduce the further decompositions $\mathcal{G}_\mathrm{b} = \mathcal{G}_{\mathrm{b},1} + \mathcal{G}_{\mathrm{b},2}$, $\tilde{\mathcal{G}}_\mathrm{b} = \tilde{\mathcal{G}}_{\mathrm{b},1} + \tilde{\mathcal{G}}_{\mathrm{b},2}$, where

$$\begin{aligned}\mathcal{G}_{\mathrm{b},1}(u) &= \varepsilon^2\mathcal{F}^{-1}\left[\int_0^1 (1-\chi(q))G\mathcal{F}[u]\,\mathrm{d}\xi\right], \\ \mathcal{G}_{\mathrm{b},2}(u) &= \mathcal{F}^{-1}\left[\int_0^1 \frac{1+\varepsilon}{Q}(1-\chi(q))\mathcal{F}[u]\,\mathrm{d}\xi\right]\end{aligned}$$

and $\tilde{\mathcal{G}}_{\mathrm{b},1}$, $\tilde{\mathcal{G}}_{\mathrm{b},2}$ are defined in the same way. We establish the required estimates for each of these operators separately, treating $\mathcal{G}_{\mathrm{b},1}$, $\tilde{\mathcal{G}}_{\mathrm{b},1}$ by singular-integral techniques together with Theorem 9 and $\mathcal{G}_{\mathrm{b},2}$, $\tilde{\mathcal{G}}_{\mathrm{b},2}$ by the scaled version of Mikhlin's theorem together with Theorem 8. In order to apply Theorem 9 to $\mathcal{G}_{\mathrm{b},2}$ and $\tilde{\mathcal{G}}_{\mathrm{b},2}$ it is necessary to verify hypothesis (101) on their Fourier transforms and hypothesis (102) on their kernels. The first of these tasks is undertaken in the following proposition.

Proposition 17. *The estimates*

$$\|\mathcal{F}[\mathcal{G}_{\mathrm{b},1}](w)\|_{L^p(0,1)} \leqq c\varepsilon^2\|w\|_{L^p(0,1)}, \quad \|\mathcal{F}[\tilde{\mathcal{G}}_{\mathrm{b},1}](w)\|_{L^p(0,1)} \leqq c\varepsilon\|w\|_{L^p(0,1)}$$

hold for each $w \in L^p(0,1)$.

Proof. A straightforward argument using the differential calculus shows that

$$q^2 - (1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2 \leqq -c_{q^\star} q^3 \tag{114}$$

for $q \geqq q_\star$, where $q_\star$ is any positive real number and $c_{q^\star}$ is a positive constant which depends only upon $q_\star$. It follows from (114), the inequality

$$\frac{\cosh qy}{\cosh q}\left\{\begin{array}{l}\cosh q(1-\xi)\\ \sinh q(\xi-1)\end{array}\right\} \leqq c\mathrm{e}^{-q(\xi-y)}, \quad y \leqq \xi$$

and the corresponding inequality for $\xi \leqq y$ obtained by interchanging y and ξ that

$$|G| \leqq \frac{c}{q^3}(1+q^2)\mathrm{e}^{-q|\xi-y|} \leqq \frac{c_{q^\star}}{q}\mathrm{e}^{-q|\xi-y|}, \quad q \geqq q_\star.$$

Using this estimate with $q = q_0$, we find that

$$\begin{aligned}
\|\mathcal{F}[\mathcal{G}_{\mathrm{b},1}](w)\|_{L^p(0,1)}^p &= \int_0^1 \left|\int_0^1 \varepsilon^2 G(1-\chi)w\,\mathrm{d}\xi\right|^p \mathrm{d}y\\
&\leqq c\int_0^1 \left|\int_0^1 \frac{\varepsilon^2}{q}(1-\chi)\mathrm{e}^{-q|y-\xi|}|w|\,\mathrm{d}\xi\right|^p \mathrm{d}y\\
&\leqq c\left(\frac{\varepsilon^2}{q}\right)^p(1-\chi)^p\int_0^1\left[\int_0^1 \mathrm{e}^{-q|y-\xi|}|w|\,\mathrm{d}\xi\right]^p \mathrm{d}y\\
&\leqq c\left(\frac{\varepsilon^2}{q}\right)^p(1-\chi)^p\left(\frac{1}{q}\right)^p\int_0^1 |w|^p\,\mathrm{d}\xi\\
&\leqq c\varepsilon^{2p}\|w\|_p^p,
\end{aligned}$$

and the estimate for $\tilde{\mathcal{G}}_{\mathrm{b},1}$ is obtained by the same method. □

Lemma 20. *Choose* $p \in (1,\infty)$. *For each* $u \in L^p(\Sigma)$ *the function* $\mathcal{G}_{\mathrm{b},1}(u)$ *belongs to* $L^p(\Sigma)$ *and satisfies the estimate*

$$\|\mathcal{G}_{\mathrm{b},1}(u)\|_p \leqq c\varepsilon^2\|u\|_p.$$

Proof. Observe that

$$\frac{\cosh qy}{\cosh q}\left\{\begin{array}{l}\cosh q(1-\xi)\\ \sinh q(\xi-1)\end{array}\right\} = \pm\frac{\mathrm{e}^{-q(\xi-y)}}{2(1+\mathrm{e}^{-2q})} + \frac{\mathrm{e}^{-q(\xi+y)}}{2(1+\mathrm{e}^{-2q})} \pm\frac{\mathrm{e}^{-q(2-\xi-y)}}{2(1+\mathrm{e}^{-2q})} + \frac{\mathrm{e}^{-q(1-(\xi-y))}}{2(\mathrm{e}^{q}+\mathrm{e}^{-q})},$$

and using this formula and the corresponding formula obtained by interchanging y and ξ, one finds that

$$\begin{aligned} G = {} & \frac{1+\varepsilon+\beta q^2-\varepsilon\mu^2/q}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q|\xi-y|} \\ & +\frac{1+\varepsilon+\beta q^2-\varepsilon\mu^2/q}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q(\xi+y)} \\ & +\frac{1+\varepsilon+\beta q^2-\varepsilon\mu^2/q}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q(2-\xi-y)} \\ & +\frac{1+\varepsilon+\beta q^2-\varepsilon\mu^2/q}{2(\mathrm{e}^{q}+\mathrm{e}^{-q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q(1-|\xi-y|)}. \end{aligned} \tag{115}$$

We now consider the first of these terms in detail; the others are handled in an analogous fashion. □

Define

$$\begin{aligned} I &= \varepsilon^2\mathcal{F}^{-1}\left[\frac{(1+\varepsilon+\beta q^2-\varepsilon\mu^2/q)(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q|\xi-y|}\right] \\ &= \frac{\varepsilon^2}{2\pi}\int_{\mathbb{R}^2}\frac{(1+\varepsilon+\beta q^2-\varepsilon\mu^2/q)(1-\chi(q))\mathrm{e}^{-q|\xi-y|}}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-\mathrm{i}\mu x}\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}\mu\,\mathrm{d}k. \end{aligned}$$

Introducing polar coordinates (q,θ) and (r,ϕ) defined by

$$\varepsilon^{\frac{1}{2}}\mu = q\cos\theta,\ \varepsilon k = q\sin\theta,\quad x = \varepsilon^{\frac{1}{2}}r\cos\phi,\quad z=\varepsilon r\sin\phi, \tag{116}$$

we find that

$$I = \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_0^{2\pi}\!\!\int_0^{\infty}\frac{q(1+\varepsilon+\beta q^2-q\cos^2(\phi+\psi))(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,$$

where $\psi=\theta-\phi$ and $\tilde{Q}=q^2\cos^2(\phi+\psi)-(1+\varepsilon+\beta q^2)q\tanh q$. Our strategy is to show that

$$|\partial_x I| \leqq \frac{c}{r^3},\quad \varepsilon^{\frac{1}{2}}|\partial_z I| \leqq \frac{c}{r^3}$$

uniformly over $\{y,\xi\in[0,1]: y\neq\xi\}$; because

$$\partial_x = \varepsilon^{-\frac{1}{2}}\cos\phi\,\partial_r - \frac{\varepsilon^{-\frac{1}{2}}}{r}\sin\phi\,\partial_\phi,\quad \varepsilon^{\frac{1}{2}}\partial_z = \varepsilon^{-\frac{1}{2}}\sin\phi\,\partial_r - \frac{\varepsilon^{-\frac{1}{2}}}{r}\cos\phi\,\partial_\phi$$

it suffices to show that

$$|\partial_r I| \leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^3}, \quad |\partial_\phi I| \leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^2}.$$

(Here, and in the remainder of this proof, all estimates hold uniformly over $\{y, \xi \in [0,1] : y \neq \xi\}$.) Let us write $I = I_1 + I_2 + I_3$, where I_2, I_3 are obtained from I by replacing the range of integration for ψ by, respectively, $(\pi/2 - \hat{\varepsilon}, \pi/2 + \hat{\varepsilon})$, $(3\pi/2 - \hat{\varepsilon}, 3\pi/2 + \hat{\varepsilon})$ and $\hat{\varepsilon}$ is a small positive constant, and consider each integral separately.

