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Local Existence for the FENE-Dumbbell Model of Polymeric Fluids

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Abstract

We study the well-posedness of a multi-scale model of polymeric fluids. The microscopic model is the kinetic theory of the finitely extensible nonlinear elastic (FENE) dumbbell model. The macroscopic model is the incompressible non-Newton fluids with polymer stress computed via the Kramers expression. The boundary condition of the FENE-type Fokker-Planck equation is proved to be unnecessary by the singularity on the boundary. Other main results are the local existence, uniqueness and regularity theorems for the FENE model in certain parameter range.

1. Introduction

This paper is concerned with the local well-posedness of the coupled hydrodynamic and kinetic model for the dynamics of a mixture of dilute polymer molecules in a solvent. By denoting u and p as the velocity and pressure fields of the fluid respectively, the hydrodynamic systems take the form:

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \frac{\gamma}{Re}\triangle\mathbf{u} + \frac{1-\gamma}{ReDe}\nabla\cdot\tau, \qquad \text{for } \mathbf{x}\in\Omega, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \text{for } \mathbf{x}\in\Omega, \tag{2}$$

where the parameters Re and De are the Reynolds and Deborah numbers respectively, and γ is the viscosity ratio between the polymer and the solvent. The first term on the right-hand side of (1) is the contribution to stress from the solvent, and the second term is the contribution from the polymer. To compute the polymer stress τ, we model the individual polymer by a dumbbell with two beads connected by a spring. In the dilute regime, the interaction between the dumbbells are neglected. The configuration of the dumbbell is completely determined by the configuration of the spring which is described by a configuration field variable $\mathbf{Q}$ governed by the following equations:

$$\frac{d\mathbf{Q}}{dt} + \mathbf{u} \cdot \nabla \mathbf{Q} = \kappa \cdot \mathbf{Q} - \frac{1}{2De}\mathbf{g}(\mathbf{Q}) + \frac{1}{\sqrt{2De}}\dot{\mathbf{W}}, \tag{3}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}, \tag{4}$$

where $\kappa = (\nabla \mathbf{u})^T$, $\dot{\mathbf{W}}$ is temporal white noises, and $\mathbf{g}(\mathbf{Q})$ is the spring force. In this paper, we will focus on a more commonly used model for the spring force, namely the finitely extensible nonlinear elastic (FENE) model:

$$\mathbf{g}(\mathbf{Q}) = f_b(|\mathbf{Q}|^2)\mathbf{Q} \text{ and } f_b(|\mathbf{Q}|^2) = \frac{1}{1 - |\mathbf{Q}|^2/b} \quad |\mathbf{Q}| < \sqrt{b}, \tag{5}$$

where the parameter b is a measure of the maximum extensibility of the dumbbell.

It is noted that (3) is a stochastic partial differential equation for the configuration field $\mathbf{Q}$, which has an equivalent Fokker-Planck form:

$$\psi_t + (\mathbf{u} \cdot \nabla)\psi = \frac{1}{2De}\triangle_{\mathbf{Q}}\psi - \nabla_{\mathbf{Q}} \cdot \left[(\kappa \cdot \mathbf{Q} - \frac{1}{2De}\mathbf{g}(\mathbf{Q}))\psi\right]. \tag{6}$$

In (6) , $\mathbf{Q} \in \mathbb{R}^d$ becomes an independent variable of the distribution function $\psi(\mathbf{x}, \mathbf{Q}, t)$, where d is the dimension of the configuration space. From a numerical calculation, (6) is more complicated than (3) as more dimensions of the variables are involved. However, from a theoretical point of view, (6) seems to allow us to use traditional analytical methods for partial differential equations with high dimensions to deal with theoretical issues such as local well-posedness.

Equations (1), (2) and (6) are finally closed by an expression for the polymer stress τ from the principle of virtual work:

$$\tau = -\mathbf{I} + \langle \mathbf{g}(\mathbf{Q}) \otimes \mathbf{Q} \rangle, \tag{7}$$

where $\langle \cdot \rangle$ denotes ensemble average with respect to $\mathbf{Q}$:

$$\langle f \rangle = \int f(\mathbf{Q})\psi(\mathbf{Q})d\mathbf{Q}. \tag{8}$$

More details about these models can be found in [2, 4, 11]. In this paper, we will work with the following non-dimensionalized system, e.g. [15]:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \frac{\gamma}{Re}\triangle \mathbf{u} + \frac{1-\gamma}{ReDe} b \nabla \cdot \tau, \text{ for } \mathbf{x} \in \Omega, \tag{9}$$

$$\nabla \cdot \mathbf{u} = 0, \text{ for } \mathbf{x} \in \Omega, \tag{10}$$

$$\tau = -\frac{1}{b}\mathbf{I} + \langle \mathbf{g}(\mathbf{Q}) \otimes \mathbf{Q} \rangle, \text{ for } \mathbf{Q} \in \mathbb{R}^d, \tag{11}$$

$$\psi_t + (\mathbf{u} \cdot \nabla)\psi = \frac{1}{2bDe}\triangle_{\mathbf{Q}}\psi - \nabla_{\mathbf{Q}} \cdot \left[(\kappa \cdot \mathbf{Q} - \frac{1}{2De}\mathbf{g}(\mathbf{Q}))\psi\right], \tag{12}$$

$$\mathbf{g}(\mathbf{Q}) = f(|\mathbf{Q}|^2)\mathbf{Q} \quad \text{and} \quad f(|\mathbf{Q}|^2) = \frac{1}{1 - |\mathbf{Q}|^2}, \quad |\mathbf{Q}| < 1. \tag{13}$$

Next we will consider (9)–(13) with the boundary condition for the hydrodynamic equation

$$\mathbf{u} = 0, \quad \text{for} \quad \mathbf{x} \in \partial\Omega, \tag{14}$$

and note that the boundary condition of FENE-type Fokker-Planck equation is unnecessary by the singularity on the boundary, and the initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, \quad \psi(\mathbf{x}, \mathbf{Q}, 0) = \psi_0(\mathbf{x}, \mathbf{Q}), \quad \mathbf{x} \in \Omega, \mathbf{Q} \in B_1. \tag{15}$$

Hereinafter, B_r denotes the ball with center 0 and radius r, $B_r^c = \{x \in \mathbb{R}^d, x \notin B_r\}$. This system satisfies the following energy identity:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \left(\frac{1}{2}|\mathbf{u}|^2 + \lambda \int_{B_1} \psi(\frac{1}{2bDe}\ln\psi + \frac{1}{2De}U)d\mathbf{Q}\right)d\mathbf{x}\\ = -\int_\Omega \left(\frac{\gamma}{Re}|\nabla\mathbf{u}|^2 + \lambda\int_{B_1}\psi|\nabla_{\mathbf{Q}}(\frac{1}{2bDe}\ln\psi + \frac{1}{2De}U)|^2 d\mathbf{Q}\right)d\mathbf{x},\end{aligned}$$

where $\lambda = \frac{\gamma-1}{Re}\cdot 2b$ and $U = -\frac{1}{2}\ln(1-|\mathbf{Q}|^2)^2$, therefore $\mathbf{g}$ in (13) can be rewritten as $\nabla_{\mathbf{Q}}U$.

Define a scale of spaces for the distribution function ψ:

$$\mathcal{X}_{n,i} = \left\{\psi : B_1 \to \mathbb{R} | \sum_{k=0}^{i}\left[\int_{B_1}\left(\sum_{j=0}^{n+5-k} f^j\right)|\nabla_{\mathbf{Q}}^k\psi(\mathbf{Q})|d\mathbf{Q}\right] < \infty, n \in \mathbb{N}\right\}, \tag{16}$$

for $n + 5 \geqq i (i \in \mathbb{N})$ with the norm

$$\|\psi\|_{\mathcal{X}_{n,i}} = \sum_{k=0}^{i}\left[\int_{B_1}\left(\sum_{j=0}^{n+5-k} f^j\right)|\nabla_{\mathbf{Q}}^k\psi(\mathbf{Q})|d\mathbf{Q}\right],$$

where $f = 1/(1-|\mathbf{Q}|^2)$ is given by (13). Later we will use $n \geqq 7$ for the solution in this paper. Our main assumptions are:

(A1) The domain $\Omega \in \mathbb{R}^3$ is bounded and $\partial\Omega$ is of class C^4.
(A2) $\mathbf{u}_0 \in H^4(\Omega)$.
(A3) For given $n \in \mathbb{N}$, $\psi_0 \in H^4(\Omega, \mathcal{X}_{n,3})$. Moreover, $\psi_0 \geqq 0$ and $\int_{B_1}\psi_0(\mathbf{x}, \mathbf{Q})d\mathbf{Q} = 1$ for every $\mathbf{x} \in \Omega$.

In addition, we need compatibility conditions between the initial data and the incompressibility and boundary conditions. So we assume the following compatibility conditions:

(C) div $\mathbf{u}_0 = 0$ and $\mathbf{u}_0 = 0$ on $\partial\Omega$.

Below we state the main result of this work.

Theorem 1. *Assume that* (**A1**)–(**A3**) *and* (**C**) *hold. Then there exists a* $T^* > 0$ *such that the problem* (9)–(15) *has a unique solution satisfying*

$$\mathbf{u} \in \bigcap_{k=0}^{2} H^k([0, T^*]; H^{4-2k}(\Omega)); \tag{17}$$

$$\tau \in \bigcap_{k=0}^{1} H^k(0, T^*]; H^{3-2k}(\Omega)); \tag{18}$$

$$\psi \in \bigcap_{k=0}^{1} H^k(0, T^*]; H^{3-2k}(\Omega, \mathcal{X}_{n,0})) \tag{19}$$

$$\psi(x, \mathbf{Q}, t) = 0, \quad a.e. \quad \mathbf{Q} \in B_1^c \tag{20}$$

provided that

$$b > \frac{4n(n+1)}{2n-d-5}. \tag{21}$$

In particular, $b > 32, n > 7$ *as* $d = 2$ *and* $b > 36, n > 8$ *as* $d = 3$.

From the regularities of (17)–(19), it is known that $\mathbf{u} \in C^1(\Omega\times[0, T^*])$, $\tau \in C(\Omega\times[0, T^*])$ and $\int_{B_1} \left[\sum_{j=0}^{n+5} (\frac{1}{1-|\mathbf{Q}|^2})^j\right] \psi(\mathbf{x}, \mathbf{Q}, t) d\mathbf{Q} \in C(\Omega\times[0, T^*])$. In particular, we point out that from (20) the distribution function ψ is almost everywhere zero on the domain of B_1^c for any $\mathbf{x} \in \Omega$ and $t \in [0, T^*)$, which is consistent with the result of JOURDAIN *et al.* [9].

The difficulty of the problem lies in the nonlinearity and the singularity of the FENE force $\mathbf{g}(\mathbf{Q})$ in (12). Equations (9) and (12) are linearized to alternatively solve the decoupled equations. For the linearized macro equations (22) and (23), the well-posedness of the solution can be obtained by the Galerkin method. For the decoupled micro equation (25), the key idea is to extend the FENE-type Fokker-Planck equation as a Cauchy problem with the regularized extensible nonlinear elastic force (52) concerned with a parameter ε. Thus the uniform estimates can be obtained in the weighted space (60) of the solution ϕ^ε to the approximation Cauchy problem (54) and (55). With these estimates of ϕ^ε being independent of ε, there exists a limit function ψ in the weighted space (16) which satisfies (12) by Lemma 3.1. Thus, it is shown that the distribution function ψ is almost everywhere zero on the domain of B_1^c and $\int_{B_1} \psi(\mathbf{x}, \mathbf{Q}, t) d\mathbf{Q} = 1$ for every $\mathbf{x} \in \Omega, t > 0$.

We now review some existing results on similar problems. For general spring force $\mathbf{g}(\mathbf{Q}) = f(|\mathbf{Q}|^2)\mathbf{Q}$ in (13), where f satisfies certain conditions, RENARDY [17] proved a local existence and uniqueness theorem in weighted spaces for solutions of Euler equation coupled with kinetic theory of polymeric fluid. Here, some technical estimates in weighted spaces were not presented, which turned out to be crucial not only for proving well-posedness but also for analyzing numerical methods. Therefore, in [13] a detailed proof is given of these technical estimates along with an extension of some of Renardy's results. It is noted that the results in [13, 17] all excluded the FENE case (5) which corresponds to the most popularly used model of FENE dumbbells [2].

BARRETT *et al.* [1] established the existence of global-in-time weak solutions to the coupled microscopic-macroscopic model (9)–(16) with Dirichlet boundary condition of FENE-type Fokker-Planck equation. In order to complete their existence proofs, the velocity field appearing in the drift term of the Fokker-Planck equation (12) had to be suitably mollified in the case of corotational microscopic-macroscopic models. In the case of general, noncorotational models, the extra-stress tensor τ_p on the right side of (11) had to be mollified also. DU *et al.* [5] showed the finite-time existence of the solution to the Fokker-Planck equation (12) with a Neumann boundary condition for general given velocity gradients, and also long-time existence of the solution in case that the velocity gradients are close to being purely symmetric or anti-symmetric. The stability of the steady-state solution for general velocity gradient was also highlighted in the analysis of the FENE-P closure approximation.

The FENE dumbbells in the stochastic model (3) are of particular interest since numerical methods based on the stochastic models are becoming more popular. In the analytical aspects, E *et al.* [7] proved the local well-posedness for the stochastic model for general nonlinear spring models with smooth potential. The one-dimensional problems were investigated in [6, 8–10]. The authors gave a complete numerical analysis of a finite element method coupled with a Monte Carlo method for the Hookean dumbbell case in [8]. In [9] JOURDAIN *et al.* studied the FENE model for the one-dimensional shear flows. In this case, the local well-posedness of the weak solution to the stochastic model was proved for $b > 6$. We would like to point out the lower bounds on the parameter b in Theorem 1.1 of this paper are obtained from the energy estimates for the Fokker-Planck equation (1.6). In [9], they stated that the local-in-time existence and uniqueness holds under the"optimal" assumption $b > 2$ for certain initial random variables (see also [10] for more details). They note that none of the assumptions on b are restrictive in practice given that b is physically of the order of 100.

