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Derivation of the Zakharov Equations

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Abstract

This article continues the study, initiated in [27, 7], of the validity of the Zakharov model which describes Langmuir turbulence. We give an existence theorem for a class of singular quasilinear equations. This theorem is valid for prepared initial data. We apply this result to the Euler–Maxwell equations which describes laser-plasma interactions, to obtain, in a high-frequency limit, an asymptotic estimate that describes solutions of the Euler–Maxwell equations in terms of WKB approximate solutions, the leading terms of which are solutions of the Zakharov equations. Due to the transparency properties of the Euler–Maxwell equations evidenced in [27], this study is carried out in a supercritical (highly nonlinear) regime. In such a regime, resonances between plasma waves, electromagnetic waves and acoustic waves could create instabilities in small time. The key of this work is the control of these resonances. The proof involves the techniques of geometric optics of Joly, Métivier and Rauch [12, 13]; recent results by Lannes on norms of pseudodifferential operators [14]; and a semiclassical paradifferential calculus.

1. Introduction

We describe solutions of initial value problems for quasilinear hyperbolic systems of the form

$$\partial_t u + \frac{1}{\varepsilon^2}\mathcal{A}(\varepsilon, \varepsilon u, \varepsilon\partial_x)u = \frac{1}{\varepsilon}\mathcal{B}(u, u), \tag{1}$$

in the high-frequency limit $\varepsilon \to 0$.

In (1), $\mathcal{A}$ is a symmetric hyperbolic, differential, or pseudodifferential operator; the singular source term $\mathcal{B}$ is bilinear. The unknown u^ε has values in $\mathbb{R}^n$. It depends on time $t \in \mathbb{R}_+$ and space $x \in \mathbb{R}^d$, and is subject to the initial condition,

$$u^\varepsilon(0, x) = a^\varepsilon(x), \tag{2}$$

where a^ε is a bounded family in $H^s(\mathbb{R}^d)$, for some large s.

In this setting, the existence, uniqueness and regularity of solutions to (1)–(2) for fixed $\varepsilon > 0$ is classical.

The limit $\varepsilon \to 0$ is singular in two ways: first, solutions develop fast oscillations in time, with frequencies of typical size $O(1/\varepsilon^2)$; second, the amplitude $O(1)$ of the initial datum is large, and hence the singular source term $\mathcal{B}/\varepsilon$ could create instabilities in small time $O(\varepsilon)$.

Under appropriate assumptions, we prove the existence of solutions to (1)–(2) over time intervals independent of ε, and their stability under initial perturbations of the form $\varepsilon^{k_0}\varphi^\varepsilon$, where k_0 is large enough, and φ^ε is bounded in a semiclassical Sobolev space. In particular, φ^ε may contain fast oscillations of the form $e^{ikx/\varepsilon}$.

We show that our assumptions are satisfied by the Euler–Maxwell equations which describe laser-plasma interactions. This implies in particular that, in a high-frequency limit, solutions of the Euler–Maxwell equations are well approximated by WKB approximate solutions, the leading terms of which are solutions of the Zakharov equations.

Our assumptions and results are precisely stated in Section 2.4.

1.1. Discussion: weakly nonlinear versus highly nonlinear geometric optics

Consider (1), and assume for instance that $\mathcal{A}$ is a differential operator of the form

$$\mathcal{A}(\varepsilon, \varepsilon u, \varepsilon\xi) = \mathcal{A}_0(\varepsilon) + \varepsilon\mathcal{A}_1(\varepsilon, \xi) + \varepsilon^2\mathcal{A}_2(\varepsilon, u, \xi), \tag{3}$$

where $\mathcal{A}_0$ is skew-Hermitian and does not depend on ξ, and where $\mathcal{A}_1$ and $\mathcal{A}_2$ are linear in ξ and Hermitian. Suppose that the family of initial data has the form $u^\varepsilon(0, x) = \varepsilon^p a^\varepsilon(x)$, where a^ε is bounded in $H^s(\mathbb{R}^d)$, uniformly in ε, for $s > 1 + \frac{d}{2}$.

The *weakly nonlinear* regime corresponds to $p = 1$. In this regime, the classical existence proof provides a maximal existence time $t^*(\varepsilon)$ that satisfies $\liminf_{\varepsilon\to 0} t^*(\varepsilon) > 0$. Indeed, if we let $v^\varepsilon = \varepsilon u^\varepsilon$, then the initial datum for v^ε is $O(1)$, and the equation in v^ε is

$$\partial_t v^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{A}(\varepsilon, \varepsilon v^\varepsilon, \varepsilon\partial_x)v^\varepsilon = \mathcal{B}(v^\varepsilon, v^\varepsilon).$$

The classical H^s estimate for quasilinear symmetric hyperbolic operators then yields

$$|v^\varepsilon(t)|_{H^s} \leqq |a^\varepsilon|_{H^s} + C\int_0^t |v^\varepsilon(t')|_{H^s}\, dt',$$

where C depends on $|v^\varepsilon|_{W^{1,\infty}}$, then, with Gronwall's lemma, the uniform bound

$$|v^\varepsilon(t)|_{H^s} \leqq |a^\varepsilon|_{H^s} e^{Ct}.$$

This uniform estimate is the key of the proof of the existence of a solution over a time interval independent of ε.

On the contrary, when $p = 0$, the initial data have a *large amplitude* $O(1)$. In this regime, the maximal existence time *a priori* satisfies $t^*(\varepsilon) = O(\varepsilon)$, and in particular $\liminf_{\varepsilon\to 0} t^*(\varepsilon) = 0$. The H^s energy estimate gives indeed

$$|u^\varepsilon(t)|_{H^s} \leqq |a^\varepsilon|_{H^s} + \frac{C}{\varepsilon}\int_0^t |u^\varepsilon(t')|_{H^s}\, dt',$$

whence,

$$|u^\varepsilon(t)|_{H^s} \leqq |a^\varepsilon|_{H^s} e^{Ct/\varepsilon}.$$

This shows that the singular term $\mathcal{B}/\varepsilon$ may cause the solution to blow-up in small time. This highly nonlinear (or supercritical) regime is called *oscillations fortes*, or *strong oscillations*, by Cheverry, Guès and Métivier in [5].

Our goal is to state conditions on (1)–(2) *that are sufficient to have* $\liminf_{\varepsilon\to 0} t^*(\varepsilon) > 0$ *in the regime* $p = 0$, *and that are satisfied by the Euler–Maxwell equations.*

1.2. The Euler–Maxwell equations and the Zakharov equations

This paper is a direct continuation of [27, 7]. The underlying physical context is the study of laser-plasma interactions; in particular, the question of the rigorous derivation of the Zakharov model from fundamental equations.

We take here as a system of fundamental equations the Maxwell equations coupled with the Euler equations [10, 22]

$$\begin{aligned}
&\partial_t B^\flat + c\,\nabla\times E^\flat = 0,\\
&\partial_t E^\flat - c\,\nabla\times B^\flat = 4\pi e((n_0+n_e^\flat)v_e^\flat - (n_0+n_i^\flat)v_i^\flat)),\\
&m_e(n_0+n_e^\flat)(\partial_t v_e^\flat + (v_e^\flat\cdot\nabla)v_e^\flat) = -\gamma_e T_e \nabla n_e^\flat - e(n_0+n_e^\flat)\left(E^\flat + \frac{1}{c}v_e^\flat\times B^\flat\right),\\
&m_i(n_0+n_i^\flat)(\partial_t v_i^\flat + (v_i^\flat\cdot\nabla)v_i^\flat) = -\gamma_i T_i \nabla n_i^\flat + e(n_0+n_e^\flat)\left(E^\flat + \frac{1}{c}v_i^\flat\times B^\flat\right),\\
&\partial_t n_e^\flat + \nabla\cdot((n_0+n_e^\flat)v_e^\flat) = 0,\\
&\partial_t n_i^\flat + \nabla\cdot((n_0+n_e^\flat)v_i^\flat) = 0.
\end{aligned}$$

The variables are $B^\flat$, $E^\flat$ the electromagnetic field, $v_e^\flat$, $v_i^\flat$ the velocities of the electrons and of the ions, and $n_e^\flat$ and $n_i^\flat$ the density fluctuations from the equilibrium n_0. The first two equations are Maxwell's equations which describe the time evolution of the electromagnetic field, the next two equations are the equations of conservation of momentum for the electrons and the ions, and the last two equations are the equations of conservation of mass for the electrons and the ions. The electric charge of the electrons is $-e$; to simplify, we assume that the charge of the ions is $+e$. The parameters are m_e and m_i the masses of both species, γ_e and γ_i the specific heat ratios of both species, T_i and T_e the temperatures of both species and n_0 the (assumed constant and isotropic) density of the plasma at equilibrium.

In the above system, Maxwell's equations are coupled to Euler's equations by the current density term in the right-hand side of the Ampère equation and by the Lorentz force in the right-hand side of the equations of conservation of momentum.

The additional divergence equations

$$\nabla \cdot B^\flat = 0, \quad \nabla \cdot E^\flat = 4\pi e(n_e^\flat - n_i^\flat), \tag{4}$$

are satisfied at all times if they are satisfied by the initial data. A brief discussion of the relevance of this model is given in [27]. Our starting point is the following, nondimensional form of these equations introduced in [27]:

$$(\mathrm{EM})^\sharp \begin{cases} \partial_t B + \nabla \times E = 0, \\ \partial_t E - \nabla \times B = \dfrac{1}{\varepsilon}(1 + n_e^\sharp)v_e - \dfrac{1}{\varepsilon}\dfrac{\theta_i}{\theta_e}(1 + n_i^\sharp)v_i, \\ \partial_t v_e + \theta_e(v_e \cdot \nabla)v_e = -\theta_e \dfrac{\nabla n_e^\sharp}{1 + n_e^\sharp} - \dfrac{1}{\varepsilon}(E + \theta_e v_e \times B), \\ \partial_t n_e^\sharp + \theta_e \nabla \cdot ((1 + n_e^\sharp)v_e) = 0, \\ \partial_t v_i + \theta_i(v_i \cdot \nabla)v_i = -\alpha^2 \theta_i \dfrac{\nabla n_i^\sharp}{1 + n_i^\sharp} + \dfrac{1}{\varepsilon}\dfrac{\theta_i}{\theta_e}(E + \theta_i v_i \times B), \\ \partial_t n_i^\sharp + \theta_i \nabla \cdot ((1 + n_i^\sharp)v_i) = 0. \end{cases}$$

In $(\mathrm{EM})^\sharp$, the variable is

$$u^\sharp = \left(B, E, v_e, n_e^\sharp, v_i, n_i^\sharp\right) \in \mathbb{R}^{14},$$

where $(B, E) \in \mathbb{R}^{3+3}$ is the electromagnetic field, $(v_e, v_i) \in \mathbb{R}^{3+3}$ are the velocities of the electrons and of the ions and $(n_e^\sharp, n_i^\sharp) \in \mathbb{R}^{1+1}$ are the fluctuations of densities of both species. The variable $u^\sharp$ depends on time $t \in \mathbb{R}_+$ and space $x \in \mathbb{R}^3$.

The change of variables for small amplitudes,

$$1 + n_e^\sharp = e^{n_e}, \quad 1 + n_i^\sharp = e^{n_i}, \tag{5}$$

leads to the system,

$$(\mathrm{EM}) \begin{cases} \partial_t B + \nabla \times E = 0, \\ \partial_t E - \nabla \times B = \dfrac{1}{\varepsilon}\left(e^{n_e} v_e - \dfrac{\theta_i}{\theta_e} e^{n_i} v_i\right), \\ \partial_t v_e + \theta_e(v_e \cdot \nabla)v_e = -\theta_e \nabla n_e - \dfrac{1}{\varepsilon}(E + \theta_e v_e \times B), \\ \partial_t n_e + \theta_e \nabla \cdot v_e + \theta_e(v_e \cdot \nabla)n_e = 0, \\ \partial_t v_i + \theta_i(v_i \cdot \nabla)v_i = -\alpha^2 \theta_i \nabla n_i + \dfrac{\theta_i}{\varepsilon\theta_e}(E + \theta_i v_i \times B), \\ \partial_t n_i + \theta_i \nabla \cdot v_i + \theta_i v_i \cdot \nabla n_i = 0. \end{cases}$$

In (EM), the variable is

$$\tilde{u} = \left(B, E, v_e, n_e, v_i, \frac{n_i}{\alpha}\right) \in \mathbb{R}^{14}.$$

The small parameter ε is

$$\varepsilon := \frac{1}{\omega_{pe} t_0},$$

where ω_{pe} is the electronic plasma frequency:

$$\omega_{pe} := \sqrt{\frac{4\pi e^2 n_0}{m_e}}, \tag{6}$$

and t_0 is the duration of the laser pulse. A typical value for ε in realistic physical applications is $\varepsilon \simeq 10^{-6}$. The parameters α, θ_e and θ_i are

$$\theta_e := \frac{1}{c}\sqrt{\frac{\gamma_e T_e}{m_e}}, \quad \theta_i := \frac{1}{c}\sqrt{\frac{\gamma_e T_e}{m_i}}, \quad \alpha := \frac{T_i}{T_e}.$$

Typically, $\theta_e \simeq 10^{-3}$. As the ions are much heavier than the electrons, θ_i and α are much smaller than θ_e. We consider the specific regime

$$\theta_i = \varepsilon, \tag{7}$$

and we look for solutions to (EM) with initial data of size $O(\varepsilon)$ defined over diffractive times $O(1/\varepsilon)$. That is, we make the ansatz

$$u(t, x) := \varepsilon \tilde{u}(\varepsilon t, x). \tag{8}$$

Written as a system of equations in the variable u, (EM) takes the form (1) (see Section 3).

The Zakharov system is a simplified model for the description of the nonlinear interactions between $\tilde{E}$, the envelope of the electric field, and $\bar{n}$, the mean mode of the ionic fluctuations of density in the plasma

$$\text{(Z)} \begin{cases} i\partial_t \tilde{E} + \Delta \tilde{E} = \bar{n}\tilde{E}, \\ \partial_t^2 \bar{n} - \Delta \bar{n} = \Delta |\tilde{E}|^2. \end{cases}$$

This model was derived by Zakharov and his collaborators in the 1970s [17]. It describes nonlinear interactions between high-frequency, electromagnetic waves and low-frequency, acoustic waves. In (Z), the Schrödinger operator is the classical three-scale approximation of Maxwell's equations [12]; the wave operator is the classical long-wave approximation of the Euler equations. The nonlinear term in the right-hand side of the equation in $\tilde{E}$ directly derives from the current density term in the Ampère equation. The term $\Delta|\tilde{E}|^2$ derives from the convective terms and the nonlinear force term in the equations of conservation of momentum. A WKB expansion of solutions of the $(\text{EM})^\sharp$ system is performed in Section 3.1. In the nonlinear regime of our interest, the limit system is (Z).

1.3. Description of the results

We extend here the results of [27, 7] by showing that, in the high-frequency limit and in a highly nonlinear regime, solutions of the Euler–Maxwell equations are well approximated by solutions of the Zakharov equations. In Section 2, under appropriate assumptions, we prove the following theorem:

Theorem 1. *The unique solution to* (1)–(2) *is defined over a time interval independent of* ε.

In Section 3, we apply Theorem 1 to the Euler–Maxwell equations, to obtain:

Theorem 2. *In the high-frequency limit* $\varepsilon \to 0$, *solutions of the Euler–Maxwell equations initiating from polarized, large-amplitude initial data, are well approximated by solutions of the Zakharov equations initiating from nearby initial data, in the sense that there exists* $t_0 > 0$ *and* $C > 0$, *independent of* ε, *such that,*

$$\sup_{0 \leqq t \leqq t_0} \sup_x \left| E - (\tilde{E} e^{i\omega_{pe} t/\varepsilon^2} + (\tilde{E})^* e^{-i\omega_{pe} t/\varepsilon^2}) \right| + \left| n^\varepsilon - \varepsilon \bar{n} \right| \leqq C\varepsilon^2,$$

for ε *small enough, where* E^ε *and* n^ε *represent the electric field and the* (*electronic or ionic*) *fluctuation of density in the solution of the* (EM) *system, and* $(\tilde{E}, \bar{n})$ *is the solution of* (Z). (*Above,* $(\tilde{E})^*$ *is the complex conjugate of* $\tilde{E}$.)

Precise statements are given in Sections 2.4 (Therorem 8) and 3.2 (Theorem 9).

1.4. Outline of the proof

The proof of Theorem 1 (Section 2.5) goes along the following lines: (a) the construction of a precise, regular, polarized *approximate solution* defined over a time interval independent of ε; (b) the *preparation* of the system; and (c) the *control of the resonant interactions of oscillating waves*.

1.4.1. Existence of an approximate solution. In Section 3, we consider the initial value problem for the (EM) system, and show that it takes the form (1)–(2). We construct a highly oscillating WKB approximate solution, initiating from prepared initial data. This WKB solution is a profile depending on the variables t, x, θ, with a periodic dependence on θ

$$\begin{aligned} u_a^\varepsilon(t,x) &= \left[\mathbf{u}_a^\varepsilon(t,x,\theta)\right]_{\theta=\omega t/\varepsilon^2} \\ &= \left[(\mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \ldots)(t,x,\theta)\right]_{\theta=\omega t/\varepsilon^2}, \end{aligned} \tag{9}$$

where the nondimensional frequency ω is defined in terms of the electronic plasma frequency introduced in Section 1.2 ($\omega = 1$ in Section 3.1).

Let us now comment on the preparation condition. For u_a^ε in the form (9) to be an approximate solution to (1)–(2), where, to simplify the discussion, the operator $\mathcal{A}$ has the form (3), it is sufficient that its representation $\mathbf{u}_a^\varepsilon$ satisfies the singular equation

$$(\varepsilon^2 \partial_t + (\omega \partial_\theta + \mathcal{A}_0(\varepsilon) + \varepsilon(\mathcal{A}_1(\varepsilon, \partial_x) + \varepsilon \mathcal{A}_2(\mathbf{u}_a^\varepsilon, \partial_x))\mathbf{u}_a^\varepsilon = \mathcal{B}(\mathbf{u}_a^\varepsilon, \mathbf{u}_a^\varepsilon) + O(\varepsilon^m).$$

The limits $\varepsilon \to 0$ yield

$$(\omega\partial_\theta + \mathcal{A}_0(0))\mathbf{u}_0 = 0. \tag{10}$$

Let us assume that, in the family of matrices $\{ip\omega + \mathcal{A}_0(0)\}_{p\in\mathbb{Z}}$, only $\pm i\omega + \mathcal{A}_0(0)$ have nontrivial kernels (as is actually the case for the (EM) system, see Section 3). Then, (10) implies

$$\mathbf{u}_0 = u_{0,-1}e^{-i\theta} + u_{0,1}e^{i\theta}, \quad u_{0,\pm 1} \in \ker(\pm i\omega + \mathcal{A}_0(0)),$$

and, if $\mathbf{u}_0$ is continuous at $(t,\theta) = (0,0)$,

$$a^0 = a^0_{-1} + a^0_1, \qquad a^0_{\pm 1} \in \ker(\pm i\omega + \mathcal{A}_0(0)). \tag{11}$$

The above condition is the preparation assumption for the initial datum a^ε. For the (EM) system, it takes the form

$$a^0 = \left(0_{\mathbb{R}^3}, \tilde{E} + (\tilde{E})^*, \frac{i}{\omega}\tilde{E} - \frac{i}{\omega}(\tilde{E})^*, 0_{\mathbb{R}}, 0_{\mathbb{R}^3}, 0_{\mathbb{R}}\right),$$

for some complex amplitude $\tilde{E}$ (above, $(\tilde{E})^*$ denotes the complex conjugate of $\tilde{E}$).

WKB solutions to the Euler–Maxwell equations, initating from highly oscillating prepared initial data, are considered in [27]. It is shown in [27] that

(1) the (EM) equations satisfy *transparency* properties, that is, null conditions for coefficients which describes constructive interactions of characteristic waves. As a result, the weakly nonlinear (in the sense of Section 1.1) approximation of the (EM) system is a *linear* transport equation;
(2) the leading terms of WKB solutions of the (EM) equations initiating from large-amplitude data, satisfy, in the high-frequency limit $\varepsilon \to 0$, systems of the form

$$(\mathrm{Z})_c \begin{cases} i(\partial_t + c\partial_z)E + \Delta_\perp E = nE, \\ (\partial_t^2 - \Delta_\perp)n = \Delta_\perp(|E|^2), \end{cases}$$

where z is the direction of propagation of the laser pulse, and $\Delta_\perp$ is the Laplace operator in the transverse directions. The parameter c is the group velocity.

The approximate solution that is constructed in Section 3 satisfies the ansatz

$$u^\varepsilon_a(t,x) = U^\varepsilon_a\left(t, x, \frac{\omega t}{\varepsilon^2}\right). \tag{12}$$

In particular, there are *three* times scales. This is consistent with the well-known fact that the Schrödinger equation is an approximation of the Maxwell equations in the diffractive limit (that is, $t = O(1)$ and oscillations with frequencies $O(1/\varepsilon^2)$). Note, however, that the wave equation, also present in the (Z) system, is an approximation of the Euler equation in the geometric optics limit (that is, $t = O(1)$ and oscillations in $O(1/\varepsilon)$). The third scale is actually built in the Euler equations by the "cold ions" assumption $\theta_i = \varepsilon$.

In (12), the profiles are purely time-oscillating. In particular, the initial data do not have fast oscillations. The limit system is (Z), that is, a Zakharov system with zero group velocity (see the characteristic variety pictured in Figure 2). Such waves are called *plasma waves* in the physical literature. The approximate solution has the form

$$u_a^\varepsilon = \left(u_{0,1}e^{i\omega t/\varepsilon^2} + u_{0,1}^* e^{-i\omega t/\varepsilon^2}\right) + \varepsilon(u_{1,0} + \dots) + \varepsilon^2(\dots), \tag{13}$$

where the components of $u_{0,1}$ and $u_{1,0}$ satisfy (Z) ($u_{0,1}$ is an envelope, corresponding to $\tilde{E}$, with the notation of Theorem 2, while $u_{1,0}$ is a mean mode, corresponding to $\bar{n}$.).