Notice that

$$\begin{aligned}
&\partial_r I_1 \\
&= \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_J\int_0^\infty \frac{\mathrm{i}q^2\cos\psi(1+\varepsilon+\beta q^2-q\cos^2(\phi+\psi))(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi \\
&= \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_J \frac{-\mathrm{i}\cos\psi}{(|\xi-y|+\mathrm{i}r\cos\psi)^3} \\
&\quad\times\int_0^\infty \partial_q^3\left(\frac{q^2(1+\varepsilon+\beta q^2-q\cos^2(\phi+\psi))(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\right)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi
\end{aligned}$$

in which $J = [0,2\pi]\setminus([\pi/2-\hat{\varepsilon}, \pi/2+\hat{\varepsilon}]\cup[3\pi/2-\hat{\varepsilon}, 3\pi/2+\hat{\varepsilon}])$ and the second line is obtained by three integrations by parts with respect to q (the requirement that $y \neq \xi$ is used at this step). Because

$$\frac{1}{\tilde{Q}} = \mathcal{O}(q^{-3}), \quad \partial_q^i\tilde{Q} = \mathcal{O}(q^{3-i}), \quad i = 0, 1, 2, \dots$$

as $q \to \infty$, the third derivative of the quantity in large parentheses in the above expression is $\mathcal{O}(q^{-2})$ as $q \to \infty$; it also vanishes near $q = 0$ and is therefore integrable. It follows from these observations that

$$\begin{aligned}
|\partial_r I_1| &\leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^3}\int_J\int_0^\infty\left|\partial_q^3\left(\frac{q^2(1+\varepsilon+\beta q^2-q\cos^2(\phi+\psi))(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\right)\right|\mathrm{d}q\,\mathrm{d}\theta \\
&\leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^3}.
\end{aligned}$$

The integral I_2 is dealt with using the substitution $\omega = \cos\psi$, so that

$$\begin{aligned}
&\partial_r I_2 = \\
&\frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\int_0^\infty \frac{\mathrm{i}q^2\omega(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega \\
&-\frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\int_0^\infty \frac{\mathrm{i}q^3\omega(\omega\cos\phi-\sqrt{1-\omega^2}\sin\phi)^2(1-\chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega,
\end{aligned} \tag{117}$$

where $\tilde{\varepsilon} = \sin \hat{\varepsilon}$ and

$$\tilde{Q} = q^2(\omega \cos \phi - \sqrt{1-\omega^2} \sin \phi)^2 - (1 + \varepsilon + \beta q^2) q \tanh q.$$

Examining the first integral on the right-hand side of equation (117), note that

$$\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \frac{q^2 \omega (1 + \varepsilon + \beta q^2)(1 - \chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1 + \mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi - y| + \mathrm{i} r\omega)} \,\mathrm{d}q \,\mathrm{d}\omega = I_2^1 + I_2^2,$$

where

$$\begin{aligned}
I_2^1 &= \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \frac{q^2 \omega (1 + \varepsilon + \beta q^2)(1 - \chi(q))}{2\tilde{Q}(1 + \mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi - y| + \mathrm{i} r\omega)} \,\mathrm{d}q \,\mathrm{d}\omega, \\
I_2^2 &= \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \mathcal{O}(\omega^3) \int_0^\infty \frac{q^2 (1 + \varepsilon + \beta q^2)(1 - \chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1 + \mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi - y| + \mathrm{i} r\omega)} \,\mathrm{d}q \,\mathrm{d}\omega \\
&= -\mathrm{i} \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \frac{\mathcal{O}(\omega^3)}{(|\xi - y| + \mathrm{i} r\omega)^3} \\
&\qquad \times \int_0^\infty \partial_q^3 \left(\frac{q^2 (1 + \varepsilon + \beta q^2)(1 - \chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1 + \mathrm{e}^{-2q})} \right) \mathrm{e}^{-q(|\xi - y| + \mathrm{i} r\omega)} \,\mathrm{d}q \,\mathrm{d}\omega \\
&= \mathcal{O}(r^{-3}).
\end{aligned}$$

Using the formulae

$$\int \omega \mathrm{e}^{-\mathrm{i} q r \omega} \,\mathrm{d}\omega = -\frac{\omega}{\mathrm{i} q r} \mathrm{e}^{-\mathrm{i} q r \omega} + \frac{1}{q^2 r^2} \mathrm{e}^{-\mathrm{i} q r \omega}$$

and

$$\partial_\omega \left(\frac{1}{\tilde{Q}} \right) = \frac{q^2}{\tilde{Q}^2} g(\omega),$$

where

$$\begin{aligned}
g(\omega) &= -2(\omega \cos \phi - \sqrt{1-\omega^2} \sin \phi) \left(\cos \phi + \frac{\omega}{\sqrt{1-\omega^2}} \sin \phi \right) \\
&= \sin 2\phi + \mathcal{O}(\omega),
\end{aligned}$$

we can integrate by parts with respect to ω to find that

$$\begin{aligned}
I_2^1 = &\int_0^\infty \left[\left(-\frac{\omega}{\mathrm{i} q r} + \frac{1}{q^2 r^2} \right) \frac{q^2 (1 + \varepsilon + \beta q^2)(1 - \chi(q))}{2\tilde{Q}(1 + \mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi - y| + \mathrm{i} r\omega)} \right]_{\omega = \tilde{\varepsilon}}^{\omega = \tilde{\varepsilon}} \mathrm{d}q \\
&- \int_0^\infty \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \left(-\frac{\omega}{\mathrm{i} q r} + \frac{1}{q^2 r^2} \right) \frac{q^4 (1 + \varepsilon + \beta q^2)(1 - \chi(q))}{2\tilde{Q}^2 (1 + \mathrm{e}^{-2q})} g(\omega) \mathrm{e}^{-q(|\xi - y| + \mathrm{i} r\omega)} \,\mathrm{d}\omega \,\mathrm{d}q
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \left[\left(-\frac{\omega}{\mathrm{i}qr} + \frac{1}{q^2r^2} \right) \frac{q^2(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \right]_{\omega=\tilde{\varepsilon}}^{\omega=\tilde{\varepsilon}} \mathrm{d}q \\
&\quad + \int_0^\infty \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \frac{\omega}{\mathrm{i}r} \sin 2\phi \, \frac{q^3(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \, \mathrm{d}\omega \, \mathrm{d}q \\
&\quad - \int_0^\infty \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \frac{1}{r^2} \sin 2\phi \, \frac{q^2(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \, \mathrm{d}\omega \, \mathrm{d}q \\
&\quad + \int_0^\infty \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \frac{\mathcal{O}(\omega^2)}{\mathrm{i}r} \frac{q^3(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \, \mathrm{d}\omega \, \mathrm{d}q \\
&\quad - \int_0^\infty \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \frac{\mathcal{O}(\omega)}{r^2} \frac{q^2(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \, \mathrm{d}\omega \, \mathrm{d}q.
\end{aligned}
$$

Integrations by parts with respect to q show that the first, fourth and fifth terms on the right-hand side of this expression are $\mathcal{O}(r^{-3})$; integrating the second and third terms on its right-hand side by parts with respect to ω and repeating the above calculation shows that they are also $\mathcal{O}(r^{-3})$.

Turning to the second integral on the right-hand side of equation (117), note that

$$
\begin{aligned}
&\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \frac{\omega(\omega\cos\phi - \sqrt{1-\omega^2}\sin\phi)^2 q^3(1-\chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \, \mathrm{d}q \, \mathrm{d}\omega \\
&= \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \frac{\omega q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \, \mathrm{d}q \, \mathrm{d}\omega \\
&\quad -2 \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \omega^2 \sin 2\phi \, \frac{q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \, \mathrm{d}q \, \mathrm{d}\omega \\
&\quad + \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \mathcal{O}(\omega^3) \frac{q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \, \mathrm{d}q \, \mathrm{d}\omega.
\end{aligned}
$$

The methods used above show that the first and third terms on the right-hand side of this expression are $\mathcal{O}(r^{-3})$. To discuss the second term, we use the formula

$$
\int \omega^2 \mathrm{e}^{-\mathrm{i}qr\omega} \, \mathrm{d}\omega = -\frac{\omega^2}{\mathrm{i}qr} + \frac{2\omega}{q^2r^2} + \frac{2}{\mathrm{i}q^3r^3}
$$

and integrate by parts with respect to ω; the result is

$$
\begin{aligned}
&\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \omega^2 \frac{q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \, \mathrm{d}q \, \mathrm{d}\omega \\
&= \int_0^\infty \left[\left(-\frac{\omega^2}{\mathrm{i}qr} + \frac{2\omega}{q^2r^2} + \frac{2}{\mathrm{i}q^3r^3} \right) \frac{q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \right]_{\omega=-\tilde{\varepsilon}}^{\omega=\tilde{\varepsilon}} \mathrm{d}q \\
&\quad - \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \left(-\frac{\omega^2}{\mathrm{i}qr} + \frac{2\omega}{q^2r^2} + \frac{2}{\mathrm{i}q^3r^3} \right) \frac{q^5(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)} \, \mathrm{d}q \, \mathrm{d}\omega,
\end{aligned}
$$

and integrations by parts with respect to q show that each term on the right-hand side of this expression is $\mathcal{O}(r^{-3})$. Altogether we have that

$$|\partial_r I_2| \leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^3},$$

and the inequality

$$|\partial_r I_3| \leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^3}$$

is obtained by the same argument.