This paper is organized as follows. In section 2, we define an iterative scheme to obtain the existence of the solution. The scheme alternates between solving an equation of the same type encountered in incompressible elasticity, and solving a linear diffusion equation. Section 3 gives the detailed proofs the main lemmas. We first give the estimates for ψ with respect to the Lagrangian variable and then obtain the corresponding estimates with respect to the Eulerian variable. The conclusions are drawn in the final section.

2. Construction of the iterative solution

In this section, we construct an iterative scheme for the system (9)–(15) to obtain the existence of the solution. Given an $\mathbf{u}^m$ we determine $\mathbf{u}^{m+1}$ by solving the equations

$$\mathbf{u}_t^{m+1} + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^{m+1} + \nabla p^{m+1} = \frac{\gamma}{Re}\Delta \mathbf{u}^{m+1} + \frac{1-\gamma}{ReDe} b \nabla \cdot \tau^m, \quad (22)$$

$$\nabla \cdot \mathbf{u}^{m+1} = 0 \quad (23)$$

with the initial condition $\mathbf{u}^{m+1}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})$ and boundary condition $\mathbf{u}^{m+1}|_{\partial\Omega}=0$, where

$$\tau^m=-\frac{1}{b}\mathbf{I}+\int_{\mathbb{R}^d}\mathbf{g}(\mathbf{Q})\otimes\mathbf{Q}\psi^{m+1}d\mathbf{Q}. \tag{24}$$

Meanwhile, for given $\mathbf{u}^m$, we determine ψ^{m+1} from the initial- value problem

$$\psi_t^{m+1}+(\mathbf{u}^m\cdot\nabla)\psi^{m+1}=-\nabla_{\mathbf{Q}}\cdot\left[(\kappa^m\cdot\mathbf{Q}-\frac{1}{De}\mathbf{g}(\mathbf{Q}))\psi^{m+1}\right]+\frac{1}{2bDe}\triangle_{\mathbf{Q}}\psi^{m+1}, \tag{25}$$

$$\psi^{m+1}(\mathbf{x},\mathbf{Q},0)=\psi_0(\mathbf{x},\mathbf{Q}), \tag{26}$$

where $\kappa^m=(\nabla\mathbf{u}^m)^T$. Our goal is to show that the mapping $\mathcal{M}:\mathbf{u}^m\mapsto\mathbf{u}^{m+1}$ is a contraction in an appropriate function space, so that the solution is the fixed point of the mapping.

We will consider the mapping $\mathcal{M}$ in the function space $S(M,T)$ with a metric $d(\cdot,\cdot)$, where $S(M,T)$ is the set of all functions $\mathbf{u}:\Omega\times[0,T]\to\mathbb{R}^3$ with the following properties:

$$\mathbf{u}\in\bigcap_{k=0}^{2}H^k([0,T];H^{4-2k}(\Omega)), \tag{27}$$

$$\|\mathbf{u}\|_{0,4}+\|\mathbf{u}\|_{1,2}+\|\mathbf{u}\|_{2,0}\leqq M, \tag{28}$$

$$\nabla\cdot\mathbf{u}=0, \tag{29}$$

$$\mathbf{u}=0 \text{ on } \partial\Omega, \tag{30}$$

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0,\ \mathbf{u}_t(\mathbf{x},0)=\mathbf{u}_1(\mathbf{x}). \tag{31}$$

Here $\|\cdot\|_{k,l}$ denotes the norm in $H^k([0,T];H^l(\Omega))$. The function $\mathbf{u}_0$ and $\mathbf{u}_1$ lie in $H^4(\Omega)$ and $H^2(\Omega)$, respectively. On $S(M,T)$, we define the metric

$$d(\mathbf{u}_1,\mathbf{u}_2)=\|\mathbf{u}_1-\mathbf{u}_2\|_{0,4}+\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}+\|\mathbf{u}_1-\mathbf{u}_2\|_{2,0}. \tag{32}$$

It is easy to verify that $S(M,T)$ is complete with the associated metric, and it is non-empty for large M, as observed in [16]. The contraction of the mapping $\mathcal{M}$ is established by using the following five lemmas.

Lemma 1. *Assume the bounds of following kind hold:*

$$\|\mathbf{u}^m\|_{0,4}+\|\mathbf{u}^m\|_{1,2}+\|\mathbf{u}^m\|_{2,0}\leqq M, \tag{33}$$

$$\|\tau^m\|_{0,3}+\|\tau^m\|_{1,1}\leqq K, \tag{34}$$

where τ^m *is defined by* (24)*. Then the system* (22)–(24) *has a solution*

$$\mathbf{u}^{m+1}\in\bigcap_{k=0}^{2}H^k([0,T];H^{4-2k}(\Omega)) \tag{35}$$

which satisfies that

$$\|\mathbf{u}^{m+1}\|_{0,4} + \|\mathbf{u}^{m+1}\|_{1,2} + \|\mathbf{u}^{m+1}\|_{2,0} \leqq \phi_1(M, T, K), \tag{36}$$

where $\phi_1(M, T, K)$ may depend on the initial values $\mathbf{u}_0$ and $\mathbf{u}_1$, and is bounded if the parameters M, T and K are bounded.

The above lemma can be established using the same method as that used in the proof for Lemma 2 of [13] when $\nabla\cdot\tau^m \in L^2([0, T]; H^2(\Omega))\cap H^1([0, T]; L^2(\Omega))$ and $\mathbf{u}_0 \in H^4(\Omega)$.

Lemma 2. *Consider the equations*

$$\mathbf{v}_t^{m+1} + (\mathbf{v}^m \cdot \nabla)\mathbf{v}^{m+1} + \nabla q^{m+1} = \frac{\gamma}{De}\Delta\mathbf{v}^{m+1} + \frac{1-\gamma}{ReDe} b\nabla\cdot\pi^m, \tag{37}$$

$$\nabla\cdot\mathbf{v}^{m+1} = 0 \tag{38}$$

with the initial and boundary conditions

$$\mathbf{v}^{m+1}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}); \quad \mathbf{v}^{m+1} = 0, \quad on\ \partial\Omega. \tag{39}$$

Assume $\mathbf{v}^m \in S(M, T)$, $\mathbf{v}^m = \mathbf{u}^m, \pi^m = \tau^m$ for $t = 0$, and that the assumptions of Lemma 1 *also hold for* (37) *(with the same constants M, K). Then the following estimate holds:*

$$\begin{aligned} \| \quad & \mathbf{u}^{m+1} - \mathbf{v}^{m+1}\|_{0,4} + \|\mathbf{u}^{m+1} - \mathbf{v}^{m+1}\|_{1,2} + \|\mathbf{u}^{m+1} - \mathbf{v}^{m+1}\|_{2,0} \\ & \leqq \phi_2(M, T, K)\cdot(\|\mathbf{u}^m - \mathbf{v}^m\|_{0,4} + \|\mathbf{u}^m - \mathbf{v}^m\|_{1,2} + \|\mathbf{u}^m - \mathbf{v}^m\|_{2,0} \\ & \quad +\|\tau^m - \pi^m\|_{0,3} + \|\tau^m - \pi^m\|_{1,1}), \end{aligned}$$

where $\mathbf{u}^{m+1}$ is defined by (22)–(23)*, $\phi_2(M, T, K)$ is similar to $\phi_1(M, T, K)$ in* Lemma 1. *Moreover,* $\lim_{T\to 0}\phi_2(M, T, K) = 0$.

Lemma 3. *If $\mathbf{u}^m \in S(M, T)$ and* (21) *holds, then there exists a unique solution of* (25)–(26) *which satisfies the regularity property:*

$$\psi^{m+1} \in \bigcap_{k=0}^{1} H^k\left([0, T]; H^{3-2k}(\Omega, \mathcal{X}_{n,2})\right). \tag{40}$$

Moreover, the solution satisfies the following estimate:

$$\|\psi^{m+1}\|_{n,2}^{1,1} \leqq K_1(n, M, T), \tag{41}$$

where K_1 is similar to ϕ_1 in Lemma 1, and $\|\cdot\|_{n,2}^{1,1}$ is the norm in

$$\bigcap_{k=0}^{1} H^k([0, T]; H^{3-2k}(\Omega, \mathcal{X}_{n,2})). \tag{42}$$

In fact, it is observed that the regularity of ψ^{m+1} is better than (40). However, only the estimate of ψ^{m+1} is needed in the norm $\|\cdot\|_{n,2}^{1,1}$, so it suffices to use (40).

Let us consider a related equation of the following form:

$$\hat{\psi}_t^{m+1} + (\mathbf{v}^m \cdot \nabla)\hat{\psi}^{m+1} = -\nabla_{\mathbf{Q}} \cdot \left[(\hat{\kappa}^m \cdot \mathbf{Q} - \frac{1}{2De}\mathbf{g}(\mathbf{Q}))\hat{\psi}^{m+1} \right] + \frac{1}{2bDe} \triangle_{\mathbf{Q}} \hat{\psi}^{m+1}, \tag{43}$$

$$\hat{\psi}^{m+1}(\mathbf{x}, \mathbf{Q}, 0) = \hat{\psi}_0(\mathbf{x}, \mathbf{Q}), \tag{44}$$

where $\hat{\kappa}^m = (\nabla \mathbf{v}^m)^T$, and we assume $\mathbf{v}^m|_{t=0} = \mathbf{u}_0$. In addition to (25), the following result holds.

Lemma 4. *Let* $\mathbf{u}^m, \mathbf{v}^m \in S(M, T)$ *be given and assume* (21) *holds. Then the following estimate holds:*

$$\|\psi^{m+1} - \hat{\psi}^{m+1}\|_{n,2}^{1,1} \leqq K_2(M, T) d(\mathbf{u}^m, \mathbf{v}^m), \tag{45}$$

where K_2 *is similar to* ϕ_2 *in* Lemma 2.

Lemma 5. *Let* τ^m *be defined by* (22)–(23). *We have*

$$\tau^m \in \bigcap_{k=0}^{1} H^k \left([0, T]; H^{3-2k}(\Omega) \right) \tag{46}$$

and

$$\|\tau^m - \pi^m\|_{0,3} + \|\tau^m - \pi^m\|_{1,1} \leqq K_2(n, M, T) d(\mathbf{u}^{m-1}, \mathbf{v}^{m-1}),$$

provided that $\psi^m, \hat{\psi}^m \in \bigcap_{k=0}^{1} H^k([0, T]; H^{3-2k}(\Omega, \mathcal{X}_{n,2}))$, *where* $\pi^m = \int_{\mathbb{R}^d} \mathbf{g}(\mathbf{Q}) \otimes \mathbf{Q} \hat{\psi}^m d\mathbf{Q}$.

By combining Lemmas 1–5, it follows easily that $\mathcal{M}$ is a contraction in $S(M, T)$, if M is chosen sufficiently large and T is chosen sufficiently small. Therefore, Theorem 1 follows immediately. The following section will be concerned with the proofs of Lemmas 3 and 4. The proofs of Lemmas 2 and 5 are similar to those for Lemmas 3 and 6 of [13], respectively, and will be omitted.

Remark 1. Now we would like to point out that the iterative scheme (22)–(25) still preserves the energy law when ψ^{m+1} possesses the regularity of (40). Similar to the arguments in [12], multiplying $\mathbf{u}^{m+1}$ to (22) and integrating it in Ω yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}^{m+1}|^2 d\mathbf{x} = -\frac{\gamma}{Re}\int_\Omega |\nabla \mathbf{u}^{m+1}|^2 d\mathbf{x} - \frac{1-\gamma}{ReDe} b \int_\Omega \int_{B_1} (\nabla \mathbf{u}^{m+1})^T \cdot \mathbf{Q} \cdot \nabla_{\mathbf{Q}} U \cdot \psi^{m+1} d\mathbf{Q} d\mathbf{x}. \tag{47}$$

For equation (25) of ψ^{m+2}, multiplying $\frac{1}{2bDe}\ln\psi^{m+2}+\frac{1}{2De}U$ and integrating it in $\Omega\times B_1$ yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega\int_{B_1}&\psi^{m+2}(\frac{1}{2bDe}\ln\psi^{m+2}+\frac{1}{2De}U)d\mathbf{Q}d\mathbf{x}\\&-\frac{1}{2De}\int_\Omega\int_{B_1}(\nabla\mathbf{u}^{m+1})^T\cdot\mathbf{Q}\cdot\nabla_{\mathbf{Q}}U\cdot\psi^{m+1}d\mathbf{Q}d\mathbf{x}\\&=-\int_\Omega\int_{B_1}\psi^{m+2}|\nabla_{\mathbf{Q}}(\frac{1}{2bDe}\ln\psi^{m+2}+\frac{1}{2De}U)|^2d\mathbf{Q}d\mathbf{x}.\end{aligned}\tag{48}$$

A combination of (47) and (48) implies the following energy law:

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(\frac{1}{2}|\mathbf{u}^{m+1}|^2+\lambda\int_{B_1}\psi^{m+2}(\frac{1}{2bDe}\ln\psi^{m+2}+\frac{1}{2De}U)d\mathbf{Q}\right)d\mathbf{x}\\&=-\int_\Omega\left(\frac{\gamma}{Re}|\nabla\mathbf{u}^{m+1}|^2+\lambda\int_{B_1}\psi^{m+2}|\nabla_{\mathbf{Q}}(\frac{1}{2bDe}\ln\psi^{m+2}+\frac{1}{2De}U)|^2d\mathbf{Q}\right)d\mathbf{x}.\end{aligned}$$

