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Higher Order Entropies

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Abstract

Higher order entropies are kinetic entropy estimators for fluid models. These quantities are quadratic in the velocity and temperature derivatives and have temperature dependent coefficients. We investigate governing equations for higher order entropies and related a priori estimates in the natural situation where viscosity and thermal conductivity depend on temperature. We establish positivity of higher order derivative source terms in these governing equations provided that $\| \log T \|_{BMO} + \| v/\sqrt{T} \|_{L^\infty}$ is small enough. The temperature factors renormalizing temperature and velocity derivatives then yield majorization of lower order convective terms only when the temperature dependence of transport coefficients is taken into account according to the kinetic theory. In this situation, we obtain entropic principles for higher order entropies of arbitrary order. As an application, we investigate a priori estimates and global existence of solutions when the initial values $\log(T_0/T_\infty)$ and $v_0/\sqrt{T_0}$ are small enough in appropriate spaces.

1. Introduction

We investigate higher order entropies for fluid models and related a priori estimates. Higher order entropies are kinetic entropy estimators for fluid models. These quantities may also be interpreted as Fisher information estimators or associated with generalized Bernstein equations. For simple fluid models, higher order entropies are quadratic or polynomial with respect to velocity and temperature derivatives and have temperature dependent coefficients. They are investigated in this paper in the situation of incompressible flows spanning the whole space with temperature dependent thermal conductivity and viscosity. The cases of compressible flows or zero Mach number flows are beyond the scope of the present paper.

As a preliminary study, we consider second order entropies for fluid models with constant transport coefficients. We derive a governing equation for second order kinetic entropy correctors and investigate when higher order derivative terms, which appear as sources, have a sign. Unconditional positivity of these source terms only holds for a restricted family of second order entropy correctors. The temperature weights renormalizing solution derivatives, however, do not yield majorization of the corresponding lower order terms arising from convection. As a consequence of this preliminary analysis, we need to investigate conditional positivity of higher order derivative source terms as well as to modify the renormalizing temperature weights by taking into account the natural temperature dependence of transport coefficients.

Temperature dependence of viscosity and thermal conductivity is a consequence of the kinetic theory of gases. Away from small temperatures, these coefficients essentially behave like a power of temperature with a common exponent $\varkappa$. In this situation, we derive a balance equation for kinetic entropy correctors of arbitrary order. These higher order kinetic entropy correctors are quadratic or polynomial in the velocity and temperature derivatives with temperature dependent coefficients. The corresponding balance equations have source terms in the form of sums of products of solution derivatives.

We then obtain conditional positivity of higher order derivative source terms when $\| \log T \|_{BMO} + \| v/\sqrt{T} \|_{L^\infty}$ is small enough. The lower order convective source terms are then majorized thanks to the temperature dependence of transport coefficients as given by the kinetic theory of gases, that is, only when $\varkappa \geqq 1/2$. In order to establish these estimates, we use the Coifman–Meyer inequalities for multilinear operators and weighted interpolation inequalities for intermediate derivatives with weights in Muckenhoupt classes. We next investigate higher order kinetic entropy estimators obtained by summing up a zeroth order entropy estimator with kinetic entropy correctors and we obtain entropic principles which are the main result of the paper.

As an example of application of higher order entropic estimates, we establish a global existence theorem provided that $\log(T_0/T_\infty)$ and $v_0/\sqrt{T_0}$ are small enough in appropriate spaces, which may be interpreted heuristically as an existence theorem for small Mach number flows.

In Section 2 we discuss the concept of higher order entropies. In Section 3, as a preliminary study, we investigate how this notion can be used by studying second order entropies for fluid models with constant transport coefficients. In Section 4, we gather material from harmonic analysis and establish various weighted inequalities. In Section 5 we establish balance equations for higher order entropy correctors in the natural situation of variable transport coefficients. In Section 6—the core of the paper—we establish that higher order entropies satisfy conditional entropic inequalities. Finally, in Section 7, as an example of application, we concentrate on global solutions.

2. Higher order entropies

The notion of entropy has been shown to be of fundamental importance in fluid modeling from both a physical and a mathematical point of view [2,5,6,11–14, 17,21,22,28,44]. We discuss heuristically in this section a concept of higher order mathematical entropies for fluid models [18].

2.1. *Entropic interpretation of the Bernstein equation*

For parabolic (or elliptic) scalar equations, a priori estimates of the solution derivatives can be obtained by using the Bernstein method [1,30]. More specifically, consider (as a simple example) the heat equation

$$\partial_t u - \Delta u = 0. \tag{2.1}$$

Defining $\zeta = |\partial_x u|^2 = \partial_x u \cdot \partial_x u$, we then have the Bernstein equation

$$\partial_t \zeta - \Delta \zeta + 2|\partial_x^2 u|^2 = 0. \tag{2.2}$$

In the Bernstein method, the higher order source term $|\partial_x^2 u|^2 = \partial_x^2 u : \partial_x^2 u = \sum_{ij} (\partial_{ij} u)^2$ is discarded so that one obtains inequalities like $\partial_t \zeta - \Delta \zeta \leqq 0$ and the maximum principle can then be used [1,30]. On the other hand, one can also directly integrate Bernstein equation (2.2) to get estimates of the integrals of $|\partial_x u|^2$ and $|\partial_x^2 u|^2$. The classical techniques which consist of multiplying the heat equation by either the laplacian Δu or the time derivative $\partial_t u$ (and then integrating by parts) are also equivalent to integrating the Bernstein equation (2.2).

Although the Bernstein method cannot be extended to systems of partial differential equations (in the absence of maximum principles), we may still try to derive an equation similar to that of the Bernstein equation. In such a generalized equation, we do not expect a second order term in the simple form $\Delta \zeta = \partial_x \cdot (\partial_x \zeta)$ since, for such systems, dissipative fluxes and gradients are not anymore related by scalar or even diagonal matrices. However, we may focus on the source term whose principal part $|\partial_x^2 u|^2$ has a sign. The structure of the Bernstein equation (2.2) then appears to be formally similar to that of an entropy balance, where ζ plays the role of a generalized entropy, even though there also exist zeroth order entropies like u^2. In the next section, we introduce a kinetic framework supporting this entropic interpretation.

2.2. *Enskog second order kinetic entropy corrector*

We consider, for the sake of simplicity, a single monatomic dilute gas. The state of the gas is described by a distribution function $f(t, x, c)$ governed by Boltzmann's equation, where t is time, x the n-dimensional spatial coordinate, and c the molecular velocity [2,5,6,11,14,17,22,44]. Approximate solutions of Boltzmann's equation are obtained from a first order Enskog (formal) expansion

$$f = f^{(0)}\big(1 + \varepsilon\phi^{(1)} + \mathscr{O}(\varepsilon^2)\big), \tag{2.3}$$

where $f^{(0)}$ is the local Maxwellian distribution, $\phi^{(1)}$ the perturbation associated with the Navier–Stokes regime and ε the usual formal expansion parameter. The perturbation $\phi^{(1)}$ depends linearly on the temperature and velocity gradients and is the solution of a linearized Boltzmann equation [6,14,17]. The compressible Navier–Stokes equations (or the zero Mach number equations) can then be obtained by taking parts of Boltzmann's equation [6,14,22].

A fundamental property is that the kinetic entropy defined by

$$S^{\mathrm{kin}} = -k_{\mathrm{B}} \int_{\mathbb{R}^n} f\big(\log f - 1\big) dc, \tag{2.4}$$

where k_{B} is the Boltzmann constant, obeys the H theorem, that is, the second principle of thermodynamics [6,11,14,17,44]. The expansion of S^{kin} induced by a second order Enskog expansion, however, can be written as

$$S^{\mathrm{kin}} = S^{(0)} + \varepsilon^2 S^{(2)} + \mathscr{O}(\varepsilon^3), \tag{2.5}$$

where $S^{(0)}$ is the usual zeroth order macroscopic entropy evaluated from Maxwellian distributions and where $S^{(2)}$ reads

$$S^{(2)} = -\frac{k_{\mathrm{B}}}{2} \int_{\mathbb{R}^n} f^{(0)} (\phi^{(1)})^2 dc, \tag{2.6}$$

so that $-S^{(2)}$ is quadratic in the temperature and velocity gradients and is a natural candidate for deriving a balance equation like (2.2). For compressible monatomic gases, after detailed calculations, one can establish that

$$S^{(2)} = -\frac{1}{\rho}\big(\overline{\lambda}|\partial_x T|^2 + \tfrac{1}{2}\overline{\eta}\,|d|^2\big), \tag{2.7}$$

where T denotes the absolute temperature, ρ the density, v the gas velocity, d the strain rate tensor $d = \partial_x v + \partial_x v^t - \frac{2}{n}(\partial_x \cdot v)\, I$ and $|d|^2 = \sum_{ij} d_{ij}^2$, and where the scalar coefficients $\overline{\lambda}$ and $\overline{\eta}$ only depend on temperature. In a first approximation, using a single term in orthogonal polynomial expansions of perturbed distribution functions, one can establish that $\overline{\lambda} = (1/2rc_p)\lambda^2/T^3$ and $\overline{\eta} = (1/2r)\eta^2/T^2$, where c_p is the constant pressure specific heat per unit mass, r the gas constant per unit mass, λ the thermal conductivity, η the shear viscosity, and the actual values of the numerical factors in front of $\overline{\lambda}$ and $\overline{\eta}$ are evaluated here for $n = 3$. In the special case of Maxwellian gases, such a calculation has already been performed by BOLTZMANN [2].

2.3. *Zeroth order entropy dissipation rate*

A second kinetic interpretation can be obtained from the zeroth order entropy balance equation

$$\partial_t S^{(0)} + \partial_x\cdot(v S^{(0)}) + \partial_x\cdot F^{(0)} = \mathfrak{v}^{(0)}, \tag{2.8}$$

where $F^{(0)}$ is the zeroth order dissipative entropy flux and $\mathfrak{v}^{(0)}$ the zeroth order entropy production given by

$$\mathfrak{v}^{(0)} = \frac{\lambda}{T^2}|\partial_x T|^2 + \tfrac{1}{2}\frac{\eta}{T}\,|d|^2. \tag{2.9}$$

This entropy production is quadratic in the macroscopic variable gradients with temperature dependent coefficients. It also appears as a natural norm of the system and a natural candidate for deriving a balance equation like (2.2). Denoting the linearized Boltzmann equation by $\mathscr{J}\phi^{(1)} = \psi^{(1)}$, the second order entropy and the entropy production are essentially in the form $\langle \phi^{(1)}, \phi^{(1)} \rangle$ and $\langle \phi^{(1)}, \psi^{(1)} \rangle$, respectively, where $\langle \xi, \zeta \rangle = \int_{\mathbb{R}^n} f^{(0)} \xi \zeta \, dc$.

2.4. Enskog second order kinetic information corrector

The logarithmic Sobolev inequality majorizes the relative entropy of f with respect to $f^{(0)}$ by the relative Fisher information of f with respect to $f^{(0)}$ [3,4,26,40,44]. Here $f^{(0)}$ is the Maxwellian distribution with the same local macroscopic properties as f

$$f^{(0)} = \frac{\rho}{\mathfrak{m}} \Big(\frac{\mathfrak{m}}{2\pi k_B T} \Big)^{\frac{n}{2}} \exp\Big(-\frac{\mathfrak{m}(c - v)^2}{2k_B T} \Big), \tag{2.10}$$

where $\mathfrak{m}$ denotes the particle mass. After a rescaling 'à la Boltzmann', the logarithmic Sobolev inequality can be written in the form

$$0 \leqq k_B \int_{\mathbb{R}^n} (f/f^{(0)}) \log(f/f^{(0)}) f^{(0)} \, dc \leqq \frac{k_B{}^2 T}{2\mathfrak{m}} \int_{\mathbb{R}^n} \frac{|\partial_c (f/f^{(0)})|^2}{f/f^{(0)}} f^{(0)} \, dc \tag{2.11}$$

and one can establish that

$$S^{\mathrm{kin}} - S^{(0)} = -k_B \int_{\mathbb{R}^n} (f/f^{(0)}) \log(f/f^{(0)}) f^{(0)} \, dc,$$

so that the relative entropy of f with respect to f^0 coincides with $S^{(0)} - S^{\mathrm{kin}}$. The relative Fisher information thus appears as an estimator of kinetic entropy deviation. One can also establish that the relative Fisher information is given by

$$\mathscr{I}^{\mathrm{kin}} - \mathscr{I}^{(0)} = k_B \int_{\mathbb{R}^n} \frac{|\partial_c (f/f^{(0)})|^2}{f/f^{(0)}} f^{(0)} \, dc,$$

where $\mathscr{I}^{\mathrm{kin}} = k_B \int_{\mathbb{R}^n} (|\partial_c f|^2/f) \, dc$ and $\mathscr{I}^{(0)} = k_B \int_{\mathbb{R}^n} (|\partial_c f^0|^2/f^0) \, dc$ denote the kinetic and zeroth order Fisher informations. Substituting a second order Enskog expansion in the logarithmic Sobolev inequality (2.11), the leading order term of the left-hand side is $-S^{(2)}$ and the leading order term $I^{(2)}$ of the right-hand side reads

$$I^{(2)} = \frac{k_B T}{2\mathfrak{m}} \mathscr{I}^{(2)} = \frac{k_B{}^2 T}{2\mathfrak{m}} \int_{\mathbb{R}^n} |\partial_c \phi|^2 f^{(0)} \, dc,$$

and is a natural candidate for deriving a balance equation like (2.2). For compressible monatomic gases, after detailed calculations, one can establish that

$$I^{(2)} = \frac{1}{\rho} \big(\overline{\overline{\lambda}} |\partial_x T|^2 + \tfrac{1}{2} \overline{\overline{\eta}} \, |d|^2 \big), \tag{2.12}$$

where again, $\overline{\overline{\lambda}}$ and $\overline{\overline{\eta}}$ only depend on temperature. In a first approximation, using a single term in orthogonal polynomial expansions of perturbed distribution functions, one can establish that $\overline{\overline{\lambda}} = \big((10r + 3c_p)/4rc_p^2\big)\lambda^2/T^3$ and $\overline{\overline{\eta}} = (1/r)\eta^2/T^2$, where the actual values of the numerical factors in front of $\overline{\overline{\lambda}}$ and $\overline{\overline{\eta}}$ are evaluated here for $n = 3$. The rescaled second order information corrector $I^{(2)}$ is thus similar to the second order entropy corrector $-S^{(2)}$ and to the zeroth order entropy production rate $\mathfrak{v}^{(0)}$.

Remark 1. Logarithmic Sobolev inequalities have been investigated in a probabilistic framework by CATTIAUX [4]. In this situation, the relative Fisher information has been shown to represent a relative entropy in a path space, that is, in the space of particle trajectories [4]. This further supports the idea that these quadratic quantities represent an entropy.

2.5. Enskog higher order entropy correctors

Higher order Enskog expansions $f/f^{(0)} = 1+\varepsilon\phi^{(1)}+\cdots+\varepsilon^{2k}\phi^{(2k)}+\mathscr{O}(\varepsilon^{2k+1})$ actually induce higher order expansions for S^{kin}

$$S^{\text{kin}} - S^{(0)} = \varepsilon^2 S^{(2)} + \varepsilon^3 S^{(3)} + \cdots + \varepsilon^{2k} S^{(2k)} + \mathscr{O}(\varepsilon^{2k+1}), \tag{2.13}$$

where $S^{(l)}$ is a sum of terms in the form $k_{\mathrm{B}} \int_{\mathbb{R}^n} \prod_{1\leqq i\leqq l}\big(\phi^{(i)}\big)^{\nu_i} f^{(0)}\, dc$ with nonnegative integers $\nu_i \geqq 0$, $1 \leqq i \leqq l$, such that $l = \sum_{1\leqq i\leqq l} i\nu_i$.

On the other hand, in the absence of external forces acting on the particles, $\phi^{(l)}$ is a sum of products of solution derivatives with a total number of l derivations $\phi^{(l)} = \big(\frac{\eta}{\rho\sqrt{rT}}\big)^l \sum_\nu \mathrm{c}_\nu \prod_{1\leqq|\alpha|\leqq l} \big(\frac{\partial^\alpha T}{T}\big)^{\nu_\alpha} \big(\frac{\partial^\alpha \rho}{\rho}\big)^{\nu'_\alpha} \big(\frac{\partial^\alpha v}{\sqrt{rT}}\big)^{\nu''_\alpha}$, where $\nu_\alpha, \nu'_\alpha, \nu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\leqq|\alpha|\leqq l}$, and the summation is over all ν such that $\sum_{1\leqq|\alpha|\leqq l} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = l$ and where the coefficients c_ν are tensors in the reduced velocity $\mathscr{C} = (c - v)/\sqrt{2rT}$ multiplied by smooth scalar functions of $|\mathscr{C}|^2$ and $\log T$. This result is established by examining the successive construction of $\phi^{(l)}$ from $\phi^{(1)}, \dots, \phi^{(l-1)}$ applying l^{th} time the (generalized) inverse of the linearized collision operator which scales as $\eta/\rho rT$ and has isotropicity properties [5,14]. After integration with respect to $\mathscr{C}$, $S^{(2k)}$ is found in the form

$$S^{(2k)} = r\rho \Big(\frac{\eta}{\rho\sqrt{rT}}\Big)^{2k} \sum_\nu c_\nu \prod_{1\leqq|\alpha|\leqq 2k} \Big(\frac{\partial^\alpha T}{T}\Big)^{\nu_\alpha} \Big(\frac{\partial^\alpha \rho}{\rho}\Big)^{\nu'_\alpha} \Big(\frac{\partial^\alpha v}{\sqrt{rT}}\Big)^{\nu''_\alpha}, \tag{2.14}$$

where $\nu_\alpha, \nu'_\alpha, \nu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\leqq|\alpha|\leqq 2k}$ must be such that $\sum_{1\leqq|\alpha|\leqq 2k} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = 2k$ and where the coefficients c_ν are smooth scalar functions of $\log T$ of order unity. After integrations by parts with respect to the spatial variables in the integrals $\int_{\mathbb{R}^n} S^{(2k)}\, dx$, in order to eliminate spatial derivatives of order strictly greater than k, and by using interpolation inequalities, one obtains that the quantity $|\int_{\mathbb{R}^n} S^{(2k)}\, dx|$ is essentially controlled by the integral of

$$\gamma^{[k]} = r\rho \Big(\frac{\eta}{\rho\sqrt{rT}}\Big)^{2k} \Big(\Big|\frac{\partial^k T}{T}\Big|^2 + \Big|\frac{\partial^k v}{\sqrt{rT}}\Big|^2 + \Big|\frac{\partial^k \rho}{\rho}\Big|^2\Big), \tag{2.15}$$

or equivalently of

$$\widetilde{\gamma}^{[k]} = r\rho \left(\frac{\eta}{\rho\sqrt{rT}}\right)^{2k} \left(|\partial^k \log T|^2 + |\partial^k (v/\sqrt{rT})|^2 + |\partial^k \log \rho|^2\right). \qquad (2.16)$$

This suggests the quantities $\gamma^{[k]}$ or $\widetilde{\gamma}^{[k]}$ as $(2k)^{\text{th}}$ order kinetic entropy correctors (or kinetic entropy deviation estimators). Furthermore, the quantity $|\int_{\mathbb{R}^n} S^{(2k-1)}\, dx|$ is controlled by $\int_{\mathbb{R}^n} \gamma^{[k]}\, dx$ and $\int_{\mathbb{R}^n} \gamma^{[k-1]}\, dx$.

One could first consider using the corrector $S^{(2k)}$ as an entropy deviation estimator. However, except in the simpler case $k = 1$, it is not clear that $S^{(2k)}$ has a sign, or even its integral $\int_{\mathbb{R}^n} S^{(2k)}\, dx$. The calculations that can be found in the literature for the perturbed distribution function $\phi^{(2)}$ associated with the Burnett regime and $S^{(3)}$ already show intricate analytical complexities. Therefore, all we know about $S^{(2k)}$ is its structure (2.14), so that for some constant c depending on the L^∞ norm of the solution we have $|\int_{\mathbb{R}^n} S^{(2k)}\, dx| \leqq c \int_{\mathbb{R}^n} \gamma^{[k]}\, dx$. As a consequence, we will investigate $\gamma^{[k]}$ instead of $S^{(2k)}$ and we are therefore looking for *majorizing entropic correctors*. The fact that entropic inequalities can be obtained for such majorizing correctors can be seen to be a consequence of the structure of the fluid equations.

A similar analysis can also be conducted for the Fisher information and suggests the same quantities $\gamma^{[k]}$ or $\widetilde{\gamma}^{[k]}$ as higher order kinetic information correctors. Moreover, denoting by $\gamma^{[0]}$ or $\widetilde{\gamma}^{[0]}$ zeroth order entropy estimators, and upon summation, we obtain the $(2k)^{\text{th}}$ order kinetic entropy estimators $\gamma^{[0]} + \cdots + \gamma^{[k]}$ and $\widetilde{\gamma}^{[0]} + \cdots + \widetilde{\gamma}^{[k]}$.

A parallel can be made with the heat equation, for which the quantity $\zeta^{[k]} = |\partial^k u|^2$ can be considered as a $(2k)^{\text{th}}$ order entropy corrector. More generally, for parabolic scalar equations with variable coefficients, Bernstein equations are associated with sums of squares of derivatives [30].

2.6. Temperature scaling

Scaling properties of scalar partial differential equations are of fundamental importance for investigating the behavior of solutions like asymptotic expansions, singular limits, boundary layers, or even the existence of solutions with the concept of renormalized solutions [12,32].

When considering systems of partial differential equations however, a possible rescaling method could be to use functions of a single scalar quantity to rescale all solution components and solution derivatives. For fluid models, a natural candidate of such a scalar quantity appears to be temperature. In particular, higher order entropies provide a natural scaling of solution derivatives in terms of powers of temperature.

Remark 2. There are also ρ factors at the denominator of the corresponding derivatives $\partial^k \rho$ in (2.15) (2.16). Similarly, for multicomponent flows, entropy production associated with diffusive processes is essentially in the form

$$\sum_{1 \leqq i \leqq n_s} \int_{\mathbb{R}^n} \frac{pD}{T} \frac{|\partial_x X_i|^2}{X_i}\, dx,$$

where D is a typical diffusion coefficient, X_i the mole fraction of the i$^{\text{th}}$ species and n_s the number of species in the mixture [17].

2.7. Persistence of kinetic entropy and small Mach numbers

Various thermodynamic theories have already considered entropies differing from that of zeroth order, that is, entropies depending on macroscopic variable gradients. These generalized entropies have been associated notably with Burnett type equations and extended thermodynamics. In both situations, new macroscopic equations are correspondingly obtained, which are of a higher order than Navier–Stokes type equations.

On the contrary, in this work, we want to investigate the properties of the solutions of a given fluid model, that is, of a given second order system of partial differential equations. In particular, we do not consider composite quantities like $S^{(0)} + S^{(2)}$ as the system entropy, since we typically deal with fluid equations for which the zeroth order entropy $S^{(0)}$ is already of fundamental importance as imposed by the hyperbolic-parabolic structure of these equations [17,28]. We only want to use quantities like $S^{(0)}$, $S^{(0)} + S^{(2)}$ and more generally like $\gamma^{[0]} + \cdots + \gamma^{[l]}$, or $\tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[l]}$, $0 \leqq l \leqq k$, as a mean to obtain further information on the solutions of the fluid model. These quantities should thus be considered as families of mathematical entropy estimators (of kinetic origin) and we will establish that they indeed satisfy conditional entropic principles for solutions of Navier–Stokes type equations.

Enskog expansion is associated with small Knudsen numbers $\mathrm{Kn} = l/L$, where l is a typical mean free path and L a hydrodynamic length. On the other hand, we are interested in fluid models which take into account dissipative effects like viscosity and heat conduction and the corresponding characteristic length L is such that the Reynolds number $\mathrm{Re} = \rho v L/\eta$ is of order unity, that is, $L = \eta/\rho v$. As a consequence, since $\rho \bar{c} l = \eta$ [14], where $\bar{c}$ is a typical sound velocity, we obtain that the Knudsen number $\mathrm{Kn} = \mathrm{Kn}\,\mathrm{Re} = \rho v l/\eta = v/\bar{c}$ is equal to a typical Mach number $\mathrm{Ma} = v/\bar{c}$. Therefore, since $\mathrm{Ma} \simeq \mathrm{Kn}$, assuming that the Mach number is small is equivalent to the underlying kinetic assumption of a small Knudsen number, we expect the Mach number to play a role in the analysis [22].

3. Preliminary study

We investigate in this section how the notion of second order entropy can be used in the simplified situation of incompressible fluids with constant transport coefficients.

3.1. Incompressible model

We consider a fluid governed by the incompressible Navier–Stokes equations

$$\partial_x \cdot v = 0, \tag{3.1}$$

$$\partial_t(\rho v) + \partial_x \cdot (\rho v \otimes v) + \partial_x p + \partial_x \cdot \Pi = 0, \tag{3.2}$$

$$\partial_t(\rho e) + \partial_x \cdot (\rho e v) + \partial_x \cdot Q = -\Pi : \partial_x v, \tag{3.3}$$

where ρ is the constant density, v the velocity, p the pressure, I the unit tensor, Π the viscous tensor, e the internal energy per unit mass, and Q the heat flux vector. The viscous tensor is given by $\Pi = -\eta d$, where $d = \partial_x v + \partial_x v^t$ is the strain rate tensor and η the shear viscosity, the heat flux by $Q = -\lambda \partial_x T$, where λ is the thermal conductivity, and the energy per unit mass e is taken for simplicity in the form $e = c_v T$, where c_v is the specific heat per unit mass. All the coefficients c_v, λ, and η, are taken to be constant in this section.

Our aim is not to study various boundary conditions and we only consider the case of functions defined on $\mathbb{R}^n$, with $n \geqq 2$, that are 'constant at infinity'. From Galilean invariance and incompressibility, we can choose that v and p vanish at infinity. We only consider smooth solutions of the Navier–Stokes equations, that is, taken into account the eventual temperature dependence of the system coefficients as in Section 5, we assume that

$$v, T - T_\infty \in C\big([0, \bar{t}\,], H^l\big) \cap C^1\big([0, \bar{t}\,], H^{l-2}\big) \cap L^2\big([0, \bar{t}\,], H^{l+1}\big), \tag{3.4}$$

where l is an integer such that $l \geqq [n/2] + 2$, that is, $l > n/2 + 1$, $\bar{t}$ is some positive time, and $T_\infty > 0$ is some fixed positive temperature. We will establish in Section 7 that these solutions are as smooth as expected from initial data. In the simpler case of constant coefficients, smoothness properties hold as soon as it is established that $v \in C([0, \bar{t}\,], L^n)$ [32]. Existence of such smooth solutions can be established locally in time, or globally in time for small initial data. We will also assume that T is positive and bounded away from zero $T \geqq T_{\min}$, where $T_{\min} > 0$ and this property is easily established as soon as it holds at initial time $T_0 \geqq T_{\min}$ thanks to the non-negativity of viscous heat dissipation [32]. We consider as usual the momentum equation as projected on the space of divergence-free L^2 functions. More specifically, we introduce the Leray projector $\mathbb{P}$ defined on $L^2(\mathbb{R}^n)^n$ by

$$\mathbb{P} = \mathbb{I} + R \otimes R, \tag{3.5}$$

where $R = (R_1, \ldots, R_n)^t$ and $R_i = (-\Delta)^{-1/2}\partial_i$, $1 \leqq i \leqq n$, are the Riesz transforms, so that $(\mathbb{P}v)_i = v_i + \sum_{1 \leqq j \leqq n} R_i R_j v_j$, $1 \leqq i \leqq n$ [31,32]. It is well known that $\mathbb{P}$ is a continuous projector in any Sobolev space H^s, $s \in \mathbb{R}$, and $\mathbb{P}$ is also continuous in L^s for $1 < s < \infty$ [31,32]. Since the viscosity η is constant, the momentum conservation equation is easily rewritten as $\partial_t(\rho v) - \eta \Delta v = -\mathbb{P}\big(\partial_x \cdot (\rho v \otimes v)\big)$, which is equivalent to defining the pressure from

$$p = \sum_{1 \leqq i, j \leqq n} R_i R_j (\rho\, v_i v_j), \tag{3.6}$$

and we have $p \in C\big([0, \bar{t}\,], H^l\big) \cap C^1\big([0, \bar{t}\,], H^{l-2}\big) \cap L^2\big([0, \bar{t}\,], H^{l+1}\big)$.

3.2. Second order entropy corrector γ

As is traditional in mathematics, we change the sign of entropy, and thus of second order entropies, and we define γ as one of the equivalent expressions $-S^{(2)}$, $\mathfrak{v}^{(0)}$ or $I^{(2)}$. Specializing formally expressions (2.7), (2.9), or (2.12) to the situation of incompressible gases, we are led to consider $\gamma = \overline{\lambda}|\partial_x T|^2 + \frac{1}{2}\overline{\eta}|d|^2$, where $d = \partial_x v + \partial_x v^t$, and we will use the coefficients $\overline{\lambda} = \mathrm{A}_\lambda / T^{1+a}$ and $\overline{\eta} = \mathrm{A}_\eta / T^a$, so that

$$\gamma = \frac{\mathrm{A}_\lambda}{T^{1+a}}|\partial_x T|^2 + \frac{1}{2}\frac{\mathrm{A}_\eta}{T^a}|d|^2, \tag{3.7}$$

where $\mathrm{A}_\lambda > 0$, $\mathrm{A}_\eta > 0$, and $a > 0$, are positive parameters at our disposal. Kinetic theory suggests values $a \in (0, 2]$, for example, for small Mach number flows or incompressible flows. On the other hand, it is necessary to assume that $a \in (0, 1]$ in order to control $\log T$ from the second order entropy corrector γ.

Remark 3. A natural scaling associated with the temperature weights of $-S^{(2)}$, $\mathfrak{v}^{(0)}$, $I^{(2)}$, and γ, is that v scales as $\sqrt{T}$.

Remark 4. The second order entropy corrector γ corresponds to $\gamma^{[k]}$ in (2.15) with $k = 1$ if we replace d by $\partial_x v$. These modifications are unessential and a similar analysis can be conducted for $\gamma^{[1]}$ as for γ.

Remark 5. In the definition of higher order entropies, we have confined ourselves to weights in the form of power functions of temperature but more general functions of temperature could also be considered as well as functions of entropy.

3.3. Balance equation for γ

We write the balance equation for γ in the form

$$\partial_t \gamma + \partial_x\cdot(v\gamma) + \partial_x\cdot\varphi + \pi + \Sigma + \omega = 0, \tag{3.8}$$

where φ represents a flux and $\pi + \Sigma + \omega$ a source term. We expect π to be non-negative and composed of higher order derivative terms, Σ to be composed of higher order derivative split terms, and ω to be composed of lower order derivative terms arising from convection.