Similar methods yield the inequalities

$$|\partial_\phi I_j| \leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^2}, \quad j = 1, 2, 3.$$

We have therefore proved that

$$\left| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}} \partial_z \end{array} \right\} \mathcal{F}^{-1}[\varepsilon^2 G(1-\chi)] \right| \leqq \frac{c}{r^3} = \frac{c\varepsilon^{\frac{3}{2}}}{(x^2 + \varepsilon^{-1} z^2)^{\frac{3}{2}}}, \quad y \neq \xi,$$

from which it follows that

$$\left\| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}} \partial_z \end{array} \right\} \mathcal{F}^{-1} \left[\int_0^1 \varepsilon^2 G(1-\chi) w \, \mathrm{d}\xi \right] \right\|_{L^p(0,1)} \leqq \frac{c\varepsilon^{\frac{3}{2}}}{(x^2 + \varepsilon^{-1} z^2)^{\frac{3}{2}}} \|w\|_{L^p(0,1)}$$

for each $w \in L^p(0, 1)$. This result, together with Proposition 17, shows that the hypotheses of Theorem 9 are met, and we conclude that

$$\left\| \mathcal{F}^{-1} \left[\int_0^1 \varepsilon^2 G(1-\chi) u \, \mathrm{d}\xi \right] \right\|_p \leqq c\varepsilon^2 \|u\|_p.$$

Lemma 21. *Choose $p \in (1, \infty)$. For each $u \in L^p(\Sigma)$ the function $\tilde{\mathcal{G}}_{\mathrm{b},1}(u)$ belongs to $L^p(\Sigma)$ and satisfies the estimate*

$$\|\tilde{\mathcal{G}}_{\mathrm{b},1}(u)\|_p \leqq c\varepsilon \|u\|_p.$$

Proof. A formula for $q^2 G$ is obtained by multiplying formula (115) by q^2; we consider the first of the terms on the right-hand side of the resulting expression in detail (the others are handled in an analogous fashion). Define

$$I_1 = \varepsilon^2 \mathcal{F}^{-1} \left[\frac{(1+\varepsilon) q^2 (1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2 - (1+\varepsilon+\beta q^2) q \tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q|\xi - y|} \right],$$

$$I_2 = \varepsilon^2 \mathcal{F}^{-1} \left[\frac{\beta q^4 (1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2 - (1+\varepsilon+\beta q^2) q \tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q|\xi - y|} \right],$$

$$I_3 = \varepsilon^2 \mathcal{F}^{-1} \left[\frac{\varepsilon \mu^2 q (1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2 - (1+\varepsilon+\beta q^2) q \tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q|\xi - y|} \right].$$

The method employed in Lemma 21 shows that

$$\left| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}} \partial_z \end{array} \right\} I_1 \right| \leqq \frac{c\varepsilon^{\frac{3}{2}}}{(x^2 + \varepsilon^{-1} z^2)^{\frac{3}{2}}}, \quad y \neq \xi,$$

from which it follows that

$$\sup_{\xi \in [0,1]} \int_0^1 \left| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}} \partial_z \end{array} \right\} I_1 \right| \, \mathrm{d}y \leqq \frac{c\varepsilon^{\frac{3}{2}}}{(x^2 + \varepsilon^{-1} z^2)^{\frac{3}{2}}}, \tag{118}$$

$$\sup_{y \in [0,1]} \int_0^1 \left| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}} \partial_z \end{array} \right\} I_1 \right| \, \mathrm{d}\xi \leqq \frac{c\varepsilon^{\frac{3}{2}}}{(x^2 + \varepsilon^{-1} z^2)^{\frac{3}{2}}}, \tag{119}$$

and out strategy is to show that (118), (119) also hold for I_2 and I_3.

Using the polar coordinates (116), observe that

$$\begin{aligned} I_2 &= (\varepsilon \partial_x^2 + \varepsilon^2 \partial_z^2) \tilde{I}_2 \\ &= \left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\phi^2 \right) \tilde{I}_2, \end{aligned}$$

where

$$\begin{aligned} \tilde{I}_2 &= \varepsilon^2 \mathcal{F}^{-1} \left[\frac{\beta q^2 (1 - \chi(q))}{2(1 + \mathrm{e}^{-2q})(q^2 - (1 + \varepsilon + \beta q^2) q \tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q|\xi - y|} \right] \\ &= \frac{\varepsilon^{\frac{1}{2}} \beta}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{q^3 (1 - \chi(q))}{2 \tilde{Q} (1 + \mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi - y| + \mathrm{i} r \cos \psi)} \, \mathrm{d}q \, \mathrm{d}\psi \end{aligned}$$

and $\psi = \theta - \phi$, $\tilde{Q} = q^2 \cos^2(\phi + \psi) - (1 + \varepsilon + \beta q^2) q \tanh q$. To show that the estimates (118), (119) also hold for I_2 it therefore suffices to examine the quantities

$$\partial_r^3 \tilde{I}_2, \quad \frac{1}{r^2} \partial_r \tilde{I}_2, \quad \frac{1}{r} \partial_r^2 \tilde{I}_2, \quad \frac{1}{r^3} \partial_\phi^2 \tilde{I}_2, \quad \frac{1}{r^2} \partial_r \partial_\phi^2 \tilde{I}_2,$$
$$\frac{1}{r} \partial_r^2 \partial_\phi \tilde{I}_2, \quad \frac{1}{r^2} \partial_r \partial_\phi \tilde{I}_2, \quad \frac{1}{r^3} \partial_\phi^3 \tilde{I}_2.$$

In order to deal with the integral

$$\begin{aligned} \partial_r^3 \tilde{I}_2 &= \frac{\varepsilon^{\frac{1}{2}} \beta}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{-\mathrm{i} q^6 (1 - \chi(q)) \cos^3 \psi}{2 \tilde{Q} (1 + \mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi - y| + \mathrm{i} r \cos \psi)} \, \mathrm{d}q \, \mathrm{d}\psi \\ &= \frac{\varepsilon^{\frac{1}{2}} \beta}{2\pi} \int_0^{2\pi} \frac{\mathrm{i} \cos^3 \psi}{(|\xi - y| + \mathrm{i} r \cos \psi)^3} \\ &\qquad \times \int_0^\infty \partial_q^3 \left(\frac{q^6 (1 - \chi(q)) \cos^3 \psi}{2 \tilde{Q} (1 + \mathrm{e}^{-2q})} \right) \mathrm{e}^{-q(|\xi - y| + \mathrm{i} r \cos \psi)} \, \mathrm{d}q \, \mathrm{d}\psi, \end{aligned}$$

where we have supposed that $y \neq \xi$ in the integration by parts, let us write

$$\partial_q^3 \left(\frac{q^6 (1 - \chi(q))}{2 \tilde{Q} (1 + \mathrm{e}^{-2q})} \right) = \ell + \mathcal{R}(q),$$

where

$$\ell = \lim_{q\to\infty} \partial_q^3 \left(\frac{q^6(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})} \right)$$

and $\mathcal{R}(q) = \mathcal{O}(q^{-2})$ as $q \to \infty$, so that

$$\partial_r^3 \tilde{I}_2 = \tilde{I}_2^1 + \tilde{I}_2^2, \quad y \neq \xi,$$

in which

$$\tilde{I}_2^1 = \frac{\varepsilon^{\frac{1}{2}}\beta}{2\pi} \int_0^{2\pi} \frac{\mathrm{i}\cos^3\psi}{(|\xi-y|+\mathrm{i}r\cos\psi)^3} \int_0^\infty \ell\, \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\, \mathrm{d}q\, \mathrm{d}\psi,$$

$$\tilde{I}_2^2 = \frac{\varepsilon^{\frac{1}{2}}\beta}{2\pi} \int_0^{2\pi} \frac{\mathrm{i}\cos^3\psi}{(|\xi-y|+\mathrm{i}r\cos\psi)^3} \int_0^\infty \mathcal{R}(q)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\, \mathrm{d}q\, \mathrm{d}\psi.$$

