Arch. Rational Mech. Anal. 183 (2007) 299–313
Digital Object Identifier (DOI) 10.1007/s00205-006-0042-4

Theory of Elastic Dielectrics Revisited

J.L. ERICKSEN

Communicated by D. KINDERLEHRER

Abstract

I develop a variational principle introduced in [2] for electromagnetic elastic bodies and discuss its consequences. Formulae for stress tensors and configurational stresses are derived by energy minimization.

1. Introduction

As an alternative to TOUPIN'S [1] theory of elastic dielectrics, I [2] sketched one based on a different proposal for the energy function. Mine is more complicated and less conventional, inducing me to express doubts about whether others would find it useful. However, after thinking more about it, I concluded that it has an important advantage. I was swayed by a remark made by BROWN [3, p. 78] in discussing a similar theory for magnetism, viz.

"Toupin [1] *and Tiersten* [2] *were interested only in equilibrium conditions (or equations of motion), not in stability tests; they therefore paid no attention to whether their stationarity conditions represent a minimum, a maximum, or something else."*

From other remarks he made, I infer that he thought "something else" is the correct interpretation and I agree. Certainly, for nonlinear theory, stability tests based on energy minimization can be very useful. I think that the theory I [2] sketched is more appropriate in this respect, so I will develop it here. Of course, we cannot expect ideas of minimum energy to apply to all loading devices, etc. I will limit the discussion to one kind of situation to which I expect it to apply. Alone in the universe are two entities, an elastic dielectric occupying a bounded region Ω_B that can be varied a little and some device producing an electric field to act on the dielectric. Generally, an electric field in a dielectric extends as a self-field throughout space, so we should be interested in minimizing energy in E_3 all of Euclidean 3-space. The usual view is that it is the exterior self-field of something else that is effective in interacting with the dielectric. It will become clear that familiar conditions for minima in the calculus of variations almost determine the form of stress tensors, configurational stresses, magnetic body forces etc., something that some workers seem not to fully appreciate.

2. Preliminary considerations

First, we have the basic equations of electromagnetism. I use the conventions presented by TRUESDELL and TOUPIN [4, Ch. F], including Lorentz' assumption that the aether relations

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E} \tag{2.1}$$

hold in matter as well as vacuum, when both free and bound charges and currents are properly accounted for. Here, ε_0 and μ_0 are positive constants that depend on the system of units used. Along with this, we have Maxwell's equations

$$\nabla \wedge \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \tag{2.2}$$

$$\nabla \cdot \mathbf{D} = Q, \tag{2.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \wedge \mathbf{E} = \mathbf{0}, \tag{2.4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.5}$$

where Q is the charge per unit volume, $\mathbf{J}$ the current, basic things that we make assumptions about. As an implication of these, we get the well-known energy equation as

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = -\nabla \cdot (\mathbf{E} \wedge \mathbf{H}) - \mathbf{J} \cdot \mathbf{E} \tag{2.6}$$

and, with (2.1), this motivates the introduction of a field energy e_F per unit volume as

$$e_F = \frac{|\mathbf{B}|^2}{2\mu_0} + \frac{|\mathbf{D}|^2}{2\varepsilon_0}, \quad \mathbf{H} = \frac{\partial e_F}{\partial \mathbf{B}}, \quad \mathbf{E} = \frac{\partial e_F}{\partial \mathbf{D}}. \tag{2.7}$$

Obviously, this is bounded below, being nonnegative. For a static theory of dielectrics, $\mathbf{J} = \mathbf{0}$ and $Q = -\nabla \cdot \mathbf{p}$, where $\mathbf{p}$ is the polarization vector. The usual assumptions are that

$$\mathbf{B} = \mathbf{H} = \mathbf{0}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E} = \mathbf{d} - \mathbf{p}, \quad \nabla \cdot \mathbf{d} = 0, \quad \mathbf{E} = -\nabla \varphi, \tag{2.8}$$

where φ is the electrostatic potential. This excludes the possibility that free charges might get into the dielectric, a possibility that is sometimes considered. For fields associated with the dielectric, these equations apply throughout space, with $\mathbf{p} = \mathbf{0}$ in the region $\Omega_E = E_3 \backslash \Omega_B$ exterior to the material body, where these fields are devoid of singularities. Henceforth, quantities associated with the dielectric will be denoted by an overbar. So, for this part of the energy,

$$\bar{e}_F = \frac{|\bar{\mathbf{D}}|^2}{2\varepsilon_0} = \frac{|\bar{\mathbf{d}} - \bar{\mathbf{p}}|^2}{2\varepsilon_0}. \tag{2.9}$$

Since we are concerned with fields throughout E_3, we require that all fields considered satisfy

$$E_F^{def} = \int_{E_3} e_F \, dv < \infty. \tag{2.10}$$

Also in the picture is some device, producing a static electric field to act on the dielectric. Quantities associated with this will be distinguished by carets. I accept the common assumption that the presence of the dielectric does not alter this, although I think that this deserves careful scrutiny. For simplicity, I assume that the device is not in physical contact with the dielectric. Exceptions such as capacitors deserve special consideration. Then, what is important for interaction with the dielectric is the electric field in Ω_B and, apart from this, I wish to say as little as possible about the device. Evaluating the quadratic field energy for the combined field gives

$$E_F = \hat{E}_F + \overline{E}_F + \int_{E_3} \frac{\overline{\mathbf{D}} \cdot \hat{\mathbf{D}}}{\varepsilon_0} \, dv. \tag{2.11}$$