For general equations of the form (1), studied in Section 2, the existence of an approximate solution is assumed (Assumption 4).

1.4.2. Preparation of the system. Given a precise approximate solution u_a^ε of the form (13), we look in Section 2.5.1 for the exact solution u^ε as a perturbation of u_a^ε

$$u^\varepsilon = u_a^\varepsilon + \varepsilon^k \dot{u}^\varepsilon.$$

The initial condition is $u^\varepsilon(t = 0) = a^\varepsilon + \varepsilon^{k_0}\varphi^\varepsilon$, where φ^ε has a high Sobolev regularity and k_0 is large enough. In the definition of $\dot{u}^\varepsilon$, k is adequately chosen in terms of k_0. We assume that u_a^ε is accurate at an order l_0, much larger that k_0. The equation in $\dot{u}^\varepsilon$ has the form

$$\partial_t \dot{u}^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{A}(\varepsilon(u_a^\varepsilon + \varepsilon^k \dot{u}^\varepsilon))\dot{u}^\varepsilon = \frac{1}{\varepsilon}(\mathcal{B}(u_a^\varepsilon, \dot{u}^\varepsilon) + \mathcal{B}(\dot{u}^\varepsilon, u_a^\varepsilon)), \tag{14}$$

The propagator $\mathcal{A}/\varepsilon^2$ is hyperbolic (Assumption 3), and thus generates highly oscillating waves, with frequencies of typical size $O(1/\varepsilon^2)$. We write the spectral decomposition of $\mathcal{A}$ as follows:

$$\mathcal{A} = \sum_{\text{kg}} i\lambda_{\text{kg}}\Pi_{\text{kg}} + \sum_{\text{ac}} i\lambda_{\text{ac}}\Pi_{\text{ac}}.$$

The real eigenvalues λ_{kg} are called Klein–Gordon modes, while the real eigenvalues λ_{ac} are called acoustic modes. The characteristic variety for the (EM) system (that is, the union of the graphs $\xi \mapsto \lambda_j(\xi)$, at $u = 0$, for $j = \text{kg}$ and $j = \text{ac}$) is pictured in Figure 2. In particular, for small frequencies $\lambda_{\text{kg}} \sim 1$, for large frequencies $\lambda_{\text{kg}} \sim O(1)\xi$, while $\lambda_{\text{ac}} \sim \varepsilon\xi$, a consequence of the cold ions hypothesis (7).

The Klein–Gordon waves generated by $\mathcal{A}$ interact with the highly oscillating approximate solution, through the convection term, and through the source term $\mathcal{B}$. These interactions create low-frequency waves, which can be seen as source terms in the equations for the components of the solutions in the directions of the acoustic modes. Thus low-frequency and high-frequency waves are propagated. The Zakharov system claims to describe how these waves interact.

Equation (14), together with an initial datum of size $O(1)$, can be likened to an ordinary differential equation

$$y' + \frac{i\alpha}{\varepsilon^2}y = \frac{1}{\varepsilon}y^2,$$

with an initial datum $y(0) = y_0$. The singular source term in the right-hand side may cause the solution to blow-up in small time $O(\varepsilon)$, but exponential cancellations are expected to happen because of the rapid oscillations created by the term in $1/\varepsilon^2$.

To investigate these exponential cancellations, it is natural to project the source term $\mathcal{B}/\varepsilon$ over the eigendirections of $\mathcal{A}$. Let the total eigenprojectors

$$\Pi_0 = \sum_{\mathrm{kg}} \Pi_{\mathrm{kg}}, \quad \Pi_s = \sum_{\mathrm{ac}} \Pi_{\mathrm{ac}}.$$

We compute

$$\begin{pmatrix} \Pi_0 \mathcal{B} \Pi_0 & \Pi_0 \mathcal{B} \Pi_s \\ \Pi_s \mathcal{B} \Pi_0 & \Pi_s \mathcal{B} \Pi_s \end{pmatrix} = \begin{pmatrix} * & O(1) \\ ** & * \end{pmatrix}.$$

As constructive interaction of waves between low and high frequency do occur for the (EM) system (see Figure 4), the $O(1)$ term in the right block of the above interaction matrix can be interpreted as an *absence of transparency.*

We then rescale the solution (Section 2.5.2) as follows,

$$v^\varepsilon := \left(\Pi_0 \, \dot{u}^\varepsilon, \frac{1}{\varepsilon} \Pi_s \, \dot{u}^\varepsilon \right).$$

The equation in v^ε has the form

$$\partial_t v^\varepsilon + \frac{1}{\varepsilon^2} A v^\varepsilon = \frac{1}{\varepsilon^2} B v^\varepsilon + \frac{1}{\varepsilon} D v^\varepsilon + O(1) v^\varepsilon,$$

with the notation

$$B = \begin{pmatrix} 0 & 0 \\ ** & 0 \end{pmatrix}, \quad D := \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

In the equation for v, the propagator A is diagonal and the leading source term B is nilpotent. The system is prepared.

1.4.3. Control of the constructive interactions of waves. In a third step (Sections 2.5.3 to 2.5.5), the singularity in the right-hand side of the equation in v^ε is reduced. Consider a change of variable in the form

$$w^\varepsilon := (\mathrm{Id} + N)^{-1} v^\varepsilon, \quad N = \begin{pmatrix} 0 & 0 \\ \underline{N} & 0 \end{pmatrix}.$$

The equation satisfied by w^ε is

$$\partial_t w^\varepsilon + \frac{1}{\varepsilon^2} A w^\varepsilon = \frac{1}{\varepsilon^2} (B - [\varepsilon^2 \partial_t + A, N]) w^\varepsilon + \frac{1}{\varepsilon} D w^\varepsilon.$$

We look for N as a solution of the *homological equation*

$$B - [\varepsilon^2 \partial_t + A, N] = O(\varepsilon^2).$$

This equation breaks down into scalar equations of the form $\Phi \underline{N} = \underline{B} + O(\varepsilon^2)$, where $\underline{N}$ represents an entry of N, and $\underline{B}$ an entry of B, and where $\Phi = \lambda_{\mathrm{kg}} - \lambda_{\mathrm{ac}} - \omega$. The equation $\Phi = 0$ is the *resonance* equation. Its solutions are pictured in Figures 3, 4 and 5. The crucial transparency assumption (Assumption 6) states that the interaction coefficient B is sufficiently small at the resonances, that is,

$$|B| \leqq C\varepsilon^2 |\Phi|. \tag{15}$$

When (15) is satisfied, the above homological equation can be solved, and the equation in w^ε becomes

$$\partial_t w^\varepsilon + \frac{1}{\varepsilon^2} A w^\varepsilon = \frac{1}{\varepsilon} D w^\varepsilon.$$

A symmetrizability assumption for D (Assumption 7) eventually allows us to perform energy estimates (Section 2.5.6), which yield uniform bounds for w^ε, and a continuation argument concludes the proof.

In Section 3, we describe the resonance equations for the Euler–Maxwell equations, and check that an estimate of the form (15) is satisfied.

1.4.4. Technical issues. The symbols that appear in the spectral decomposition of $\mathcal{A}$ are necessarily pseudodifferential operators, even when $\mathcal{A}$ is differential. They also depend on the solution u^ε, because the equations are nonlinear. We are naturally led to consider pseudodifferential operators of the form

$$q(\varepsilon, x, \xi) = p(\varepsilon, v(x), \xi),$$

where v has a Sobolev regularity. The questions of the bounds of the corresponding operators, in Sobolev spaces, and of the existence of a symbolic calculus, naturally arise. LANNES recently gave optimal bounds in [14]. For an operator of order m, these bounds have the form

$$\|\mathrm{op}(p(v, \xi)u\|_{H^s} \leqq C(|v|_{L^\infty})(\|v\|_{H^s}\|u\|_{H^{s_0}} + \|u\|_{H^{s+m}}),$$

where s_0 depends only on d and m. A symbolic calculus is available; the operator $\mathrm{op}(p_1(v))\mathrm{op}(p_2(v))$ has the symbol

$$p_1(v)p_2(v) + \sum_{|\alpha|=1} \partial_\xi^\alpha(p_1(v))\partial_x^\alpha(p_2(v)) + \dots$$

When the symbols depend on x through the solution u^ε, the subprincipal symbol depends on $\partial_x u^\varepsilon$. Consider for instance the case when p_1 is order 1 and p_2 is order 0. The subprincipal symbol is order 0; it maps H^s to H^s. The above inequality states that its norm depends on $\|u^\varepsilon\|_{H^{s+1}}$. That is, compositions of such operators naturally lead to losses of derivatives.

To overcome this difficulty, it is classical to differentiate the equation up to order s, and then perform energy estimates in L^2.

We now explain why we chose a different approach.

In the perturbation equations (14), all the derivatives are ε-derivatives (see below). The equation in $\varepsilon\partial_x\dot{u}^\varepsilon$ (we let $d = 1$ in this discussion to simplify the notation) has a singular source term in $\dot{u}^\varepsilon$. The variable $U^\varepsilon := (\dot{u}^\varepsilon, (\varepsilon\partial_x)\dot{u}^\varepsilon)$, solves the equation

$$\partial_t U^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{A}(\varepsilon u^\varepsilon)U^\varepsilon = \frac{1}{\varepsilon}\begin{pmatrix} \mathcal{B} & 0 \\ \varepsilon\mathcal{B}' & \mathcal{B} \end{pmatrix} U^\varepsilon - \frac{1}{\varepsilon^2}\begin{pmatrix} 0 \\ [\varepsilon\partial_x, \mathcal{A}(\varepsilon u^\varepsilon)] \end{pmatrix} U^\varepsilon,$$

where $\mathcal{B}$ is short for $\mathcal{B}(u_a^\varepsilon)$ and $\mathcal{B}'$ is short for $\mathcal{B}(\partial_x u_a^\varepsilon)$. Assume that $\mathcal{A}$ has the form (3), where $\mathcal{A}_2(\varepsilon, u, \xi) = \mathcal{A}_2(\varepsilon, u)\xi$. Then,

$$\frac{1}{\varepsilon^2}[\varepsilon\partial_x, \mathcal{A}(\varepsilon u^\varepsilon)] = \mathcal{A}_2(\partial_x(u_a^\varepsilon + \varepsilon^k\dot{u}^\varepsilon))(\varepsilon\partial_x)u^\varepsilon.$$

Following the approach of Section 1.4.2, we rescale the solution, by letting

$$V^\varepsilon = \left(\Pi_0\dot{u}^\varepsilon, \Pi_0\varepsilon\partial_x\dot{u}^\varepsilon, \frac{1}{\varepsilon}\Pi_s\dot{u}^\varepsilon, \frac{1}{\varepsilon}\Pi_s\varepsilon\partial_x\dot{u}^\varepsilon\right).$$

The equation becomes

$$\partial_t V^\varepsilon + \frac{1}{\varepsilon^2}A(\varepsilon u^\varepsilon)V^\varepsilon = \frac{1}{\varepsilon^2}\underline{B}V^\varepsilon + \frac{1}{\varepsilon}\underline{D}V^\varepsilon + O(1)V^\varepsilon,$$

with the notation

$$\underline{B} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Pi_s\mathcal{B}\Pi_0 & 0 & 0 & 0 \\ \varepsilon\Pi_s\mathcal{B}'\Pi_0 & \Pi_s(\mathcal{B} - \varepsilon\mathcal{A}_2(\partial_x u_a^\varepsilon))\Pi_0 & 0 & 0 \end{pmatrix},$$

and

$$\underline{D} := \begin{pmatrix} \Pi_0\mathcal{B}\Pi_0 & 0 & 0 & 0 \\ 0 & \Pi_s\mathcal{B}\Pi_s & 0 & 0 \\ 0 & 0 & \Pi_0\mathcal{B}\Pi_0 & 0 \\ 0 & 0 & 0 & \Pi_s\mathcal{B}\Pi_s \end{pmatrix}.$$

If $\Pi_s\mathcal{B}\Pi_0$ is assumed to be small enough at the resonances, then $\Pi_s\mathcal{B}'\Pi_0$ is small as well, but the commutator term $\varepsilon\Pi_s\mathcal{A}_2(\partial_x u_a^\varepsilon)\Pi_0$ might well *not* be transparent. This commutator term is not present in the original perturbation equation in $\dot{u}^\varepsilon$. That is, an extra transparency assumption would in general be necessary for the method of differentiating the perturbation equations to be applied. (In the case of the (EM) system, the term $\varepsilon\Pi_s\mathcal{A}_2(\partial_x u_a^\varepsilon)\Pi_0$ can actually be shown to satisfy a transparency estimate of the form (15).)

That is, the method of differentiating the equations has a less general scope than the one we chose, namely paradifferential smoothing. Arguably, it is also conceptually simpler.

The paradifferential smoothing of Bony [2] is another classical way to overcome artificial losses of derivatives occurring in compositions of operators. We denote paradifferential operators by op^ψ, where ψ is an admissible cut-off (see Section 2.2). It is classical that, for an operator of order m,

$$\|\mathrm{op}^\psi(p(v),\xi)u\|_{H^s} \leqq C(|v|_{L^\infty})\|u\|_{H^{s+m}}.$$

It is also classical that the norm of the subprincipal symbol in the composition of two paradifferential operators depends on v through $|v|_{W^{1,\infty}}$. No derivatives are lost in this context. Another issue, however, arises. It is to control the remainder terms generated by the smoothing procedure. We check in detail that these terms can effectively be controlled in the setting of our interest.

This setting is semiclassical, in the sense that the operators depend on ξ through $\varepsilon\xi$. It is easy to check that the above bounds and symbolic calculus can be adapted to this setting (Section 2.2). In a semiclassical setting, subprincipal symbols come up in the compositions with a prefactor ε. As the singularity is in $1/\varepsilon^2$, this implies that we need only consider the principal and the subprincipal symbols. The perturbation of the initial data is accordingly assumed to have a semiclassical Sobolev regularity. In particular, it can take the form $\varphi(x)e^{ikx/\varepsilon}$, with $\varphi \in H^s$. The final asymptotic estimate (49) is formulated in H^s_ε. It implies an estimate in L^∞.

We finally mention a technical point, associated with a lack of regularity of the operators involved in the changes of variables described in Sections 1.4.2 and 1.4.3, caused by the fact that the spectral decomposition of the hyperbolic operator in the (EM) equations becomes singular for small frequencies.

The wave equation in (Z), that comes up as a geometric optics approximation of the Euler equations, is associated with symbols in $\pm\varepsilon|\xi|$. In particular, these symbols are only bounded at the origin. As resonances between Klein–Gordon and acoustic modes occur precisely for small frequencies $|\xi| \sim \varepsilon$, a smoothing procedure, or the introduction of a cut-off, would create large error terms. At this point, we make a crucial use of the fact that all the symbols depend on the solution u^ε, only through $\varepsilon u^\varepsilon$, and that we work on *perturbation* equations: $u^\varepsilon = u^\varepsilon_a + \varepsilon^k \dot{u}^\varepsilon$, where k is large enough. As we need to handle symbols only up to $O(\varepsilon^2)$, we can approximate $p(\varepsilon u^\varepsilon, \varepsilon\xi)$ by $p(0,\varepsilon\xi) + \varepsilon\partial_u p(0,\varepsilon\xi)\cdot u^\varepsilon_a$. This approximate symbol is easier to handle for two reasons. First, it depends on x only through u^ε_a, the approximate solution. Second, it has the simple form $p_1(v)p_2(\varepsilon\xi)$. In Section 2.3, we describe how operators with nonsmooth symbols of this form operate in semiclassical Sobolev spaces. Classically, norms of pseudodifferential operators depend on derivatives in x and in ξ of the symbol, and, because x and ξ play somehow symmetric roles, derivatives in ξ can be shifted to derivatives in x. Here we can afford to lose derivatives in x. The eventual energy estimate in H^s_ε involves $\|u^\varepsilon_a\|_{H^{s'}}$, with $s' > s$. This does not harm the proof if the initial datum is assumed to have enough Sobolev regularity.

1.5. Background and references

The (Z) system was derived from kinetic models by ZAKHAROV and his collaborators in the 1970s [17].

The initial value problem for the (Z) equations has received much attention. Global existence of smooth solutions in one space dimension (and of weak solutions in two and three space dimensions, for small initial data) was proved by SULEM and SULEM [21]. Global existence of smooth solutions in two space dimensions, for small initial data, was proved by ADDED and ADDED [1]. SCHOCHET and WEINSTEIN [23] and OZAWA and TSUTSUMI [18] showed existence of local-in-time smooth Sobolev solutions. COLLIANDER and BOURGAIN [3] and GINIBRE, TSUTSUMI and VELO [11] studied critical regularity issues for local solutions. For large initial data, no evidence of singularity in finite time is known in a space dimension greater than one.

In their book on the Schrödinger equation [22], SULEM and SULEM show how the Zakharov equation can be formally derived from the Euler–Maxwell equations; the WKB asymptotics of Section 3.1 is based on their description, and on discussions with TIKHONCHUK and COLIN.

To our knowledge, the results of [27] and [7] were the first results establishing rigorous links between Euler–Maxwell and Zakharov.

Formal WKB expansions, carried out in [27], have shown how the weakly nonlinear limit of (EM) fails to describe nonlinear interactions; such a phenomenon had been observed by JOLY, MÉTIVIER and RAUCH in the context of the Maxwell–Bloch equations. JOLY, MÉTIVIER and RAUCH's paper [13], which describes large-amplitude solutions of semilinear systems of Maxwell–Bloch type by means of normal form reductions, is the main inspiration of the present work.

In [7], Klein–Gordon waves systems were formally derived from Euler–Maxwell, and the Zakharov equations were rigorously derived as a high-frequency limit of these Klein–Gordon waves systems. The stationary phase arguments of [7], where solutions were represented, through Fourier analysis, in the form $\varepsilon^{-1} \int_0^t e^{it\Phi/\varepsilon} B(t')dt'$, are analogous to the normal form reductions of the present work. The above integrals are bounded if the ratio B/Φ is bounded, which echoes the transparency condition (15).

Highly oscillating large-amplitude solutions of quasilinear systems were considered by SERRE in [19], and by CHEVERRY, GUÈS and MÉTIVIER in [5]. These papers deal with conservation laws, in particular, nondispersive systems, unlike the Euler–Maxwell system. In [4], CHEVERRY studies the parabolic relaxation of the instabilities put in evidence in [5] and applies his results to the equations of the large-scale motions in the atmosphere.

In [9], GRENIER studies a class of singular equations of the form (1), with $\mathcal{A}$ of the form (3), and $\mathcal{B} = 0$. He proves the existence of solutions over time intervals independent of ε, under the assumption that $\mathcal{A}$ possesses a "good" symmetrizer, in the sense that no singular terms are created by subprincipal symbols occurring in the symmetrization process. Grenier is naturally led to study operators depending on x through $v(x)$, where v has a Sobolev regularity. He does not assume that the initial data are well prepared, and studies the high-frequency behaviour of the solutions.

Lannes recently gave precise bounds for the norms of pseudodifferential operators depending on x through $v(x)$, where v has a Sobolev regularity, and for the norms of commutators of such operators. These questions had previously been considered by TAYLOR [25], and by GRENIER in [9] mentioned above. We use a consequence of Lannes' description of the paradifferential remainder (formulated as Proposition 1; it is used in Sections 2.5.1 and 2.5.5), and his description of the norm of a pseudodifferential operator with Sobolev regularity in x (formulated as Lemma 1; it is used in the proof of Proposition 4).

In the approximate solution u_a^ε to (EM) that is constructed in Section 3.1, the envelope of the electric field is $O(1)$, while the fluctuation of density has size $O(\varepsilon)$, in particular, it vanishes in the high-frequency limit. However, in the (Z) system, the fluctuation of density has a finite effect on the electric field. This means that there is a strong coupling between E and n in the Euler–Maxwell equations – the nontransparent condition (44) being evidence of this. Such a phenomenon was called "ghost effect" by the Kyoto school of Sone, Aoki and Takata. These authors extensively studied this phenomenon in the context of small Knudsen number analysis of rarefied gases; a good reference is SONE's book [20], and TAKATA and AOKI [24]. As SONE explains in [20], ghosts effects are characteristic of situations where large temperature variations are recorded. It would be interesting to show a formal similarity between their formal Hilbert expansions (describing the continuum limit) and the WKB expansions of highly nonlinear geometric optics (describing high-frequency limits).

COLIN and COLIN propose a generalization of the Zakharov system in [6]. Their system consists of four Schrödinger equations coupled with quasilinear terms and a wave equation. It describes three-wave interactions, in particular, the generation of a Raman backscattered field. It is an interesting question to know whether or not the result of this paper could be generalized to their extension of the (Z) system.

We conclude this introduction by mentioning open questions and directions for future work.

It is natural to ask whether or not the result still holds when the initial condition is assumed to be oscillatory, that is, has the form $a^\varepsilon(x)e^{ik/\varepsilon^2}$, where a^ε is a bounded family in H^s. It is shown in [27] that, if $k \neq 0$, the limit system is $(\mathrm{Z})_c$, where $c = \omega'(k)$, and ω is a local parametrization on the characteristic variety (see figure 2). LINARES, PONCE and SAUT prove in [15] that this system is well posed in Sobolev spaces [15]; COLIN and MÉTIVIER prove in [8] that it is ill-posed in L^∞.

Another interesting direction for future work is to consider the case of large perturbations of the initial data, of the form $\varepsilon^{k_0}\varphi^\varepsilon$, with $k_0 < 3 + \frac{d}{2}$. Our guess is that the strong coupling between the electric field and the mean mode of the fluctuation of density would then create strong instabilities in short time.

2. A class of singular equations

2.1. Symbols

We consider profiles $u, v, \ldots$ depending on $x \in \mathbb{R}^d$, with values in $\mathbb{C}^n$, and symbols $p, q, \ldots$ depending on $(\varepsilon, v, \xi) \in (0, 1] \times \mathbb{C}^n \times \mathbb{R}^d$, or on ε, x, ξ, with values in the $n \times n$ matrices with complex entries.