Proposition 1. *Let (v, T) be a smooth solution of the incompressible Navier–Stokes equations. Then we may take*

$$\pi = \frac{2\mathrm{A}_\lambda\lambda}{\rho c_v}\frac{|\partial_x^2 T|^2}{T^{1+a}} + \frac{(1+a)(2+a)\mathrm{A}_\lambda\lambda}{\rho c_v}\frac{|\partial_x T|^4}{T^{3+a}} + \frac{\mathrm{A}_\eta\eta}{\rho}\frac{|\partial_x d|^2}{T^a} + \frac{a\mathrm{A}_\eta\eta}{4\rho c_v}\frac{|d|^4}{T^{1+a}}$$
$$+ \Big(\frac{(1+a)\mathrm{A}_\lambda\eta}{2\rho c_v} + \frac{a(1+a)\mathrm{A}_\eta\lambda}{2\rho c_v}\Big)\frac{|d|^2\,|\partial_x T|^2}{T^{2+a}},$$

$$\Sigma = -\frac{4(1+a)\mathrm{A}_\lambda\lambda}{\rho c_v}\frac{\partial_x^2 T{:}\partial_x T\otimes\partial_x T}{T^{2+a}} - \Big(\frac{2\mathrm{A}_\lambda\eta}{\rho c_v} + \frac{a\mathrm{A}_\eta\lambda}{\rho c_v} + \frac{a\mathrm{A}_\eta\eta}{\rho}\Big)\frac{\partial_x d{:}d\otimes\partial_x T}{T^{1+a}},$$

$$\omega = \mathrm{A}_\lambda \frac{d\!:\!\partial_x T \otimes \partial_x T}{T^{1+a}} + 2\frac{\mathrm{A}_\eta}{\rho}\frac{d\!:\!\partial_x^2 p}{T^a} + 2\mathrm{A}_\eta \frac{d\!:\!(\partial_x v\cdot\partial_x v)}{T^a},$$

$$\varphi = \frac{a\mathrm{A}_\eta\lambda}{2\rho c_v}\frac{|d|^2\,\partial_x T}{T^{1+a}} - \frac{\mathrm{A}_\eta\eta}{\rho}\frac{d\!:\!\partial_x d}{T^a} + \frac{(1+a)\mathrm{A}_\lambda\lambda}{\rho c_v}\frac{|\partial_x T|^2\,\partial_x T}{T^{2+a}} - \frac{2\mathrm{A}_\lambda\lambda}{\rho c_v}\frac{\partial_x^2 T\cdot\partial_x T}{T^{1+a}}.$$

In this proposition, for the sake of conciseness, we have introduced a conveniently compact notation. For a and b vectors, we denote by $a\otimes b$ the matrix with elements $a_i b_j$, $1 \leqq i, j \leqq n$. For a matrix and b vector we denote by $a\otimes b$ the third order tensor with elements $a_{ij}b_k$, $1 \leqq i, j, k \leqq n$. For a and b matrices we denote by $a\!:\!b$ the quantity $\sum_{ij} a_{ij}b_{ij}$ and $|a|^2 = a\!:\!a$. For a and b third order tensors, like $\partial_x d$ or $d\otimes\partial_x T$, we denote by $a\!:\!b$ the quantity $\sum_{ijk} a_{ijk}b_{ijk}$ and we define $|a|^2 = a\!:\!a$. Some expressions would be ambiguous for general tensors, but these ambiguities are easily resolved thanks to symmetry properties of multiple derivatives.

Of course, the decomposition (3.8) is not unique since various integrations by parts may be performed and terms may be exchanged between φ, π, and Σ. In particular, all expressions involving tensor full contractions—for instance $\partial_x^2 T\!:\!\partial_x^2 T$—can be replaced by similar expressions involving only partial tensor contractions— for instance $(\Delta T)^2$. Some of these expressions are derived in Section 3.4 where we investigate the sign of the higher order derivative terms $\int_{\mathbb{R}^n}(\pi+\Sigma)\,dx$. In the decomposition of Proposition 1, we have tried to put in π all available non-negative higher order derivative contributions and the remaining higher order derivative terms have been put in Σ.

Proof. In order to derive a balanced equation for γ, we evaluate its time differential in terms of temperature and velocity gradients. To this aim, letting $\overline{\lambda} = \mathrm{A}_\lambda/T^{1+a}$, and $\overline{\eta} = \mathrm{A}_\eta/T^a$, we write that

$$\partial_t\gamma + \sum_{1\leqq l\leqq n} v_l\partial_l\gamma - \big(\partial_T\overline{\lambda}|\partial_x T|^2 + \tfrac{1}{2}\partial_T\overline{\eta}|d|^2\big)\left(\partial_t T + \sum_{1\leqq l\leqq n} v_l\partial_l T\right)$$
$$-2\overline{\lambda}\sum_{1\leqq i\leqq n}\partial_i T\left(\partial_t\partial_i T + \sum_{1\leqq l\leqq n} v_l\partial_l\partial_i T\right)$$
$$-\overline{\eta}\sum_{1\leqq i,j\leqq n} d_{ij}\left(\partial_t d_{ij} + \sum_{1\leqq l\leqq n} v_l\partial_l d_{ij}\right) = 0.$$

Upon using the governing equations we obtain

$$\partial_t\gamma \;+ \sum_{1\leqq l\leqq n} v_l\partial_l\gamma - \Big(\partial_T\overline{\lambda}|\partial_x T|^2 + \tfrac{1}{2}\partial_T\overline{\eta}|d|^2\Big)\frac{1}{\rho c_v}\Big(\lambda\partial_x\!\cdot\partial_x T + \tfrac{1}{2}\eta|d|^2\Big)$$
$$-2\overline{\lambda}\left(\sum_{1\leqq i\leqq n}\partial_i T\,\partial_i\left(\frac{1}{\rho c_v}(\lambda\partial_x\!\cdot\partial_x T + \tfrac{1}{2}\eta|d|^2)\right) - \sum_{1\leqq i,l\leqq n}\partial_i T\,\partial_i v_l\,\partial_l T\right)$$

$$
- 2\overline{\eta} \left(\sum_{1 \leqq i,j \leqq n} d_{ij} \partial_j \left(\frac{1}{\rho} (\eta \partial_x \cdot \partial_x v_i - \partial_i p) \right) - \sum_{1 \leqq i,j,l \leqq n} d_{ij} \partial_j v_l \partial_l v_i \right) = 0.
\tag{3.9}
$$

The governing equation for γ is then obtained after various integrations by parts. More specifically, let us denote by $\mathscr{T}^{\partial}$, $\mathscr{T}^{\lambda}$, and $\mathscr{T}^{\eta}$, the three last terms appearing in the left-hand side of Equation (3.9). The contributions in $\mathscr{T}^{\partial}$ in the form $|\partial_x T|^2 |d|^2$ and $|d|^4$ are left unchanged whereas the contributions in the form $|\partial_x T|^2\, \partial_x \cdot \partial_x T$ and $|d|^2\, \partial_x \cdot \partial_x T$ are integrated by parts. This yields in particular a term in the form $|\partial_x T|^4$. The two first terms of $\mathscr{T}^{\lambda}$ and $\mathscr{T}^{\eta}$ are integrated by parts, thereby eliminating third order derivatives, whereas the second term of $\mathscr{T}^{\lambda}$ is left unchanged. Finally, the third term of $\mathscr{T}^{\lambda}$ and the second and third of $\mathscr{T}^{\eta}$ yield the lower order convective contributions of ω.

3.4. Unconditional positivity of higher order derivative source terms

Integrating the γ balance equation (3.8), thanks to assumptions (3.4), all the flux terms are eliminated, and we obtain the identity

$$
\partial_t \int_{\mathbb{R}^n} \gamma \, dx + \int_{\mathbb{R}^n} (\pi + \Sigma) \, dx = - \int_{\mathbb{R}^n} \omega \, dx. \tag{3.10}
$$

Our aim in this section is to study the sign of $\int_{\mathbb{R}^n} (\pi + \Sigma) \, dx$, where π and Σ are as in Proposition 1. More specifically, we investigate the inequality

$$
\frac{1}{c} \int_{\mathbb{R}^n} \pi \, dx \leqq \int_{\mathbb{R}^n} (\pi + \Sigma) \, dx \leqq c \int_{\mathbb{R}^n} \pi \, dx, \tag{3.11}
$$

where c denotes a positive constant. This inequality implies in particular that

$$
\partial_t \int_{\mathbb{R}^n} \gamma \, dx + \frac{1}{c} \int_{\mathbb{R}^n} \pi \, dx \leqq \int_{\mathbb{R}^n} |\omega| \, dx, \tag{3.12}
$$

which is a natural first step towards entropic type inequalities. Majorization of the convective contribution $\int_{\mathbb{R}^n} |\omega| \, dx$ in terms of $\int_{\mathbb{R}^n} \pi \, dx$ and $\int_{\mathbb{R}^n} \gamma \, dx$ is discussed in Section 3.5. We investigate inequality (3.11) for $v, T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, and $T \geqq T_{\min} > 0$, where $A(\mathbb{R}^n) \subset \mathscr{C}_0(\mathbb{R}^n)$ denotes the Wiener algebra [36,37]

$$
A(\mathbb{R}^n) = \{ \, \hat{f} \in \mathscr{C}_0(\mathbb{R}^n); \exists f \in L^1, \;\, \hat{f} = \mathscr{F} f \, \}.
$$

We have denoted by $\mathscr{F} g$ the Fourier transform $\mathscr{F} g\,(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}\xi \cdot x} g(x) \, dx$ of $g \in L^1(\mathbb{R}^n)$, and the Wiener algebra (which naturally appears in multilinear derivative estimates [36,37]) is equipped with the norm $\|\hat{f}\|_A = \|f\|_{L^1}$. The Wiener algebra $A(\mathbb{R}^n)$ is a (dense) subalgebra of $\mathscr{C}_0(\mathbb{R}^n)$, the algebra of all continuous functions on $\mathbb{R}^n$ which vanish at infinity. We remind the reader that, unless explicitly stated, it is always assumed in the following that $n \geqq 2$.

Proposition 2. *Assume that the parameter a associated with γ is such that*

$$0 < a < \inf\left(\frac{4n-1}{2n^2+1}, 2\left(\frac{\lambda}{\eta c_v} + \frac{\eta c_v}{\lambda}\right)^{-1}\right). \tag{3.13}$$

Then there exist positive constants A_λ and A_η such that inequality (3.11) holds for any $v, T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, and $T \geqq T_{\min} > 0$. On the other hand, there exists $a^* < 1$ so that for any $a > a^*$, there exists $v, T - T_\infty \in \mathscr{D}(\mathbb{R}^n)$ such that the integral $\int_{\mathbb{R}^n}(\pi + \Sigma)\,dx$ is negative and a fortiori (3.11) does not hold.

Proof. By a density argument, and thanks to classical interpolation inequalities, it is sufficient to consider the situation where $v, T - T_\infty \in \mathscr{D}(\mathbb{R}^n), T \geqq T_{\min} > 0$. We first consider the terms of $\int_{\mathbb{R}^n}(\pi + \Sigma)\,dx$ which only involve temperature gradients. Regrouping these terms, we have to investigate the sign of

$$z^{[T]} = \int_{\mathbb{R}^n}\Big(2\frac{|\partial_x^2 T|^2}{T^{1+a}} - 4(1+a)\frac{\partial_x^2 T:\partial_x T \otimes \partial_x T}{T^{2+a}} + (1+a)(2+a)\frac{|\partial_x T|^4}{T^{3+a}}\Big)dx. \tag{3.14}$$

We use the polar decomposition of the Hessian matrix $\partial_x^2 T$. Defining for short

$$z^{[2]} = \frac{\partial_x^2 T}{T^{(1+a)/2}}, \qquad z^{[1]} = \frac{\partial_x T \otimes \partial_x T}{T^{(3+a)/2}}, \tag{3.15}$$

we have

$$z^{[T]} = \int_{\mathbb{R}^n}\big(2|z^{[2]}|^2 - 4(1+a)z^{[2]}:z^{[1]} + (1+a)(2+a)|z^{[1]}|^2\big)\,dx. \tag{3.16}$$

On the other hand, using integrations by parts, one easily establishes that

$$\int_{\mathbb{R}^n}\big(\mathrm{tr} z^{[2]}\big)^2\,dx = \int_{\mathbb{R}^n}\big(|z^{[2]}|^2 - 3(1+a)z^{[2]}:z^{[1]} + (1+a)(2+a)|z^{[1]}|^2\big)\,dx, \tag{3.17}$$

$$\int_{\mathbb{R}^n}\mathrm{tr} z^{[2]}\mathrm{tr} z^{[1]}\,dx = \int_{\mathbb{R}^n}\big(-2z^{[2]}:z^{[1]} + (2+a)|z^{[1]}|^2\big)\,dx, \tag{3.18}$$

and we also have $z^{[1]}:z^{[1]} = (\mathrm{tr} z^{[1]})^2$. We have denoted by $\mathrm{tr}A$ the trace of a matrix A and we define

$$\widehat{A} = A - (\mathrm{tr}A/n)\mathbb{I}, \tag{3.19}$$

where $\mathbb{I}$ is the unit matrix and $\mathrm{tr}(\widehat{A}) = 0$. After some manipulation, using (3.17) (3.18) and polar decompositions, we obtain

$$\Big(1-\frac{1}{n}\Big)\int_{\mathbb{R}^n} |z^{[2]}|^2\,dx = \int_{\mathbb{R}^n}\Big(|\widehat{z}^{[2]}|^2 - \frac{3(1+a)}{n+2}\widehat{z}^{[2]}{:}\widehat{z}^{[1]} + \frac{(1+a)(2+a)}{n+2}|\widehat{z}^{[1]}|^2\Big)\,dx,$$

$$\Big(1+\frac{2}{n}\Big)\int_{\mathbb{R}^n} z^{[2]}{:}z^{[1]}\,dx = \int_{\mathbb{R}^n}\Big(\widehat{z}^{[2]}{:}\widehat{z}^{[1]} + \frac{(2+a)}{n-1}|\widehat{z}^{[1]}|^2\Big)\,dx,$$

$$\Big(1-\frac{1}{n}\Big)\int_{\mathbb{R}^n} |z^{[1]}|^2\,dx = \int_{\mathbb{R}^n} |\widehat{z}^{[1]}|^2\,dx.$$

These relations imply (after some algebra) that

$$\Big(1-\frac{1}{n}\Big)\Big(1+\frac{2}{n}\Big)z^{[T]} = \int_{\mathbb{R}^n}\Big(2\Big(1+\frac{2}{n}\Big)|\widehat{z}^{[2]}|^2 - (1+a)\Big(4+\frac{2}{n}\Big)\widehat{z}^{[2]}{:}\widehat{z}^{[1]} + (1+a)(2+a)|\widehat{z}^{[1]}|^2\Big)\,dx, \tag{3.20}$$

so that

$$\Big(1-\frac{1}{n}\Big)\Big(1+\frac{2}{n}\Big)z^{[T]} = \int_{\mathbb{R}^n}\sum_{1\leqq i,j\leqq n} P\big(\widehat{z}^{[2]}_{ij},\widehat{z}^{[1]}_{ij}\big)\,dx,$$

where

$$P(X,Y) = 2\Big(1+\frac{2}{n}\Big)X^2 - (1+a)\Big(4+\frac{2}{n}\Big)XY + (1+a)(2+a)Y^2.$$

From the binomial formula, there exists $\delta > 0$ such that

$$\delta\Big(\int_{\mathbb{R}^n}|\widehat{z}^{[2]}|^2\,dx + \int_{\mathbb{R}^n}|\widehat{z}^{[1]}|^2\,dx\Big) \leqq z^{[T]}, \tag{3.21}$$

provided that

$$(1+a)^2\Big(4+\frac{2}{n}\Big)^2 - 8\Big(1+\frac{2}{n}\Big)(1+a)(2+a) < 0,$$

that is, provided $a < (4n-1)/(2n^2+1)$. The inequality (3.21) then implies that for some positive constant δ we have

$$\delta\Big(\int_{\mathbb{R}^n}|z^{[2]}|^2\,dx + \int_{\mathbb{R}^n}|z^{[1]}|^2\,dx\Big) \leqq z^{[T]}. \tag{3.22}$$

Note that we have used polar decompositions in (3.20) instead of (3.14) since the discriminant of the corresponding second order polynomial $2X^2 - 4(1+a)XY + (1+a)(2+a)Y^2$ is always positive.

We now have to consider the remaining terms of $\int_{\mathbb{R}^n}(\pi + \Sigma)\,dx$ involving velocity gradients

$$
\begin{aligned}
z^{[v]} =& \frac{\mathrm{A}_\eta \eta}{\rho} \int_{\mathbb{R}^n} \frac{|\partial_x d|^2}{T^a} - \Big(2\frac{\mathrm{A}_\lambda \eta}{\rho c_v} + \frac{a \mathrm{A}_\eta \lambda}{\rho c_v} + \frac{a \mathrm{A}_\eta \eta}{\rho} \Big) \int_{\mathbb{R}^n} \frac{\partial_x d \colon d \otimes \partial_x T}{T^{1+a}} \\
&+ \Big(\frac{(1+a)\mathrm{A}_\lambda \eta}{2\rho c_v} + \frac{a(1+a)\mathrm{A}_\eta \lambda}{2\rho c_v} \Big) \int_{\mathbb{R}^n} \frac{|d|^2\,|\partial_x T|^2}{T^{2+a}} + \frac{a\mathrm{A}_\eta \eta}{4\rho c_v} \int_{\mathbb{R}^n} \frac{|d|^4}{T^{1+a}},
\end{aligned}
$$

which can be written

$$
z^{[v]} = \int_{\mathbb{R}^n} \sum_{1 \leqq i,j,k \leqq n} Q\Big(\frac{\partial_k d_{ij}}{T^{a/2}}, \frac{d_{ij}\partial_k T}{T^{1+a/2}} \Big)\,dx + \frac{a\mathrm{A}_\eta \eta}{4\rho c_v} \int_{\mathbb{R}^n} \frac{|d|^4}{T^{1+a}},
$$

where

$$
\begin{aligned}
Q(X,Y) =& \frac{\mathrm{A}_\eta \eta}{\rho} X^2 - \Big(2\frac{\mathrm{A}_\lambda \eta}{\rho c_v} + \frac{a\mathrm{A}_\eta \lambda}{\rho c_v} + \frac{a\mathrm{A}_\eta \eta}{\rho} \Big) XY \\
&+ \Big(\frac{(1+a)\mathrm{A}_\lambda \eta}{2\rho c_v} + \frac{a(1+a)\mathrm{A}_\eta \lambda}{2\rho c_v} \Big) Y^2.
\end{aligned}
$$

Using the binomial formula, the existence of $\delta > 0$ such that

$$
\delta \Big(\int_{\mathbb{R}^n} \frac{|\partial_x d|^2}{T^a} + \int_{\mathbb{R}^n} \frac{|d|^2\,|\partial_x T|^2}{T^{2+a}} + \int_{\mathbb{R}^n} \frac{|d|^4}{T^{1+a}} \Big) \leqq z^{[v]},
$$

is a consequence of

$$
\Big(2\frac{\mathrm{A}_\lambda \eta}{\rho c_v} + \frac{a\mathrm{A}_\eta \lambda}{\rho c_v} + \frac{a\mathrm{A}_\eta \eta}{\rho} \Big)^2 - 4\frac{\mathrm{A}_\eta \eta}{\rho} \Big(\frac{(1+a)\mathrm{A}_\lambda \eta}{2\rho c_v} + \frac{a(1+a)\mathrm{A}_\eta \lambda}{2\rho c_v} \Big) < 0.
$$

Defining $\zeta = \mathrm{A}_\eta c_v / 2\mathrm{A}_\lambda$ and $\xi = \lambda/\eta c_v$, this is equivalent to

$$
a^2\zeta^2(1+\xi^2) + a\xi\zeta\big(\xi^{-1} + 2(1-\zeta)\big) + 1 - \zeta < 0, \tag{3.23}
$$

and it implies that $\zeta = \mathrm{A}_\eta c_v / 2\mathrm{A}_\lambda > 1$. In this situation $\zeta > 1$, there are two roots of the left-hand side of (3.23), one negative $\underline{a}(\xi,\zeta)$ and one positive $\overline{a}(\xi,\zeta)$ given by

$$
\overline{a}(\xi,\zeta) = \frac{-\big(\xi^{-1} + 2(1-\zeta)\big) + \xi^{-1}\big(4\xi^2\zeta^2 + \zeta(4 - 4\xi - 4\xi^2) + 4\xi - 3\big)^{1/2}}{2\zeta(\xi + \xi^{-1})}.
$$

Keeping in mind that a has to be positive, we must have $0 < a < \overline{a}(\xi,\zeta)$. Noting that $\overline{a}(\xi,1) = 0$ and $\overline{a}(\xi,\infty) = 2/(\xi + \xi^{-1})$, we obtain for large ζ the sufficient condition $0 < a < 2/(\xi + \xi^{-1})$ and the first part of Proposition 2 is proved.

We now assume that $a = 1$ and establish that $\int_{\mathbb{R}^n}(\pi + \Sigma)\,dx$ can be negative, keeping in mind that $n \geqq 2$. To this purpose, it is sufficient to let $v = 0$ and to only consider the terms $z^{[T]}$ involving temperature derivatives. Denoting $\tau = \log T$, it is easily checked that

$$z^{[T]} = \int_{\mathbb{R}^n} \Big(2|\partial_x^2 \tau|^2 - 4\partial_x^2 \tau : \partial_x \tau \otimes \partial_x \tau \Big) dx. \tag{3.24}$$

Let $\psi \in \mathscr{D}(\mathbb{R})$ be a C^∞ function with $\psi(s) > 0$ for $|s| < 1$ and $\psi(s) = 0$ for $|s| \geqq 1$, and consider $\zeta(x) = \prod_i \psi(x_i)$. Since the contributions $\int_{\mathbb{R}^n} \partial_k^2 \zeta \, (\partial_k \zeta)^2 \, dx$, $1 \leqq k \leqq n$, in the sum $\int_{\mathbb{R}^n} \partial_x^2 \zeta : \partial_x \zeta \otimes \partial_x \zeta \, dx$ vanish and since $\partial_{jk}\zeta \, \partial_j \zeta \, \partial_k \zeta$ is nonnegative and nonzero for $j \neq k$, it is easily checked that $\int_{\mathbb{R}^n} \partial_x^2 \zeta : \partial_x \zeta \otimes \partial_x \zeta \, dx$ is strictly positive. Since the two terms scale differently in (3.24), letting $\tau - \tau_\infty = \Lambda \zeta$, that is, $T = T_\infty \exp(\Lambda \zeta)$, there exists $\Lambda > 0$ large enough such that the quantity $z^{[T]}$ is negative, and we also have $v, T - T_\infty \in \mathscr{D}(\mathbb{R}^n)$. By a continuity argument with respect to a, using the same fixed τ, $z^{[T]}$ remains negative for a close to unity.

We next assume that $a > 1$ and establish that $z^{[T]}$ can be negative in any dimension $n \geqq 1$. We first consider the one dimensional situation $n = 1$. The main idea is to cancel the weight T^{1-a} by a change of variable and to rescale again the logarithm of temperature. To this end we introduce $\widehat{\tau}(\xi) = \tau(x) = \log T(x)$, where $\xi(x)$ is a new variable to be determined, and we have

$$\frac{d_x T}{T} = d_x \tau = d_\xi \widehat{\tau} \, d_x \xi$$

$$\frac{d_x^2 T}{T} = d_x^2 \tau + (d_x \tau)^2 = d_\xi^2 \widehat{\tau} \, (d_x \xi)^2 + (d_\xi \widehat{\tau})^2 (d_x \xi)^2 + d_\xi \widehat{\tau} \, d_x^2 \xi.$$

It is then easily found that, after the change of variable from x to ξ, the new weight in $z^{[T]}$ is given by $T^{1-a}(d_x \xi)^3$ since the factor $(d_x \xi)^4$ naturally appears and since one power of $d_x \xi$ is used for the change of variable in the integral. Therefore, we impose that

$$\xi = \int_0^x d_x \xi \, dx, \qquad d_x \xi = \exp\big(-(1-a)\tau/3 \big),$$

so that

$$x = \int_0^\xi d_\xi x \, d\xi, \qquad d_\xi x = \exp\big((1-a)\widehat{\tau}/3 \big), \tag{3.25}$$

and we further obtain that

$$\frac{d_x^2 \xi}{(d_x \xi)^2} = -\tfrac{1-a}{3} \, d_\xi \widehat{\tau}.$$

Finally, all calculations done, we obtain that

$$z^{[T]} = \int_{\mathbb{R}} \big(2(d_\xi^2 \widehat{\tau})^2 - \tfrac{1}{9}(a+2)(a-1)(d_\xi \widehat{\tau})^4 \big) \, d\xi, \tag{3.26}$$

keeping in mind that terms like $\int_{\mathbb{R}} d_\xi^2 \widehat{\tau} (d_\xi \widehat{\tau})^2 \, d\xi$ vanish when $\widehat{\tau} - \tau_\infty \in \mathscr{D}(\mathbb{R}^n)$. Since the two terms scale differently in (3.26), we write $\widehat{\tau} - \tau_\infty = \Lambda \widehat{\psi}$, where $\widehat{\psi} \in \mathscr{D}(\mathbb{R})$ is taken as an even function of ξ such that $\widehat{\psi} > 0$ on $(-1, 1)$ and $\widehat{\psi} = 0$ elsewhere, and there exists $\Lambda > 0$ large enough such that $z^{[T]}$ is negative. From the relations (3.25) we can then evaluate x as a function of ξ and map $\widehat{\tau}$ and $\widehat{\psi}$ as functions of x denoted by $\tau(x) = \widehat{\tau}(\xi)$ and $\psi(x) = \widehat{\psi}(\xi)$. Letting $\psi_1 = \Lambda\psi$, we finally have $\psi_1 > 0$ on an interval in the form $(-\bar{x}_1, \bar{x}_1)$, $\psi_1 = 0$ on the complementary set, $T_1 = T_\infty \exp(\psi_1)$, $T_1 - T_\infty \in \mathscr{D}(\mathbb{R})$ and $z^{[T]}(T_1) < 0$. Incidentally, changing the back of the variable from ξ to x in relation (3.26) (or after a few integrations by parts in (3.14)) it is easily checked that for $n = 1$, inequality (3.22) holds for $0 < a < 1$, and the bound 1 coincides with the bound $(4n-1)/(2n^2+1)$. Furthermore, in the degenerate situation $a = 1$ and $n = 1$, $z^{[T]}$ is non-negative with $z^{[T]} = \int_{\mathbb{R}^n} 2(d_x^2\tau)^2\, dx$ but (3.22) does not hold since $\delta \int_{\mathbb{R}^n} (d_x\tau)^4\, dx \leqq \int_{\mathbb{R}^n} \pi\, dx$ for δ small enough.

We now generalize this counter example to any dimension by induction. We assume that we have $\psi_l \in \mathscr{D}(\mathbb{R}^l)$ with $\psi_l > 0$ on $\prod_{1\leqq i\leqq l}(-\bar{x}_i, \bar{x}_i)$ and $\psi_l = 0$ on the complementary set, such that for $\tau_l - \tau_\infty = \psi_l$ and $T_l = T_\infty \exp(\psi_l)$ we have $z^{[T]}(T_l) < 0$. In order to build $\psi_{l+1} \in \mathscr{D}(\mathbb{R}^{l+1})$, τ_{l+1}, and T_{l+1}, with similar properties, we define

$$\psi_{l+1}(x_1, \ldots, x_l, x_{l+1}) = \psi_l(x_1, \ldots, x_l)\phi_{l+1}(x_{l+1}),$$

where ϕ_{l+1} is even $\phi_{l+1}(x_{l+1}) = \phi_{l+1}(-x_{l+1})$, $\phi_{l+1}(x_{l+1}) = 1$ over $[0, \bar{x}_{l+1} - 1]$ and $\phi_{l+1}(x_{l+1}) = \Phi\big(x_{l+1} - (\bar{x}_{l+1} - 1)\big)$ for $x_{l+1} \geqq \bar{x}_{l+1} - 1$, where $\bar{x}_{l+1} > 1$ is to be determined and where $\Phi \in C^\infty[0, \infty)$ is such that $\Phi(s) = 1$ for $0 \leqq s \leqq 1/2$, $0 < \Phi(s) < 1$ for $1/2 < s < 1$ and $\Phi(s) = 0$ for $s \geqq 1$. We have in particular $\phi_{l+1} = 1$ over $(-\bar{x}_{l+1} + 1, \bar{x}_{l+1} - 1)$, $\phi_{l+1} = 0$ on $\mathbb{R}\backslash(-\bar{x}_{l+1}, \bar{x}_{l+1})$, and we define $\tau_{l+1} - \tau_\infty = \psi_{l+1}$ and $T_{l+1} = T_\infty \exp(\psi_{l+1})$. In order to evaluate $z^{[T]}(T_{l+1})$, we use symmetry and we divide the integration over the x_{l+1} variable into the intervals $(0, \bar{x}_{l+1} - 1)$, and $(\bar{x}_{l+1} - 1, \bar{x}_{l+1})$. Keeping in mind that $\phi_{l+1} = 1$ over $[0, \bar{x}_{l+1} - 1)$ we obtain that

$$z^{[T]}(T_{l+1}) = 2(\bar{x}_{l+1} - 1)z^{[T]}(T_l) + 2r_{l+1},$$

where r_{l+1} denotes the integral over $\prod_{1\leqq i\leqq l}(-\bar{x}_i, \bar{x}_i) \times (\bar{x}_{l+1} - 1, \bar{x}_{l+1})$, so that r_{l+1} is independent of $\bar{x}_{l+1}$ after the change of variable from x_{l+1} to $x_{l+1} - (\bar{x}_{l+1} - 1)$. Since by induction we have $z^{[T]}(T_l) < 0$ it is possible to take $\bar{x}_{l+1}$ large enough such that $z^{[T]}(T_{l+1}) < 0$ and the proof is complete.

Remark 6. Many refinements of Proposition 2 are feasible but are beyond the scope of this work. Note that we have investigated inequality (3.11) independently of the fact that (v, T) is a solution of the governing equations and independently of any constraint on (v, T). This is in contrast with Section 6, where we will impose constraints on the norms of $\log T$ and $v/\sqrt{T}$. The ratio $(4n - 1)/(2n^2 + 1)$ can be written $1 - 2(n - 1)^2/(2n^2 + 1)$, which is always smaller than unity (keeping in mind that $n \geqq 2$—and is $11/19$ for $n = 3$).

Remark 7. Inequalities like

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^4}{T^{3+a}}\, dx \leqq c \int_{\mathbb{R}^n} \frac{|\partial_x^2 T|^2}{T^{1+a}}\, dx, \tag{3.27}$$

hold whenever $a \neq -2$, $T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ and $T \geqq T_{\min} > 0$. It can be established by considering $\partial_x\cdot\big(|\partial_x T|^2 \partial_x T / T^{(2+a)}\big)$ and using a density argument.