With the external fields considered as fixed, the first term on the right just contributes an unimportant constant, so I discard it. Using (2.1), (2.8) and bearing in mind (2.4), the last term takes the form

$$\int_{\Omega_B} (\bar{\mathbf{d}} - \overline{\mathbf{p}}) \cdot \hat{\mathbf{E}} \, dv + \int_{\Omega_E} \bar{\mathbf{d}} \cdot \hat{\mathbf{E}} \, dv, \quad \nabla \wedge \hat{\mathbf{E}} = \mathbf{0}, \tag{2.12}$$

wherein the first integral is useful, but we would like to dispose of the second. In [2], I did not deal with fields in Ω_E. At least in the union of the variable regions Ω_B and on $\partial\Omega_B$, we make the usual assumption that $\hat{\mathbf{E}}$ is given as a smooth field, satisfying the vacuum equations

$$\hat{\mathbf{E}} = -\nabla\hat{\varphi}, \quad \nabla^2\hat{\varphi} = 0, \quad \hat{\mathbf{d}} = \varepsilon_0\hat{\mathbf{E}}. \tag{2.13}$$

Elsewhere, it should be at least a weak solution of $(2.12)_2$ typically suffering jump discontinuities on one or more surfaces. When I call fields smooth, I mean that they are at least smooth enough to qualify as classical solutions of the related equations. For example, $\mathbf{B}$, $\bar{\mathbf{d}}$ and $\bar{\mathbf{p}}$ are smooth if they are at least continuously differentiable. To proceed, we need to present some theory.

3. Energetics

At least as weak solutions, fields $\bar{\mathbf{d}}$ to be considered are to satisfy

$$\nabla \cdot \bar{\mathbf{d}} = 0. \tag{3.1}$$

To this end, I introduce the vector potentials

$$\bar{\mathbf{d}} = \nabla \wedge \mathbf{A}, \quad \mathbf{A} \to \mathbf{0} \text{ at } \infty. \tag{3.2}$$

This is subject to the usual gauge transformations

$$\mathbf{A} \to \mathbf{A} + \nabla\chi. \tag{3.3}$$

I will be using the calculus of variations, so the set of these fields will be broader than those occurring as equilibrium fields, but they will be taken as smooth in Ω_E. Instead of using the last equation in (2.8), I will use

$$\nabla \wedge (\overline{\mathbf{d}} - \overline{\mathbf{p}}) = \mathbf{0} \nabla \wedge (\nabla \wedge \mathbf{A}) = \nabla^2 \mathbf{A} - \nabla\nabla \cdot \mathbf{A} = \nabla \wedge \overline{\mathbf{p}.} \tag{3.4}$$

With this, we can use a gauge condition, a convenient one being

$$\nabla \cdot \mathbf{A} = 0. \tag{3.5}$$

Here, I will not consider using (3.5) until I cover matters of gauge invariance. First, consider the unwanted integral in (2.12), now written as

$$\int_{\Omega_E} \hat{\mathbf{E}} \cdot \nabla \wedge \mathbf{A} \, dv. \tag{3.6}$$

Suppose that $\hat{\mathbf{E}}$ is smooth except on some surface $\sum$, with unit normal $\mathbf{n}$, where it suffers a jump discontinuity. Workers analyzing this would impose the standard jump condition

$$\left[\hat{\mathbf{E}}\right] = \alpha \mathbf{n}, \tag{3.7}$$

where the square brackets denote the jump and α is some scalar, as well as assuming that

$$\nabla \wedge \hat{\mathbf{E}} = \mathbf{0} \tag{3.8}$$

except on $\sum$. By an elementary calculation,

$$\hat{\mathbf{E}} \cdot \nabla \wedge \mathbf{A} = \nabla \cdot (\mathbf{A} \wedge \hat{\mathbf{E}}) + \mathbf{A} \cdot \nabla \wedge \hat{\mathbf{E}}. \tag{3.9}$$

Integrate this over Ω_E, use (3.7), (3.8), the divergence theorem and assume that $\mathbf{A} \wedge \hat{\mathbf{E}} \to \mathbf{0}$ sufficiently fast at ∞, to get

$$\int_{\Omega_E} \hat{\mathbf{E}} \cdot \nabla \wedge \mathbf{A} \, dv = - \int_{\partial\Omega_B} \mathbf{A} \wedge \hat{\mathbf{E}} \cdot d\mathbf{S}, \tag{3.10}$$

where $d\mathbf{S}$ denotes the vector element of area, taken outward with respect to Ω_B, not Ω_E. Obviously, we would get the same result if we replaced $\sum$ by a finite number of such discontinuity surfaces. This is my plausibility argument for replacing the left side of (3.10) by the right. So, if we keep this in mind, we can take the field energy as

$$E_F = \int_{\Omega_E} \frac{|\overline{\mathbf{d}}|^2}{2\varepsilon_0} \, dv + \int_{\Omega_B} (\overline{\mathbf{d}} - \overline{\mathbf{p}}) \cdot \hat{\mathbf{E}} \, dv. \tag{3.11}$$

Then, in Ω_B, we need a contribution covering elastic deformation. Introduce the usual reference configuration occupying a fixed region Ω_R, referred to material coordinates X^α. When I use coordinates, interpret them as rectangular Cartesian. Instead of describing deformations as mappings of Ω_R onto Ω_{B}, I use the inverse mappings, given by functions of the form

$$X^\alpha = X^\alpha(x^i), \tag{3.12}$$

where the x^i are spatial coordinates. This is because forms of the electromagnetic equations are referred to these. As a general rule, the theory of material symmetry can be borrowed from elasticity theory, in the common cases where, in equilibrium, $\bar{\mathbf{p}} = \mathbf{0}$ in the absence of external fields. Dealing with exceptions such as those that occur in ferroelectric materials can be tricky to deal with properly. Later, I will relate this format to the more conventional one used by Toupin. So, the range of this is fixed, but the domain is not given. I will only deal with homogeneous materials, referred to homogeneous reference configurations. For the remaining energy function e_M I use the equivalent of that employed by Toupin,

$$e_M = e_M\,(\bar{\mathbf{p}}, \nabla\mathbf{X}) \quad \text{in } \Omega_{\mathrm{B}}, = 0 \quad \text{in } \Omega_E, \tag{3.13}$$

to be invariant under rotations,

$$e_M(\mathbf{R}\overline{\mathbf{p}}, \mathbf{R}\nabla\mathbf{X}) = e_M(\overline{\mathbf{p}}, \nabla\mathbf{X}), \ \ \mathbf{R} \in SO(3). \tag{3.14}$$