For $0 < \varepsilon \leqq 1$ and $s \in \mathbb{R}$, we let

$$\|v\|_{\varepsilon,s} := \|(1 + |\varepsilon\xi|^2)^{s/2}\hat{v}(\xi)\|_{L^2(\mathbb{R}^d_\xi)}.$$

The symbol ˆ denotes the Fourier transform in x. In particular, $\|\cdot\|_{1,s}$ denotes the classical Sobolev norm. A profile v is said to belong to $H^s_\varepsilon(\mathbb{R}^d)$ when $\|v\|_{\varepsilon,s}$ is finite. The space $H^s_1(\mathbb{R}^d)$, or simply $H^s(\mathbb{R}^d)$, is the classical Sobolev space. Remark that

$$\|h_\varepsilon v\|_{1,s} = \|v\|_{\varepsilon,s},$$

where

$$h_\varepsilon v(x) := \varepsilon^{d/2} v(\varepsilon x).$$

For $k \in \mathbb{N}$, let

$$|v|_{k,\infty} := \sum_{0 \leqq |\alpha| \leqq k} \sup_{x \in \mathbb{R}^d} |\partial_x^\alpha v|.$$

A profile v is said to belong to $W^{k,\infty}(\mathbb{R}^d)$ when $|v|_{k,\infty}$ is finite. Let $d_0 > \frac{d}{2}$. For all $k \in \mathbb{N}$, the embedding $H_\varepsilon^{k+d_0} \hookrightarrow W^{k,\infty}$ holds:

$$|v|_{k,\infty} \leqq C\varepsilon^{-k-d/2}\|v\|_{\varepsilon,k+d_0}.$$

We now define, and somehow adapt to our context, the class of symbols studied by Lannes in [14] (see also Taylor [25] and Grenier [9]).

A symbol $p(\varepsilon, v, \xi)$ defined in $(0, 1) \times \mathbb{C}^n \times \mathbb{R}^d$, is said to belong to the class $C^\infty\mathcal{M}^m$, $m \in \mathbb{N}$, when there exists $\varepsilon_0 > 0$ such that

(i) $p_{|\{|\xi| \leqq 1\}} \in C^\infty((0, \varepsilon_0) \times \mathbb{R}^d, L^\infty(\{|\xi| \leqq 1\})$, and
(ii) for all α, β, there exists a nondecreasing function $C_{\alpha,\beta}$ such that for all v,

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \sup_{|\xi| \geqq 1/4} \langle\xi\rangle^{|\beta|-m} |\partial^\alpha_{\varepsilon,v}\partial^\beta_\xi p(\varepsilon, v, \xi)| \leqq C_{\alpha,\beta}(|v|). \tag{16}$$

In particular, if $p \in C^\infty\mathcal{M}^m$, then for all α, $\partial^\alpha_{\varepsilon,v} p \in C^\infty\mathcal{M}^m$.

A symbol $p \in C^\infty\mathcal{M}^m$ is said to be *k-regular at the origin* when

$$p_{|\{|\xi| \leqq 1\}} \in C^\infty((0, \varepsilon_0) \times \mathbb{R}^d, W^{k,\infty}(\{|\xi| \leqq 1\}).$$

A symbol is said to be smooth at the origin if it is k-regular for all k. Symbols in $C^\infty\mathcal{M}^m$ that depend only on ε, ξ, are called Fourier multipliers.

If $p \in C^\infty\mathcal{M}^m$ is evaluated at $v \in H^s_\varepsilon$, $s > \frac{d}{2}$, then Moser's inequality implies that

$$\sup_{|\xi| \leqq 1} \|p(v(\cdot), \xi) - p(0, \xi)\|_{\varepsilon,s} \leqq C_s(|v|_{0,\infty})\|v\|_{\varepsilon,s},$$

and for all β

$$\sup_{|\xi| \geqq 1/4} (1 + |\xi|^2)^{|\beta|-m} \|\partial^\beta_\xi (p(v(\cdot), \xi) - p(0, \xi))\|_{\varepsilon,s} \leqq C_{\beta,s}(|v|_{0,\infty})\|v\|_{\varepsilon,s},$$

for some nondecreasing functions $C_s, C_{\beta,s}$.

2.2. Paradifferential operators

The class $\mathcal{S}_k^m$, $m \in \mathbb{R}, k \in \mathbb{N}$, is defined as the space of symbols $q(\varepsilon, x, \xi)$ such that there exists $0 < \varepsilon_0 < 1$, such that, for all $0 < \varepsilon < \varepsilon_0$, for all α, β, with $|\alpha| \leqq k$, there exists $C_{\varepsilon,\alpha,\beta}$ such that for all x, ξ,

$$\langle \xi \rangle^{|\beta|-m} |\partial_x^\alpha \partial_\xi^\beta q(\varepsilon, x, \xi)| \leqq C_{\varepsilon,\alpha,\beta}.$$

With this definition, if $p \in C^\infty \mathcal{M}^m$ is smooth at the origin, and if v is a profile in $W^{k,\infty}(\mathbb{R}^d)$, then $q := p(v)$ belongs to $\mathcal{S}_k^m$.

To $q \in \mathcal{S}_k^m$, we associate the pseudodifferential operator $\mathrm{op}_{\varepsilon'}(q)$, $0 < \varepsilon' \leqq 1$, formally defined by its action as

$$(\mathrm{op}_{\varepsilon'}(q)z)(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} q(\varepsilon, x, \varepsilon'\xi)\hat{z}(\xi)d\xi.$$

With this definition

$$\mathrm{op}_\varepsilon(q) := (h_\varepsilon)^{-1} \mathrm{op}_1(\tilde{q}) h_\varepsilon,$$

where $\tilde{q}(\varepsilon, x, \xi) := q(\varepsilon, \varepsilon x, \xi)$.

Symbols in $\mathcal{S}_k^m$ are smoothed into paradifferential symbols as follows. Let $\chi_0 : \mathbb{R}_+ \to \mathbb{R}$ be a smooth function, such that $0 \leqq \chi_0 \leqq 1$, and

$$\chi_0(\lambda) = 1, \quad \text{for } \lambda \leqq 1.1; \quad \chi_0(\lambda) = 0, \quad \text{for } \lambda \geqq 1.9.$$

Let $\varphi_0(\xi) := \chi((1 + |\xi|^2)^{1/2})$, and for $k \geqq 1$, define $\varphi_k : \mathbb{R}^d \to \mathbb{R}$ by

$$\varphi_k(\xi) := \chi_0(2^{-k}(1 + |\xi|^2)^{1/2}) - \chi_0(2^{-(k-1)}(1 + |\xi|^2)^{1/2}).$$

With this notation for all $\xi \in \mathbb{R}^d$,

$$1 = \sum_{k \geqq 0} \varphi_k(\xi).$$

Let $\chi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function, $0 \leqq \chi \leqq 1$, and such that

$$\chi(\eta) = 1, \quad \text{for } |\eta| \leqq 1.1; \quad \chi(\eta) = 0, \quad \text{for } |\eta| \geqq 1.9.$$

Let $N \geqq 3$, and $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be defined by

$$\psi(\eta, \xi) = \sum_{k \geqq 0} \chi(2^{-k+N}\eta)\varphi_k(\xi). \tag{17}$$

Then ψ satisfies

$$\psi(\eta, \xi) = \begin{cases} 1, & |\eta| \leqq \delta_1(1 + |\xi|^2)^{1/2}, \\ 0, & |\eta| \geqq \delta_2(1 + |\xi|^2)^{1/2}, \end{cases}$$

with $\delta_1 = 2^{-N-1}$, and $\delta_2 = 2^{-N+1}$. We let $N = 3$ in the following. In particular,

$$\psi(\eta, \xi) = 1, \quad \text{for all } \xi, \text{ for all } |\eta| \leqq 2^{-5}. \tag{18}$$

We let,

$$q^{\psi}(x,\xi) := (\check{\psi}(\cdot,\xi) * q(\cdot,\xi))(x), \tag{19}$$

where $\check{\psi}$ denotes the inverse Fourier transform of ψ in its first variable, and $*$ denotes convolution in $x \in \mathbb{R}^d$. The paradifferential operator associated with q is

$$\mathrm{op}^{\psi}_{\varepsilon'}(q) := \mathrm{op}_{\varepsilon'}(q^{\psi}).$$

An important subclass of symbols $\sigma(x,\xi)$ consists of the set of symbols such that, there exists $\delta \in (0,1)$, such that

$$\text{for all } \xi, \quad \mathrm{supp}\, \hat{\sigma}(\cdot,\xi) \subset \left\{ \eta \in \mathbb{R}^d, |\eta| \leqq \delta(1+|\xi|^2)^{1/2} \right\}, \tag{20}$$

where $\mathrm{supp}\, \hat{\sigma}(\cdot,\xi)$ is the support of the partial Fourier transform in x of σ.

The paradifferential symbol q^{ψ} associated with any symbol q satisfies the spectral condition (20), with $\delta = 2^{-2}$.

The following proposition describes how well the action of a pseudodifferential operator is approximated by the action of its associated paradifferential operator, a classical result in the case of differential symbols and when $\varepsilon = 1$, of which Lannes gave an extension to pseudodifferential symbols in [14]. We check below that the result of Lannes extends to $0 < \varepsilon < 1$; in particular, that the action of the paradifferential remainder, in H^s_ε, is very small with respect to ε, when s is large.

In the following statements, C denotes nondecreasing functions, and d_0 is a real number such that $[\frac{d}{2}] < d_0 \leqq [\frac{d}{2}] + 1$.

Following [14], we define the seminorms

$$M_{k,m}(\sigma) := \sup_{|\beta| \leqq k} \sup_{\xi \in \mathbb{R}^d} (1+|\xi|^2)^{(|\beta|-m)/2} \|\partial_\xi^\beta \sigma(\cdot,\xi)\|_{L^\infty}, \tag{21}$$

$$N_{k,m,s}(\sigma) := \sup_{|\beta| \leqq k} \sup_{\xi \in \mathbb{R}^d} (1+|\xi|^2)^{(|\beta|-m)/2} \|\partial_\xi^\beta \sigma(\cdot,\xi)\|_{1,s}. \tag{22}$$

Proposition 1 (Remainder)**.** *Let $v \in H^s_\varepsilon$, $s > \frac{d}{2}$, and $p \in C^\infty \mathcal{M}^m$ be smooth at the origin. Then*

$$\mathrm{op}_\varepsilon(p(v)) = \mathrm{op}^{\psi}_{\varepsilon}(p(v)) + \mathrm{op}_\varepsilon(R_{p(v)}),$$

and, for all $u \in H^{m+d_0}_\varepsilon$,

$$\|\mathrm{op}_\varepsilon(R_{p(v)})u\|_{\varepsilon,s} \leqq \varepsilon^{s-d/2} C(|v|_{0,\infty}) \|v\|_{\varepsilon,s} \|u\|_{\varepsilon,m+d_0}. \tag{23}$$

Proof. We indicate how (23) follows from Propositions 20 and 23 of [14]. Let $q := p(v) - p(0)$, and, for any symbol σ, let $R_\sigma := \sigma - \sigma^{\psi}$. The operation of paradifferential smoothing is a convolution in x, and thus $R_q = R_{p(v)}$. The Fourier transform of the symbol of R_q is

$$\widehat{R_q} = (1-\psi(\eta,\xi))\hat{q}(\eta,\xi).$$

The point is that because of (18), the above symbol is identically zero for small η. Let $\tilde\varphi_0 := \chi$, and for $k \geqq 1$,

$$\tilde\varphi_k(\eta) := \chi(2^{-k}\eta) - \chi(2^{-(k-1)}\eta).$$

Then, for all $\eta \in \mathbb{R}^d$,

$$1 = \sum_{k \geqq 0} \tilde\varphi_k(\eta).$$

We can write

$$\hat q(\eta, \xi) = \sum_{|k-k'| \geqq 3} \tilde\varphi_{k'}(\eta)\hat q(\eta, \xi)\varphi_k(\xi) + \sum_{|k-k'|<3} \tilde\varphi_{k'}(\eta)\hat q(\eta, \xi)\varphi_k(\xi),$$

where the sums run over integers $k, k' \geqq 0$. The first sum in $\hat q$ is further decomposed into $\hat q_1 + \hat q_2$, where

$$\hat q_1 := \sum_{k \geqq 3}\sum_{k' \leqq k-3} \tilde\varphi_{k'}(\eta)\hat q(\eta, \xi)\varphi_k(\xi), \quad \hat q_2 := \sum_{k' \geqq 3}\sum_{k \leqq k'-3} \tilde\varphi_{k'}(\eta)\hat q(\eta, \xi)\varphi_k(\xi).$$

Remark that

$$(\psi \hat q)(\eta, \xi) = \hat q_1(\eta, \xi) + \sum_{k<3} \chi(2^{-k+3}\eta)\hat q(\eta, \xi)\varphi_k(\xi).$$

Thus we have $R_q = q_2 + q_{r,1} + q_{r,2}$, where

$$\hat q_{r,1} := \sum_{k' \geqq 0, |k-k'|<3} \tilde\varphi_{k'}(\eta)\hat q(\eta, \xi)\varphi_k(\xi),$$
$$\hat q_{r,2} := \chi(\eta)\hat q(\eta, \xi)\chi_0(2^{-3}(1+|\xi|^2)^{1/2}) - \sum_{k<3} \chi(2^{-k+3}\eta)\hat q(\eta, \xi)\varphi_k(\xi).$$

The symbols $\hat q_2$, $\hat q_{r,1}$, $\hat q_{r,2}$ correspond to the symbols σ_{II}, $\sigma_{R,1}$ and $\sigma_{R,2}$ in [14]. We want to bound

$$\|\mathrm{op}_\varepsilon(R_q)u\|_{\varepsilon,s} = \|\mathrm{op}_1(\tilde q_2 + \tilde q_{r,1} + \tilde q_{r,2})h_\varepsilon u\|_{1,s},$$

where $\tilde q(x) := q(\varepsilon x)$. Propositions 20 and 23 of [14] imply that

$$\|\mathrm{op}_1(\tilde q_2 + \tilde q_{r,1})h_\varepsilon u\|_{1,s} \leqq N_{\gamma_0,m,s}(\tilde q)\|u\|_{\varepsilon,m+d_0},$$

where the notation N was introduced in (22), and γ_0 depends only on d. Now, owing to (18), q can be replaced by $(1 - \mathrm{op}_1(\bar\chi))q$ in the symbol of R_q, the function $\bar\chi$ being smooth, identically equal to one for $|\eta| \leqq 2^{-6}$, and identically equal to zero for $|\eta| \geqq 2^{-5}$. For all $w \in H^s$, for all $k \leqq s$, there holds

$$\|(1 - \mathrm{op}_1(\bar\chi))w\|_{1,s} \leqq C\|\nabla_x^k w\|_{1,s-k}. \tag{24}$$

We can bound $N_{\gamma_0,m,s}(\tilde{q})$ with (24), and this implies that the contribution of $\tilde{q}_2+\tilde{q}_{r,1}$ to the operator norm of R_q is bounded by

$$C\varepsilon^{s-d/2} \sup_{|\gamma|\leqq\gamma_0} \sup_{\xi\in\mathbb{R}^d} (1+|\xi|^2)^{(|\gamma|-m)/2} \|\partial_\xi^\gamma (p(v)-p(0))\|_{\varepsilon,s}, \tag{25}$$

which, in turn, is bounded by $\varepsilon^{s-d/2}C(|v|_{0,\infty})\|v\|_{\varepsilon,s}$. The support of $\hat{q}_{r,2}$ is included in a ball $|\eta|+|\xi|\leqq A$. Thus we can use Lemma 36 of [14] to estimate the contribution of $\mathrm{op}_\varepsilon(q_{r,2})$. Using (24) again, we see that, up to a multiplicative constant, it is also bounded by (25). □

The following proposition corresponds to Lemma 19 of [14].

Proposition 2. *Let q such that for some m, $M_{d,m}(q)<\infty$. Assume that q satisfies the spectral condition* (20)*, for some $\delta\in(0,1)$. Then $\mathrm{op}_\varepsilon(q)$ maps H_ε^{s+m} to H_ε^s, for all s, with a norm bounded by a constant times $M_{d,m}(q)$.*

A direct consequence is the following classical result (see [14] or, for instance, [16] Proposition 2.21 in Appendix B).

Proposition 3 (Action)**.** *Let $v\in L^\infty$ and $p\in C^\infty\mathcal{M}^m$ be smooth at the origin. The operator $\mathrm{op}_\varepsilon^\psi(p(v))$ maps H_ε^{s+m} to H_ε^s, for all s, and for all $u\in H_\varepsilon^{s+m}$,*

$$\|\mathrm{op}_\varepsilon^\psi(p(v))u\|_{\varepsilon,s} \leqq C(|v|_{0,\infty})\|u\|_{\varepsilon,s+m}.$$

We will use the following result that gives a precise description of the norm of a pseudodifferential operator with limited regularity in x. It is due to Lannes (the estimate given in Theorem 1 of [14] is actually more precise than the one that follows; in particular, Lannes gives a tame estimate).

Lemma 1. *Let $s>\frac{d}{2}$, and $\sigma(x,\xi)$ such that, for all $\beta\in\mathbb{N}^d$, $N_{\beta,s,m}(\sigma)<\infty$, for some m. Then $\mathrm{op}_1(\sigma)$ maps H^{s+m} to H^s, with a norm bounded by a constant times $N_{\gamma_0,s,m}(\sigma)$, where γ_0 depends only on d.*

Next we state and prove in detail a proposition describing the composition of two paradifferential operators.

Proposition 4 (Composition)**.** *Let $p_1\in C^\infty\mathcal{M}_1^m$ and $p_2\in C^\infty\mathcal{M}_2^m$ be smooth at the origin, and let*

$$p_1\sharp p_2 := \sum_{|\alpha|=1}(-i)\partial_\xi^\alpha p_1 \partial_x^\alpha p_2.$$

Let $s>\frac{d}{2}+2$, $s'=s+m_1+m_2-1$, and assume that p_1, p_2 are evaluated at $v_1, v_2\in H_\varepsilon^s$.

(i) *For all $u\in H_\varepsilon^{s'}$,*

$$\begin{aligned}&\| (\mathrm{op}_\varepsilon^\psi(p_1)\mathrm{op}_\varepsilon^\psi(p_2)-\mathrm{op}_\varepsilon^\psi(p_1p_2)u \|_{\varepsilon,s}\\ &\quad\leqq \varepsilon C|v_2|_{1,\infty}\|u\|_{\varepsilon,s'}+\varepsilon^{s-d/2}C\|v_1,v_2\|_{\varepsilon,s}\|u\|_{\varepsilon,m_1+m_2+d_0},\end{aligned}$$

where C depends on $|v_1,v_2|_{0,\infty}$.

(ii) *For all* $u \in H^{s'-1}_\varepsilon$,

$$\begin{aligned}&\| \, (\mathrm{op}^\psi_\varepsilon(p_1)\mathrm{op}^\psi_\varepsilon(p_2) - \mathrm{op}^\psi_\varepsilon(p_1 p_2 + \varepsilon p_1 \sharp p_2))u \, \|_{\varepsilon,s} \\ &\leqq \varepsilon^2 C(|v_1|_{1,\infty} + |v_2|_{2,\infty} + \|v_1, v_2\|_{\varepsilon,d_0+2})\|u\|_{\varepsilon,s'-1} \\ &\quad + \varepsilon^{s-d/2} C \|v_1, v_2\|_{\varepsilon,s} \|u\|_{\varepsilon,m_1+m_2+d_0},\end{aligned}$$

where C *depends on* $|v_1, v_2|_{1,\infty}$.