3. Proof of Lemmas 3 and 4

3.1. Estimates of ψ with respect to Lagrangian variables

We will first obtain the estimate for ψ with respect to the Lagrangian variables, and then translate them to the Eulerian variables. Consider the flow map

$$\frac{\partial}{\partial t}\mathbf{x}(\alpha,t)=\mathbf{u}^m(\mathbf{x}(\alpha,t),t),\quad \mathbf{x}(\alpha,0)=\alpha,\tag{49}$$

where α denotes the Lagrangian coordinates, and define $\phi(\alpha,\mathbf{Q},t)=\psi^{m+1}(\mathbf{x}(\alpha,t),\mathbf{Q},t)$. Then (25) can be rewritten in the form

$$\frac{\partial}{\partial t}\phi(\alpha,\mathbf{Q},t)=-\nabla_{\mathbf{Q}}\cdot\left[(\kappa\cdot\mathbf{Q}-\frac{1}{2De}\mathbf{g}(\mathbf{Q}))\phi\right]+\frac{1}{2bDe}\triangle_{\mathbf{Q}}\phi.\tag{50}$$

In this section we still denote $(\nabla\mathbf{u}^m(\mathbf{x}(\alpha,t),t))^T$ by κ. Now we will construct an approximate equation for (50). Firstly, let η be a C^∞ function $[0,1]\to[0,1]$ such that $\eta(s)=1$ in a neighborhood of 0 and $\eta(s)=0$ in a neighborhood of 1, and let $\eta_\varepsilon(\mathbf{Q})=\eta(\frac{2}{\varepsilon}[|\mathbf{Q}|-(1-\varepsilon)])$ for $0<\varepsilon<1/2$. We then define

$$f_\varepsilon(|\mathbf{Q}|^2)=\begin{cases}f(|\mathbf{Q}|^2), & \text{if } |\mathbf{Q}|\leqq 1-\varepsilon,\\ f((1-\varepsilon/2)^2), & \text{if } |\mathbf{Q}|\geqq 1-\varepsilon/2,\\ \eta_\varepsilon(\mathbf{Q})f(|\mathbf{Q}|^2)+(1-\eta_\varepsilon(\mathbf{Q}))f((1-\frac{\varepsilon}{2})^2), & \text{if } 1-\varepsilon<|\mathbf{Q}|<1-\varepsilon/2.\end{cases}\tag{51}$$

Now we set

$$\mathbf{g}_\varepsilon(\mathbf{Q})=f_\varepsilon(|\mathbf{Q}|^2)\mathbf{Q},\tag{52}$$

and the extension of $\psi_0(\mathbf{x}, \mathbf{Q})$ is given by

$$\bar{\psi_0}(\mathbf{x}, \mathbf{Q}) = \begin{cases} \psi_0(\mathbf{x}, \mathbf{Q}), & \text{if } \mathbf{Q}| \leqq 1, \\ 0, & \text{if } |\mathbf{Q}| > 1. \end{cases} \tag{53}$$

Next we will solve the following Cauchy problem

$$\frac{\partial}{\partial t}\phi^{\varepsilon}(\alpha, \mathbf{Q}, t) = -\nabla_{\mathbf{Q}} \cdot \left[(\kappa \cdot \mathbf{Q} - \frac{1}{2De}\mathbf{g}_{\varepsilon}(\mathbf{Q}))\phi^{\varepsilon} \right] + \frac{1}{2bDe}\triangle_{\mathbf{Q}}\phi^{\varepsilon} \tag{54}$$

$$\phi^{\varepsilon}(\alpha, \mathbf{Q}, 0) = \bar{\psi_0}(\alpha, \mathbf{Q}). \tag{55}$$

Since the coefficients of the equation (54) are unbounded, standard existence arguments for parabolic equations cannot be used. Motivated by the work of [17], we will use a sequence of approximating problems with bounded coefficients to derive uniform estimates.

Let $\chi(\mathbf{Q})$ be a C^{∞} function such that $\chi(\mathbf{Q}) = 1$ in B_1, where χ is a monotone decreasing function of $|\mathbf{Q}|$ satisfying $\chi(\mathbf{Q})|\mathbf{Q}| \geqq 1$ in $B_1^c = \mathbb{R}^d \backslash B_1$ and $\chi(\mathbf{Q}) = |\mathbf{Q}|^{-1}$ for large $|\mathbf{Q}|$. Obviously, such an $\chi(\mathbf{Q})$ exists. For $N \in \mathbb{N}$, let $\chi_N(\mathbf{Q}) = \chi(\mathbf{Q}/N)$. We now consider (54)–(55) by the following approximate problem

$$\begin{aligned} \frac{\partial}{\partial t}\phi_{N,\varepsilon}(\alpha, \mathbf{Q}, t) = -\nabla_{\mathbf{Q}} \cdot &\left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - \frac{1}{2De}\mathbf{g}_{\varepsilon}(\mathbf{Q}))\phi_{N,\varepsilon} \right] \\ &+ \frac{1}{2bDe}\triangle_{\mathbf{Q}}\phi_{N,\varepsilon}, \end{aligned} \tag{56}$$

$$\phi_{N,\varepsilon}(\alpha, \mathbf{Q}, 0) = \bar{\psi_0}(\alpha, \mathbf{Q}). \tag{57}$$

The proof of solutions to the Cauchy problem (56)–(57) is straightforward since the coefficients of (56) are bounded. It follows from the maximum principle that positivity is preserved, and by integrating both sides of (56) we find that

$$\int_{\mathbb{R}^d} \phi_{N,\varepsilon}(\alpha, \mathbf{Q}, t)d\mathbf{Q} = \int_{\mathbb{R}^d} \phi_{N,\varepsilon}(\alpha, \mathbf{Q}, 0)d\mathbf{Q} = \int_{B_1} \psi(\alpha, \mathbf{Q}, 0)d\mathbf{Q}{=}1, \quad \text{for all } t. \tag{58}$$

We will consider the next regularity estimates of $\phi_{N,\varepsilon}$ and its derivatives with respect to α and $\mathbf{Q}$. For ease of notation, we denote

$$\phi_{N,\varepsilon}^{(i_1,i_2,\dots,i_m;j_1,j_2,\dots,j_n)} = \partial^m_{Q_{i_1},Q_{i_2},\dots,Q_{i_m}} \partial^n_{\alpha_{j_1},\alpha_{j_2},\dots,\alpha_{j_n}} \phi_{N,\varepsilon}, \tag{59}$$

and in particular,

$$\phi_{N,\varepsilon}^{(i_1,i_2,\dots,i_m;0)} = \partial^m_{Q_{i_1},Q_{i_2},\dots,Q_{i_m}} \phi_{N,\varepsilon}, \qquad \phi_{N,\varepsilon}^{(0;j_1,j_2,\dots,j_n)} = \partial^n_{\alpha_{j_1},\alpha_{j_2},\dots,\alpha_{j_n}} \phi_{N,\varepsilon}. \tag{60}$$

Define a group of spaces similar to the one in (16):

$$\mathcal{X}^{\varepsilon}_{n,i} = \left\{ \phi_{\varepsilon} : \mathbb{R}^d \to \mathbb{R} | \ \sum_{k=0}^{i} \left[\int_{\mathbb{R}^d} \left(\sum_{j=0}^{n+5-k} w^j_{\varepsilon} \right) |\nabla^k_{\mathbf{Q}}\phi_{\varepsilon}(\mathbf{Q})| d\mathbf{Q} \right] < \infty, n \in \mathbb{N} \right\} \tag{61}$$

with the norm

$$\|\phi_\varepsilon\|_{\mathcal{X}^\varepsilon_{n,i}} = \sum_{k=0}^{i} \left[\int_{\mathbb{R}^d} \left(\sum_{j=0}^{n+5-k} w_\varepsilon^j \right) |\nabla^k_{\mathbf{Q}} \phi_\varepsilon(\mathbf{Q})| d\mathbf{Q} \right],$$

where

$$w_\varepsilon(\mathbf{Q}) = \begin{cases} f(|\mathbf{Q}|^2) = \frac{1}{1-|\mathbf{Q}|^2}, & \text{if } |\mathbf{Q}| \leqq 1-\varepsilon, \\ A_1 + A_2\left[|\mathbf{Q}| - (1-\varepsilon)\right], & \text{if } |\mathbf{Q}| > 1-\varepsilon. \end{cases} \tag{62}$$

In (62), $A_1 = \frac{1}{\varepsilon(2-\varepsilon)}$ and $A_2 = \frac{2(1-\varepsilon)}{\varepsilon^2(2-\varepsilon)^2}$. The function w_ε and its gradient are continuous at $|\mathbf{Q}| = 1-\varepsilon$, and w_ε has linear growth at infinity. By the definition of $w_\varepsilon(\mathbf{Q})$, we can obtain the following result.

Lemma 6. *Suppose that ϕ^ε satisfies* (54) *for every $\varepsilon \in (0, 1/2)$ and $\|\phi^\varepsilon\|_{\mathcal{X}^\varepsilon_{n,i+1}}$ is bounded being independent of ε, then there exists a function $\phi \in \mathcal{X}_{n,i}$ which satisfies* (50) *for $\mathbf{Q} \in B_1$, and ϕ is almost everywhere zero on the domain of B_1^c and $\|\phi\|_{\mathcal{X}_{n,i}} \leqq \lim_{\varepsilon\to 0} \|\phi^\varepsilon\|_{\mathcal{X}^\varepsilon_{n,i+1}}$.*

Proof. For $\varepsilon_1 \in (0, 1/2)$ fixed, by the properties of ϕ^ε, we know that ϕ^ε satisfies (54) for $\mathbf{Q} \in B_{1-\varepsilon_1}$and every $\varepsilon \in (0, \varepsilon_1]$ and

$$\begin{aligned} &\sum_{k=0}^{i+1} \left[\int_{B_{1-\varepsilon_1}} \left(\sum_{j=0}^{n+5-k} w_\varepsilon^j \right) |\nabla^k_{\mathbf{Q}} \phi^\varepsilon(\mathbf{Q})| d\mathbf{Q} \right] \\ &\quad = \sum_{k=0}^{i+1} \left[\int_{B_{1-\varepsilon_1}} \left(\sum_{j=0}^{n+5-k} f^j \right) |\nabla^k_{\mathbf{Q}} \phi^\varepsilon(\mathbf{Q})| d\mathbf{Q} \right] \\ &\quad \leqq \|\phi^\varepsilon\|_{\mathcal{X}^\varepsilon_{n,i+1}}, \end{aligned}$$

here we have used the definition (62) of $w_\varepsilon(\mathbf{Q})$. The uniform bound of $\|\phi^\varepsilon\|_{\mathcal{X}^\varepsilon_{n,i+1}}$ implies that

$$\sum_{k=0}^{i+1} \left[\int_{B_{1-\varepsilon_1}} \left(\sum_{j=0}^{n+5-k} f^j \right) |\nabla^k_{\mathbf{Q}} \phi^\varepsilon(\mathbf{Q})| d\mathbf{Q} \right] \tag{63}$$

is uniformly bounded with respect to ε. Therefore, there is a convergent subsequence in $B_{1-\varepsilon_1}$ still denoted by $\{\phi^\varepsilon\}$ with the loss of confusion. We now denote its limit by ϕ for $\mathbf{Q} \in B_{1-\varepsilon_1}$ as $\varepsilon \to 0$.

Here we should note that (63) is based in L^1 type norms and the limit of a convergent bounded sequence in L^1 may not be a L^1 function but a singular measure. To overcome the difficulty, we can utilize the technique in [17] to obtain the weak regular limit function by using high-order regular sequence (63). That is, ϕ has the bound of

$$\sum_{k=0}^{i} \left[\int_{B_{1-\varepsilon_1}} \left(\sum_{j=0}^{n+5-k} f^j \right) |\nabla^k_{\mathbf{Q}} \phi(\mathbf{Q})| d\mathbf{Q} \right] \tag{64}$$

being independent of $\varepsilon_1 \in (0, 1/2)$. Now we see that the regularity index in (64) is i instead of $i+1$ in (63). Moreover, ϕ also satisfies (54) for $\mathbf{Q} \in B_{1-\varepsilon_1}$ because of $\mathbf{g}_\varepsilon(\mathbf{Q})) = f(|\mathbf{Q}|^2)\mathbf{Q}$ for $\mathbf{Q} \in B_{1-\varepsilon_1}$ being no singularity. By the property of (64), we know that when $\varepsilon_1 \to 0$,

$$\sum_{k=0}^{i} \left[\int_{B_{1-\varepsilon_1}} \left(\sum_{j=0}^{n+5-k} f^j \right) |\nabla_{\mathbf{Q}}^k \phi(\mathbf{Q})| d\mathbf{Q} \right]$$

converge to a singular integral

$$\sum_{k=0}^{i} \left[\int_{B_1} \left(\sum_{j=0}^{n+5-k} f^j \right) |\nabla_{\mathbf{Q}}^k \phi(\mathbf{Q})| d\mathbf{Q} \right].$$

Hence, ϕ satisfies (50) in the sense of the weighted integral with respect to the variable $\mathbf{Q} \in B_1$ and $\phi \in \mathcal{X}_{n,i}$ as $\varepsilon_1 \to 0$. Using the same argument and the definition of $w_\varepsilon(\mathbf{Q})$, we have

$$\sum_{k=0}^{i} \left[\int_{B_R \cap B_1^c} |\nabla_{\mathbf{Q}}^k \phi(\mathbf{Q})| d\mathbf{Q} \right] = 0, \quad \forall R > 1,$$

then ϕ is almost everywhere zero on the domain of B_1^c. □

Next we state and prove a crucial lemma which will be used frequently in the paper.

