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The Singular Set of Lipschitzian Minima of Multiple Integrals

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Abstract

The singular set of any Lipschitzian minimizer of a general quasiconvex functional is uniformly porous and hence its Hausdorff dimension is strictly smaller than the space dimension.

1. Introduction and main result

The present paper is about partial regularity of minimizers of integral functionals of the general type

$$\mathcal{F}[v] := \int_\Omega F(x, v(x), Dv(x))\, dx \tag{1.1}$$

defined for maps $v\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of the appropriate Sobolev class. We are concerned with the multi-dimensional case $n, N \geqq 2$ and throughout the discussion the set Ω is a fixed bounded open domain in $\mathbb{R}^n$. The integrand $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ will be subjected to a set of conditions (listed below) that by now are standard in the calculus of variations. Among these conditions we single out the condition of quasiconvexity as introduced by Morrey in [31]. We recall that the integrand is said to be quasiconvex provided

$$\int_{(0,1)^n} [F(x_0, y_0, z_0 + D\varphi(x)) - F(x_0, y_0, z_0)]\, dx \geqq 0\,, \tag{1.2}$$

for every $\varphi \in C_c^\infty((0,1)^n, \mathbb{R}^N)$ i.e. for every smooth and compactly supported map, and for all $x_0 \in \Omega$, $y_0 \in \mathbb{R}^N$, $z_0 \in \mathbb{R}^{N\times n}$. This condition replaces the usual convexity condition in the multi-dimensional calculus of variations, and is essentially equivalent to sequential lower semicontinuity in the weak topology of appropriate Sobolev spaces [31, 1]. As such it is intimately linked to the existence of minimizers for $\mathcal{F}$ on Dirichlet classes of Sobolev maps and, as we recall below and which forms the main theme here, quasiconvexity in a strict form allows us to prove partial regularity of minimizers too. Hence quasiconvexity plays a role at many levels, and moreover we note that certain aspects of the mathematical foundations of non-linear elasticity and materials science [3, 9, 32] lead to models with quasiconvex, non-convex integrands.

We next turn to the precise statements and a description of our results, and start with the hypotheses for the integrand, listed below in (1.3), (1.4), (1.5), (1.6), (1.8) and (1.10). As we confine attention to the case where the integrand exhibits quadratic/super-quadratic growth (i.e. $p \geqq 2$), our hypotheses are identical to those of [20] Chapter 9 (cf. (9.31)–(9.34) and (9.66) there). These conditions are standard in the calculus of variations.

Our first assumption is that for each fixed $x \in \Omega$ and $y \in \mathbb{R}^N$ the partial function $F(x, y, \cdot)$ is twice continuously differentiable:

$$z \mapsto F(x, y, z) \quad \text{is} \quad C^2. \tag{1.3}$$

The second assumption is a growth and coercivity condition:

$$f(z) \leqq F(x, y, z) \leqq L(1 + |z|^p) \tag{1.4}$$

for all $x \in \Omega$, $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^{N\times n}$, where $L > 0$ and $p \geqq 2$ are constants and $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is a continuous function with p-growth, i.e.

$$|f(z)| \leqq L(1 + |z|^p)$$

and, for some positive constant $\nu \in (0, L]$,

$$\int_{(0,1)^n} f(D\varphi(x))\, dx \geqq \nu \int_{(0,1)^n} |D\varphi(x)|^p\, dx,$$

for all $z \in \mathbb{R}^{N\times n}$ and all $\varphi \in C_c^\infty((0, 1)^n, \mathbb{R}^N)$. The third condition is a reinforced version of the quasiconvexity condition in (1.2). We assume that F is uniformly strictly quasiconvex in the sense that

$$\begin{aligned} \nu \int_{(0,1)^n} &(1 + |D\varphi(x)|^2)^{\frac{p-2}{2}} |D\varphi(x)|^2\, dx \\ &\leqq \int_{(0,1)^n} [F(x_0, y_0, z_0 + D\varphi(x)) - F(x_0, y_0, z_0)]\, dx \end{aligned} \tag{1.5}$$

for all $x_0 \in \Omega$, $y_0 \in \mathbb{R}^N$, $z_0 \in \mathbb{R}^{N\times n}$, and every $\varphi \in C_c^\infty((0, 1)^n, \mathbb{R}^N)$. Condition (1.5) was first considered in the context of regularity theory by Evans in [14].

Regarding the dependence upon "the coefficients" (x, y) we shall assume the following (non-uniform) continuity condition

$$|F(x_1, y_1, z) - F(x_2, y_2, z)| \leqq \tilde{L}\theta(|y_1|, |x_1 - x_2|^2 + |y_1 - y_2|^2)(1 + |z|^p) \tag{1.6}$$

for all $x_1, x_2 \in \Omega$, $y, y_1, y_2 \in \mathbb{R}^N$ and $z \in \mathbb{R}^{N\times n}$. Here $\tilde{L} \geqq 1$ is a constant and the function $\theta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to be of the form

$$\theta(s, t) = \min\{1, \varrho_0(s)\omega(t)\}\,, \tag{1.7}$$

where $\varrho_0 \colon \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing and $\omega \colon \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus of continuity, by which we mean a non-decreasing concave function such that $\omega(0) = 0$. Finally, following [2], we shall assume a no growth condition on the second derivatives of F with respect to the gradient variable z, F_{zz}, but only the existence of a (non-uniform) modulus of continuity, that is, a separately increasing function $\gamma \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, such that $\gamma(\cdot, 0) = 0$ and

$$|F_{zz}(x, y, z_1) - F_{zz}(x, y, z_2)| \leqq \gamma(|y| + |z_1| + |z_2|, |z_1 - z_2|) \tag{1.8}$$

for every $x \in \Omega$, $y \in \mathbb{R}^N$ and $z_1, z_2 \in \mathbb{R}^{N\times n}$. For later convenience, we define the "growth function" of F_{zz} as

$$G(M) := \sup_{|y|+|z|\leqq M, x\in\Omega} \frac{|F_{zz}(x, y, z)|}{1 + |z|^{p-2}}, \quad M \geqq 0 . \tag{1.9}$$

Let us briefly comment on the last three assumptions. The condition (1.6) quantifies the continuity of the integrand $F(x, y, z)$ with respect to (x, y). In particular, when we assume that

$$\omega(r) \leqq r^{\alpha/2}, \quad \alpha \in (0, 1] \tag{1.10}$$

then $(x, y) \mapsto F(x, y, z)/(1 + |z|^p)$ is Hölder continuous with exponent α uniformly in z. Assumption (1.8) is mild; for example, it is automatically satisfied by product type functionals of the form

$$v \mapsto \int_\Omega c(x, v) g(Dv)\, dx , \tag{1.11}$$

where $c = c(x, y)$ is a Hölder continuous function with values in the interval $[\nu, L]$ and $g = g(z)$ satisfies the quasiconvexity condition (1.5) and has p-growth.

The crucial point, mentioned briefly above, is that the quasiconvexity condition (1.5), together with the other conditions, allows us to prove the so-called partial regularity of minimizers, as first shown in [14]. To be more precise, define

$$\begin{aligned} \Omega_u := \{x \in \Omega \ :\ & u \in C^{1,\sigma}(A, \mathbb{R}^N), \\ & \text{for some } \sigma > 0 \text{ and some neighborhood } A \text{ of } x\} \end{aligned} \tag{1.12}$$

as the set of regular points of u. Then it can be shown that the open set Ω_u has full measure

$$|\Omega \setminus \Omega_u| = 0 \tag{1.13}$$

and that, assuming (1.10), $u \in C^{1,\alpha/2}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^N)$ (see [14, 2, 13]). The closed set

$$\Sigma_u := \Omega \setminus \Omega_u \tag{1.14}$$

is called the *singular set* of the minimizer u. It is in general non-empty, already in the classical case where $F \equiv F(z)$ is strongly convex, as shown by celebrated examples [7, 34, 38]. The above results motivate the quest for better size bounds of the singular set Σ_u of a minimizer u. A natural and well-established way of measuring the size of a set is by its Hausdorff dimension. In the following we denote the Hausdorff dimension of a set $A \subset \mathbb{R}^n$ by $\dim_{\mathcal{H}}(A)$. The first step is to exclude that Σ_u could be n-dimensional. In the special case where $F \equiv F(z)$ is a strongly convex function, the estimate $\dim_{\mathcal{H}}(\Sigma_u) \leqq n-2$ is a classical result dating back to the 1970s. We recall that it is obtained by using the difference-quotient method to the Euler–Lagrange system of the functional, whereby it is shown that $u \in W^{2,2}_{\mathrm{loc}}$. General results about $W^{2,2}_{\mathrm{loc}}$ maps then imply the desired singular set estimate (see [20] Chapter 2 for details). The problem for the general case when $z \mapsto F(x,y,z)$ is strongly convex was raised later (see for instance the open problems in 3 page 269 from [16] and (a) page 117 from [19]) and, essentially under the assumptions considered above, it was settled in [24], where we proved that

$$\dim_{\mathcal{H}}(\Sigma_u) \leqq n - \min\{\alpha, p(s-1)\} . \tag{1.15}$$

Here $\alpha > 0$ is the Hölder continuity exponent of $(x,y) \mapsto F(x,y,z)$ appearing in (1.10), and s is the higher-integrability exponent of the gradient whose existence is guaranteed by Gehring's lemma,

$$|Du|^p \in L^s_{\mathrm{loc}}(\Omega) \quad s \equiv s(n,N,p,L/\nu) > 1 \tag{1.16}$$

(see again [24] for details). Similar results hold for weak solutions to elliptic systems [28, 29]; for an account of results on partial regularity and singular sets of minima we refer to [30]. The proof in [24] relies on a localization technique that makes once again essential use of the Euler–Lagrange systems of certain convex comparison functionals, and finally leads to us establish a higher (fractional) differentiability of Du, which again by general results entails (1.15). An approach based on the Euler–Lagrange system and some kind of difference-quotient method seems to require convexity. In fact, for the quasiconvex case, the Euler–Lagrange system in itself cannot yield regularity results. This was recently shown by Müller & Šverák in [33], where they demonstrated even the absence of partial regularity for critical non-minimizing points of uniformly strictly quasiconvex integrals

$$v \mapsto \int_\Omega F(Dv(x))\,dx . \tag{1.17}$$

That the situation is equally hopeless when the above quasiconvexity condition on the integrand F is strengthened to polyconvexity was demonstrated more recently by Székelyhidi in [39]. Therefore the task of estimating the size of the singular set of minimizers in the quasiconvex (and polyconvex) case appears to require different methods. Hence, despite attracting some attention (see [17] Section 4.2), it has remained an open problem since [14] as to whether the singular set of a minimizer of a uniformly strictly quasiconvex (or polyconvex) integral (1.17) could be n-dimensional.