Proof.
(i) Let $\mathrm{op}_\varepsilon(R) := \mathrm{op}^\psi_\varepsilon(p_1)\mathrm{op}^\psi_\varepsilon(p_2) - \mathrm{op}^\psi_\varepsilon(p_1 p_2)$, $q_1 := \tilde{p}_1$, and $q_2 := \tilde{p}_2$. To evaluate the norm of $\mathrm{op}_\varepsilon(R)$, we bound the norm of

$$\mathrm{op}_1(\tilde{R}) = \mathrm{op}^\psi_1(q_1)\mathrm{op}^\psi_1(q_2) - \mathrm{op}^\psi_1(q_1 q_2 + q_1 \sharp q_2),$$

by decomposing R into $R_1 + R_2$, such that

$$\begin{aligned}\tilde{R}_1 &:= q_1^\psi q_2^\psi - (q_1^\psi q_2^\psi), \\ \tilde{R}_2 &:= q_1^\psi q_2^\psi - (q_1 q_2)^\psi = R_{q_1} q_2^\psi - q_1 R_{q_2} + R_{q_1 q_2}.\end{aligned}$$

Proposition 31(ii) of [14] asserts that the norm of $\mathrm{op}_1(\tilde{R}_1)$, as an operator mapping $H^{s'}$ to H^s, is controlled by the product of $M_{d,m}(\nabla_x q_2)$, by a seminorm of q_1 involving a large number of derivatives in ξ and a sup norm in x. Thus,

$$\|\mathrm{op}_1(\tilde{R}_1) h_\varepsilon u\|_{\varepsilon,s} \leqq \varepsilon C |\nabla v_2|_{0,\infty} \|u\|_{\varepsilon,s'}.$$

An argument already used in the proof of Proposition 2.1 shows that, for all β,

$$N_{\beta,s,m_1+d_0-s}(R_{q_1}) \leqq \varepsilon^{s-d/2} C \|v_1\|_{\varepsilon,s}.$$

Besides, $N_{\beta,s,m_2}(q_2^\psi) \leqq C N_{\beta,s,m_2}(q_2)$, where C depends only on ψ. Let $m' = m_1 + m_2 + d_0$. By Moser's inequality,

$$N_{\beta,s,m'-s}(R_{q_1} q_2^\psi) \leqq N_{\beta,s,m_1+d_0-s}(R_{q_1}) N_{\beta,s,m_2}(q_2^\psi).$$

Lemma 1 asserts that $\mathrm{op}_1(R_{q_1} q_2^\psi)$ maps $H^{m'}$ to H^s, with a norm controlled by $N_{\gamma_0,s,m'-s}(R_{q_1} q_2^\psi)$. The other terms in $\tilde{R}_2$ are bounded in the same way, so that

$$\|\mathrm{op}_1(\tilde{R}_2) h_\varepsilon u\|_{\varepsilon,s} \leqq \varepsilon^{s-d/2} C_s \|u\|_{\varepsilon,m_1+m_2+d_0},$$

which, together with the above bound for $\mathrm{op}_1(\tilde{R}_1)$, yields the estimate in the first part of the proposition.
(ii) Let $\mathrm{op}_\varepsilon(R') := \mathrm{op}^\psi_\varepsilon(p_1)\mathrm{op}^\psi_\varepsilon(p_2) - \mathrm{op}^\psi_\varepsilon(p_1 p_2 + \varepsilon p_1 \sharp p_2)$. To evaluate the norm of $\mathrm{op}_\varepsilon(R')$, we bound the norm of

$$\mathrm{op}_1(\tilde{R}') = \mathrm{op}^\psi_1(q_1)\mathrm{op}^\psi_1(q_2) - \mathrm{op}^\psi_1(q_1 q_2 + q_1 \sharp q_2),$$

by decomposing $\tilde{R}'$ into $\tilde{R}'_1 + \tilde{R}_2 + \tilde{R}_3$, such that

$$\tilde{R}'_1 := q_1^\psi q_2^\psi - (q_1^\psi q_2^\psi - q_1^\psi \sharp q_2^\psi), \quad \tilde{R}_3 := q_1^\psi \sharp q_2^\psi - (q_1 \sharp q_2)^\psi,$$

and $\tilde{R}_2$ as in the proof of (i). Proposition 31(ii) of [14] asserts that the norm of $\mathrm{op}_1(\tilde{R}_1')$, as an operator mapping $H^{s'}$ to H^s, is controlled by the product of $M_{d,m}(\nabla_x^2 q_2)$ by a seminorm of q_1. Thus,

$$\|\mathrm{op}_1(\tilde{R}_1')h_\varepsilon u\|_{\varepsilon,s} \leqq \varepsilon^2 C(|\nabla_x^2 v_2|_{0,\infty} + |\nabla_x v_2|_{0,\infty}^2)\|u\|_{\varepsilon,s'}.$$

Let ψ' be another admissible cut-off function, defined just like ψ in (17), but with $N = 4$. Then ψ' vanishes identically for $|\eta| \geqq 2^{-3}\langle\xi\rangle$. We denote by $\sigma^{\psi'}$ the regularization of σ by ψ'. The term $\tilde{R}_3$ is split into $-\tilde{R}_{3,1} + \tilde{R}_{3,2}$, such that

$$\tilde{R}_{3,1} := q_1^{\psi'} \sharp q_2^{\psi'} - q_1^{\psi} \sharp q_2^{\psi}, \quad \tilde{R}_{3,2} := q_1^{\psi'} \sharp q_2^{\psi'} - (q_1 \sharp q_2)^{\psi}.$$

The symbol $\tilde{R}_{3,1}$ is a sum of terms of the form

$$\left(\sigma_1^{\psi'} - \sigma_1^{\psi}\right)\sigma_2^{\psi'} + \sigma_1^{\psi}(\sigma_2^{\psi'} - \sigma_2^{\psi}), \tag{26}$$

where $|\alpha| = 1$, $\sigma_1 := \partial_\xi^\alpha q_1$, $\sigma_2 := \partial_x^\alpha q_2$. The symbols in (26) satisfy the spectral condition (20) with $\delta \leqq 2^{-2} + 2^{-2} < 1$. Thus we can apply Proposition 2 to control the norm of $\mathrm{op}_1(\tilde{R}_{3,1})$. It is classical (see for instance Proposition 2.3 in the Appendix B of [16]) that

$$M_{d,\bar{m}-1}(\sigma^{\psi'} - \sigma^{\psi}) \leqq M_{d,\bar{m}}(\nabla_x \sigma),$$

for all symbol σ that is smooth in ξ and such that $M_{\bar{m},d}(\nabla_x \sigma) < \infty$. Thus, $M_{d,m_1+m_2-2}(\tilde{R}_{3,1})$ is controlled by

$$M_{d,m_1-1}(\nabla_\xi \nabla_x q_1) M_{d,m_2}(\nabla_x q_2) + M_{d,m_1-1}(\nabla_\xi q_1) M_{d,m_2}(\nabla_x^2 q_2),$$

and

$$\|\mathrm{op}_1(\tilde{R}_{3,1})h_\varepsilon u\|_{\varepsilon,s} \leqq \varepsilon^2 C_2 \|u\|_{\varepsilon,s'-1},$$

where

$$C_2 = C(|\nabla v_1|_{0,\infty}|\nabla v_2|_{0,\infty} + |\nabla^2 v_2|_{0,\infty} + |\nabla v_2|_{0,\infty}^2).$$

The proof of Lemma 40 of [14] implies that $\tilde{R}_{3,2}$ satisfies the spectral condition (20), and that $M_{d,m_1+m_2-2}(\tilde{R}_{3,2})$ is controlled by

$$M_{d+1,m_1}((1 - \mathrm{op}(\bar{\chi}))q_1) M_{d,m_2}(\nabla_x q_2) + M_{d+1,m_1}(q_1) M_{d,m_2}((1 - \mathrm{op}(\bar{\chi}))\nabla_x q_2).$$

We use (24) again, to obtain

$$M_{d,m_1+m_2+2}(\tilde{R}_{3,2}) \leqq \varepsilon^2 C\|v_1, v_2\|_{\varepsilon,d_0+2}.$$

Then Proposition 2 and the above bound conclude the proof of the second part of the proposition. □

The following proposition can be proved in the same way as Proposition 4.

Proposition 5 (Adjoint)**.** *Let $p \in C^\infty \mathcal{M}^m$ be smooth at the origin, and*

$$r_*(p) := (-i) \sum_{|\alpha|=1} \partial_\xi^\alpha \partial_x^\alpha p^*,$$

where p^* denotes the complex transpose of the matrix p. Let $s > \frac{d}{2}+2$, and $v \in H^s$. Let $\mathrm{op}_\varepsilon^\psi(p(v))^*$ denote the adjoint of $\mathrm{op}_\varepsilon^\psi(p(v))$ in L^2. For all $u \in H_\varepsilon^{s+m-2}$,

$$\begin{aligned} &\| (\mathrm{op}_\varepsilon^\psi(p(v))^* - \mathrm{op}_\varepsilon^\psi(p(v)^*) - \varepsilon \mathrm{op}_\varepsilon^\psi(r_*(p(v))))u \|_{\varepsilon,s} \\ &\leqq \varepsilon^2 C(|v|_{2,\infty} + \|v\|_{\varepsilon,d_0+2}) \|u\|_{\varepsilon,s+m-2} \\ &\quad + \varepsilon^{s-d/2} C \|v\|_{\varepsilon,s} \|u\|_{\varepsilon,m+d_0}, \end{aligned}$$

where C depends on $|v|_{1,\infty}$.

2.3. Pseudodifferential operators with limited regularity

We now consider nonsmooth symbols that have the simple product structure:

$$p(\varepsilon, v, \xi) = p_1(v) p_2(\varepsilon, \xi). \tag{27}$$

If p is a symbol in $C^\infty \mathcal{M}^m$, with the structure (27), then, in particular, p_1 is a smooth map, and $p_2 \in C^\infty \mathcal{M}^m$. Matrix-valued symbols are said to have the structure (27) when every entry can be written as a sum of terms of the form (27).

We will use the following lemma:

Lemma 2. *Let $u, v \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2}$, and assume that $\hat{v}$ has compact support. Then*

$$\|uv\|_{1,s} \leqq C \|u\|_{1,s} |v|_{0,\infty},$$

where C depends only on s and on the space dimension.

Proof. This estimate follows easily from a dyadic decomposition. More details can be found in Lemma 17 of [14], for instance. □

In the following propositions, C denotes nondecreasing functions, and $[\frac{d}{2}] < d_0 \leqq [\frac{d}{2}] + 1$.

Proposition 6. *Let $p \in C^\infty \mathcal{M}^m$, of the form* (27)*, and $v \in H_\varepsilon^s$, where $s > 1 + \frac{d}{2}$. For all $u \in H_\varepsilon^{s+m}(\mathbb{R}^d)$,*

$$\|\mathrm{op}_\varepsilon(p(v))u\|_{\varepsilon,s} \leqq C(\|u\|_{\varepsilon,s+m} + \|v\|_{1,d_0} \|u\|_{L^2} + \varepsilon^{[s]-d/2} \|v\|_{1,[s]+1} \|u\|_{\varepsilon,m+d_0}),$$

where C depends on $|v|_{1,\infty}$.

Proof. If p does not depend on v, the result obviously holds, and so, changing p_1 to $p_1 - p_1(0)$ if necessary, we are reduced to the case $p_1(0) = 0$.

We use the smooth truncation χ introduced at the beginning of Section 2.2. As p_2 is smooth for $|\xi| \geqq 1/4$, the action of $p_1(v)(1 - \mathrm{op}_\varepsilon(\chi))\mathrm{op}_\varepsilon(p_2)$ can be estimated with Propositions 1 and 3. It remains to bound

$$\|\mathrm{op}_\varepsilon(p_1(v)\chi p_2)u\|_{\varepsilon,s} = \|p_1(\tilde{v})\mathrm{op}_1(\chi p_2)h_\varepsilon u\|_{1,s}, \tag{28}$$

where $\tilde{v}(x) = v(\varepsilon x)$. We can write $p_1(\tilde{v})$ as the sum of $\mathrm{op}_1(\chi)p_1(\tilde{v})$ and $(1 - \mathrm{op}_1(\chi))p_1(\tilde{v})$, and apply the above Lemma. Thus we need to bound

$$|\mathrm{op}_1(\chi)p_1(\tilde{v})|_{0,\infty}\|\mathrm{op}_1(\chi p_2)h_\varepsilon u\|_{1,s}, \tag{29}$$

and

$$\|(1 - \mathrm{op}_1(\chi))p_1(\tilde{v})\|_{1,s}|\mathrm{op}_1(\chi p_2)h_\varepsilon u|_{0,\infty}. \tag{30}$$

The first factor in (29) is bounded by

$$\begin{aligned}\sup_x \left| \int e^{i\varepsilon x\xi}\chi(\varepsilon\xi)\widehat{p_1(v)}(\xi)d\xi \right| &\leqq C\|\widehat{p_1(v)}\|_{L^1} \\ &\leqq C(|v|_{0,\infty})\|v\|_{1,d_0}.\end{aligned}$$

To bound the first factor in (30), we use (24), and obtain

$$\|(1 - \mathrm{op}_1(\chi))p(\tilde{v})\|_{1,s} \leqq C\|\nabla_x^{[s]}(p(\tilde{v}))\|_{1,s-[s]}.$$

Then,

$$\|\nabla_x^{[s]}(p(\tilde{v}))\|_{1,s-[s]} = \varepsilon^{[s]-d/2}\|\nabla_x^{[s]}p(v)\|_{\varepsilon,s-[s]}.$$

The $H_\varepsilon^{s-[s]}$ norm is bounded by the H^1 norm, so that

$$\|(1 - \mathrm{op}_1(\chi))p(\tilde{v})\|_{1,s} \leqq \varepsilon^{[s]-d/2}C(|v|_{0,\infty})\|v\|_{1,[s]+1}.$$

The second factor in (29) is controlled by the L^2 norm of u. The second factor in (30) is bounded by the L^1 norm of $\chi p_2\mathcal{F}(h_\varepsilon u)$. As χp_2 is uniformly bounded, the second factor in (30) is controlled by the L^1 norm of $\mathcal{F}h_\varepsilon u$, which in turn is controlled by the $H_\varepsilon^{d_0}$ norm of u. These bounds, when combined, yield the estimate in the proposition. □

Next we describe the composition of two operators of the form (27). As the composition of two such operators involves remainder terms that do not have the form (27), we need the following notation and lemmas.

Let q be defined on $\mathbb{R}^d_\xi$, and p be smooth on $\mathbb{R}^d_x$. Introduce the notation,

$$Q^\varepsilon(\xi,\xi') := q(\varepsilon\xi + \xi') - q(\xi'),$$

and, formally,

$$\rho(q,p)_{|(\varepsilon,x,\xi)} := (\check{Q}^\varepsilon(\cdot,\xi) * p)(x),$$

where $\check{Q}^\varepsilon$ denotes the inverse Fourier transform of Q^ε in its first variable, and $*$ is a convolution in $x \in \mathbb{R}^d$.

With this notation, $\mathrm{op}_\varepsilon(\rho(q, p))u$ is, formally, the inverse Fourier transform of

$$\int (q(\varepsilon\xi) - q(\varepsilon\xi'))\widehat{p(v)}(\xi - \xi')\hat{u}(\xi')d\xi'.$$

Lemma 3. *Let $s > \frac{d}{2}$; let p be smooth on $\mathbb{R}^n$, such that $p(0) = 0$, and χ be the smooth truncation introduced at the beginning of* Section 2.2.

(i) *If $q_{\{|\xi|\leqq 1\}} \in L^\infty$ and $v \in H^{s+d_0}$,*

$$\|\mathrm{op}_\varepsilon(\rho(\chi q, p(v))u\|_{\varepsilon,s} \leqq C(|v|_{0,\infty})\|v\|_{1,s+d_0}\|u\|_{L^2}$$

(ii) *If $q_{\{|\xi|\leqq 1\}} \in W^{1,\infty}$ and $v \in H^{s+d_0+1}$, then*

$$\|\mathrm{op}_\varepsilon\rho(\chi q, p(v))u\|_{\varepsilon,s} \leqq \varepsilon C(|v|_{0,\infty})\|v\|_{1,s+d_0+1}\|u\|_{L^2}.$$

(iii) *If $q \in C^\infty\mathcal{M}^m$ is a Fourier multiplier, if $v \in H^{s+d_0+m}$, then*

$$\|\mathrm{op}_\varepsilon(\rho((1-\chi)q, p(v)))u\|_{\varepsilon,s} \leqq \varepsilon C(|v|_{0,\infty})\|v\|_{1,s+m+d_0}\|u\|_{\varepsilon,s+m-1}.$$

The third estimate is not tame, but it will be sufficient for our purposes.

Proof.

(i) Let $q_0 := \chi q$, and $w_0 := \mathrm{op}_\varepsilon\rho^\varepsilon(q_0, p(v))u$. The H^s_ε norm of w_0 is equal to the L^2_ξ norm of

$$(1 + |\varepsilon\xi|^2)^{s/2}\int (q_0(\varepsilon\xi') - q_0(\varepsilon\xi))\widehat{p(v)}(\xi - \xi')\hat{u}(\xi')d\xi', \tag{31}$$

With Peetre's inequality and because q_0 is compactly supported, the L^2_ξ norm of (31) is bounded by

$$|q_0|_{0,\infty}\left\|\int (1 + |\varepsilon(\xi - \xi')|^2)^{s/2}|\widehat{p(v)}(\xi - \xi')||\hat{u}(\xi')|d\xi'\right\|_{L^2_\xi}.$$

The above integral is a convolution. Thus

$$\begin{aligned}\|w_0\|_{\varepsilon,s} &\leqq C\|\,\mathcal{F}^{-1}|(1 + |\varepsilon\xi|^2)^{s/2}\widehat{p(v)}|\ \mathcal{F}^{-1}|\hat{u}|\,\|_{L^2_x}\\ &\leqq C|\,\mathcal{F}^{-1}|(1 + |\varepsilon\xi|^2)^{s/2}\widehat{p(v)}|\,|_{0,\infty}\,\|u\|_{L^2_x}.\end{aligned}$$

Now the L^∞ norm of $\mathcal{F}^{-1}(1 + |\varepsilon\xi|^2)^{s/2}|\widehat{p(v)}|$ is bounded by the L^1 norm of $(1 + |\varepsilon\xi|^2)^{s/2}\widehat{p(v)}$, which in turn is bounded by $C(|v|_{0,\infty})\|v\|_{1,s+d_0}$. This yields the desired estimate.

(ii) If $\partial_x^\gamma q_0$ is bounded, then w is the inverse Fourier transform of

$$\varepsilon\sum_{|\gamma|=1}\int\int_0^1 \partial_\xi^\gamma q_0(\varepsilon\xi + \varepsilon t(\xi' - \xi))\,dt\ \widehat{\partial_x^\gamma p(v)}(\xi - \xi')\hat{u}(\xi')d\xi',$$

and the bounds that led to (i) are easily adapted to obtain (ii).

(iii) Let $q_1 := (1-\chi)q$, and $w_1 = \mathrm{op}_\varepsilon(\rho((1-\chi)q, p(v)))u$. The H^s_ε norm of w_1 is equal to the L^2_ξ norm of

$$\varepsilon \sum_{|\gamma|=1} (1+|\varepsilon\xi|^2)^{s/2} \int\int_0^1 \partial_\xi^\gamma q_1(\varepsilon\xi + \varepsilon t(\xi'-\xi))\, dt\, \widehat{\partial_x^\gamma p(v)}(\xi-\xi')\hat{u}(\xi')d\xi'.$$

There exists C such that, for all ε, ξ, ξ',

$$|\partial_\xi^\gamma q_1(\varepsilon\xi + \varepsilon t(\xi'-\xi))| \leqq C((1+|\varepsilon\xi'|^2)^{(m-1)/2} + (1+|\varepsilon(\xi-\xi')|^2)^{(m-1)/2}).$$

The rest of the proof of (iii) is similar to the proof of (i). □

Given two symbols of the form (27):

$$p(\varepsilon, v, \xi) = p_1(v)p_2(\varepsilon, \xi), \quad q(\varepsilon, v, \xi) = q_1(v)q_2(\varepsilon, \xi), \tag{32}$$

let

$$\mathtt{r}(p, q) := q_1\rho(\chi q_2, p_1)p_2. \tag{33}$$

Lemma 4. *Let p, q of the form* (32)*, where p_1, q_1 are smooth and vanish at $v = 0$, $\chi q_2 \in L^\infty$, and $p_2 \in C^\infty\mathcal{M}^m$. Let $\mathtt{r} := \mathtt{r}(p, q)$. If p_1 and q_1 are evaluated at $v \in H^{s+d_0+1}$, for all $u \in H^m_\varepsilon$,*

$$\|\mathrm{op}_\varepsilon(\mathtt{r})u\|_{\varepsilon,s} \leqq C(\|v\|_{1,s+d_0})\|u\|_{\varepsilon,m}.$$

and, if $a \in C^\infty\mathcal{M}^1$ is a Fourier multiplier, for all $u \in H^m_\varepsilon$,

$$\|(\mathrm{op}_\varepsilon(a)\mathrm{op}_\varepsilon(\mathtt{r}) - \mathrm{op}_\varepsilon(a\mathtt{r}))u\|_{\varepsilon,s} \leqq \varepsilon C(\|v\|_{1,s+d_0+1})\|u\|_{\varepsilon,m}.$$

Proof. Let $z := \mathrm{op}_\varepsilon(p_2)u$, $p_1 = p_1(v)$, and $q_1 = q_1(v)$. The Fourier transform of $\mathrm{op}_\varepsilon(\mathtt{r})u$ is a convolution in ξ

$$\int \widehat{q_1}(\xi-\xi')((\chi q_2)(\varepsilon\xi'') - (\chi q_2)(\varepsilon\xi'))\widehat{p_1}(\xi'-\xi'')\hat{z}(\xi'')d\xi''d\xi',$$

and, because $\chi q_2 \in L^\infty$, the first estimate in the lemma is obtained in the same way as Lemma 3(i).

Let $w := (\mathrm{op}_\varepsilon(a)\mathrm{op}_\varepsilon(\mathtt{r}) - \mathrm{op}_\varepsilon(a\mathtt{r}))u$. The Fourier transform of w is the sum, over $|\gamma| = 1$, of

$$\varepsilon \int a_\gamma(\varepsilon, \varepsilon\xi, \varepsilon\xi')\widehat{\partial_x^\gamma q_1}(\xi-\xi')((\chi q_2)(\varepsilon\xi'') - (\chi q_2)(\varepsilon\xi'))\widehat{p_1}(\xi'-\xi'')\hat{z}(\xi'')d\xi''d\xi',$$

and

$$\varepsilon \int a_\gamma(\varepsilon, \varepsilon\xi', \varepsilon\xi'')\widehat{q_1}(\xi-\xi')((\chi q_2)(\varepsilon\xi'') - (\chi q_2)(\varepsilon\xi'))\widehat{\partial_x^\gamma p_1}(\xi'-\xi'')\hat{z}(\xi'')d\xi''d\xi',$$

where $a_\gamma(\varepsilon, \eta, \eta') := \int_0^1 \partial_\xi^\gamma a(\varepsilon, \eta + t(\eta'-\eta))dt$. Again, because $a_\gamma, \chi q_2 \in L^\infty$, these convolutions can be bounded in the same way as in the proof of Lemma 3 to yield the second estimate. □

We can now state a proposition that describes the composition of two symbols of the form (27).