Lemma 7. *For* $1 - \varepsilon < \delta \leqq 1$, *there exists a positive constant* ε_0 *such that*

$$\int_{B_\delta} \left(\frac{1}{1-|\mathbf{Q}|^2} \right)^{n+1} \phi_{N,\varepsilon} d\mathbf{Q} \leqq \mu \int_{B_\delta} |\mathbf{Q}|^2 \left(\frac{1}{1-|\mathbf{Q}|^2} \right)^{n+2} \phi_{N,\varepsilon} d\mathbf{Q} + C \int_{B_\delta} \left(\frac{1}{1-|\mathbf{Q}|^2} \right)^{n} \phi_{N,\varepsilon} d\mathbf{Q}, \tag{65}$$

for small ε, *where* $\mu > \varepsilon_0$ *and* C *only depends on* ε_0.

Proof. Choosing $\varepsilon < a < 1/4$ gives

$$\begin{aligned} &\int_{B_\delta} \left(\frac{1}{1-|\mathbf{Q}|^2} \right)^{n+1} \phi_{N,\varepsilon} d\mathbf{Q} \\ &= \int_{|\mathbf{Q}| \leqq \delta - a} \left(\frac{1}{1-|\mathbf{Q}|^2} \right)^{n+1} \phi_{N,\varepsilon} d\mathbf{Q} \\ &\quad + \int_{\delta - a < |\mathbf{Q}| \leqq \delta} (1-|\mathbf{Q}|^2) \left(\frac{1}{1-|\mathbf{Q}|^2} \right)^{n+2} \phi_{N,\varepsilon} d\mathbf{Q} \\ &\leqq \frac{1}{1-(\delta-a)^2} \int_{|\mathbf{Q}| \leqq \delta - a} \left(\frac{1}{1-|\mathbf{Q}|^2} \right)^{n} \phi_{N,\varepsilon} d\mathbf{Q} \\ &\quad + (1-(\delta-a)^2) \int_{\delta - a < |\mathbf{Q}| \leqq \delta} \left(\frac{1}{1-|\mathbf{Q}|^2} \right)^{n+2} \phi_{N,\varepsilon} d\mathbf{Q}. \end{aligned}$$

Since $a < 1/4$ and $\delta > 1 - \varepsilon > 1/2$ by $\varepsilon \in (0, 1/2)$ and the fact

$$\begin{aligned} 1-(\delta-a)^2 &\leqq [1+a-(1-\varepsilon)]\,[1+\delta-a] && (\text{since } \delta > 1-\varepsilon) \\ &\leqq 2\,[1+a-(1-\varepsilon)] && (\text{since } \delta \leqq 1) \\ &= 2\,[a+\varepsilon)] \\ &\leqq 4a && (\text{since } a > \varepsilon). \end{aligned}$$

Using $\delta - a < 1/4$, then $4a/(\delta-a)^2 \leqq 64a < \mu$, where we choose $\mu > \varepsilon_0 = 64a$. Thus $1-(\delta-a)^2 \leqq 4a \leqq \mu(\delta-a)^2$. Therefore,

$$\begin{aligned} &(1-(\delta-a)^2)\int_{\delta-a<|\mathbf{Q}|\leqq\delta}\left(\frac{1}{1-|\mathbf{Q}|^2}\right)^{n+2}\phi_{N,\varepsilon}d\mathbf{Q} \\ &\leqq \mu\int_{\delta-a<|\mathbf{Q}|\leqq\delta}|\mathbf{Q}|^2\left(\frac{1}{1-|\mathbf{Q}|^2}\right)^{n+2}\phi_{N,\varepsilon}d\mathbf{Q} \\ &\leqq \mu\int_{B_\delta}|\mathbf{Q}|^2\left(\frac{1}{1-|\mathbf{Q}|^2}\right)^{n+2}\phi_{N,\varepsilon}d\mathbf{Q}. \end{aligned}$$

Meanwhile, we choose $C \geqq \frac{1}{a(1-a)}$, then we have

$$\begin{aligned} &\frac{1}{1-(\delta-a)^2}\int_{|\mathbf{Q}|\leqq\delta-a}\left(\frac{1}{1-|\mathbf{Q}|^2}\right)^{n}\phi_{N,\varepsilon}d\mathbf{Q} \\ &\leqq C\int_{|\mathbf{Q}|\leqq\delta-a}\left(\frac{1}{1-|\mathbf{Q}^2|}\right)^{n}\phi_{N,\varepsilon}d\mathbf{Q} \\ &\leqq C\int_{B_\delta}\left(\frac{1}{1-|\mathbf{Q}|^2}\right)^{n}\phi_{N,\varepsilon}d\mathbf{Q}. \end{aligned}$$

Combining the above inequalities yields the result (65). □

3.1.1. The estimate of $\boldsymbol{\phi_{N,\varepsilon}}$. Let $\delta = 1-\varepsilon$ and denote $D_1 = 1/(2De)$, $D_2 = 1/(2bDe)$. Multiplying (56) with $(w_\varepsilon(\mathbf{Q}))^n$ (also denote by w_ε^n) and integrating by parts yields

$$\begin{aligned} &\frac{\partial}{\partial t}\int_{\mathbb{R}^d} w_\varepsilon^n\phi_{N,\varepsilon}d\mathbf{Q} = \frac{\partial}{\partial t}\left(\int_{B_\delta}+\int_{B_\delta^c}\right)w_\varepsilon^n\phi_{N,\varepsilon}d\mathbf{Q} \\ &= D_2\left(\int_{B_\delta}+\int_{B_\delta^c}\right)\phi_{N,\varepsilon}\triangle_{\mathbf{Q}}w_\varepsilon^n d\mathbf{Q} \\ &\quad+\left(\int_{B_\delta}+\int_{B_\delta^c}\right)\left[\chi_N(\mathbf{Q})\phi_{N,\varepsilon}\left\{\kappa\cdot\mathbf{Q}-D_1\mathbf{g}_\varepsilon(\mathbf{Q})\right\}\cdot\nabla_{\mathbf{Q}}w_\varepsilon^n\right]d\mathbf{Q} \end{aligned}$$

$$
\begin{aligned}
&= D_2 \int_{B_\delta} \phi_{N,\varepsilon} \left[4n(n+1)|\mathbf{Q}|^2 f^{n+2} + 2nd\ f^{n+1} \right] d\mathbf{Q} \\
&\quad + \int_{B_\delta} \chi_N(\mathbf{Q}) \phi_{N,\varepsilon} \{\kappa \cdot \mathbf{Q} - D_1 f \mathbf{Q}\} \cdot 2n f^{n+1} \mathbf{Q} d\mathbf{Q} \\
&\quad + D_2 \int_{B_\delta^c} n A_2 w_\varepsilon^{n-1} \phi_{N,\varepsilon} [(d-1)/|\mathbf{Q}|]\, d\mathbf{Q} \\
&\quad + D_2 \int_{B_\delta^c} n(n-1) A_2^2 w_\varepsilon^{n-2} \phi_{N,\varepsilon} d\mathbf{Q} \\
&\quad + \int_{B_\delta^c} n A_2 w_\varepsilon^{n-1} \chi_N(\mathbf{Q}) \phi_{N,\varepsilon} \{\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q})\} \cdot [\mathbf{Q}/|\mathbf{Q}|]\, d\mathbf{Q},
\end{aligned}
$$

where we have used the continuity of w_ε and its gradient at $|\mathbf{Q}| = 1 - \varepsilon$. Since $\chi_N(\mathbf{Q}) = 1$ in B_1 for all $N \in \mathbb{N}$, we obtain by virtue of Lemma 7 that

$$
\begin{aligned}
&\frac{\partial}{\partial t} \int_{\mathbb{R}^d} w_\varepsilon^n \phi_{N,\varepsilon} d\mathbf{Q} + 2nD_1 \left(1 - \frac{2(n+1)}{b}\right) \int_{B_\delta} \phi_{N,\varepsilon} |\mathbf{Q}|^2 f^{n+2} d\mathbf{Q} \\
&\qquad + \int_{B_\delta^c} n A_2 D_1 \chi_N(\mathbf{Q}) \phi_{N,\varepsilon} w_\varepsilon^{n-1} f_\varepsilon(|\mathbf{Q}|^2) |\mathbf{Q}| d\mathbf{Q} \\
&\leqq \mu \int_{B_\delta} \phi_{N,\varepsilon} |\mathbf{Q}|^2 f^{n+2} d\mathbf{Q} \\
&\qquad + C D_2 \int_{B_\delta} \phi_{N,\varepsilon} 2nd\ f^n d\mathbf{Q} + C|\kappa| \int_{B_\delta} \chi_N(\mathbf{Q}) \phi_{N,\varepsilon} 2n f^n d\mathbf{Q} \\
&\qquad + D_2 \int_{B_\delta^c} n A_2 w_\varepsilon^{n-1} \phi_{N,\varepsilon} [(d-1)/|\mathbf{Q}|]\, d\mathbf{Q} \\
&\qquad + D_2 \int_{B_\delta^c} n(n-1) A_2^2 w_\varepsilon^{n-2} \phi_{N,\varepsilon} d\mathbf{Q} \\
&\qquad + |\kappa| \int_{B_\delta^c} n A_2 \chi_N(\mathbf{Q}) \phi_{N,\varepsilon} w_\varepsilon^{n-1} |\mathbf{Q}| d\mathbf{Q}. \qquad (66)
\end{aligned}
$$

By the definitions of A_2, B_δ^c, $f_\varepsilon(|\mathbf{Q}|^2)$ and $\chi_N(\mathbf{Q})$, we claim that when we choose a suitably small ε, the last three terms on the right-hand side of (66) will be bounded by the term $\int_{B_\delta^c} n A_2 D_1 \chi_N(\mathbf{Q}) \phi_{N,\varepsilon} w_\varepsilon^{n-1} f_\varepsilon(|\mathbf{Q}|^2) |\mathbf{Q}| d\mathbf{Q}$, i.e. the last term on the left-hand side of (66). Now we will show this one by one.

(i) Firstly, choose a constant $0 < \varrho < \frac{1}{2}$, $\varrho f_\varepsilon(|\mathbf{Q}|^2)|\mathbf{Q}| \geqq \varrho \frac{1-\varepsilon}{\varepsilon(2-\varepsilon)}$ as $|\mathbf{Q}| > 1-\varepsilon$. So for every $b > 0$, we can choose $\varepsilon > 0$ such that $\varrho f_\varepsilon(|\mathbf{Q}|^2)|\mathbf{Q}| \geqq 2\frac{d-1}{b}$. Thus,

$$
\begin{aligned}
&\varrho \int_{B_\delta^c} n A_2 D_1 \chi_N(\mathbf{Q}) \phi_{N,\varepsilon} w_\varepsilon^{n-1} f_\varepsilon(|\mathbf{Q}|^2) |\mathbf{Q}| d\mathbf{Q} \\
&\qquad \geqq D_2 \int_{B_\delta^c} n A_2 w_\varepsilon^{n-1} \phi_{N,\varepsilon} [(d-1)/|\mathbf{Q}|]\, d\mathbf{Q}. \qquad (67)
\end{aligned}
$$

(ii) Similarly, for the uniform upper bound of $|\kappa|$, we can also choose $\varepsilon > 0$ such that $\varrho f_\varepsilon(|\mathbf{Q}|^2) \geqq \varrho \frac{1}{\varepsilon(2-\varepsilon)} \geqq \frac{|\kappa|}{D_1}$ for $|\mathbf{Q}| > 1-\varepsilon$. Thus

$$\varrho \int_{B_\delta^c} nA_2 D_1 \chi_N(\mathbf{Q})\phi_{N,\varepsilon} w_\varepsilon^{n-1} f_\varepsilon(|\mathbf{Q}|^2)|\mathbf{Q}|d\mathbf{Q} \geqq |\kappa| \int_{B_\delta^c} nA_2 \chi_N(\mathbf{Q})\phi_{N,\varepsilon} w_\varepsilon^{n-1}|\mathbf{Q}|d\mathbf{Q}. \tag{68}$$

(iii) Below we will prove the following inequality:

$$(1-2\varrho)\int_{B_\delta^c} nA_2 D_1 \chi_N(\mathbf{Q})\phi_{N,\varepsilon} w_\varepsilon^{n-1} f_\varepsilon(|\mathbf{Q}|^2)|\mathbf{Q}|d\mathbf{Q} \geqq D_2 \int_{B_\delta^c} n(n-1)A_2^2 w_\varepsilon^{n-2}\phi_{N,\varepsilon} d\mathbf{Q}. \tag{69}$$

It will be established by considering the following three possible cases.

(i) For $1-\varepsilon < |\mathbf{Q}| \leqq 1$, we know $\chi_N(\mathbf{Q}) = 1$ and

$$(1-2\varrho)|\mathbf{Q}| f_\varepsilon(|\mathbf{Q}|^2) w_\varepsilon \geqq (1-2\varrho)\frac{1-\varepsilon}{\varepsilon(2-\varepsilon)} w_\varepsilon \geqq (1-2\varrho)\frac{1-\varepsilon}{\varepsilon(2-\varepsilon)}\frac{1}{\varepsilon(2-\varepsilon)}.$$

Therefore,

$$(1-2\varrho)\chi_N(\mathbf{Q})|\mathbf{Q}| f_\varepsilon(|\mathbf{Q}|^2) w_\varepsilon \geqq \frac{n-1}{b} A_2 \tag{70}$$

provided that

$$\frac{2(n-1)}{b} \leqq 1-2\varrho. \tag{71}$$

This implies that when b, n satisfy (71), we have (69) for $1-\varepsilon < |\mathbf{Q}| \leqq 1$.