In the present paper we take the first step, by considering the case of Lipschitzian minimizers of the general quasiconvex functional $\mathcal{F}$ in (1.1), and proving

Theorem 1.1. *Let $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be a minimizer of the functional $\mathcal{F}$ under the assumptions* (1.3)–(1.10). *Then there exists a positive number $\delta > 0$, depending only on $n, N, p, \nu, L, \tilde{L}, \|u\|_{W^{1,\infty}}$, the modulus of continuity $\gamma(\cdot)$ in* (1.8)*, and the functions $\varrho_0(\cdot)$, $G(\cdot)$ in* (1.7)–(1.9)*, but otherwise independent of u, of the exponent α in* (1.10)*, and of the integrand F, such that*

$$\dim_{\mathcal{H}}(\Sigma_u) \leqq n - \delta. \tag{1.18}$$

In fact we obtain the stronger result that the singular set is uniformly porous, see Theorem 5.1 and Section 2 for terminology. Using porosity properties to get dimension bounds is a strategy followed also in other contexts, and our approach undoubtedly has similarities to some of the methods developed in the parametric setting [4, 6, 36]. An important point, cf. Remark 3 below, is that the number δ can be *bounded explicitly in terms of the data*, and, on the contrary to the bound in (1.15), it is independent of the Hölder continuity exponent α. For Lipschitzian solutions this improves the results in [23, 24] for minimizers of convex integrands when the Hölder exponent α is small. Finally, by the method employed here, the best result we can hope for is getting $\delta = 1$, cf. Remark 1 below. This is in accordance with the results for functionals with convex dependence on the gradient variable, see in particular Theorem 1.2 in [24]. Further results and extensions are proposed throughout the paper; in particular, the result of Theorem 1.1 also holds for the so-called ω-minima, see Section 3, and for solutions to quasimonotone systems, see Section 6. Moreover, in Section 4 we shall prove regularity properties of the gradient of minima stated in terms of certain Carleson-type estimates. The derivation of these estimates relies crucially on the Caccioppoli inequalities.

Finally, let us comment on the *a priori* Lipschitz continuity condition on the minimizer in Theorem 1.1. This is a restrictive hypothesis, which is not always satisfied, even in the classical basic case (1.17) when the integrand F is strongly convex (see [38]). However, it nevertheless happens to be automatically satisfied for large classes of quasiconvex functionals of the type (1.17). More precisely, starting from the work of Chipot & Evans [5], it is known that in order to prove the Lipschitz continuity of minimizers only the behavior of $F(z)$ for large $|z|$ matters; more precisely, assuming that $F(z)$ is suitably close in a C^2 sense to the model integrand $z \mapsto |z|^p$, for large values of $|z|$, we can prove that any local minimizer of the functional in (1.17) is locally Lipschitz continuous. Then an estimate of the type (1.18) holds in the interior of Ω, that is, for any subset $\Sigma_u \cap \Omega'$, where $\Omega' \subset\subset \Omega$, and with δ depending also on $\mathrm{dist}(\Omega', \partial\Omega)$. We refer to Section 6 for a brief discussion and further references on these matters.

A brief outline of the organization of the paper is as follows. In Section 2 we have collected some preliminary material, notably the connection between porosity and Hausdorff dimension, and a special case of a square-function characterization of Sobolev maps. Section 3 is devoted to an integral characterization of regular points. Section 4 combines the square-function characterization of Sobolev maps with a Caccioppoli inequality to obtain a Carleson condition for the excess. This condition is used together with the integral characterization of regular points in Section 5 to prove that the singular set is uniformly porous. Theorem 1.1 follows then by the discussion in Section 2. Finally, in Section 6 we discuss various extensions and, as mentioned above, the Lipschitz condition imposed on the minimizer.

2. Preliminaries

In this paper we shall adopt the usual convention of denoting by c a general, positive and finite constant, that may vary from line to line. Special occurences will be denoted by c_1, $\tilde{c}$ and so on. The most relevant dependencies will be indicated. With $x_0 \in \mathbb{R}^n$ and $R > 0$, we denote by $B_R \equiv B(x_0, R) := \{x \in \mathbb{R}^n \, : \, |x - x_0| < R\}$ the open (Euclidean) ball with radius R and center x_0. Often it is clear from the context that the balls under consideration all have the same center and in such cases we merely write B_R etc.

The space of real $N \times n$ matrices, denoted by $\mathbb{R}^{N\times n}$, is considered with the usual Frobenius (i.e. Euclidean) norm and its elements are often denoted by z, z_0 etc.

We use standard notation for maps and function spaces (e.g. as in [20]). In particular, for an integrable map $g\colon B(x_0, R) \to \mathbb{H}$, where $\mathbb{H}$ is a finite dimensional inner-product space, we denote its average by one of the following symbols

$$(g)_B \equiv (g)_{x_0,R} := -\!\!\!\!\!\!\int_{B(x_0,R)} g(x)\,dx := \frac{1}{|B(x_0, R)|}\int_{B(x_0,R)} g(x)\,dx\,,$$

where $|B(x_0, R)|$ denotes the Lebesgue measure of $B(x_0, R)$.

Set porosity and Hausdorff dimension. For a subset A of $\mathbb{R}^n$, a point x in $\mathbb{R}^n$ and a positive number $r > 0$ we define

$$p(A, x, r) := \sup\{\varrho \, : \, B(y, \varrho) \subset B(x, r) \setminus A \text{ for some } y \in \mathbb{R}^n\}. \tag{2.1}$$

Note that $0 \leqq p(A, x, r) \leqq r$ for all x and that $0 \leqq p(A, x, r) \leqq r/2$ for $x \in A$.

Definition 1. For numbers $\kappa > 0$, $\lambda \in [0, 1/2]$ we say that the subset $A \subset \mathbb{R}^n$ is (λ, κ)-porous provided $p(A, x, r) \geqq \lambda r$ holds for all $x \in \mathbb{R}^n$ and all $r \in (0, \kappa)$.

This terminology is taken from [37]. We refer the reader to [27], page 156, for further background and references on this subject. Besides the above notion of uniform porosity we also consider another aimed at describing "non-uniform porosity" of sets, where the "size of the holes" in a ball with radius r is allowed to depend on r itself.

Definition 2. Let $\kappa > 0$ and $m\colon (0, \kappa) \to (0, 1/2]$ be a function. We say that the subset $A \subset \mathbb{R}^n$ is $(m(\cdot), \kappa)$-porous provided $p(A, x, r) \geqq m(r)r$ holds for all $x \in \mathbb{R}^n$ and all $r \in (0, \kappa)$.

For our purposes it is important that (λ, κ)-porous sets have dimensions strictly below n. The main point of interest for us is that porosity allows us to reduce the Hausdorff dimension. The following is an elementary result whose proof can be found for instance in the Introduction of [37].

Lemma 2.1. *There exists a strictly decreasing and continuous function $d_n : \mathbb{R}^+ \to \mathbb{R}^+$ with $d_n(0) = n$, such that whenever A is a non-empty bounded (λ, κ)-porous subset of $\mathbb{R}^n$, where $\kappa > 0$ and $\lambda \in (0, 1/2)$, then*

$$\dim_{\mathcal{H}}(A) \leqq d_n(\lambda) < n \,. \tag{2.2}$$

Note that in the previous proposition the dimensional bound is actually independent of the number κ, a fact that we shall use later. The following upper bound is an immediate consequence of the proof in [37]:

$$d_n(\lambda) \leqq \frac{\log\Big(\big(4\sqrt{n}/\lambda\big)^n + 1\Big)}{\log\Big(4\sqrt{n}/\lambda + 2\Big)} \,.$$

Note that the right-hand side is strictly less than n and that it increases to n when $\lambda \searrow 0$.

Remark 1. The notion of porosity as considered here cannot bound the dimension beyond $n - 1$. Indeed a theorem of Mattila [27] states that

$$\lim_{\lambda \nearrow \frac{1}{2}} d_n(\lambda) = n - 1 \,,$$

and actually a more precise result is due to Salli [37]:

$$\dim_{\mathcal{H}}(A) \leqq n - 1 + \frac{c}{\ln(1/(1 - 2\lambda))} \,, \tag{2.3}$$

where the constant $c \equiv c(n)$ only depends on n. As observed in [37] this bound exhibits the correct asymptotic behavior of $\dim_{\mathcal{H}}(A)$ as $\lambda \nearrow 1/2$. As our bounds on the dimension of the singular sets are derived from porosity, the number δ appearing in (1.18) cannot exceed 1, cf. Section 5. We finally note that not only the Hausdorff dimension but also the (potentially larger) upper Minkowski dimension can be bounded using uniform porosity (see [27, 37]). We have chosen not to highlight this in a theorem for the reason that uniform porosity is an even stronger property (which does not follow from dimensional bounds, Hausdorff, Minkowski or otherwise). The result about uniform porosity of the singular set is stated as Theorem 5.1.

A result of Dorronsoro, revisited. The following result about general Sobolev maps is instrumental to our proof of Theorem 1.1. We state it in a slightly elaborate form which is the one required in Section 4. The result can be inferred from [10] when performing a certain localization argument. For the convenience of the reader we present an elementary proof that avoids the more sophisticated machinery required for the general results presented in [10].