The particular case $a = -1$ has been investigated by LIONS and VILLANI [33]. Different inequalities will be established in Section 4 with powers of $\| \log T \|_{BMO}$ as multiplicative factors in the right-hand side of (3.27).

Proposition 2 shows that unconditional positiveness of higher order derivative source terms $\int_{\mathbb{R}^n} (\pi + \Sigma)\, dx$ only holds for a restricted family of second order entropy correctors. In particular, unconditional positiveness does not hold for the natural logarithmic scaling $a = 1$. An inescapable consequence is that only conditional positiveness of higher order derivative source terms will allow stronger and more satisfactory results. Conditional positivity will be investigated in Section 6 for generalized entropies of arbitrary order with temperature dependent transport coefficients.

3.5. Estimates of convective terms

In order to estimate convective terms, we need to express velocity gradients in terms of the strain rate tensor.

Proposition 3. *For any $v \in H^1$ and any index pair (i, j) we have* [42]

$$2\partial_j v_i = d_{ij} - \sum_{1 \leqq l \leqq n} R_l R_j d_{li} + \sum_{1 \leqq l \leqq n} R_l R_i d_{lj}, \tag{3.28}$$

where $R_i = (-\Delta)^{-1/2}\partial_i$ are the Riesz transforms, $1 \leqq i \leqq n$, and we also have

$$2\partial_j \partial_k v_i = \partial_k d_{ij} + \partial_j d_{ik} - \partial_i d_{jk}. \tag{3.29}$$

In the following proposition, we obtain a typical estimate of $\int_{\mathbb{R}^n} |\omega|\, dx$ in terms of $\int_{\mathbb{R}^n} \pi\, dx$ and $\int_{\mathbb{R}^n} |d|^2\, dx$.

Proposition 4. *Assume that $v, T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $T \geqq T_{\min} > 0$, and that $a \leqq 1/3$. Then the following estimate holds*

$$\int_{\mathbb{R}^n} |\omega|\, dx \leqq c\Big(\int_{\mathbb{R}^n} \pi\, dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n} |d|^2\, dx\Big)^{1/2} \sup_{\mathbb{R}^n} T^{(1-a)/2}. \tag{3.30}$$

Proof. We have $\omega = \overline{\lambda} d{:}\partial_x T \otimes \partial_x T + 2(\overline{\eta}/\rho) d{:}\partial_x^2 p + 2\overline{\eta} d{:}\partial_x v{\cdot}\partial_x v$, where $\overline{\lambda} = \mathrm{A}_\lambda / T^{1+a}$ and $\overline{\eta} = \mathrm{A}_\eta / T^a$ and we examine each term at a time. The first term $\overline{\lambda} d{:}\partial_x T \otimes \partial_x T$ can directly be estimated by using the Holder inequality

$$\int_{\mathbb{R}^n} \frac{|d{:}\partial_x T \otimes \partial_x T|}{T^{1+a}}\, dx \leqq c\Big(\int_{\mathbb{R}^n} \pi\, dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n} |d|^2\, dx\Big)^{1/2} \sup T^{(1-a)/2}.$$

In order to estimate $|2\overline{\eta} d{:}\partial_x v{\cdot}\partial_x v|$, we use the expression of $\partial_x v$ in terms of d, and we obtain a sum of terms in the form

$$\int_{\mathbb{R}^n} \frac{|d|\, \big|\mathscr{R}(d)\big|\, \big|\mathscr{R}'(d)\big|}{T^a}\, dx,$$

where $\mathscr{R}$ and $\mathscr{R}'$ are products of Riesz transforms. Upon introducing temperature factors as

$$\int_{\mathbb{R}^n} \frac{|d|}{T^{\frac{1+a}{4}}} \left|\mathscr{R}\left(\frac{d}{T^{\frac{1+a}{4}}} T^{\frac{1+a}{4}}\right)\right| \left|\mathscr{R}'(d)\right| T^{(1-3a)/4}\, dx, \tag{3.31}$$

and applying the Holder inequality with exponents

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{\infty} + \frac{1}{2} + \frac{1}{\infty} = 1,$$

we obtain the desired estimate provided that $a \leqq 1/3$. Upon using the expression of $\partial_x^2 p$ in terms of velocity gradients and the expression of velocity gradients in terms of the strain rate tensor, we can finally express $2(\overline{\eta}/\rho)d{:}\partial_x^2 p$ as a sum of terms in the form

$$\int_{\mathbb{R}^n} \frac{d\, \mathscr{R}\big(\mathscr{R}'(d)\, \mathscr{R}''(d)\big)}{T^a}\, dx,$$

where $\mathscr{R}$, $\mathscr{R}'$, and $\mathscr{R}''$ are products of Riesz transforms so that the pressure term can be treated as the term $|2\overline{\eta} d{:}\partial_x v\cdot \partial_x v|$.

3.6. Temperature weights

The main difficulties of the Navier–Stokes equations arise from the nonlinear convective terms $v\cdot\partial_x \phi$, where ϕ stands for v or T. These terms introduce nonlinearities through a multiplication by the velocity v of the gradient $\partial_x \phi$ appearing in the ϕ equation. On the other hand, the natural scaling associated with the temperature weights of γ is that v scales as $\sqrt{T}$. Therefore, we expect extra temperature factors in the form $\sqrt{T}$ to appear when estimating $\int |\omega|\, dx$ in terms of $\int \pi\, dx$ and $\int \gamma\, dx$, as inherited from original nonlinearities. Indeed, a direct consequence of Proposition 4 is that

$$\int_{\mathbb{R}^n} |\omega|\, dx \leqq c\Big(\int_{\mathbb{R}^n} \pi\, dx\Big)^{1/2} \Big(\int_{\mathbb{R}^n} \gamma\, dx\Big)^{1/2} \sup_{\mathbb{R}^n} T^{1/2}, \tag{3.32}$$

and we are now presented with the problem of controlling these $\sup_{\mathbb{R}^n} T$ factors. A first possibility could be to use (3.30) with $a = 1$ and the natural kinetic energy estimates of $\int_0^t \int_{\mathbb{R}^n} |d|^2\, dxdt$, assuming that we can eliminate the limitation $a \leqq 1/3$ in Proposition 4. However, this seems hopeless since inequality (3.11) does not hold unconditionally for $a = 1$. More generally, larger values of a promote majorization of convective terms, but prevent inequality (3.11), and, conversely, smaller values of a promote inequality (3.11), but prevent majorization of convective terms. Another possibility could be to estimate $\sup_{\mathbb{R}^n} T$ in terms of $\int_{\mathbb{R}^n} \pi\, dx$ and $\int_{\mathbb{R}^n} \gamma\, dx$, but this is not possible since $T_\infty > 0$ and only $T - T_\infty$ can be estimated in this manner, as for instance for $n = 3$

$$\sup_{\mathbb{R}^n} |T - T_\infty| \leqq c\Big(\int_{\mathbb{R}^n} \pi\, dx \int_{\mathbb{R}^n} \gamma\, dx\Big)^{\frac{1}{2(1-a)}}. \tag{3.33}$$

Therefore, it appears that, in the estimates of Proposition 4, the powers of the solution derivatives are straightforward thanks to the terms $|\partial_x T|^4$, $|d|^4$, $|\partial_x^2 T|^2$, and

$|\partial_x d|^2$, but difficulties arise however, in the temperature exponents appearing at the denominator of the convective term ω, which are too small in comparison to those appearing in the higher derivative terms π associated with transport fluxes.

A natural phenomenon which reduces the temperature exponents appearing at the denominator of the higher order derivative terms of π is the temperature dependence of transport coefficients. When λ and η scale as $T^{\varkappa}$, all temperature exponents at the denominators of π are decreased by $\varkappa$ whereas those of ω are unchanged. The corresponding system of partial differential equations is investigated in the following sections using somewhat different methods. The direct techniques used in this section do not apply anymore because of a new pressure term $p_\eta = -\sum_{1\leqq i,j\leqq n} R_i R_j(\eta d_{ij})$ due to the derivatives of viscosity with respect to temperature, which vanishes for constant η. This pressure term introduces extra contributions in Σ which are not simply controlled by those of π. Furthermore, the simple direct method of this section cannot be used for the higher order entropy correctors $\gamma^{[k]}$ or $\tilde{\gamma}^{[k]}$ when $k \geqq 2$.

Remark 8. The same discussion can be conducted in a periodic framework and yields the same conclusions. In this situation, we also have estimates in the form

$$\int_\Omega |\omega|\, dx \leqq c\Big(\int_\Omega \pi\, dx\Big)^{3/4}\Big(\int_\Omega T^{3-a}\, dx\Big)^{1/4}, \tag{3.34}$$

where the periodic domain Ω is a product of intervals, but the quantity $\int_\Omega T^{3-a}\, dx$ cannot be estimated in terms of $\int_\Omega \pi\, dx$ and $\int_\Omega \gamma\, dx$. Only the difference $T - \overline{T}$, where $\overline{T}$ denotes the average of T over the periodic domain Ω, can be estimated in terms of $\int_\Omega \pi\, dx$ and $\int_\Omega \gamma\, dx$.

Remark 9. Assuming that $\partial_x T/T \in L^2 \cap L^4$ when $n = 3$ implies that $\log T$ has a finite limit at infinity [15] so that T_∞ must be positive. In other words, it does not make sense to try to rescale with $T_\infty = 0$.

Remark 10. The convective term ω, after a few integrations by parts, can also be written as a sum of terms proportional to the velocity v. This does not improve the estimates of $\int_{\mathbb{R}^n} |\omega|\, dx$ since the gains obtained with the v factor are compensated by the loss of one derivative factor.

4. Weighted inequalities

We collect in this section various weighted inequalities that we will use in our investigation of the situation of temperature dependent transport coefficients.

4.1. *Differential identities*

Let α_i, $1 \leqq i \leqq n$, be non-negative integers and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be the corresponding multiindex. We denote by ∂^α the differential operator $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and by $|\alpha|$ its order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The derivative of superpositions has been investigated in particular by VOL'PERT and HUDJAEV [43] and the following proposition is established by induction on $|\alpha|$.

Lemma 1. *Let f and g be smooth functions and $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multiindex. Then we have*

$$\partial^\alpha (fg) = \sum_{0 \leqq \beta \leqq \alpha} c_{\alpha\beta}\, \partial^\beta f \partial^{(\alpha-\beta)} g, \tag{4.1}$$

where $c_{\alpha\beta} = \alpha!/\beta!(\alpha-\beta)!$ are non-negative integer coefficients, $\beta! = \beta_1! \cdots \beta_n!$, and where we write $0 \leqq \beta \leqq \alpha$ when $0 \leqq \beta_i \leqq \alpha_i$, $1 \leqq i \leqq n$.

Furthermore, let f and g be smooth scalar functions, and let α be a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| \geqq 1$. The partial derivatives of the superposition $g \circ f$ can be written in the form

$$\partial^\alpha (g \circ f) = \sum_{\sigma\mu} c_{\sigma\mu}\, \partial^\sigma g \prod_{1 \leqq |\beta| \leqq |\alpha|} \big(\partial^\beta f\big)^{\mu_\beta}, \tag{4.2}$$

where $c_{\sigma\mu}$ are non-negative integer coefficients, and the sum is over $1 \leqq \sigma \leqq |\alpha|$, $\mu = (\mu_\beta)_{1 \leqq |\beta| \leqq |\alpha|}$ with $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, such that

$$\sum_{1 \leqq |\beta| \leqq |\alpha|} \mu_\beta = \sigma, \qquad \sum_{1 \leqq |\beta| \leqq |\alpha|} \beta\ \mu_\beta = \alpha, \tag{4.3}$$

so that we have in particular $\sum_{1 \leqq |\beta| \leqq |\alpha|} |\beta|\ \mu_\beta = |\alpha|$.

A natural scaling induced by higher order entropies is that v scales as $\sqrt{T}$. As a consequence, we introduce the rescaled unknowns τ and w defined by

$$\tau = \log T, \qquad w = \frac{v}{\sqrt{T}}, \tag{4.4}$$

which will naturally appear in higher order entropy estimates. In particular, we will need the following differential identities, easily established by induction on $|\alpha|$.

Lemma 2. *Let T be smooth and positive and α be a multiindex. Then we have*

$$\frac{\partial^\alpha T}{T} = \sum_\mu c_\mu \prod_{1 \leqq |\beta| \leqq |\alpha|} \big(\partial^\beta \tau\big)^{\mu_\beta} = \partial^\alpha \tau + \sum_\mu c_\mu \prod_{1 \leqq |\beta| \leqq |\alpha|-1} \big(\partial^\beta \tau\big)^{\mu_\beta}, \tag{4.5}$$

where $\mu = (\mu_\beta)_{1 \leqq |\beta| \leqq |\alpha|}$ with $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and c_μ are non-negative integer coefficients. The sum is extended over the μ such that

$$\sum_{1 \leqq |\beta| \leqq |\alpha|} \beta\ \mu_\beta = \alpha,$$

so that we have in particular $\sum_{1 \leqq |\beta| \leqq |\alpha|} |\beta|\ \mu_\beta = |\alpha|$, and the only term with $|\beta| = |\alpha|$ corresponds to $\partial^\alpha \tau$. Conversely, we have

$$\partial^\alpha \tau = \sum_\mu c'_\mu \prod_{1 \leqq |\beta| \leqq |\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta} = \frac{\partial^\alpha T}{T} + \sum_\mu c'_\mu \prod_{1 \leqq |\beta| \leqq |\alpha|-1} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta}, \tag{4.6}$$

where c'_μ are integer coefficients and the sum is extended over the same set of μ.

Lemma 3. *Let T and v be smooth, T be positive, i with $1 \leqq i \leqq n$, and α be a multiindex. Then we have*

$$\frac{\partial^\alpha v_i}{\sqrt{T}} = \sum_{\mu\tilde{\alpha}} c_{\mu\tilde{\alpha}} \prod_{1\leqq|\beta|\leqq|\alpha|} \big(\partial^\beta \tau\big)^{\mu_\beta} \, \partial^{\tilde{\alpha}} w_i, \tag{4.7}$$

where $\mu = (\mu_\beta)_{1\leqq|\beta|\leqq|\alpha|}$, $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, $\tilde{\alpha} \in \mathbb{N}^n$, $c_{\mu\tilde{\alpha}}$ are non-negative integer coefficients, and the sum is extended over the μ and $\tilde{\alpha}$, such that

$$0 \leqq \tilde{\alpha} \leqq \alpha, \qquad \sum_{1\leqq|\beta|\leqq|\alpha|} \beta \, \mu_\beta + \tilde{\alpha} = \alpha.$$

More precisely, isolating the only term $\partial^\alpha w_i$ corresponding to $\tilde{\alpha} = \alpha$ and all the terms corresponding to $\tilde{\alpha} = (0, \ldots, 0)$, we have

$$\frac{\partial^\alpha v_i}{\sqrt{T}} = \partial^\alpha w_i + \sum_{\mu\tilde{\alpha}} c_{\mu\tilde{\alpha}} \prod_{1\leqq|\beta|\leqq|\alpha|} \big(\partial^\beta \tau\big)^{\mu_\beta} \, \partial^{\tilde{\alpha}} w_i \; + \sum_{\mu} c_{\mu 0} \prod_{1\leqq|\beta|\leqq|\alpha|} \big(\partial^\beta \tau\big)^{\mu_\beta} \, w_i, \tag{4.8}$$

where the $\tilde{\alpha}$ in the middle sum are such that $1 \leqq |\tilde{\alpha}| < |\alpha|$. Conversely, we have

$$\partial^\alpha w_i = \sum_{\mu\tilde{\alpha}} c'_{\mu\tilde{\alpha}} \prod_{1\leqq|\beta|\leqq|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta} \, \frac{\partial^{\tilde{\alpha}} v_i}{\sqrt{T}}, \tag{4.9}$$

and more precisely

$$\partial^\alpha w_i = \frac{\partial^\alpha v_i}{\sqrt{T}} + \sum_{\mu\tilde{\alpha}} c'_{\mu\tilde{\alpha}} \prod_{1\leqq|\beta|\leqq|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta} \, \frac{\partial^{\tilde{\alpha}} v_i}{\sqrt{T}} \; + \sum_{\mu} c'_{\mu 0} \prod_{1\leqq|\beta|\leqq|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta} \, \frac{v_i}{\sqrt{T}}, \tag{4.10}$$

where $c'_{\mu\tilde{\alpha}}$ are integer coefficients and the sums are extended over the same sets.

4.2. Weighted operators

We investigate the norm of weighted Calderón–Zygmund operators in Lebesgue spaces [8,16,36,37]. A natural condition associated with weights has been shown to be the Muckenhoupt property A_p, where $1 \leqq p \leqq \infty$ [8,9,16,23–25,35].

Definition 1. Let $\mathscr{G} : \mathscr{D}(\mathbb{R}^n) \longrightarrow \mathscr{D}'(\mathbb{R}^n)$ be a continuous linear operator and denote by K the restriction of the distribution kernel associated with $\mathscr{G}$ to the open set $x \neq y$ of $\mathbb{R}^n \times \mathbb{R}^n$. We say that $\mathscr{G}$ is a Calderón–Zygmund operator when the following properties are satisfied [35]:

(i) K is a locally integrable function and there exists c_0 such that for $x \neq y |K(x, y)| \leqq c_0|x - y|^{-n}$.
(ii) There exists $\delta \in (0, 1]$ and c_1 such that for $x \neq y$ and $|x' - x| \leqq \frac{1}{2}|x - y|$ we have $|K(x', y) - K(x, y)| \leqq c_1|x' - x|^\delta |x - y|^{-n-\delta}$.

(iii) Similarly if $x \neq y$ and $|y' - y| \leqq \frac{1}{2}|x - y|$ we have $|K(x, y') - K(x, y)| \leqq c_1 |y' - y|^\delta |x - y|^{-n-\delta}$.

(iv) $\mathscr{G}$ can be extended into a continuous linear operator over $L^2(\mathbb{R}^n)$ with a norm lower or equal to c_2.

Definition 2. Let $g \in L^1_{loc}(\mathbb{R}^n)$ be positive and locally integrable and $1 < p < \infty$. We say that g satisfies the Muckenhoupt condition A_p if

$$[g]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q g \, dx \right) \left(\frac{1}{|Q|} \int_Q g^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty, \tag{4.11}$$

where the supremum is taken over all cubes.

For detailed studies about the Muckenhoupt property we refer to the book of Garcia-Cuerva and Rubio de Francia [16]. We have in particular $A_p \cap A_q = A_{\min(p,q)}$ and the weights of A_p have their logarithms in *BMO* [16,35]. A locally summable function f belongs to the space $BMO(\mathbb{R}^n)$ if

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q \left| f(x) - \bar{f}_Q \right| dx < \infty,$$

where the supremum is taken over all cubes Q and $\bar{f}_Q = 1/|Q| \int_Q f(x) \, dx$ denotes the average of f over Q [27,34]. The function space *BMO* has been introduced by John and Nirenberg [27] and naturally arises when estimating the norms of the weighted operators $T^\theta R_i T^{-\theta}$, where $R_i = (-\Delta)^{-1/2} \partial_i$, $1 \leqq i \leqq n$, are Riesz transforms, or when using the Coifman and Meyer inequalities [36,37]. The space *BMO* and its dual $\mathscr{H}^1$ have already been used in the context of the Navier–Stokes equations [29,31,32].

Theorem 1. *Let $\mathscr{G}$ be a Calderón–Zygmund operator, $1 < p < \infty$, and g be a weight in A_p. Then the operator $\mathscr{G}$ is bounded in $L^p(g dx)$, or equivalently, the operator $g^{1/p} \mathscr{G} g^{-1/p}$ is bounded in L^p, with a norm lower than $\mathscr{C}(c_0, c_1, c_2, n, p, [g]_{A_p})$, where $\mathscr{C}$ only depends on c_0, c_1, c_2, n, p, and $[g]_{A_p}$.*

Proof. We refer to Meyer [34,35], Garcia-Cuerva and Rubio de Francia [16], and Coifman and Fefferman [8]. A careful examination of the above mentioned references reveals that the constant $\mathscr{C}$ only depends on c_0, c_1, c_2, n, p, and $[g]_{A_p}$.

Theorem 2. *There exist constants $b(n)$ and $B(n)$ such that for any $\theta \in \mathbb{R}$, any $u \in BMO$, and any $1 < p < \infty$, the condition*

$$|\theta| \|u\|_{BMO} < \tfrac{1}{2} b(n) \min(1, p-1),$$

implies that $\exp(\theta u) \in A_p$ *and*

$$\left[\exp(\theta u)\right]_{A_p} \leqq \left(1 + B(n)\right)^p.$$

Proof. From a result of JOHN and NIRENBERG [27], there exist positive constants $b(n)$ and $B(n)$, depending only on n, such that for any $u \in BMO$, any cube Q and any positive s the following inequality holds

$$\mu_Q(s) = \frac{1}{|Q|}\mathrm{mes}\Big\{\ x \in Q,\ \ |u(x) - \bar{u}_Q| > s\ \Big\} \leqq B(n)\exp\Big(-\frac{s\,b(n)}{\|u\|_{BMO}}\Big),$$

where $\bar{u}_Q$ denotes the average of u over Q. Using the identity

$$\frac{1}{|Q|}\int_Q f(|u - \bar{u}_Q|)\,dx = \int_0^\infty \mu_Q(s)\,df(s),$$

valid for increasing continuously differentiable functions f such that $f(0) = 0$, with $f(s) = \exp(b's) - 1$ and $0 < b'\|u\|_{BMO} < b(n)/2$, we obtain that

$$\frac{1}{|Q|}\int_Q \exp\big(b'|u - \bar{u}_Q|\big)\,dx \leqq 1 + \frac{B(n)b'\|u\|_{BMO}}{b(n) - b'\|u\|_{BMO}} \leqq 1 + B(n).$$

Therefore, we deduce that

$$\begin{aligned}
&\sup_Q\left(\frac{1}{|Q|}\int_Q \exp(\theta u)\,dx\right)\left(\frac{1}{|Q|}\int_Q \exp(-\frac{\theta u}{p-1})\,dx\right)^{p-1}\\
&\leqq \sup_Q\left(\frac{1}{|Q|}\int_Q \exp(|\theta|\,|u - \bar{u}_Q|)\,dx\right)\left(\frac{1}{|Q|}\int_Q \exp(\frac{|\theta|}{p-1}\,|u - \bar{u}_Q|)\,dx\right)^{p-1}\\
&\leqq \big(1 + B(n)\big)^p,
\end{aligned}$$

provided that $|\theta|\|u\|_{BMO} < \frac{1}{2}b(n)\min(1, p-1)$.

As a direct application of Theorems 1 and 2, we investigate operators with weights in the form $\exp(\theta u)$, where $\theta \in \mathbb{R}$ and $u \in BMO$.

Corollary 1. *Let $\mathscr{G}$ be a Calderón–Zygmund operator and $1 < p < \infty$. There exist constants $\delta(n, p)$ and $\mathscr{C}(c_0, c_1, c_2, n, p)$, depending respectively on (n, p) and (c_0, c_1, c_2, n, p), such that for any $\theta \in \mathbb{R}$ and $u \in BMO$, the condition $|\theta|\|u\|_{BMO} < \delta(n, p)$ implies that the operator $\mathscr{G}$ is bounded in $L^p\big(\exp(\theta u)dx\big)$, or equivalently, that the operator $\exp(\theta u/p)\mathscr{G}\exp(-\theta u/p)$ is bounded in L^p, with a norm lower than $\mathscr{C}(c_0, c_1, c_2, n, p)$.*

4.3. Multilinear estimates

We investigate weighted multilinear estimates, with weights in A_p, and we denote by $A(\mathbb{R}^n)$ the Wiener algebra in $\mathbb{R}^n$ [36,37,39].

Theorem 3. *Let k, l be positive integers, and α^j, $1 \leqq j \leqq l$, be multiindices such that $|\alpha^j| \geqq 1$, $1 \leqq j \leqq l$, and $k = \sum_{1\leqq j\leqq l}|\alpha^j|$. Let $1 < p < \infty$, $g \in A_p$, and $u_1, \ldots, u_l$, be such that there exist constants $u_{j,\infty}$ with $u_j - u_{j,\infty} \in$*

$H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, and such that $g^{\frac{1}{p}}\partial^k u_j \in L^p$, $1 \leqq j \leqq l$. There exists a constant $c = c(k, n, p, [g]_{A_p})$ only depending on k, n, p, and $[g]_{A_p}$, such that

$$\Big\| g^{1/p} \prod_{1\leqq j\leqq l} \partial^{\alpha^j} u_j \Big\|_{L^p} \leqq c\Big(\sum_{1\leqq j\leqq l} \|u_j\|_{BMO}\Big)^{l-1} \Big(\sum_{1\leqq j\leqq l} \big\| g^{1/p}\partial^k u_j \big\|_{L^p}\Big), \tag{4.12}$$

where we define

$$\big\| g^{1/p}\partial^k \mathsf{v}\big\|_{L^p}^p = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\int_{\mathbb{R}^n} g\big|\partial^\alpha \mathsf{v}\big|^p\,dx,$$

using the multinomial coefficients [10,41]

$$\binom{k}{\alpha} = \frac{k!}{\alpha!} = \frac{k!}{\alpha_1!\cdots\alpha_n!}.$$

Proof. We use the Coifman–Meyer theory of multilinear operators [36,37]. We can first assume that $u_{j,\infty} = 0$, $1 \leqq j \leqq l$, since these constants do not modify the norms in (4.12). Since the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, we can assume that $u_j \in \mathscr{S}(\mathbb{R}^n)$, $1 \leqq j \leqq l$. In this situation, we can write that

$$\prod_{1\leqq j\leqq l} \partial^{\alpha^j} u_j(x) = \mathrm{Cte}\int \prod_{1\leqq j\leqq l} \exp\big(\mathrm{i}x\cdot\xi^j\big)(\xi^j)^{\alpha^j}\hat{u}_j(\xi^j)\,d\xi^j,$$

where $\xi^j \in \mathbb{R}^n$, $1 \leqq j \leqq l$, and $\hat{u}_j$ denotes the Fourier transform of u_j. We introduce $\Phi \in C^\infty[0,\infty)$ such that $\mathrm{supp}(\Phi) \subset [0,1]$, $0 \leqq \Phi \leqq 1$, $\Phi = 1$ over $[0, 1/2]$, and we set $\Psi = 1-\Phi$. We further define $\Phi_i = \Phi(\sum_{i+1\leqq j\leqq l} |\xi^j|^2/|\xi^i|^2)$ and $\Psi_i = 1 - \Phi_i$, for $1 \leqq i \leqq l-1$, and we have the partition of unity

$$1 = \Phi_1 + \Psi_1\Phi_2 + \Psi_1\Psi_2\Phi_3 + \cdots + \prod_{1\leqq j\leqq l-2}\Psi_j\ \Phi_{l-1} + \prod_{1\leqq j\leqq l-1}\Psi_j.$$

We multiply this partition of unity by the product $\prod_{1\leqq j\leqq l}(\xi^j)^{\alpha^j}$ and each factor is rewritten in the form

$$\prod_{1\leqq j\leqq l}(\xi^j)^{\alpha^j}\prod_{1\leqq j\leqq i-1}\Psi_j\ \Phi_i = |\xi^i|^k \prod_{1\leqq j\leqq l}\Big(\frac{\xi^j}{|\xi^i|}\Big)^{\alpha^j}\prod_{1\leqq j\leqq i-1}\Psi_j\ \Phi_i,$$

with the convention that $\Phi_l = 1$. Denoting by $\mathscr{H}_i$ the multilinear operator associated with the kernel

$$\zeta_i = \prod_{1\leqq j\leqq l}\Big(\frac{\xi^j}{|\xi^i|}\Big)^{\alpha^j}\prod_{1\leqq j\leqq i-1}\Psi_j\ \Phi_i,$$

we have obtained that

$$\prod_{1\leqq j\leqq l}\partial^{\alpha^j}u_j = \sum_{1\leqq i\leqq l}\mathscr{H}_i\big(u_1,\ldots,u_{i-1},(-\Delta)^{\frac{k}{2}}u_i,u_{i+1},\ldots,u_l\big).$$

We claim that the operator $\mathscr{H}_i$, where $1 \leqq i \leqq l$ is fixed, satisfies the assumptions of the Coifman and Meyer Theorem (Theorem 2, page 434, Section III.XIII.4 of [36,37]), so that it can be extended into a continuous operator over $BMO^{i-1} \times L^2 \times BMO^{l-i}$. Indeed, for $1 \leqq i \leqq l-1$, the kernel ζ_i of $\mathscr{H}_i$ is nonzero only when $\prod_{1\leqq j\leqq i-1} \Psi_j \Phi_i$ is nonzero, that is, only when

$$\begin{cases} \displaystyle\sum_{i+1\leqq j\leqq l} |\xi^j|^2 \leqq |\xi^i|^2, \\ |\xi^{i-k}|^2 \quad \leqq \quad 2 \displaystyle\sum_{i-k+1\leqq j\leqq l} |\xi^j|^2, \qquad 1 \leqq k \leqq i-1. \end{cases}$$

These conditions imply $|\xi^{i-k}|^2 \leqq 4^k |\xi^i|^2$, $1 \leqq k \leqq i-1$, and $\sum_{j\neq i} |\xi^j|^2 \leqq 4^i |\xi^i|^2$. On the other hand, for $i = l$, we have $\Phi_l = 1$, and the kernel ζ_l of $\mathscr{H}_l$ is nonzero only when $\prod_{1\leqq j\leqq l-1} \Psi_j$ is nonzero, that is, only when

$$|\xi^{l-k}|^2 \leqq 2 \sum_{l-k+1\leqq j\leqq l} |\xi^j|^2, \qquad 1 \leqq k \leqq l-1,$$

and these conditions imply that $|\xi^{l-k}|^2 \leqq 4^k |\xi^l|^2$, $1 \leqq k \leqq l-1$, and $\sum_{j\neq l} |\xi^j|^2 \leqq 4^l |\xi^l|^2$. As a consequence, for any $1 \leqq i \leqq l$, the kernel ζ_i of $\mathscr{H}_i$ is bounded and smooth for $(\xi_1, \dots, \xi_l) \neq (0, \dots, 0)$. Furthermore, for $\beta = (\beta^1, \dots, \beta^l)$, $\beta^j \in \mathbb{N}^n$, we have

$$|\partial^\beta \zeta_i| \leqq C\big(|\xi^1| + \cdots + |\xi^l|\big)^{-|\beta|},$$

where $|\beta| = |\beta^1| + \cdots + |\beta^l|$. Finally, we have $\zeta_i(\xi_1, \dots, \xi_l) = 0$ whenever $\xi_j = 0$ for any $j \neq i$. Therefore, from the Coifman and Meyer theorem we obtain that for $\mathsf{v} \in L^2(\mathbb{R}^n)$

$$\big\| \mathscr{H}_i(u_1, \dots, u_{i-1}, \mathsf{v}, u_{i+1}, \dots, u_l) \big\|_{L^2} \leqq c \prod_{j\neq i} \|u_j\|_{BMO} \; \|\mathsf{v}\|_{L^2},$$

and that for u_j, $j \neq i$ fixed, the operator

$$\mathsf{v} \longrightarrow \mathscr{H}_i(u_1, \dots, u_{i-1}, \mathsf{v}, u_{i+1}, \dots, u_l),$$

is a Calderón–Zygmund operator. From the results of Coifman–Meyer (Theorem 2, page 434, Section III.XIII.4 of [36,37]), we also obtain that the distribution kernel K_i associated with $\mathscr{H}_i$ is such that

$$|K_i(x, y)| \leqq c \prod_{j\neq i} \|u_j\|_{BMO} |x - y|^{-n},$$

with similar inequalities for the derivatives. Therefore, the operator $\mathscr{H}_i$ rescaled by $\prod_{j\neq i} \|u_j\|_{BMO}$ satisfies the properties (i)–(iv) of Definition 1 with $\delta = 1$ and with constants c_0, c_1 and c_2 depending only on k and n. As a consequence, as soon as the weight g satisfies the Muckenhoupt condition A_p, we have

$$\big\| g^{1/p} \mathscr{H}_i(u_1, \dots, u_{i-1}, \mathsf{v}, u_{i+1}, \dots, u_l) \big\|_{L^p} \leqq c \prod_{j\neq i} \|u_j\|_{BMO} \; \|g^{1/p} \mathsf{v}\|_{L^p},$$

where c only depends on n, k, p and $[g]_{A_p}$. Summing over i, we have obtained that

$$\Big\|g^{1/p}\prod_{1\leqq j\leqq l}\partial^{\alpha^j}u_j\Big\|_{L^p}\leqq c\sum_{1\leqq i\leqq l}\ \prod_{\substack{1\leqq j\leqq l\\ j\neq i}}\|u_j\|_{BMO}\ \Big\|g^{1/p}(-\Delta)^{\frac{k}{2}}u_i\Big\|_{L^p}.$$

The proof is then complete upon noting that there exists a constant c only depending on n, k, p and $[g]_{A_p}$ such that

$$\Big\|g^{1/p}(-\Delta)^{\frac{k}{2}}u_i\Big\|_{L^p}\leqq c\Big\|g^{1/p}\partial^k u_i\Big\|_{L^p}.$$

This is obvious when k is even since then $k=2l$ and $(-\Delta)^{k/2}=(-\Delta)^l$ whereas for k odd $k=2l+1$ we have $(-\Delta)^{k/2}=(-\Delta)^l(-\Delta)^{1/2}$ and from $\sum_{1\leqq j\leqq n}R_j^2=-I$ we obtain that $-(-\Delta)^{k/2}=\sum_{1\leqq j\leqq n}(-\Delta)^l(-\Delta)^{1/2}R_j^2=\sum_{1\leqq j\leqq n}R_j(-\Delta)^l\partial_j$ and $\big\|g^{1/p}(-\Delta)^{k/2}\phi\big\|_{L^p}\leqq\sum_{1\leqq j\leqq n}\big\|g^{1/p}(-\Delta)^l\partial_j\phi\big\|_{L^p}$ since $g^{1/p}R_jg^{-1/p}$ is continuous in L^p, $1\leqq j\leqq n$, thanks to $g\in A_p$, as established in Theorem 1.