It follows from the fact that $\mathcal{R}(q)$ is integrable that $|\tilde{I}_2^2| \leqq c\varepsilon^{\frac{1}{2}}/r^3$ and hence that

$$\sup_{\xi\in[0,1]} \int_0^1 |\tilde{I}_2^2|\, \mathrm{d}y \leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^3}.$$

Furthermore, one has that

$$\begin{aligned}
&\sup_{\xi\in[0,1]} \int_0^1 |\tilde{I}_2^1|\, \mathrm{d}y \\
&= \varepsilon^{\frac{1}{2}}\beta \sup_{\xi\in[0,1]} \int_0^1 \left| \int_0^{2\pi} \frac{\cos^3\psi}{(|\xi-y|+\mathrm{i}r\cos\psi)^3} \int_0^\infty \ell\, \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\, \mathrm{d}q\, \mathrm{d}\psi \right| \mathrm{d}y \\
&= \varepsilon^{\frac{1}{2}}\beta \sup_{\xi\in[0,1]} \int_0^1 \left| \int_0^{2\pi} \frac{-\ell\cos^3\psi}{(|\xi-y|-\mathrm{i}r\cos\psi)^4}\, \mathrm{d}\psi \right| \mathrm{d}y \\
&= \varepsilon^{\frac{1}{2}}\beta \sup_{\xi\in[0,1]} \int_0^1 \left| \int_0^{2\pi} -\ell\cos^3\psi \frac{(|\xi-y|-\mathrm{i}r\cos\psi)^4}{(|\xi-y|^2+r^2\cos^2\psi)^4}\, \mathrm{d}\psi \right| \mathrm{d}y \\
&\leqq c\varepsilon^{\frac{1}{2}} \sup_{\xi\in[0,1]} \int_0^1 \int_0^{2\pi} |\cos\psi|^3 \frac{|\xi-y|^4+r^4\cos^4\psi}{(|\xi-y|^2+r^2\cos^2\psi)^4}\, \mathrm{d}\psi\, \mathrm{d}y \\
&\leqq c\varepsilon^{\frac{1}{2}} \int_0^1 \int_0^{2\pi} |\cos\psi|^3 \frac{|w|^4+r^4\cos^4\psi}{(|w|^2+r^2\cos^2\psi)^4}\, \mathrm{d}\psi\, \mathrm{d}w \\
&= c\varepsilon^{\frac{1}{2}} \int_0^{2\pi} \int_0^{\frac{1}{r|\cos\psi|}} \frac{t^4+1}{r^3(t^2+1)^4}\, \mathrm{d}t\, \mathrm{d}\psi \\
&\leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^3} \int_0^\infty \frac{t^4+1}{(t^2+1)^4}\, \mathrm{d}t \\
&\leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^3}.
\end{aligned}$$

A similar argument shows that

$$\sup_{\xi\in[0,1]}\int_0^1 |\partial_r \tilde{I}_2|\,\mathrm{d}y \leqq \frac{c\varepsilon^{\frac{1}{2}}}{r}, \quad \sup_{\xi\in[0,1]}\int_0^1 |\partial_r^2 \tilde{I}_2|\,\mathrm{d}y \leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^2}.$$

Direct calculations yield the formulae

$$\begin{aligned}\partial_\phi^2 \tilde{I}_2 = \frac{\varepsilon^{\frac{1}{2}}\beta}{2\pi}\int_0^{2\pi}\int_0^\infty &\left(\frac{2q^7h^2}{\tilde{Q}^3}-\frac{q^5h_\phi}{\tilde{Q}^2}\right)\\ &\times\frac{1-\phi(q)}{2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,\\ \partial_\phi^3 \tilde{I}_2 = \frac{\varepsilon^{\frac{1}{2}}\beta}{2\pi}\int_0^{2\pi}\int_0^\infty &\left(\frac{-6q^9h^3}{\tilde{Q}^4}+\frac{6q^7hh_\phi}{\tilde{Q}^3}-\frac{q^5h_{\phi\phi}}{\tilde{Q}^2}\right)\\ &\times\frac{(1-\chi(q))}{2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,\end{aligned}$$

where $h(\phi,\psi)=\sin 2(\phi+\psi)$. Observe that

$$\begin{aligned}&\sup_{\xi\in[0,1]}\int_0^1\left|\int_0^{2\pi}\int_0^\infty \frac{q^5(1-\chi(q))}{\tilde{Q}^2(1+\mathrm{e}^{-2q})}\left\{\begin{matrix}h_\phi\\ h_{\phi\phi}\end{matrix}\right\}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi\right|\,\mathrm{d}y\\ &\leqq c\int_0^1\int_0^\infty\left|\frac{q^5(1-\chi(q))}{\tilde{Q}^2(1+\mathrm{e}^{-2q})}\right|\mathrm{e}^{-qw}\,\mathrm{d}q\,\mathrm{d}w\\ &\leqq c\int_0^\infty\frac{1}{q}\left|\frac{q^5(1-\chi(q))}{\tilde{Q}^2(1+\mathrm{e}^{-2q})}\right|\,\mathrm{d}q,\\ &\leqq c,\end{aligned}$$

and since

$$\frac{q^7(1-\chi(q))}{2\tilde{Q}^3(1+\mathrm{e}^{-2q})}, \quad \frac{q^9(1-\chi(q))}{2\tilde{Q}^4(1+\mathrm{e}^{-2q})}$$

are integrable this result also holds for the remaining terms in the formulae for $\partial_\phi^2\tilde{I}_2$, $\partial_\phi^3\tilde{I}_3$, which therefore satisfy

$$\sup_{\xi\in[0,1]}\int_0^1 |\partial_\phi^2 \tilde{I}_2|\,\mathrm{d}y \leqq c\varepsilon^{\frac{1}{2}}, \quad \sup_{\xi\in[0,1]}\int_0^1 |\partial_\phi^3 \tilde{I}_2|\,\mathrm{d}y \leqq c\varepsilon^{\frac{1}{2}}.$$

We find from the calculation

$$\partial_r\partial_\phi\tilde{I}_2 = \frac{\varepsilon^{\frac{1}{2}}\beta}{2\pi}\int_0^{2\pi}\int_0^\infty \frac{-\mathrm{i}q^6\cos\psi\, h(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi$$

$$= -\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}\beta}{2\pi}\int_0^{2\pi}\frac{\cos\psi\, h}{|\xi-y|+\mathrm{i}r\cos\psi}$$
$$\times\int_0^\infty \partial_q\left(\frac{q^6(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\right)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,$$

which is valid for $y \neq \xi$, that

$$\begin{aligned}\sup_{\xi\in[0,1]}\int_0^1 |\partial_r\partial_\phi\tilde{I}_2|\,\mathrm{d}y &\leqq \frac{c\varepsilon^{\frac{1}{2}}}{r}\int_0^1\int_0^\infty\left|\partial_q\left(\frac{q^6(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\right)\right|\mathrm{e}^{-qw}\,\mathrm{d}q\,\mathrm{d}w\\ &\leqq \frac{c\varepsilon^{\frac{1}{2}}}{r}\int_0^\infty\frac{1}{q}\left|\partial_q\left(\frac{q^6(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\right)\right|\mathrm{e}^{-qw}\,\mathrm{d}q\\ &\leqq \frac{c\varepsilon^{\frac{1}{2}}}{r},\end{aligned}$$

and similar arguments show that

$$\sup_{\xi\in[0,1]}\int_0^1|\partial_r\partial_\phi^2\tilde{I}_2|\,\mathrm{d}y \leqq \frac{c\varepsilon^{\frac{1}{2}}}{r},\qquad \sup_{\xi\in[0,1]}\int_0^1|\partial_r^2\partial_\phi\tilde{I}_2|\,\mathrm{d}y \leqq \frac{c\varepsilon^{\frac{1}{2}}}{r^2}.$$

We have therefore demonstrated that I_2 satisfies (118), and a similar technique shows that the same is true of I_3. Furthermore, we may clearly interchange the roles of y and ξ in the above arguments and hence conclude that I_2 and I_3 also satisfy (119). Altogether we have that

$$\sup_{\xi\in[0,1]}\int_0^1\left|\left\{\begin{array}{c}\partial_x\\ \varepsilon^{\frac{1}{2}}\partial_z\end{array}\right\}\mathcal{F}^{-1}[\varepsilon^2q^2G(1-\chi)]\right|\,\mathrm{d}y \leqq \frac{c}{r^3},$$
$$\sup_{y\in[0,1]}\int_0^1\left|\left\{\begin{array}{c}\partial_x\\ \varepsilon^{\frac{1}{2}}\partial_z\end{array}\right\}\mathcal{F}^{-1}[\varepsilon^2q^2G(1-\chi)]\right|\,\mathrm{d}\xi \leqq \frac{c}{r^3},$$

from which it follows that

$$\left\|\left\{\begin{array}{c}\partial_x\\ \varepsilon^{\frac{1}{2}}\partial_z\end{array}\right\}\mathcal{F}^{-1}\left[\int_0^1\varepsilon^2q^2G(1-\chi)w\,\mathrm{d}\xi\right]\right\|_{L^p(0,1)} \leqq \frac{c\varepsilon^{\frac{3}{2}}}{(x^2+\varepsilon^{-1}z^2)^{\frac{3}{2}}}\|w\|_{L^p(0,1)}$$

for each $w \in L^p(0,1)$ (see, for example, [22, Corollary 2.5.4]); using this result and Proposition 17, we find from Theorem 9 that

$$\left\|\mathcal{F}^{-1}\left[\int_0^1\varepsilon^2q^2G(1-\chi)u\,\mathrm{d}\xi\right]\right\|_p \leqq c\varepsilon^2\|u\|_p.$$

□

The estimates for $\mathcal{G}_{\mathrm{b},2}$ and $\tilde{\mathcal{G}}_{\mathrm{b},2}$ are presented in the next two lemmata, the second of which is proved in the same way as the first.