Lemma 2.2. *There exists a constant $c \equiv c(n)$ depending only on n such that for any ball $B(x_0, 2R) \subset \mathbb{R}^n$ and any $w \in W^{1,2}(B(x_0, 2R), \mathbb{R}^N)$ the inequality*

$$\int_{B(x_0,R)} \int_0^R \fint_{B(x,r)} \left| \frac{w(y) - (w)_{x,r} - (Dw)_{x,r}(y - x)}{r} \right|^2 dy \,\frac{dr}{r}\, dx \leqq c \int_{B(x_0,2R)} |Dw|^2 \, dx \tag{2.4}$$

holds.

Proof. We start the proof with some simplifications. First, by considering the map

$$\tilde{w}(y) := \frac{1}{R} w(x_0 + Ry), \quad y \in B(0, 2),$$

instead of w and making appropriate substitutions in (2.4) we see that it suffices to consider the case where $x_0 = 0$ and $R = 1$. Next, as (2.4) remains unchanged under translations $w \mapsto w + k$ we may also assume that

$$(w)_{0,2} = 0. \tag{2.5}$$

For notational convenience we write B for $B(0, 1)$ and rB for $B(0, r)$ in the following. Let $\rho\colon \mathbb{R}^n \to [0, 1]$ be a C^1 cut-off function verifying

$$\rho \equiv 1 \text{ on } 2B, \;\; \rho \equiv 0 \text{ on } \mathbb{R}^n \setminus (3B) \;\text{ and }\; |D\rho| \leqq 2.$$

Define

$$v(x) := \begin{cases} w(x) & \text{if } x \in 2B \\ \\ w\left(\frac{2x}{|x|^2}\right)\rho(x) & \text{if } x \in \mathbb{R}^n \setminus (2B). \end{cases}$$

Then $w \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^N)$ is supported in $3B$ and since for $|x| > 2$,

$$Dv(x) = Dw\left(\frac{2x}{|x|^2}\right) 2 \frac{\mathrm{Id}|x|^2 - 2x \otimes x}{|x|^4} \rho(x) + w\left(\frac{2x}{|x|^2}\right) \otimes D\rho(x),$$

where Id denotes the $n \times n$ identity matrix and $\otimes$ denotes the vector tensor product, we find after a routine estimation

$$\int_{\mathbb{R}^n \setminus (2B)} |Dv|^2 \, dx \leqq c \int_{2B} \left(|w|^2 + |Dw|^2\right) dx.$$

In view of (2.5) Poincaré's inequality yields $\int_{2B} |w|^2 dx \leqq c \int_{2B} |Dw|^2 dx$ so we arrive at

$$\int_{\mathbb{R}^n} |Dv|^2 \, dx \leqq c \int_{2B} |Dw|^2 \, dx \;, \tag{2.6}$$

where $c \equiv c(n)$. Now

$$\begin{aligned} &\int_B \int_0^1 \fint_{B(x,r)} |w(y) - (w)_{x,r} - (Dw)_{x,r}(y - x)|^2 \, dy \, \frac{dr}{r^3} \, dx \\ &\leqq \int_{\mathbb{R}^n} \int_0^\infty \fint_{B(x,r)} |v(y) - (v)_{x,r} - (Dv)_{x,r}(y - x)|^2 \, dy \, \frac{dr}{r^3} \, dx \\ &= \int_0^\infty \fint_{rB} \int_{\mathbb{R}^n} |v(x + y) - (v)_{x,r} - (Dv)_{x,r} y|^2 \, dx \, dy \, \frac{dr}{r^3} \;, \end{aligned}$$

where in the last equality we changed the variables and integration orders. For each fixed $r > 0$ and $y \in rB$ the map $x \in \mathbb{R}^n \mapsto v(x+y) - (v)_{x,r} - (Dv)_{x,r}\,y$ is square integrable. Recall that the Fourier transformation on L^2 is defined as

$$\hat{h}(\xi) := \lim_{k\to\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{|x|\leqq k} h(x)e^{-ix\cdot\xi}\,dx,$$

where the convergence is in the L^2 sense. Hence by Plancherel's theorem and further elementary properties of the Fourier transform we have

$$\begin{aligned}
&\int_0^\infty \fint_{rB}\int_{\mathbb{R}^n} |v(x+y) - (v)_{x,r} - (Dv)_{x,r}\,y|^2\,dx\,dy\,\frac{dr}{r^3}\\
&= \int_0^\infty \fint_{rB}\int_{\mathbb{R}^n} \Big|\hat{v}(\xi)e^{iy\cdot\xi} - \fint_{rB}\hat{v}(\xi)e^{iz\cdot\xi}\,dz\\
&\quad - \fint_{rB}\hat{v}(\xi)\otimes(i\xi)e^{iz\cdot\xi}\,dz\cdot y\Big|^2\,d\xi\,dy\,\frac{dr}{r^3}\\
&= \int_{\mathbb{R}^n} m(\xi)|\xi|^2|\hat{v}(\xi)|^2\,d\xi,
\end{aligned}$$

where

$$m(\xi) := \frac{1}{|\xi|^2}\int_0^\infty \fint_B \Big|e^{iy\cdot r\xi} - \fint_B e^{iz\cdot r\xi}\,dz - iy\cdot r\xi \fint_B e^{iz\cdot r\xi}\,dz\Big|^2\,dy\,\frac{dr}{r^3}.$$

We assert that this multiplier is in fact a (finite) constant: $m(\xi) \equiv m_0$ for all $\xi \neq 0$. This will be obvious once we have shown that $m(\xi)$ is finite and well defined for all $\xi \neq 0$. To this end we start with the observation

$$\fint_B e^{iz\cdot r\xi}\,dz = \frac{c}{(r|\xi|)^{\frac{n}{2}}} J_{\frac{n}{2}}(r|\xi|)$$

where $J_{\frac{n}{2}}$ denotes the Bessel function of the first kind of order $n/2$ and $c \equiv c(n)$ is a constant that only depends on n. Hence by standard properties of Bessel functions we get for some (new) constant $c \equiv c(n)$ that

$$\Big|\fint_B e^{iz\cdot r\xi}\,dz - 1\Big| \leqq cr^2|\xi|^2 \qquad \text{for}\quad r|\xi| \leqq 1 \tag{2.7}$$

and

$$\Big|\fint_B e^{iz\cdot r\xi}\,dz\Big| \leqq \frac{c}{(r|\xi|)^{\frac{n+1}{2}}}\quad \text{for}\ \ r|\xi| \geqq 1. \tag{2.8}$$

Consequently, by splitting the r-integral in the expression for $m(\xi)$ as

$$|\xi|^2 m(\xi) = \int_0^\infty \Big(\cdots\Big)\frac{dr}{r^3} = \int_0^{\frac{1}{|\xi|}} \Big(\cdots\Big)\frac{dr}{r^3} + \int_{\frac{1}{|\xi|}}^\infty \Big(\cdots\Big)\frac{dr}{r^3}, \tag{2.9}$$

using the inequalities

$$\left|e^{iy\cdot r\xi} - \fint_B e^{iz\cdot r\xi}\,dz - iy\cdot r\xi \fint_B e^{iz\cdot r\xi}\,dz\right|^2 \leqq 2\left|e^{iy\cdot r\xi} - 1 - iy\cdot r\xi\right|^2$$
$$+2|1+iy\cdot r\xi|^2 \left|\fint_B e^{iz\cdot r\xi}\,dz - 1\right|^2,$$

$$|e^{iy\cdot r\xi} - 1 - iy\cdot r\xi| \leqq cr^2|\xi|^2 \qquad (|y| \leqq 1 \text{ and } r|\xi| \leqq 1),$$

and (2.7) on the first integral in (2.9), and the inequality

$$\left|e^{iy\cdot r\xi} - \fint_B e^{iz\cdot r\xi}\,dz - iy\cdot r\xi \fint_B e^{iz\cdot r\xi}\,dz\right|^2 \leqq 2 + 2\left|\fint_B e^{iz\cdot r\xi}\,dz\right|^2 (r|\xi|)^2$$

and (2.8) on the second integral in (2.9), we get

$$\sup_\xi |m(\xi)| \leqq c(n) < \infty.$$

Now we can check that $m(O\xi) = m(\xi)$ and $m(t\xi) = m(\xi)$ for all orthogonal matrices $O \in \mathrm{O}(n)$, numbers $t > 0$ and $\xi \neq 0$ by making the appropriate substitutions in the integral defining m. The multiplier m is therefore constant as asserted. The proof is concluded by use of Plancherel's theorem and (2.6).

3. An ε-regularity result

In the remainder of the paper, unless mentioned otherwise, instead of merely considering minimizers, we shall consider the more general ω-minimizers. These intervene in many contexts when dealing with regularity problems in the calculus of variations [8, 13, 20, 23].

Definition 3. A map $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is an ω-minimizer of the functional $\mathcal{F}$ if and only if

$$\int_{B_R} F(x, u(x), Du(x))\,dx \leqq \left[1 + \omega(R^2)\right] \int_{B_R} F(x, v(x), Dv(x))\,dx \quad (3.1)$$

for any $v \in W^{1,p}(B_R, \mathbb{R}^N)$ such that $u - v \in W_0^{1,p}(B_R, \mathbb{R}^N)$, where $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing concave function satisfying $\omega(0) = 0$, and $B_R \subset\subset \Omega$ is an arbitrary ball with radius R.

For notational convenience we used the function ω from (1.6) also in (3.1). It will however be clear from the proof that different choices are possible as well. Of course, a minimizer is also an ω-minimizer (with $\omega \equiv 0$), while ω-minima enjoy the same partial regularity properties of minima described in Section 1 when assuming (1.10) for some $\alpha > 0$, see [13, 20, 23].