Proposition 7. *Given* $p \in C^\infty \mathcal{M}^{m_1}$ *and* $q \in C^\infty \mathcal{M}^{m_2}$,

(i) *if* p *and* q *have the form* (32)*, if* p *is 1-regular at the origin, if* p *and* q *are estimated at* $v \in H^{s+d_0+1}$*, for all* $u \in H_\varepsilon^{s+m_1+m_2-1}$,

$$\begin{aligned}\| \, [\mathrm{op}_\varepsilon(p), \mathrm{op}_\varepsilon(q)]u - \mathrm{op}_\varepsilon([p,q] + \mathtt{r}(p,q))u \, \|_{\varepsilon,s} \\ \leqq \varepsilon C(\|v\|_{1,s+d_0+1}) \|u\|_{\varepsilon,s+m_1+m_2-1};\end{aligned}$$

(ii) *if* p *is a Fourier multiplier and is* 2*-regular at the origin, if* q *has the form* (27) *and is estimated at* $v \in H^{s+d_0+2}$*, for all* $u \in H_\varepsilon^{s+m_1+m_2-2}$,

$$\begin{aligned}\| \, [\mathrm{op}_\varepsilon(p), \mathrm{op}_\varepsilon(q)]u - \mathrm{op}_\varepsilon([p,q] + \varepsilon(p\sharp q))u \, \|_{\varepsilon,s} \\ \leqq \varepsilon^2 C(\|v\|_{1,s+d_0+2}) \|u\|_{\varepsilon,s+m_1+m_2-2}.\end{aligned}$$

Proof.

(i) We compute

$$\begin{aligned}\mathrm{op}_\varepsilon(p)\mathrm{op}_\varepsilon(q)u &= p_1 \mathrm{op}_\varepsilon(p_2)(q_1 \mathrm{op}_\varepsilon(p_2)u) \\ &= p_1(\mathrm{op}_\varepsilon(p_2 q_1) - \mathrm{op}_\varepsilon(\rho(p_2,q_1)))\mathrm{op}_\varepsilon(p_2)u.\end{aligned}$$

Thus,

$$\begin{aligned}&[\mathrm{op}_\varepsilon(p), \mathrm{op}_\varepsilon(q)] - \mathrm{op}_\varepsilon([p,q] + \mathtt{r}(p,q)) \\ &\quad = p_1 \mathrm{op}_\varepsilon(\rho(p_2,q_1))\mathrm{op}_\varepsilon(q_2) - q_1 \mathrm{op}_\varepsilon(\rho((1-\chi)q_2, p_1)\mathrm{op}_\varepsilon(p_2),\end{aligned}$$

and the action of the right-hand side on u is estimated with Lemma 3(ii) and (iii).

(ii) Let $w := ([\mathrm{op}_\varepsilon(p), \mathrm{op}_\varepsilon(q)] - \mathrm{op}_\varepsilon[p,q] + \varepsilon \mathrm{op}_\varepsilon(p\sharp q))u$. The Fourier transform of w is

$$\varepsilon^2 \sum_{|\gamma|=2} \int p_\gamma(\varepsilon\xi, \varepsilon\xi') \widehat{\partial_x^\gamma q_1(v)}(\xi - \xi') q_2(\xi') \hat{u}(\xi') d\xi',$$

where $p_\gamma(\varepsilon\xi, \varepsilon\xi') := \int_0^1 (1-t)\partial_\xi^\gamma p(\varepsilon\xi' + \varepsilon t(\xi - \xi'))dt$, and we obtain the second estimate as above. □

2.4. Assumptions and results

We use the profile and symbol spaces introduced in Section 2.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{G}^\varepsilon$ such that

(i) $\mathcal{A}$ is a smooth symbol in $C^\infty \mathcal{M}^1$;
(ii) $\mathcal{B}$ is a bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$;
(iii) $\mathcal{G}^\varepsilon$ is a family of smooth maps: $\mathbb{R}^n \to \mathbb{R}^n$, such that $\mathcal{G}^\varepsilon(0) = 0$.

Following the notation of Section 2.1, we denote by $\mathrm{op}_\varepsilon(\mathcal{A}(\varepsilon, \varepsilon v))$ the semiclassical pseudodifferential operator with symbol $\mathcal{A}(\varepsilon, \varepsilon v, \xi)$. The Taylor expansion of $\mathcal{A}$ in v is:

$$\mathcal{A}(\varepsilon, \varepsilon v) = \mathcal{A}^{(0)}(\varepsilon, 0) + \varepsilon \mathcal{A}^{(1)}(v) + \varepsilon^2 \mathcal{A}^{(2)}(\varepsilon, v),$$

where $\mathcal{A}^{(j)} \in C^\infty \mathcal{M}^1$, for $j = 0, 1, 2$, and where $\mathcal{A}^{(1)}(v) := \partial_v \mathcal{A}(0, 0) \cdot v$ is linear in v and has the form (27).

We study the initial value problem

$$\begin{cases} \partial_t u^\varepsilon + \dfrac{1}{\varepsilon^2}\mathrm{op}_\varepsilon(\mathcal{A}(\varepsilon, \varepsilon u^\varepsilon))u^\varepsilon = \dfrac{1}{\varepsilon}\mathcal{B}(u^\varepsilon, u^\varepsilon) + \mathcal{G}^\varepsilon(u^\varepsilon), \\ \qquad\qquad u^\varepsilon(0, x) = a^\varepsilon(x) + \varepsilon^{k_0}\varphi^\varepsilon(x), \end{cases} \tag{34}$$

where the initial datum a^ε belongs to $H^\sigma(\mathbb{R}^d)$, for some Sobolev index σ, much larger than $\frac{d}{2}$ (we will actually need $\sigma > 6 + d$), uniformly with respect to ε:

$$\sup_{0<\varepsilon<\varepsilon_0} \|a^\varepsilon\|_{1,\sigma} < \infty.$$

The perturbation $\varepsilon^{k_0}\varphi^\varepsilon$ is such that

$$k_0 > 3 + \frac{d}{2}, \tag{35}$$

and φ^ε belongs to $H^s_\varepsilon(\mathbb{R}^d)$, uniformly with respect to ε:

$$\sup_{0<\varepsilon<\varepsilon_0} \|\varphi^\varepsilon\|_{\varepsilon,s} < \infty,$$

for some large Sobolev index s, smaller than σ.

Our first assumption is a hyperbolicity assumption that implies in particular the local well posedness of the initial value problem (34).

Assumption 3 (Hyperbolicity)**.** *For all ε, v, ξ, the matrix $\mathcal{A}(\varepsilon, v, \xi)$ is Hermitian. Let*

$$\mathcal{A} = \sum_{1 \leqq j \leqq n_0} i\lambda_j \Pi_j + \sum_{n_0+1 \leqq k \leqq n} i\lambda_k \Pi_k, \tag{36}$$

be its spectral decomposition, where the eigenvalues λ_j, λ_k are real and the eigenprojectors Π_j, Π_k are orthogonal. We assume:

(i) *the eigenvalues can be ordered as follows: for all ε, v, ξ,*

$$\sup_{n_0+1 \leqq k} |\lambda_k(\varepsilon, v, \xi)| < \sup_{j \leqq n_0} |\lambda_j(\varepsilon, v, \xi)|;$$

(ii) *for all $1 \leqq m \leqq n$, $\lambda_m(\varepsilon, 0, \xi) \in C^\infty\mathcal{M}^1$, and $\Pi_m(\varepsilon, 0, \xi) \in C^\infty\mathcal{M}^0$;*
(iii) *for all $n_0 + 1 \leqq k \leqq n$, $\lambda_k(0, 0, \xi) = 0$, for all ξ.*

In reference to the Euler–Maxwell equations (see Section 3), the eigenvalues λ_j for $1 \leqq j \leqq n_0$ are called Klein–Gordon modes, while the eigenvalues λ_k, for $n_0 + 1 \leqq k \leqq n$, are called acoustic modes. Condition (i) in Assumption 3 states that the acoustic modes do not cross the Klein–Gordon modes. Condition (ii) is a regularity assumption, and condition (iii) amounts to saying that the acoustic velocities are $O(\varepsilon)$, a consequence of (7).

Condition (ii) implies in particular that the spectral decomposition is smooth for $|\xi| \geqq 1/4$. It might become singular for small frequencies. In the case of the (EM) equations, eigenvalues do cross for small frequencies (see Figure 2 and Section 3.2.1).

Let

$$\Pi_0 := \sum_{1 \leqq k \leqq n_0} \Pi_k, \qquad \Pi_s := \sum_{n_0+1 \leqq j \leqq n} \Pi_j. \tag{37}$$

In addition to Assumption 3, we will assume that

$$\Pi_0, \Pi_s(\varepsilon, v, \xi) \in C^\infty \mathcal{M}^0. \tag{38}$$

When $\mathcal{A}$ depends analytically on ε, v, ξ, (38) follows from standard considerations, as detailed in Section 3.

Assumption 4 (Approximate solution)**.** *For all* $l_0 < \sigma - 2 - \frac{d}{2}$, *there exists* $t^*(l_0) > 0$, *independent of* ε, *and a family of profiles* u_a^ε, *such that*

$$\begin{cases} \partial_t u_a^\varepsilon + \dfrac{1}{\varepsilon^2}\mathrm{op}_\varepsilon(\mathcal{A}(\varepsilon, \varepsilon u_a^\varepsilon))u_a^\varepsilon = \dfrac{1}{\varepsilon}\mathcal{B}(u_a^\varepsilon, u_a^\varepsilon) + \mathcal{G}^\varepsilon(u_a^\varepsilon) + \varepsilon^{l_0} R_a^\varepsilon, \\ u_a^\varepsilon(0, x) = a^\varepsilon(x), \end{cases} \tag{39}$$

where R_a^ε *is uniformly bounded with respect to* ε,

$$\sup_{0<\varepsilon<\varepsilon_0} \sup_{0 \leqq t \leqq t^*(l_0)} \|R_a^\varepsilon(t)\|_{1,\sigma-l_0-2} < \infty.$$

There exists a characteristic frequency $\omega \neq 0$, *a finite set* $\mathcal{R}^* \subset \mathbb{Z}$ *and profiles* $\{u_{a,p}^\varepsilon\}_{p\in\mathcal{R}^*}$ *and* v_a^ε, *such that* u_a^ε *decomposes as*

$$u_a^\varepsilon(t, x) = \sum_{p\in\mathcal{R}^*} e^{ip\omega t/\varepsilon^2} u_{a,p}^\varepsilon(t, x) + \varepsilon v_a^\varepsilon(t, x),$$

with the uniform bounds,

$$\sup_{0<\varepsilon<\varepsilon_0} \sup_{0 \leqq t \leqq t^*(l_0)} (\|u_{a,p}^\varepsilon(t)\|_{1,\sigma} + \|\partial_t u_{a,p}^\varepsilon(t)\|_{1,\sigma-2}) < \infty,$$

$$\sup_{0<\varepsilon<\varepsilon_0} \sup_{0 \leqq t \leqq t^*(l_0)} (\|v_a^\varepsilon(t)\|_{1,\sigma} + \|\varepsilon^2 \partial_t v_a^\varepsilon(t)\|_{1,\sigma-1}) < \infty.$$

For the Euler–Maxwell system of equations, such an approximate solution is explicitly constructed in Section 3, under an assumption of preparedness for the initial datum. In the following assumption, $\mathcal{R}^*$ refers to the set of characteristic harmonics introduced in Assumption 4.

Assumption 5 (Resonances)**.** *There exists* $0 < c_l < c_m < C_m$*, such that the resonance equations in* $\xi \in \mathbb{R}^d$ *and* $p, p' \in \mathcal{R}^*$,

$$\Phi_{j,k,p}(\varepsilon) := \lambda_j(\varepsilon, 0, \xi) - \lambda_k(\varepsilon, 0, \xi) + p\omega = 0, \tag{40}$$

$$\Psi_{j,p,p'} := \lambda_j(0, 0, \xi) - (p + p')\omega = 0, \tag{41}$$

satisfy,

(0-0) $1 \leqq j, k \leqq n_0$*: the solutions* ξ, p *of* (40) *at* $\varepsilon = 0$, *are located in the interval* $c_m \leqq |\xi| \leqq C_m$*; outside this interval,* $\Phi_{j,k,p}(0)$ *is bounded away from* 0, *uniformly in* ξ, *for all* $p \in \mathcal{R}^*$;

(0-s) $1 \leqq j \leqq n_0$, $n_0 + 1 \leqq k \leqq n$*: for* ε *small enough, the solutions* ξ, p *of* (40) *are located in the interval* $|\xi| \leqq c_l$*; outside this interval,* $\Phi_{j,k,p}(\varepsilon)$ *is bounded away from* 0, *uniformly in* ξ *and* ε, *for all* $p \in \mathcal{R}^*$;

(0-0-s) $1 \leqq j \leqq n_0$*: for all* $p, p' \in \mathcal{R}^*$, $\Psi_{j,p,p'}$ *is bounded away from* 0, *uniformly in* $|\xi| \in [0, c_m]$.

Examples of resonances for the (EM) system are shown in Figures 3, 4 and 5.

Next we state the assumptions that describe the interaction coefficients at the resonances. Introduce first the notation

$$\mathcal{B}(u_a^\varepsilon)z := \mathcal{B}(u_a^\varepsilon, z) + \mathcal{B}(z, u_a^\varepsilon) - \mathrm{op}_\varepsilon(\mathcal{A}^{(1)}(z))u_a^\varepsilon, \tag{42}$$

$$\mathcal{D}(\varepsilon, u_a^\varepsilon)z := (\mathcal{G}^\varepsilon)'(u_a^\varepsilon) \cdot z + \mathrm{op}_\varepsilon(\partial_v \mathcal{A}^{(2)}(\varepsilon, u_a^\varepsilon) \cdot z)u_a^\varepsilon, \tag{43}$$

where u_a^ε is the approximate solution given by Assumption 4. Thus defined, $\mathcal{B}(u_a^\varepsilon)$ and $\mathcal{D}(\varepsilon, u_a^\varepsilon)$ belong to $C^\infty \mathcal{M}^0$.

We assume that for some $j \leqq n_0 < k$, there exists p, ξ_0 and $\eta > 0$, such that $\Phi_{j,k,p}(\xi_0) = 0$, and

$$|\Pi_j(0,0)\mathcal{B}(u_{a,p}^\varepsilon)\Pi_k(0,0)| > \eta, \quad \text{uniformly in } \xi \sim \xi_0, \tag{44}$$

Inequality (44) means that the interaction coefficient $\Pi_j \mathcal{B} \Pi_k$ is *not transparent* for resonances between Klein–Gordon and acoustic modes.

Let

$$\rho(\varepsilon, u_a^\varepsilon) := (\Pi_s \mathcal{A}^{(0)})\sharp\Pi_0 + (\Pi_s \sharp \mathcal{A}^{(1)})\Pi_0 + \Pi_s \mathcal{A}^{(0)}(\Pi_0 \sharp \Pi_0) - (\Pi_s \sharp \mathcal{B})\Pi_0, \tag{45}$$

where the projectors Π_0, Π_s are evaluated at $(\varepsilon, \varepsilon u_a^\varepsilon)$, and $\mathcal{A}^{(1)}$ is evaluated at u_a^ε. Let

$$B^{\mathrm{r}}(\varepsilon, u_a^\varepsilon) := \Pi_s(\mathcal{B} + \partial_u \Pi_s \cdot (\varepsilon^2 \partial_t u_a^\varepsilon) + \varepsilon\rho)\Pi_0, \tag{46}$$

where ρ is evaluated at $(\varepsilon, u_a^\varepsilon)$, $\mathcal{B}$ is evaluated at u_a^ε, and the projectors and their derivatives are evaluated at $(\varepsilon, \varepsilon u_a^\varepsilon)$. Remark that $B^{\mathrm{r}}(\varepsilon, 0) = 0$, and

$$B^{\mathrm{r}}(\varepsilon, u_a^\varepsilon) = \partial_u B^{\mathrm{r}}(\varepsilon, 0) \cdot u_a^\varepsilon + \varepsilon \partial_u^2 B^{\mathrm{r}}(\varepsilon, 0) \cdot (u_a^\varepsilon, u_a^\varepsilon) + O(\varepsilon^2).$$

The linear term $\partial_u B^{\mathrm{r}}(\varepsilon, 0) \cdot u_a^\varepsilon$ is the crucial interaction coefficient. The following transparency assumption states that it is sufficiently small at the resonances between Klein–Gordon and acoustic modes.

Assumption 6 (Transparency). *There exists* $\varepsilon_0 > 0$ *and* $C > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$, *for* $j \leqq n_0 < k$, *for all* $p \in \mathcal{R}^*$,

$$|\Pi_k(\varepsilon, 0)\mathcal{D}(0, u_a^\varepsilon)\Pi_j(\varepsilon, 0)| \leqq C\varepsilon, \tag{47}$$

and

$$|\Pi_k(\varepsilon, 0)(\partial_u B^{\Upsilon}(\varepsilon, 0) \cdot u_a^\varepsilon)\Pi_j(\varepsilon, 0)| \leqq C(\varepsilon^2 + |\Phi_{j,k,p}(\varepsilon)|), \tag{48}$$

uniformly in $|\xi| \leqq c_l$, $x \in \mathbb{R}^d$ *and* $t \in [0, t^*(l_0))$.

Introduce finally

$$E := \begin{pmatrix} \Pi_0\mathcal{B}\Pi_0 & 0 \\ 0 & \Pi_s\mathcal{B}\Pi_s \end{pmatrix}, \quad iA := \begin{pmatrix} \Pi_0\mathcal{A} & 0 \\ 0 & \Pi_s\mathcal{A} \end{pmatrix},$$

where $\mathcal{A}$ and the projectors are seen as symbols depending on $(\varepsilon, \varepsilon u)$, and $\mathcal{B}$ as a symbol depending on (ε, u).

Assumption 7 (Symmetrizability). *There exists* S, *a smooth Fourier multiplier in* $C^\infty\mathcal{M}^0$, *such that*

$$\frac{1}{\gamma}\|u\|_{\varepsilon,s}^2 \leqq (\mathrm{op}_\varepsilon(S)u, u)_{\varepsilon,s} \leqq \gamma\|u\|_{\varepsilon,s}^2,$$

for all $u \in H_\varepsilon^s$ *and for some* $\gamma > 0$, *and*

$$\frac{1}{\varepsilon}(SE + (SE)^*) \in C^\infty\mathcal{M}^0, \quad \frac{1}{\varepsilon^2}(iSA + (iSA)^*) \in C^\infty\mathcal{M}^0.$$

In the following theorem, u_a^ε is the approximate solution at order l_0, for some $3 + k_0 \leqq l_0 \leqq \sigma - 2 - \frac{d}{2}$, whose existence is guaranteed by Assumption 4, $t^* = t^*(l_0)$ is its maximal existence time, independent of ε, and s is a Sobolev index, such that $1 + \frac{d}{2} < s < \sigma - l_0 - 2$.

Theorem 8. *Under Assumptions* 3, 4, 5, 6 *and* 7, *there exists a unique solution* $u^\varepsilon \in C^0([0, t_0], H_\varepsilon^s(\mathbb{R}^d))$ *to the initial value problem* (34), *for all* $0 \leqq t_0 < t^*$; *there exists* $C > 0$ *and* $\varepsilon_0 > 0$, *such that, for all* $0 < \varepsilon < \varepsilon_0$, *for all* $0 \leqq t_0 < t^*$,

$$\sup_{0 \leqq t \leqq t_0} \|(u^\varepsilon - u_a^\varepsilon)(t)\|_{\varepsilon,s} \leqq C\varepsilon^{k_0 - 1}. \tag{49}$$

In particular,

$$\sup_{0 \leqq t \leqq t_0} |(u^\varepsilon - u_a^\varepsilon)(t)|_{0,\infty} \leqq C\varepsilon^{k_0 - 1 - d/2}.$$

In the error estimate (49), C depends on a Sobolev norm of the initial data and on t_0.

2.5. Proof of Theorem 8

In the proof below, we often drop the epsilons as we write u for u^ε, u_a for u_a^ε, etc. We use the notation and results of Section 2.1 to describe symbols and operators.

We start with (34). Let $l_0 \geqq k_0 + 3$, and let u_a be the approximate solution at order l_0 given by Assumption 4. An existence time for u_a is $t^* > 0$, independent of ε.

2.5.1. The perturbation equations. The exact solution u is sought as a perturbation of u_a:

$$u = u_a + \varepsilon^{k_0-1}\dot{u}. \tag{50}$$

The symbol $\mathcal{A}^{(0)}$ does not depend on v. This implies, with Proposition 1,

$$\mathrm{op}_\varepsilon(\mathcal{A}(\varepsilon, \varepsilon u)) = \mathrm{op}_\varepsilon^\psi(\mathcal{A}(\varepsilon, \varepsilon u)) + \varepsilon^{s+1-d/2}\mathrm{op}_\varepsilon(R_{\mathcal{A}^{(1)}(u)} + \varepsilon R_{\mathcal{A}^{(2)}(\varepsilon,u)}).$$

The perturbation equations are

$$\begin{cases} \partial_t \dot{u} + \dfrac{1}{\varepsilon^2}\mathrm{op}_\varepsilon^\psi(\mathcal{A}(\varepsilon, \varepsilon u))\dot{u} = \dfrac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\mathcal{B}(u_a))\dot{u} + \mathrm{op}_\varepsilon(\mathcal{D}(0, u_a))\dot{u} + \varepsilon R^\varepsilon, \\ \dot{u}(0,x) = \varepsilon\varphi^\varepsilon(x), \end{cases} \tag{51}$$

where $\mathcal{B}$ and $\mathcal{D}$ are given by (42) and (43), and where

$$\begin{aligned} R^\varepsilon := {} & \varepsilon^{k_0-3}\mathcal{B}(\dot{u}, \dot{u}) + \varepsilon^{k_0-2}\int_0^1 (1-t)(\mathcal{G}^\varepsilon)''(u_a + t\varepsilon^{k_0-1}\dot{u}) \cdot (\dot{u}, \dot{u})dt \\ & - \varepsilon^{k_0-2}\mathrm{op}_\varepsilon\left(\int_0^1 (1-t)\partial_v^2\mathcal{A}^{(2)}(\varepsilon, u_a + t\varepsilon^{k_0-1}\dot{u}) \cdot (\dot{u}, \dot{u})\right) u_a \\ & - \varepsilon^{s-2-s/2}\mathrm{op}_\varepsilon\left(R_{\mathcal{A}^{(1)}(u)} + \varepsilon R_{\mathcal{A}^{(2)}(\varepsilon,u)} - R_{\mathcal{B}(u_a)} - \varepsilon R_{\mathcal{D}(0,u_a)}\right) \\ & - \varepsilon^{l_0-k_0-2}R_a^\varepsilon + \int_0^1 \partial_\varepsilon\mathcal{D}(t\varepsilon, u_a)dt. \end{aligned}$$

Under Assumption 3, standard hyperbolic theory provides the existence of a unique solution $\dot{u}$ to (51) over a small time interval $[0, t_*(\varepsilon)]$, with the uniform estimate

$$\sup_{0<\varepsilon<\varepsilon_0}\ \sup_{0\leqq t\leqq t_*(\varepsilon)} \|\dot{u}(t)\|_{\varepsilon,s} \leqq \delta. \tag{52}$$

The term R^ε is a remainder, in the sense that its H_ε^s norm can be bounded in terms of δ, uniformly in ε. The H_ε^s norm of the terms in the first line in the definition of R^ε is indeed bounded by $\varepsilon^{k_0-2-d/2}C\|\dot{u}\|_{\varepsilon,d_0}\|\dot{u}\|_{\varepsilon,s}$; the term in the second line of R^ε is bounded by

$$\varepsilon^{k_0-1-d}C(\|u_a\|_{\varepsilon,s+1}\|\dot{u}\|_{\varepsilon,d_0}^2 + |u_a|_{1,\infty}\|\dot{u}\|_{\varepsilon,d_0}\|\dot{u}\|_{\varepsilon,s}),$$

and the other terms are bounded by

$$\varepsilon^{s-1-d/2} C \|\dot{u}\|_{\varepsilon,1+d_0} \|u\|_{\varepsilon,s} + \varepsilon^{l_0-k_0-2} \|R_a^\varepsilon\|_{\varepsilon,s}.$$

In all these estimates, C depends on $|u|_{0,\infty}$.