Lemma 22. *Choose* $p \in (1, \infty)$. *For each* $u \in L^p(\Sigma)$ *the function* $\mathcal{G}_{\mathrm{b},2}(u)$ *belongs to* $L^p(\Sigma)$ *and satisfies the estimate*

$$\|\mathcal{G}_{\mathrm{b},2}(u)\|_p \leqq c\varepsilon^2 \|u\|_p.$$

Proof. Direct calculations show that each of

$$\left|\frac{1}{Q}\right|, \ \varepsilon^{-\frac{1}{2}}\left|\partial_\mu\left(\frac{1}{Q}\right)\right|, \ \varepsilon^{-1}\left|\partial_k\left(\frac{1}{Q}\right)\right|, \ \varepsilon^{-1}q^2\left|\partial_\mu^2\left(\frac{1}{Q}\right)\right|,$$
$$\varepsilon^{-2}q^2\left|\partial_k^2\left(\frac{1}{Q}\right)\right|, \ \varepsilon^{-\frac{3}{2}}q^2\left|\partial_\mu\partial_k\left(\frac{1}{Q}\right)\right|$$

is bounded by $c\varepsilon^2$ for $q \geqq q_0$, while the 'cut-off' function χ has the property that

$$|\chi|, \ \varepsilon^{-\frac{1}{2}}q|\partial_\mu\chi|, \ \varepsilon^{-1}q|\partial_k\chi|, \ \varepsilon^{-1}q^2|\partial_\mu^2\chi|, \ \varepsilon^{-2}q^2|\partial_k^2\chi|, \ \varepsilon^{-\frac{3}{2}}q^2|\partial_\mu\partial_k\chi| \ \leqq \ c$$

for $q \geqq q_0$; these estimates clearly also hold for $(1-\chi)$. The multiplier $(1-\chi)/Q$ therefore satisfies the hypotheses of Lemma 14 uniformly for $y, \xi \in [0, 1]$, and it follows from Theorem 8 that

$$\left\|\mathcal{F}^{-1}\left[\int_0^1 \frac{1+\varepsilon}{Q}(1-\chi)\mathcal{F}[u]\,\mathrm{d}\xi\right]\right\|_p \leqq c\varepsilon^2\|u\|_p.$$

□

Lemma 23. *Choose* $p \in (1, \infty)$. *For each* $u \in L^p(\Sigma)$ *the function* $\tilde{\mathcal{G}}_{\mathrm{b},2}(u)$ *belongs to* $L^p(\Sigma)$ *and satisfies the estimate*

$$\|\tilde{\mathcal{G}}_{\mathrm{b},2}(u)\|_p \leqq c\varepsilon\|u\|_p.$$

Lemmata 18, 19 and 20–23 show that

$$\|\mathcal{G}(u)\|_p \leqq c\varepsilon\|u\|_p, \quad \|\tilde{\mathcal{G}}(u)\|_p \leqq c\varepsilon\|u\|_p,$$

and we can deduce the remaining estimate for $\partial_y^2\mathcal{G}$ from them.

Corollary 1. *Choose* $p \in (1, \infty)$. *For each* $u \in L^p(\Sigma)$ *the function* $\partial_y^2\mathcal{G}(u)$ *belongs to* $L^p(\Sigma)$ *and satisfies the estimate*

$$\|\partial_y^2\mathcal{G}(u)\|_p \leqq c\varepsilon^2\|u\|_p.$$

Proof. Observe that

$$\begin{aligned}
\partial_y^2\mathcal{G}_{\mathrm{a}}(u) &= \mathcal{F}^{-1}\left[\int_0^1 \partial_y^2 G_1\chi\mathcal{F}[u]\,\mathrm{d}\xi\right] + \mathcal{F}^{-1}\left[\int_0^1 \partial_y^2 G_1(1-\chi)\mathcal{F}[u]\,\mathrm{d}\xi\right] \\
&= \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2\partial_y^2\tilde{G}_1\chi\mathcal{F}[u]\,\mathrm{d}\xi\right] + \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2\partial_y^2 G(1-\chi)\mathcal{F}[u]\,\mathrm{d}\xi\right] \\
&= \mathcal{F}^{-1}\left[\int_0^1 \frac{\varepsilon^2(1+\varepsilon)\chi q^2(\tilde{G}+1)}{q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2}\mathcal{F}[u]\,\mathrm{d}\xi\right] \\
&\quad + \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2 q^2 G(1-\chi)\mathcal{F}[u]\,\mathrm{d}\xi\right],
\end{aligned}$$

where we have used (106), (113) and the facts that

$$\partial_y^2 \tilde{G} = q^2(\tilde{G}+1), \quad \partial_y^2 G = q^2 G.$$

The assertion therefore follows from the estimate

$$\left\| \mathcal{F}^{-1}\left[\int_0^1 \frac{\varepsilon^2(1+\varepsilon)\chi q^2(\tilde{G}+1)}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2 k^2}\mathcal{F}[u]\,\mathrm{d}\xi\right]\right\|_p \leqq c\varepsilon^2\|u\|_p,$$

which is obtained by noting that $\partial_q^i(q^2(\tilde{G}+1)) = \mathcal{O}(q^{4-i})$, $i = 0, 1, 2$ as $q \to 0$ uniformly for $y, \xi \in [0, 1]$ and repeating the first part of the proof of Lemma 19, and the estimate

$$\left\| \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2 q^2 G(1-\chi)\mathcal{F}[u]\,\mathrm{d}\xi\right]\right\|_p \leqq c\varepsilon^2\|u\|_p,$$

which is obtained in the proof of Lemma 21. □

The above theory establishes the basic estimate

$$\|\mathcal{G}(u)\|_{2,p,\varepsilon} \leqq c\varepsilon\|u\|_p, \tag{120}$$

and we now complete our analysis by showing how Lemma 4(i), (ii) follow from this result.

Corollary 2. *Choose* $\delta \in [0, 1]$ *and* $p \in (1, \infty)$. *For each* $u \in W_\varepsilon^{\delta,p}(\Sigma)$ *the functions*

$$\mathcal{G}_4(u) = \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\mu G_1\mathcal{F}[u]\,\mathrm{d}\xi\right], \quad \mathcal{G}_5(u) = \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\varepsilon^{\frac{1}{2}}kG_1\mathcal{F}[u]\,\mathrm{d}\xi\right]$$

belong to $W_\varepsilon^{1+\delta,p}(\Sigma)$ *and satisfy the estimate*

$$\|\mathcal{G}_j(u)\|_{\delta,p,\varepsilon} \leqq c\varepsilon\|u\|_{\delta,p,\varepsilon}, \quad j = 4, 5.$$

Proof. Observe that

$$\|\mathcal{G}_4(u)\|_{1,p,\varepsilon} = \|\partial_x\mathcal{G}(u)\|_{1,p,\varepsilon} \leqq \|\mathcal{G}(u)\|_{2,p,\varepsilon} \leqq \|u\|_p,$$

and

$$\|\mathcal{G}_4(u)\|_{2,p,\varepsilon} = \|\mathcal{G}(u_x)\|_{2,p,\varepsilon} \leqq c\varepsilon\|u_x\|_p \leqq c\varepsilon\|u\|_{1,p,\varepsilon}.$$

Interpolating between the previous two inequalities, we find that

$$\|\mathcal{G}_4(u)\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon\|u\|_{\delta,p,\varepsilon},$$

and we similarly find that

$$\|\mathcal{G}_5(u)\|_{1+\delta,p,\varepsilon} \leqq c\varepsilon\|u\|_{\delta,p,\varepsilon}.$$

□

Parts (ii)–(v) of Lemma 4 are established in an analogous fashion.