In the sequel we refer to points in Ω_u (defined at (1.12)) as regular points of the fixed ω-minimizer u of the functional $\mathcal{F}$. The main result of this section is an integral characterization of such points, stated as Proposition 3.2 below. As the specialist will realize, the result is not really new; rather it is a careful re-reading/re-adjustment of the proof of Proposition 9.3 in [20] under the additional assumption that the ω-minimizer is Lipschitz continuous. We follow the proof given in [20] (which in turn is taken from [18]) mainly for two reasons. First, we want a good estimate of the constant δ in Theorem 1.1, so we have to keep control of all the constants appearing in the estimates, that is, we have to refer to a direct proof of partial regularity of ω-minimizers (not involving indirect arguments such as the blow-up technique [2, 14] and the A-harmonic approximation [13] etc.). Second, the direct proof of [20] provides a decay estimate that allows us to use in an optimal and direct way our Lipschitz regularity assumption. However, the proof in [20, 18] is unfortunately affected by a few inaccuracies, detectable in Theorem 9.1 and Theorem 9.5 from [20]. We shall briefly sketch how to amend these and simultaneously take the opportunity to put certain estimates in a particular form that we need later (in particular, the Caccioppoli inequality in Proposition 3.1 below).

Following the exposition in [20] the computations are conveniently carried out in terms of the auxiliary map

$$V(z) := (1 + |z|^2)^{\frac{p-2}{4}} z, \quad z \in \mathbb{R}^{N\times n}. \tag{3.2}$$

The corresponding excess functional is then

$$E(x_0, R) := \fint_{B(x_0,R)} |V(Du(x)) - V((Du)_{x_0,R})|^2 \, dx. \tag{3.3}$$

However, in view of the inequalities

$$c^{-1}\Big(1 + |z_1|^2 + |z_2|^2\Big)^{\frac{p-2}{2}} \leqq \frac{|V(z_2) - V(z_1)|^2}{|z_2 - z_1|^2} \leqq c\Big(1 + |z_1|^2 + |z_2|^2\Big)^{\frac{p-2}{2}} \tag{3.4}$$

that are valid for all matrices $z_1, z_2 \in \mathbb{R}^{N\times n}$, where $c = c(n, N, p)$ is a constant, see Chapter 9 of [20] (or [21]), it is easily seen that for Lipschitzian maps u the excess in (3.3) is equivalent to

$$H(x_0, R) := \fint_{B(x_0,R)} |Du(x) - (Du)_{x_0,R}|^2 \, dx. \tag{3.5}$$

More precisely, for each $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ we have for all balls $B(x_0, R) \subset \Omega$ that

$$c^{-1} E(x_0, R) \leqq H(x_0, R) \leqq c E(x_0, R), \tag{3.6}$$

for some constant $c = c(n, N, p, \|u\|_{W^{1,\infty}})$. We start with the following Caccioppoli-type inequality (following the terminology of [20] a "Caccioppoli inequality of the second kind"):

Proposition 3.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be an ω-minimizer of the functional $\mathcal{F}$ under the assumptions* (1.3)–(1.10)*. For every polynomial map $a(x) := u_0 + \langle z_0, x - x_0\rangle$, with $u_0 \in \mathbb{R}^N$, $z_0 \in \mathbb{R}^{N\times n}$ such that $|u_0| + |z_0| \leqq M/2$ for some $M > 0$, and every ball $B(x_0, 2R) \subset \Omega$, we have*

$$\begin{aligned}
&\int_{B(x_0,R)} \Big(|Du - Da|^2 + |Du - Da|^p\Big)\, dx \qquad (3.7)\\
&\leqq \frac{c}{R^2}\int_{B(x_0,2R)} |u-a|^2\, dx + \frac{c}{R^p}\int_{B(x_0,2R)} |u-a|^p\, dx\\
&\quad + c\int_{B(x_0,2R)} \Big[\omega(4R^2) + \theta\Big(|u_0|, 4R^2 + |u-u_0|^2\Big)\Big]\big(1 + |Du|^p\big)\, dx,
\end{aligned}$$

where the constant c only depends on n, N, p, ν, L, $\tilde{L}$, $G(M)$ and M.

Proof. Let us set, for any $z \in \mathbb{R}^{N\times n}$

$$g(z) := F(x_0, u_0, z_0 + z) - F(x_0, u_0, z_0) - \langle F_z(x_0, u_0, z_0), z\rangle,$$

and $w(x) := u(x) - a(x)$. Then, with $B(x_0, s) \equiv B_s$, and using mean value theorem as in [2], we find a constant $c \equiv c(n, N, p, L, G(M)) > 0$ such that

$$g(z) \leqq cM^{p-2}|V(z)|^2 \qquad (3.8)$$

and, using quasiconvexity of F,

$$\nu\int_{B_s} |V(D\varphi)|^2\, dx \leqq \int_{B_s} g(D\varphi)\, dx \qquad (3.9)$$

for any $\varphi \in C_c^\infty(B_s, \mathbb{R}^N)$. Moreover we have

$$\begin{aligned}
&\int_{B_s} g(Dw)\, dx \leqq \int_{B_s} g(Dw + D\varphi)\, dx \qquad (3.10)\\
&\quad + c\int_{B_s}\Big[\omega(4R^2) + \theta\Big(|u_0|, 4R^2 + |u-u_0|^2\Big)\Big]\big(1 + |Du|^p + |D\varphi|^p\big)\, dx,
\end{aligned}$$

for every $s \leqq 2R$, and $\varphi \in C_c^\infty(B_s, \mathbb{R}^N)$, while c depends here on $n, N, p, L, \tilde{L}$. In order to prove the previous inequality we use (3.1) and the very definition of w to compute

$$
\begin{aligned}
&\int_{B_s} g(Dw)\,dx \\
&\quad = \int_{B_s} \big[F(x,u,Du) - F(x_0,u_0,z_0) - \langle F_z(x_0,u_0,z_0), Dw\rangle\big]\,dx \\
&\qquad + \int_{B_s} \big[F(x_0,u_0,Du) - F(x,u,Du)\big]\,dx \\
&\quad \leqq \int_{B_s} \big[F(x,u,Du+D\varphi) - F(x_0,u_0,z_0) - \langle F_z(x_0,u_0,z_0), Dw + D\varphi\rangle\big]\,dx \\
&\qquad + \omega(4s^2)\int_{B_s} F(x,u,Du+D\varphi)\,dx \\
&\qquad + \int_{B_s} \big[F(x_0,u_0,Du) - F(x,u,Du)\big]\,dx \\
&\quad \leqq \int_{B_s} g(Dw+D\varphi)\,dx + c\omega(4R^2)\int_{B_s} \big(1+|Du|^p+|D\varphi|^p\big)\,dx \\
&\qquad + \int_{B_s} \big[F(x_0,u_0,Du) - F(x,u,Du)\big]\,dx \\
&\qquad + \int_{B_s} \big[F(x,u,Du+D\varphi) - F(x_0,u_0,Du+D\varphi)\big]\,dx
\end{aligned}
$$

and then (3.10) follows by estimating the last two integrals by means of (1.6) and using the properties of θ. Now let us take $R < t < s < 2R$, and let $\eta \in C_c^\infty(B_s)$ be a cut-off function such that $\eta \equiv 1$ on B_t, and $|D\eta| \leqq 4/(s-t)$. Let us set $\phi_1 := \eta w$ and $\phi_2 := (1-\eta)w$, so that $D\phi_1 + D\phi_2 = Dw$. Then using first (3.9), and then (3.10), we have

$$
\begin{aligned}
\nu \int_{B_s} |V(D\phi_1)|^2\,dx &\leqq \int_{B_s} g(D\phi_1)\,dx = \int_{B_s} g(Dw - D\phi_2)\,dx \\
&= \int_{B_s} g(Dw)\,dx + \int_{B_s} \big[g(Dw-D\phi_2) - g(Dw)\big]\,dx \\
&\leqq \int_{B_s\setminus B_t} g(D\phi_2)\,dx + \int_{B_s\setminus B_t} \big[g(Dw-D\phi_2) - g(Dw)\big]\,dx \\
&\quad + c(M)\int_{B_s} \Big[\omega(4R^2) + \theta\Big(|u_0|, 4R^2 + |u-u_0|^2\Big)\Big]\big(1+|Du|^p\big)\,dx,
\end{aligned}
$$

where we used (3.10), and, at the end, the fact that $|D\phi_1|^p \leqq c(|Du|^p + |z_0|^p) \leqq c(M)\big(1+|Du|^p\big)$. Using (3.8) and the fact that $|V(z)|^2 \approx |z|^2 + |z|^p$, we easily get

$$
\begin{aligned}
\int_{B_t} |V(Dw)|^2\,dx &\leqq c\int_{B_s\setminus B_t} |V(Dw)|^2\,dx + c\int_{B_s} \left|\frac{w}{s-t}\right|^2 + \left|\frac{w}{s-t}\right|^p\,dx \\
&\quad + c\int_{B_s} \Big[\omega(4R^2) + \theta\Big(|u_0|, 4R^2 + |u-u_0|^2\Big)\Big]\big(1+|Du|^p\big)\,dx.
\end{aligned}
$$

The assertion now easily follows by "filling the hole", that is adding the quantity $c\int_{B_t} |V(Dw)|^2\,dx$ to both sides of the previous inequality, and then applying a standard iteration lemma, i.e. Lemma 6.1 from [20]. See [2] for further details. In such a proof, all the constants can be estimated explicitly in terms of the data.

Remark 2. Let us assume $p = 2$, and let us deal with genuine minimizers. By a careful examination of the previous proof, and especially the derivation of (3.8), it follows that in the case $F \equiv F(z)$, and assuming a growth condition of the type

$$|F_{zz}(z)| \leqq L \tag{3.11}$$

we can replace (3.7) by

$$\int_{B(x_0,R)} |Du - Da|^2\, dx \leqq \frac{c}{R^2} \int_{B(x_0,2R)} |u - a|^2\, dx\ , \tag{3.12}$$

where the constant c only depends on n, N, ν, L.