With the estimates for u_a given in Assumption 4, the bound for $\dot{u}$ given in (52), and the form of the equation (51), for $\alpha + |\beta| \leqq 2$,

$$\sup_{0 \leqq t \leqq t_*(\varepsilon)} |(\varepsilon^2 \partial_t)^\alpha \partial_x^\beta u|_{0,\infty} \leqq c_a + \varepsilon^{k_0-3-d/2} \delta, \tag{53}$$

where c_a does not depend on ε. The size of the perturbation of the initial data in (34), namely $O(\varepsilon^{k_0})$ in H^s_ε, where k_0 satisfies (35), was chosen in order that the estimate (53) be uniform in ε.

With this notation, the above estimates give, for ε_0 small enough, and $s > 1 + \frac{d}{2}$,

$$\|R^\varepsilon\|_{\varepsilon,s} \leqq \tilde{C} \|\dot{u}\|_{\varepsilon,s} + \varepsilon^{l_0-k_0-2} \|R_a^\varepsilon\|_{\varepsilon,s},$$

where $\tilde{C}$ is a nondecreasing function of $\varepsilon_0^{k_0-3-d/2}\delta$, $\|u_a\|_{\varepsilon,s+1}$, s and d.

We generically denote by $R_{(0)}$ any pseudo or paradifferential operator, possibly depending on the solution u, such that, for all $z \in H^s_\varepsilon(\mathbb{R}^d)$,

$$\|R_{(0)} z\|_{\varepsilon,s} \leqq \tilde{C}(\|z\|_{\varepsilon,s} + \varepsilon^{s-d/2-2} \|u\|_{\varepsilon,s} \|z\|_{\varepsilon,1+d_0}), \tag{54}$$

uniformly in $t \in [0, t_*(\varepsilon)]$, where $\tilde{C}$ is nondecreasing, and

$$\tilde{C} = \tilde{C}(\varepsilon_0^{k_0-3-d/2}\delta, \|u_a\|_{1,s+d_0+2}, s, d), \tag{55}$$

and where $[\frac{d}{2}] < d_0 \leqq [\frac{d}{2}] + 1$.

We denote by $O(\varepsilon^k)$ symbols associated with pseudo or paradifferential operators of the form $\varepsilon^k R_{(0)}$.

In the next section, we are led to study compositions of paradifferential symbols of the form $p_j(u_{\alpha,\beta})$, where $p_j \in C^\infty \mathcal{M}^{m_j}$ is smooth, and $u_{\alpha,\beta} = (\varepsilon^2 \partial_t)^\alpha \partial_x^\beta u$. It follows from Propositions 3 and 4 that:

(i) If $m_j \leqq 0$, and $\alpha + |\beta| \leqq 2$, then $\mathrm{op}_\varepsilon^\psi(p_j(u_{\alpha,\beta})) = R_{(0)}$.

(ii) If $m_1 + m_2 \leqq 1$, if $p_1 = p_1(u_{\alpha,\beta})$, and $p_2 = p_2(u_{\alpha',\beta'})$, with $\alpha + \alpha' + |\beta| + |\beta'| \leqq 1$, then

$$\mathrm{op}_\varepsilon^\psi(p_1)\mathrm{op}_\varepsilon^\psi(p_2) - \mathrm{op}_\varepsilon^\psi(p_1 p_2) = \varepsilon R_{(0)}.$$

(iii) If $m_1 + m_2 \leqq 2$, if $p_1 = p_1(u)$ and $p_2 = p_2(u)$, then

$$\mathrm{op}_\varepsilon^\psi(p_1)\mathrm{op}_\varepsilon^\psi(p_2) - \mathrm{op}_\varepsilon^\psi(p_1 p_2) - \varepsilon \mathrm{op}_\varepsilon^\psi(p_1 \sharp p_2) = \varepsilon^2 R_{(0)}.$$

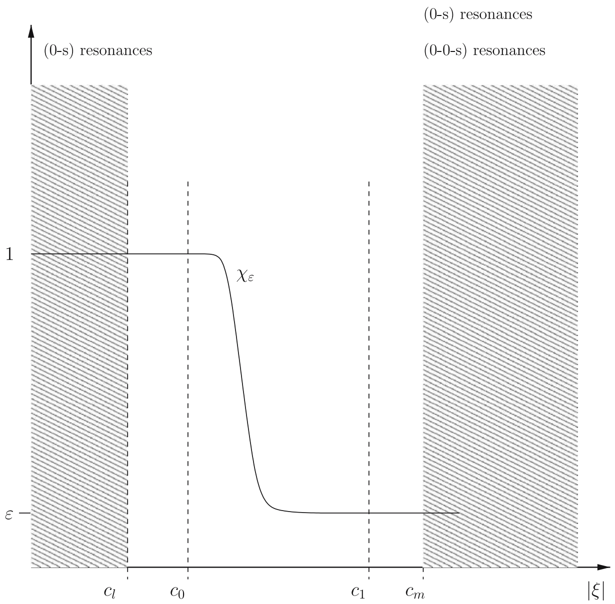

Fig. 1. Truncation in frequency

2.5.2. Projection and rescaling. We start by introducing some notation.

(i) We denote by $\chi_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ a smooth function, such that $\varepsilon \leqq \chi_\varepsilon \leqq 1$, and such that χ_ε is identically equal to 1 for $|\xi| \leqq c_0$, and identically equal to ε for $|\xi| \geqq c_1$, where $0 < c_0 < c_l < c_1 < c_m$, with the notation of Assumption 2.12 (see Figure 1).

(ii) The notation $\sharp$ was introduced in Section 2.2, where it is used to denote the subprincipal symbol in the composition of two operators. We use it in this section in two distinct ways. Let σ_1 and σ_2 denote two symbols, depending on x through some function $z(x)$. If $\sigma_2(x, \xi) = \Sigma_2(\varepsilon z(x), \xi)$, for some symbol Σ_2, then we let $\sigma_1 \sharp \sigma_2$ denote the symbol,

$$-i \sum_{|\alpha|=1} (\partial_\xi^\alpha \sigma_1)(x, \xi)\, (\partial_v \Sigma_2)(\varepsilon z(x), \xi) \cdot (\partial_x^\alpha z)(x).$$

If $\sigma_2(x, \xi) = \Sigma_2(z(x), \xi)$, then we let $\sigma_1 \sharp \sigma_2$ denote the symbol,

$$-i \sum_{|\alpha|=1} (\partial_\xi^\alpha \sigma_1)(x, \xi)\, (\partial_v \Sigma_2)(z(x), \xi) \cdot (\partial_x^\alpha z)(x).$$

We will always specify below if σ_2 is to be understood as an operator depending in x through $z(x)$ or through $\varepsilon z(x)$, so that no confusion should be possible. For instance, Proposition 2.3 implies that

$$\mathrm{op}_\varepsilon^\psi(\Pi_0)\mathrm{op}_\varepsilon^\psi(\Pi_0) = \mathrm{op}_\varepsilon^\psi(\Pi_0) + \varepsilon^2 \mathrm{op}_\varepsilon^\psi(\Pi_0 \sharp \Pi_0) + \varepsilon^3 R_{(0)},$$

where Π_0 is evaluated at εu. Similarly, when Π_0 is evaluated at εu, and $\mathcal{B}$ is evaluated at u,

$$\mathrm{op}_\varepsilon^\psi(\Pi_0)\mathrm{op}_\varepsilon^\psi(\mathcal{B}) = \mathrm{op}_\varepsilon^\psi(\Pi_0\mathcal{B}) + \varepsilon\mathrm{op}_\varepsilon^\psi(\Pi_0\sharp\mathcal{B}) + \varepsilon^2 R_{(0)},$$

(iii) We let $a \equiv b$ to mean that $a - b = \varepsilon^2 R_{(0)}\dot{u}$.
Introduce the change of variables,

$$v_0 := \mathrm{op}_\varepsilon^\psi(\Pi_0)\dot{u}, \quad v_s := \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon)\mathrm{op}_\varepsilon^\psi(\Pi_s)\dot{u}, \quad v := (v_0, v_s),$$

where Π_0, Π_s are evaluated at εu. Then

$$\dot{u} = v_0 + \mathrm{op}_\varepsilon^\psi(\varepsilon\chi_\varepsilon^{-1})v_s. \tag{56}$$

With the above notation, Proposition 4 and the orthogonality of Π_0 and Π_s,

$$\begin{aligned}
\mathrm{op}_\varepsilon^\psi(\Pi_0)v_0 &= v_0 + \varepsilon^2\mathrm{op}_\varepsilon^\psi(\Pi_0\sharp\Pi_0)\dot{u} + \varepsilon^3 R_{(0)}\dot{u},\\
\mathrm{op}_\varepsilon^\psi(\Pi_s)v_s &= v_s + \varepsilon\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon(\Pi_s\sharp\Pi_s) + (\chi_\varepsilon\sharp\Pi_s)\Pi_s)\dot{u} + \varepsilon^2 R_{(0)}\dot{u}.
\end{aligned}$$

We multiply (51) by $\mathrm{op}_\varepsilon^\psi(\Pi_0)$ to the left to find the equation satisfied by v_0, and we multiply (51) by $\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon)\mathrm{op}(\Pi_s)$ to the left to find the equation satisfied by v_s. We use Proposition 4 to spell out the compositions. In the symbolic computations below, the projectors Π_0 and Π_s, and the operator $\mathcal{A}$, are evaluated at εu. The symbol $\mathcal{B}$ is evaluated at u_a, and $\mathcal{D}$ is evaluated at $(0, u_a)$.

The terms in ∂_t are

$$\begin{aligned}
\mathrm{op}_\varepsilon^\psi(\Pi_0)\partial_t\dot{u} &= \partial_t v_0 - \mathrm{op}_\varepsilon^\psi(\partial_t\Pi_0)\dot{u}\\
&= \partial_t v_0 - \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi((\varepsilon\partial_t\Pi_0)\Pi_0)v_0\\
&\quad -\frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\varepsilon\chi_\varepsilon^{-1}(\varepsilon\partial_t\Pi_0)\Pi_s)v_s + R_{(0)}v;
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon)\mathrm{op}_\varepsilon^\psi(\Pi_s)\partial_t\dot{u} &= \partial_t v_s - \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon)\mathrm{op}_\varepsilon^\psi(\partial_t\Pi_s)\dot{u}\\
&= \partial_t v_s - \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi((\varepsilon\partial_t\Pi_s)\Pi_s)v_s\\
&\quad -\frac{1}{\varepsilon^2}\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon(\varepsilon\partial_t\Pi_s)\Pi_0)v_0\\
&\quad -\frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi((\chi_\varepsilon\sharp\varepsilon\partial_t\Pi_s)\Pi_0)v_0 + R_{(0)}v.
\end{aligned}$$

The terms of order one are

$$\begin{aligned}
\mathrm{op}_\varepsilon^\psi(\Pi_0)\mathrm{op}_\varepsilon^\psi(\mathcal{A})\dot{u} &\equiv \mathrm{op}_\varepsilon^\psi(\Pi_0\mathcal{A})\dot{u}\\
&\equiv \mathrm{op}_\varepsilon^\psi(\Pi_0\mathcal{A})v_0 - \mathrm{op}_\varepsilon^\psi(\varepsilon\chi_\varepsilon^{-1}\Pi_0\mathcal{A}(\chi_\varepsilon\sharp\Pi_s)\Pi_s)\dot{u}\\
&\equiv \mathrm{op}_\varepsilon^\psi(\Pi_0\mathcal{A})v_0 - \mathrm{op}_\varepsilon^\psi((\varepsilon\chi_\varepsilon^{-1})^2\Pi_0\mathcal{A}(\chi_\varepsilon\sharp\Pi_s)\Pi_s)v_s,
\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon)\mathrm{op}_\varepsilon^\psi(\Pi_s)\mathrm{op}_\varepsilon^\psi(\mathcal{A})\dot{u} &\equiv \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon)\mathrm{op}_\varepsilon^\psi(\Pi_s\mathcal{A})\dot{u} + \varepsilon\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon)\mathrm{op}_\varepsilon^\psi(\Pi_s\mathcal{A})\dot{u}\\ &\equiv \mathrm{op}_\varepsilon^\psi(\Pi_s\mathcal{A})v_s + \varepsilon\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon(\varepsilon\chi_\varepsilon^{-1}\sharp(\Pi_s\mathcal{A})))v_s\\ &\quad +\varepsilon\mathrm{op}_\varepsilon^\psi(\rho_{s0})v_0,\end{aligned}$$

where

$$\rho_{s0} := \chi_\varepsilon((\Pi_s\mathcal{A})\sharp\Pi_0 + (\Pi_s\sharp\mathcal{A})\Pi_0 - \Pi_s\mathcal{A}(\Pi_0\sharp\Pi_0).$$

The singular terms in the right-hand sides are

$$\begin{aligned}\mathrm{op}_\varepsilon^\psi(\Pi_0)\mathrm{op}_\varepsilon(\mathcal{B})\dot{u} &= \mathrm{op}_\varepsilon^\psi(\Pi_0\mathcal{B}\Pi_0)v_0 + \mathrm{op}_\varepsilon^\psi(\varepsilon\chi_\varepsilon^{-1}\Pi_0\mathcal{B}\Pi_s)v_s\\ &\quad +\varepsilon R_{(0)}v;\\ \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon)\mathrm{op}_\varepsilon^\psi(\Pi_s)\mathrm{op}_\varepsilon^\psi(\mathcal{B})\dot{u} &= \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon\Pi_s\mathcal{B}\Pi_0)v_0\\ &\quad +\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon(\Pi_s\sharp\mathcal{B})\Pi_0 + \chi_\varepsilon\sharp(\Pi_s\mathcal{B})\Pi_0)v_0\\ &\quad +\mathrm{op}_\varepsilon^\psi(\Pi_s\mathcal{B}\Pi_s)v_s + \mathrm{op}_\varepsilon^\psi(\varepsilon\chi_\varepsilon^{-1}(\chi_\varepsilon\sharp(\Pi_s\mathcal{B}))\Pi_s)v_s\\ &\quad +\varepsilon R_{(0)}v.\end{aligned}$$

The other singular source term is

$$\frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon)\mathrm{op}_\varepsilon^\psi(\Pi_s)\mathrm{op}_\varepsilon^\psi(\mathcal{D})\dot{u} = \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\chi_\varepsilon\Pi_s\mathcal{D}\Pi_0)v_0 + R_{(0)}v.$$

The equation in v thus takes the form

$$\partial_t v + \frac{1}{\varepsilon^2}\mathrm{op}_\varepsilon^\psi(iA)v = \frac{1}{\varepsilon^2}\mathrm{op}_\varepsilon^\psi(\underline{B})v + \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\underline{D})v + R_{(0)}v + r_a^\varepsilon,$$

where

(i) $A \in C^\infty\mathcal{M}^1$ is evaluated at εu, and is defined as

$$iA := \begin{pmatrix}\mathcal{A}\Pi_0 & 0\\ 0 & \mathcal{A}\Pi_s\end{pmatrix};$$

(ii) $\underline{B} = \underline{B}(\varepsilon, u_a) \in C^\infty\mathcal{M}^0$ is defined as

$$\underline{B} := \begin{pmatrix}0 & 0\\ \underline{B}_{s0} & 0\end{pmatrix},$$

with the notation

$$\begin{aligned}\underline{B}_{s0} :=&\chi_\varepsilon(\Pi_s(\mathcal{B}+\varepsilon\mathcal{D})\Pi_0 + \varepsilon(\partial_t\Pi_s)\Pi_0 + \varepsilon(\rho_{s0} - (\Pi_s\sharp\mathcal{B})\Pi_0)\\ &- \varepsilon\chi_\varepsilon\sharp(\varepsilon\partial_t\Pi_s - \Pi_s\mathcal{B})\Pi_0,\end{aligned}$$

where $\Pi_0, \Pi_s, \mathcal{A}$ are evaluated at $(\varepsilon, \varepsilon u_a)$; $\mathcal{B}$ is evaluated at u_a, and $\mathcal{D}$ is evaluated at $(0, u_a)$;

(iii) $\underline{D} = \underline{D}(\varepsilon, u_a) = \underline{B}' + \underline{B}'' + E + \underline{F} \in C^\infty\mathcal{M}^0$ is defined as

$$\underline{B}' := \begin{pmatrix} 0 & 0 \\ 0 & \underline{B}_s \end{pmatrix}, \quad \underline{B}'' := \begin{pmatrix} 0 & \underline{B}_{0s} \\ 0 & 0 \end{pmatrix},$$

with the notation,

$$\begin{aligned} \underline{B}_s &:= -\chi_\varepsilon(\varepsilon\chi_\varepsilon^{-1}\sharp(\Pi_s\mathcal{A})) + \varepsilon\chi_\varepsilon^{-1}(\chi_\varepsilon\sharp(\Pi_s\mathcal{B}))\Pi_s, \\ \underline{B}_{0s} &:= \varepsilon\chi_\varepsilon^{-1}(\Pi_0\mathcal{B}\Pi_s + \varepsilon\partial_t\Pi_0 - (\varepsilon\chi_\varepsilon^{-1})\Pi_0\mathcal{A}(\chi_\varepsilon\sharp\Pi_s))\Pi_s, \end{aligned}$$

$$E := \begin{pmatrix} \Pi_0\mathcal{B}\Pi_0 & 0 \\ 0 & \Pi_s\mathcal{B}\Pi_s \end{pmatrix},$$

where $\Pi_0, \Pi_s, \mathcal{A}$ are evaluated at $(0,0)$, $\mathcal{B}$ is evaluated at u_a, $\mathcal{D}$ is evaluated at $(0, u_a)$, and

$$\underline{F} := \begin{pmatrix} (\varepsilon\partial_t\Pi_0)\Pi_0 & 0 \\ 0 & (\varepsilon\partial_t\Pi_s)\Pi_s \end{pmatrix},$$

where $\varepsilon\partial_t\Pi_j$ is short for $\partial_v\Pi_j(0,0)\cdot(\varepsilon^2\partial_t u_a)$, $j = 0, s$, and where Π_0, Π_s are evaluated at $(0,0)$;

(iv) $r_a^\varepsilon := \varepsilon^{l_0 - k_0 - 2}(\mathrm{op}_\varepsilon^\psi(\Pi_0)R_a^\varepsilon, \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(\Pi_s)R_a^\varepsilon)$.

Next we polarize the source terms, by letting

$$B_{s0} := \Pi_s\underline{B}_{s0}\Pi_0, \quad B_s := \Pi_s\underline{B}_s\Pi_s, \quad B_{0s} := \Pi_0\underline{B}_{0s}\Pi_s,$$

and

$$B := \begin{pmatrix} 0 & 0 \\ B_{s0} & 0 \end{pmatrix}, \quad B' := \begin{pmatrix} 0 & 0 \\ 0 & B_s \end{pmatrix}, \quad B'' := \begin{pmatrix} 0 & B_{0s} \\ 0 & 0 \end{pmatrix}.$$

We let also

$$F := \underline{F} + \begin{pmatrix} 0 & \underline{B}_{0s} - B_{0s} \\ \underline{B}_{s0} - B_{s0} & \underline{B}_s - B_s \end{pmatrix},$$

and

$$D := B' + B'' + E + F.$$

The equation is now

$$\partial_t v + \frac{1}{\varepsilon^2}\mathrm{op}_\varepsilon^\psi(iA)v = \frac{1}{\varepsilon^2}\mathrm{op}_\varepsilon^\psi(B)v + \frac{1}{\varepsilon}\mathrm{op}_\varepsilon^\psi(D)v + R_{(0)}v + r_a^\varepsilon. \tag{57}$$

In (57), the variables v_0 and v_s are coupled only by zero-order terms, the leading singular term is polarized and has a nilpotent structure. The system is prepared. All the symbols in (57) are smooth.

In the next sections, the terms B'', B' and B will be eliminated by normal form reductions. In the subsequent H_ε^s energy estimate, the term E/ε will be symmetrized, while the nonpolarized term F/ε will be seen to contribute to $O(1)$.