Copying the argument used by Toupin gives an identity

$$\frac{\partial e_M}{\partial \overline{p}_i}\overline{p}_j + \frac{\partial e_M}{\partial X^\alpha_{,i}} X^\alpha_{,j} = \frac{\partial e_M}{\partial \overline{p}_j}\overline{p}_i + \frac{\partial e_M}{\partial X^\alpha_{,j}} X^\alpha_{,i}, \tag{3.15}$$

equivalent to one he deduces. The total energy can then be put in the form

$$E = \int\limits_{\Omega_E} e_F dv + \int\limits_{\Omega_B} (e_F + e_M)\, dv, \tag{3.16}$$

where

$$e_F = e_F(\nabla\mathbf{A}, \overline{\mathbf{p}}, \hat{\mathbf{E}}) = \frac{|\overline{\mathbf{d}} - \overline{\mathbf{p}}|^2}{2\varepsilon_0} + (\overline{\mathbf{d}} - \overline{\mathbf{p}}) \cdot \hat{\mathbf{E}} \quad \text{with } \overline{\mathbf{d}} = \nabla \wedge \mathbf{A} \text{ in } \Omega_B. \tag{3.17}$$

In Ω_E, recalling (3.10), use this with $\bar{\mathbf{p}} = \mathbf{0}$, $\bar{\mathbf{d}} \cdot \hat{\mathbf{E}} = 0$. For this, a calculation gives the analogue of (3.15) as

$$r_{ij} = \frac{\partial e_F}{\partial \overline{p}_i}\overline{p}_j + \frac{\partial e_F}{\partial \hat{E}_i}\hat{E}_j + \frac{\partial e_F}{\partial A_{k,i}}(A_{k,j} - A_{j,k}) = r_{ji}. \tag{3.18}$$

Bear in mind that I am not considering possible mechanical loading devices, although I will cover the possibility of restraining all or part of Ω_B from moving freely. Under these conditions, it seems to me reasonable to use minimization of E as a way of defining stable equilibria, acknowledging that, for minimizers to exist, it will at least be necessary to impose some restrictions on the function e_M. I will not rehash results in the calculus of variations that are of some help in assessing this.

4. A variational treatment

Here, we use the calculus of variations to get various equations and jump conditions that, at least naively, should be satisfied by energy minimizers. Largely, this is a matter of showing how familiar results are obtained by less familiar reasoning. Prescription of forces and couples in material bodies subject to electromagnetic effects has long been a controversial matter, so I will not even mention them again until I have in hand the relevant equations. Let us consider special kinds of variations of E, as described in (3.13), (3.16) and (3.17). In crystals, interior surfaces of discontinuity are rather common, for example those associated with twins.

First, as generally interpreted, $\nabla \cdot \bar{\mathbf{d}} = 0$ includes the condition that, at a surface of jump discontinuity, $\bar{\mathbf{d}} \cdot \mathbf{n}$ be continuous. To cover this, assume that $\mathbf{A}$ is continuous, with jumps in its first derivatives. With square brackets denoting jumps, the usual conditions of compatibility give

$$\left[A_{i,j}\right] = a_i n_j \Rightarrow \left[\bar{\mathbf{d}}\right] = [\nabla \wedge \mathbf{A}] = \mathbf{n} \wedge \mathbf{a} \tag{4.1}$$

for some vector $\mathbf{a}$. With this, we can evaluate the right side of (3.10) on either side of $\partial\Omega_B$ and I will use it on the interior side.

Varying only $\bar{\mathbf{p}}$, with $\delta\bar{\mathbf{p}}$ taken as an arbitrary smooth function of position and setting the variation equal to zero gives the equation

$$-\frac{(\bar{\mathbf{d}} - \bar{\mathbf{p}})}{\varepsilon_0} - \hat{\mathbf{E}} + \frac{\partial e_M}{\partial \bar{\mathbf{p}}} = \mathbf{0}, \quad \frac{\partial e}{\partial \bar{\mathbf{p}}} = \mathbf{0} \quad \text{in } \Omega_B, \tag{4.2}$$

which, with $(2.7)_3$ is equivalent to an equation deduced by Toupin,

$$\mathbf{E} = \bar{\mathbf{E}} + \hat{\mathbf{E}} = \frac{\partial e_M}{\partial \bar{\mathbf{p}}}. \tag{4.3}$$

Here is where effective interaction between $\hat{\mathbf{E}}$ and the dielectric occurs. Next, varying only $\mathbf{A}$ with $\delta\mathbf{A}$ any smooth function gives

$$\int_{\Omega E} \frac{\nabla \wedge \mathbf{A}}{\varepsilon_0} \cdot \nabla \wedge \delta\mathbf{A}\, dv + \int_{\Omega_B} \left(\frac{\nabla \wedge \mathbf{A} - \bar{\mathbf{p}}}{\varepsilon_0} + \hat{\mathbf{E}}\right) \cdot \nabla \wedge \delta\mathbf{A}\, dv - \int_{\partial\Omega_B} \delta\mathbf{A} \cdot (\hat{\mathbf{E}} \wedge d\mathbf{S}) = \mathbf{0}, \tag{4.4}$$

the last term coming from (3.10). We could take this as a definition of weak solutions of an equation for $\mathbf{A}$ but, later, I will give a simpler proposal for this. To make equations to be deduced better match familiar equations in the calculus of variations, I will use

$$\int_{E_3} \frac{\partial e_F}{\partial \mathbf{A}_{i,j}} \delta\mathbf{A}_{i,j}\, dv - \int_{\partial\Omega_{\mathrm{B}}} \delta\mathbf{A} \cdot \left(\hat{\mathbf{E}} \wedge d\mathbf{S}\right) = 0. \tag{4.5}$$