5.3. Convergence properties

Our final piece of analysis is the proof of Lemma 12, which relates to the operators

$$\mathcal{G}_i^{N,m} = \chi_N \mathcal{G}_i (1 - \chi_{R_m}), \quad i = 1, \dots, 6, 8, \dots, 11.$$

We begin by examining $\mathcal{G}_1^{N,m}, \mathcal{G}_2^{N,m}, \mathcal{G}_3^{N,m} : W_\varepsilon^{\delta,p}(\mathbb{R}^2) \to W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$.

Lemma 24. *Choose $N > 0$, suppose that $\{R_m\}$ is a sequence of positive, real numbers such that $R_m \to \infty$ as $m \to \infty$ and let $\chi_N : \mathbb{R}^2 \to \mathbb{R}$, $\chi_{R_m} : \mathbb{R}^2 \to \mathbb{R}$ be smooth 'cut-off' functions whose support is contained in respectively $\bar{B}_N(0)$ and $\bar{B}_{R_m}(0)$. The functions*

$$\mathcal{G}_i^{N,m}(u) = \chi_N \mathcal{G}_i((1 - \chi_{R_m})u), \quad i = 1, 2, 3$$

satisfy

$$\|\mathcal{G}_i^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leqq c_\varepsilon^{N,m} \|u\|_{\delta,p,\varepsilon}, \quad i = 1, 2, 3$$

for each $\delta \in [0, 1]$ and each sufficiently large value of p, in which the symbol $c_\varepsilon^{N,m}$ denotes a quantity that, for each fixed value of N and ε, tends to zero as $m \to \infty$.

Proof. Suppose that $f(\mu, k)$ is one of

$$\frac{1}{1+\varepsilon+\beta q^2}, \quad \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}, \quad \frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{1+\varepsilon+\beta q^2},$$

$$\frac{-\mu^2}{1+\varepsilon+\beta q^2}, \quad \frac{-\varepsilon k^2}{1+\varepsilon+\beta q^2}, \quad \frac{-\varepsilon^{\frac{1}{2}}\mu k}{1+\varepsilon+\beta q^2}$$

and define

$$\mathcal{G}^{N,m}(u) = \chi_N \mathcal{F}^{-1}[f(\mu, k)\mathcal{F}[(1 - \chi_{R_m})u]],$$

so that

$$\begin{aligned} &\mathcal{G}^{N,m}(u)(x, z) \\ &\quad = \chi_N(x, z) \int_{\mathbb{R}^2} K(x - x_1, z - z_1)(1 - \chi_{R_m}(x_1, z_1))u(x_1, z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1, \end{aligned}$$

where $K(x, z) = \mathcal{F}^{-1}[f(\mu, k)]$. (Note that $f \notin L^1(\mathbb{R}^2)$, so that K is only well defined as part of the above convolution.) The $L^p(\mathbb{R}^2)$-norm of $\mathcal{G}^{N,m}(u)$ is given by

$$\|\mathcal{G}^{N,m}(u)\|_p = \left(\int_{N_1^{x,z}} \left| \int_{N_2^{x_1,z_1}} K(x - x_1, z - z_1)u(x_1, z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1 \right|^p \mathrm{d}x\,\mathrm{d}z \right)^{\frac{1}{p}},$$

where

$$N_1^{x,z} = \{(x_1, z_1) : x^2 + z^2 \leqq N\}, \quad N_2^{x_1,z_1} = \{(x_1, z_1) : x^2 + z^2 \geqq R_m\},$$

and using the generalised version of Hölder's inequality [21, Theorem 188], one finds that

$$\begin{aligned}
&\|\mathcal{G}^{N,m}(u)\|_p \\
&= \left(\int_{N_1^{x,z}} \left| \int_{N_2^{x_1,z_1}} \frac{|x-x_1|^2+|z-z_1|^2}{|x-x_1|^2+|z-z_1|^2} K(x-x_1, z-z_1) u(x_1,z_1)\, \mathrm{d}x_1\, \mathrm{d}z_1 \right|^p \mathrm{d}x\, \mathrm{d}z \right)^{\frac{1}{p}} \\
&\leqq \|u\|_p \left(\int_{N_1^{x,z}} \left(\int_{N_2^{x_1,z_1}} \left(\frac{1}{|x-x_1|^2+|z-z_1|^2} \right)^{q_1} \mathrm{d}x_1\, \mathrm{d}z_1 \right)^{\frac{p}{q_1}} \right. \\
&\qquad \left. \times \left(\int_{N_2^{x_1,z_1}} ((|x-x_1|^2+|z-z_1|^2)|K(x-x_1,z-z_1)|)^{q_2}\, \mathrm{d}x_1\, \mathrm{d}z_1 \right)^{\frac{p}{q_2}} \mathrm{d}x\, \mathrm{d}z \right)^{\frac{1}{p}},
\end{aligned}$$

where

$$\frac{1}{p}+\frac{1}{q_1}+\frac{1}{q_2}=1, \quad 1<q_1<2, \quad q_2>2$$

(choices of q_1, q_2 in the indicated ranges are possible for sufficiently large values of p).

A direct calculation shows that $\partial_\mu^2 f$, $\partial_k^2 f$ are bounded as $q \to 0$ and $\mathcal{O}(q^{-2})$ as $q \to \infty$; they therefore belong to $L^s(\mathbb{R}^2)$ for all $s>1$. Using this fact, we find that

$$\begin{aligned}
&\left(\int_{N_2^{x_1,z_1}} ((|x-x_1|^2+|z-z_1|^2)|K(x-x_1,z-z_1)|)^{q_2}\, \mathrm{d}x_1\, \mathrm{d}z_1 \right)^{\frac{1}{q_2}} \\
&\qquad \leqq \left(\int_{\mathbb{R}^2} ((|x-x_1|^2+|z-z_1|^2)|K(x-x_1,z-z_1)|)^{q_2}\, \mathrm{d}x_1\, \mathrm{d}z_1 \right)^{\frac{1}{q_2}} \\
&\qquad = \left(\int_{\mathbb{R}^2} ((|x|^2+|z|^2)|K(x,z)|)^{q_2}\, \mathrm{d}x\, \mathrm{d}z \right)^{\frac{1}{q_2}} \\
&\qquad \leqq \left(\int_{\mathbb{R}^2} |x^2 K(x,z)|^{q_2}\, \mathrm{d}x\, \mathrm{d}z \right)^{\frac{1}{q_2}} + \left(\int_{\mathbb{R}^2} |z^2 K(x,z)|^{q_2}\, \mathrm{d}x\, \mathrm{d}z \right)^{\frac{1}{q_2}} \\
&\qquad \leqq \left(\int_{\mathbb{R}^2} |\partial_\mu^2 f(\mu,k)|^{q_2'}\, \mathrm{d}\mu\, \mathrm{d}k \right)^{\frac{1}{q_2'}} + \left(\int_{\mathbb{R}^2} |\partial_k^2 f(\mu,k)|^{q_2'}\, \mathrm{d}\mu\, \mathrm{d}k \right)^{\frac{1}{q_2'}} \\
&\qquad \leqq c_\varepsilon,
\end{aligned}$$

where q_2' is the conjugate index to q_2 and we have used the Hausdorff–Young inequality

$$\|u\|_{q_2} \leqq \|\mathcal{F}[u]\|_{q_2'}, \quad q_2>2$$

(see, for example, HARDY et al. [21, Sections 8.5, 8.17]). It follows that

$$\|\mathcal{G}^{N,m}(u)\|_p \leqq c_\varepsilon \left(\int_{N_1^{x,z}} \left(\int_{N_2^{x_1,z_1}} \left(\frac{1}{|x-x_1|^2+|z-z_1|^2} \right)^{q_1} \mathrm{d}x_1\, \mathrm{d}z_1 \right)^{\frac{p}{q_1}} \mathrm{d}x\, \mathrm{d}z \right)^{\frac{1}{p}} \|u\|_p,$$

and this inequality and the calculation

$$\begin{aligned}
&\left(\int_{N_1^{x,z}}\left(\int_{N_2^{x_1,z_1}}\left(\frac{1}{|x-x_1|^2+|z-z_1|^2}\right)^{q_1}\,\mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{p}{q_1}}\,\mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{p}}\\
&\leqq \left(\int_{N_1^{x,z}}\left(\int_{N_3^{x_1,z_1}}\left(\frac{1}{|x_1|^2+|z_1|^2}\right)^{q_1}\,\mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{p}{q_1}}\,\mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{p}}\\
&= (\pi N^2)^{\frac{1}{p}}\left(\frac{2\pi}{2q_1-2}\right)^{\frac{1}{q_1}}(R_m-N)^{-2+\frac{2}{q_1}}\\
&\to 0 \qquad (121)
\end{aligned}$$

as $m \to \infty$, where $N_3^{x_1,z_1} = \{(x_1, z_1) : x_1^2 + z_1^2 \geqq R_m - N\}$, imply that