Under the assumptions of the previous proposition, using inequality (3.7), and arguing as in the proof of Theorem 9.5 from [20] with minor technical modifications, we arrive at the following reverse Hölder inequality, valid for any ball $B_{2R} \subset\subset \Omega$, and for any $z_0 \in \mathbb{R}^{N\times n}$:

$$\begin{aligned}\left(\fint_{B_R} |V(Du) - V(z_0)|^{2q}\, dx\right)^{\frac{1}{q}} &\leqq c \fint_{B_{2R}} |V(Du) - V(z_0)|^2\, dx \\ &\quad + cR^{\mu}\left(\fint_{B_{2R}} \left(1 + |Du|^p\right) dx\right)^{s+\frac{\mu}{p}} .\end{aligned} \tag{3.13}$$

Here the constants c and $q > 1$ depend on $n, N, \nu, L, \tilde{L}, G(M)$ and M, with $|z_0| \leqq M$, and

$$\mu = \frac{\alpha}{2}\left(1 - \frac{1}{q}\right) > 0\ . \tag{3.14}$$

The number $s > 1$ is the higher-integrability exponent appearing in (1.16). Concerning the dependence of q we have

$$\lim_{M\nearrow\infty} q(M) = 1 \quad q \in (1, s)\ . \tag{3.15}$$

With (3.13) in our hands the proof of Proposition 9.3 from [20] can be carried out; there the exponent r must be substituted by q defined in (3.13), while μ is the number defined at (3.14). Proposition 9.3 from [20] is all that we need to go on in the following:

Proposition 3.2. *Let $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be an ω-minimizer of the functional $\mathcal{F}$, under the assumptions* (1.3)–(1.10). *Then there exists a positive universal constant $\varepsilon > 0$, depending only on $n, N, p, \nu, L, \tilde{L}$, $\|u\|_{W^{1,\infty}}$, the modulus of continuity $\gamma(\cdot)$ in* (1.8)*, and the function $G(\cdot)$ in* (1.9)*, but otherwise independent of u, of the exponent α, and of the integrand F, such that a point $x_0 \in \Omega$ is regular if and only if there exists a radius $R > 0$ such that $B(x_0, 2R) \subset\subset \Omega$ and*

$$\fint_{B(x_0,2R)} |Du(x) - (Du)_{x_0,2R}|^2\, dx < \varepsilon\ . \tag{3.16}$$

The constant ε is in particular independent of x_0.

Proof. We start by recalling the result of Proposition 9.3 from [20] and describing it in some detail. Given $R > 0$ such that $B(x_0, 2R) \subset\subset \Omega$, we have the following inequality, valid for any $0 < \varrho < R$, $\beta > 1$, and any $\sigma > 0$:

$$E(x_0, \varrho) \leqq A\left\{ \left(\frac{\varrho}{R}\right)^2 + \left(\frac{R}{\varrho}\right)^n \times \left[\sigma + \frac{c(\sigma)}{\beta^2} + c(\sigma)\chi(E(x_0, 2R))\right]\right\}$$
$$E(x_0, 2R) + H\left(\frac{R}{\varrho}\right)^n R^{\mu}. \tag{3.17}$$

Such result holds for $W^{1,p}$ ω-minimizers, without requiring their Lipschitz regularity. In the previous inequality we have that

$$A \equiv A\Big(n, N, p, \nu, L, \tilde{L}, |(u)_{x_0,R}| + |(Du)_{x_0,R}|\Big),$$

is a computable non-decreasing function of any of its arguments. The expression $c(\sigma)$ is such that

$$\lim_{\sigma \searrow 0} c(\sigma) = \infty\,,$$

and it is bounded on the intervals $[a, \infty)$ for each fixed $a > 0$. The function χ also depends on β, and is defined as follows:

$$\begin{aligned} \chi(E(x_0, 2R)) &\equiv \chi\Big(|(u)_{x_0,R}| + |(Du)_{x_0,R}| + \beta,\, E(x_0, 2R)\Big) \\ &\equiv \gamma\Big(|(u)_{x_0,R}| + 2|(Du)_{x_0,R}| + \beta,\, E(x_0, 2R)\Big)^{1-\frac{1}{q}} \\ &\quad + E(x_0, 2R)^{1-\frac{1}{q}}\,. \end{aligned} \tag{3.18}$$

The number μ is defined in (3.14), while the exponent $q > 1$ is the one appearing in (3.13), when applied with $z_0 := (Du)_{x_0,R}$. Recall that the function $\gamma(\cdot)$ is the local modulus of continuity defined at (1.8). Therefore, χ is also an increasing function of each of its arguments. The function H is

$$H \equiv H\Big(n, N, p, \nu, L, \tilde{L}, |(u)_{x_0,R}| + |(Du)_{x_0,R}| + E(x_0, 2R)\Big),$$

and it is also an increasing function. Note that the functions A, H and χ depend on the quantity $|(u)_{x_0,R}| + |(Du)_{x_0,R}|$ also via $G(|(u)_{x_0,R}| + |(Du)_{x_0,R}|)$, and without loss of generality we may suppose that $G(\cdot)$ is a non-decreasing function. We can now start with the proof, which relies on (3.17), and is a variant of the standard iteration argument used in partial regularity proofs. We make essential use of the Lipschitz continuity of u, and start by setting

$$M := \|u\|_{L^\infty(\Omega)} + 3\Big(1 + \|Du\|^2_{L^\infty(\Omega)}\Big)^{\frac{p}{2}}. \tag{3.19}$$

As a first consequence we have that the number q in (3.13) is uniformly bounded away from 1 when x_0 and R vary, see (3.15); according to (3.14) μ stays uniformly bounded away from 0. Therefore, we can consider the values of $q > 1$ and $\mu > 0$

fixed for the rest of the proof, independently of the point $x_0 \in \Omega$, and the radius $R > 0$; their dependence is exactly as the one explained in the statement for the number ε. Now, $E(x_0, 2R) \leqq M$, and hence for any possible point x_0, and radius R, we have that

$$A + H \leqq A(n, N, p, \nu, L, \tilde{L}, M) + H(n, N, p, \nu, L, \tilde{L}, M) =: A_0 + H_0.$$

Similarly, we have

$$\begin{aligned} \chi(E(x_0, 2R)) &\leqq \left[\gamma\Big(M + \beta, E(x_0, 2R)\Big)\right]^{1-\frac{1}{q}} + [E(x_0, 2R)]^{1-\frac{1}{q}} \\ &=: \chi_0(\beta, E(x_0, 2R)). \end{aligned} \tag{3.20}$$

Merging the last two inequalities with (3.17) yields

$$\begin{aligned} E(x_0, \varrho) \leqq A_0 \bigg\{ & \left(\frac{\varrho}{R}\right)^2 + \left(\frac{R}{\varrho}\right)^n \\ & \times \left[\sigma + \frac{c(\sigma)}{\beta^2} + c(\sigma)\chi_0(\beta, E(x_0, 2R))\right]\bigg\} E(x_0, 2R) \\ & + H_0 \left(\frac{R}{\varrho}\right)^n R^{\mu}. \end{aligned} \tag{3.21}$$

Now, we observe that

$$\begin{aligned} E(x_0, 2R) &\leqq c(n, p) \fint_{B(x_0,2R)} \Big(1 + |Du(x)|^2 + |(Du)_{x_0,2R}|^2\Big)^{\frac{p-2}{2}} \\ &\quad \times |Du - (Du)_{x_0,2R}|^2 \, dx \\ &\leqq c(n, p) M \fint_{B(x_0,2R)} |Du - (Du)_{x_0,2R}|^2 \, dx \; . \end{aligned} \tag{3.22}$$

At this point we can conclude in a standard way. We start taking a number $\tau \equiv \tau(n, N, p, \nu, L, \tilde{L}, M, G(M)) \in (0, 1)$ small enough in order to have

$$\tau^{1/2} A_0 \leqq \frac{1}{4} \; . \tag{3.23}$$

Next, insert $\varrho := \tau R$ in (3.21), to get, after an elementary estimation,

$$\begin{aligned} E(x_0, \tau R) \leqq & \bigg\{ \frac{1}{4}\tau^{3/2} + \frac{1}{4}\tau^{-n-1/2} \\ & \times \left[\sigma + \frac{c(\sigma)}{\beta^2} + c(\sigma)\chi_0(\beta, E(x_0, 2R))\right]\bigg\} E(x_0, 2R) \\ & + H_0 \tau^{-n} R^{\mu}. \end{aligned} \tag{3.24}$$

Taking into account the dependence upon the various quantities in τ, we take $\sigma \equiv \sigma(n, N, p, \nu, L, \tilde{L}, M, G(M)) = \tau^{n+2}/3 > 0$. In turn this determines $c(\sigma) \equiv c(n, N, p, \nu, L, \tilde{L}, M, G(M)) < \infty$; then we choose

$$\beta \equiv \beta(n, N, p, \nu, L, \tilde{L}, M, G(M)) := \sqrt{\frac{3c(\sigma)}{\tau^{n+2}}} < \infty \; .$$

Finally we are ready to choose ε appearing in the statement of the proposition; it is at this point that the dependence on the continuity modulus $\gamma(\cdot)$ from (1.8) appears. Using (3.16), keeping in mind (3.22), the dependence upon the various constants in $c(\sigma)$ and τ, and the definition of χ_0 given in (3.20), we take $\varepsilon \equiv \varepsilon(n, N, p, \nu, L, \tilde{L}, M, G(M), \gamma(\cdot)) > 0$ such that

$$\chi_0(\beta, c(n,p)M\varepsilon) < \frac{\tau^{n+2}}{3c(\sigma)}, \tag{3.25}$$

where $c(n, p)$ is the constant appearing in (3.22). With these choices (3.24) can be recast as

$$E(x_0, \tau R) \leqq \frac{\tau^{3/2}}{2} E(x_0, R) + BR^{\mu},$$

and $B \equiv B(\tau) \equiv B(n, N, p, \nu, L, \tilde{L}, M, G(M))$ is an absolute constant. Observe that $E(x_0, t) \leqq \tau^{-n} E(x_0, \tau^k R)$, whenever $t \in (\tau^{k+1}R, \tau^k R)$, for every $k \in \mathbb{N}$. Therefore, assuming with no loss of generality that $\mu \leqq 3/2$, we may apply Lemma 7.3 from [20] with the choice $\varphi(t) := E(x_0, t)$, whereby

$$E(x_0, \varrho) \leqq C \left(\frac{\varrho}{R}\right)^{\mu} E(x_0, R) + CB\varrho^{\mu}, \tag{3.26}$$

for a constant C that only depends on $n, N, p, \nu, L, \tilde{L}, \|u\|_{W^{1,\infty}}$, the modulus of continuity $\gamma(\cdot)$ and the growth function $G(\cdot)$; in particular, it is independent of the point x_0. It is clear from the above proof that (3.26) holds as soon as (3.16) does, for the choice of ε made in (3.25). In turn, by continuity, if (3.16) holds at x_0, then it holds for all points in a small ball $B(x_0, r)$, and therefore so does (3.26). At this stage we may refer to a well-known integral characterization of Hölder continuity due to Campanato and Meyers (see Theorem 2.9 and Chapter 9 of [20]) to deduce that Du is Hölder continuous with exponent $\mu/2$ in $B(x_0, r/2)$; x_0 is therefore a regular point and the proof is concluded.