I note that

$$\frac{\partial e_F}{\partial A_{i,j}} = \frac{\partial e_F}{\partial \bar{d}_k} \varepsilon_{kji} = \left(\frac{\nabla \wedge \mathbf{A} - \bar{\mathbf{p}}}{\varepsilon_0} + \hat{\mathbf{E}}\right)_k \varepsilon_{kji} \text{ in } \Omega_{\mathrm{B}}, \tag{4.6}$$

this with $\bar{\mathbf{p}} = \mathbf{0}$ and, because of (3.10), with $\hat{\mathbf{E}}$ replaced by $\mathbf{0}$ in Ω_{E}. From this and (2.13), it follows that

$$\nabla \cdot \left(\frac{\partial e_F}{\partial \mathbf{A}} \right) = \mathbf{0} \quad (3.4). \tag{4.7}$$

By the traditional manipulations (4.5) gives this except at singularities. If $\sum$ is a surface of jump discontinuity in Ω_B dividing it into two parts, we get the usual jump condition

$$\left[\frac{\partial e_F}{\partial \nabla \mathbf{A}} \right] d\mathbf{S} = \mathbf{0}. \tag{4.8}$$

In (4.7) and (4.8), the contribution from $\hat{\mathbf{E}}$ drops out, since this is continuous and (2.13) holds. It also drops out in the analogue for $\partial \Omega_B$ because of the last term in (4.5). Conditions of this kind are also called corner conditions. In either case, (4.8) reduces to

$$\left[\frac{\nabla \wedge \mathbf{A} - \bar{\mathbf{p}}}{\varepsilon_0} \right] \wedge \mathbf{n} = [\bar{\mathbf{E}}] \wedge \mathbf{n} = \mathbf{0}, \tag{4.9}$$

where $\mathbf{n}$ is the unit normal, the usual jump condition for electric fields. Also, with (4.1), we get the other standard condition for $\bar{\mathbf{E}}$ as

$$\left[\varepsilon_0 \bar{\mathbf{E}} + \bar{\mathbf{p}} \right] \cdot \mathbf{n} = 0. \tag{4.10}$$

I note that, with our assumption that $\mathbf{A}$ is continuous at jump discontinuities, $\bar{\mathbf{d}} \cdot \hat{\mathbf{E}}$ contributes nothing to equilibrium equations or jump conditions, leading me to

$$\text{replace } e_F \text{ by } \tilde{e}_F = \frac{|\nabla \wedge \mathbf{A} - \bar{\mathbf{p}}|^2}{2\varepsilon_0} - \bar{\mathbf{p}} \cdot \hat{\mathbf{E}} \tag{4.11}$$

and

$$e \text{ by } \tilde{e} = \tilde{e}_F + e_M. \tag{4.12}$$

With this, (4.4) reduces to my proposal for a definition of weak solutions of (3.4) as

$$\int_{\Omega_{\mathrm{E}}} \frac{\nabla \wedge \mathbf{A}}{\varepsilon_0} \cdot \nabla \wedge \delta \mathbf{A} \, dv + \int_{\Omega_B} \left(\frac{(\nabla \wedge \mathbf{A} - \bar{\mathbf{p}})}{\varepsilon_0} \right) \cdot \nabla \wedge \delta \mathbf{A} \, dv = 0 \tag{4.13}$$

for smooth fields $\delta \mathbf{A}$. A useful result can be obtained as follows. Using the fact that $\nabla \wedge \bar{\mathbf{E}} = \mathbf{0}$ except at singularities, we have

$$\bar{\mathbf{d}} \cdot \bar{\mathbf{E}} = (\nabla \wedge \mathbf{A}) \cdot \bar{\mathbf{E}} = \nabla \cdot \left(\mathbf{A} \wedge \bar{\mathbf{E}} \right). \tag{4.14}$$

Integrate this over E_3 and use the divergence theorem. At a surface of jump discontinuity, with $\mathbf{A}$ continuous and (4.9), $\mathbf{A} \wedge \bar{\mathbf{E}} \cdot \mathbf{n}$ is continuous. Assuming $\mathbf{A} \wedge \bar{\mathbf{E}} \to \mathbf{0}$ fast enough at ∞ this gives

$$\int_{E_3} \overline{\mathbf{d}} \cdot \overline{\mathbf{E}} \, dv = \int_{E_3} \left(\varepsilon_0 \overline{\mathbf{E}} + \overline{\mathbf{p}} \right) \cdot \overline{\mathbf{E}} dv = 0 \tag{4.15}$$

or

$$\varepsilon_0 \int_{E_3} |\overline{\mathbf{E}}|^2 dv = - \int_{\Omega_B} \overline{\mathbf{p}} \cdot \overline{\mathbf{E}}\, dv. \tag{4.16}$$