Remark 11. The definition of $\|\partial^k\mathsf{v}\|_{L^p}^p$ using the multinomial coefficients yields in particular that for $p=2$

$$\|\partial^k\mathsf{v}\|_{L^2}^2=\sum_{|\alpha|=k}\frac{k!}{\alpha!}\int_{\mathbb{R}^n}\big(\partial^\alpha\mathsf{v}\big)^2\,dx=\sum_{1\leqq i_1,\ldots,i_k\leqq n}\|\partial_{i_1}\cdots\partial_{i_k}\mathsf{v}\|_{L^2}^2,\tag{4.13}$$

so that it is compatible with the classical definition $|\partial^2\mathsf{v}|^2=\sum_{ij}(\partial_i\partial_j\mathsf{v})^2$ already used in Section 3.3. This natural definition also simplifies the analytic form of higher order entropies governing equations.

Remark 12. The space of smooth functions with compact support $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n)\cap BMO(\mathbb{R}^n)$ (for the norm $\|\cdot\|_{H^k}+\|\cdot\|_{BMO}$ of course) if and only if $k\geqq n/2$. Indeed, for $k<n/2$, $\mathscr{D}(\mathbb{R}^n)$ is not even dense in $H^k(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and counterexamples are classically found in the form of a series of needles as for instance

$$\varphi=\sum_{i\geqq 1}\varphi_{x_i,\varepsilon_i},\qquad \varphi_{y,\delta}(x)=\Big\{\Big(1-\frac{\|x-y\|^2}{\delta^2}\Big)^+\Big\}^k,$$

where $\delta>0$, $x_i=(i,0,\ldots,0)$, and the sequence $\{\varepsilon_i\}$ is such that $0<\varepsilon_i<1/2$ and $\sum_{i\geqq 1}\varepsilon_i^{n-2k}<\infty$. On the other hand, for $k=n/2$, we have $H^k(\mathbb{R}^n)\cap BMO(\mathbb{R}^n)=H^k(\mathbb{R}^n)$, whereas for $k>n/2$, $H^k(\mathbb{R}^n)$ is included in the Wiener algebra $A(\mathbb{R}^n)$. We have introduced the natural simplifying assumption $u_j-u_{j,\infty}\in H^k(\mathbb{R}^n)\cap A(\mathbb{R}^n)$ since it will be sufficient for our applications and since for $k<n/2$, $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n)\cap A(\mathbb{R}^n)$ and $A(\mathbb{R}^n)\subset L^\infty(\mathbb{R}^n)\subset BMO(\mathbb{R}^n)$. Extending inequalities (4.12) to $H^k(\mathbb{R}^n)\cap BMO(\mathbb{R}^n)$ when $k<n/2$ by using the notion of strict convergence [36,37] is not relevant to our study.

4.4. Weighted interpolation inequalities

We generalize here some Nirenberg interpolation inequalities for intermediate derivatives [38] with weights satisfying the Muckenhoupt properties.

Theorem 4. *Let k, j be non-negative integers, let $1 < q < \infty$, $1 < r < \infty$, and assume that $k \geqq 1$ and $0 \leqq j \leqq k$. Further assume that g is a weight of the* MUCKENHOUPT *class $A_r \cap A_q = A_{\min(q,r)}$ and let p be such that*

$$\frac{1}{p} = \frac{k-j}{k}\frac{1}{q} + \frac{j}{k}\frac{1}{r}. \tag{4.14}$$

Then for any v such that $\mathsf{v} \in L^q(g dx)$ and $\partial^k \mathsf{v} \in L^r(g dx)$, the intermediate derivative $\partial^j \mathsf{v}$ is in $L^p(g dx)$ and there exists a constant $\mathscr{C}$ only depending on n, k, q, r, $[g]_{A_q}$ and $[g]_{A_r}$ such that

$$\Big(\int_{\mathbb{R}^n} g|\partial^j \mathsf{v}|^p \, dx\Big)^{\frac{1}{p}} \leqq \mathscr{C}\Big(\int_{\mathbb{R}^n} g|\mathsf{v}|^q \, dx\Big)^{(1-\frac{j}{k})\frac{1}{q}} \Big(\int_{\mathbb{R}^n} g|\partial^k \mathsf{v}|^r \, dx\Big)^{\frac{j}{k}\frac{1}{r}}. \tag{4.15}$$

Proof. By induction on k, the proof of (4.15) is easily reduced to the special case $j = 1$ and $k = 2$. In this situation, we have $2/p = 1/q + 1/r$ and defining $p' = p/2$ we have $1/2 < p' < \infty$ and

$$\frac{1}{p'} = \frac{1}{q} + \frac{1}{r}. \tag{4.16}$$

Inequality (4.15) can then be rewritten in terms of the square of the gradient

$$\Big(\int_{\mathbb{R}^n} g\big(|\partial \mathsf{v}|^2\big)^{p'} \, dx\Big)^{\frac{1}{p'}} \leqq c\Big(\int_{\mathbb{R}^n} g|\mathsf{v}|^q \, dx\Big)^{\frac{1}{q}} \Big(\int_{\mathbb{R}^n} g|\partial^2 \mathsf{v}|^r \, dx\Big)^{\frac{1}{r}}. \tag{4.17}$$

In order to estimate the square of the gradient $|\partial \mathsf{v}|^2$ we consider any pair of indices i_1 and i_2, any functions u_1, u_2, in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, and we write (as in the proof of Theorem 3) that

$$\partial_{i_1} u_1(x)\ \partial_{i_2} u_2(x) = \mathrm{Cte} \int \exp\big(\mathrm{i}x \cdot (\xi^1 + \xi^2)\big) \xi^1_{i_1} \xi^2_{i_2} \hat{u}_1(\xi^1) \hat{u}_2(\xi^2) \, d\xi^1 d\xi^2, \tag{4.18}$$

where $\xi^1, \xi^2 \in \mathbb{R}^n$, and $\hat{u}_j$ is the Fourier transform of u_j, $j = 1, 2$. Introducing again a partition of unity $\Phi_1 + \Psi_1 = 1$ as in Theorem 3, we can write that

$$\partial_{i_1} u_1\ \partial_{i_2} u_2 = \mathscr{H}_1\big(u_1, (-\Delta)u_2\big) + \mathscr{H}_2\big((-\Delta)u_1, u_2\big), \tag{4.19}$$

where $\mathscr{H}_1$ is the multilinear operator with symbol $\Phi_1\ \xi^1_{i_1}\xi^2_{i_2}/|\xi^1|^2$ and $\mathscr{H}_2$ the multilinear operator with symbol $\Psi_1\ \xi^1_{i_1}\xi^2_{i_2}/|\xi^2|^2$. Using the results of GRAFAKOS and TORRES [23–25] we deduce that the operators $\mathscr{H}_1$ et $\mathscr{H}_2$ are multilinear Calderón–Zygmund operators (Proposition 6 of [23] or Section 2 of [25]). On the other hand, the weight g also belongs to the class A_∞, that is, there exists constants $C > 0$ and $\varepsilon \in (0, 1]$ such that for any cube Q and any measurable set $E \subset Q$ we have

$$\frac{g(E)}{g(Q)} \leqq C\Big(\frac{|E|}{|Q|}\Big)^{\varepsilon},$$

where $g(E) = \int_E g(x)\,dx$ and $|E|$ denotes the Lebesgue measure of E. More specifically, for any $1 < s < \infty$ and $g \in A_s$, we have $g \in A_\infty$, where the constants C and ε only depend on s and $[g]_{A_s}$ [8,16]. As a consequence, we can use the weighted inequalities established by Grafakos and Torres (Corollary 3 of [24] or Corollary 5 of [25]) taking into account that $1 < r < \infty$, $1 < q < \infty$, $1/2 < p' < \infty$, (4.16) and letting $u_1 = u_2 = v$, and the interpolation constant $\mathscr{C}$ depending finally only on n, k, q, r, $[g]_{A_q}$ and $[g]_{A_r}$.

We now consider the case $q = \infty$ by combining the interpolation inequality of Theorem 4 with the multilinear estimates of Theorem 3.

Theorem 5. *Let k, j be non-negative integers, $1 < r < \infty$, assume that $k \geqq 1$, and $1 \leqq j \leqq k$. Further assume that g is a weight in the Muckenhoupt class A_r and let p be such that*

$$\frac{1}{p} = \frac{j}{kr}. \tag{4.20}$$

Then for any v such that $\mathsf{v} - \mathsf{v}_\infty \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, where v_∞ is a constant, and such that $\partial^k \mathsf{v} \in L^r(g dx)$, the intermediate derivative $\partial^j \mathsf{v}$ is in $L^p(g dx)$ and there exists a constant $\mathscr{C}$ only depending on n, k, r, and $[g]_{A_r}$ such that

$$\Big(\int_{\mathbb{R}^n} g |\partial^j \mathsf{v}|^{\frac{rk}{j}}\, dx \Big)^{\frac{j}{rk}} \leqq \mathscr{C}\, \big\| \mathsf{v} \big\|_{BMO}^{1-\frac{j}{k}} \Big(\int_{\mathbb{R}^n} g |\partial^k \mathsf{v}|^r\, dx \Big)^{\frac{j}{rk}}. \tag{4.21}$$

Proof. Letting $p = r$, $|\alpha^j| = 1$, and $k = l$ in Theorem 3, we deduce that

$$\Big(\int_{\mathbb{R}^n} g |\partial \mathsf{v}|^{rk}\, dx \Big)^{\frac{1}{rk}} \leqq c \big\| \mathsf{v} \big\|_{BMO}^{1-\frac{1}{k}} \Big(\int_{\mathbb{R}^n} g |\partial^k \mathsf{v}|^r\, dx \Big)^{\frac{1}{rk}}, \tag{4.22}$$

and this yields (4.21) for $j = 1$. For $1 < j < k$ we can then interpolate $\partial^j \mathsf{v}$ between $\partial^1 \mathsf{v}$ and $\partial^k \mathsf{v}$ and combine inequality (4.15) of Theorem 4 with (4.22).

Remark 13. The elegant elementary proofs of Nirenberg [38] are easily adapted to weights in A_p^* classes [16] but not to weights in Muckenhoupt classes A_p. The A_p^* classes are defined as in Definition 2 with (4.11) but with n-dimensional products of intervals instead of cubes. Nirenberg elementary proofs are indeed first given in one dimension and then extended to higher dimensions, and only weights in A_p^* classes are such that the induced weights in smaller dimensions (obtained by freezing some of the coordinates) satisfy A_p conditions in the smaller dimensions, uniformly with respect to the frozen variables [16].

4.5. Weighted products of derivatives

We first investigate products of derivatives of the rescaled unknowns τ and w, with powers of temperature as natural weights.

Theorem 6. *Let $k \geqq 1$ be an integer, $\bar{\theta} > 0$ be positive, $1 < p < \infty$, τ be such that $\tau - \tau_\infty \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ for some constant τ_∞. There exist positive constants $\delta(n, k, \bar{\theta}, p)$ and $c(n, k, p)$, only depending on $(n, k, \bar{\theta}, p)$ and (n, k, p), respectively, such that if $\|\tau\|_{BMO} < \delta$, then for any real θ with $|\theta| \leqq \bar{\theta}$, any integer $l \geqq 1$, and any multiindices α^j, $1 \leqq j \leqq l$, with $|\alpha^j| \geqq 1$, $1 \leqq j \leqq l$, and $\sum_{1\leqq j\leqq l} |\alpha^j| = k$, whenever $\exp(\theta\tau/p)\partial^k\tau \in L^p(\mathbb{R}^n)$, we have the estimates*

$$\Big\| e^{\frac{\theta\tau}{p}} \prod_{1\leqq j\leqq l} \big(\partial^{\alpha^j}\tau\big)\Big\|_{L^p} \leqq c\|\tau\|_{BMO}^{l-1} \, \|e^{\frac{\theta\tau}{p}}\partial^k\tau\|_{L^p}. \tag{4.23}$$

Further assuming that $w \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $e^{\theta\tau/p}\partial^k w \in L^p(\mathbb{R}^n)$, and $0 \leqq \bar{l} \leqq l$, we have

$$\Big\| e^{\frac{\theta\tau}{p}} \prod_{1\leqq j\leqq \bar{l}} \big(\partial^{\alpha^j}\tau\big) \prod_{\bar{l}+1\leqq j\leqq l} \big(\partial^{\alpha^j}w\big)\Big\|_{L^p} \leqq c\Big(\|\tau\|_{BMO} + \|w\|_{BMO}\Big)^{l-1} \times \Big(\|e^{\frac{\theta\tau}{p}}\partial^k\tau\|_{L^p} + \|e^{\frac{\theta\tau}{p}}\partial^k w\|_{L^p}\Big), \tag{4.24}$$

where we have naturally defined

$$\|e^{\frac{\theta\tau}{p}}\partial^k w\|_{L^p}^p = \sum_{1\leqq i\leqq n} \|e^{\frac{\theta\tau}{p}}\partial^k w_i\|_{L^p}^p = \sum_{\substack{|\alpha|=k\\ 1\leqq i\leqq n}} \frac{k!}{\alpha!}\int_{\mathbb{R}^n} e^{\theta\tau}\big|\partial^\alpha w_i\big|^p\,dx,$$

and where, in the left-hand member of (4.24), with a slight abuse of notation, we have denoted by w any of its components $w_1, \ldots, w_n$.

Proof. This is an application of Theorems 2 and 3 since for $\bar{\theta}\|\tau\|_{BMO} < b(n)/2$ and $\bar{\theta}\|\tau\|_{BMO} < b(n)(p-1)/2$ we have $[e^{\theta\tau}]_{A_p} \leqq \big(1 + B(n)\big)^p$.

We now estimate products of derivatives of temperature and velocity components rescaled by the proper temperature factors.

Theorem 7. *Let $k \geqq 1$ be an integer, $\bar{\theta} > 0$ be positive, $1 < p < \infty$, T be such that $T \geqq T_{\min} > 0$ and $T - T_\infty \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ for some positive T_∞. There exists positive constants $\delta(n, k, \bar{\theta}, p)$ and $c(n, k, p)$, only depending on $(n, k, \bar{\theta}, p)$ and (n, k, p), respectively, such that if $\|\log T\|_{BMO} < \delta$, then for any real θ such that $|\theta| \leqq \bar{\theta}$, any integer $l \geqq 1$, and any multiindices α^j, $1 \leqq j \leqq l$, with $|\alpha^j| \geqq 1$, $1 \leqq j \leqq l$, and $\sum_{1\leqq j\leqq l} |\alpha^j| = k$, whenever $T^{\theta/p}(\partial^k T)/T \in L^p(\mathbb{R}^n)$, we have the estimates*

$$\Big\| T^{\frac{\theta}{p}} \prod_{1\leqq j\leqq l} \Big(\frac{\partial^{\alpha^j}T}{T}\Big)\Big\|_{L^p} \leqq c\|\log T\|_{BMO}^{l-1} \, \|T^{\frac{\theta}{p}}\frac{\partial^k T}{T}\|_{L^p}. \tag{4.25}$$

Further assuming $v \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $\| \log T \|_{BMO} + \| v/\sqrt{T} \|_{L^\infty} < \delta(n, k, \bar{\theta}, p)$, *whenever* $T^{\theta/p}(\partial^k v)/\sqrt{T} \in L^p(\mathbb{R}^n)$, *we have for* $0 \leqq \bar{l} \leqq l$

$$\Big\| T^{\frac{\theta}{p}} \prod_{1 \leqq j \leqq \bar{l}} \Big(\frac{\partial^{\alpha^j} T}{T}\Big) \prod_{\bar{l}+1 \leqq j \leqq l} \Big(\frac{\partial^{\alpha^j} v}{\sqrt{T}}\Big) \Big\|_{L^p} \leqq c \Big(\| \log T \|_{BMO} + \Big\| \frac{v}{\sqrt{T}} \Big\|_{L^\infty} \Big)^{l-1}$$

$$\times \Big(\Big\| T^{\frac{\theta}{p}} \frac{\partial^k T}{T} \Big\|_{L^p} + \Big\| T^{\frac{\theta}{p}} \frac{\partial^k v}{\sqrt{T}} \Big\|_{L^p} \Big), \tag{4.26}$$

where, in the left hand member, with a slight abuse of notation, we have denoted by v *any of its components* $v_1, \ldots, v_n$.

Proof. Assume that $\delta < b(n)/2\bar{\theta}$ and $\delta < (p-1)b(n)/2\bar{\theta}$ so that all the weights $T^\theta = \exp(\theta \log T)$ satisfy the Muckenhoupt condition A_p as soon as $\| \log T \|_{BMO} < \delta$, and are such that $[T^\theta]_{A_p} \leqq \big(1 + B(n)\big)^p$. Let $l \geqq 1$ be an integer, and α^j, $1 \leqq j \leqq l$, be multiindices with $|\alpha^j| \geqq 1$, $1 \leqq j \leqq l$, and $\sum_{1 \leqq j \leqq l} |\alpha^j| = k$. From Theorem 6 applied with $\tau = \log T$ we have

$$\Big\| T^{\frac{\theta}{p}} \prod_{1 \leqq j \leqq l} \partial^{\alpha^j} \tau \Big\|_{L^p} \leqq c \| \log T \|_{BMO}^{l-1} \ \Big\| T^{\frac{\theta}{p}} \partial^k \tau \Big\|_{L^p},$$

where $c = c(n, k, p)$, and, thus, we only have to estimate integrals like

$$\Big\| T^{\frac{\theta}{p}} \left(\prod_{1 \leqq j \leqq l} \Big(\frac{\partial^{\alpha^j} T}{T}\Big) - \prod_{1 \leqq j \leqq l} \partial^{\alpha^j} \tau \right) \Big\|_{L^p}.$$

Thanks to the differential identities established in Lemma 2, we can write that

$$\prod_{1 \leqq j \leqq l} \Big(\frac{\partial^{\alpha^j} T}{T}\Big) = \prod_{1 \leqq j \leqq l} \partial^{\alpha^j} \tau + \sum_{\mu^1 \cdots \mu^l} \prod_{1 \leqq j \leqq l} c_{\mu^j} \prod_{1 \leqq |\beta| \leqq |\alpha^j|} (\partial^\beta \tau)^{\mu^j_\beta}, \tag{4.27}$$

where $\mu^j = (\mu^j_\beta)_{1 \leqq |\beta| \leqq |\alpha^j|}$, with $\mu^j_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and where c_{μ^j} are non-negative integer coefficients. The μ^j are also such that $\sum_{1 \leqq |\beta| \leqq |\alpha^j|} |\beta| \ \mu^j_\beta = |\alpha^j|$, so that we have in particular

$$\sum_{\substack{1 \leqq |\beta| \leqq |\alpha^j| \\ 1 \leqq j \leqq l}} |\beta| \ \mu^j_\beta = \sum_{1 \leqq j \leqq l} |\alpha^j| = k.$$

When $k = l$, all derivatives must be of first order so that the sum in the right-hand side of (4.27) is absent. On the other hand, when $l < k$, in each term of this sum, there are always at least $l + 1$ derivative factors in the product, since the only term with exactly l factors has been isolated, and at most k derivative factors. From the multilinear estimates of Theorem 3 applied to each of these terms we obtain

$$\begin{aligned}&\Big\| T^{\frac{\theta}{p}}\Big(\prod_{1\leqq j\leqq l}(\frac{\partial^{\alpha^j}T}{T}) - \prod_{1\leqq j\leqq l}\partial^{\alpha^j}\tau\Big)\Big\|_{L^p}\\ &\qquad\leqq c\left(\|\log T\|_{BMO}^{l}+\cdots+\|\log T\|_{BMO}^{k-1}\right)\|T^{\frac{\theta}{p}}\partial^k\tau\|_{L^p}.\end{aligned}$$

Therefore, assuming $\delta < 1$, we have established for any $1 \leqq l \leqq k$ that

$$\Big\| T^{\frac{\theta}{p}}\Big(\prod_{1\leqq j\leqq l}(\frac{\partial^{\alpha^j}T}{T}) - \prod_{1\leqq j\leqq l}\partial^{\alpha^j}\tau\Big)\Big\|_{L^p} \leqq c\,\|\log T\|_{BMO}^{l}\,\|T^{\frac{\theta}{p}}\partial^k\tau\|_{L^p}. \tag{4.28}$$

We now consider the special case $l = 1$ and we sum the above estimates (4.28) over all α with $|\alpha| = k$. This yields

$$\Big\| T^{\frac{\theta}{p}}\Big(\frac{\partial^k T}{T} - \partial^k\tau\Big)\Big\|_{L^p} \leqq c\|\log T\|_{BMO}\,\|T^{\frac{\theta}{p}}\partial^k\tau\|_{L^p},$$

where $c = c(n, k, p)$ so that for $c(n, k, p)\|\log T\|_{BMO} < 1/2$ we have

$$\tfrac{1}{2}\,\|T^{\frac{\theta}{p}}\partial^k\tau\|_{L^p} \ \leqq\ \|T^{\frac{\theta}{p}}\frac{\partial^k T}{T}\|_{L^p} \ \leqq\ \tfrac{3}{2}\,\|T^{\frac{\theta}{p}}\partial^k\tau\|_{L^p}. \tag{4.29}$$

Then reinserting inequality (4.29) in inequality (4.28) completes the proof of (4.25).

The proof of inequality (4.26) is similar, and it is found in particular that when $\|\log T\|_{BMO} + \|v/\sqrt{T}\|_{L^\infty} < \delta(n, k, \bar{\theta}, p)$, we have

$$\begin{aligned}&\Big\| T^{\frac{\theta}{p}}\Big(\frac{\partial^k v}{\sqrt{T}} - \partial^k w\Big)\Big\|_{L^p}\\ &\qquad\leqq\ c\left(\|\log T\|_{BMO} + \|\frac{v}{\sqrt{T}}\|_{L^\infty}\right)\left(\|T^{\frac{\theta}{p}}\partial^k\tau\|_{L^p} + \|T^{\frac{\theta}{p}}\partial^k w\|_{L^p}\right),\end{aligned} \tag{4.30}$$

where the terms proportional to w in relations (4.8) have been taken into account with the factors $\|v/\sqrt{T}\|_{L^\infty}$.

Remark 14. As a special case of Theorem 7, we obtain that for $T - T_\infty \in H^2(\mathbb{R}^n)\cap A(\mathbb{R}^n)$, $T \geqq T_{\min} > 0$ and $\|\log T\|_{BMO}$ small enough, we have

$$\int_{\mathbb{R}^n}\frac{|\partial_x T|^4}{T^{3+a}}\,dx \leqq c\|\log T\|_{BMO}^2\int_{\mathbb{R}^n}\frac{|\partial_x^2 T|^2}{T^{1+a}}\,dx. \tag{4.31}$$

This inequality differs from that of Remark 7 by the factors $\|\log T\|_{BMO}$.

Remark 15. The inequalities obtained in Theorems 6 and 7 will be used in Section 6 in order to establish positiveness of source terms in higher order entropy governing equations. As a typical example we use here inequality (4.31) in order to investigate the solutions of the scalar parabolic equation

$$\partial_t T - \partial_x \cdot (\lambda \partial_x T) = 0.$$

We assume that $T - T_\infty \in C\big([0,\bar{t}\,], H^l\big) \cap C^1\big([0,\bar{t}\,], H^{l-2}\big) \cap L^2\big([0,\bar{t}\,], H^{l+1}\big)$, where l is an integer such that $l \geqq [n/2]+2$, $\bar{t}$ is some positive time, $T_\infty > 0$ is a fixed positive temperature, and we assume that $\lambda = T^{\varkappa}$. After a few integrations by parts, it is easily established that

$$\partial_t \int_{\mathbb{R}^n} \frac{|\partial_x T|^2}{T^{1+a}}\, dx + 2\int_{\mathbb{R}^n} \frac{|\partial_x^2 T|^2}{T^{1+a-\varkappa}}\, dx - \big(4(1+a)-2\varkappa\big)\int_{\mathbb{R}^n} \frac{\partial_x^2 T : \partial_x T \otimes \partial_x T}{T^{2+a-\varkappa}}\, dx$$
$$+ (1+a)(2+a-2\varkappa)\int_{\mathbb{R}^n} \frac{|\partial_x T|^4}{T^{3+a-\varkappa}}\, dx = 0,$$

and we already know from Proposition 2 that, when $\varkappa = 0$, there exists T with $T - T_\infty \in \mathscr{D}(\mathbb{R}^n)$ such that the sum of the three last terms is negative, either assuming $n \geqq 1$ and $a > 1$, or assuming $n \geqq 2$ and $a > a^*$, where $a^* < 1$. Even more, the second counter example given in the proof of Proposition 2 can easily be extended to the situation where $\varkappa > 0$ and yields T with $T - T_\infty \in \mathscr{D}(\mathbb{R}^n)$ such that the sum of the three last terms is negative, for $n \geqq 1$ and $a > 1 - 2\varkappa$. In particular, starting from such an initial temperature field, the corresponding solution of the heat equation will be such that $\int_{\mathbb{R}^n} (|\partial_x T|^2 / T^{1+a})\, dx$ is increasing for some time interval.

On the other hand, we obtain from the Cauchy inequality that

$$\partial_t \int_{\mathbb{R}^n} \frac{|\partial_x T|^2}{T^{1+a}}\, dx + \int_{\mathbb{R}^n} \frac{|\partial_x^2 T|^2}{T^{1+a-\varkappa}}\, dx \leqq C(a,\varkappa)\int_{\mathbb{R}^n} \frac{|\partial_x T|^4}{T^{3+a-\varkappa}}\, dx,$$

so that from inequality (4.31), applied with a and replaced by $a-\varkappa$ we have

$$\partial_t \int_{\mathbb{R}^n} \frac{|\partial_x T|^2}{T^{1+a}}\, dx + (1 - c\|\log T\|_{BMO}^2)\int_{\mathbb{R}^n} \frac{|\partial_x^2 T|^2}{T^{1+a-\varkappa}}\, dx \leqq 0,$$

for some constant c, and this yields a priori estimates as long as $\|\log T\|_{BMO}$ is small enough.

5. Higher order entropies governing equations

We first discuss the temperature dependence of transport coefficients as obtained from the kinetic theory of gases. We then derive a governing equation for kinetic entropy correctors of arbitrary order in the situation of incompressible flows spanning the whole space. The case of compressible flows or zero Mach number flows are beyond the scope of the present paper [19].