$$\|\mathcal{G}^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_p.$$

Clearly

$$\|\mathcal{G}_i^{N,m}(u)\|_{1,p,\varepsilon} \leqq \|\mathcal{G}_i^{N,m}(u)\|_p + \|\partial_x\mathcal{G}_i^{N,m}(u)\|_p + \|\partial_z\mathcal{G}_i^{N,m}(u)\|_p, \quad i = 1, 2, 3,$$

and because $\mathcal{G}_i^{N,m}(u) = \chi_N\mathcal{F}^{-1}[f_i(\mu, k)\mathcal{F}[(1-\chi_{R_m})u]]$, $i = 1, 2, 3$, where $f_i(\mu, k)$ is the ith choice for $f(\mu, k)$, we have that

$$\|\mathcal{G}_i^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_p.$$

Furthermore, the above argument shows that both terms on the right-hand side of the inequalities

$$\begin{aligned}
\|\partial_x\mathcal{G}_i^{N,m}(u)\|_p &\leqq \|\partial_x\chi_N\mathcal{F}^{-1}[f_i(\mu, k)\mathcal{F}[(1-\chi_{R_m})u]]\|_p\\
&\quad+\|\chi_N\mathcal{F}^{-1}[\mathrm{i}\mu f_i(\mu, k)\mathcal{F}[(1-\chi_{R_m})u]]\|_p,\\
\varepsilon^{\frac{1}{2}}\|\partial_z\mathcal{G}_i^{N,m}(u)\|_p &\leqq \varepsilon^{\frac{1}{2}}\|\partial_z\chi_N\mathcal{F}^{-1}[f_i(\mu, k)\mathcal{F}[(1-\chi_{R_m})u]]\|_p\\
&\quad+\|\chi_N\mathcal{F}^{-1}[\mathrm{i}\varepsilon^{\frac{1}{2}}kf_i(\mu, k)\mathcal{F}[(1-\chi_{R_m})u]]\|_p
\end{aligned}$$

are bounded by $c_\varepsilon^{N,m}\|u\|_p$, the first because $\partial_x\chi_N$, $\partial_z\chi_N$ have the same support as χ_N and the second because each of $\mathrm{i}\mu f_i(\mu, k)$ and $\mathrm{i}\varepsilon^{\frac{1}{2}}kf_i(\mu, k)$ is one of the fourth, fifth or sixth choices for $f(\mu, k)$. Altogether we have that

$$\|\mathcal{G}_i^{N,m}(u)\|_{1,p,\varepsilon} \leqq c_\varepsilon^{N,m}\|u\|_p, \qquad (122)$$

and a similar argument shows that

$$\|\partial_{xx}\mathcal{G}_i^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon}, \quad \varepsilon\|\partial_{zz}\mathcal{G}_i^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon},$$

so that

$$\|\mathcal{G}_i^{N,m}(u)\|_{2,p,\varepsilon} \leqq c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon}. \qquad (123)$$

Interpolating between (122) and (123), one finds that

$$\|\mathcal{G}_i^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leqq c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}.$$

□

The corresponding results for $\mathcal{G}_4^{N,m}$, $\mathcal{G}_5^{N,m}$, $\mathcal{G}_6^{N,m}$, and $\mathcal{G}_8^{N,m}$, …, $\mathcal{G}_{11}^{N,m}$, are obtained by combining elements of the proof of Lemma 24 with the methods used to establish the mapping properties of $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$ and $\mathcal{G}_8$, …, $\mathcal{G}_{11}$ in Section 5.2. We give the details for $\mathcal{G}_4^{N,m}$ and $\mathcal{G}_5^{N,m}$; the remaining operators are treated in an analogous fashion.

Lemma 25. *Choose $N > 0$, suppose that $\{R_m\}$ is a sequence of positive, real numbers such that $R_m \to \infty$ as $m \to \infty$ and let $\chi_N : \mathbb{R}^2 \to \mathbb{R}$, $\chi_{R_m} : \mathbb{R}^2 \to \mathbb{R}$ be smooth 'cut-off' functions whose support is contained in, respectively, $\bar{B}_N(0)$ and $\bar{B}_{R_m}(0)$. The functions*

$$\mathcal{G}_4^{N,m}(u) = \chi_N \mathcal{G}_4((1-\chi_{R_m})u), \quad \mathcal{G}_5^{N,m}(u) = \chi_N \mathcal{G}_5((1-\chi_{R_m})u)$$

satisfy

$$\|\mathcal{G}_4^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leqq c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}, \quad \|\mathcal{G}_5^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leqq c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}$$

for each $\delta \in [0, 1]$ and each sufficiently large value of p, in which the symbol $c_\varepsilon^{N,m}$ denotes a quantity that, for each fixed value of N and ε, tends to zero as $m \to \infty$.

Proof. The first step is to show that

$$\begin{aligned} \|\mathcal{G}_4^{N,m}(u)\|_p &\leqq c_\varepsilon^{N,m}\|u\|_p, \\ \|\bar{\mathcal{G}}_4^{N,m}(u)\|_p &\leqq c_\varepsilon^{N,m}\|u\|_p, \\ \|\hat{\mathcal{G}}_4^{N,m}(u)\|_p &\leqq c_\varepsilon^{N,m}\|u\|_p, \end{aligned} \tag{124}$$

where $\bar{\mathcal{G}}_4^{N,m}$ and $\hat{\mathcal{G}}_4^{N,m}$ are the operators obtained by replacing $\mathrm{i}\mu$ with, respectively, $-\mu^2$ and $-\varepsilon^{\frac{1}{2}}\mu k$ in the definition of $\mathcal{G}_4^{N,m}$; using the argument given at the end of Lemma 24 we immediately deduce that

$$\|\partial_x \mathcal{G}_4^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_p, \quad \varepsilon^{\frac{1}{2}}\|\partial_z \mathcal{G}_4^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_p \tag{125}$$

and

$$\|\partial_{xx} \mathcal{G}_4^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon}, \quad \varepsilon\|\partial_{zz} \mathcal{G}_4^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon}. \tag{126}$$

To this end we use the decompositions

$$\mathcal{G}_4^{N,m} = \mathcal{G}_{4\mathrm{a}}^{N,m} + \mathcal{G}_{4\mathrm{b}}^{N,m}, \quad \bar{\mathcal{G}}_4^{N,m} = \bar{\mathcal{G}}_{4\mathrm{a}}^{N,m} + \bar{\mathcal{G}}_{4\mathrm{b}}^{N,m}, \quad \hat{\mathcal{G}}_4^{N,m} = \hat{\mathcal{G}}_{4\mathrm{a}}^{N,m} + \hat{\mathcal{G}}_{4\mathrm{b}}^{N,m}$$

and

$$\mathcal{G}_{4\mathrm{b}}^{N,m} = \mathcal{G}_{4\mathrm{b},1}^{N,m} + \mathcal{G}_{4\mathrm{b},2}^{N,m}, \quad \bar{\mathcal{G}}_{4\mathrm{b}}^{N,m} = \bar{\mathcal{G}}_{4\mathrm{b},1}^{N,m} + \bar{\mathcal{G}}_{4\mathrm{b},2}^{N,m}, \quad \hat{\mathcal{G}}_{4\mathrm{b}}^{N,m} = \hat{\mathcal{G}}_{4\mathrm{b},1}^{N,m} + \hat{\mathcal{G}}_{4\mathrm{b},2}^{N,m},$$

which are defined using respectively the 'cut-off' function χ (see the explanation above Lemma 18) and the expression (113) (see the explanation above Proposition 17).

Let us write

$$G_1 = \varepsilon^2\tilde{G}_1 + \varepsilon^2\tilde{G}_2 + \varepsilon^2\tilde{G}_3$$

(see equation (106)). Calculations similar to those presented in Lemma 18 show that

$$\partial_\mu^2(\mathrm{i}\mu\tilde{G}_i), \quad \partial_k^2(\mathrm{i}\mu\tilde{G}_i), \quad i = 1, 2, 3$$

are bounded at the origin, so that $\partial_\mu^2(\chi\mathrm{i}\mu G_1)$ and $\partial_k^2(\chi\mathrm{i}\mu G_1)$ belong to $L^s(\mathbb{R}^2)$ for each $s > 1$. Noting that all estimates are uniform for $y, \xi \in [0, 1]$, we may apply the method explained in Lemma 24 to find that

$$\|\mathcal{G}_{4\mathrm{a}}^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_p,$$

and the same argument shows that

$$\|\bar{\mathcal{G}}_{4\mathrm{a}}^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_p, \quad \|\hat{\mathcal{G}}_{4\mathrm{a}}^{N,m}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_p.$$

To obtain the corresponding estimates for $\mathcal{G}_{4\mathrm{b},1}^{N,m}$ we use the expression (115) derived in Lemma 20. We consider the first of the four terms in this expression in detail; the others are handled in an analogous fashion. Define

$$I = \varepsilon^2\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu(1+\varepsilon+\beta q^2-\varepsilon\mu^2/q)(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q|\xi-y|}\right].$$