(ii) When $|\mathbf{Q}| \geqq N$, by virtue of $\chi_N(\mathbf{Q}) = \chi(\mathbf{Q}/N)$ and $\chi(\mathbf{Q}) = |\mathbf{Q}|^{-1}$ for large $|\mathbf{Q}|$, we have $\chi_N(\mathbf{Q})|\mathbf{Q}| = N$. Moreover, $f_\varepsilon(|\mathbf{Q}|^2) \geqq \frac{4}{\varepsilon(4-\varepsilon)}$. So for any fixed b and n, it is easy to see that

$$(1-2\varrho)\chi_N(\mathbf{Q})|\mathbf{Q}| f_\varepsilon(|\mathbf{Q}|^2) w_\varepsilon \geqq (1-2\varrho)N\frac{4}{\varepsilon(4-\varepsilon)}[A_1 + (N-1)A_2] \geqq \frac{n-1}{b} A_2$$

is true for large N. This shows that (69) is true when $|\mathbf{Q}| \gg 1$.

(iii) When $1 < |\mathbf{Q}| < N$, $\chi_N(\mathbf{Q})|\mathbf{Q}| \geqq 1$ (since $\chi(\mathbf{Q})|\mathbf{Q}| \geqq 1$ in $B_1^c = \mathbb{R}^d \backslash B_1$),

$$\begin{aligned}(1-2\varrho)\chi_N(\mathbf{Q})|\mathbf{Q}|f_\varepsilon(|\mathbf{Q}|^2)w_\varepsilon &\geqq (1-2\varrho)\frac{4}{\varepsilon(4-\varepsilon)}\left[A_1+\varepsilon A_2\right]\\ &\geqq (1-2\varrho)\frac{4}{\varepsilon(4-\varepsilon)}\left[\frac{1}{\varepsilon(2-\varepsilon)}+\frac{2(1-\varepsilon)}{\varepsilon(2-\varepsilon)^2}\right]\\ &\geqq \frac{n-1}{b}\frac{2(1-\varepsilon)}{\varepsilon^2(2-\varepsilon)^2}\end{aligned}$$

for $b \geqq 2(n-1)$ fixed and for suitable $0 < \varrho < 1/2$ and small ε. Thus (69) is true. This completes the proof for (69).

The results (67)–(69) confirm our earlier claim by choosing ε suitably and by assuming (71). Now we will weaken the condition (71) by replacing it with

$$b > 2(n-1). \tag{72}$$

By choosing suitably a $0 < \varrho \ll 1$ we can prove (71) for $b > 2(n-1)$. This is also why we introduce a parameter ϱ. By the above analysis, we have shown that the inequality (66) can be rewritten as

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d} w_\varepsilon^n \phi_{N,\varepsilon} d\mathbf{Q} \leqq C(n,d,D_2,\kappa)\int_{\mathbb{R}^d} w_\varepsilon^n \phi_{N,\varepsilon} d\mathbf{Q} \tag{73}$$

provided that

$$b > 2(n+1), \tag{74}$$

where (74) implies (72) and $2nD_1\left(1-\frac{2(n+1)}{b}\right) > \mu$, here we choose $\mu = \frac{3}{2}\varepsilon_0$, $\varepsilon_0 = nD_1(1-2(n+1)/b)$. The positive constant $C(n,d,D_2,\kappa)$ in (73) is independent of ε. The Gronwall inequality yields

$$\int_{\mathbb{R}^d} w_\varepsilon^n \phi_{N,\varepsilon} d\mathbf{Q} \in L^\infty([0,T], L^\infty(\Omega)) \tag{75}$$

under the assumptions of (74) and

$$\int_{\mathbb{R}^d} w_\varepsilon^n \bar{\psi}_0 d\mathbf{Q} \in L^\infty(\Omega)). \tag{76}$$

By the definition (53) of $\bar{\psi}_0$ and the definition (62) of w_ε we know that

$$\int_{\mathbb{R}^d} w_\varepsilon^n \bar{\psi}_0 d\mathbf{Q} = \int_{B_1} w_\varepsilon^n \psi_0 d\mathbf{Q} \leqq \int_{B_1} f^n \psi_0 d\mathbf{Q}. \tag{77}$$

This shows that $\int_{\mathbb{R}^d} w_\varepsilon^n \bar{\psi}_0 d\mathbf{Q}$ is bounded by being independent of ε if the condition **(A3)** is satisfied. In the following the other uniform bounds of the derivatives to $\bar{\psi}_0$ follow a similar argument.

3.1.2. The estimate of $\nabla_{\mathbf{Q}}^m \phi_{N,\varepsilon}$ $(m = 1, 2, 3)$. Differentiating (56) with respect to Q_i gives the following equation for $\phi_{N,\varepsilon}^{(i;0)}$:

$$\frac{\partial}{\partial t}\phi_{N,\varepsilon}^{(i;0)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q}))\phi_{N,\varepsilon}^{(i;0)}\right] + D_2 \triangle_{\mathbf{Q}} \phi_{N,\varepsilon}^{(i;0)} + \Theta_i \tag{78}$$

$$\phi_{N,\varepsilon}^{(i;0)}(\alpha, \mathbf{Q}, 0) = \partial_{Q_i} \bar{\psi}_0(\alpha, \mathbf{Q}), \tag{79}$$

where

$$\begin{aligned}\Theta_i &= -\left\{\partial_{Q_i}\chi_N\left[\kappa_{jk} \cdot Q_k - D_1 g_{\varepsilon j}(\mathbf{Q})\right]\right. \\ &\quad \left. + \chi_N(\mathbf{Q})\left[\kappa_{ji} - D_1 \partial_{Q_i} g_{\varepsilon j}(\mathbf{Q})\right]\right\}\phi_{N,\varepsilon}^{(j;0)} \\ &\quad - \nabla_{\mathbf{Q}} \cdot \left\{\partial_{Q_i}\left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q}))\right]\right\}\phi_{N,\varepsilon} \\ &= \Theta_i^+ + \Theta_i^-. \end{aligned} \tag{80}$$

Here $\Theta_i^+ = \Theta_i \vee 0$ and $\Theta_i^- = -(\Theta_i \wedge 0)$. Now we decompose $\phi_{N,\varepsilon}^{(i;0)} = \phi_{N+,\varepsilon}^{(i;0)} - \phi_{N-,\varepsilon}^{(i;0)}$, where $\phi_{N+,\varepsilon}^{(i;0)}$ and $\phi_{N-,\varepsilon}^{(i;0)}$ are the solutions of the following problems,

$$\frac{\partial}{\partial t}\phi_{N+,\varepsilon}^{(i;0)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q}))\phi_{N+,\varepsilon}^{(i;0)}\right] + D_2 \triangle_{\mathbf{Q}} \phi_{N+,\varepsilon}^{(i;0)} + \Theta_i^+ \tag{81}$$

$$\phi_{N+,\varepsilon}^{(i;0)}(\alpha, \mathbf{Q}, 0) = \max\{\partial_{Q_i} \bar{\psi}_0(\alpha, \mathbf{Q}), 0\}, \tag{82}$$

and

$$\frac{\partial}{\partial t}\phi_{N-,\varepsilon}^{(i;0)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q}))\phi_{N-,\varepsilon}^{(i;0)}\right] + D_2 \triangle_{\mathbf{Q}} \phi_{N-,\varepsilon}^{(i;0)} + \Theta_i^- \tag{83}$$

$$\phi_{N-,\varepsilon}^{(i;0)}(\alpha, \mathbf{Q}, 0) = \max\{-\partial_{Q_i} \bar{\psi}_0(\alpha, \mathbf{Q}), 0\}, \tag{84}$$

respectively. Since $\phi_{N+,\varepsilon}^{(i;0)}$ and $\phi_{N-,\varepsilon}^{(i;0)}$ are both positive, we can now proceed with the estimate of $\phi_{N,\varepsilon}$. We obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\mathbb{R}^d} w_\varepsilon^n \phi_{N+,\varepsilon}^{(i;0)} d\mathbf{Q} + 2nD_1\left(1 - \frac{2(n+1)}{b}\right)\int_{B_\delta} \phi_{N+,\varepsilon}^{(i;0)} |\mathbf{Q}|^2 f^{n+2} d\mathbf{Q} \\ &\quad + \int_{B_\delta^c} nD_1 A_2 \chi_N(\mathbf{Q}) \phi_{N+,\varepsilon}^{(i;0)} w_\varepsilon^{n-1} f_\varepsilon(|\mathbf{Q}|^2)|\mathbf{Q}| d\mathbf{Q} \\ &\leqq \mu \int_{B_\delta} \phi_{N+,\varepsilon}^{(i;0)} |\mathbf{Q}|^2 f^{n+2} d\mathbf{Q}\end{aligned}$$

$$
\begin{aligned}
&+CD_2\int_{B_\delta}\phi_{N+,\varepsilon}^{(i;0)}2nd\,f^n d\mathbf{Q}+C|\kappa|\int_{B_\delta}\chi_N(\mathbf{Q})\phi_{N+,\varepsilon}^{(i;0)}2nf^n d\mathbf{Q}\\
&+D_2\int_{B_\delta^c}nA_2w_\varepsilon^{n-1}\phi_{N+,\varepsilon}^{(i;0)}\left[(d-1)/|\mathbf{Q}|\right]d\mathbf{Q}\\
&+D_2\int_{B_\delta^c}n(n-1)A_2^2w_\varepsilon^{n-2}\phi_{N+,\varepsilon}^{(i;0)}d\mathbf{Q}\\
&+|\kappa|\int_{B_\delta^c}nA_2\chi_N(\mathbf{Q})\phi_{N+,\varepsilon}^{(i;0)}w_\varepsilon^{n-1}|\mathbf{Q}|d\mathbf{Q}+\int_{\mathbb{R}^d}w_\varepsilon^n\Theta_i^+d\mathbf{Q}. \qquad (85)
\end{aligned}
$$

By using (80), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d}w_\varepsilon^n\sum_i(\Theta_i^++\Theta_i^-)d\mathbf{Q}\\
&\leqq\int_{\mathbb{R}^d}\sum_i\left(w_\varepsilon^n\phi_{N,\varepsilon}\left|\nabla_{\mathbf{Q}}\cdot\{\partial_{Q_i}\left[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-D_1\mathbf{g}_\varepsilon(\mathbf{Q}))\right]\}\right|\right)d\mathbf{Q}\\
&\quad+\int_{\mathbb{R}^d}\sum_i\Bigg(w_\varepsilon^n(\phi_{N+,\varepsilon}^{(j;0)}+\phi_{N-,\varepsilon}^{(j;0)})\left\{|\partial_{Q_i}\chi_N|(|\kappa_{jk}||Q_k|+D_1|g_{\varepsilon j}(\mathbf{Q})|)\right.\\
&\quad\left.+\chi_N(\mathbf{Q})(|\kappa_{ji}|+\underbrace{D_1|\partial_{Q_i}g_{\varepsilon j}(\mathbf{Q})|}))\right\}\Bigg)d\mathbf{Q}. \qquad (86)
\end{aligned}
$$

Hereinafter the term underlined "$\underbrace{}$" will affect the condition imposed on b (later it will be seen from the condition on b). Now we start to estimate the right-hand side of (86) in domains B_δ and B_δ^c, respectively:

$$
\begin{aligned}
\int_{B_\delta}\bullet\,d\mathbf{Q}\leqq\ &C|\kappa|\int_{B_\delta}\phi_{N,\varepsilon}f^n d\mathbf{Q}+C\int_{B_\delta}\phi_{N,\varepsilon}f^{n+1}d\mathbf{Q}\\
&+C\int_{B_\delta}\phi_{N,\varepsilon}\left[f^{n+2}+f^{n+3}\right]d\mathbf{Q}\\
&+C\int_{B_\delta}\left(\sum_i|\phi_{N,\varepsilon}^{(i;0)}|\right)\left(|\kappa|f^n+d\ f^{n+1}\right)d\mathbf{Q}\\
&+\int_{B_\delta}\left(\sum_i|\phi_{N,\varepsilon}^{(i;0)}|\right)\left(|\kappa|f^n+d\ f^{n+1}+D_1(d+1)\,|\mathbf{Q}|^2f^{n+2}\right)d\mathbf{Q}.
\end{aligned}
\qquad (87)
$$

The last term in the fifth integral is from the term underlined "$\underbrace{}$" in (86). The first three terms on the right-hand side of (87) can be estimated by a constant independent of ε times $\int_{\mathbb{R}^d}(\sum_{i=n}^{n+3}w_\varepsilon^i)\phi_{N,\varepsilon}d\mathbf{Q}$, by using the same arguments as that in Section 3.1.1. The terms with the factor f^{n+1} in the last two integrals of (87) can be estimated by utilizing Lemma 7 with $\phi_{N,\varepsilon}$ replaced by $\sum_i|\phi_{N,\varepsilon}^{(i;0)}|$. The last term in the fifth integral can be controlled by putting some conditions on b (the restriction (91) below). For the estimate of $\int_{B_\delta^c} \bullet\, d\mathbf{Q}$, observe

$$\int_{B_\delta^c} \bullet\, d\mathbf{Q} \leqq (|\kappa| + D_1 f_\varepsilon(|\mathbf{Q}|^2)) \int_{B_\delta^c} w_\varepsilon^n \phi_{N,\varepsilon} d\mathbf{Q}$$
$$+(|\kappa| + D_1 f_\varepsilon(|\mathbf{Q}|^2)) \int_{B_\delta^c} w_\varepsilon^n \left(\sum_i |\phi_{N,\varepsilon}^{(i;0)}|\right) d\mathbf{Q},$$

as N is sufficiently large. The first integral can be controlled by $\int_{\mathbb{R}^d} (\sum_{i=n}^{n+3} w_\varepsilon^i)\phi_{N,\varepsilon}\, d\mathbf{Q}$ from Section 3.1.1. Note when ε is sufficiently small, the second integral may be controlled by $\int_{B_\delta^c} nD_1 A_2 \chi_N(\mathbf{Q})(\sum_i |\phi_{N,\varepsilon}^{(i;0)}|) w_\varepsilon^{n-1} f_\varepsilon(|\mathbf{Q}|^2)|\mathbf{Q}| d\mathbf{Q}$ related to the third integral on the left-hand side of (85) since $n \geqq 7$. Similar estimates can be done for all $\phi_{N-,\varepsilon}^{(i;0)}$. Summing over all the indices i and the $+, -$ equations, we obtain an inequality for $\int_{\mathbb{R}^d} w_\varepsilon^n \sum_i |\phi_{N,\varepsilon}^{(i;0)}| d\mathbf{Q}$. More precisely, we obtain

$$\int_{\mathbb{R}^d} w_\varepsilon^n \sum_i |\phi_{N,\varepsilon}^{(i;0)}| d\mathbf{Q} \in L^\infty([0,T]; L^\infty(\Omega)) \tag{88}$$

provided that

$$\int_{\mathbb{R}^d} w_\varepsilon^n \sum_i |\partial_{Q_i} \bar{\psi}_0| d\mathbf{Q} \in L^\infty(\Omega) \tag{89}$$

$$\int_{\mathbb{R}^d} \left(\sum_{i=n}^{n+3} w_\varepsilon^i\right) \bar{\psi}_0 d\mathbf{Q} \in L^\infty(\Omega) \tag{90}$$

when

$$\frac{2n}{2De}\left(1 - \frac{2(n+1)}{b}\right) > \frac{d+1}{2De} \text{ i.e. } b > \frac{4n(n+1)}{2n-d-1}. \tag{91}$$