2.5.3. First reduction. In this section, the nonresonant term B'' is eliminated from (57).

Proposition 8. *Under Assumptions* 5 *and* 6*, there exists a smooth symbol* $L \in C^\infty\mathcal{M}^{-1}$ *such that,*

$$[\varepsilon^2\partial_t + \mathrm{op}_\varepsilon^\psi(iA), \mathrm{op}_\varepsilon^\psi(L(u_a)] = \mathrm{op}_\varepsilon^\psi(B'') + \varepsilon R_{(0)}. \tag{58}$$

Proof. The leading term in the symbol of the source term in (58) is linear in u_a:

$$B_{0s} = \partial_u B_{0s}(0,0)\cdot u_a + O(\varepsilon),$$

and satisfies $\Pi_0 B_{0s}\Pi_s = B_{0s}$. We look for a solution $L \in C^\infty\mathcal{M}^{-1}$ to (58) in the same form: $L = \begin{pmatrix} 0 & L_{0s} \\ 0 & 0 \end{pmatrix}$, where $\Pi_0 L_{0s}\Pi_s = L_{0s}$, and

$$L_{0s} = \sum_{p\in\mathcal{R}^*} e^{ip\omega t/\varepsilon^2} L_{0s,p}(u_{a,p}),$$

where $L_{0s,p}$ is linear in $u_{a,p}$. Then

$$[\mathrm{op}_\varepsilon^\psi(iA), \mathrm{op}_\varepsilon^\psi(L)] = [\mathrm{op}_\varepsilon^\psi(iA(0,0), \mathrm{op}_\varepsilon^\psi(L)] + \Gamma,$$

where Γ is the commutator,

$$[\mathrm{op}_\varepsilon^\psi(A - A(0,0)), \mathrm{op}_\varepsilon^\psi(L)],$$

which, because A depends on u through εu, and because L is assumed to belong to $C^\infty\mathcal{M}^{-1}$, has the form $\varepsilon R_{(0)}$. If we suppose in addition that L is smooth at the origin, we can use Proposition 4 to obtain

$$[\mathrm{op}_\varepsilon^\psi(iA(0,0), \mathrm{op}_\varepsilon^\psi(L)] = \mathrm{op}_\varepsilon^\psi[iA(0,0), L] + \varepsilon R_{(0)}.$$

Thus to solve (58), it suffices to solve the equation

$$\varepsilon^2\partial_t L_{0s} + [iA(0,0), L]_{0s} = \partial_u B_{0s}(0,0)\cdot u_a, \tag{59}$$

up to $O(\varepsilon)$. We compute

$$\varepsilon^2\partial_t L_{0s} + [iA(0,0), L]_{0s} = \sum_{j\leqq n_0<k}\sum_{p\in\mathcal{R}^*} e^{ip\omega t/\varepsilon^2}\Phi_{j,k,p}(0)\Pi_j L_{0s,p}\Pi_k,$$

where the projectors Π_j, Π_k are evaluated at $(0,0)$, and where $\Phi_{j,k,p}(0)$ is the phase defined in (40), evaluated at $\varepsilon = 0$. Let χ_L be a smooth function on $\mathbb{R}^d$, identically equal to 0 for $|\xi| \leqq c_l$, and identically equal to 1 for $|\xi| \geqq c_0$. Let then

$$L_{0s,p} := \chi_L \sum_{j\leqq n_0<k} \Phi_{j,k,p}^{-1}(0)\Pi_j(\partial_u B_{0s}(0,0)\cdot u_{a,p})\Pi_k.$$

As $\Phi_{j,k,p}^{-1}$, for $j \leqq n_0 < k$, is uniformly bounded for $|\xi| \geqq c_l$, the above defines a symbol $L \in C^\infty\mathcal{M}^{-1}$. Besides, this symbol is smooth, and solves (59) up to the error term $(1-\chi_L)B_{0s}$. As $|\varepsilon\chi_\varepsilon^{-1}\chi_L| \leqq \varepsilon$, this error is $O(\varepsilon)$. □

Proposition 3 implies that the norm of $\mathrm{op}_\varepsilon^\psi(L)$, as an operator from H_ε^s to H_ε^{s+1}, is bounded by $C(|u_a|_{0,\infty})$.

Consider now the change of variables

$$\tilde{v} := (\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon^\psi(L))^{-1} v, \tag{60}$$

Then,

$$\partial_t v = (\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon^\psi(L)) \partial_t \tilde{v} + \mathrm{op}_\varepsilon^\psi(\varepsilon \partial_t L)\tilde{v}.$$

As L is order -1 and smooth at the origin

$$\begin{aligned}(\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon^\psi(L))^{-1} \mathrm{op}_\varepsilon^\psi(A)(\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon^\psi(L)) &= \mathrm{op}_\varepsilon^\psi(A) + \varepsilon[\mathrm{op}_\varepsilon^\psi(A), \mathrm{op}_\varepsilon^\psi(L)] \\ &= \mathrm{op}_\varepsilon^\psi(A) + \varepsilon \mathrm{op}_\varepsilon^\psi[A, L],\end{aligned}$$

up to error terms of the form $\varepsilon^2 R_{(0)}$. Similarly,

$$(\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon^\psi(L))^{-1} \mathrm{op}_\varepsilon^\psi(B)(\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon^\psi(L)) = \mathrm{op}_\varepsilon^\psi(B) + \varepsilon \mathrm{op}_\varepsilon^\psi[B, L] + \varepsilon^2 R_{(0)}$$

and

$$(\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon^\psi(L))^{-1} \mathrm{op}_\varepsilon^\psi(D)(\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon^\psi(L)) = \mathrm{op}_\varepsilon^\psi(D) + \varepsilon R_{(0)}.$$

The leading term in ε in the commutator

$$[B, L] = \begin{pmatrix} -L B_{s0} & 0 \\ 0 & B_{s0} L \end{pmatrix},$$

is $O(\chi_\varepsilon) O(\varepsilon \chi_\varepsilon^{-1}) = O(\varepsilon)$. Thus, choosing L as in Proposition 8, we find that the equation satisfied by $\tilde{v}$ is

$$\partial_t \tilde{v} + \frac{1}{\varepsilon^2} \mathrm{op}_\varepsilon^\psi(iA)\tilde{v} = \frac{1}{\varepsilon^2} \mathrm{op}_\varepsilon^\psi(B)\tilde{v} + \frac{1}{\varepsilon} \mathrm{op}_\varepsilon^\psi(\tilde{D})\tilde{v} + R_{(0)}\tilde{v} + \tilde{r}_a^\varepsilon, \tag{61}$$

where $\tilde{r}_a^\varepsilon := (\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon^\psi(L))^{-1} r_a^\varepsilon$, and

$$\tilde{D} = B' + E + F.$$

2.5.4. Second reduction. In this section, the nonresonant term B' is eliminated from (61).

Proposition 9. *Under Assumptions* 5 *and* 6*, there exists a smooth symbol* $M \in C^\infty \mathcal{M}^{-1}$ *such that*

$$[\varepsilon^2 \partial_t + \mathrm{op}_\varepsilon^\psi(iA), \mathrm{op}_\varepsilon^\psi(M(u_a))] = \mathrm{op}_\varepsilon^\psi(B') + \varepsilon R_{(0)}. \tag{62}$$

Proof. The leading term in the symbol of the source term is linear in u_a:

$$B_s = \partial_u B_s(0,0) \cdot u_a + O(\varepsilon),$$

and satisfies $\Pi_s B_s \Pi_s = B_s$. We look for a solution of (62) in the form of a smooth symbol $M \in C^\infty \mathcal{M}^{-1}$, such that $M = \begin{pmatrix} 0 & 0 \\ 0 & M_s \end{pmatrix}$, where $\Pi_s M_s \Pi_s = M_s$, and

$$M_s = \sum_{p \in \mathcal{R}^*} e^{(ip\omega t)/\varepsilon^2} M_{s,p}(u_{a,p}),$$

where $M_{s,p}$ is linear in $u_{a,p}$. We check as in the proof of Proposition 8 that in order to solve (62), it suffices to solve up to $O(\varepsilon)$ the equation

$$\varepsilon^2 \partial_t M_s + [iA(0,0), M]_s = \partial_u B_s(0,0) \cdot u_a.$$

Let

$$M_{s,p} := \sum_{n_0 < j,k} \Phi^{-1}_{j,k,p}(0) \Pi_j (\partial_u B_s(0,0) \cdot u_{a,p}) \Pi_k,$$

where the projectors are evaluated at $(0,0)$. Condition (iii) in Assumption 3 implies that $\Phi^{-1}_{j,k,p}(0) = (ip\omega)^{-1}$, for $j,k > n_0$. As B_s depends only on symbols depending on derivatives of χ_ε and $\varepsilon \chi_\varepsilon^{-1}$, the support of B_s is included $[c_0, c_1]$. Thus $M_{s,p}$ is a smooth symbol with compact support, and M solves (62). □

Proposition 3 implies that the norm of $\mathrm{op}^\psi_\varepsilon(M)$, as an operator from H^s_ε to H^{s+1}_ε, is bounded by $C(|u_a|_{0,\infty})$.

Consider now the change of variables

$$\check{v} := (\mathrm{Id} + \varepsilon \mathrm{op}^\psi_\varepsilon(M))^{-1} \tilde{v}, \tag{63}$$

With the above proposition, the equation satisfied by $\check{v}$ is

$$\partial_t \check{v} + \frac{1}{\varepsilon^2} \mathrm{op}^\psi_\varepsilon(iA)\check{v} = \frac{1}{\varepsilon^2} \mathrm{op}^\psi_\varepsilon(\check{B})\check{v} + \frac{1}{\varepsilon} \mathrm{op}^\psi_\varepsilon(E+F)\check{v} + R_{(0)}\check{v} + \check{r}^\varepsilon_a, \tag{64}$$

where $\check{r}^\varepsilon_a := (\mathrm{Id} + \varepsilon \mathrm{op}^\psi_\varepsilon(M))^{-1} \tilde{r}^\varepsilon_a$, and

$$\check{B} := B + \varepsilon [B, M].$$

2.5.5. Third reduction. In this section, the resonant term $\check{B}$ is eliminated from (64), under the transparency condition (48).

Proposition 10. *Under Assumptions* 5 *and* 6*, there exists* $N \in C^\infty \mathcal{M}^{-1}$*, such that*

$$[\varepsilon^2 \partial_t + \mathrm{op}^\psi_\varepsilon(iA) - \varepsilon \mathrm{op}^\psi_\varepsilon(E), \mathrm{op}^\psi_\varepsilon(N(u_a))] = \mathrm{op}^\psi_\varepsilon(\check{B}) + \varepsilon^2 R_{(0)}, \tag{65}$$

Proof. The source term is $\check{B} = \begin{pmatrix} 0 & 0 \\ \check{B}_{s0} & 0 \end{pmatrix}$, where

$$\check{B}_{s0} = \chi_\varepsilon (B^{\mathtt{r}} + \varepsilon \Pi_s \mathcal{D} \Pi_0) + \varepsilon (B^{\mathtt{nr}} - \chi_\varepsilon M_s B^{\mathtt{r}}) + O(\varepsilon^2).$$

The notation $B^{\mathtt{r}}$ is introduced in (46), and

$$B^{\mathtt{nr}}(\varepsilon, u_a) := -\Pi_s (\chi_\varepsilon \sharp (\varepsilon \partial_t \Pi_s - \Pi_s \mathcal{B})) \Pi_0.$$

The transparency assumption (47) implies that the term $\varepsilon \chi_\varepsilon \Pi_s \mathcal{D} \Pi_0$ is $O(\varepsilon^2)$. The Taylor expansions of $B^{\mathtt{r}}$ and $B^{\mathtt{nr}}$ in their second variables are

$$B^{\mathtt{r}}(\varepsilon, u_a) = \partial_u B^{\mathtt{r}}(\varepsilon, 0) \cdot u_a + \varepsilon \partial_u^2 B^{\mathtt{r}}(\varepsilon, 0) \cdot (u_a, u_a) + O(\varepsilon^2),$$

and

$$B^{\mathtt{nr}}(\varepsilon, u_a) = \partial_u B^{\mathtt{nr}}(\varepsilon, 0) \cdot u_a + O(\varepsilon).$$

That is, up to $O(\varepsilon^2)$, the source $\check{B}_{s0}$ is the sum of a linear term in u_a,

$$\chi_\varepsilon \partial_u B^{\mathtt{r}}(\varepsilon, 0) \cdot u_a + \varepsilon \partial_u B^{\mathtt{nr}}(\varepsilon, 0) \cdot u_a, \tag{66}$$

and of a bilinear term in u_a,

$$\varepsilon \chi_\varepsilon (\partial_u^2 B^{\mathtt{r}}(0,0) \cdot (u_a, u_a) - M_s(u_a) \partial_u B^{\mathtt{r}}(0,0) \cdot u_a). \tag{67}$$

All the terms in (66)–(67) have the product structure (27). Accordingly, we look for N in the form

$$N = N^{(0)} + \varepsilon (N^{(1)} + N^{(2)} + N^{(3)}), \tag{68}$$

with

$$N^{(j)} = \begin{pmatrix} 0 & 0 \\ N_{s0}^{(j)} & 0 \end{pmatrix} \in C^\infty \mathcal{M}^{-1}, \quad \text{for all } j,$$

and where

(a) $N^{(0)}$, $N^{(1)}$ and $N^{(2)}$ have the structure (27);
(b) all the entries of $N^{(3)}$ have the form $a(\xi) \mathtt{r}(p, q) b(\xi)$, for some Fourier multiplier a and b, and some symbols p, q in the form (27) (the notation $\mathtt{r}$ is introduced in (33));
(c) $N^{(0)}$ and $N^{(1)}$ are linear in u_a, and $N^{(2)}$ and $N^{(3)}$ are bilinear in u_a:

$$N^{(j)} = \sum_{p \in \mathcal{R}^*} e^{ip\omega t/\varepsilon^2} N_p^{(j)}(u_{a,p}), \quad j = 0, 1,$$
$$N^{(j)} = \sum_{p, p' \in \mathcal{R}^*} e^{i(p+p')\omega t/\varepsilon^2} N_{p,p'}^{(2)}(u_{a,p}, u_{a,p'}), \quad j = 2, 3.$$

We now describe the symbols of the commutators

$$[\mathrm{op}_\varepsilon^\psi(iA - E), \mathrm{op}_\varepsilon(N^{(j)})], \quad j = 0, 1, 2, 3,$$

using the results of Section 2.3.

As the symbol $N^{(0)}$ is assumed to belong to $C^\infty\mathcal{M}^{-1}$ and to depend on x only through u_a,

$$[\mathrm{op}_\varepsilon^\psi(iA), \mathrm{op}_\varepsilon(N^{(0)})] = [\mathrm{op}_\varepsilon(iA(\varepsilon, \varepsilon u_a)), \mathrm{op}_\varepsilon(N^{(0)})] + \varepsilon^2 R_{(0)},$$

and the commutator in the right-hand side of the above equation is

$$[\mathrm{op}_\varepsilon(iA(\varepsilon, 0)), \mathrm{op}_\varepsilon(N^{(0)})] + \varepsilon[\mathrm{op}_\varepsilon(i\partial_v A(0,0)\cdot u_a), \mathrm{op}_\varepsilon(N^{(0)})],$$

up to $\varepsilon^2 R_{(0)}$. As $N^{(0)}$ is assumed to have the form (27) and because $A(\varepsilon, 0)$ is smooth, Proposition 7(ii) implies that

$$[\mathrm{op}_\varepsilon(iA(\varepsilon, 0), \mathrm{op}_\varepsilon(N^{(0)})] = \mathrm{op}_\varepsilon[iA(\varepsilon, 0), N^{(0)}] + \varepsilon \mathrm{op}_\varepsilon(iA(0,0)\sharp N^{(0)}),$$

up to a remainder in ε^2. As $N^{(0)}$ depends only on u_a, Proposition 7(ii) implies that this remainder has the form $\varepsilon^2 R_{(0)}$. Remark that

$$E(\varepsilon, u_a) = \partial_u E(0,0)\cdot u_a + O(\varepsilon).$$

The symbol $i\partial_u A - E$ is smooth, and thus Proposition 7(i) implies that the commutator

$$[\mathrm{op}_\varepsilon(\partial_u iA(0,0)\cdot u_a - E), \mathrm{op}_\varepsilon(N^{(0)})],$$

is equal to

$$\mathrm{op}_\varepsilon[\partial_v(iA - E)(0,0)\cdot u_a, N^{(0)}] + \mathrm{op}_\varepsilon \mathtt{r}(i\partial_v(A - E)(0,0)\cdot u_a, N^{(0)}),$$

up to a remainder in $O(\varepsilon)$. With Proposition 7 (i), and because $N^{(0)}$ depends only on u_a, this remainder has the form $\varepsilon^2 R_{(0)}$.

Similarly, for $j = 1, 2, 3$,

$$[\mathrm{op}_\varepsilon(iA), \mathrm{op}_\varepsilon(N^{(j)})] = [\mathrm{op}_\varepsilon(iA(0,0)), \mathrm{op}_\varepsilon(N^{(j)})] + \varepsilon R_{(0)}.$$

For $j = 1, 2$, Proposition 7(ii) implies that

$$[\mathrm{op}_\varepsilon(iA(0,0)), \mathrm{op}_\varepsilon(N^{(j)})] = \mathrm{op}_\varepsilon[iA(0,0), N^{(j)}] + \varepsilon R_{(0)}.$$

Lemma 4 implies that

$$[\mathrm{op}_\varepsilon(iA(0,0)), \mathrm{op}_\varepsilon(N^{(3)})] = \mathrm{op}_\varepsilon[iA(0,0), N^{(3)}] + \varepsilon R_{(0)}.$$

These symbolic computations show that in order to solve (65), it is sufficient to solve the system

$$\varepsilon^2\partial_t N_{s0}^{(0)} + [iA(\varepsilon, 0), N^{(0)}]_{s0} = \chi_\varepsilon \partial_u B^{\mathrm{r}}(\varepsilon, 0)\cdot u_a, \tag{69}$$

$$\varepsilon^2\partial_t N_{s0}^{(1)} + [iA(0,0), N^{(1)}]_{s0} = \partial_u B^{\mathrm{nr}}(0,0)\cdot u_a - iA_s(0,0)\sharp N_{s0}^{(0)}, \tag{70}$$

$$\varepsilon^2\partial_t N_{s0}^{(2)} + [iA(0,0), N^{(2)}]_{s0} = B^{\mathrm{bl}}(u_a, u_a), \tag{71}$$

$$\varepsilon^2\partial_t N_{s0}^{(3)} + [iA(0,0), N^{(3)}]_{s0} = \mathtt{r}(\partial_v(iA - E)(0,0)\cdot u_a, N^{(0)})_{s0}, \tag{72}$$

with the notation

$$B^{\mathrm{bl}}(u_a, u_a) := \chi_\varepsilon(\partial_u^2 B^{\mathrm{r}}(0,0)\cdot(u_a,u_a) - M_s(u_a)\partial_u B^{\mathrm{r}}(0,0)\cdot u_a) \\ + [\partial_v(iA - E)(0,0)\cdot u_a, N^{(0)}]_{s0}.$$

We now solve (69). For $j \leqq n_0 < k$, for $0 < \varepsilon < \varepsilon_0$, let

$$\overline{\Phi}_{j,k,p}(\varepsilon) := \begin{cases} \Phi_{j,k,p}(\varepsilon), & \text{if } |\Phi_{j,k,p}(\varepsilon)| \geqq \varepsilon^2/2, \\ \varepsilon^2/2, & \text{otherwise.} \end{cases}$$

These new phases are not continuous, but they are bounded in ξ, uniformly in ε, for $|\xi| \leqq c_l$. Let then

$$N^{(0)}_{s0,p} := \chi_\varepsilon \sum_{j \leqq n_0 < k} \overline{\Phi}_{j,k,p}(\varepsilon)^{-1}\Pi_k(\varepsilon,0)(\partial_u B_0^{\mathrm{r}}(\varepsilon,0)\cdot u_{a,p})\Pi_j(\varepsilon,0). \quad (73)$$

Then every entry of $N^{(0)}_{s0}$ is a product $p(u_a)q(\varepsilon,\xi)$, where p is smooth, and the transparency condition (48) in Assumption 6 ensures that $q(\varepsilon,\xi)$ is bounded in ξ, uniformly in ε. Besides, by definition, q is compactly supported. Thus $N^{(0)} \in C^\infty\mathcal{M}^{-1}$, and it solves (69).

In (70) the source term $A_s(0,0)\sharp N^{(0)}_{s0}$ vanishes identically (a consequence of Assumption 3 (iii)). Let

$$N^{(1)}_{s0,p} := \sum_{j \leqq n_0 < k} \Phi_{j,k,p}(0)^{-1}\Pi_k(0,0)(\partial_u B^{\mathrm{nr}}(0,0)\cdot u_{a,p})\Pi_j(0,0).$$

As B^{nr} is supported in $|\xi| \in [c_0, c_1]$, far from the resonances between Klein–Gordon and acoustic modes, the above defines a symbol in $C^\infty\mathcal{M}^{-1}$. Then $N^{(1)}_{s0}$ has the structure (27), is linear in u_a, and solves (70).

Let χ_N be a smooth truncation function on $\mathbb{R}^d$, identically equal to 1 for $|\xi| \leqq c_1$, and identically equal to 0 for $|\xi| \geqq c_m$. Let then

$$N^{(2)}_{s0,p,p'} := \chi_N \sum_{j \leqq n_0} \Psi^{-1}_{k,p,p'}\Pi_s(0,0)B^{\mathrm{bl}}(u_{a,p}, u_{a,p'})\Pi_j(0,0).$$

The phases $\Psi_{k,p,p'}$ are uniformly bounded away from 0 for $|\xi| \leqq c_m$ (Assumption 5 (0-0-s)). Thus the above defines a symbol in $C^\infty\mathcal{M}^{-1}$. Then $N^{(2)}$ has the structure (27), just like the source term B^{bl}, is bilinear in u_a, and solves (71), up to the error $(1-\chi_N)B^{\mathrm{bl}}$. As $|(1-\chi_N)\chi_\varepsilon| \leqq \varepsilon$, the error is $O(\varepsilon)$.