Remark 3. As also observed at the beginning of the section, in the previous proof all the constants can be bounded explicitly in terms of the data by carefully keeping track of the various estimates involved; therefore also the constant ε can be bounded explicitly in the terms of the data, and this will finally reflect in the possibility of an explicit bound for the number δ in Theorem 1.1 in terms of the data.

Remark 4. The result of Proposition 3.2 remains true without assuming any *a priori* Lipschitz regularity, in the situation considered in Remark 2, and provided condition (1.8) is strengthened to

$$|F_{zz}(z_1) - F_{zz}(z_2)| \leqq \gamma(|z_1 - z_2|). \tag{3.27}$$

Here $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded concave and increasing function such that $\gamma(0) = 0$. Under these assumptions the number ε appearing in (3.16) depends only on n, N, ν, L and $\gamma(\cdot)$. The partial regularity proof of minimizers in this case is simpler, and it can be found in Paragraph 9.4 from [20], where this model case is considered. Other conditions on $F(z)$ are discussed briefly in Section 6.

4. A Carleson condition for the excess

In this section a regularity property for the gradient of an ω-minimizer is established. It can be viewed as a quantitative version of Lebesgue's differentiation theorem expressed in terms of a Carleson condition. We believe the result could be of interest in itself and have therefore highlighted its statement in a theorem. We formulate it in terms of the excess $H(x,r)$ defined in (3.5).

Theorem 4.1. *Let $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be an ω-minimizer of the functional $\mathcal{F}$, under the assumptions* (1.3)–(1.10). *There exist two constants*

$$C_0 = C_0(n, N, p, \nu, L, \tilde{L}, G(\cdot), \|u\|_{W^{1,\infty}}), \qquad R_0 = R_0(C_0, \alpha), \tag{4.1}$$

such that for all balls $B(x_0, 4R) \subset \Omega$ with radii $R \leqq R_0$, the inequality

$$\fint_{B(x_0,R)} \int_0^R \fint_{B(x,r)} |Du(y) - (Du)_{x,r}|^2 \, dy \, \frac{dr}{r} \, dx \leqq C_0 \tag{4.2}$$

holds. In particular, C_0 is independent of the Hölder continuity exponent α in (1.10)*, and both C_0 and R_0 are independent of $x_0 \in \Omega$.*

The proof of Theorem 4.1 has two ingredients. The first is Lemma 2.2, which expresses a general property of Sobolev maps. The second ingredient is a Caccioppoli inequality of the second kind, which is specific to minimizers. In fact, we stress that the proof of this Caccioppoli inequality makes essential use of minimality and uniform strict quasiconvexity.

Proof of Theorem 4.1. For a ball $B(x, 2r) \subset \Omega$, define the affine map $a(y) := (u)_{x,2r} + (Du)_{x,2r}(y - x)$ and note that

$$\sup_{B(x,2r)} |u - (u)_{x,2r}| \leqq 2mr, \quad m := \|u\|_{W^{1,\infty}}. \tag{4.3}$$

By the very definition of H in (3.5) we get

$$H(x,r) \leqq \fint_{B(x,r)} |Du - Da|^2 \, dy,$$

so that, by Proposition 3.1

$$\begin{aligned} H(x,r) &\leqq \fint_{B(x,r)} \Big(|Du - Da|^2 + |Du - Da|^p \Big) \, dy \\ &\leqq \frac{c(m)}{r^2} \fint_{B(x,2r)} |u - a|^2 \, dy + \frac{c(m)}{r^p} \fint_{B(x,2r)} |u - a|^p \, dy \\ &\quad + c(m) \fint_{B(x,2r)} \Big[\omega(4r^2) + \theta\Big(|(u)_{x,2r}|, 4r^2 + |u - (u)_{x,2r}|^2 \Big) \Big] \\ &\quad \times (1 + |Du|^p) \, dy. \end{aligned} \tag{4.4}$$

The constant c, here as in the following, also depends on n, N, ν, L, $\tilde{L}$. Here we may estimate

$$\begin{aligned}
&\frac{c(m)}{r^2} \fint_{B(x,2r)} |u-a|^2\,dy + \frac{c(m)}{r^p} \fint_{B(x,2r)} |u-a|^p\,dy \\
&\quad \leqq \frac{c(m)}{r^2}\Big(1+(2m)^{p-2}\Big) \fint_{B(x,2r)} |u-a|^2\,dy \\
&\quad = \frac{c(m)}{r^2} \fint_{B(x,2r)} |u-a|^2\,dy
\end{aligned}$$

and, recalling that $\theta(s,t)=\min\{1,\varrho_0(s)\omega(t)\}$, it follows that

$$\begin{aligned}
&c(m) \fint_{B(x,2r)} \Big[\omega(4r^2)+\theta\Big(|(u)_{x,2r}|, 4r^2+|u-(u)_{x,2r}|^2\Big)\Big](1+|Du|^p)\,dy \\
&\leqq c(m)\big(1+\varrho_0(m)\big) \fint_{B(x,2r)} \omega\Big(4r^2+|u-(u)_{x,2r}|^2\Big)(1+|Du|^p)\,dy.
\end{aligned}$$

Since $\omega(\cdot)$ is concave we have, by Jensen's inequality and (4.3),

$$\begin{aligned}
&\fint_{B(x,2r)} \omega\Big(4r^2+|u-(u)_{x,2r}|^2\Big)\,dy \\
&\quad \leqq \omega\left(\fint_{B(x,2r)} \Big(4r^2+|u-(u)_{x,2r}|^2\Big)\,dy\right) \\
&\quad \leqq \omega(4(1+m^2)r^2) \leqq \omega(c(m)r^2)\ .
\end{aligned}$$

Collecting the above estimates we arrive at

$$H(x,r) \leqq \frac{c}{r^2} \fint_{B(x,2r)} |u-a|^2\,dy + c\omega(cr^2) \tag{4.5}$$

for some constant $c=c(n,N,p,\nu,L,\tilde{L},\varrho_0,m,G(m))$. We remark that the constant c in particular is independent of the modulus of continuity $\omega(\cdot)$. At this stage we invoke Lemma 2.2. Fix a ball $B(x_0,R)$ with $B(x_0,4R)\subset\Omega$. Integrating (4.5) we get

$$\begin{aligned}
&\int_{B(x_0,R)}\int_0^R H(x,r)\,\frac{dr}{r}\,dx \leqq c\int_0^R \omega(cr^2)\,\frac{dr}{r}\,|B(x_0,R)| \\
&\quad + c\int_{B(x_0,R)}\int_0^R \fint_{B(x,2r)} \Big|u(y)-(u)_{x,2r}-(Du)_{x,2r}(y-x)\Big|^2\,dy\,\frac{dr}{r^3}\,dx.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^R \fint_{B(x,2r)} |u(y)-(u)_{x,2r}-(Du)_{x,2r}(y-x)|^2\,dy\,\frac{dr}{r^3} \\
&\quad = 4\int_0^{2R} \fint_{B(x,r)} |u(y)-(u)_{x,2r}-(Du)_{x,2r}(y-x)|^2\,dy\,\frac{dr}{r^3}
\end{aligned}$$

we get by use of Lemma 2.2

$$\begin{aligned}\int_{B(x_0,R)}\int_0^R H(x,r)\,\frac{dr}{r}\,dx &\leqq c\int_0^R \omega(cr^2)\,\frac{dr}{r}\,|B(x_0,R)| \\ &\quad + c\int_{B(x_0,4R)} |Du|^2\,dx \\ &\leqq \left(c\int_0^{R_0}\omega(cr^2)\,\frac{dr}{r} + 4^n cm^2\right)|B(x_0,R)|.\end{aligned}$$

Under the assumption (1.10) we have

$$\int_0^{R_0}\omega(cr^2)\,\frac{dr}{r} \leqq \frac{c^{\frac{\alpha}{2}}}{\alpha}\Big(R_0\Big)^{\alpha} \leqq 1, \tag{4.6}$$

where the last inequality follows taking $R_0 \equiv R_0(c,\alpha) > 0$ small enough. The proof is complete.

Remark 5. We do not need the condition (1.10) on $\omega(\cdot)$ to conclude with inequality (4.6). It suffices to have the following Dini condition:

$$\int_0 \frac{\omega(r^2)}{r}\,dr < \infty\,. \tag{4.7}$$

Indeed, under this condition we can still take $R_0 \equiv R_0(c,\omega(\cdot)) > 0$ such that

$$\int_0^{R_0}\omega(r^2)\,\frac{dr}{r} = \frac{1}{\sqrt{c}}\int_0^{\sqrt{c}R_0}\omega(r^2)\,\frac{dr}{r}\,dr \leqq 1.$$

This observation opens the way for an extension of our results to elliptic systems and quasiconvex functionals with appropriately Dini-continuous coefficients. Indeed, in such cases it is still possible to prove partial regularity of weak solutions/minimizers [11–13].