The requirement (91) is similar to (74) and in fact, (91) includes (74). The regularity requirement (90) is required to estimate $\int_{\mathbb{R}^d} (\sum_{i=n}^{n+3} w_\varepsilon^i)\phi_{N,\varepsilon} d\mathbf{Q}$ in Section 3.1.1. It can be shown that (89) and (90) are satisfied if

$$\psi_0 \in L^\infty(\Omega, \mathcal{X}_{n,1}). \tag{92}$$

Similarly, we can differentiate (78) with respect to Q_m to obtain the following equation for $\phi_{N,\varepsilon}^{(i,m;0)}$:

$$\frac{\partial}{\partial t}\phi_{N,\varepsilon}^{(i,m;0)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q}))\phi_{N,\varepsilon}^{(i,m;0)}\right]$$
$$+D_2 \triangle_{\mathbf{Q}} \phi_{N,\varepsilon}^{(i,m;0)} + \Xi_{im} \tag{93}$$

$$\phi_{N,\varepsilon}^{(i;0)}(\alpha, \mathbf{Q}, 0) = \partial_{Q_i} \bar{\psi}_0(\alpha, \mathbf{Q}), \tag{94}$$

where

$$\begin{aligned}
\Xi_{im} &= -\{\nabla_{\mathbf{Q}}\cdot[\,\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-\underbrace{D_1\mathbf{g}_\varepsilon(\mathbf{Q})}))]\,\}\phi_{N,\varepsilon}^{(i,m;0)} \\
&\quad -\partial_{Q_m}\{\nabla_{\mathbf{Q}}\cdot[\,\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-D_1\mathbf{g}_\varepsilon(\mathbf{Q}))]\,\}\phi_{N,\varepsilon}^{(i;0)}+\partial_{Q_m}\Theta_i \\
&= -\{\nabla_{\mathbf{Q}}\cdot[\,\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-\underbrace{D_1\mathbf{g}_\varepsilon(\mathbf{Q})}))]\,\}\phi_{N,\varepsilon}^{(i,m;0)} \\
&\quad -\partial_{Q_m}\{\nabla_{\mathbf{Q}}\cdot[\,\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-D_1\mathbf{g}_\varepsilon(\mathbf{Q}))]\,\}\phi_{N,\varepsilon}^{(i;0)} \\
&\quad -\partial_{Q_m}\{\partial_{Q_i}\chi_N[\,\kappa_{jk}\cdot Q_k-D_1 g_{\varepsilon j}(\mathbf{Q})] \\
&\quad +\chi_N(\mathbf{Q})[\,\kappa_{ji}-D_1\partial_{Q_i}g_{\varepsilon j}(\mathbf{Q})]\,\}\phi_{N,\varepsilon}^{(j;0)} \\
&\quad -\{\partial_{Q_i}\chi_N[\,\kappa_{jk}\cdot Q_k-D_1 g_{\varepsilon j}(\mathbf{Q})] \\
&\quad +\chi_N(\mathbf{Q})[\,\kappa_{ji}-\underbrace{D_1\partial_{Q_i}g_{\varepsilon j}(\mathbf{Q})}]\,\}\phi_{N,\varepsilon}^{(j,m;0)} \\
&\quad -\partial_{Q_m}\nabla_{\mathbf{Q}}\cdot\{\partial_{Q_i}[\,\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-D_1\mathbf{g}_\varepsilon(\mathbf{Q}))]\,\}\phi_{N,\varepsilon} \\
&\quad -\nabla_{\mathbf{Q}}\cdot\{\partial_{Q_i}[\,\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-D_1\mathbf{g}_\varepsilon(\mathbf{Q}))]\,\}\partial_{Q_m}\phi_{N,\varepsilon} \\
&= \Xi^+ + \Xi^-.
\end{aligned} \tag{95}$$

Using the same techniques used above, we can obtain

$$\int_{\mathbb{R}^d} w_\varepsilon^n \sum_{i,m} |\phi_{N,\varepsilon}^{(i,m;0)}|d\mathbf{Q} \in L^\infty([\,0,T]\,;L^\infty(\Omega)) \tag{96}$$

provided that

$$\int_{\mathbb{R}^d} w_\varepsilon^n \sum_{i,m} |\partial_{Q_i}\partial_{Q_m}\bar{\psi}_0|d\mathbf{Q} \quad \in L^\infty(\Omega) \tag{97}$$

$$\int_{\mathbb{R}^d} \left(\sum_{i=n}^{n+3} w_\varepsilon^i\right)|\partial_{Q_j}\bar{\psi}_0|d\mathbf{Q} \quad \in L^\infty(\Omega) \tag{98}$$

$$\int_{\mathbb{R}^d} \left(\sum_{i=n}^{n+4} w_\varepsilon^i\right)\bar{\psi}_0 d\mathbf{Q} \quad \in L^\infty(\Omega) \tag{99}$$

when

$$\frac{2n}{2De}\left(1-\frac{2(n+1)}{b}\right) > \frac{d+3}{2De} \text{ i.e. } b > \frac{4n(n+1)}{2n-d-3}. \tag{100}$$

The regularity requirements (98)–(99) are used to estimate $\int_{\mathbb{R}^d}(\sum_{i=n}^{n+3} w_\varepsilon^i)|\partial_{Q_j}\phi_{N,\varepsilon}|\,d\mathbf{Q}$ as earlier in this section, and to estimate $\int_{\mathbb{R}^d}(\sum_{i=n}^{n+4} w_\varepsilon^i)\phi_{N,\varepsilon}d\mathbf{Q}$ as in Section 3.1.1, respectively. The conditions (97)–(99) are satisfied if

$$\psi_0 \in L^\infty(\Omega,\mathcal{X}_{n,2}). \tag{101}$$

Similarly, we can obtain

$$\int_{\mathbb{R}^d} w_\varepsilon^n \sum_{i,j,m} |\phi_{N,\varepsilon}^{(i,j,m;0)}|d\mathbf{Q} \in L^\infty([\,0,T]\,;L^\infty(\Omega)) \tag{102}$$

provided that

$$\psi_0 \in L^\infty(\Omega, \mathcal{X}_{n,3}) \tag{103}$$

when

$$\frac{2n}{2De}\left(1 - \frac{2(n+1)}{b}\right) > \frac{d+5}{2De} \quad \text{i.e.} \quad b > \frac{4n(n+1)}{2n-d-5}. \tag{104}$$

3.1.3. The estimate of $(\phi_{N,\varepsilon})_t$. We take the absolute value of the two sides of (56) and multiply it by w_ε^n, and integrate the resulting equation in $\mathbf{Q}$ space. It can be shown that

$$\int_{\mathbb{R}^d} w_\varepsilon^n |(\phi_{N,\varepsilon})_t| d\mathbf{Q} \in L^\infty([0,T]; L^\infty(\Omega)) \tag{105}$$

provided that (100) and (101) are satisfied.

3.1.4. The estimate of $\nabla_\alpha \phi_{N,\varepsilon}$. In the following we will show that $\int_{\mathbb{R}^d} w_\varepsilon^n |\nabla_\alpha \phi_{N,\varepsilon}| \, d\mathbf{Q}$ is bounded. Differentiating (56) with respect to α_i yields the equation

$$\frac{\partial}{\partial t}\phi_{N,\varepsilon}^{(0;i)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q}))\phi_{N,\varepsilon}^{(0;i)}\right] + D_2 \triangle_{\mathbf{Q}} \phi_{N,\varepsilon}^{(0;i)} + \Psi_i \tag{106}$$

$$\phi_N^{(0;i)}(\alpha, \mathbf{Q}, 0) = \partial_{\alpha_i} \bar{\psi}_0(\alpha, \mathbf{Q}), \tag{107}$$

where

$$\begin{aligned}
\Psi_i &= -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q}) \partial_{\alpha_i} \kappa \cdot \mathbf{Q} \phi_{N,\varepsilon}\right] \\
&= -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q}) \partial_{\alpha_i} \kappa \cdot \mathbf{Q}\right] \phi_{N,\varepsilon} - \left[\chi_N(\mathbf{Q}) \partial_{\alpha_i} \kappa \cdot \mathbf{Q}\right] \cdot \nabla_{\mathbf{Q}} \phi_{N,\varepsilon} \\
&= \Psi_i^+ + \Psi_i^-.
\end{aligned} \tag{108}$$

Here Ψ_i^+ and Ψ_i^- have a similar definition as before. Let $\phi_{N,\varepsilon}^{(0;i)} = \phi_{N+,\varepsilon}^{(0;i)} - \phi_{N-,\varepsilon}^{(0;i)}$, where $\phi_{N+,\varepsilon}^{(0;i)}$ and $\phi_{N-,\varepsilon}^{(0;i)}$ are the solutions of the following problems

$$\frac{\partial}{\partial t}\phi_{N\pm,\varepsilon}^{(0;i)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q}))\phi_{N\pm,\varepsilon}^{(0;i)}\right] + D_2 \triangle_{\mathbf{Q}} \phi_{N\pm,\varepsilon}^{(0;i)} + \Psi_i^\pm \tag{109}$$

$$\phi_{N\pm,\varepsilon}^{(0;i)}(\alpha, \mathbf{Q}, 0) = (\partial_{\alpha_i} \bar{\psi}_0)^\pm. \tag{110}$$

Since $\phi_{N+,\varepsilon}^{(0;i)}$ and $\phi_{N-,\varepsilon}^{(0;i)}$ are positive, we can proceed as before. By observing that there are only $\phi_{N,\varepsilon}$ and $\phi_{N,\varepsilon}^{(j;0)}$ terms involved in Ψ_i, and by using $\nabla_\alpha \kappa \in L^2([0,T], L^\infty(\Omega))$, we can obtain

$$\int_{\mathbb{R}^d} w_\varepsilon^n \sum_i |\phi_{N,\varepsilon}^{(0;i)}| d\mathbf{Q} \in L^\infty([0,T]; L^\infty(\Omega)) \tag{111}$$

under the assumptions (91) and

$$\psi_0 \in H^3(\Omega, \mathcal{X}_{n,2}). \tag{112}$$

3.1.5. The estimate of $\nabla_{\mathbf{Q}}\nabla_\alpha\phi_{N,\varepsilon}$ and $\nabla^2_{\mathbf{Q}}\nabla_\alpha\phi_{N,\varepsilon}$. Next we will estimate the mixed derivatives of $\phi_{N,\varepsilon}$. Differentiating (106) with respect to Q_j yields the equation of $\phi^{(j;i)}_{N,\varepsilon}$,

$$\begin{aligned}\frac{\partial}{\partial t}\phi^{(j;i)}_{N,\varepsilon} &= -\nabla_{\mathbf{Q}}\cdot\Big[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-D_1\mathbf{g}_\varepsilon(\mathbf{Q}))\phi^{(j;i)}_{N,\varepsilon}\Big]\\ &\quad +D_2\triangle_{\mathbf{Q}}\phi^{(j;i)}_{N,\varepsilon}+\Lambda_{ji} \qquad (113)\end{aligned}$$

$$\phi^{(j;i)}_{N,\varepsilon}(\alpha,\mathbf{Q},0)=\partial_{Q_j}\partial_{\alpha_i}\bar{\psi}_0(\alpha,\mathbf{Q}), \qquad (114)$$

where

$$\begin{aligned}\Lambda_{ji} &= -\{\partial_{Q_j}\chi_N[\kappa_{lk}\cdot Q_k - D_1 g_{\varepsilon l}(\mathbf{Q})]+\chi_N(\mathbf{Q})[\kappa_{lj}-\underbrace{D_1\partial_{Q_j}g_{\varepsilon l}(\mathbf{Q})}]\}\phi^{(l;i)}_{N,\varepsilon}\\ &\quad -\nabla_{\mathbf{Q}}\cdot\{\partial_{Q_j}[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-D_1\mathbf{g}_\varepsilon(\mathbf{Q}))]\}\phi^{(0;i)}_{N,\varepsilon}\\ &\quad -\nabla_{\mathbf{Q}}\cdot[\partial_{Q_j}\chi_N\partial_{\alpha_i}\kappa\cdot\mathbf{Q}\phi_{N,\varepsilon}]-\partial_{Q_l}[\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa_{lj}\phi_{N,\varepsilon}]\\ &\quad -\nabla_{\mathbf{Q}}\cdot[\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa\cdot\mathbf{Q}\phi^{(j;0)}_{N,\varepsilon}]\\ &= \Lambda^+_{ji}+\Lambda^-_{ji}, \qquad (115)\end{aligned}$$