5.1. Temperature dependent coefficients

Thermal conductivity and viscosity of a gas depend on temperature

$$\lambda = \lambda(T), \qquad \eta = \eta(T), \tag{5.1}$$

as shown by the kinetic theory [6,14,17]. When one term Sonine polynomial expansions are used to evaluate perturbed distribution functions, the coefficients λ/c_v and η are found in the form

$$\lambda/c_v = \frac{\mathfrak{a}_\lambda T^{1/2}}{\Omega^{(2,2)*}}, \qquad \eta = \frac{\mathfrak{a}_\eta T^{1/2}}{\Omega^{(2,2)*}},$$

where $\mathfrak{a}_\lambda$ and $\mathfrak{a}_\eta$ are constants and $\Omega^{(2,2)*}$ is a reduced collision integral, and the ratio $\lambda/c_v\eta$ is then a constant. For the rigid sphere model for instance, we have exactly $\lambda/c_v = \mathfrak{a}_\lambda T^{1/2}$ and $\eta = \mathfrak{a}_\eta T^{1/2}$. Similarly, for particles interacting as point centers of repulsion with an interaction potential $V = c/\mathrm{r}^\nu$, where r is the distance between two particles, one establishes that $\Omega^{(2,2)*}$ is proportional to $T^{-2/\nu}$ so that we have $\lambda/c_v = \mathfrak{a}_\lambda T^{\varkappa}$, and $\eta = \mathfrak{a}_\eta T^{\varkappa}$ with $\varkappa = 1/2 + 2/\nu$, [6,14]. The temperature exponent $\varkappa$ then varies from $\varkappa = 1/2$ for rigid spheres with $\nu = \infty$ up to $\varkappa = 1$ for Maxwell molecules with $\nu = 4$. More generally, consider particles interacting with a Lennard–Jones ν-ν' potential

$$V = 4\varepsilon\Big(\big(\frac{\sigma}{\mathrm{r}}\big)^{\nu} - \big(\frac{\sigma}{\mathrm{r}}\big)^{\nu'}\Big),$$

where V denotes the interaction potential, σ the collision diameter, ε the potential well depth, and ν, ν' are integers with $\nu > \nu'$ and typical values $\nu = 12$, $\nu' = 6$ [6,14]. Collision integrals like $\Omega^{(2,2)*}$ then only depend on the reduced temperature $k_{\mathrm{B}}T/\varepsilon$, and, when $k_{\mathrm{B}}T/\varepsilon$ is large, the repulsive part $\mathrm{r}^{-\nu}$ is dominant, whereas when $k_{\mathrm{B}}T/\varepsilon$ is small the attractive part $\mathrm{r}^{-\nu'}$ is dominant [6]. As a consequence, like for point centers of repulsion, collision integrals behave like $T^{s'}$ with $s' = 1/2 + 2/\nu'$ for small T and like T^s with $s = 1/2 + 2/\nu$ for large T [6]. In particular, the logarithm $\log \Omega^{(2,2)*}$ has linear asymptotes as functions of $\log T$, and $d^k \log \Omega^{(2,2)*}/d(\log T)^k$ is bounded for any $k \geqq 1$. As a consequence, $\log \eta$ and $\log \lambda$ have parallel linear asymptotes as functions of $\log T$, and $d^k \log \eta/d(\log T)^k$ and $d^k \log \lambda/d(\log T)^k$ are bounded for any $k \geqq 1$, or equivalently, $(1/\eta)T^k d^k\eta/dT^k$ and $(1/\lambda)T^k d^k\lambda/dT^k$ are bounded for any $k \geqq 1$.

Similar results are also obtained when more than one term is taken into account in orthogonal polynomial expansions of perturbed distribution functions. Indeed, all collision integrals $\Omega^{(i,j)*}$, i, $j \geqq 1$, have a common temperature behavior, that is, all ratios of collision integrals are bounded, as for instance for Lennard–Jones or Stockmayer potentials [14,17]. These collision integrals are then used to define the coefficients of the transport linear systems which thus share a common temperature scaling. As a consequence, the transport coefficients, which are obtained through solutions of transport linear systems, inherit a common temperature scaling [17]. The same conclusion is also reached with polyatomic molecules when Wang–Chang–Uhlenbeck–Sonine polynomial expansions are used [17]. As a consequence, the relevant mathematical assumptions are that all transport coefficients have a common temperature scaling in such a way that $\lambda/c_v\eta$ remains positive and bounded, and $d^k \log\eta/d(\log T)^k$ and $d^k \log\lambda/d(\log T)^k$ are bounded for any $k \geqq 1$.

On the other hand, in our particular application, using the maximum principle for temperature yields a uniform lower bound for T, only depending on initial data. Therefore, we may assume that $T \geqq T_{\min}$, where $T_{\min}$ is fixed and positive. In this situation, the behavior of transport coefficients for small temperatures is not relevant. In other words, only the repulsive part of the interaction potential between particles plays a role and we may assume that such behavior is asymptotically that of point centers of repulsion as we have discussed for Lennard–Jones potentials. Therefore, from a mathematical point of view, since we are not interested in small temperatures, we may simplify the assumptions about the temperature dependence of transport coefficients and assume that λ and η are $C^\infty(0,\infty)$, that there exist $\varkappa$, $\underline{\mathfrak{a}} > 0$, and $\overline{\mathfrak{a}} > 0$ with

$$\underline{\mathfrak{a}}\, T^{\varkappa} \leqq \lambda/c_v \leqq \overline{\mathfrak{a}}\, T^{\varkappa}, \qquad \underline{\mathfrak{a}}\, T^{\varkappa} \leqq \eta \leqq \overline{\mathfrak{a}}\, T^{\varkappa}, \tag{5.2}$$

and that, for any integer $\sigma \geqq 1$, there exists $\overline{\mathfrak{a}}_\sigma > 0$ with

$$T^\sigma\, |\partial_T^\sigma \lambda| \leqq \overline{\mathfrak{a}}_\sigma\, T^{\varkappa}, \qquad T^\sigma\, |\partial_T^\sigma \eta| \leqq \overline{\mathfrak{a}}_\sigma\, T^{\varkappa}. \tag{5.3}$$

Kinetic theory suggests that $1/2 \leqq \varkappa \leqq 1$ but the situations where $0 \leqq \varkappa < 1/2$ or $\varkappa > 1$ are still interesting to investigate from a mathematical point of view.

Remark 16. Assumptions on transport coefficients valid for all temperatures may be written

$$\underline{c}\,\zeta \leqq \lambda/c_v \leqq \overline{c}\,\zeta, \qquad \underline{c}\,\zeta \leqq \eta \leqq \overline{c}\,\zeta,$$

$$T^\sigma\, |\partial_T^\sigma \lambda| \leqq \overline{c}_\sigma\, \zeta, \qquad T^\sigma\, |\partial_T^\sigma \eta| \leqq \overline{c}_\sigma\, \zeta, \qquad \sigma \geqq 1,$$

where $\underline{c}$, $\overline{c}$, and $\overline{c}_\sigma$, $\sigma \geqq 1$, are positive constants. The function ζ is a smooth function of T such that $T^\sigma\, |\partial_T^\sigma \zeta| \leqq \overline{c}_\sigma\, \zeta$, $\sigma \geqq 1$. For Lennard–Jones ν-ν' potentials, we can take for instance $\zeta = T^s$ for large T and $\zeta = T^{s'}$ for small T, with $s = 1/2 + 2/\nu$ and $s' = 1/2 + 2/\nu'$ [6]. We have made in this paper the simpler choice $\zeta = T^s = T^{\varkappa}$ since we can exclude small temperatures. It is interesting to note that with an interaction potential which is infinite at small interparticle distances, we always have $\nu = \infty$ so that $\zeta = T^{1/2}$ for large temperatures.

5.2. Fluid governing equations

With variable transport coefficients, the fluid governing equations can be written

$$\partial_x\!\cdot v = 0, \tag{5.4}$$

$$\partial_t(\rho v) + \partial_x\!\cdot(\rho v\otimes v + pI) - \partial_x\!\cdot\big(\eta(T)\, d\big) = 0, \tag{5.5}$$

$$\partial_t(\rho e) + \partial_x\!\cdot(\rho e v) - \partial_x\!\cdot\big(\lambda(T)\,\partial_x T\big) = \tfrac{1}{2}\eta(T) d{:}d, \tag{5.6}$$

where ρ is the constant density, v the velocity, p the pressure, $d = \partial_x v + \partial_x v^t$ the strain rate tensor, $\eta(T)$ the viscosity, e the internal energy per unit mass, and $\lambda(T)$

the thermal conductivity. The energy per unit mass e is still taken for simplicity in the form $e = c_v T$, where c_v is a constant.

We again consider the case of functions defined on $\mathbb{R}^n$ with $n \geqq 2$, that are 'constant at infinity', and we only consider solutions such that

$$v, T - T_\infty \in C\big([0, \bar{t}\,], H^l\big) \cap C^1\big([0, \bar{t}\,], H^{l-2}\big) \cap L^2\big([0, \bar{t}\,], H^{l+1}\big), \qquad (5.7)$$

where l is an integer such that $l \geqq [n/2] + 2$, that is, $l > n/2 + 1$, $\bar{t}$ is some positive time, and $T_\infty > 0$ is a fixed positive temperature. We also assume that T is positive and bounded away from zero $T \geqq T_{\min}$ where $T_{\min}$ is positive. It will be shown in Section 7 that these solutions are smooth when the initial data is smooth, whenever they exist. Since the viscosity η is no longer a constant, the momentum conservation equation is rewritten in the form $\partial_t(\rho v) = \mathbb{P}\big(\partial_x \cdot (-\rho v \otimes v + \eta(T) d)\big)$, which is equivalent to defining the pressure by

$$p = \sum_{1 \leqq i,j \leqq n} R_i R_j \big(\rho v_i v_j - \eta d_{ij}\big). \qquad (5.8)$$

We also have $p \in C\big([0, \bar{t}\,], H^l\big) \cap C^1\big([0, \bar{t}\,], H^{l-2}\big) \cap L^2\big([0, \bar{t}\,], H^{l+1}\big)$ from (5.5) and from the identity $\partial_k p = \sum_{1 \leqq i,j \leqq n} R_k R_j \big(\rho v_i \partial_i v_j - 2\partial_T \eta \partial_i T \partial_j v_i\big)$.

Remark 17. In the special case where $\lambda = \mathfrak{a}_\lambda T^{\varkappa}$, $\eta = \mathfrak{a}_\eta T^{\varkappa}$, and c_v is constant, if $v(t, x)$ and $T(t, x)$ are a solution of the Navier–Stokes equations (5.4)– (5.6), then

$$\xi\, v(\xi^{2(1-\varkappa)} t, \xi^{(1-2\varkappa)} x), \qquad \xi^2\, T(\xi^{2(1-\varkappa)} t, \xi^{(1-2\varkappa)} x), \qquad (5.9)$$

are also a solution for any positive ξ. The special situation $\varkappa = 0$ corresponds to the usual rescaled solutions [7,31]. Note that space and time are not stretched in the same direction when $1/2 < \varkappa < 1$.

Remark 18. All the results obtained in this section and the following are also valid if the internal energy e per unit mass is taken to be $e = e_0 + \int_0^T c_v(s)\, ds$ with a heat capacity coefficient c_v depending on temperature in such a way that

$$\underline{c} \leqq c_v \leqq \overline{c}, \qquad T^\sigma\, |\partial_T^\sigma c_v| \leqq \overline{c}_\sigma, \qquad \sigma \geqq 1,$$

where $\underline{c} > 0$, $\overline{c} > 0$, and $\overline{c}_\sigma > 0$, $\sigma \geqq 1$, are positive constants. For the sake of simplicity we will not explicate the corresponding results.

5.3. Higher order kinetic entropy estimators

Specializing formally expression (2.15) to the situation of an incompressible gas, we define the $(2k)^{\text{th}}$ order kinetic entropy corrector $\gamma^{[k]}$ by

$$\gamma^{[k]} = \mathrm{A}_\lambda^{[k]} \frac{|\partial^k T|^2}{T^{1+a_k}} + \mathrm{A}_\eta^{[k]} \frac{|\partial^k v|^2}{T^{a_k}}, \qquad (5.10)$$

with

$$|\partial^k T|^2 = \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\partial^\alpha T)^2, \qquad |\partial^k v|^2 = \sum_{1 \leqq i \leqq n} |\partial^k v_i|^2,$$

where $k!/\alpha!$ are the multinomial coefficients [10,41], and $\mathrm{A}_\lambda^{[k]} > 0$, $\mathrm{A}_\eta^{[k]} > 0$, $a_k \in \mathbb{R}$, are parameters at our disposal. We do not assume anymore that a_k is positive since some negative values will naturally appear in the discussion. Similarly, following (2.16), we also define the $(2k)^{\text{th}}$ order kinetic entropy corrector $\tilde{\gamma}^{[k]}$ by

$$\tilde{\gamma}^{[k]} = \exp\big((1-a_k)\tau\big)\Big(\mathrm{A}_\lambda^{[k]}|\partial^k \tau|^2 + \mathrm{A}_\eta^{[k]}|\partial^k w|^2\Big), \tag{5.11}$$

where $\tau = \log T$ and $w = v/\sqrt{T}$. The entropy correctors $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ will be shown to have similar properties and both may be used to derive a priori estimates.

In order to recast the zeroth order entropy balance into a more convenient form, we also define $\gamma^{[0]} = \tilde{\gamma}^{[0]}$, for $0 < a_0 \leqq 1$, by

$$\gamma^{[0]} = \tilde{\gamma}^{[0]} = (\mathrm{A}_\lambda^{[0]} + \mathrm{A}_\eta^{[0]})\zeta^{[0]}, \tag{5.12}$$

where $\mathrm{A}_\lambda^{[0]} > 0$, $\mathrm{A}_\eta^{[0]} > 0$, and

$$\zeta^{[0]} = \begin{cases} \dfrac{T - T_\infty}{T_\infty} - \log\left(\dfrac{T}{T_\infty}\right) + \frac{1}{2}\dfrac{v^2}{c_v T_\infty}, & \text{if } a_0 = 1, \\[2ex] \dfrac{T - T_\infty}{T_\infty^{a_0}} - \dfrac{T^{1-a_0} - T_\infty^{1-a_0}}{1 - a_0} + \frac{1}{2}\dfrac{v^2}{c_v T_\infty^{a_0}}, & \text{if } 0 < a_0 < 1. \end{cases}$$

Finally, we introduce the $(2k)^{\text{th}}$ order kinetic entropy estimators defined by

$$\Gamma^{[k]} = \gamma^{[0]} + \cdots + \gamma^{[k]}, \qquad k \geqq 0, \tag{5.13}$$

$$\widetilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[k]}, \qquad k \geqq 0, \tag{5.14}$$

which will play an important role. Strictly speaking, we should term $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ "mathematical $(2k)^{\text{th}}$ order partial entropies" or "$(2k)^{\text{th}}$ order kinetic entropy correctors" or "$(2k)^{\text{th}}$ order kinetic entropy deviation estimators" and $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ "mathematical $(2k)^{\text{th}}$ order entropies", or "$(2k)^{\text{th}}$ order kinetic entropy estimators". We have also seen in Section 2 that all these quantities can also be associated with Fisher information. However, we will often informally term $\gamma^{[k]}$, $\tilde{\gamma}^{[k]}$, $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ "mathematical $(2k)^{\text{th}}$ order entropies" or simply "higher order entropies". Our aim is now to establish balance equations for $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$. In Section 6, we will use these equations to derive a priori estimates and to establish that $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ satisfy conditional entropic principles.

Remark 19. Replacing $\partial_x v$ by d in the definition of $\gamma^{[k]}$ would yield

$$\widehat{\gamma}^{[k]} = \mathrm{A}_\lambda^{[k]}\frac{|\partial^k T|^2}{T^{1+a_k}} + \frac{1}{2}\mathrm{A}_\eta^{[k]}\frac{|\partial^{k-1} d|^2}{T^{a_k}}, \tag{5.15}$$

which coincides for $k = 1$ with the quantity γ introduced in Section 3. However, the definitions (5.10) and (5.15) are equivalent for $k \geqq 2$ from (3.29) and yield similar results for $k = 1$ from the expression (3.28) of $\partial_x v$ in terms of d and the continuity of $T^\theta R_i T^{-\theta}$ for $\|\log T\|_{BMO}$ small enough.

5.4. Balance equation for $\gamma^{[k]}$

We investigate the $\gamma^{[k]}$ balance equation for incompressible fluids with temperature dependent transport coefficients.

Proposition 5. *Let $k \geqq 1$ be an integer and (v, T) be a smooth solution of the incompressible Navier–Stokes equations (5.4)– (5.6). Then the following balance equation holds*

$$\partial_t \gamma^{[k]} + \partial_x \cdot (v\gamma^{[k]}) + \partial_x \cdot \varphi_\gamma^{[k]} + \pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]} = 0, \tag{5.16}$$

where $\varphi_\gamma^{[k]}$ is a flux and $\pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]}$ a source term. The quantity $\pi_\gamma^{[k]}$ contains higher order derivative non-negative terms, $\Sigma_\gamma^{[k]}$ higher order derivative split terms, and $\omega_\gamma^{[k]}$ lower order derivative terms of convective origin. The term $\pi_\gamma^{[k]}$ can be taken as

$$\pi_\gamma^{[k]} = \frac{2\lambda \mathrm{A}_\lambda^{[k]}}{\rho c_v} \frac{|\partial^{k+1} T|^2}{T^{1+a_k}} + \frac{2\eta \mathrm{A}_\eta^{[k]}}{\rho} \frac{|\partial^{k+1} v|^2}{T^{a_k}}, \tag{5.17}$$

in such a way that

$$2\underline{b}_k\, \gamma^{[k+1]} \leqq \pi_\gamma^{[k]}\, T^{-(a_{k+1}-a_k+\varkappa)} \leqq 2\overline{b}_k\, \gamma^{[k+1]}, \tag{5.18}$$

$\rho \underline{b}_k = \underline{\mathfrak{a}} \min(\mathrm{A}_\lambda^{[k]}/\mathrm{A}_\lambda^{[k+1]}, \mathrm{A}_\eta^{[k]}/\mathrm{A}_\eta^{[k+1]})$, $\rho \overline{b}_k = \overline{\mathfrak{a}} \max(\mathrm{A}_\lambda^{[k]}/\mathrm{A}_\lambda^{[k+1]}, \mathrm{A}_\eta^{[k]}/\mathrm{A}_\eta^{[k+1]})$. *The term $\Sigma_\gamma^{[k]}$ is in the form*

$$\begin{aligned}\Sigma_\gamma^{[k]} = &\sum_{\sigma\nu\mu} T^{\sigma-\varkappa}(c_{\sigma\nu\mu}\partial_T^\sigma \lambda + c'_{\sigma\nu\mu}\partial_T^\sigma \eta)\; \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)} \\ &+ \sum_{\sigma\nu\mu\mathscr{R}} c_{\sigma\nu\mu\mathscr{R}} \Pi_\nu^{(k+1)} \mathscr{R}\big(T^{\sigma-\varkappa}\partial_T^\sigma \eta\; \Pi_\mu^{(k+1)}\big),\end{aligned} \tag{5.19}$$

where the sums are over $0 \leqq \sigma \leqq k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1\leqq|\alpha|\leqq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\leqq|\alpha|\leqq k+1}$, $\nu_\alpha, \nu'_\alpha, \mu_\alpha, \mu'_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and for $\mathscr{R}$ singular operator in the form $T^{-\theta} R_i R_j T^\theta$ with $\theta = (a_k + \varkappa)/2$ and $1 \leqq i, j \leqq n$. The products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by

$$\Pi_\nu^{(k+1)} = T^{(1-a_k+\varkappa)/2} \prod_{1\leqq|\alpha|\leqq k+1} \Big(\frac{\partial^\alpha T}{T}\Big)^{\nu_\alpha} \prod_{1\leqq|\alpha|\leqq k+1} \Big(\frac{\partial^\alpha v}{\sqrt{T}}\Big)^{\nu'_\alpha}, \tag{5.20}$$

where v denotes (with a slight abuse of notation) any of its components $v_1, \dots, v_n$, and μ and ν must be such that

$$\sum_{1\leqq|\alpha|\leqq k+1} |\alpha|(\nu_\alpha + \nu'_\alpha) = k+1, \qquad \sum_{1\leqq|\alpha|\leqq k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1,$$

$$\sum_{|\alpha|=k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) \leqq 1,$$

so that there is at most one derivative of order $(k+1)$ *in the product* $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$. *In particular, one of the terms* $\Pi_\nu^{(k+1)}$ *or* $\Pi_\mu^{(k+1)}$ *is always split between two or more derivative factors. Furthermore the term* $\omega_\gamma^{[k]}$ *is given by*

$$\omega_\gamma^{[k]}\, T^{-(1-2\varkappa+a_{k-1}-a_k)/2} = \sum_{\nu\mu} c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + \sum_{\nu\mu\mathscr{R}} c_{\nu\mu\mathscr{R}}\Pi_\nu^{(k)}\mathscr{R}\big(\Pi_\mu^{(k+1)}\big), \tag{5.21}$$

where we use similar notation for $\Pi_\nu^{(k)}$ *as for* $\Pi_\mu^{(k+1)}$ *and the summation extends over*

$$\sum_{1\leqq|\alpha|\leqq k} |\alpha|(\nu_\alpha + \nu'_\alpha) = k, \qquad \sum_{1\leqq|\alpha|\leqq k} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1,$$

so that in particular $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha) = 0$ *and there are always at least two factors in the product* $\Pi_\mu^{(k+1)}$, *and where the singular operator* $\mathscr{R}$ *is in the form* $T^{-\theta}R_iR_jT^\theta$ *with* $\theta = (1+a_k-\varkappa)/2$ *and* $1 \leqq i, j \leqq n$. *Finally the flux* $\varphi_\gamma^{[k]} = (\varphi_{\gamma 1}^{[k]}, \ldots, \varphi_{\gamma n}^{[k]})$ *is given by the following formula with* $\mathscr{R}$ *taken as in* (5.19)

$$\begin{aligned}\varphi_{\gamma l}^{[k]}\, T^{-(a_{k-1}-a_k)/2} = &\sum_{\sigma\nu\mu} T^{\sigma-\varkappa}(c_{\sigma\nu\mu l}\partial_T^\sigma\lambda + c'_{\sigma\nu\mu l}\partial_T^\sigma\eta)\ \Pi_\nu^{(k)}\Pi_\mu^{(k+1)}\\ &+ \sum_{\sigma\nu\mu\mathscr{R}} c_{\sigma\nu\mu\mathscr{R}l}\Pi_\nu^{(k)}\mathscr{R}\big(T^{\sigma-\varkappa}\partial_T^\sigma\eta\ \Pi_\mu^{(k+1)}\big).\end{aligned} \tag{5.22}$$

Proof. The proof (given in Appendix A) is lengthy and tedious but presents no serious difficulties other than notational.

5.5. Balance equation for $\tilde{\gamma}^{[k]}$

We will need in the following the $\tilde{\gamma}^{[k]}$ balance equation that we correspondingly write in terms of the auxiliary variables w and τ.

Proposition 6. *Let* $k \geqq 1$ *be an integer and* (v, T) *be a smooth solution of the incompressible Navier–Stokes equations* (5.4)– (5.6). *Then the following balance equation holds*

$$\partial_t\tilde{\gamma}^{[k]} + \partial_x\cdot(v\tilde{\gamma}^{[k]}) + \partial_x\cdot\varphi_{\tilde{\gamma}}^{[k]} + \pi_{\tilde{\gamma}}^{[k]} + \Sigma_{\tilde{\gamma}}^{[k]} + \omega_{\tilde{\gamma}}^{[k]} = 0, \tag{5.23}$$

where $\varphi_{\tilde{\gamma}}^{[k]}$ *is a flux and* $\pi_{\tilde{\gamma}}^{[k]} + \Sigma_{\tilde{\gamma}}^{[k]} + \omega_{\tilde{\gamma}}^{[k]}$ *a source term. The term* $\pi_{\tilde{\gamma}}^{[k]}$ *can be taken as*

$$\pi_{\tilde{\gamma}}^{[k]} = e^{(1-a_k)\tau}\Big(\frac{2\lambda \mathrm{A}_\lambda^{[k]}}{\rho c_v}\,|\partial^{k+1}\tau|^2 + \frac{2\eta \mathrm{A}_\eta^{[k]}}{\rho}|\partial^{k+1}w|^2\Big), \tag{5.24}$$

in such a way that

$$2\underline{b}_k\,\tilde{\gamma}^{[k+1]} \leqq \pi_{\tilde{\gamma}}^{[k]}\, e^{-(a_{k+1}-a_k+\varkappa)\tau} \leqq 2\overline{b}_k\,\tilde{\gamma}^{[k+1]}, \tag{5.25}$$

where $\underline{b}_k$ *and* $\overline{b}_k$ *are as in Proposition 5. The term* $\Sigma_{\tilde{\gamma}}^{[k]}$ *is in the form*

$$\begin{aligned}\Sigma_{\tilde{\gamma}}^{[k]} =& \sum_{\sigma\nu\mu} e^{-\varkappa\tau}\,(c_{\sigma\nu\mu}\partial_\tau^\sigma\lambda + c'_{\sigma\nu\mu}\partial_\tau^\sigma\eta)\;\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}\\ &+ \sum_{\sigma\nu\mu\iota\mathscr{R}} c_{\sigma\nu\mu\iota\mathscr{R}}\Pi_\nu^{(k+1)}\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)} + \frac{\mathrm{A}_\eta^{[k]}}{\rho}\Big(\frac{\lambda}{c_v} - \eta\Big)e^{(1-a_k)\tau}\,w\cdot\partial^{k+1}w\,\partial^{k+1}\tau,\end{aligned} \tag{5.26}$$

where we have isolated the new terms which contain two derivatives of order $(k+1)$

$$w\cdot\partial^{k+1}w\,\partial^{k+1}\tau = \sum_{\substack{|\alpha|=k+1\\ 1\leqq i\leqq n}} \frac{(k+1)!}{\alpha!}\,w_i\,\partial^\alpha w_i\,\partial^\alpha\tau.$$

For the two first contributions in (5.26) composed of strictly differential terms, the sums are over $\nu = (\nu_\alpha, \nu'_\alpha)_{0\leqq|\alpha|\leqq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{0\leqq|\alpha|\leqq k+1}$, $0 \leqq \sigma \leqq k$, *and* $\nu_\alpha, \nu'_\alpha, \mu_\alpha, \mu'_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, *with the products* $\Pi_\nu^{(k+1)}$ *and* $\Pi_\mu^{(k+1)}$ *defined by*

$$\Pi_\nu^{(k+1)} = e^{(1-a_k+\varkappa)\tau/2}\prod_{1\leqq|\alpha|\leqq k+1}\big(\partial^\alpha\tau\big)^{\nu_\alpha}\prod_{0\leqq|\alpha|\leqq k+1}\big(\partial^\alpha w\big)^{\nu'_\alpha}, \tag{5.27}$$

where w *denotes (with an abuse of notation) any of its components* $w_1, \dots, w_n$. *As for* $\Sigma_\gamma^{[k]}$, μ *and* ν *are such that* $\sum_{1\leqq|\alpha|\leqq k+1}|\alpha|(\nu_\alpha+\nu'_\alpha) = \sum_{1\leqq|\alpha|\leqq k+1}|\alpha|(\mu_\alpha + \mu'_\alpha) = k+1$, $\sum_{|\alpha|=k+1}(\nu_\alpha+\nu'_\alpha+\mu_\alpha+\mu'_\alpha) \leqq 1$, *so that there is at most one derivative of* $(k+1)^{\text{th}}$ *order in the product* $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$. *In particular, one of the terms* $\Pi_\nu^{(k+1)}$ *or* $\Pi_\mu^{(k+1)}$ *is always split between two or more derivative factors. Note also that the products over the velocity factors extend up to* $|\alpha| = 0$ *in contrast with the* $\gamma^{[k]}$ *balance equation. The non strictly differential terms* $\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)}$ *are defined by*

$$\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)} = \widetilde{\Pi}_\mu^{(k+1,l)}\mathscr{R}\Big(e^{-\varkappa\tau}\partial_\tau^\sigma\eta\;\widetilde{\Pi}_\iota^{(k+1,k+1-l)}\Big) \tag{5.28}$$

with

$$\begin{cases}\widetilde{\Pi}_\mu^{(k+1,l)} = e^{(1-a_k+\varkappa)\frac{l\tau}{2(k+1)}}\displaystyle\prod_{1\leqq|\alpha|\leqq k+1}\big(\partial^\alpha\tau\big)^{\mu_\alpha}\prod_{0\leqq|\alpha|\leqq k+1}\big(\partial^\alpha w\big)^{\mu'_\alpha},\\[2ex] \widetilde{\Pi}_\iota^{(k+1,k+1-l)} = e^{(1-a_k+\varkappa)\frac{(k+1-l)\tau}{2(k+1)}}\displaystyle\prod_{1\leqq|\alpha|\leqq k+1}\big(\partial^\alpha\tau\big)^{\iota_\alpha}\prod_{0\leqq|\alpha|\leqq k+1}\big(\partial^\alpha w\big)^{\iota'_\alpha}.\end{cases} \tag{5.29}$$

The sums are over $0 \leqq \sigma \leqq k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1\leqq|\alpha|\leqq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\leqq|\alpha|\leqq k}$, *and* $\iota = (\iota_\alpha, \iota_\alpha)_{1\leqq|\alpha|\leqq k}$, *where* ν, μ, *and* ι *must be such that*

$$\sum_{1\leqq|\alpha|\leqq k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = l, \qquad \sum_{1\leqq|\alpha|\leqq k+1} |\alpha|(\iota_\alpha + \iota'_\alpha) = k+1-l,$$

$$\sum_{|\alpha|=k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha + \iota_\alpha + \iota'_\alpha) \leqq 1,$$

for some $0 \leqq l \leqq k$, *so that there is at most one derivative of* $(k+1)^{\text{th}}$ *order in the product* $\Pi_\nu^{(k+1)} \Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)}$, *and* $\mathscr{R}$ *singular operator in the form* $e^{-\theta\tau} R_i R_j e^{\theta\tau}$, $1 \leqq i, j \leqq n$, *with*

$$\theta = \frac{a_k + \varkappa}{2} + \frac{l}{2(k+1)}(1 - a_k + \varkappa).$$

Note that, in contrast with the $\gamma^{[k]}$ *balance equation, the integral operator* $\mathscr{R}$ *breaks the term* $\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)}$ *into two pieces. Furthermore the term* $\omega_{\tilde{\gamma}}^{[k]}$ *is given by*

$$\omega_{\tilde{\gamma}}^{[k]}\, e^{-(1-2\varkappa+a_{k-1}-a_k)\tau/2} = \sum_{\nu\mu} c_{\nu\mu} \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} + \sum_{\nu\mu\iota\mathscr{R}} c_{\nu\mu\iota\mathscr{R}} \Pi_\nu^{(k)} \Pi_{\mu\iota\mathscr{R}}^{(k+1)}, \tag{5.30}$$

where we use similar notation for $\Pi_\nu^{(k)}$ *as for* $\Pi_\mu^{(k+1)}$ *and where* $\Pi_{\mu l\mathscr{R}}^{(k+1)}$ *is given by*

$$\Pi_{\mu\iota\mathscr{R}}^{(k+1)} = \widetilde{\Pi}_\mu^{(k+1,l)} \mathscr{R}\Big(\widetilde{\Pi}_\iota^{(k+1,k+1-l)}\Big). \tag{5.31}$$

The summations are over $\sum_{1\leqq|\alpha|\leqq k} |\alpha|(\nu_\alpha + \nu'_\alpha) = k$, $\sum_{1\leqq|\alpha|\leqq k} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1$, *and* $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha) = 0$ *for the strictly differential terms* $\Pi_\nu^{(k)} \Pi_\mu^{(k+1)}$, *so that there are always at least two derivative factors in the product* $\Pi_\mu^{(k+1)}$. *For the non strictly differential terms* $\Pi_\nu^{(k)} \Pi_{\mu\iota\mathscr{R}}^{(k+1)}$ *we have* $\sum_{1\leqq|\alpha|\leqq k} |\alpha|(\nu_\alpha + \nu'_\alpha) = k$, $\sum_{1\leqq|\alpha|\leqq k} |\alpha|(\mu_\alpha + \mu'_\alpha) = l$, $\sum_{1\leqq|\alpha|\leqq k} |\alpha|(\iota_\alpha + \iota'_\alpha) = k+1-l$, *and* $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha + \iota_\alpha + \iota'_\alpha) = 0$ *for some* $0 \leqq l \leqq k$, *so that there are always at least two derivative factors in the product* $\Pi_{\mu l\mathscr{R}}^{(k+1)}$, *and the singular operator* $\mathscr{R}$ *is in the form* $e^{-\theta\tau} R_i R_j T^{\theta\tau}$ *with* $1 \leqq i, j \leqq n$ *and*

$$\theta = 1 - \frac{k+1-l}{2(k+1)}(1 - a_k + \varkappa).$$

Finally the flux $\varphi_{\tilde{\gamma}}^{[k]} = (\varphi_{\tilde{\gamma},1}^{[k]}, \ldots, \varphi_{\tilde{\gamma},n}^{[k]})$ *is given by the following formula with* $\mathscr{R}$ *taken as in* (5.27)

$$\varphi_{\tilde{\gamma},l}^{[k]}\, e^{-(a_{k-1}-a_k)\tau/2} = \sum_{\sigma\nu\mu} e^{-\varkappa\tau} (c_{\sigma\nu\mu l} \partial_\tau^\sigma \lambda + c'_{\sigma\nu\mu l} \partial_\tau^\sigma \eta)\, \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu\iota\mathscr{R}} c_{\sigma\nu\mu\iota\mathscr{R}l} \Pi_\nu^{(k)} \Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)}. \tag{5.32}$$

Proof. The proof is lengthy and tedious but similar to that of Proposition 5. The new complications arise from commutators between temperature weights and differential operators.