Note that, in this, we did not enforce (4.3). In fact, there seems to be no easy way of enforcing it. My theory is set up so that we could minimize energy without assuming it holds. There is another interesting result. It is common to introduce self-fields for dielectrics, by thinking of $\bar{\mathbf{p}}$ as some definite function of position. With likely prescriptions of constitutive equations, the expectation is that we can use the equations and jump conditions now determined to calculate self-fields $\bar{\mathbf{E}} = \mathbf{E}_S$ and $\bar{\mathbf{d}} = \bar{\mathbf{d}}_S = \nabla \wedge \mathbf{A}_S$. Add to the latter a field $\nabla \wedge \delta\mathbf{A}$ where this need not be regarded as a variation, but is, like the $\delta\mathbf{A}$ in (4.3), smooth everywhere. Now calculate the total field energy for the sum of these fields, denoted by E. Expand this, use (4.13) and you get

$$\tilde{E} = E_S(\mathbf{A}_S) + \int_{E_3} \frac{|\nabla \wedge \delta\mathbf{A}|^2}{2\varepsilon_0}\, dv \geqq E_S(\mathbf{A_S}), \tag{4.17}$$

where $E_S(\mathbf{A}_S)$ is the self-field energy, the equality holding only if $\nabla \wedge \delta\mathbf{A} = \mathbf{0}$.

Then there are equations involving e_M, obtained by varying $\mathbf{X}$. Here, we must account for the fact that $\mathbf{X}$ is restricted to the fixed region Ω_R and that points on $\partial\Omega_B$ correspond to points on its boundary. As before, I will allow jump discontinuities on an interior surface $\sum$ and on $\partial\Omega_B$. Formally, such variations give

$$\int_{\Omega_B} \frac{\partial e_M}{\partial X^\alpha_{,i}} \delta X^\alpha_{,i}\, dv = - \int_{\Omega_B} \left(\frac{\partial e_M}{\partial X^\alpha_{,i}} \right)_{,i} \delta X^\alpha dv + \int_{\Sigma} \left[\frac{\partial e_M}{\partial X^\alpha_{,i}} \right] \delta X^\alpha dS_i + \int_{\Omega_B} \frac{\partial e_M}{\partial X^\alpha} \delta X^\alpha dS_i \tag{4.18}$$

where δX^α is subject to some restrictions. This can be thought of as varying the deformation, so a material particle at $\mathbf{x}$ gets replaced by a different one. Generally, the surface $\sum$ need not be a material surface, so we can let matter on one side move to the other, as long as mass is conserved. With the usual assumptions of elasticity theory, the present mass density ρ and the fixed reference mass density ρ_R are related by

$$\rho = \rho_R J,\ J = |\det \nabla \mathbf{X}|, \tag{4.19}$$

so

$$\int_{\Omega_B} \frac{\rho}{\rho_R} dv = \int_{\Omega_B} J dv = \int_{\Omega_R} dv, \tag{4.20}$$

a fixed number. We have Nanson's identity,

$$\nabla \cdot \left(\frac{\partial J}{\partial \nabla \mathbf{X}} \right) = 0 \tag{4.21}$$

so

$$\delta \int\limits_{\Omega_B} J\, dv = \int\limits_{\Sigma} \left[\frac{\partial J}{\partial X^\alpha_{,i}} \right] \delta X^\alpha dS_i + \int\limits_{\delta \Omega_B} \frac{\partial J}{\partial X^\alpha_{,i}} \delta X^\alpha dS_i. \tag{4.22}$$

First, set $\delta \mathbf{X} = \mathbf{0}$ on $\partial \Omega_B$ and equate the variational integrals to zero. This gives

$$\nabla \cdot \left(\frac{\partial e_M}{\partial \nabla \mathbf{X}} \right) = \mathbf{0} \quad \text{in } \Omega_B \tag{4.23}$$

and

$$\left[\frac{\partial e_M}{\partial \nabla \mathbf{X}} - \lambda \frac{\partial J}{\partial \nabla \mathbf{X}} \right] \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \Sigma, \tag{4.24}$$

where λ is a Lagrange multiplier. This can be simplified by making an assumption that I accept, namely that $\mathbf{X}$ is continuous, so

$$[\mathbf{X}] = \mathbf{0}, \left[\nabla X^\alpha \right] = F^\alpha \mathbf{n}, \tag{4.25}$$

for some vector $\mathbf{F}$. By an elementary calculation,

$$(4.25) \Rightarrow \left[\frac{\partial J}{\partial X^\alpha_{,i}} \right] n_i = 0 \Rightarrow \left[\frac{\partial e_M}{\partial X^\alpha_{,i}} \right] n_i = 0. \tag{4.26}$$

In elasticity theory, the tensor inside the parentheses in (4.23) is the configurational stress, referred to spatial coordinates. Actually, it is better to regard this as an equivalence class: adding a linear function of J to the energy just adds a constant to the total energy, changing this tensor without affecting (4.23) or (4.26) . I will call it the *elastic configurational stress*, to distinguish it from other configurational stresses sometimes used in electromagnetic theory. Here, (4.23) is the conservation law associated with invariance of e_M under the translations $\mathbf{X} \to \mathbf{X} + const$. On $\partial \Omega_B$, a material surface, we cannot move matter across it, but material particles on it can be moved to different positions. So, consider an infinitesimal point transformation $\mathbf{x} \to \mathbf{x} + \varepsilon \delta \mathbf{x}$, where ε is the parameter associated with variations. The condition that this map the boundary to itself requires that

$$\delta \mathbf{x} \cdot \mathbf{n} = 0. \tag{4.27}$$

Requiring that the value of $\mathbf{X}$ be the same at a point and its image gives a variation satisfying

$$\delta X^\alpha = -X^\alpha_{,i} \delta x^i \tag{4.28}$$

and requiring the last term in (4.18) to vanish for all such variations gives

$$\frac{\partial e_M}{\partial X^\alpha_{,i}} X^\alpha_{,j} n_i = \mu n_j \quad \text{on } \partial \Omega_B, \tag{4.29}$$

where μ is an undetermined function of position. If clamping devices make such variations impossible, we should not use (4.29). For dielectrics, I have not seen