Here Λ^+_{ji} and Λ^-_{ji} have similar meanings as before. Let $\phi^{(j;i)}_{N,\varepsilon}=\phi^{(j;i)}_{N+,\varepsilon}-\phi^{(j;i)}_{N-,\varepsilon}$, where $\phi^{(j;i)}_{N+,\varepsilon}$ and $\phi^{(j;i)}_{N-,\varepsilon}$ are the solutions of the following problems

$$\begin{aligned}\frac{\partial}{\partial t}\phi^{(j;i)}_{N\pm,\varepsilon} &= -\nabla_{\mathbf{Q}}\cdot\Big[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-D_1\mathbf{g}_\varepsilon(\mathbf{Q}))\phi^{(j;i)}_{N\pm,\varepsilon}\Big]\\ &\quad +D_2\triangle_{\mathbf{Q}}\phi^{(j;i)}_{N\pm,\varepsilon}+\Lambda^\pm_{ji} \qquad (116)\end{aligned}$$

$$\phi^{(j;i)}_{N\pm,\varepsilon}(\alpha,\mathbf{Q},0)=(\partial_{Q_j}\partial_{\alpha_i}\bar{\psi}_0)^\pm. \qquad (117)$$

Since $\phi^{(j;i)}_{N+,\varepsilon}$ and $\phi^{(j;i)}_{N-,\varepsilon}$ are positive, we can proceed as before. The terms involving $\phi^{(l;i)}_{N,\varepsilon}$ in Λ_{ji} can be controlled by the condition (100); the other terms have been estimated in previous sections. Thus, using the fact that $\nabla_\alpha\kappa$ and κ belong to $L^2([0,T],L^\infty(\Omega))$, we find

$$\int_{\mathbb{R}^d} w^n_\varepsilon\sum_{ij}|\phi^{(j;i)}_{N,\varepsilon}|d\mathbf{Q}\in L^\infty([0,T];L^\infty(\Omega)) \qquad (118)$$

under the assumptions (100) and (112).

Similarly, differentiating (113) with respect to Q_m yields the equation for $\phi^{(j,m;i)}_{N,\varepsilon}$,

$$\begin{aligned}\frac{\partial}{\partial t}\phi^{(j,m;i)}_{N,\varepsilon} &= -\nabla_{\mathbf{Q}}\cdot\Big[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-D_1\mathbf{g}_\varepsilon(\mathbf{Q}))\phi^{(j,m;i)}_{N,\varepsilon}\Big]\\ &\quad +D_2\triangle_{\mathbf{Q}}\phi^{(j,m;i)}_{N,\varepsilon}+\daleth_{ji} \qquad (119)\end{aligned}$$

$$\phi^{(j,m;i)}_{N,\varepsilon}(\alpha,\mathbf{Q},0)=\partial_{Q_m}\partial_{Q_j}\partial_{\alpha_i}\bar{\psi}_0(\alpha,\mathbf{Q}), \qquad (120)$$

where

$$
\begin{aligned}
\beth_{ji} &= -\big\{\partial_{Q_j}\chi_N\left[\kappa_{lk}\cdot Q_k - D_1 g_{\varepsilon l}(\mathbf{Q})\right] + \chi_N(\mathbf{Q})\left[\kappa_{lj} - D_1\partial_{Q_j} g_{\varepsilon l}(\mathbf{Q})\right]\big\}\phi_{N,\varepsilon}^{(j,l;i)} \\
&\quad -\nabla_{\mathbf{Q}}\cdot\big\{\partial_{Q_j}\left[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q} - D_1\mathbf{g}_\varepsilon(\mathbf{Q}))\right]\big\}\phi_{N,\varepsilon}^{(j;i)} + \partial_{Q_m}\Lambda_{ji} \\
&= -\big\{\partial_{Q_j}\chi_N\left[\kappa_{lk}\cdot Q_k - D_1 g_{\varepsilon l}(\mathbf{Q})\right] + \chi_N(\mathbf{Q})\big[\kappa_{lj} - \underbrace{D_1\partial_{Q_j} g_{\varepsilon l}(\mathbf{Q})}\,\big]\big\}\phi_{N,\varepsilon}^{(j,l;i)} \\
&\quad -\nabla_{\mathbf{Q}}\cdot\big\{\partial_{Q_j}\left[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q} - D_1\mathbf{g}_\varepsilon(\mathbf{Q}))\right]\big\}\phi_{N,\varepsilon}^{(j;i)} \\
&\quad -\big\{\partial_{Q_j}\chi_N\left[\kappa_{lk}\cdot Q_k - D_1 g_{\varepsilon l}(\mathbf{Q})\right] + \chi_N(\mathbf{Q})\big[\kappa_{lj} - \underbrace{D_1\partial_{Q_j} g_{\varepsilon l}(\mathbf{Q})}\,\big]\big\}\phi_{N,\varepsilon}^{(l,m;i)} \\
&\quad -\partial_{Q_m}\big\{\partial_{Q_j}\chi_N\left[\kappa_{lk}\cdot Q_k - D_1 g_{\varepsilon l}(\mathbf{Q})\right] + \chi_N(\mathbf{Q})\left[\kappa_{lj} - D_1\partial_{Q_j} g_{\varepsilon l}(\mathbf{Q})\right]\big\}\phi_{N,\varepsilon}^{(l;i)} \\
&\quad +\partial_{Q_m}\Big\{-\nabla_{\mathbf{Q}}\cdot\big\{\partial_{Q_j}\left[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q} - D_1\mathbf{g}_\varepsilon(\mathbf{Q}))\right]\big\}\phi_{N,\varepsilon}^{(0;i)} \\
&\quad -\nabla_{\mathbf{Q}}\cdot\left[\partial_{Q_j}\chi_N\partial_{\alpha_i}\kappa\cdot\mathbf{Q}\phi_{N,\varepsilon}\right] - \partial_{Q_l}\left[\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa_{lj}\phi_{N,\varepsilon}\right] \\
&\quad -\nabla_{\mathbf{Q}}\cdot\Big[\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa\cdot\mathbf{Q}\phi_{N,\varepsilon}^{(j;0)}\Big]\Big\}.
\end{aligned} \tag{121}
$$

Similarly, we can obtain the following estimate

$$
\int_{\mathbb{R}^d} w_\varepsilon^n \sum_{i,j,m} |\phi_{N,\varepsilon}^{(j,m;i)}|d\mathbf{Q} \in L^\infty([0,T];L^\infty(\Omega)) \tag{122}
$$

provided that (100) and (112) are satisfied.

3.1.6. The estimate of $(\nabla_\alpha\phi_{N,\varepsilon})_t$. We take the absolute value of both sides of (106) and multiply it by w_ε^n and then integrate the resulting equation in $\mathbf{Q}$. Moreover, we integrate it with respect to α given the factor $\partial_{\alpha_i}\kappa$ of Ψ_i in (106) only belongs to $L^2([0,T], L^\infty(\Omega))$. By using (118) and (122), it can be shown that

$$
\int_{\mathbb{R}^d} w_\varepsilon^n |(\nabla_\alpha\phi_{N,\varepsilon})_t| d\mathbf{Q} \in L^\infty([0,T];L^\infty(\Omega)) \tag{123}
$$

provided (100) and (112) are satisfied.

3.1.7. The estimate of $\nabla_\alpha^2\phi_{N,\varepsilon}$. Next we estimate the high-order derivatives of $\phi_{N,\varepsilon}$ with respect to α. Differentiating (106) with respect to α_j yields the equations

$$
\begin{aligned}
\frac{\partial}{\partial t}\phi_{N,\varepsilon}^{(0;i,j)} &= -\nabla_{\mathbf{Q}}\cdot\Big[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q} - D_1\mathbf{g}_\varepsilon(\mathbf{Q}))\phi_{N,\varepsilon}^{(0;i,j)}\Big] \\
&\quad + D_2\triangle_{\mathbf{Q}}\phi_{N,\varepsilon}^{(0;i,j)} + \Pi_{ij}
\end{aligned} \tag{124}
$$

$$
\phi_{N,\varepsilon}^{(0;i,j)}(\alpha,\mathbf{Q},0) = \partial_{\alpha_i\alpha_j}^2\psi_0^\varepsilon(\alpha,\mathbf{Q}), \tag{125}
$$

where

$$
\begin{aligned}
\Pi_{ij} &= -\nabla_{\mathbf{Q}}\cdot\Big[\chi_N(\mathbf{Q})\partial_{\alpha_i\alpha_j}^2\kappa\cdot\mathbf{Q}\phi_{N,\varepsilon}\Big] - \nabla_{\mathbf{Q}}\cdot\Big[\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa\cdot\mathbf{Q}\phi_{N,\varepsilon}^{(0;j)}\Big] \\
&\quad -\nabla_{\mathbf{Q}}\cdot\Big[\chi_N(\mathbf{Q})\partial_{\alpha_j}\kappa\cdot\mathbf{Q}\phi_{N,\varepsilon}^{(0;i)}\Big] \\
&= \Pi_{ij}^+ + \Pi_{ij}^-.
\end{aligned} \tag{126}
$$

Here Π_{ij}^{+} and Π_{ij}^{-} have similar definitions as before. Again we decompose $\phi_{N,\varepsilon}^{(0;i,j)} = \phi_{N+,\varepsilon}^{(0;i,j)} - \phi_{N-,\varepsilon}^{(0;i,j)}$, where $\phi_{N+,\varepsilon}^{(0;i,j)}$ and $\phi_{N-,\varepsilon}^{(0;i,j)}$ satisfy

$$\frac{\partial}{\partial t}\phi_{N\pm,\varepsilon}^{(0;i,j)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q}))\phi_{N\pm,\varepsilon}^{(0;i,j)}\right] + D_2 \triangle_{\mathbf{Q}} \phi_{N\pm,\varepsilon}^{(0;i,j)} + \Pi_{ij}^{\pm} \tag{127}$$

$$\phi_{N\pm,\varepsilon}^{(0;i,j)}(\alpha, \mathbf{Q}, 0) = (\partial_{\alpha_i\alpha_j}^2 \bar{\psi}_0)^{\pm}. \tag{128}$$

It follows from (126) that

$$\begin{aligned}\int_{\mathbb{R}^d} w_\varepsilon^n \Pi_{ij}^{+} d\mathbf{Q} \leqq |\nabla_\alpha^2 \kappa| \left(\int_{\mathbb{R}^d} w_\varepsilon^n \phi_{N,\varepsilon} d\mathbf{Q} + \int_{\mathbb{R}^d} |\mathbf{Q}| w_\varepsilon^n \sum_k |\phi_{N,\varepsilon}^{(k;0)}| d\mathbf{Q}\right) \\ + |\nabla_\alpha \kappa| \left(\int_{\mathbb{R}^d} w_\varepsilon^n \sum_l |\phi_{N,\varepsilon}^{(0;l)}| d\mathbf{Q} + \int_{\mathbb{R}^d} |\mathbf{Q}| w_\varepsilon^n \sum_{k,l} |\phi_{N,\varepsilon}^{(k;l)}| d\mathbf{Q}\right).\end{aligned} \tag{129}$$

By multiplying both sides of (127) by w_ε^n, and integrating in $\mathbf{Q}$ and summing them altogether, we obtain

$$\begin{aligned}\frac{\partial}{\partial t} \int_{\mathbb{R}^d} w_\varepsilon^n |\phi_{N,\varepsilon}^{(0;i,j)}| d\mathbf{Q} &\leqq C(n, d, D_2, \kappa) \int_{\mathbb{R}^d} w_\varepsilon^n |\phi_{N,\varepsilon}^{(0;i,j)}| d\mathbf{Q} \\ &+ |\nabla_\alpha^2 \kappa| \left(\int_{\mathbb{R}^d} w_\varepsilon^n \phi_{N,\varepsilon} d\mathbf{Q} + \int_{\mathbb{R}^d} |\mathbf{Q}| w_\varepsilon^n \sum_k |\phi_{N,\varepsilon}^{(k;0)}| d\mathbf{Q}\right) \\ &+ |\nabla_\alpha \kappa| \left(\int_{\mathbb{R}^d} w_\varepsilon^n \sum_l |\phi_{N,\varepsilon}^{(0;l)}| d\mathbf{Q} + \int_{\mathbb{R}^d} |\mathbf{Q}| w_\varepsilon^n \sum_{k,l} |\phi_{N,\varepsilon}^{(k;l)}| d\mathbf{Q}\right).\end{aligned} \tag{130}$$

By multiplying (130) by $\int_{\mathbb{R}^d} w_\varepsilon^n |\phi_{N,\varepsilon}^{(0;i,j)}| d\mathbf{Q}$, and integrating the resulting inequality in the α space gives

$$\begin{aligned}&\frac{\partial}{\partial t} \int_\Omega \left(\int_{\mathbb{R}^d} w_\varepsilon^n |\phi_{N,\varepsilon}^{(0;i,j)}| d\mathbf{Q}\right)^2 d\alpha \\ &\leqq C \int_\Omega \left(\int_{\mathbb{R}^d} w_\varepsilon^n |\phi_{N,\varepsilon}^{(0;i,j)}| d\mathbf{Q}\right)^2 d\alpha + C_1 \int_\Omega |\nabla_\alpha^2 \kappa|^2 d\alpha + C_2 \|\nabla_\alpha \kappa\|_{L^\infty}^2.\end{aligned} \tag{131}$$