6. Higher order entropy estimates

In this section (the core of the paper) we investigate higher order entropy estimates for incompressible flows spanning the whole space in the natural situation of temperature dependent viscosity and thermal conductivity. We establish conditional entropic inequalities for higher order kinetic entropy estimators, that is, entropic inequalities that hold whenever $\| \log T \|_{BMO} + \|v/\sqrt{T}\|_{L^\infty}$ is small enough.

6.1. Positiveness of higher order derivative source terms

We first investigate the control of $\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}|\, dx$ by $\int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx$, using the weighted inequalities established in Section 4. We will denote by χ the quantity

$$\chi = \| \log T \|_{BMO} + \| \frac{v}{\sqrt{T}} \|_{L^\infty} = \|\tau\|_{BMO} + \|w\|_{L^\infty}, \tag{6.1}$$

which will play a fundamental role in the analysis. We will establish in particular that entropic inequalities hold for $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ when χ is small enough. Note that this quantity invariant under the change of scales (5.9) described in Remark is 17. It can also be interpreted as involving the natural variables $\log T$ and $v/\sqrt{rT}$ appearing in Maxwellian distributions [5]. Since we have formally $v/\sqrt{rT} = \mathcal{O}(\mathrm{Ma})$ and $\log(T/T_\infty) = \mathcal{O}(\mathrm{Ma})$, the constraint that χ remains small may also be interpreted as a small Mach number constraint, which is consistent with the Enskog expansion [22].

Proposition 7. *Let $k \geqq 1$ be an integer and (v, T) be a smooth solution of the incompressible Navier–Stokes equations* (5.4)– (5.6)*. There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi = \| \log T \|_{BMO} + \|v/\sqrt{T}\|_{L^\infty} < \delta$, we have the estimates*

$$\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}|\, dx \leqq c\, \chi \int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx. \tag{6.2}$$

Proof. From (5.19), (5.20), since $T^{\sigma-\varkappa}\partial_T^\sigma \lambda$ and $T^{\sigma-\varkappa}\partial_T^\sigma \eta$ are uniformly bounded from assumptions (5.2) and (5.3), and since the operators $T^\theta R_i R_j T^{-\theta}$ are continuous over L^2 for $\| \log T \|_{BMO}$ small enough, we only have to estimate the L^2 norm of the products $\Pi_v^{(k+1)}$ and $\Pi_\mu^{(k+1)}$. However, using Theorem 7 with $p = 2$, we obtain when $\| \log T \|_{BMO} + \|v/\sqrt{T}\|_{L^\infty} < \delta(k, n)$, the weighted inequalities

$$\|\Pi_v^{(k+1)}\|_{L^2} \leqq c\chi^{N_v-1} \Big(\|T^{\frac{\theta}{2}} \frac{\partial^{k+1} T}{T}\|_{L^2} + \|T^{\frac{\theta}{2}} \frac{\partial^{k+1} v}{\sqrt{T}}\|_{L^2} \Big),$$

with $\theta = 1 - a_k + \varkappa$, $c = c(k, n)$, and

$$N_\nu = \sum_{1 \leqq |\alpha| \leqq k+1} (\nu_\alpha + \nu'_\alpha).$$

As a consequence, we have

$$\| \Pi_\nu^{(k+1)} \|_{L^2} \| \Pi_\mu^{(k+1)} \|_{L^2} \leqq c \chi^{N_\nu + N_\mu - 2} \int_{\mathbb{R}^n} \pi_\gamma^{[k]} \, dx,$$

and the proof is complete from

$$N_\nu + N_\mu - 2 = \sum_{1 \leqq |\alpha| \leqq k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) - 2 \geqq 1,$$

since at least one of the products $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is split into two or more derivative factors.

We have a similar result for $\tilde{\gamma}^{[k]}$ which is more technical to establish because of the special structure of the products $\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)}$ in (5.28).

Proposition 8. *Let $k \geqq 1$ be an integer and (v, T) be a smooth solution of the incompressible Navier–Stokes equations (5.4)– (5.6). There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi = \|\tau\|_{BMO} + \|w\|_{L^\infty} < \delta$ we have the estimates*

$$\int_{\mathbb{R}^n} |\Sigma_{\tilde{\gamma}}^{[k]}| \, dx \leqq c \, \chi \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]} \, dx. \tag{6.3}$$

Proof. We use the expression (5.26)– (5.27) in order to estimate $\Sigma_{\tilde{\gamma}}^{[k]}$. On one hand, for strictly differential terms, the proof is similar to that for $\gamma^{[k]}$. Indeed, the terms $e^{-\varkappa\tau} \partial_\tau^\sigma \lambda \, \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)}$ or $e^{-\varkappa\tau} \partial_\tau^\sigma \eta \, \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)}$ are easily majorized since the quantities $e^{-\varkappa\tau} \partial_\tau^\sigma \lambda$ and $e^{-\varkappa\tau} \partial_\tau^\sigma \eta$ are uniformly bounded and since the L^2 norm of the products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ is directly obtained from the multilinear estimates of Theorem 6. The fact that there is always a factor χ in the upper bound (6.3) results from the fact that one of the two products $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is always split. On the other hand, the special contributions involving two derivatives of $(k+1)^{\text{th}}$ order are rewritten in the form

$$e^{-\varkappa\tau} \Big(\frac{\lambda}{c_v} - \eta \Big) e^{(1 - a_k + \varkappa)\tau} w \cdot \partial^{k+1} w \, \partial^{k+1} \tau,$$

and are easily taken into account with a $\|w\|_{L^\infty}$ factor.

The new difficulty is to evaluate the L^2 norm of $\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)}$. These terms $\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)}$ are in the form

$$\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)} = \widetilde{\Pi}_\mu^{(k+1,l)} \mathscr{R}\Big(e^{-\varkappa\tau} \partial_\tau^\sigma \eta \, \widetilde{\Pi}_\iota^{(k+1,k+1-l)} \Big) \tag{6.4}$$

where $\mathscr{R} = e^{-\theta\tau} R_i R_j e^{\theta\tau}$ and θ is given in Proposition 6, and with

$$\widetilde{\Pi}_\mu^{(k+1,l)} = e^{(1-a_k+\varkappa)\frac{l\tau}{2(k+1)}} \prod_{1\leqq|\alpha|\leqq k+1} \left(\partial^\alpha \tau\right)^{\mu_\alpha} \prod_{0\leqq|\alpha|\leqq k+1} \left(\partial^\alpha w\right)^{\mu'_\alpha}. \tag{6.5}$$

The sums are over $0 \leqq \sigma \leqq k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1\leqq|\alpha|\leqq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\leqq|\alpha|\leqq k+1}$, $\iota = (\iota_\alpha, \iota'_\alpha)_{1\leqq|\alpha|\leqq k+1}$, $0 \leqq l \leqq k$, $\mathscr{R}$ singular operator as described in Proposition 6 and ν, μ and ι must be such that

$$\sum_{1\leqq|\alpha|\leqq k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = l, \qquad \sum_{1\leqq|\alpha|\leqq k+1} |\alpha|(\iota_\alpha + \iota'_\alpha) = k+1-l,$$

$$\sum_{|\alpha|=k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha + \iota_\alpha + \iota'_\alpha) \leqq 1,$$

so that there is at most one derivative of order $(k+1)$ in the product $\Pi_\nu^{(k+1)} \Pi_{\mu l \mathscr{R}\sigma}^{(k+1)}$.

We first split the exponential term in $\widetilde{\Pi}_\mu^{(k+1,l)}$ over each derivative factor thanks to the relation $\sum_{1\leqq|\alpha|\leqq k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = l$ and we obtain

$$\widetilde{\Pi}_\mu^{(k+1,l)} = \prod_{1\leqq|\alpha|\leqq k+1} e^{\frac{(1-a_k+\varkappa)\tau|\alpha|\mu_\alpha}{2(k+1)}} \left(\partial^\alpha \tau\right)^{\mu_\alpha} \prod_{0\leqq|\alpha|\leqq k+1} e^{\frac{(1-a_k+\varkappa)\tau|\alpha|\mu'_\alpha\tau}{2(k+1)}} \left(\partial^\alpha w\right)^{\mu'_\alpha}. \tag{6.6}$$

Letting $p_\alpha = 2(k+1)/\mu_\alpha|\alpha|$, and $p'_\alpha = 2(k+1)/\mu'_\alpha|\alpha|$, we have

$$\sum_{1\leqq|\alpha|\leqq k+1} \left(\frac{1}{p_\alpha} + \frac{1}{p'_\alpha}\right) = \frac{l}{2(k+1)},$$

and we can use the Hölder inequality to estimate $\left\| \widetilde{\Pi}_\mu^{(k+1,l)} \right\|_{L^{\frac{2(k+1)}{l}}}$. To this purpose, from the weighted interpolation inequalities of intermediate derivatives established in Theorem 5 applied with $r = 2$, $j = |\alpha|$, and k replaced by $k+1$, we obtain

$$\begin{aligned} &\left\| e^{\frac{(1-a_k+\varkappa)\tau|\alpha|\mu_\alpha}{2(k+1)}} \left(\partial^\alpha \phi\right)^{\mu_\alpha} \right\|_{L^{p_\alpha}} \\ &\qquad \leqq c\chi^{\mu_\alpha(1-\frac{|\alpha|}{k+1})} \left(\|e^{\frac{\theta\tau}{2}} \partial^{k+1} \tau\|_{L^2} + \|e^{\frac{\theta\tau}{2}} \partial^{k+1} w\|_{L^2} \right)^{\frac{\mu_\alpha|\alpha|}{k+1}}, \end{aligned}$$

where ϕ denotes τ or w. Upon multiplying these inequalities for $1 \leqq |\alpha| \leqq k+1$, and from the Hölder inequality, we deduce that

$$\left\| \widetilde{\Pi}_\mu^{(k+1,l)} \right\|_{L^{\frac{2(k+1)}{l}}} \leqq c\chi^{N_\mu - \frac{l}{k+1}} \left(\|e^{\frac{\theta\tau}{2}} \partial^{k+1} \tau\|_{L^2} + \|e^{\frac{\theta\tau}{2}} \partial^{k+1} w\|_{L^2} \right)^{\frac{l}{k+1}},$$

where $\theta = 1 - a_k + \varkappa$, $c = c(k,n)$, and

$$N_\mu = \sum_{1\leqq|\alpha|\leqq k+1} (\mu_\alpha + \mu'_\alpha).$$

We can treat similarly the factor $\widetilde{\Pi}_{\iota}^{(k+1,k+1-l)}$ to obtain that

$$\|\widetilde{\Pi}_{\iota}^{(k+1,k+1-l)}\|_{L^{\frac{2(k+1)}{k+1-l}}} \leqq c\chi^{N_{\iota}-\frac{k+1-l}{k+1}}\Big(\|e^{\frac{\theta\tau}{2}}\partial^{k+1}\tau\|_{L^2}+\|e^{\frac{\theta\tau}{2}}\partial^{k+1}w\|_{L^2}\Big)^{\frac{k+1-l}{k+1}}.$$

Since the operator $\mathscr{R}$ is continuous in $L^{\frac{2(k+1)}{k+1-l}}$ for $\|\log T\|_{BMO}$ small enough, we deduce that

$$\|\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)}\|_{L^2} \leqq c\|\widetilde{\Pi}_{\mu}^{(k+1,l)}\|_{L^{\frac{2(k+1)}{l}}}\|\widetilde{\Pi}_{\iota}^{(k+1,k+1-l)}\|_{L^{\frac{2(k+1)}{k+1-l}}}$$

so that

$$\|\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)}\|_{L^2} \leqq c\chi^{N_{\mu}+N_{\iota}-1}\Big(\|e^{\frac{\theta\tau}{2}}\partial^{k+1}\tau\|_{L^2}+\|e^{\frac{\theta\tau}{2}}\partial^{k+1}w\|_{L^2}\Big),$$

and finally

$$\|\Pi_{\nu}^{(k+1)}\|_{L^2}\|\Pi_{\mu\iota\mathscr{R}\sigma}^{(k+1)}\|_{L^2} \leqq c\chi^{N_{\nu}+N_{\mu}+N_{\iota}-2}\int_{\mathbb{R}^n}\pi_{\tilde{\gamma}}^{[k]}\,dx,$$

and the end of the proof is similar to that of Proposition 7.

Corollary 2. *Let $k \geqq 1$ be an integer and (v,T) be a smooth solution of the incompressible Navier–Stokes equations (5.4)– (5.6). There exists positive constants $\delta(k,n)$ and $c(k,n)$, only depending on (k,n), such that for $\chi < \delta$ the following inequalities hold*

$$\partial_t\int_{\mathbb{R}^n}\gamma^{[k]}\,dx+(1-c\,\chi)\int_{\mathbb{R}^n}\pi_{\gamma}^{[k]}\,dx \leqq \int_{\mathbb{R}^n}|\omega_{\gamma}^{[k]}|\,dx, \tag{6.7}$$

$$\partial_t\int_{\mathbb{R}^n}\tilde{\gamma}^{[k]}\,dx+(1-c\,\chi)\int_{\mathbb{R}^n}\pi_{\tilde{\gamma}}^{[k]}\,dx \leqq \int_{\mathbb{R}^n}|\omega_{\tilde{\gamma}}^{[k]}|\,dx. \tag{6.8}$$

In particular, when $\chi \leqq 1/2c(k,n)$, we have made a first step towards entropic inequalities for $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$.

We have established in Corollary 2 that inequalities (6.7) and (6.8) hold as long as the quantity χ is small enough. In order to obtain global estimates, we will have to ensure that this quantity χ remains small if it is initially small. The inequality (6.2) also implies that $(1-c\,\chi)\int_{\mathbb{R}^n}\pi_{\gamma}^{[k]}\,dx \leqq \int_{\mathbb{R}^n}(\pi_{\gamma}^{[k]}+\Sigma_{\gamma}^{[k]})\,dx \leqq (1+c\,\chi)\int_{\mathbb{R}^n}\pi_{\gamma}^{[k]}\,dx$ and (6.3) implies $(1-c\,\chi)\int_{\mathbb{R}^n}\pi_{\tilde{\gamma}}^{[k]}\,dx \leqq \int_{\mathbb{R}^n}(\pi_{\tilde{\gamma}}^{[k]}+\Sigma_{\tilde{\gamma}}^{[k]})\,dx \leqq (1+c\,\chi)\int_{\mathbb{R}^n}\pi_{\tilde{\gamma}}^{[k]}\,dx$. These inequalities, as well as (3.11), could be informally termed 'entropicity inequalities' since they induce the first steps (6.7) (6.8) and (3.12) towards entropic inequalities.

Remark 20. For the heat equation (2.1), the quantity $\zeta^{[k]}=|\partial^k u|^2$ can also be considered as a $(2k)^{\text{th}}$ order entropy corrector. The corresponding balance equations can then be written

$$\partial_t\zeta^{[k]}-\Delta\zeta^{[k]}+2|\partial_x^{k+1}u|^2=0.$$

In contrast with the Navier–Stokes system, we observe that for the heat equation, unconditional positiveness of the source terms hold in the $\zeta^{[k]}$ balance equation.

6.2. Estimates of convective terms and of $\gamma^{[0]} = \tilde{\gamma}^{[0]}$

We estimate the lower order convective terms $\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx$ by the higher order dissipative terms $\int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx$ and $\int_{\mathbb{R}^n} \pi_\gamma^{[k-1]}\, dx$.

Proposition 9. *Let $k \geqq 1$ be an integer and (v, T) be a smooth solution of the incompressible Navier–Stokes equations (5.4)– (5.6). There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi < \delta$ we have the estimates*

$$\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx \leqq c\chi \sup_{\mathbb{R}^n}\{T^{(1-2\varkappa+a_{k-1}-a_k)/2}\} \Big(\int_{\mathbb{R}^n} \pi_\gamma^{[k-1]}\, dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx\Big)^{\frac{1}{2}}, \tag{6.9}$$

$$\int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k]}|\, dx \leqq c\chi \sup_{\mathbb{R}^n}\{e^{(1-2\varkappa+a_{k-1}-a_k)\tau/2}\} \Big(\int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k-1]}\, dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]}\, dx\Big)^{\frac{1}{2}}. \tag{6.10}$$

Proof. From the expression (5.21) and the continuity of the operators $T^\theta R_i R_j T^{-\theta}$ for $\|\log T\|_{BMO}$ small enough, we deduce that

$$\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx \leqq c \sup_{\mathbb{R}^n}\{T^{(1-2\varkappa+a_{k-1}-a_k)/2}\}\ \|\Pi_\nu^{(k)}\|_{L^2}\ \|\Pi_\mu^{(k+1)}\|_{L^2}, \tag{6.11}$$

and the estimate (6.9) is a direct consequence of the inequalities established in Section 4 and in the proof of Proposition 7, since there are at least two factors in the product $\Pi_\mu^{(k+1)}$. The proof of (6.10) is similar *mutatis mutandis* since the terms $\Pi_{\mu l \mathscr{R}}^{(k+1)}$ can be estimated by using the inequalities of Proposition 8.

We now recast the classical zeroth order entropic estimate in a convenient form that will be needed to investigate entropic principles associated with $\Gamma^{[k]}$.

Proposition 10. *Let $0 < a_0 \leqq 1$ and let $\gamma^{[0]}$ be given by (5.12). Then $\gamma^{[0]} \geqq 0$ and there exist positive constants $\delta_0 > 0$ and $\underline{b}_0'$ such that for $\chi < \delta_0$ small enough*

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + \underline{b}_0' \int_{\mathbb{R}^n} \pi_\gamma^{[0]}\, dx \leqq 0, \tag{6.12}$$

where we define from (5.17)

$$\pi_\gamma^{[0]} = \frac{2\lambda \mathrm{A}_\lambda^{[0]}}{\rho c_v} \frac{|\partial^1 T|^2}{T^{1+a_0}} + \frac{2\eta \mathrm{A}_\eta^{[0]}}{\rho} \frac{|\partial^1 v|^2}{T^{a_0}}.$$

Equivalently, there exists a positive constant $\underline{b}_0$ such that for $\chi < \delta_0$, we have

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + 2\underline{b}_0 \int_{\mathbb{R}^n} T^{\varkappa+a_1-a_0} \gamma^{[1]}\, dx \leqq 0. \tag{6.13}$$

Proof. We only consider the case $0 < a_0 < 1$ since the case $a_0 = 1$ is similar. It is first easily established that the temperature part of $\gamma^{[0]}$ is non-negative so that $\gamma^{[0]} \geqq 0$. Dividing the temperature equation by T^{a_0} and integrating over $\mathbb{R}^n$ we obtain after some algebra

$$-\partial_t \int_{\mathbb{R}^n} \frac{T^{1-a_0} - T_\infty^{1-a_0}}{1-a_0}\, dx + \frac{a_0}{\rho c_v} \int_{\mathbb{R}^n} \frac{\lambda |\partial_x T|^2}{T^{1+a_0}}\, dx + \frac{1}{2\rho c_v} \int_{\mathbb{R}^n} \frac{\eta |d|^2}{T^{a_0}}\, dx = 0.$$

On the other hand, dividing the total energy conservation equation by $T_\infty^{a_0}$ and integrating over $\mathbb{R}^n$ we obtain

$$\partial_t \int_{\mathbb{R}^n} \Big(\frac{T - T_\infty}{T_\infty^{a_0}} + \tfrac{1}{2} \frac{v^2}{c_v T_\infty^{a_0}} \Big)\, dx = 0.$$

Finally, from the relations (3.28), we obtain the inequality

$$\int_{\mathbb{R}^n} \frac{|\partial_x v|^2}{T^{a_0-\varkappa}}\, dx \leqq c \int_{\mathbb{R}^n} \frac{|d|^2}{T^{a_0-\varkappa}}\, dx,$$

for $\| \log T \|_{BMO}$ small enough and combining these estimates completes the proof.

We also recast the classical zeroth order entropic estimate in a convenient form that will be needed to investigate entropic principles associated with $\widetilde{\Gamma}^{[k]}$.

Proposition 11. *Let* $0 < a_0 \leqq 1$ *and let* $\tilde{\gamma}^{[0]}$ *by given be* (5.12). *Then* $\tilde{\gamma}^{[0]} \geqq 0$ *and there exist positive constants* $\delta_0 > 0$, $\underline{b}_0'$, *and* c *such that for* $\chi < \delta_0$

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[0]}\, dx + (\underline{b}_0' - c\chi) \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]}\, dx \leqq 0, \tag{6.14}$$

where we define from (5.24)

$$\pi_{\tilde{\gamma}}^{[0]} = e^{(1-a_0)\tau} \Big(\frac{2\lambda \mathrm{A}_\lambda^{[0]}}{\rho c_v} |\partial^1 \tau|^2 + \frac{2\eta \mathrm{A}_\eta^{[0]}}{\rho} |\partial^1 w|^2 \Big).$$

Equivalently, there exist positive constants $\delta_0 > 0$, $\underline{b}_0$, *and* c *such that for* $\chi < \delta_0$

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[0]}\, dx + (2\underline{b}_0 - c\chi) \int_{\mathbb{R}^n} e^{(\varkappa + a_1 - a_0)\tau} \tilde{\gamma}^{[1]}\, dx \leqq 0. \tag{6.15}$$

Proof. This is a direct consequence of Proposition 10 and of the differential relations

$$\frac{\partial_i v}{\sqrt{T}} = \partial_i w + \tfrac{1}{2} w \partial_i \tau,$$

which yield that $\int_{\mathbb{R}^n} \pi_\gamma^{[0]}\, dx$ is minorized by $(1 - c\chi) \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]}\, dx$.

6.3. Natural scale of temperature weights

The estimates established in the previous sections are valid for any positive $\mathrm{A}_\lambda^{[k]}$ and $\mathrm{A}_\eta^{[k]}$, $k \geqq 0$, and we now set for simplicity

$$\mathrm{A}_\lambda^{[k]} = 1, \qquad \mathrm{A}_\eta^{[k]} = \frac{1}{r}, \qquad k \geqq 0. \tag{6.16}$$

With this simple choice, we note that the constants $\underline{b}_k = \mathfrak{a}/\rho$ in Proposition 5 and Proposition 6 are independent of $k \geqq 1$, and we correspondingly denote $\underline{b} = \min(\underline{b}_0, \mathfrak{a}/\rho)$.

In order to combine the estimates of Corollary 2 and Propositions 9, 10 and 11, obtained for various values of $k \geqq 0$, we now need to specify the scale of temperature weights a_k, $k \geqq 0$, used to renormalize the successive derivatives of T and v. In this section, we impose that the $\sup_{\mathbb{R}^n} T^{1-2\varkappa+a_{k-1}-a_k}$ factors appearing in the convective term estimates of Proposition 9 disappear, by letting $1-2\varkappa+a_{k-1}-a_k = 0$, $k \geqq 1$, in such a way that

$$a_k = a_0 + k(1-2\varkappa), \qquad k \geqq 0. \tag{6.17}$$

This scale fulfills the natural requirement that estimates for $\pi_\gamma^{[k-1]}$ and inequality (6.7) for $\gamma^{[k]}$ yield estimates for $\pi_\gamma^{[k]}$. This scale of temperature weights also corresponds to the scale given by the kinetic theory of gases with (2.15), (2.16) since the factor $(\eta/\rho\sqrt{rT})^{2k} = (\eta^2/\rho^2 rT)^k$ yields the temperature exponent $k(1-2\varkappa)$ from assumptions (5.2), (5.3). Therefore, this scale $a_k = a_0 + k(1-2\varkappa)$, $k \geqq 0$, will be termed the natural scale of temperature weights, and this scale also arises in the compressible case [19]. It is interesting to note that with this scale, a_k is decreasing with k for physical values of $\varkappa$, that is, for values such that $\varkappa \geqq 1/2$. On the other hand, a_k is increasing with k for unphysical values of $\varkappa$, that is, for values such that $0 \leqq \varkappa < 1/2$. This means in particular that, in the unphysical situation $0 \leqq \varkappa < 1/2$, larger powers of T are needed in order to renormalize higher derivatives.

As a direct consequence of the preceding sections, we obtain the following estimates concerning higher order entropies. A similar proposition can also be established for $\tilde{\gamma}^{[k]}$ but the details are omitted.

Proposition 12. *Let (v, T) be a smooth solution of the incompressible Navier–Stokes equations (5.4)– (5.6). Assume that $a_l = a_0 + l(1-2\varkappa)$, $l \geqq 0$, and let $k \geqq 1$ be fixed. There exist positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi < \delta$ we have the estimates*

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \big(2\underline{b} - c\chi\big) \int_{\mathbb{R}^n} T^{1-\varkappa} \gamma^{[k+1]}\, dx \leqq c\chi \int_{\mathbb{R}^n} T^{1-\varkappa} \gamma^{[k]}\, dx. \tag{6.18}$$

Proof. These estimates are direct consequences of Corollary 2 and Proposition 9 since $\pi_\gamma^{[k]} \geqq 2\underline{b} T^{1-\varkappa} \gamma^{[k+1]}$ for $\underline{b} = \min(\underline{b}_0, \mathfrak{a}/\rho)$.

After some algebra, it is easily checked that inequality (6.18) can also be written with factors χ replaced by χ^2. We can now combine the inequalities obtained for $k = 1, \dots, l$ in Proposition 12 together with the inequality obtained for $k = 0$ in Proposition 10, in order to estimate the $(2k)^{\text{th}}$ order kinetic entropy estimator $\Gamma^{[k]} = \gamma^{[0]} + \cdots + \gamma^{[k]}$.

Theorem 8. *Let (v, T) be a smooth solution of the incompressible Navier–Stokes equations (5.4)– (5.6). Assume that $a_l = a_0 + l(1 - 2\varkappa)$, $l \geqq 0$, and let $k \in \mathbb{N}$ be fixed. There exist positive constants $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$ and $\delta_{\mathrm{N}}(k, n)$ such that for $\chi < \delta_{\mathrm{N}}$ we have*

$$\partial_t \int_{\mathbb{R}^n} \Big(\gamma^{[0]} + \gamma^{[1]} + \cdots + \gamma^{[k]}\Big)\, dx + \underline{b} \int_{\mathbb{R}^n} T^{1-\varkappa} \Big(\gamma^{[1]} + \gamma^{[2]} + \cdots + \gamma^{[k+1]}\Big)\, dx \leqq 0. \tag{6.19}$$

Proof. This results upon summing the estimates of Propositions 12 and 10.

This theorem shows that the $(2k)^{\text{th}}$ order kinetic entropy estimator $\Gamma^{[k]}$ obeys an entropic principle and similar results also hold for $\widetilde{\Gamma}^{[k]}$.

Remark 21. Note that, from a practical point of view, only the situation where $\varkappa \geqq 1/2$ seems interesting since the sequence a_k, $k \geqq 0$, is then decreasing so that the weights $1/T^{a_k - 1}$ are minorized by a common weight $1/T^{a_0 - 1}$ and the derivatives can be estimated with this common weight. Assuming that $\varkappa \geqq 1/2$, we indeed have inequalities such as

$$c\Big(\gamma^{[0]} + \sum_{1 \leqq k \leqq l} \Big(\frac{|\partial^k T|^2}{T^2} + \frac{|\partial^k v|^2}{T}\Big) \frac{1}{T^{a_0 - 1}}\Big) \leqq \Gamma^{[l]}, \tag{6.20}$$

where c depends on $T_{\min}$ and the resulting estimates are similar to the estimates obtained in the next section with uniform scales of temperature weights.

6.4. Uniform scale of temperature weights

In this section we still use the simple values $\mathrm{A}_\lambda^{[k]} = 1$ and $\mathrm{A}_\eta^{[k]} = 1/r$, for $k \geqq 0$. On the other hand, in contrast with Section 6.3, we impose that the temperature weights are all equal

$$a_k = a_0, \qquad k \geqq 0. \tag{6.21}$$

This scale of temperature weights will be termed the uniform scale. It is important to note that if $1 - 2\varkappa + a_{k-1} - a_k > 0$, the $\sup_{\mathbb{R}^n} T^{1-2\varkappa + a_{k-1} - a_k}$ factors of the right members of (6.9) and (6.10) in Proposition 9 cannot be majorized in terms of the solution derivatives since $T_\infty > 0$. As a consequence, taking into account the natural lower bound for temperature in terms of initial data $T \geqq T_{\min} > 0$, controlling these $\sup_{\mathbb{R}^n} T^{1-2\varkappa + a_{k-1} - a_k}$ factors requires the negative exponents in (6.9) and (6.10). Therefore, we must have $1 - 2\varkappa + a_{k-1} - a_k \leqq 0$, $k \geqq 1$, and thus $a_0 + k(1 - 2\varkappa) \leqq a_k$, for $k \geqq 0$, and the natural scale of temperature weights appears to be a lower bound among all the useful scales. In particular, selecting a uniform scale requires that $k(1-2\varkappa) \leqq$ Cte, so that we must have $\varkappa \geqq 1/2$. In other words, the transport coefficients have to follow the temperature dependence indicated by the kinetic theory in order to use a uniform scale. With this scale, higher order entropy estimates directly yield estimates for higher order derivatives of $\log T$ and $v/\sqrt{T}$. This scale will be used in Section 7 in order to investigate asymptotic stability of equilibrium states for incompressible flows.