Fix an open subset $\Omega' \subset\subset \Omega$, and denote $d := \operatorname{dist}(\Omega', \partial\Omega) > 0$. We recall that a measure μ on $\Omega' \times [0,d)$ is called a *Carleson measure* provided there exist constants $c \equiv c(\mu(\cdot)) < \infty$ and $R_0 \in (0, d/4)$ such that

$$\mu(B(x,R)\times[0,R)) \leqq cR^n \quad \forall\; R \in (0,R_0) \quad \forall\, x \in \Omega'. \tag{4.8}$$

The result of Theorem 4.1 can be restated in this terminology as

Proposition 4.2. *Let $u \in W^{1,\infty}(\Omega,\mathbb{R}^N)$ be an ω-minimizer of the functional $\mathcal{F}$ under the assumptions* (1.3)–(1.10). *There exist two constants c and R_0* (*as in* (4.1)) *such that*

$$H(x,r)\,\frac{dr}{r}\,dx$$

is a Carleson measure in the sense of (4.8).

Something can still be said when minimizers are not Lipschitz. Indeed, a certain integral measure of the convergence rate of $Du - (Du)_{x,r}$ in the sense of (4.2) can be obtained in terms of powers of radii, essentially as a consequence of higher integrability. We shall restrict ourselves now to the situation considered in Remark 2. Starting with (4.4) in the quadratic case $p = 2$, and using (3.12) instead of (3.7) we get

$$H(x,r) \leqq \frac{c}{r^2} \fint_{B(x,2r)} |u-a|^2\, dy,$$

and proceeding as after (4.5) we arrive at

$$\int_{B(x_0,R)} \int_0^R H(x,r)\, \frac{dr}{r}\, dx \leqq c \int_{B(x_0,4R)} |Du|^2\, dx. \tag{4.9}$$

Now fix $t \geqq 1$; regardless of whether the right-hand side is infinite, Hölder's inequality gives

$$\int_{B(x_0,4R)} |Du|^2\, dx \leqq c \left(\int_{B(x_0,4R)} |Du|^{2t}\, dx \right)^{\frac{1}{t}} R^{n\left(1-\frac{1}{t}\right)}.$$

Therefore, combining the last inequality with (4.9) we obtain the following weaker Carleson-type decay estimate:

Proposition 4.3. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimizer of the functional $\mathcal{F}$, with $F \equiv F(z)$, under the assumptions* (1.3)–(1.5) *with $p = 2$ and* (3.11). *There exists a constant $C_1 = C_1(n, N, \nu, L)$ such that for all balls $B(x_0, 4R) \subset \Omega$, the following inequality holds:*

$$\int_{B(x_0,R)} \int_0^R \fint_{B(x,r)} |Du(y) - (Du)_{x,r}|^2\, dy\, \frac{dr}{r}\, dx$$
$$\leqq C_1 \left(\int_{B(x_0,4R)} |Du|^{2t}\, dx \right)^{\frac{1}{t}} R^{n\left(1-\frac{1}{t}\right)}, \tag{4.10}$$

where $t \geqq 1$. In particular, (4.10) *is non-trivial taking $t = s$, where $s > 1$ is the higher-integrability exponent of Du defined in* (1.16).

Though not sufficient to establish an estimate like (1.18), the previous result still implies a certain degree of porosity of the singular set. The precise statement is in Proposition 5.2 below.

5. Porosity of the singular set and proof of Theorem 1.1

Theorem 1.1, and its more general version concerning ω-minima, is an immediate consequence of Lemma 2.1 and the following result.

Theorem 5.1. *Let $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be an ω-minimizer of the functional $\mathcal{F}$ under the assumptions* (1.3)–(1.10). *Then there exists a number*

$$\lambda = \lambda(n, N, p, \nu, L, \tilde{L}, \gamma(\cdot), G(\cdot), \varrho_0(\cdot), \|u\|_{W^{1,\infty}}) \in (0, 1/2) \tag{5.1}$$

such that for each $\Omega' \subset\subset \Omega$ there is a $\kappa > 0$ for which $\Omega' \cap \Sigma_u$ is (λ, κ)-porous.

Proof. The proof is based on Proposition 3.2 and Theorem 4.1. Fix $\Omega' \subset\subset \Omega$ and let $A := \Omega' \cap \Sigma_u$. Define

$$\kappa := \frac{1}{8} \min\{\mathrm{dist}(\Omega', \partial\Omega), R_0\}, \tag{5.2}$$

where $R_0 > 0$ is defined in Theorem 4.1. Let $x_0 \in \mathbb{R}^n$ and note that

$$\mathrm{dist}(x_0, \Omega') \geqq \frac{\mathrm{dist}(\Omega', \partial\Omega)}{8} \implies p(A, x_0, R) \geqq R/2 \ \text{ if } \ R \in (0, \kappa],$$

since, trivially, $B(x_0, R/2) \cap \Omega' = \emptyset$. Hence for the rest of the proof we may assume without loss of generality that $\mathrm{dist}(x_0, \Omega') < \mathrm{dist}(\Omega', \partial\Omega)/8$. In this situation $B(x_0, 4R) \subset \Omega$ when $R \leqq \kappa$, so by virtue of Theorem 4.1,

$$\int_{B(x_0,R)} \int_0^R H(x, r) \frac{dr}{r}\, dx \leqq C_0 |B(x_0, R)|, \tag{5.3}$$

where $H(x, r)$ is the excess defined in (3.5). For each $\Lambda \in (0, 1)$ we consider the set

$$E := \{x \in B(x_0, R/2) : \inf_{\Lambda R < r < R} H(x, r) \geqq 2^{-n}\varepsilon\},$$

where the number $\varepsilon > 0$ is defined in Proposition 3.2. By routine estimations

$$|E| \frac{\varepsilon}{2^n} \ln \frac{1}{\Lambda} \leqq \int_E \int_{\Lambda R}^R H(x, r) \frac{dr}{r}\, dx \leqq C_0 |B(x_0, R)| \tag{5.4}$$

and thus

$$\frac{|E|}{|B(x_0, R/2)|} \leqq \frac{2^{2n} C_0}{\varepsilon \ln \frac{1}{\Lambda}}.$$

If therefore we take

$$\Lambda := \exp\left(-\frac{2^{2n+1} C_0}{\varepsilon}\right), \tag{5.5}$$

then $|E|/|B(x_0, R/2)| \leqq 1/2$ and it follows that we can find $x \in B(x_0, R/2) \setminus E$, that is, $x \in B(x_0, R/2)$ and $H(x, r) < 2^{-n}\varepsilon$ for some $r \in (\Lambda R, R)$. Now if $x' \in B(x, r/2) \subset B(x_0, R)$, then $B(x', r/2) \subset B(x, r)$ and

$$H(x', r/2) \leqq \fint_{B(x', \frac{r}{2})} |Du - (Du)_{x,r}|^2\, dy \leqq 2^n H(x, r) < \varepsilon,$$

and therefore any such x' is a regular point by virtue of Proposition 3.2; as a consequence we have $B(x, r/2) \cap \Sigma_u = \emptyset$. To conclude the proof we take $\lambda := \Lambda/2$, and note that then $B(x, \lambda R) \subset B(x, r/2) \subset B(x_0, R)\backslash A$, that is $p(A, x_0, R) \geqq \lambda R$ for $R \in (0, \kappa]$, and the (λ, κ)-porosity follows. Note that by (5.5) the dependence on the various parameters of Λ is inherited by the ones of ε and C_0; this implies the dependence stated in (5.1).

In the general case where minimizers are not assumed to be Lipschitz continuous the singular set turns out to be non-uniformly porous in the sense of Definition 2; we shall see this under the assumptions of Remarks 2 and 4.

Proposition 5.2. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimizer of the functional $\mathcal{F}$, with $F \equiv F(z)$, under the assumptions* (1.3)–(1.5) *with $p = 2$,* (3.11)*, and* (3.27). *For every $\Omega' \subset\subset \Omega$ there is a $\kappa > 0$ for which $\Omega' \cap \Sigma_u$ is $(m(\cdot), \kappa)$-porous, where*

$$m(r) := \frac{1}{2} \exp\left(-\frac{C_2}{r^{\frac{n}{s}}}\right), \tag{5.6}$$

the number C_2 is a constant depending only on $n, N, \nu, L, \gamma(\cdot)$, and where s is the one from (1.16). *Moreover, in* (5.6) *the exponent s can be replaced by any $t > 1$ such that $Du \in L^{2t}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$.*

The significance of the previous result is clarified by the following observation: whenever $B(x_0, R) \subset\subset \Omega$ is a ball, by Lebesgue's differentiation theorem it is always possible to find a smaller ball $B(x', \lambda R) \subset B(x_0, R)$ such that $H(x', \lambda R)$ is small enough and, consequently, $B(x', \lambda R/2) \subset \Omega\backslash\Sigma_u$. On the other hand, there is no *a priori* lower bound on the size of λ. Proposition 5.2 provides such a bound in terms of the function $m(\cdot)$, and on the structural data $n, N, \nu, L, \gamma(\cdot)$. This is another weaker and geometric way to say that the singular set is small. Also note that the integrability properties of the gradient play a relevant role in this situation, exactly as in the convex case (see [24]).

Proof of Proposition 5.2. We follow the proof and the notation of Theorem 5.1. Instead of using Theorem 4.1, we use Proposition 4.3, and instead of (5.3) we find

$$\int_{B(x_0,R)} \int_0^R H(x, r)\, \frac{dr}{r}\, dx \leqq \frac{C_1}{\omega_n} |B(x_0, R)|^{1-\frac{1}{s}}, \tag{5.7}$$

where s appears in (1.16), and C_1 in Proposition 4.3. Estimating as in (5.4), and denoting $\omega_n := |B(0, 1)|$, we get

$$\frac{|E|}{|B(x_0, R/2)|} \leqq \frac{2^{2n} C_1}{\omega_n \varepsilon \ln \frac{1}{\Lambda} |B(x_0, R)|^{\frac{1}{s}}}\,.$$

We can conclude again that $|E|/|B(x_0, R/2)| \leqq 1/2$ provided this time we take

$$\Lambda(R) := \exp\left(-\frac{2^{2n+1} C_1}{\omega_n^2 \varepsilon R^{\frac{n}{s}}}\right). \tag{5.8}$$

The rest of the proof follows exactly as for Theorem 5.1, just taking

$$C_2 := \frac{2^{2n+1}C_1}{\omega_n^2 \varepsilon}$$

and keeping into account of the dependence on the parameters of C_1 stated in Proposition 4.3, and of the number ε, stated in Remark 4. Also observe that the number κ can be just defined as $\kappa := 1/8\mathrm{dist}(\Omega', \partial\Omega)$.