Here we have used the fact that

$$\begin{matrix}\int_{\mathbb{R}^d} w_\varepsilon^n \phi_{N,\varepsilon} d\mathbf{Q}, & \int_{\mathbb{R}^d} w_\varepsilon^{n+1} \sum_k |\phi_{N,\varepsilon}^{(k;0)}| d\mathbf{Q} \\ \int_{\mathbb{R}^d} w_\varepsilon^n \sum_l |\phi_{N,\varepsilon}^{(0;l)}| d\mathbf{Q}, & \int_{\mathbb{R}^d} w_\varepsilon^{n+1} \sum_{k,l} |\phi_{N,\varepsilon}^{(k;l)}| d\mathbf{Q}\end{matrix} \tag{132}$$

belong to $L^\infty([0,T] \times \Omega)$ and the boundedness of Ω. By using $|\nabla_\alpha \kappa| \in L^2([0,T], L^\infty(\Omega))$ and the Gronwall's inequality, we obtain

$$\phi_{N,\varepsilon} \in L^\infty([0,T]; H^2(\Omega, \mathcal{X}_{n,0}^\varepsilon)), \quad \text{if} \quad \psi_0 \in H^3(\Omega, \mathcal{X}_{n,2}). \tag{133}$$

3.1.8. The estimate of $\nabla_\alpha^3 \phi_{N,\varepsilon}$. Differentiating (124) with respect to α, yields the equations

$$\frac{\partial}{\partial t}\phi_{N,\varepsilon}^{(0;i,j,k)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q}))\phi_{N,\varepsilon}^{(0;i,j,k)}\right] + D_2 \triangle_{\mathbf{Q}} \phi_{N,\varepsilon}^{(0;i,j,k)} + \Upsilon_{ijk} \tag{134}$$

$$\phi_{N,\varepsilon}^{(0;i,j,k)}(\alpha, \mathbf{Q}, 0) = \partial^3_{\alpha_i\alpha_j\alpha_k}\bar{\psi}_0(\alpha, \mathbf{Q}), \tag{135}$$

where

$$\begin{aligned}\Upsilon_{ijk} &= -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})\partial^3_{\alpha_i\alpha_j\alpha_k}\kappa \cdot \mathbf{Q}\phi_{N,\varepsilon}\right] - \nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})\partial^2_{\alpha_i\alpha_j}\kappa \cdot \mathbf{Q}\phi_{N,\varepsilon}^{(0;k)}\right] \\ &\quad -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})\partial^2_{\alpha_i\alpha_k}\kappa \cdot \mathbf{Q}\phi_{N,\varepsilon}^{(0;j)}\right] - \nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})\partial^2_{\alpha_j\alpha_k}\kappa \cdot \mathbf{Q}\phi_{N,\varepsilon}^{(0;i)}\right] \\ &\quad -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa \cdot \mathbf{Q}\phi_{N,\varepsilon}^{(0;j,k)}\right] - \nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})\partial_{\alpha_j}\kappa \cdot \mathbf{Q}\phi_{N,\varepsilon}^{(0;i,k)}\right] \\ &\quad -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})\partial_{\alpha_k}\kappa \cdot \mathbf{Q}\phi_{N,\varepsilon}^{(0;i,j)}\right] \\ &= \Upsilon_{ijk}^+ + \Upsilon_{ijk}^-. \end{aligned} \tag{136}$$

Here Υ_{ijk}^+ and Υ_{ijk}^- have similar definitions as before. Again we decompose $\phi_{N,\varepsilon}^{(0;i,j,k)} = \phi_{N+,\varepsilon}^{(0;i,j,k)} - \phi_{N-,\varepsilon}^{(0;i,j,k)}$, where $\phi_{N+,\varepsilon}^{(0;i,j,k)}$ and $\phi_{N-,\varepsilon}^{(0;i,j,k)}$ satisfy

$$\frac{\partial}{\partial t}\phi_{N\pm,\varepsilon}^{(0;i,j,k)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - D_1 \mathbf{g}_\varepsilon(\mathbf{Q}))\phi_{N\pm,\varepsilon}^{(0;i,j,k)}\right] + D_2 \triangle_{\mathbf{Q}} \phi_{N\pm,\varepsilon}^{(0;i,j,k)} + \Upsilon_{ijk}^\pm \tag{137}$$

$$\phi_{N\pm,\varepsilon}^{(0;i,j,k)}(\alpha, \mathbf{Q}, 0) = (\partial^3_{\alpha_i\alpha_j\alpha_k}\bar{\psi}_0)^\pm. \tag{138}$$

It follows from (136) that

$$\begin{aligned}&\int_{\mathbb{R}^d} w_\varepsilon^n \Upsilon_{ijk}^+ d\mathbf{Q} \\ &\leqq |\nabla_\alpha^3\kappa| \left(\int_{\mathbb{R}^d} w_\varepsilon^n \phi_{N,\varepsilon} d\mathbf{Q} + \int_{\mathbb{R}^d} |\mathbf{Q}| w_\varepsilon^n \sum_l |\phi_{N,\varepsilon}^{(l;0)}| d\mathbf{Q}\right) \\ &\quad + |\nabla_\alpha^2\kappa| \left(\int_{\mathbb{R}^d} w_\varepsilon^n \sum_l |\phi_{N,\varepsilon}^{(0;l)}| d\mathbf{Q} + \int_{\mathbb{R}^d} |\mathbf{Q}| w_\varepsilon^n \sum_{i,j} |\phi_{N,\varepsilon}^{(i;j)}| d\mathbf{Q}\right) \\ &\quad + |\nabla_\alpha\kappa| \left(\int_{\mathbb{R}^d} w_\varepsilon^n \sum_{i,j} |\phi_{N,\varepsilon}^{(0;i,j)}| d\mathbf{Q} + \int_{\mathbb{R}^d} |\mathbf{Q}| w_\varepsilon^n \sum_{l,i,j} |\phi_{N,\varepsilon}^{(l;i,j)}| d\mathbf{Q}\right). \end{aligned} \tag{139}$$

By multiplying both sides of (137) by w_ε^n, and integrating in $\mathbf{Q}$ and summing them altogether, yields

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^d} w_\varepsilon^n|\phi_{N,\varepsilon}^{(0;i,j,k)}|d\mathbf{Q}\\
&\quad\leqq C(n,d,D_2,\kappa)\int_{\mathbb{R}^d} w_\varepsilon^n|\phi_{N,\varepsilon}^{(0;i,j,k)}|d\mathbf{Q}\\
&\qquad+|\nabla_\alpha^3\kappa|\left(\int_{\mathbb{R}^d} w_\varepsilon^n\phi_{N,\varepsilon}d\mathbf{Q}+\int_{\mathbb{R}^d}|\mathbf{Q}|w_\varepsilon^n\sum_l|\phi_{N,\varepsilon}^{(l;0)}|d\mathbf{Q}\right)\\
&\qquad+|\nabla_\alpha^2\kappa|\left(\int_{\mathbb{R}^d} w_\varepsilon^n\sum_l|\phi_{N,\varepsilon}^{(0;l)}|d\mathbf{Q}+\int_{\mathbb{R}^d}|\mathbf{Q}|w_\varepsilon^n\sum_{i,j}|\phi_{N,\varepsilon}^{(i;j)}|d\mathbf{Q}\right)\\
&\qquad+|\nabla_\alpha\kappa|\left(\int_{\mathbb{R}^d} w_\varepsilon^n\sum_{i,j}|\phi_{N,\varepsilon}^{(0;i,j)}|d\mathbf{Q}+\int_{\mathbb{R}^d}|\mathbf{Q}|w_\varepsilon^n\sum_{l,i,j}|\phi_{N,\varepsilon}^{(l;i,j)}|d\mathbf{Q}\right). \qquad (140)
\end{aligned}$$

Now we multiply $\int_{\mathbb{R}^d} w_\varepsilon^n|\phi_{N,\varepsilon}^{(0;i,j,k)}|d\mathbf{Q}$ to (140) and integrate it with respect to α. It can be shown that

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega\left(\int_{\mathbb{R}^d} w_\varepsilon^n|\phi_{N,\varepsilon}^{(0;i,j,k)}|d\mathbf{Q}\right)^2 d\alpha\\
&\quad\leqq C\int_\Omega\left(\int_{\mathbb{R}^d} w_\varepsilon^n|\phi_{N,\varepsilon}^{(0;i,j,k)}|d\mathbf{Q}\right)^2 d\alpha\\
&\qquad+C_1\int_\Omega(|\nabla_\alpha^3\kappa|^2+|\nabla_\alpha^2\kappa|^2)d\alpha+C_2\|\nabla_\alpha\kappa\|_{L^\infty}^2. \qquad (141)
\end{aligned}$$

Here we have also used the fact that functions in (132) belong to $L^\infty([0,T];L^\infty(\Omega))$ and the boundedness of Ω. An application of Gronwall's inequality, together with the fact $\nabla_\alpha\kappa\in L^2([0,T];L^\infty(\Omega))$, implies that

$$\phi_{N,\varepsilon}\in L^\infty([0,T];H^3(\Omega,\mathcal{X}_{n,0}^\varepsilon)),\qquad\text{if}\qquad\psi_0\in H^3(\Omega,\mathcal{X}_{n,2}). \qquad (142)$$

3.2. Estimate of ψ with respect to the Eulerian variable

It is known that we need the estimates for the derivatives of ψ with respect to $\mathbf{x}$, but we now only have estimates for ψ with respect to α. Equation (49) implies that the desired estimates can be obtained if we can estimate $\mathbf{x}(\alpha,t)$ in (49). Since $\mathbf{u}^m\in S(M,T)$, we can obtain the following estimate of flow map from (49) (see e.g. [13])

$$\nabla\alpha\in L^\infty([0,T]\times\Omega),\quad \nabla^2\alpha,\nabla^3\alpha\in L^\infty([0,T];L^2(\Omega)). \qquad (143)$$

Using (143) and the fact

$$\nabla\phi_{N,\varepsilon}(\mathbf{x},\mathbf{Q},t)=\nabla_\alpha\phi_{N,\varepsilon}\cdot\nabla\alpha, \qquad (144)$$

we have

$$\phi_{N,\varepsilon}(\mathbf{x}, \mathbf{Q}, t) \in \cap_{k=0}^{1} H^k([0, T]; H^{3-2k}(\Omega, \mathcal{X}_{n,2}^{\varepsilon})) \tag{145}$$

when $\mathbf{u}^m \in S(M, T)$ and

$$\phi_{N,\varepsilon}(\alpha, \mathbf{Q}, t) \in \cap_{k=0}^{1} H^k([0, T]; H^{3-2k}(\Omega, \mathcal{X}_{n,2}^{\varepsilon})). \tag{146}$$

3.3. The limit of $\phi_{N,\varepsilon}$ when $N \to \infty$ and $\varepsilon \to 0$

All of the estimates above are uniform in N and the restriction to the initial data is $\psi_0 \in H^3(\Omega, \mathcal{X}_{n,2})$. However, when N goes to ∞, the limit of the sequence $\phi_{N,\varepsilon}$ may not be in $\cap_{k=0}^{1} H^k([0, T]; H^{3-2k}(\Omega, \mathcal{X}_{n,2}^{\varepsilon}))$ since $\mathcal{X}_{n,2}^{\varepsilon}$ are based in L^1 type norms, and it is known that the limit of a distributionally convergent bounded sequence in L^1 may not be a L^1 function but a singular measure. To overcome this difficulty, we can use the technique in [17] to improve the regularity of ψ_0. This is the reason for assuming $\psi_0 \in H^4(\Omega, \mathcal{X}_{n,3})$ in the condition (**A3**) of Theorem 1. The details are omitted since they are similar to that in [17]. Moreover, since all of the estimates above are uniform in ε, when $\varepsilon \to 0$, we know that $\phi \in \cap_{k=0}^{1} H^k([0, T]; H^{3-2k}(\Omega, \mathcal{X}_{n,2}))$ by using Lemma 6, which is the solution of (50) with the initial data $\psi_0 \in H^4(\Omega, \mathcal{X}_{n,3})$. In addition, when $\varepsilon \to 0$, we have $\phi(\mathbf{x}, \mathbf{Q}, t) \geqq 0$ and (58) becomes

$$\int_{B_1} \phi(\mathbf{x}, \mathbf{Q}, t) d\mathbf{Q} = \int_{B_1} \psi_0(\mathbf{x}, \mathbf{Q}) d\mathbf{Q} = 1.$$

Lemma 4 can be established by applying the same type of estimates to the function $\psi - \hat{\psi}$. The details will be omitted here.

4. Conclusion

A detailed well-posedness analysis for the FENE dumbbell model of polymeric fluids is carried out in this paper. The model under consideration is a coupled system for the fluid velocity $\mathbf{u}$ and the distribution density ψ for polymeric fluids. A rigorous analysis is focused on the uniform estimate ψ and its derivative with respect to space variable α in L^1 weighted norm. It is demonstrated that the uniform estimates to the L^∞ norm of $\int_{\mathbb{R}^d} w_\varepsilon^n \phi_{N,\varepsilon} d\mathbf{Q}$, $\int_{\mathbb{R}^d} w_\varepsilon^n |\nabla_{\mathbf{Q}} \phi_{N,\varepsilon}| d\mathbf{Q}$, $\int_{\mathbb{R}^d} w_\varepsilon^{n+1} |\nabla_{\mathbf{x}} \phi_{N,\varepsilon}| d\mathbf{Q}$ and $\int_{\mathbb{R}^d} w_\varepsilon^{n+1} |\nabla_{\mathbf{x}} \nabla_{\mathbf{Q}} \phi_{N,\varepsilon}| d\mathbf{Q}$ are essential in establishing the main results. Importantly, the boundary condition of the FENE-type Fokker-Planck equation is proved to be unnecessary by the singularity on the boundary. We point out that this work is crucial for the numerical analysis of the recently proposed multiscale methods [3, 14] for solving (9)–(15).

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