(4.23), (4.26) or (4.29) deduced elsewhere. Other conditions that have been derived are known to at least some experts. What are generally called Weierstrass–Erdmann corner conditions include not only the jump conditions that have been derived, but others which allow the discontinuity surfaces to move to different positions. For these, the standard reasoning gives

$$\left[\tilde{e}\delta_{ij} - X^{\alpha}_{,i}\frac{\partial e_M}{\partial X^{\alpha}_{,j}} - A_{k,i}\frac{\partial \tilde{e}_F}{\partial A_{k,j}}\right] n_j = 0 \quad \text{on } \Sigma. \tag{4.30}$$

Commonly, this is applied to interior surfaces but, since the fields extend outside the body, it makes sense to treat $\partial\Omega_B$ as an interior surface. Then, granted that clamping devices do not prevent parts of $\partial\Omega_B$ from moving freely, (4.30) also applies to these parts, with $e_M = 0$ and $\bar{\mathbf{p}} = \mathbf{0}$ in Ω_E.

Often, workers introduce principles of virtual work to get conditions comparable to (4.28), at least for $\partial\Omega_B$. and it might be useful to do so when we allow mechanical loading devices, for example. For the present theory, a proposal of this kind should be consistent with (4.30).

In TOUPIN'S [1] theory of dielectrics, the energy function associated with fields is equivalent to

$$t_F = -\frac{\varepsilon_0|\nabla\varphi_S|^2}{2} + \left(\nabla\varphi_s - \hat{\mathbf{E}}\right)\cdot\bar{\mathbf{p}}, \tag{4.31}$$

where φ_S is the electrostatic potential associated with a self-field. Like others, he interprets the self-fields as solutions of the Poisson equation $-\varepsilon_0\nabla\cdot\bar{\mathbf{E}} = \varepsilon_0\nabla^2\bar{\varphi} = \nabla\cdot\bar{\mathbf{p}}$ in E_3, with $\bar{\mathbf{p}}$ considered as some definite function of position. With the negative quadratic term, it is clear that we cannot minimize his energy, and I do not see another good possibility for adding a stability test to his theory. In his theory, equilibria are represented as something more like saddle points. In my analyses, I have only used first variations, so minimizers are not distinguished from other extremals. Toupin also used first variations in dealing with (4.29).

5. Forces and torques

While I have deduced relevant equations without introducing assumptions about forces or torques, they are of physical interest. To avoid getting inconsistent theory, any equations satisfied by these should be implied by equations now in hand. First, I will calculate and rearrange terms in $\nabla\tilde{e}$ in Ω_B. First, by (4.2), the term involving $\partial\tilde{e}/\partial\bar{\mathbf{p}}$ drops out. Using (4.23), another term is

$$\frac{\partial e_M}{\partial X^{\alpha}_{,j}}X^{\alpha}_{,ji} = \left(X^{\alpha}_{,i}\frac{\partial e_M}{\partial X^{\alpha}_{,j}}\right)_{,j}. \tag{5.1}$$

Then, using (4.11),

$$\frac{\partial\tilde{e}_F}{\partial\hat{E}_j}\hat{E}_{j,i} = -\bar{p}_j\hat{E}_{j,i}. \tag{5.2}$$

Finally,

$$\frac{\partial \tilde{e}_F}{\partial A_{k,j}} A_{k,ji} = \left(A_{k,i} \frac{\partial \tilde{e}_F}{\partial A_{k,j}} \right)_{,j}. \tag{5.3}$$

Putting these together, we get

$$S_{ij,j} + f_i = 0,\ f_i = \bar{p}_j \hat{E}_{j,i}, \tag{5.4}$$

where

$$S_{ij} = \tilde{e}\delta_{ij} - X^{\alpha}_{,i} \frac{\partial e_M}{\partial X^{\alpha}_{,j}} - A_{k,i} \frac{\partial \tilde{e}_F}{\partial A_{k,j}}. \tag{5.5}$$

Here, **s** is the tensor occurring in the Weierstrass–Erdmann corner condition (4.30). We might like **s** as a description of a stress tensor, since (5.4) looks reasonable as an equilibrium equation and the corner condition (4.30) agrees with what would commonly be used as a jump condition for it. However, it has an unpleasant feature, not being invariant under gauge transformations. This can be fixed by defining the stress tensor **t** by

$$t_{ij} = S_{ij} + A_{i,k} \frac{\partial \tilde{e}_F}{\partial A_{k,j}}. \tag{5.6}$$

With this, I have covered matters of gauge invariance, so I now have no objection to using the gauge condition (3.5), although I have no need to use it here. It might help to recall (4.6). It is then easy to show that

$$\left(A_{I,k} \frac{\partial \tilde{e}_F}{\partial A_{k,j}} \right)_{,j} = 0, \tag{5.7}$$

so (5.4) is equivalent to

$$t_{ij,j} + f_i = 0. \tag{5.8}$$

At a jump discontinuity surface with normal **n**

$$\left[A_{i,k} \frac{\partial \tilde{e}_F}{\partial A_{k,j}} \right] n_j = \left[A_{i,k} \right] \frac{\partial \tilde{e}_F}{\partial A_{k,j}} n_j \tag{5.9}$$

by (4.8) and, by (4.1) and (4.6), the whole quantity is continuous. Thus

$$\left[t_{ij} \right] n_j = 0 \ (4.30). \tag{5.10}$$

Recall that this also applies to parts of $\partial \Omega_B$ that can deform freely. Of course, these conditions are consistent with the integral version

$$\int_{\partial \Omega} \mathbf{t} d\mathbf{S} + \int_{\Omega} \mathbf{f}\, dv = 0,\ \Omega \subset \Omega_B. \tag{5.11}$$