In terms of the polar coordinates (116), one has that

$$I = I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi}\int_0^{2\pi}\int_0^\infty \frac{(1+\varepsilon)q^2(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\cos(\phi+\psi)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,$$

$$I_2 = \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi}\int_0^{2\pi}\int_0^\infty \frac{q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\cos^3(\phi+\psi)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,$$

$$I_3 = \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi}\int_0^{2\pi}\int_0^\infty \frac{\beta q^4(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\cos(\phi+\psi)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,$$

$\psi = \theta - \phi$ and $\tilde{Q} = q^2\cos^2(\phi+\psi) - (1+\varepsilon+\beta q^2)q\tanh q$, and the method used in the proof of Lemma 20 shows that

$$|I_1| \leqq \frac{c_\varepsilon}{r^2}, \quad |I_2| \leqq \frac{c_\varepsilon}{r^2}, \quad |I_3| \leqq \frac{c_\varepsilon}{r^3}, \quad y \neq \xi.$$

The above calculation indicates that

$$\mathcal{F}^{-1}[\mathcal{G}_{4\mathrm{b},1}^{N,m}(\mu, k; y, \xi)] = \sum K_i(x, z; y, \xi),$$

where each summand (of which there are a finite number) satisfies the inequality

$$|K_i(x, y; y, \xi)| \leqq \frac{c_\varepsilon}{r^{n_i}}, \quad n_i \geqq 2$$

for $y \neq \xi$. Observe that

$$
\begin{aligned}
&\left(\int_{N_1^{x,z}}\int_0^1 \left|\int_{N_2^{x_1,z_1}}\int_0^1 K_i(x-x_1, z-z_1; y, \xi)u(x_1, \xi, z_1)\,\mathrm{d}\xi\,\mathrm{d}x_1\,\mathrm{d}z_1\right|^p \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{p}}\\
&\leqq \|u\|_p \int_{N_1^{x,z}}\int_0^1 \left(\int_{N_2^{x_1,z_1}}\int_0^1 |K_i(x-x_1, z-z_1; y, \xi)|\,\mathrm{d}\xi\,\mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{p}{p'}} \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z\\
&\leqq c_\varepsilon \|u\|_p \int_{N_1^{x,z}} \left(\int_{N_2^{x_1,z_1}} \left(\frac{1}{|x-x_1|^2+|z-z_1|^2}\right)^{\frac{p'n_i}{2}} \mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{p}{p'}}\\
&\leqq c_\varepsilon (\pi N^2)^{\frac{1}{p}} \left(\frac{2\pi}{p'n_i-2}\right)^{\frac{1}{p'}} (R_m - N)^{-n_i+\frac{2}{p'}} \|u\|_p\\
&\to 0
\end{aligned}
$$

as $m \to \infty$, where p' is the conjugate index to p and we have used Hölder's inequality and the calculation (121). It follows that

$$
\begin{aligned}
&\|\mathcal{G}^{N,m}(u)\|_p^p\\
&= \int_{N_1^{x,z}}\int_0^1 \left|\int_{N_2^{x_1,z_1}}\int_0^1 \sum K_i(x-x_1, z-z_1; y, \xi)u(x_1, \xi, z_1)\,\mathrm{d}\xi\,\mathrm{d}x_1\,\mathrm{d}z_1\right|^p \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z\\
&\leqq c_\varepsilon^{N,m} \|u\|_p^p.
\end{aligned}
$$

This technique also yields the estimates for $\bar{\mathcal{G}}_{1\mathrm{b},1}^{N,m}$ and $\hat{\mathcal{G}}_{1\mathrm{b},1}^{N,m}$; here we have to estimate

$$
I = \varepsilon^2 \mathcal{F}^{-1}\left[\left\{\begin{matrix} -\mu^2 \\ -\varepsilon^{\frac{1}{2}}\mu k \end{matrix}\right\} \frac{(1+\varepsilon+\beta q^2-\varepsilon\mu^2/q)(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2 k^2)} \mathrm{e}^{-q|\xi-y|}\right]
$$

(and three other terms with slightly different exponential factors), and hence

$$
\begin{aligned}
I_1 = \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi}\int_0^{2\pi}\int_0^\infty &\frac{(1+\varepsilon)q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\\
&\times \left\{\begin{matrix} \cos^2(\phi+\psi) \\ \cos(\phi+\psi)\sin(\phi+\psi) \end{matrix}\right\} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,\\
I_2 = \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi}\int_0^{2\pi}\int_0^\infty &\frac{q^4(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\\
&\times \left\{\begin{matrix} \cos^4(\phi+\psi) \\ \cos^3(\phi+\psi)\sin(\phi+\psi) \end{matrix}\right\} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,\\
I_3 = \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi}\int_0^{2\pi}\int_0^\infty &\frac{\beta q^5(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\\
&\times \left\{\begin{matrix} \cos^2(\phi+\psi) \\ \cos(\phi+\psi)\sin(\phi+\psi) \end{matrix}\right\} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi.
\end{aligned}
$$

We find that

$$|I_1| \leqq \frac{c_\varepsilon}{r^3}, \quad |I_2| \leqq \frac{c_\varepsilon}{r^3}, \quad |I_3| \leqq \frac{c_\varepsilon}{r^4}, \quad y \neq \xi,$$

and the argument given above therefore yields the inequalities

$$\|\bar{\mathcal{G}}^{N,m}_{4\mathrm{b},1}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_p, \quad \|\hat{\mathcal{G}}^{N,m}_{4\mathrm{b},1}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_p.$$

Calculations similar to those presented in Lemma 22 show that $\partial_\mu^2(\mathrm{i}\mu/Q)$, $\partial_k^2(\mathrm{i}\mu/Q)$ are $O(q^{-3})$ as $q \to \infty$, so that $\partial_\mu^2((1-\chi)\mathrm{i}\mu/Q)$, $\partial_k^2((1-\chi)\mathrm{i}\mu/Q)$ belong to $L^s(\mathbb{R}^2)$ for all $s > 1$. Noting that all estimates are uniform for y, $\xi \in [0, 1]$, we may apply the method used in Lemma 24 to find that

$$\|\mathcal{G}^{N,m}_{4\mathrm{b},2}(u)\|_p \leqq c_\varepsilon^{N,m}\|u\|_p,$$

and the same method yields the corresponding estimates for $\bar{\mathcal{G}}^{N,m}_{4\mathrm{b},2}$ and $\hat{\mathcal{G}}^{N,m}_{4\mathrm{b},2}$. Finally, we obtain the estimates

$$\begin{aligned}\|\partial_y \mathcal{G}_4^{N,m}(u)\|_p &= \left\|\chi_N \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\mu\partial_y G_1 \mathcal{F}[(1-\chi_{R_m})u]\,\mathrm{d}\xi\right]\right\|_p \\ &\leqq c_\varepsilon^{N,m}\|u\|_p, \end{aligned} \tag{127}$$

$$\begin{aligned}\|\partial_y^2 \mathcal{G}_4^{N,m}(u)\|_p &= \left\|\chi_N \mathcal{F}^{-1}\left[\int_0^1 \partial_y^2 G_1(\mathcal{F}[\partial_x((1-\chi_{R_m})u)])\,\mathrm{d}\xi\right]\right\|_p \\ &\leqq c_\varepsilon^{N,m}\|\partial_x((1-\chi_{R_m})u)\|_p \\ &\leqq c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon} \end{aligned} \tag{128}$$

using the method given above for $\bar{\mathcal{G}}_4^{N,m}$, noting that $\partial_\mu^2(\mathrm{i}\mu\partial_y G_1)$, $\partial_k^2(\mathrm{i}\mu\partial_y G_1)$ and $\partial_\mu^2(\partial_y^2 G_1)$, $\partial_k^2(\partial_y^2 G_1)$ are bounded at the origin and the polar-coordinate representation of their kernels differ from those of $\bar{G}_4^{N,m}$ only in the form of the trigonometric factor.

It follows from (124)–(128) that

$$\|\mathcal{G}_4^{N,m}(u)\|_{1,p,\varepsilon} \leqq c_\varepsilon^{N,m}\|u\|_p, \quad \|\mathcal{G}_4^{N,m}(u)\|_{2,p,\varepsilon} \leqq c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon},$$

and interpolating between these inequalities, we find that

$$\|\mathcal{G}_4^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leqq c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}.$$

The same method yields the corresponding estimate for $\mathcal{G}_5^{N,m}$. □

Acknowledgements. M. D. Groves wishes to record his thanks for the warm hospitality shown by the Mathematisches Institut I, Universität Karlsruhe during an extended visit in which the research reported in this paper was undertaken. S.-M. Sun was partly supported by the National Science Foundation under grant DMS-0309160.

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(*Received August 9, 2005 / Accepted November 6, 2006*)
Published online October 5, 2007 – © Springer-Verlag (2007)