Proposition 13. *Let (v, T) be a smooth solution of the incompressible Navier–Stokes equations (5.4)– (5.6). Assume that $\varkappa \geqq 1/2$, that $a_l = a_0$, $l \geqq 0$, let $k \geqq 1$ be fixed, and assume that $T_{\min} \leqq T$. There exist positive constants $\delta(k, n, T_{\min})$ and $c(k, n, T_{\min})$ such that for $\chi < \delta$, we have the estimates*

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \big(2\underline{b} - c\chi\big) \int_{\mathbb{R}^n} T^{\varkappa} \gamma^{[k+1]}\, dx \leqq c\chi \int_{\mathbb{R}^n} T^{\varkappa} \gamma^{[k]}\, dx, \tag{6.22}$$

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[k]}\, dx + \big(2\underline{b} - c\chi\big) \int_{\mathbb{R}^n} T^{\varkappa} \tilde{\gamma}^{[k+1]}\, dx \leqq c\chi \int_{\mathbb{R}^n} T^{\varkappa} \tilde{\gamma}^{[k]}\, dx. \tag{6.23}$$

Proof. The proof is similar to that of Proposition 12 and the $T_{\min}$ dependence arises from the negative powers of the $\sup_{\mathbb{R}^n} T$ factors.

After some algebra, it is easily checked that inequalities (6.22) and (6.23) can also be written with factors χ replaced by χ^2.

Theorem 9. *Let (v, T) be a smooth solution of the incompressible Navier–Stokes equations (5.4)– (5.6). Assume that $\varkappa \geqq 1/2$, that $a_l = a_0$, $l \geqq 0$, let $k \in \mathbb{N}$ be fixed, and assume that $T_{\min} \leqq T$. There exist positive constants $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$ and $\delta_{\mathrm{U}}(k, n, T_{\min})$ such that for $\chi < \delta_{\mathrm{U}}$ we have the estimates*

$$\partial_t \int_{\mathbb{R}^n} \Big(\gamma^{[0]} + \gamma^{[1]} + \cdots + \gamma^{[k]}\Big)\, dx + \underline{b} \int_{\mathbb{R}^n} T^{\varkappa} \Big(\gamma^{[1]} + \gamma^{[2]} + \cdots + \gamma^{[k+1]}\Big)\, dx \leqq 0. \tag{6.24}$$

$$\partial_t \int_{\mathbb{R}^n} \Big(\tilde{\gamma}^{[0]} + \tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[k]}\Big)\, dx + \underline{b} \int_{\mathbb{R}^n} T^{\varkappa} \Big(\tilde{\gamma}^{[1]} + \tilde{\gamma}^{[2]} + \cdots + \tilde{\gamma}^{[k+1]}\Big)\, dx \leqq 0. \tag{6.25}$$

Proof. This is a direct consequence of Propositions 10, 11 and 13.

Theorem 9 shows that the $(2k)^{\text{th}}$ order kinetic entropy estimators $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ obey entropic principles. These estimates will be used in the next section in the situation of logarithmic scaling $a_k = 1$, $k \geqq 0$. Note that the estimates obtained with a uniform scale are similar to the estimates obtained with the natural scale combined with inequalities such as (6.20).

7. Global solutions

We present in this section an example of the application of higher order entropy estimates. We first establish a local existence theorem for incompressible flows spanning the whole space with temperature dependent transport coefficients and also establish that these solutions are smooth depending on initial data. We next combine the local existence theorem with higher order entropy estimates in order to obtain global existence and asymptotic stability when $\log(T_0/T_\infty)$ and $v_0/\sqrt{T_0}$ are small in appropriate spaces. We assume throughout this section that the scale of temperature weights is uniform with $a_k = 1$, $k \geqq 0$, and that the transport coefficients λ and η satisfy assumptions (5.2), (5.3) with $\varkappa \geqq 1/2$. We only investigate strong solutions and (to the author's knowledge) assumptions (5.2), (5.3) were not previously used.

7.1. Local existence

We denote by v the combined unknown $\mathrm{v} = (v, T)$, keeping in mind that the momentum conservation equation is considered as projected on the space of divergence free functions. We denote accordingly by v_∞ the equilibrium point $\mathrm{v}_\infty = (0, T_\infty)$ with $v_\infty = 0$ and $T_\infty > 0$. We denote by $\mathscr{O}_\mathrm{v} = \mathbb{R}^n \times (0, \infty)$ the natural domain for the variable v, where $n \geqq 2$.

Theorem 10. *Let $n \geqq 2$ and $l \geqq [n/2]+2$ be integers and let $b > 0$ be given. Let $\mathscr{O}_0$ be an open bounded convex set such that $\overline{\mathscr{O}}_0 \subset \mathscr{O}_\mathrm{v}$, d_1 with $0 < d_1 < d(\overline{\mathscr{O}}_0, \partial\mathscr{O}_\mathrm{v})$, and define $\mathscr{O}_1 = \{\mathrm{v} \in \mathscr{O}_\mathrm{v};\ d(\mathrm{v}, \overline{\mathscr{O}}_0) < d_1\}$. There exists $\bar{t} > 0$ small enough, which only depend on $\mathscr{O}_0$, d_1, and b, such that for any v_0 with $\|\mathrm{v}_0 - \mathrm{v}_\infty\|_{H^l} < b$ and $\mathrm{v}_0 \in \overline{\mathscr{O}}_0$, there exists a unique local solution $\mathrm{v} = (v, T)$ to the system (5.4)–(5.6) with the initial condition*

$$\mathrm{v}(0, x) = \mathrm{v}_0(x), \tag{7.1}$$

such that

$$\mathrm{v}(t, x) \in \mathscr{O}_1, \tag{7.2}$$

and

$$\mathrm{v} - \mathrm{v}_\infty \in C^0\big([0, \bar{t}\,], H^l(\mathbb{R}^n)\big) \cap C^1\big([0, \bar{t}\,], H^{l-2}(\mathbb{R}^n)\big) \cap L^2\big((0, \bar{t}\,), H^{l+1}(\mathbb{R}^n)\big). \tag{7.3}$$

In addition, denoting for short $\mathrm{v}(t) = \mathrm{v}(t, \cdot)$, there exists $C > 0$ which only depends on $\mathscr{O}_0$, d_1, and b, such that

$$\sup_{0 \leqq s \leqq \bar{t}} \|\mathrm{v}(s) - \mathrm{v}_\infty\|_{H^l}^2 + \int_0^{\bar{t}} \|\mathrm{v}(s) - \mathrm{v}_\infty\|_{H^{l+1}}^2\, ds \leqq C \|\mathrm{v}_0 - \mathrm{v}_\infty\|_{H^l}^2. \tag{7.4}$$

Proof. The following proof is adapted from Kawashima to the situation of incompressible flows [17,20,28]. Solutions to the nonlinear system (5.4)– (5.6) are fixed points $\widetilde{\mathrm{v}} = \mathrm{v}$ of the linear system of equations in $\widetilde{\mathrm{v}} = (\widetilde{v}, \widetilde{T})$

$$\begin{cases} \partial_t(\rho\widetilde{v}) - \mathbb{P}\big(\partial_x\cdot\big(\eta(T)\,\partial_x\widetilde{v}\big)\big) = \mathbb{P}\big(f_v(\mathrm{v}, \partial_x\mathrm{v})\big), \\ \partial_t(\rho c_v\widetilde{T}) - \partial_x\cdot\big(\lambda(T)\,\partial_x\widetilde{T}\big) = f_T(\mathrm{v}, \partial_x\mathrm{v}), \end{cases} \tag{7.5}$$

with $f_v = -\partial_x \cdot (\rho v \otimes v) + \partial_T \eta(T) \partial_x T \cdot \partial_x v^t$ and $f_T = -\partial_x \cdot (\rho c_v T v) + \frac{1}{2}\eta(T) d{:}d$. Fixed points $\widetilde{\mathrm{v}} = \mathrm{v}$ are investigated in the function space $\mathsf{X}_{\bar{t}}(\mathscr{O}_1, M, M_1)$, that is defined by $\mathrm{v} \in \mathsf{X}_{\bar{t}}(\mathscr{O}_1, M, M_1)$ if $\mathrm{v}(t,x) \in \mathscr{O}_1$,

$$\mathrm{v} - \mathrm{v}_\infty \in C^0\big([0,\bar{t}\,], H^l(\mathbb{R}^n)\big) \cap L^2\big((0,\bar{t}\,), H^{l+1}(\mathbb{R}^n)\big),$$

$$\partial_t \mathrm{v} \in C^0\big([0,\bar{t}\,], H^{l-2}(\mathbb{R}^n)\big) \cap L^2\big((0,\bar{t}\,), H^{l-1}(\mathbb{R}^n)\big),$$

$$\sup_{0 \leqq s \leqq \bar{t}} \|\mathrm{v}(s) - \mathrm{v}_\infty\|_{H^l}^2 + \int_0^{\bar{t}} \|\mathrm{v}(s) - \mathrm{v}_\infty\|_{H^{l+1}}^2\, ds \leqq M^2,$$

and

$$\int_0^{\bar{t}} \|\partial_t \mathrm{v}(s)\|_{H^{l-1}}^2\, ds \leqq M_1^2.$$

For v in $\mathsf{X}_{\bar{t}}(\mathscr{O}_1, M, M_1)$, $1 \leqq k \leqq l$, and $f = (f_v, f_T)$, we have the estimates

$$\|\widetilde{\mathrm{v}}(t) - \mathrm{v}_\infty\|_{H^k}^2 + \int_0^t \|\widetilde{\mathrm{v}}(s) - \mathrm{v}_\infty\|_{H^{k+1}}^2\, ds \leqq C_1^2 \exp\big(C_2(t + M_1\sqrt{t}\,)\big) \times \Big(\|\mathrm{v}_0 - \mathrm{v}_\infty\|_{H^k}^2 + C_2 \int_0^t \|f(s)\|_{H^{k-1}}^2\, ds\Big), \tag{7.6}$$

where $C_1 = C_1(\mathscr{O}_1)$ depends on $\mathscr{O}_1$ and $C_2 = C_2(\mathscr{O}_1, M)$ depends on $\mathscr{O}_1$ and M, and is an increasing function of M. These a priori estimates for solutions of the linear equations (7.5) are obtained by deriving the governing equations, multiplying by the derivative of the solution, and using the properties of the Leray projector $\mathbb{P}$ [28,32]. On the other hand, existence of such solutions $\widetilde{\mathrm{v}}$ to the linear equations are obtained from a priori estimates by standard arguments like Galerkin approximations.

Furthermore, using the classical estimates

$$\|\psi(\phi) - \psi(0)\|_{H^k} \leqq C_0 \|\psi\|_{C^k(\overline{\mathscr{O}}_\phi)} (1 + \|\phi\|_{L^\infty})^{k-1} \|\phi\|_{H^k}, \tag{7.7}$$

where $\mathscr{O}_\phi$ is a convex open set with $\phi(x) \in \mathscr{O}_\phi$, $x \in \mathbb{R}^n$, and increasing eventually the constant $C_2(\mathscr{O}_1, M)$ of (7.6), we obtain for $f = (f_v, f_T)$

$$\|f(t)\|_{H^{l-1}}^2 \leqq C_2 M^2, \qquad 0 \leqq t \leqq \bar{t}. \tag{7.8}$$

From the governing equations, we also deduce that

$$\int_0^t \|\partial_t \widetilde{\mathrm{v}}(s)\|_{H^{l-1}}^2\, ds \leqq C_3^2\big(\widetilde{M}^2 + t(M^2 + \widetilde{M}^2)\big), \tag{7.9}$$

where $\widetilde{M}$ is defined for $\widetilde{\mathrm{v}}$ as M for v and C_3 depends on $\mathscr{O}_1$ and M, and is an increasing function of M. We now define for any $\alpha \in (0, b]$

$$M_\alpha = 2C_1(\mathscr{O}_1)\alpha, \qquad M_{1\alpha} = 2C_3(\mathscr{O}_1, M_b) 2C_1(\mathscr{O}_1)\alpha.$$

Then let $\bar{t} \leqq 3/2$ be small enough such that

$$\exp\big(C_2(\mathscr{O}_1, M_b)(\bar{t} + M_{1b}\sqrt{\bar{t}}\ )\big) \leqq 2,$$

$$C_2^2(\mathscr{O}_1, M_b)\bar{t}\big(2C_1(\mathscr{O}_1)\big)^2 \leqq 1,$$

and $C_0 M_{1b}\sqrt{\bar{t}}\ < d_1$, where C_0 is such that $\|\phi\|_{L^\infty} \leqq C_0\|\phi\|_{H^{l-1}}$. Then, for any $\alpha \in (0, b]$, any $\mathrm{v} \in \mathsf{X}_{\bar{t}}\big(\mathscr{O}_1, M_\alpha, M_{1\alpha}\big)$, any $\mathrm{v}_0(x)$, such that $\mathrm{v}_0 - \mathrm{v}_\infty \in H^l(\mathbb{R}^d)$, $\mathrm{v}_0 \in \overline{\mathscr{O}}_0$, and $\|\mathrm{v}_0 - \mathrm{v}_\infty\|_{H^l} \leqq \alpha$, the solution $\widetilde{\mathrm{v}}$ to the linearized equations with the initial data v_0 stays in the same space $\mathsf{X}_{\bar{t}}\big(\mathscr{O}_1, M_\alpha, M_{1\alpha}\big)$. More specifically, we obtain from (7.6) and (7.8) that

$$\widetilde{M}^2 \leqq 2C_1^2\alpha^2\big(1 + 4C_1^2C_2^2 t\big) \leqq 4C_1^2\alpha^2 = M_\alpha^2,$$

and from (7.9) we deduce that $\widetilde{M}_1^2 \leqq C_3^2 M_\alpha^2(1 + 2t) \leqq M_{1\alpha}^2$. Finally, we also have $\|\widetilde{\mathrm{v}} - \mathrm{v}_0\|_{L^\infty} \leqq C_0 M_{1\alpha}\sqrt{\bar{t}}\ < d_1$, since $C_0 M_{1b}\sqrt{\bar{t}}\ < d_1$, so that $\widetilde{\mathrm{v}} \in \mathscr{O}_1$.

In order to obtain fixed points, we establish that for $\bar{t}$ small enough, the map $\mathrm{v} \to \widetilde{\mathrm{v}}$ is a contraction in all the spaces $\mathsf{X}_{\bar{t}}\big(\mathscr{O}_1, M_\alpha, M_{1\alpha}\big)$, $\alpha \in (0, b]$. Let v and $\widehat{\mathrm{v}}$ be in $\mathsf{X}_{\bar{t}}\big(\mathscr{O}_1, M_b, M_{1b}\big)$, let $\mathrm{v}^0(x)$ and $\widehat{\mathrm{v}}^0(x)$ such that $\mathrm{v}_0 - \mathrm{v}_\infty \in H^l(\mathbb{R}^d)$, $\widehat{\mathrm{v}}_0 - \mathrm{v}_\infty \in H^l(\mathbb{R}^d)$, $\mathrm{v}_0, \widehat{\mathrm{v}}_0 \in \overline{\mathscr{O}}_0$, $\|\mathrm{v}_0 - \mathrm{v}_\infty\|_l < \alpha$, $\|\widehat{\mathrm{v}}_0 - \mathrm{v}_\infty\|_l < \alpha$, and define $\delta\mathrm{v} = \mathrm{v} - \widehat{\mathrm{v}}$ and $\widetilde{\delta\mathrm{v}} = \widetilde{\mathrm{v}} - \widetilde{\widehat{\mathrm{v}}}$. Forming the difference between the linearized equations, we obtain for $\delta\mathrm{v} = (\delta v, \delta T)$ and $\widetilde{\delta\mathrm{v}} = (\widetilde{\delta v}, \widetilde{\delta T})$

$$\begin{cases} \partial_t(\rho\widetilde{\delta v}) - \mathbb{P}\big(\partial_x\cdot\big(\eta(T)\,\partial_x\widetilde{\delta v}\big)\big) = \mathbb{P}\big(\delta f_v\big), \\ \partial_t(\rho c_v\widetilde{\delta T}) - \partial_x\cdot\big(\lambda(T)\,\partial_x\widetilde{\delta T}\big) = \delta f_T, \end{cases} \tag{7.10}$$

where

$$\delta f_v = f_v(\mathrm{v}, \partial_x\mathrm{v}) - f_v(\widehat{\mathrm{v}}, \partial_x\widehat{\mathrm{v}}) + \partial_x\cdot\big((\eta(T) - \eta(\widehat{T}))\partial_x\widetilde{\widehat{v}}\big),$$
$$\delta f_T = f_T(\mathrm{v}, \partial_x\mathrm{v}) - f_T(\widehat{\mathrm{v}}, \partial_x\widehat{\mathrm{v}}) + \partial_x\cdot\big((\lambda(T) - \lambda(\widehat{T}))\partial_x\widetilde{\widehat{T}}\big).$$

These expressions now imply that $\|\delta f\|_{H^{l-2}} \leqq C_4\|\delta\mathrm{v}\|_{H^{l-1}}$, where the constant C_4 depends on $\mathscr{O}_1$ and b, since v, $\widetilde{\mathrm{v}}$, $\widehat{\mathrm{v}}$ and $\widetilde{\widehat{\mathrm{v}}}$ are in the space $\mathsf{X}_{\bar{t}}\big(\mathscr{O}_1, M_b, M_{1b}\big)$, and thanks to estimates in the form

$$\|\psi(\phi) - \psi(\hat{\phi})\|_{H^k} \leqq C_0\|\psi\|_{C^{k+1}(\overline{\mathscr{O}}_\phi)}(1 + \|\phi\|_{H^k} + \|\hat{\phi}\|_{H^k})^k\, \|\phi - \hat{\phi}\|_{H^k}, \tag{7.11}$$

where $\mathscr{O}_\phi$ is a convex open set with $\phi(x) \in \mathscr{O}_\phi$, $\hat{\phi}(x) \in \mathscr{O}_\phi$, $x \in \mathbb{R}^n$, and k is such that $k \geqq [n/2] + 1$. As a consequence, defining

$$|\!|\!|\widetilde{\delta\mathrm{v}}|\!|\!|_{l-1}^2 = \sup_{0\leqq s\leqq\bar{t}} \|\widetilde{\delta\mathrm{v}}(s)\|_{H^{l-1}}^2 + \int_0^{\bar{t}} \|\widetilde{\delta\mathrm{v}}(s)\|_{H^l}^2\, ds,$$

we obtain that

$$|\!|\!|\widetilde{\delta\mathrm{v}}|\!|\!|_{l-1}^2 \ \leqq\ C_5\|\mathrm{v}_0 - \widehat{\mathrm{v}}_0\|_{H^{l-1}}^2 +\ C_5\bar{t} \sup_{0\leqq s\leqq\bar{t}} \|\delta\mathrm{v}(s)\|_{H^{l-1}}^2,$$

where the constant C_5 only depends on $\mathscr{O}_1$ and b. Now if $\bar{t}$ is small enough so that $C_5\bar{t} < 1/4$, by letting $\mathrm{v}_0 = \widehat{\mathrm{v}}_0$, we obtain that $\|\widetilde{\delta \mathrm{v}}\|_{l-1} \leqq \frac{1}{2}\|\delta\mathrm{v}\|_{l-1}$ so that the map $\mathrm{v} \to \widetilde{\mathrm{v}}$ is a contraction in all the spaces $\mathsf{X}_{\bar{t}}\big(\mathscr{O}_1, M_\alpha, M_{1\alpha}\big)$, $\alpha \in (0, b]$. Introducing the iterates v^n starting at the initial condition $\mathrm{v}^0 = \mathrm{v}_0$ and such that $\mathrm{v}^{n+1} = \widetilde{\mathrm{v}}^n$, that is, v^{n+1} is obtained as the solution of linearized equations, then the sequence $\{\mathrm{v}^n\}_{n\geqq 0}$ is easily shown to be convergent to a local solution of the nonlinear equations satisfying the estimates (7.4) at order $l-1$. Finally, the estimates (7.4) at order l are recovered since for any $\alpha \in (0, b]$, the space $\mathsf{X}_{\bar{t}}\big(\mathscr{O}_1, M_\alpha, M_{1\alpha}\big)$ is invariant, and the proof is complete.

7.2. Properties of the solutions

We establish in this section that the solutions constructed in Theorem 10 are as smooth as expected from initial data.

Theorem 11. *The solutions obtained in Theorem 10 inherit the regularity of* v_0*, that is, for any* $k \geqq l$ *such that* $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k$*, we have*

$$\mathrm{v} - \mathrm{v}_\infty \in C^0\big([0, \bar{t}\,], H^k(\mathbb{R}^n)\big) \cap C^1\big([0, \bar{t}\,], H^{k-2}(\mathbb{R}^n)\big) \cap L^2\big((0, \bar{t}\,), H^{k+1}(\mathbb{R}^n)\big). \tag{7.12}$$

In particular, v *is smooth when* $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k(\mathbb{R}^n)$ *for any* $k \in \mathbb{N}$.

Proof. Let $k \geqq l$ such that $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k$ and denote by $e^{[k]}$ the quantity $e^{[k]} = |\partial^k T|^2 + |\partial^k v|^2$. We have to estimate $e^{[k]}$ in order to establish (7.12).

A balance equation for $e^{[k]}$ can easily be derived (and is much simpler than that of $\gamma^{[k]}$ of $\tilde{\gamma}^{[k]}$) and written in the form

$$\partial_t e^{[k]} + \partial_x\cdot(v e^{[k]}) + \partial_x\cdot\varphi_e^{[k]} + \pi_e^{[k]} + \Sigma_e^{[k]} + \omega_e^{[k]} = 0. \tag{7.13}$$

The term $\pi_e^{[k]}$ is given by

$$\pi_e^{[k]} = \frac{2\lambda}{\rho c_v}\,|\partial^{k+1}T|^2 + \frac{2\eta}{\rho}|\partial^{k+1}v|^2, \tag{7.14}$$

in such a way that $2\underline{b}\, e^{[k+1]} \leqq \pi_e^{[k]}\, T^{-\varkappa} \leqq 2\overline{b}\, e^{[k+1]}$, where $\underline{b}$ and $\overline{b}$ are positive constants. The term $\Sigma_e^{[k]}$ is in the form

$$\Sigma_e^{[k]} = \sum_{\sigma\nu\mu}(c_{\sigma\nu\mu}\partial_T^\sigma\lambda \ + c'_{\sigma\nu\mu}\partial_T^\sigma\eta)\ \Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}, \tag{7.15}$$

where the sums are over $0 \leqq \sigma \leqq k$ $\nu = (\nu_\alpha, \nu'_\alpha)_{1\leqq|\alpha|\leqq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\leqq|\alpha|\leqq k+1}$, $\nu_\alpha, \nu'_\alpha, \mu_\alpha, \mu'_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by

$$\Pi_\nu^{(k+1)} = \prod_{1\leqq|\alpha|\leqq k+1}\big(\partial^\alpha T\big)^{\nu_\alpha} \prod_{1\leqq|\alpha|\leqq k+1}\big(\partial^\alpha v\big)^{\nu'_\alpha}, \tag{7.16}$$

where v denotes any of its components $v_1, \dots, v_n$, and μ and ν must be such that $\sum_{1\leqq|\alpha|\leqq k+1}|\alpha|(\nu_\alpha + \nu'_\alpha) = k+1$, $\sum_{1\leqq|\alpha|\leqq k+1}|\alpha|(\mu_\alpha + \mu'_\alpha) = k+1$,

$\sum_{|\alpha|=k+1}(\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) \leqq 1$, so that there is at most one derivative of $(k+1)^{\text{th}}$ order in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$. Furthermore the term $\omega_e^{[k]}$ is given by

$$\omega_e^{[k]} = \sum_{\nu\mu} c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}, \tag{7.17}$$

where we use similar notation for $\Pi_\nu^{(k)}$ as for $\Pi_\mu^{(k+1)}$ and the summation extends over $\sum_{1\leqq|\alpha|\leqq k}|\alpha|(\nu_\alpha + \nu'_\alpha) = k$, $\sum_{1\leqq|\alpha|\leqq k}|\alpha|(\mu_\alpha + \mu'_\alpha) = k+1$ so that in particular $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha) = 0$ and there are always at least two factors in the product $\Pi_\mu^{(k+1)}$. Finally the flux $\varphi_e^{[k]} = (\varphi_{e1}^{[k]}, \ldots, \varphi_{en}^{[k]})$ is given by the following formula

$$\varphi_{el}^{[k]} = \sum_{\sigma\nu\mu l}(c_{\sigma\nu\mu l}\partial_T^\sigma\lambda \;+ c'_{\sigma\nu\mu l}\partial_T^\sigma\eta)\;\Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + \sum_\alpha c_\alpha\partial^\alpha v_l\partial^\alpha p. \tag{7.18}$$

Now instead of regrouping the term $\Sigma_e^{[k]}$ with $\pi_e^{[k]}$, as in Corollary 2, Propositions 12 and 13, we regroup it with $\omega_e^{[k]}$ [43], thanks to the L^∞ a priori estimates for gradients, available from $l > n/2+1$. Whenever the product $\Pi_\nu^{(k+1)}$ is split, we indeed have estimates in the form [43]

$$\|\Pi_\nu^{(k+1)}\|_{L^2}^2 \leqq c\big(1 + \|\partial T\|_{L^\infty} + \|\partial v\|_{L^\infty}\big)^{2(k-1)}\int_{\mathbb{R}^n}\big(e^{[1]} + \cdots + e^{[k]}\big)\,dx,$$

so that from

$$\partial_t\int_{\mathbb{R}^n} e^{[k]}\,dx + \int_{\mathbb{R}^n}\pi_e^{[k]}\,dx \leqq \int_{\mathbb{R}^n}\big(|\Sigma_e^{[k]}| + |\omega_e^{[k]}|\big)\,dx,$$

we obtain that

$$\partial_t\int_{\mathbb{R}^n} e^{[j]}\,dx + \delta\int_{\mathbb{R}^n} e^{[j+1]}\,dx \leqq c\int_{\mathbb{R}^n}\big(e^{[1]} + \cdots + e^{[k]}\big)\,dx, \qquad 1 \leqq j \leqq k,$$

where δ and c depend on L^∞ estimates of v and ∂v. We can then sum up these inequalities and use the Gronwall lemma to conclude that $\int_{\mathbb{R}^n} e^{[k]}\,dx$ and $\int_0^t\int_{\mathbb{R}^n} e^{[k+1]}\,dx\,dt$ remain uniformly bounded over the whole time interval under consideration $[0,\bar t\,]$. Finally, when $\mathrm{v}_0 - \mathrm{v}_\infty$ is in H^k for any $k \geqq 0$, $\mathrm{v} - \mathrm{v}_\infty$ is in $C^0([0,\bar t\,], H^k)$ for any k, and we recover the regularity with respect to time from the governing equations so that v is smooth.

Remark 22. Boundedness of first order spatial derivatives is sufficient to establish the estimates of Theorem 11 because the coefficients of the time derivative terms in (5.5)– (5.6) are constants. For general quasilinear parabolic systems, one further needs to control second order spatial derivatives, whereas for general nonlinear parabolic systems, it is necessary to control third order spatial derivatives [43].

In the next propositions, we reformulate for convenience the local existence theorem in terms of the combined unknown $\mathrm{w} = (w, \tau)$ associated with the renormalized variables w and τ.

Lemma 3. *Denote by* $\mathscr{F} : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}^{n+1}$ *the application defined by* $\mathscr{F}(\mathrm{v}) = \mathrm{w}$, *that is,* $\mathscr{F}(v, T) = (w, \tau) = (v/\sqrt{T}, \log T)$. *Then* $\mathscr{F}$ *is a* C^∞ *diffeomorphism and its jacobian matrix reads*

$$\partial_{\mathrm{v}} \mathscr{F} = \begin{pmatrix} \frac{\mathbb{I}}{\sqrt{T}} & -\frac{1}{2}\frac{v}{T^{3/2}} \\ 0 & \frac{1}{T} \end{pmatrix}.$$

Moreover, for any $\mathrm{M}_w > 0$, $\mathrm{M}_\tau > 0$, *defining* $\widetilde{\mathscr{O}} = (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau)$, *the corresponding open set* $\mathscr{O} = \mathscr{F}^{-1}(\widetilde{\mathscr{O}})$ *is convex.*

Proof. The fact that $\mathscr{F}$ is a C^∞ diffeomorphism is straightforward to establish. Let then $\mathrm{M}_w > 0$, $\mathrm{M}_\tau > 0$, and assume that $(v, T) \in \mathscr{O}$, $(v', T') \in \mathscr{O}$. By definition, we have $|v_i| < \mathrm{M}_w\sqrt{T}$, $|v_i'| < \mathrm{M}_w\sqrt{T'}$, for $1 \leqq i \leqq n$, and $e^{-\mathrm{M}_\tau} < T < e^{\mathrm{M}_\tau}$, $e^{-\mathrm{M}_\tau} < T' < e^{\mathrm{M}_\tau}$. For $0 < \alpha < 1$, we easily obtain that $e^{-\mathrm{M}_\tau} < \alpha T + (1-\alpha)T' < e^{\mathrm{M}_\tau}$ and

$$|\alpha v_i + (1-\alpha)v_i'| < \mathrm{M}_w(\alpha\sqrt{T} + (1-\alpha)\sqrt{T'}),$$

but we have $\alpha\sqrt{T} + (1-\alpha)\sqrt{T'} \leqq \sqrt{\alpha T + (1-\alpha)T'}$ from concavity properties so that $\mathscr{O}$ is convex.

Proposition 14. *Let* $\mathrm{M}_w > 0$, $\mathrm{M}_\tau > 0$, $\widetilde{\mathscr{O}}_0 = (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau)$ *and* $\mathscr{O}_0 = \mathscr{F}^{-1}(\widetilde{\mathscr{O}}_0)$. *Let* $0 < d_1 < d(\overline{\mathscr{O}}_0, \partial\mathscr{O}_{\mathrm{v}})$, $\mathscr{O}_1 = \{\mathrm{v} \in \mathscr{O}_{\mathrm{v}};\ d(\mathrm{v}, \overline{\mathscr{O}}_0) < d_1\}$, *and select an arbitrary* $b > 0$. *From Theorem 10 we have a local solution built with the paramaters* $\mathscr{O}_0$, d_1, *and* b. *This solution is then*

$$\mathrm{w} - \mathrm{w}_\infty \in C^0\big([0, \bar{t}\,], H^l(\mathbb{R}^n)\big) \cap C^1\big([0, \bar{t}\,], H^{l-2}(\mathbb{R}^n)\big) \cap L^2\big((0, \bar{t}\,), H^{l+1}(\mathbb{R}^n)\big), \tag{7.19}$$

and there exists $C > 0$ *which only depends on* $\mathscr{O}_0$, d_1, *and* b, *such that*

$$\sup_{0 \leqq s \leqq \bar{t}} \|\mathrm{w}(s) - \mathrm{w}_\infty\|_{H^l}^2 + \int_0^{\bar{t}} \|\mathrm{w}(s) - \mathrm{w}_\infty\|_{H^{l+1}}^2\, ds \leqq C\|\mathrm{w}_0 - \mathrm{w}_\infty\|_{H^l}^2. \tag{7.20}$$

Moreover, the kinetic estimators are such that $\Gamma^{[l]}, \widetilde{\Gamma}^{[l]} \in C\big([0, \bar{t}\,], L^1(\mathbb{R}^n)\big)$.