Further, by a calculation based on (3.15) and (3.18),

$$t_{ij} - t_{ji} = \bar{p}_i \hat{E}_j - \bar{p}_j \hat{E}_i. \tag{5.12}$$

For this, it is important that **t** be invariant under gauge transformations. With these results, we get the integral formulation

$$\int_{\partial\Omega} \mathbf{x} \wedge (\mathbf{t}d\mathbf{S}) + \int_{\Omega} \left(\mathbf{x} \wedge \mathbf{f} + \overline{\mathbf{p}} \wedge \hat{\mathbf{E}}\right) dv = \mathbf{0}, \Omega \subset \Omega_B, \tag{5.13}$$

which includes the body couple that agrees with the elementary estimate of it.

By a routine calculation, we get

$$\left(A_{i,k} - A_{k,i}\right) \frac{\partial \tilde{e}_F}{\partial A_{k,j}} = \overline{E}_i \overline{d}_j - \bar{\mathbf{d}} \cdot \overline{\mathbf{E}} \delta_{ij}. \tag{5.14}$$

In Ω_E, where $\bar{\mathbf{d}} = \varepsilon \bar{\mathbf{E}}$, the stress tensor reduces to the familiar symmetric Maxwell stress tensor $\mathbf{t}_{CM}$, with zero divergence,

$$\mathbf{t}_{CM} = \varepsilon_0 \left(\overline{\mathbf{E}} \otimes \overline{\mathbf{E}} - \frac{|\overline{\mathbf{E}}|^2}{2} 1 \right). \tag{5.15}$$

Then (5.11) and (5.13) extend to

$$\int_{\partial\Omega} \mathbf{t}d\mathbf{S} + \int_{\Omega} \mathbf{f}\, dv = 0, \Omega_B \subset \Omega \tag{5.16}$$

and

$$\int_{\partial\Omega} \mathbf{x} \wedge (\mathbf{t}d\mathbf{S}) + \int_{\Omega} \left(\mathbf{x} \wedge \mathbf{f} + \overline{\mathbf{p}} \wedge \hat{\mathbf{E}}\right) dv = \mathbf{0}, \Omega_B \subset \Omega. \tag{5.17}$$

In these, the integrals over Ω can obviously be replaced by integrals over Ω_B. This is in line with the common view that the self-field in Ω_E does not affect the forces or torques acting on the body. Although the prescription for stress is different in Toupin's theory, it is consistent with all of these integral versions. For at least some analyses, it is not really necessary to use (3.4), although it has been important in establishing some results.

There is an alternative formulation that is more like Toupin's. If we set

$$\hat{e}_M = \frac{\rho_R}{\rho} e_M, \tag{5.18}$$

a calculation gives

$$e_M \delta_{ij} - X^{\alpha}_{,i} \frac{\partial e_M}{\partial X^{\alpha}_{,j}} = -\frac{\rho}{\rho_R} X^{\alpha}_{,i} \frac{\partial \hat{e}_M}{\partial X^{\alpha}_{,j}}. \tag{5.19}$$

Then, we can change the variables and replace $\nabla\mathbf{X}$ by its inverse, the deformation gradient commonly used in elasticity theory. This gives the representation of this part of the stress used by Toupin. We can also refer this part to material coordinates, using

$$\int_{\Omega_B} \frac{\rho}{\rho_R} \hat{e} dv = \int_{\Omega_R} \hat{e} dv, \tag{5.20}$$

and copy this for material subregions. I shall not belabor the routine bookkeeping involved in adapting formulae to this formulation.

As the remainder of the stress, Toupin [1] first proposed the equivalent of

$$t_{ij}^T = \overline{E}_i \overline{d}_j - \frac{\varepsilon_0 |\overline{\mathbf{E}}|^2}{2} \delta_{ij} \Rightarrow t_{ij.j}^T = \overline{p}_j \overline{E}_{i,j}, \tag{5.21}$$

at least when the fields satisfy his conditions for self-fields, borrowing this from Maxwell. Brown [3, pp. 61–62] criticizes this, pointing out that, even in Maxwell's time, this was not accepted by some other workers. Toupin also assumes what the body force and couple should be. Using (5.6) and (5.14), my version can be reduced to the form

$$t_{ij}^E = \overline{E}_i \overline{d}_j - \left\{ \frac{\varepsilon_0 |\overline{\mathbf{E}}|^2}{2} + \overline{\mathbf{p}} \cdot \mathbf{E} \right\} \delta_{i,j}. \tag{5.22}$$

For this,

$$t_{ij,j}^E = -\bar{p}_{j,i} E_j - \bar{p}_j \hat{E}_{j,i}, \tag{5.23}$$

where the first term gets canceled with a term from e_M by (4.3) and the second gives the body force $\mathbf{f}$. In a later work, Toupin [5] relies more on the calculus of variations to deduce different formulae for stress tensors, one being of the same form as (5.22), but with the fields replaced by total fields. For this, he modifies his older energy function (4.31) by replacing the self-field by the total field, so there is no separation of the total field into parts. This kind of modification could be useful if we want to study the possible change in what I call $\hat{\mathbf{E}}$ induced by the self-field of our body. Again, his modified energy is not bounded below, so we cannot use minimization of it as a test for stability. In this work, he does get the jump condition $[\mathbf{t}]\mathbf{n} = \mathbf{0}$ and the standard conditions on fields for jump discontinuities but not (4.23), (4.26) or (4.29).