6. Additional results

6.1. Asymptotically regular integrands

For certain classes of quasiconvex integrals, where the integrand $F(x, y, z) \equiv F(z)$ has a sufficiently regular behavior for large values of $|z|$, ω-minimizers are automatically locally Lipschitz. In the case of minima, this has been first pointed out by CHIPOT & EVANS in [5] when $p = 2$, and subsequently extended by RAYMOND in [35] to the cases $p > 2$ and more general integrands of the form $F = F_0(z) + a(x, y)$. More precisely, letting

$$H(z) := (1 + |z|^2)^{p/2} , \tag{6.1}$$

the integrand $F(z)$ is assumed to satisfy

$$\lim_{|z|\to\infty} \frac{|F_{zz}(z) - H_{zz}(z)|}{|z|^{p-2}} = 0 . \tag{6.2}$$

A typical model case consists of the functionals

$$v \mapsto \int_\Omega (1 + |Dv|^2)^{\frac{p}{2}} + g(Dv)\, dx , \tag{6.3}$$

where $g\colon \mathbb{R}^{N\times n} \to \mathbb{R}^+$ is a C^2 and quasiconvex function (not necessarily strictly), such that $g_{zz}(z)/|z|^{p-2} \to 0$ when $|z| \to \infty$. For such functionals we have local Lipschitz continuity of $W^{1,p}$ ω-minimizers, and therefore it follows from Theorem 1.1 that for every $\Omega' \subset\subset \Omega$, there exists a positive number

$$\delta' \equiv \delta'(n, N, p, \nu, L, G(\cdot), \mathrm{dist}(\Omega', \partial\Omega)) > 0,$$

such that

$$\dim_{\mathcal{H}}(\Sigma_u \cap \Omega') \leqq n - \delta'. \tag{6.4}$$

Finally, when assuming (6.2), and $\partial\Omega$ sufficiently smooth, e.g. C^2, minimizing $\mathcal{F}$ in a prescribed Dirichlet class $u_0 + W_0^{1,p}(\Omega, \mathbb{R}^N)$, with $u_0 \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$ for some $\beta \in (0, 1]$, that is, solving

$$\begin{cases} \displaystyle\min_w \int_\Omega F(Dw(x))\, dx \\ w \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^{N\times n}) , \end{cases}$$

we obtain a globally Lipschitz continuous minimizer $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$. In this case estimate (1.18) remains valid with a δ depending only on the "data", that is $n, N, p, \nu, L, \tilde{L}, \gamma(\cdot), G(\cdot), \partial\Omega, \|u_0\|_{C^{1,\beta}}$. For such global regularity results, as well as for their validity in the case of ω-minima we refer to a recent paper by Foss [15]. Observe also that the proofs in [5] are indirect, i.e. they rely on a blow-up argument. Nevertheless, using the direct proofs developed [35] and [15], it is possible to quantify the constants in the Lipschitz estimates, and therefore once again the constant δ in (1.18).

We end this brief discussion of Lipschitz estimates with the remark that other choices, different from (6.1), are possible too. More precisely, what is required for the comparison integrand H is that minimizers of

$$v \mapsto \int_\Omega H(Dv)\,dx$$

are fully regular. This is the case, for instance, when $H(z)$ has the so-called "Uhlenbeck structure", i.e. $H(z) := g(|z|)$, where the function $g\colon \mathbb{R} \to \mathbb{R}$ is of class $C^0(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ and satisfies suitable growth and monotonicity conditions; see [30] Section 4.7. Another possibility is to use $H(z) := |z|^{p-2}\langle Az, z\rangle$, where A is a constant tensor, satisfying $\langle Az, z\rangle \geqq \nu|z|^2$. When $p = 2$, this is the original example considered in [5]. Another source of examples with this kind of behavior for large $|z|$ is [22].

6.2. Quasimonotone systems

The result of Theorem 1.1 also extends to weak solutions of so-called quasimonotone systems (see [26, 40])

$$\operatorname{div} a(x, Du) = 0, \tag{6.5}$$

where the vector field $a\colon \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is C^1 in the last (gradient) variable, and satisfies for some constants $0 < \nu < L$ the growth condition

$$|a(x, z)| \leqq L(1 + |z|^2)^{\frac{p-1}{2}}, \tag{6.6}$$

for all $x \in \Omega$, $z \in \mathbb{R}^{N\times n}$, and the *uniform strict quasimonotonicity* condition:

$$\begin{aligned} \nu \int_{(0,1)^n} &(1 + |D\varphi(x)|^2)^{\frac{p-2}{2}} |D\varphi(x)|^2\,dx \\ &\leqq \int_{(0,1)^n} \langle a(x_0, z_0 + D\varphi) - a(x_0, z_0), D\varphi\rangle\,dx, \end{aligned} \tag{6.7}$$

for every $x_0 \in \Omega$, $z_0 \in \mathbb{R}^{N\times n}$ and $\varphi \in C_c^\infty((0, 1)^n, \mathbb{R}^N)$. The last condition is clearly analogous to (1.5). Finally we assume a Hölder continuous dependence on the coefficient x, that is

$$|a(x_1, z) - a(x_2, z)| \leqq \tilde{L}|x_1 - x_2|^\alpha (1 + |z|^2)^{\frac{p-1}{2}}, \tag{6.8}$$

for all $x_1, x_2 \in \Omega$ and $z \in \mathbb{R}^{N\times n}$, where $\tilde{L} \geqq 1$ and $\alpha \in (0,1]$, and as for (1.8) and the existence of the modulus of continuity $\gamma\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as in (1.8), such that

$$|a_z(x, z_1) - a_z(x, z_2)| \leqq \gamma(|z_1| + |z_2|, |z_1 - z_2|), \tag{6.9}$$

for every $x \in \Omega$ and $z_1, z_2 \in \mathbb{R}^{N\times n}$. According to (1.9), we define

$$G(M) := \sup_{|z|\leqq M, x\in\Omega} \frac{|a_z(x,z)|}{1+|z|^{p-2}}, \quad M \geqq 0\,. \tag{6.10}$$

A weak solution to (6.5), under the assumption (6.6), is of course a map $u \in W^{1,p}(\Omega, \mathbb{R}^{N\times n})$ such that

$$\int_\Omega \langle a(x, Du), D\varphi\rangle\, dx = 0$$

for every $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$. Under the assumptions (6.6)–(1.8) the partial $C^{1,\alpha}$-regularity in the sense of (1.12)–(1.13) of weak solutions to (6.5) has been established by HAMBURGER, see Theorem 1.1 from [21], and this time it turns out that $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^N)$. On the other hand, condition (6.7) is too weak to allow for the application of any type of difference-quotient method, and therefore no dimension estimate for the singular sets of weak solutions is available. Nevertheless we have

Theorem 6.1. *Let $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be a weak solution to* (6.5) *under the assumptions* (6.6)–(6.9)*, and let $\Sigma_u := \Omega \setminus \Omega_u$ denote its singular set. Then there exists a positive number $\delta > 0$, depending only on n, N, p, ν, L, $\tilde{L}$, $\|u\|_{W^{1,\infty}}$, the modulus of continuity $\gamma(\cdot)$ in* (6.9)*, and the growth function $G(\cdot)$ in* (6.10)*, but otherwise independent of u, of the exponent α, and of the vector field a, such that* $\dim_{\mathcal{H}}(\Sigma_u) \leqq n - \delta$.

The proof of the previous theorem can be obtained by the procedure in Section 3, once the analogues of Propositions 3.1 and 3.2 have been established. In turn this can be done having as a starting point a suitable analogue of estimate (3.17), that is estimate (5.1) from [21]. Some remarks; the number δ here is once again explicitly computable, since the methods in [21] are not indirect. The result of Theorem 6.1 obviously applies to elliptic systems, that are a particular case of strictly quasimonotone systems as considered above, and in some cases improves the results in [29], where MINGIONE proved the estimate $\dim_{\mathcal{H}}(\Sigma_u) \leqq n - 2\alpha$. Here, in the case of Lipschitzian solutions the dimension estimate does not get lost when $\alpha \to 0$. Moreover, the result of Theorem 6.1 also extends to more general systems of the type $\operatorname{div} a(x, u, Du) = 0$, under the natural assumptions considered in [21]. Finally, restricting to the autonomous case $a(x,z) \equiv a(z)$, the assumption $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ considered in the last theorem is automatically satisfied assuming the following condition, similar to the one in (6.11):

$$\lim_{|z|\to\infty} \frac{|a_z(z) - K_z(z)|}{|z|^{p-2}} = 0\,, \tag{6.11}$$

where this time $K(z) = (1 + |z|^2)^{p-2/2} z$ similarly to Subsection 6.1. A typical model example is given by $a(z) := (1 + |z|^2)^{(p-2)/2} z + b(z)$, where the C^1-vector field $b\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is quasimonotone and satisfies the condition $b_z(z)/|z|^{p-2} \to 0$ when $|z| \to \infty$. This follows by use of arguments similar to those applied in the case of functionals. Also for quasimonotone systems we have a singular set estimate in the interior: we have that given a weak solution to (6.5) $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, then for every $\Omega' \subset\subset \Omega$, there exists a positive number $\delta' > 0$ depending on the objects $n, N, p, \nu, L, \tilde{L}, \gamma(\cdot), G(\cdot), \mathrm{dist}(\Omega', \partial\Omega)$, such that $\dim_{\mathcal{H}}(\Sigma_u \cap \Omega') \leqq n - \delta'$. Again, as for the case of functionals we can have different choices for $K(z)$ in (6.11). For instance we can take any vector field $K(z) := h(|z|)z$, where h satisfies suitable growth and monotonicity conditions; see again [30] Section 4.7.

Another possible extension of our results concerns the so-called $W^{1,q}$-local minima (see [25]). As the Caccioppoli inequality in such cases only has been established in a very weak form, the necessary modifications are somewhat more involved and we plan to report on it elsewhere.

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