Proof. The set $\mathscr{O}_0 = \mathscr{F}^{-1}(\widetilde{\mathscr{O}}_0)$ is convex and from Theorem 10, there exists a local solution built with $\mathscr{O}_0$, d_1 and b. We then have estimates in the form

$$\underline{c}_{\mathrm{v}}\|\mathrm{w} - \mathrm{w}_\infty\|_{H^l} \leqq \|\mathrm{v} - \mathrm{v}_\infty\|_{H^l} \leqq \overline{c}_{\mathrm{v}}\|\mathrm{w} - \mathrm{w}_\infty\|_{H^l}, \tag{7.21}$$

where $\underline{c}_{\mathrm{v}}$ and $\overline{c}_{\mathrm{v}}$ only depend on $\mathscr{O}_1$ and l thanks to the estimates (7.7). Similarly, the regularity properties are direct consequences of the estimates (7.11). The properties $\Gamma^{[l]}, \widetilde{\Gamma}^{[l]} \in C\big([0, \bar{t}\,], L^1(\mathbb{R}^n)\big)$ are then straightforward to establish.

Lemma 4. *There exists a constant* $\underline{c}_\Gamma$ *only depending on* $T_{\min}$ *such that for any* $k \geqq 0$ *and any* w *with* $\mathrm{w} - \mathrm{w}_\infty \in H^k$ *and* $\log T_{\min} \leqq \tau$ *we have*

$$\underline{c}_\Gamma\|\mathrm{w} - \mathrm{w}_\infty\|_{H^k}^2 \leqq \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]}\, dx. \tag{7.22}$$

Proof. This is a direct consequence of $w^2 \leqq v^2/T_{\min}$ and of the inequality

$$\frac{T_{\min}}{2T_\infty}|\zeta - \tau_\infty|^2 \leqq \exp(\zeta - \tau_\infty) - 1 - (\zeta - \tau_\infty),$$

valid for $\tau_{\min} = \log T_{\min} \leqq \zeta$, where $\tau_\infty = \log T_\infty$ and $T_{\min} \leqq T_\infty$.

7.3. Global existence

In this section, we investigate global existence of solutions $\mathrm{v} = (v, T)$ for which the quantity $\chi = \| \log T\|_{BMO} + \|v/\sqrt{T}\|_{L^\infty}$ remains small.

Theorem 12. *Let $n \geqq 2$ and $l \geqq [n/2]+2$ be integers. Assume that the coefficients λ and η satisfy (5.2) (5.3) with $\varkappa \geqq 1/2$. There exists $\delta_\Gamma(l, n, T_{\min}) > 0$ such that for T_0 and v_0 satisfying $T_{\min} \leqq \inf_{\mathbb{R}^n} T_0$, $\partial_x \cdot v_0 = 0$, $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k$, $k \in \mathbb{N}$, and*

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \leqq \delta_\Gamma, \tag{7.23}$$

where $\widetilde{\Gamma}_0^{[l]}$ denotes the functional $\widetilde{\Gamma}^{[l]}$ evaluated at initial conditions, there exists a unique global solution $\mathrm{v} = (v, T)$ such that

$$\begin{cases} \mathrm{v} - \mathrm{v}_\infty, \mathrm{w} - \mathrm{w}_\infty \in C\big([0, \infty), H^l(\mathbb{R}^n)\big) \cap C^1\big([0, \infty), H^{l-2}(\mathbb{R}^n)\big), \\ \partial_x \mathrm{v}, \partial_x \mathrm{w} \in L^2\big((0, \infty), H^l(\mathbb{R}^n)\big), \end{cases} \tag{7.24}$$

and we have the estimates

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx + \underline{b} \int_0^t \int_{\mathbb{R}^n} T^{\varkappa}\big(\widetilde{\Gamma}^{[l+1]} - \tilde{\gamma}^{[0]}\big)\, dxdt \leqq \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx, \tag{7.25}$$

where $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$. Furthermore, this solution is smooth and we have

$$\lim_{t\to\infty} \|\mathrm{v}(t, \cdot) - \mathrm{v}_\infty\|_{L^\infty} = 0. \tag{7.26}$$

Proof. Letting $l_0 = [n/2] + 1$, we have the inequalities

$$\chi = \|\tau\|_{BMO} + \|w\|_{L^\infty} \leqq \|\tau - \tau_\infty\|_{L^\infty} + \|w\|_{L^\infty} \leqq c_0\|\mathrm{w} - \mathrm{w}_\infty\|_{H^{l_0}}.$$

In order to obtain a value of δ_Γ small enough, so that the higher order entropic estimates of Theorem 9 hold, we set

$$\delta_\Gamma = \frac{\delta_{\mathrm{U}}^2}{4c_0^2}\underline{c}_\Gamma,$$

where δ_{U} is defined in Theorem 9 and $\underline{c}_\Gamma$ in Lemma 4, and this value will indeed ensure that $\chi \leqq \delta_{\mathrm{U}}/2$. Corresponding to this value of δ_Γ, we have estimates in the forms $\|\mathrm{w}-\mathrm{w}_\infty\|_{L^\infty} \leqq c_0(\delta_\Gamma/\underline{c}_\Gamma)^{1/2}$ and $\|\mathrm{w}-\mathrm{w}_\infty\|_{H^l} \leqq (\delta_\Gamma/\underline{c}_\Gamma)^{1/2}$. We now select $\mathrm{M}_w > 0$ and $\mathrm{M}_\tau > 0$ large enough such that

$$\{ z \in \mathbb{R}^{n+1};\ \|z-\mathrm{w}_\infty\| \leqq c_0(\delta_\Gamma/\underline{c}_\Gamma)^{1/2}+1 \} \subset (-\mathrm{M}_w,\mathrm{M}_w)^n \times (-\mathrm{M}_\tau,\mathrm{M}_\tau).$$

We next define $\widetilde{\mathscr{O}}_0 = (-\mathrm{M}_w,\mathrm{M}_w)^n \times (-\mathrm{M}_\tau,\mathrm{M}_\tau)$, $\mathscr{O}_0 = \mathscr{F}^{-1}(\widetilde{\mathscr{O}}_0)$, and we know that $\mathscr{O}_0$ is convex. Let $0 < d_1 < d(\overline{\mathscr{O}}_0, \partial\mathscr{O}_\mathrm{v})$, and define $\mathscr{O}_1 = \{ z \in \mathscr{O}_\mathrm{v};\ d(z,\mathscr{O}_0) < d_1 \}$ and $\widetilde{\mathscr{O}}_1 = \mathscr{F}(\mathscr{O}_1)$. Now for functions taking their values in $\mathscr{O}_1$ we have inequalities in the form $\|\mathrm{v}-\mathrm{v}_\infty\|_{H^k} \leqq \overline{c}_\mathrm{v}\|\mathrm{w}-\mathrm{w}_\infty\|_{H^k}$, where $\overline{c}_\mathrm{v}$ only depends on k and $\mathscr{O}_1$. We thus obtain the a priori estimate $\|\mathrm{v}-\mathrm{v}_\infty\|_{H^l} \leqq \overline{c}_\mathrm{v}(\delta_\Gamma/\underline{c}_\Gamma)^{1/2}$. We now set $b = \overline{c}_\mathrm{v}(\delta_\Gamma/\underline{c}_\Gamma)^{1/2}+1$ and from Theorem 10 and Proposition 14 we have local solutions over a time interval $[0,\bar{t}\,]$ built with the paramaters $\mathscr{O}_0$, d_1, and b.

Let now T_0 and v_0 satisfy $T_{\min} \leqq \inf_{\mathbb{R}^n} T_0$, $\partial_x \cdot v_0 = 0$, $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k$, $k \in \mathbb{N}$, and $\int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \leqq \delta_\Gamma$. Then by construction $\mathrm{v}_0 \in \mathscr{O}_0$ and $\|\mathrm{v}-\mathrm{v}_\infty\|_{H^l} < b$, and we have a local solution over the time interval $[0,\bar{t}\,]$. Letting $\chi(t) = \|\tau(t,\cdot)\|_{BMO} + \|w(t,\cdot)\|_{L^\infty}$ we also have by construction $\chi(0) \leqq \delta_\mathrm{U}/2$ and we claim that for any $t \in [0,\bar{t}\,]$ we also have $\chi(t) \leqq \delta_\mathrm{U}/2$. We introduce the set

$$\mathscr{E} = \{ s \in (0,\bar{t}\,];\ \forall t \in [0,s],\ \chi(t) \leqq (2/3)\delta_\mathrm{U} \},$$

which is not empty since $t \to \chi(t)$ is continuous and $\chi(0) \leqq \delta_\mathrm{U}/2$. Denoting $\overline{a} = \sup \mathscr{E}$ we have $\chi(t) \leqq (2/3)\delta_\mathrm{U}$ over $[0,\overline{a}]$ so that the entropic estimates of Theorem 9 hold and we have

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx \leqq \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \leqq \delta_\Gamma, \qquad 0 \leqq t \leqq \overline{a}.$$

This now implies that $\chi(t) \leqq \delta_\mathrm{U}/2$ over $[0,\overline{a}]$ so that $\overline{a} = \bar{t}$. From the above a priori estimates, we also obtain that for $t \in [0,\bar{t}\,]$ we have $\|\mathrm{w}(t)-\mathrm{w}_\infty\|_{L^\infty} \leqq c_0(\delta_\Gamma/\underline{c}_\Gamma)^{1/2}$, so that $\mathrm{v}(t) \in \mathscr{O}_0$, and $\|\mathrm{v}(t)-\mathrm{v}_\infty\|_{H^l} \leqq b-1 < b$, in particular at $t = \bar{t}$. We may now use again the local existence theorem over $[\bar{t}, 2\bar{t}\,]$ and an easy induction shows that the solution is a global solution.

The asymptotic stability is obtained by letting $\Phi(t) = \|\partial_x \mathrm{w}(t,\cdot)\|_{H^{l-2}}^2$ and establishing that

$$\int_0^\infty |\Phi(t)|\, dt + \int_0^\infty |\partial_t \Phi(t)|\, dt \leqq C \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx.$$

This shows that $\lim_{t\to\infty} \|\partial_x \mathrm{w}(t,\cdot)\|_{H^{l-2}} = 0$, and using the interpolation inequality

$$\|\phi\|_{C^0} \leqq C_0\, \|\partial_x^{l-1}\phi\|_{L^2}^a\, \|\phi\|_{L^2}^{1-a},$$

where $n/a = 2(l-1)$ we conclude that $\lim_{t\to\infty} \|\mathrm{w}(t,\cdot)-\mathrm{w}_\infty\|_{C^0} = 0$, and next that $\lim_{t\to\infty} \|\mathrm{v}(t,\cdot)-\mathrm{v}_\infty\|_{C^0} = 0$. Thanks to $\|\phi\|_{C^{l-([n/2]+2)}} \leqq C_0\, \|\partial_x^{l-1}\phi\|_{L^2}^a\, \|\phi\|_{H^l}^{1-a}$ we also have $\lim_{t\to\infty} \|\mathrm{w}(t,\cdot)-\mathrm{w}_\infty\|_{C^{l-([n/2]+2)}} = \lim_{t\to\infty} \|\mathrm{v}(t,\cdot)-\mathrm{v}_\infty\|_{C^{l-([n/2]+2)}} = 0$.

Remark 23. It is also possible to obtain global existence by assuming that both $\chi(0)$ and $\int_{\mathbb{R}^n} \Gamma_0^{[k]}\, dx$ are small enough.

Remark 24. Asymptotic stability of constant equilibrium states is usually obtained for $\mathrm{v}_0 - \mathrm{v}_\infty$ small enough in appropriate spaces. Assuming that $\log(T_0/T_\infty)$ and $v_0/\sqrt{T_0}$ are small enough seems more natural since these quantities are scale invariant and since the Knudsen and Mach numbers are of the same order of magnitude. In addition, the corresponding a priori estimates have a natural thermodynamic interpretation with higher order entropies. A complete analysis of the asymptotic expansions for small Mach and Knudsen numbers, however, is out of the scope of the present paper.

8. Conclusion

We have investigated higher order kinetic entropy estimators for incompressible fluid models in the natural situation where viscosity and thermal conductivity depend on temperature. We have established that entropic inequalities hold for such estimators provided that $\| \log T \|_{BMO} + \| v/\sqrt{T} \|_{L^\infty}$ is small enough. Domination of lower order convective terms has been obtained when the temperature dependence of transport coefficients is that suggested by the kinetic theory, that is, essentially a power of temperature with a common exponent $\varkappa \geqq 1/2$. As an illustration, we have established a global existence theorem provided that the initial values $\log(T_0/T_\infty)$ and $v_0/\sqrt{T_0}$ are small enough in appropriate spaces. Similar ideas can be introduced for compressible fluid models as well as zero Mach number models *mutatis mutandis* [19].

Appendix A. Proof of Proposition 5

We investigate the $\gamma^{[k]}$ balance equation for incompressible fluids with temperature dependent transport coefficients. The proof is lengthy and tedious but presents no serious difficulties other than notational.

Proof. Differentiating the expression of $\gamma^{[k]}$ we obtain

$$\partial_t \gamma^{[k]} + \left((1 + a_k) \mathrm{A}_\lambda^{[k]} \frac{|\partial^k T|^2}{T^{2+a_k}} + a_k \mathrm{A}_\eta^{[k]} \frac{|\partial^k v|^2}{T^{1+a_k}} \right) \partial_t T$$
$$- 2\mathrm{A}_\lambda^{[k]} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^\alpha T \partial^\alpha \partial_t T}{T^{1+a_k}} - 2\mathrm{A}_\eta^{[k]} \sum_{\substack{1 \leqq i \leqq n \\ |\alpha|=k}} \frac{k!}{\alpha!} \frac{\partial^\alpha v_i \partial^\alpha \partial_t v_i}{T^{a_k}} = 0,$$

so that from the governing equations

$$\begin{aligned}\partial_t \gamma^{[k]} &+ (1+a_k)\mathrm{A}_\lambda^{[k]} \frac{|\partial^k T|^2}{T^{2+a_k}} \Big(\frac{1}{\rho c_v} (\partial_x\cdot(\lambda \partial_x T) + \tfrac{1}{2}\eta |d|^2) - v\cdot\partial_x T \Big) \\ &+ a_k \mathrm{A}_\eta^{[k]} \frac{|\partial^k v|^2}{T^{1+a_k}} \Big(\frac{1}{\rho c_v} (\partial_x\cdot(\lambda \partial_x T) + \tfrac{1}{2}\eta |d|^2) - v\cdot\partial_x T \Big) \\ &- 2\mathrm{A}_\lambda^{[k]} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^\alpha T}{T^{1+a_k}} \partial^\alpha \Big(\frac{1}{\rho c_v} (\partial_x\cdot(\lambda \partial_x T) + \tfrac{1}{2}\eta |d|^2) - v\cdot\partial_x T \Big) \\ &- 2\mathrm{A}_\eta^{[k]} \sum_{\substack{1\leqq i\leqq n\\ |\alpha|=k}} \frac{k!}{\alpha!} \frac{\partial^\alpha v_i}{T^{a_k}} \partial^\alpha \Big(\frac{1}{\rho} (\partial_x\cdot(\eta d_{i\cdot}) - \partial_i p) - v\cdot\partial_x v_i \Big) = 0,\end{aligned}$$

and we denote by $\mathscr{T}^\partial$ the two first sums of the left-hand side, by $\mathscr{T}^\lambda$ the third sum and by $\mathscr{T}^\eta$ the last sum. We first examine separately higher order derivative contributions associated with each sum $\mathscr{T}^\partial$, $\mathscr{T}^\lambda$, and $\mathscr{T}^\eta$. The lower order convective terms in $\mathscr{T}^\partial$, $\mathscr{T}^\lambda$, and $\mathscr{T}^\eta$, are examined all together at the end.

The terms in $\mathscr{T}^\partial$ associated with $\partial_x\cdot(\lambda\partial_x T)$ are integrated by parts. They yield flux contributions and source terms in the form

$$-\sum_{1\leqq l\leqq n} \partial_l \left(\frac{(1+a_k)\mathrm{A}_\lambda^{[k]}}{\rho c_v} \frac{|\partial^k T|^2}{T^{2+a_k}} + \frac{a_k \mathrm{A}_\eta^{[k]}}{\rho c_v} \frac{|\partial^k v|^2}{T^{1+a_k}} \right) \lambda \partial_l T,$$

which are easily rewritten as sums of terms like $c_{\sigma\nu\mu} T^{\sigma-\varkappa} \partial_T^\sigma \lambda \;\, \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)^{\text{th}}$ order. On the other hand, the terms of $\mathscr{T}^\partial$ associated with $|d|^2$ are left unchanged and have the same structure.

We now consider the term $\mathscr{T}^\lambda$ with each contribution at a time. The most important contribution in $\mathscr{T}^\lambda$ is that associated with the terms $\partial^\alpha \partial_l(\lambda\partial_l T)$, $1\leqq l\leqq n$. These terms are integrated by parts and yield sources in the form

$$+\frac{2\mathrm{A}_\lambda^{[k]}}{\rho c_v} \sum_{\substack{1\leqq l\leqq n\\ |\alpha|=k}} \frac{k!}{\alpha!} \partial_l \Big(\frac{\partial^\alpha T}{T^{1+a_k}} \Big) \partial^\alpha \big(\lambda \partial_l T\big).$$

After expanding the derivatives, the above sum can be written

$$\begin{aligned}&\sum_{\substack{1\leqq l\leqq n\\ |\alpha|=k}} \frac{k!}{\alpha!} \frac{\partial^\alpha \partial_l T}{T^{1+a_k}} \Big(\lambda \partial^\alpha \partial_l T + \sum_{\tilde{\alpha}\sigma\nu} c_{\alpha\tilde{\alpha}\nu} \partial^{\alpha-\tilde{\alpha}} \partial_l T \partial_T^\sigma \lambda \prod_\beta (\partial^\beta T)^{\nu_\beta} \Big) \\ &- \sum_{\substack{1\leqq l\leqq n\\ |\alpha|=k}} \frac{k!}{\alpha!} (1+a_k) \frac{\partial^\alpha T \partial_l T}{T^{2+a_k}} \Big(\lambda \partial^\alpha \partial_l T + \sum_{\tilde{\alpha}\sigma\nu} c_{\alpha\tilde{\alpha}\nu} \partial^{\alpha-\tilde{\alpha}} \partial_l T \partial_T^\sigma \lambda \prod_\beta (\partial^\beta T)^{\nu_\beta} \Big),\end{aligned}$$

where the summations and products extend over $1 \leqq l \leqq n$, $|\alpha| = k$, $0 \leqq \tilde{\alpha} \leqq \alpha$, $\tilde{\alpha} \neq 0$, $1 \leqq \sigma \leqq |\tilde{\alpha}|$, $\sum_\beta |\beta|\nu_\beta = |\tilde{\alpha}|$, $1 \leqq |\beta| \leqq |\tilde{\alpha}|$, and $\sum_\beta \nu_\beta = \sigma$. We can now extract for $\pi_\gamma^{[k]}$ the term in the form $\lambda(\partial^\alpha \partial_l T)^2$ which can be written

$$\frac{2\lambda \mathrm{A}_\lambda^{[k]}}{\rho c_v} \sum_{\substack{1 \leqq l \leqq n \\ |\alpha| = k}} \frac{k!}{\alpha!} \frac{(\partial^\alpha \partial_l T)^2}{T^{1+a_k}} = \frac{2\lambda \mathrm{A}_\lambda^{[k]}}{\rho c_v} \sum_{|\alpha| = k+1} \frac{(k+1)!}{\alpha!} \frac{(\partial^\alpha T)^2}{T^{1+a_k}},$$

thanks to the properties of multinomial coefficients [10,41]. All other terms are of admissible form for $\Sigma_\gamma^{[k]}$, that is, in the form $c_{\sigma\nu\mu} T^{\sigma-\varkappa} \partial_T^\sigma \lambda \; \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)^{\text{th}}$ order since $\sum_\beta |\beta|\nu_\beta + 1 + |\alpha - \tilde{\alpha}| = k+1$. More specifically, we can factorize T^{-a_k} in the first factors, $T^{1+\varkappa}$ in the parenthesis, and all the terms involving derivatives of $\partial_T^\sigma \lambda$ are multiplied and divided by T^σ thanks to $\sum_\beta \nu_\beta = \sigma$.

The contributions in $\mathscr{T}^\lambda$ associated with $|d|^2$ are treated in a similar way. Indeed, we decompose each multiindex α with $|\alpha| = k$ into $\alpha = \tilde{\alpha} + e_{i_\alpha}$, where $|\tilde{\alpha}| = k-1$, i_α is chosen arbitrarily with $\alpha_{i_\alpha} \neq 0$, and $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{N}^n$, so that we have $\partial^\alpha = \partial^{\tilde{\alpha}} \partial_{i_\alpha}$. We can then integrate these terms by parts and obtain sources in the form

$$+\frac{\mathrm{A}_\lambda^{[k]}}{\rho c_v} \sum_{\substack{1 \leqq i,j \leqq n \\ |\alpha| = k}} \partial_{i_\alpha} \Big(\frac{\partial^\alpha T}{T^{1+a_k}} \Big) \partial^{\tilde{\alpha}} \big(\eta d_{ij}^2 \big).$$

Upon expending the derivatives with the help of the differential identities established in the previous section, all these terms are of the admissible form for $\Sigma_\gamma^{[k]}$.

We now consider the sum $\mathscr{T}^\eta$ and its most important contribution is that corresponding to $\partial^\alpha \partial_x \cdot (\eta d)$ which reads

$$\frac{2\mathrm{A}_\eta^{[k]}}{\rho} \sum_{\substack{1 \leqq i,l \leqq n \\ |\alpha| = k}} \frac{k!}{\alpha!} \frac{\partial^\alpha v_i \partial^\alpha \partial_l (\eta d_{il})}{T^{a_k}}.$$

We then use the identity $\sum_l \partial_l(\eta d_{il}) = \sum_l \partial_l(\eta \partial_l v_i) + \sum_l \partial_l \eta \partial_i v_l$ and focus on the contributions of the terms $\partial_l(\eta \partial_l v_i)$. The contributions associated with $\partial_l \eta \partial_i v_l$ are of admissible form for $\Sigma_\gamma^{[k]}$ after one integration by parts using $\alpha = \tilde{\alpha} + e_{i_\alpha}$ and the corresponding details are omitted. After integration by parts, we obtain sources in the form

$$+\frac{2\mathrm{A}_\eta^{[k]}}{\rho} \sum_{\substack{1 \leqq i,l \leqq n \\ |\alpha| = k}} \frac{k!}{\alpha!} \partial_l \Big(\frac{\partial^\alpha v_i}{T^{a_k}} \Big) \partial^\alpha (\eta \partial_l v_i),$$

and after expanding the derivatives, the sum can be written

$$\sum_{\substack{1\leqq i,l\leqq n\\|\alpha|=k}} \frac{k!}{\alpha!}\frac{\partial^\alpha\partial_l v_i}{T^{a_k}}\Big(\eta\partial^\alpha\partial_l v_i + \sum_{\tilde\alpha\sigma\nu} c_{\alpha\tilde\alpha\nu}\partial^{\alpha-\tilde\alpha}\partial_l v_i\,\partial_T^\sigma\eta\prod_\beta(\partial^\beta T)^{\nu_\beta}\Big)$$
$$-\sum_{\substack{1\leqq i,l\leqq n\\|\alpha|=k}} \frac{k!}{\alpha!}a_k\frac{\partial^\alpha v_i\partial_l T}{T^{1+a_k}}\Big(\eta\partial^\alpha\partial_l v_i + \sum_{\tilde\alpha\sigma\nu} c_{\alpha\tilde\alpha\nu}\partial^{\alpha-\tilde\alpha}\partial_l v_i\,\partial_T^\sigma\eta\prod_\beta(\partial^\beta T)^{\nu_\beta}\Big),$$

where the summations and products extend over $1\leqq i,l\leqq n$, $|\alpha|=k$, $0\leqq\tilde\alpha\leqq\alpha$, $\tilde\alpha\neq 0$, $1\leqq\sigma\leqq|\tilde\alpha|$, $\sum_\beta|\beta|\nu_\beta=|\tilde\alpha|$, $1\leqq|\beta|\leqq|\tilde\alpha|$, and $\sum_\beta\nu_\beta=\sigma$. We can now extract the term in the form $\eta(\partial^\alpha\partial_l v_i)^2$ for $\pi_\gamma^{[k]}$ which is rewritten as

$$\frac{2\eta\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\leqq i,l\leqq n\\|\alpha|=k}}\frac{k!}{\alpha!}\frac{(\partial^\alpha\partial_l v_i)^2}{T^{a_k}} = \frac{2\eta\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\leqq i\leqq n\\|\alpha|=k+1}}\frac{(k+1)!}{\alpha!}\frac{(\partial^\alpha v_i)^2}{T^{a_k}},$$

thanks to the properties of multinomial coefficients. All the other terms are of admissible form for $\Sigma_\gamma^{[k]}$, that is, in the form $c_{\sigma\nu\mu}T^{\sigma-\varkappa}\partial_T^\sigma\eta\;\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)^{\text{th}}$ order. The pressure can be divided as (5.8) and we consider here the contribution $-\sum_{i,j}R_iR_j\big(\eta d_{ij}\big)$. After one integration by parts, we obtain the sources

$$+\frac{2\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\leqq i,l,m\leqq n\\|\alpha|=k}}\frac{k!}{\alpha!}\partial_i\Big(\frac{\partial^\alpha v_i}{T^{a_k}}\Big)\partial^\alpha R_lR_m(\eta d_{lm}).$$

Since Riesz transforms and derivatives commute, the sum can be written, after expanding derivatives

$$-\sum_{\substack{1\leqq i,l,m\leqq n\\|\alpha|=k}}\frac{k!}{\alpha!}a_k\frac{\partial^\alpha v_i\partial_i T}{T^{1+a_k}}R_lR_m\left(\sum_{\tilde\alpha\sigma\nu}c_{\alpha\tilde\alpha\nu}\partial^{\alpha-\tilde\alpha}d_{lm}\partial_T^\sigma\eta\prod_\beta(\partial^\beta T)^{\nu_\beta}\right),$$

where the summations and products extend over $1\leqq i,l,m\leqq n$, $|\alpha|=k$, $0\leqq\tilde\alpha\leqq\alpha$, $1\leqq\sigma\leqq|\tilde\alpha|$, $\sum_\beta|\beta|\nu_\beta=|\tilde\alpha|$, $1\leqq|\beta|\leqq|\tilde\alpha|$, and $\sum_\beta\nu_\beta=\sigma$. All these terms can be written as $c_{\sigma\nu\mu\mathscr{R}}\Pi_\nu^{(k+1)}\mathscr{R}\big(T^{\sigma-\varkappa}\partial_T^\sigma\eta\;\Pi_\mu^{(k+1)}\big)$, where $\mathscr{R}=T^{-\theta}R_lR_mT^\theta$ with $\theta=(a_k+\varkappa)/2$.

Lower order convective terms first yield the contributions

$$-(1+a_k)\mathrm{A}_\lambda^{[k]}\frac{|\partial^kT|^2}{T^{2+a_k}}\big(v\cdot\partial_xT\big)-a_k\mathrm{A}_\eta^{[k]}\frac{|\partial^kv|^2}{T^{1+a_k}}\big(v\cdot\partial_xT\big)$$
$$+2\mathrm{A}_\lambda^{[k]}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha T}{T^{1+a_k}}\partial^\alpha\big(v\cdot\partial_xT\big)+2\mathrm{A}_\eta^{[k]}\sum_{\substack{1\leqq i\leqq n\\|\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}}\partial^\alpha\big(v\cdot\partial_xv_i\big),$$

and the terms proportional to v are easily recast in the form $v\cdot\partial_x\gamma^{[k]}$, so that the only remaining contributions are the sources

$$+\,2\mathrm{A}_\lambda^{[k]}\sum_{\substack{|\alpha|=k\\1\leqq l\leqq n}}\ \sum_{\substack{0\leqq\beta\leqq\alpha\\1\leqq|\beta|}} c_{\alpha\beta}\frac{k!}{\alpha!}\frac{\partial^\alpha T}{T^{1+a_k}}\partial^\beta v_l\,\partial^{(\alpha-\beta)}\partial_l T$$

$$+\,2\mathrm{A}_\eta^{[k]}\sum_{\substack{1\leqq i,l\leqq n\\|\alpha|=k}}\ \sum_{\substack{0\leqq\beta\leqq\alpha\\1\leqq|\beta|}} c_{\alpha\beta}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}}\partial^\beta v_l\,\partial^{(\alpha-\beta)}\partial_l v_i,$$

which are easily rewritten in the form $c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}T^{(1-2\varkappa+a_{k-1}-a_k)/2}$. We finally have to consider the contributions to $\omega_\gamma^{[k]}$ due to the pressure term $\sum_{i,j} R_iR_j\big(\rho v_iv_j\big)$, which read

$$+\frac{2\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\leqq i,l,m\leqq n\\|\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}}\partial^\alpha\partial_i\big(R_lR_m(v_lv_m)\big).$$

We now use $\partial_i\partial_{i_\alpha}R_lR_m = R_iR_{i_\alpha}\partial_m\partial_n$, where $\alpha=\tilde\alpha+e_{i_\alpha}$, and $\sum_{mn}\partial_m\partial_n(v_mv_n) = \sum_{mn}\partial_m v_n\partial_n v_m$, in order to obtain

$$+\frac{2\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\leqq i,l,m\leqq n\\|\alpha|=k}}\frac{\partial^\alpha v_i}{T^{a_k}}R_iR_{i_\alpha}\big(\partial^{\tilde\alpha}(\partial_m v_n\partial_m v_n)\big),$$

and these terms can be written $c_{\nu\mu\mathscr{R}}T^{(1-2\varkappa+a_{k-1}-a_k)/2}\Pi_\nu^{(k)}\mathscr{R}\big(\Pi_\mu^{(k+1)}\big)$ with $\mathscr{R} = T^{-\theta}R_iR_jT^\theta$ and $\theta=(1+a_k-\varkappa)/2$.

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