Consider adding the scalar term in $\mathbf{t}^E$ to $\tilde{e}$, to get what is essentially equivalent to (4.31), what Toupin [1] originally used as an energy function, viz,

$$t = -\frac{\varepsilon_0 |\overline{\mathbf{E}}|^2}{2} - \overline{\mathbf{p}} \cdot (\overline{\mathbf{E}} + \hat{\mathbf{E}}) + e_M. \tag{5.24}$$

In terms of this, the formula for my stress tensor takes the form

$$t_{ij} = t\delta ij + \overline{E}_i \overline{d}_j - X_{,i}^\alpha \frac{\partial e_M}{\partial X_{,j}^\alpha}. \tag{5.25}$$

Taking the Legendre transform of t with respect to $\bar{\mathbf{E}}$ gives

$$t - \frac{\partial t}{\partial \overline{\mathbf{E}}}.\overline{\mathbf{E}} = \frac{\varepsilon_0 |\overline{\mathbf{E}}|^2}{2} - \overline{\mathbf{p}} \cdot \hat{\mathbf{E}} + e_M = \tilde{e} \tag{5.26}$$

and, conversely, with $\tilde{e}$ expressed as a function of $\bar{\mathbf{d}}$, $\bar{\mathbf{p}}$ and $\hat{\mathbf{E}}$,

$$\tilde{e} - \frac{\partial \tilde{e}}{\partial \bar{\mathbf{d}}} \cdot \bar{\mathbf{d}} = t. \tag{5.27}$$

Now take t, writing $\bar{\mathbf{E}} = -\nabla\bar{\varphi}$ as in (4.29) and use a formula for stress tensor $\hat{\mathbf{t}}$ that TOUPIN [5] presents,

$$\hat{t}_{ij} = t\delta_{ij} - \overline{\varphi}_{,i} \frac{\partial t}{\partial \overline{\varphi}_{,j}} - X^{\alpha}_{,i} \frac{\partial t}{\partial X^{\alpha}_{,j}}, \tag{5.28}$$

except that in (5.28) as well as the energy, he here replaces the self-field potential by the potential for the total field. However, for his older theory, (5.28) works out to be equivalent to my stress tensor. Again, analogues of (4.23), (4.26) and (4.29) are not mentioned. So, had Toupin done this, as he obviously could have, our theories would agree on the prescription for stress. I do not rule out the possibility of constructing a theory of stability fitting Toupin's theory, as modified here, to get a theory comparable to mine, but this has not been done, as far as I know.

To determine stress boundary conditions on $\partial\Omega_B$, first take the stress vector $\mathbf{t}_{CM}\mathbf{n}$ associated with (5.15), using (4.9) and (4.10) to express this in terms of interior limits of the fields. This gives

$$\mathbf{t}_{CM}\mathbf{n} = \overline{\mathbf{d}} \cdot \mathbf{n}\overline{\mathbf{E}} + \left(\frac{(\overline{\mathbf{p}} \cdot \mathbf{n})^2}{2\varepsilon_0} - \frac{\varepsilon_0 |\overline{\mathbf{E}}|^2}{2} \right) \mathrm{n}. \tag{5.29}$$

Equate this to the stress vector obtained as a limit from the interior and, after canceling some terms, we get

$$[\mathbf{t}]\,\mathbf{n} = \mathbf{0}, \left(e_M - \overline{\mathbf{p}} \cdot \mathbf{E} - \frac{(\overline{\mathbf{p}} \cdot \mathbf{n})^2}{2\varepsilon_0} \right) n_i - X^{\alpha}_{,i} \frac{\partial e_M}{\partial X^{\alpha}_{,j}} n_j = 0. \tag{5.30}$$

Note that this agrees with (4.29) and it gives a formula for the μ occurring there. So, this covers my proposal for describing stress, in general terms.

TOUPIN [1] is one of many writers to discuss the analysis of an ellipsoid with constant $\bar{\mathbf{p}}$ but he allows for homogeneous deformation and for rather general forms of the function e_M appropriate for isotropic materials. Of course, homogeneous deformations take ellipsoids to ellipsoids. His formulae for fields refer to the deformed ellipsoid. BROWN [3, Section 10] discusses this a bit more, using a similar theory of magnetism. He explains why such theory cannot be consistent with (5.30), this being prevented by the term cubic in the normal, and he discusses attempts to correct this using linear theory, which is not an easy matter. Neither of these authors tries to find conditions e_M that get other conditions to hold or to make any attempt to assess stability. Brown does indicate the possibility of getting simple solutions satisfying (5.30) for infinite cylinders, assuming certain equations can be satisfied.

Toupin [1] also treats shear of an infinite slab of an isotropic material, simply illustrating the nonlinear effect of electrostriction. Again, this is for constant $\bar{\mathbf{p}}$ and $\nabla\mathbf{X}$. It should be feasible to do more with the analysis of this. It would be good to find other simple solutions illustrating nonlinear effects, even if they do not satisfy all of the necessary conditions exactly.

References

1. Toupin, R. A.: The elastic dielectric. *J. Rational Mech. Anal.* **5**, 849–915 (1956)
2. Ericksen, J. L.: Electromagnetic effects in thermoelastic materials. *Math. Mech. Solids* **7**, 165–189 (2002)
3. Brown, W. F.: *Magnetoelastic Interactions*. Springer -Verlag, New York, 1966
4. Truesdell, C. Toupin, R. A.: *The classical field theories*. Handbuch der Physik, ed. S. Flügge, III/I, 226–793, 1993
5. Toupin, R. A.: Stress tensors in elastic dielectrics. *Arch. Ration. Mech. Anal.* **5**, 440–552 (1960)

5378 Buckskin Bob Rd.
Florence,
OR 97439,
USA.

(*Received January 10, 2005 / Accepted January 10, 2005*)