Finally, let

$$N^{(3)}_{s0,p,p'} := \chi_N \sum_{j \leqq n_0} \Psi^{-1}_{k,p,p'}\Pi_s(0,0)\mathtt{r}(u_{a,p}, u_{a,p'})\Pi_j(0,0),$$

with the notation

$$\mathtt{r}(u_{a,p}, u_{a,p'}) := \mathtt{r}(\partial_v(iA-E)(0,0)\cdot u_{a,p}, N^{(0)}_p(u_{a,p'}))_{s0}.$$

All the entries of $N^{(3)}$ have the form $a(\xi)\mathtt{r}(p,q)b(\xi)$, where a is smooth and p, q have the form (27). The symbol $N^{(3)}$ solves (72), up to the error $(1-\chi_N)\mathtt{r}$, which, because $N^{(0)}$ is $O(\varepsilon)$ for $|\xi| \geqq c_1$, is $O(\varepsilon)$ as well.

Finally N defined by (68) satisfies the assumptions (a), (b), (c), on which the symbolic computations were based, and solves (65). □

Proposition 6 and Lemma 4 imply that the norm of $\mathrm{op}_\varepsilon(N)$, as an operator from H^s_ε to H^{s+1}_ε, is bounded by $C(\|u_a\|_{1,s+d_0+2})$.

Consider now the change of variables

$$w := (\mathrm{Id} + \mathrm{op}_\varepsilon(N))^{-1}\check{v}. \tag{74}$$

Then,

$$\partial_t \check{v} = (\mathrm{Id} + \mathrm{op}_\varepsilon(N))\partial_t w + \mathrm{op}_\varepsilon(\partial_t N)w.$$

As N is block triangular

$$(\mathrm{Id} + \mathrm{op}_\varepsilon(N))^{-1}\mathrm{op}^\psi_\varepsilon(A)(\mathrm{Id} + \mathrm{op}_\varepsilon(N)) = \mathrm{op}^\psi_\varepsilon(A) + [\mathrm{op}^\psi_\varepsilon(A), \mathrm{op}_\varepsilon(N)];$$

$$(\mathrm{Id} + \mathrm{op}_\varepsilon(N))^{-1}\mathrm{op}^\psi_\varepsilon(B)(\mathrm{Id} + \mathrm{op}_\varepsilon(N)) = \mathrm{op}^\psi_\varepsilon(B),$$

and

$$\mathrm{op}_\varepsilon(N)\mathrm{op}_\varepsilon(\partial_t N) = 0, \quad \mathrm{op}_\varepsilon(N)\mathrm{op}^\psi_\varepsilon(E)\mathrm{op}_\varepsilon(N) = 0.$$

Thus, with the above proposition, the equation satisfied by w is

$$\partial_t w + \frac{1}{\varepsilon^2}\mathrm{op}^\psi_\varepsilon(iA)w = \frac{1}{\varepsilon}\mathrm{op}^\psi_\varepsilon(E + \check{F})w + R_{(0)}w + \bar{r}^\varepsilon_a, \tag{75}$$

where $\bar{r}^\varepsilon_a := (\mathrm{Id} + \mathrm{op}_\varepsilon(N))^{-1}\tilde{r}^\varepsilon_a$, and

$$\mathrm{op}^\psi_\varepsilon(\check{F}) := (\mathrm{Id} + \mathrm{op}_\varepsilon(N))^{-1}\mathrm{op}^\psi_\varepsilon(F)(\mathrm{Id} + \mathrm{op}_\varepsilon(N)).$$

2.5.6. Uniform Sobolev estimates. We perform energy estimates on (75), using the symmetrizer S whose existence is granted by Assumption 7.

We evaluate

$$\begin{aligned}\partial_t(\mathrm{op}_\varepsilon(S)w, w)_{\varepsilon,s} &= (\mathrm{op}_\varepsilon(S)\partial_t w, w)_{\varepsilon,s} + (\mathrm{op}_\varepsilon(S)w, \partial_t w)_{\varepsilon,s}\\ &= 2\Re(\mathrm{op}_\varepsilon(S)\partial_t w, w)_{\varepsilon,s}.\end{aligned}$$

We apply Proposition 5 to find

$$\mathrm{op}^\psi_\varepsilon(iSA)^* = \mathrm{op}^\psi_\varepsilon((iSA)^*) + \varepsilon^2 R_{(0)},$$

because A depends on u through εu. Similarly,

$$\mathrm{op}^\psi_\varepsilon(iSE)^* = \mathrm{op}^\psi_\varepsilon((iSE)^*) + \varepsilon R_{(0)},$$

and this implies

$$\frac{1}{\varepsilon^2}\Re(\mathrm{op}_\varepsilon(S)\mathrm{op}_\varepsilon^\psi(iA)w, w)_{\varepsilon,s} = \left(\mathrm{op}_\varepsilon^\psi\left(\frac{1}{\varepsilon^2}(iSA + (iSA)^*\right)w, w\right)_{\varepsilon,s} + \varepsilon^2(R_{(0)}w, w)_{\varepsilon,s};$$

$$\frac{1}{\varepsilon}\Re\left(\mathrm{op}_\varepsilon(S)\mathrm{op}_\varepsilon^\psi(E)w, w\right)_{\varepsilon,s} = \left(\mathrm{op}_\varepsilon\left(\frac{1}{\varepsilon}(iSE + (iSE)^*\right)w, w\right)_{\varepsilon,s} + \varepsilon^2(R_{(0)}w, w)_{\varepsilon,s}.$$

Now Assumption 7 implies that the symbols in the right-hand sides of the above equations all belong to $C^\infty\mathcal{M}^0$, and Proposition 3 implies that these symbols have the form $R_{(0)}$. The other source term contributes to

$$\frac{1}{\varepsilon}\Re(\mathrm{op}_\varepsilon(S)\mathrm{op}_\varepsilon^\psi(\check{F})w, w)_{\varepsilon,s}. \tag{76}$$

Introduce the notation

$$\Pi := \begin{pmatrix} \Pi_0 & 0 \\ 0 & \Pi_s \end{pmatrix}.$$

It follows from the definitions of the above changes of variables that

$$w = \mathrm{op}_\varepsilon^\psi(\Pi)w + \varepsilon R_{(0)}w,$$

and that, up to a term of the form $\varepsilon R_{(0)}$,

$$\mathrm{op}_\varepsilon^\psi(\Pi)(\mathrm{Id} - \mathrm{op}_\varepsilon(N)) = (\mathrm{Id} - \mathrm{op}_\varepsilon(N))\mathrm{op}_\varepsilon^\psi(\Pi).$$

Besides, up to a term of the form $\varepsilon R_{(0)}$,

$$\mathrm{op}_\varepsilon^\psi(\Pi)\mathrm{op}_\varepsilon(S) = \mathrm{op}_\varepsilon(S)\mathrm{op}_\varepsilon^\psi(\Pi).$$

Thus, up to a term of the form $\varepsilon(R_{(0)}w, w)_{\varepsilon,s}$, (76) is equal to

$$\frac{1}{\varepsilon}\Re\big(\mathrm{op}_\varepsilon(S)(\mathrm{Id} - \mathrm{op}_\varepsilon(N))\mathrm{op}_\varepsilon^\psi(\tilde{F})(\mathrm{Id} + \mathrm{op}_\varepsilon(N))w, w\big)_{\varepsilon,s},$$

where $\tilde{F} := \Pi F \Pi$. As Π_0 and Π_s are projectors

$$\Pi_0\partial_t\Pi_0\Pi_0 = 0, \quad \Pi_s\partial_t\Pi_s\Pi_s = 0,$$

and it follows from the definition of F (given in Section 2.5.2) that $\tilde{F} = O(\varepsilon)$. Gathering the above estimates, we find that

$$\partial_t(\mathrm{op}_\varepsilon^\psi(S)w, w)_{\varepsilon,s} = (R_{(0)}w, w)_{\varepsilon,s} + 2\Re(\mathrm{op}_\varepsilon^\psi(S)\bar{r}_a^\varepsilon, w)_{\varepsilon,s}. \tag{77}$$

The changes of variables of Sections 2.5.3 to 2.5.5 define a normal form $\Psi^\varepsilon(u_a)$, such that

$$w = (\mathrm{Id} + \Psi^\varepsilon(u_a))v,$$

and for ε_0 small enough and $0 < \varepsilon < \varepsilon_0$, both $\mathrm{Id}+\Psi^\varepsilon(u_a)$ and $(\mathrm{Id}+\Psi^\varepsilon(u_a))^{-1}$ are uniformly bounded as operators $H^s_\varepsilon \to H^s_\varepsilon$, with norms depending on $\|u_a\|_{1,s+d_0+2}$. From (77), we obtain

$$\|v(t)\|^2_{\varepsilon,s} \leqq C\|\varphi^\varepsilon\|^2_{\varepsilon,s} + C\int_0^t (\|v(t')\|_{\varepsilon,s} + \varepsilon^{l_0-k_0-2}\|R^\varepsilon_a(t')\|_{\varepsilon,s})\|v(t')\|_{\varepsilon,s}dt',$$

where the constant C depends on $\varepsilon_0^{k_0-3-d/2}\delta$, $\|u_a\|_{1,s+d_0+2}$, s, d, and on γ. As (56) implies that $\dot{u} = R_{(0)}v$, Gronwall's lemma finally yields

$$\|\dot{u}(t)\|_{\varepsilon,s} \leqq C_1\|\varphi^\varepsilon\|_{\varepsilon,s}e^{C_1 t}. \tag{78}$$

A classical continuation argument shows that, for ε_0 small enough, the bound (78) is valid over a time interval $[0, t_0]$, independent of ε. Then (50) yields the asymptotic estimate (49).

3. Application to the Euler–Maxwell equations

We show in this section that the Euler–Maxwell equations satisfy the assumptions of Theorem 8. The system we consider is (EM), introduced in Section 1.2, in the specific regime (7)–(8).

For the unknown

$$u^\varepsilon(t,x) = \left(B, E, v_e, n_e, v_i, \frac{n_i}{\alpha}\right), \tag{79}$$

the system takes the form

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{A}(\varepsilon, \varepsilon u^\varepsilon, \varepsilon\partial_x)u^\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(u^\varepsilon, u^\varepsilon) + \mathcal{G}^\varepsilon(u^\varepsilon), \tag{80}$$

where

$$\mathcal{A}(\varepsilon, \varepsilon u, \varepsilon\partial_x) = \mathcal{A}_0(\varepsilon, \varepsilon\partial_x) + \varepsilon\mathcal{A}_1(\varepsilon, u, \varepsilon\partial_x),$$

with the notation

$$\mathcal{A}_0(\varepsilon, \xi) := \begin{pmatrix} 0 & \xi\times & 0 & 0 & 0 & 0 \\ -\xi\times & 0 & i & 0 & -i\frac{\varepsilon}{\theta_e} & 0 \\ 0 & -i & 0 & \theta_e\xi & 0 & 0 \\ 0 & 0 & \theta_e\xi\cdot & 0 & 0 & 0 \\ 0 & i\frac{\varepsilon}{\theta_e} & 0 & 0 & 0 & \varepsilon\alpha\xi \\ 0 & 0 & 0 & 0 & \varepsilon\alpha\xi\cdot & 0 \end{pmatrix},$$

$$\mathcal{A}_1(\varepsilon, u, \xi) := \mathrm{diag}(0_{\mathbb{C}^3}, 0_{\mathbb{C}^3}, \theta_e(v_e\cdot\xi), \theta_e(v_e\cdot\xi), \varepsilon(v_i\cdot\xi), \varepsilon(v_i\cdot\xi)),$$

and

$$\mathcal{B}(u, u') := (0_{\mathbb{C}^3}, n_e v'_e, -\theta_e v'_e \times B, 0_{\mathbb{C}}, 0_{\mathbb{C}^3}, 0_{\mathbb{C}}),$$

$$\mathcal{G}^\varepsilon(u) := \frac{1}{\theta_e}\left(0, f^\varepsilon(n_e)v_e - \frac{1}{\theta_e}(\alpha n_i + \varepsilon f^\varepsilon(\alpha n_i))v_i, 0, 0, \frac{1}{\theta_e}v_i \times B, 0\right),$$

with the notation, $f^\varepsilon(x) := \varepsilon^{-2}(e^{\varepsilon x} - 1 - \varepsilon x)$.

3.1. WKB approximate solution

Let the initial datum

$$a = (0, E^0, v_e^0, 0, 0, 0) \in H^\sigma, \tag{81}$$

for some large index σ, where the electric field satisfies

$$\nabla \cdot E^0 = 0,$$

in accordance with (4). We assume that a is polarized (or well prepared), in the sense that

$$E^0 = \tilde{E} + (\tilde{E})^*, \quad v_e^0 = \frac{i}{\omega}\tilde{E} - \frac{i}{\omega}(\tilde{E})^*,$$

for some fundamental frequency ω, defined in terms of ω_{pe}, and some complex amplitude $\tilde{E}$ (above, $(\tilde{E})^*$ denotes the complex conjugate of $\tilde{E}$).

As the conservative form of the convective terms in the equations of conservations of mass in $(\mathrm{EM})^\sharp$ allows simple formal computations, we carry out the WKB expansion on $(\mathrm{EM})^\sharp$ rather than on (EM).

Consider $(\mathrm{EM})^\sharp$ in the regime (7). We look for an approximate solution u^ε_{app} in the form of a profile

$$u^\varepsilon_{app}(t, x) = \varepsilon[\mathbf{u}^\varepsilon_{app}(\varepsilon t, x, \theta)]_{\theta = \omega t/\varepsilon}, \tag{82}$$

where $\mathbf{u}^\varepsilon_{app}$ has a WKB expansion,

$$u^\varepsilon_{app} = \sum_{m=0}^{M} \varepsilon^m \mathbf{u}_m,$$

such that for all m, $\mathbf{u}_m$ is a trigonometric polynomial in θ,

$$\mathbf{u}_m(t, x, \theta) = \sum_{p \in \mathcal{R}_m} e^{ip\theta} u_{m,p}(t, x).$$

The sets $\mathcal{R}_m \subset \mathbb{Z}$ are finite and the Fourier coefficients $u_{m,p}$ are assumed to satisfy

$$u_{m,p} \in W^{1,\infty}([0, t_0], W^{k(m),\infty}(\mathbb{R}^3)),$$

for some $t_0 > 0$, independent of m, and some large $k(m)$. We plug this ansatz in $(\mathrm{EM})^\sharp$ and find a cascade of WKB equations, which we now describe. We sometimes use the notation $(v)_p$ to denote the p^{th} harmonic in θ of a trigonometric polynomial $\mathbf{v}(t, x, \theta)$.

Equations for the terms in $O(1/\varepsilon^2)$.

$$\omega\partial_\theta \mathbf{E}_0 = \mathbf{v}_{e0},$$
$$\omega\partial_\theta \mathbf{v}_{e0} = -\mathbf{E}_0,$$
$$\omega\partial_\theta(\mathbf{B}_0, \mathbf{n}^\sharp_{e0}, \mathbf{v}_{i0}, \mathbf{n}^\sharp_{i0}) = 0.$$

The dispersion relation is

$$\omega^2 - 1 = 0.$$

We choose $\omega = 1$. With this choice, and because ε was set to be equal to $(\omega_{pe}t_0)^{-1}$, we find that $e^{i\omega t/\varepsilon^2} = e^{i\omega_{pe}T}$, where T represents physical time, $t = \varepsilon t_0 T$, and ω_{pe} the electronic plasma frequency (6). Thus the waves we consider are oscillating at the electronic plasma frequency. These waves are called *plasma waves* in the physical literature. The set of characteristic harmonics is

$$\mathcal{R}^* = \{-1, 1\}.$$

The first term of the expansion satisfies

$$E_{0,0} = 0,\ v_{e0,0} = 0,\ \mathbf{B}_0 = B_{0,0},\ \mathbf{n}^\sharp_{e0} = n^\sharp_{e0,0},\ \mathbf{v}_{i0} = v_{i0,0},\ \mathbf{n}^\sharp_{i0} = n^\sharp_{i0,0},$$

and

$$\mathbf{E}_0 = E_{0,1}e^{i\theta} + E_{0,-1}e^{-i\theta}, \quad \mathbf{v}_{e0} = \frac{i}{\omega}E_{0,1}e^{i\theta} - \frac{i}{\omega}E_{0,-1}e^{-i\theta}. \tag{83}$$

Equations for the terms in $O(1/\varepsilon)$.

$$\omega\partial_\theta \mathbf{B}_1 + \nabla\times\mathbf{E}_0 = 0,$$
$$\omega\partial_\theta \mathbf{E}_1 - \nabla\times\mathbf{B}_0 = \mathbf{v}_{e1} + \mathbf{n}^\sharp_{e0}\mathbf{v}_{e0} - \frac{1}{\theta_e}\mathbf{v}_{i0},$$
$$\omega\partial_\theta \mathbf{v}_{e1} = -\theta_e\nabla\mathbf{n}^\sharp_{e0} - (\mathbf{E}_1 + \theta_e\mathbf{v}_{e0}\times\mathbf{B}_0),$$
$$\omega\partial_\theta \mathbf{n}^\sharp_{e1} + \theta_e\nabla\cdot\mathbf{v}_{e0} = 0,$$
$$\omega\partial_\theta \mathbf{v}_{i1} = \frac{1}{\theta_e}\mathbf{E}_0,$$
$$\omega\partial_\theta \mathbf{n}^\sharp_{i1} = 0.$$

Let

$$L_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^6).$$

The frequency ω was chosen so that det $(ip\omega + L_0) = 0$. In $\mathbb{C}^6$,

$$(ip\omega + L_0)a = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

implies the compatibility condition

$$ip\omega b_1 + b_2 = 0.$$

The oscillating terms satisfy

$$(ip\omega + L_0)\begin{pmatrix}\mathbf{E}_1\\ \mathbf{v}_{e1}\end{pmatrix}_p = \begin{pmatrix}\mathbf{n}^\sharp_{e0}\mathbf{v}_{e0}\\ -\theta_e\mathbf{v}_{e0}\times\mathbf{B}_0\end{pmatrix}_p, \quad p=-1,1,$$

and this implies

$$ip\omega(\mathbf{n}^\sharp_{e0}\mathbf{v}_{e0})_p - \theta_e(\mathbf{v}_{e0}\times\mathbf{B}_0)_p = 0. \tag{84}$$

Besides, we find the polarization conditions,

$$ip\omega B_{1,p} = -\nabla\times E_{0,p},\quad ip\omega v_{i1,p} = \frac{1}{\theta_e}E_{0,p},\quad n^\sharp_{i1,p}=0,\quad n^\sharp_{e1p} = \frac{-\theta_e}{(p\omega)^2}\nabla\cdot E_{0p}.$$

The nonoscillating terms satisfy

$$-\nabla\times B_{0,0} = v_{e1,0} - \frac{1}{\theta_e}v_{i0,0},\quad E_{1,0} = \theta_e\nabla n^\sharp_{e0,0}. \tag{85}$$

Equations for the terms in $O(1)$.

$$\begin{aligned}
\omega\partial_\theta\mathbf{B}_2 + \partial_t\mathbf{B}_0 &+\nabla\times\mathbf{E}_1 = 0,\\
\omega\partial_\theta\mathbf{E}_2 + \partial_t\mathbf{E}_0 &-\nabla\times\mathbf{B}_1 = \mathbf{v}_{e2} + \mathbf{n}^\sharp_{e1}\mathbf{v}_{e0} + \mathbf{n}^\sharp_{e0}\mathbf{v}_{e1} - \frac{1}{\theta_e}(\mathbf{v}_{i1} + \mathbf{n}^\sharp_{i0}\mathbf{v}_{i0}),\\
\omega\partial_\theta\mathbf{v}_{e2} + \partial_t\mathbf{v}_{e0} &+\theta_e(\mathbf{v}_{e0}\cdot\nabla)\mathbf{v}_{e0} = -\theta_e\nabla\mathbf{n}^\sharp_{e1} + \theta_e\mathbf{n}^\sharp_{e0}\nabla\mathbf{n}^\sharp_{e0}\\
&-(\mathbf{E}_2 + \theta_e(\mathbf{v}_{e1}\times\mathbf{B}_0 + \mathbf{v}_{e0}\times\mathbf{B}_1)),\\
\omega\partial_\theta\mathbf{n}^\sharp_{e2} &+\partial_t\mathbf{n}^\sharp_{e0} + \theta_e\nabla\cdot(\mathbf{v}_{e1} + \mathbf{n}^\sharp_{e0}\mathbf{v}_{e0}) = 0,\\
\omega\partial_\theta\mathbf{v}_{i2} &+\partial_t\mathbf{v}_{i0} = -\alpha^2\nabla\mathbf{n}^\sharp_{i0} + \frac{1}{\theta_e}(\mathbf{E}_1 + \mathbf{v}_{i0}\times\mathbf{B}_0),\\
\omega\partial_\theta\mathbf{n}^\sharp_{i2} &+\partial_t\mathbf{n}^\sharp_{i0} + \nabla\cdot\mathbf{v}_{i0} = 0.
\end{aligned}$$

As (85) implies that $E_{1,0}$ is a gradient, the first equation implies that $\partial_t B_{0,0} = 0$. This yields $\mathbf{B}_0 = 0$, and, with (84), we find that $(\mathbf{n}^\sharp_{e0}\mathbf{v}_{e0})_p = 0$. As $\mathbf{v}_{e0,p}$ is assumed to be nonidentically zero (see the form of the initial condition (81)), this implies $n^\sharp_{e0,0} = 0$, and finally $\mathbf{n}^\sharp_{e0} = 0$. That is, the electronic fluctuation of density is $O(\varepsilon)$, in accordance with the rescaling of Section 2.5.2. For $p\in\{-1,1\}$,

$$(ip + L_0)\begin{pmatrix}\mathbf{E}_2\\ \mathbf{v}_{e2}\end{pmatrix}_p = \begin{pmatrix}-\partial_t\mathbf{E}_0 + \nabla\times\mathbf{B}_1 + (\mathbf{n}^\sharp_{e1}\mathbf{v}_{e0} + \mathbf{n}^\sharp_{e0}\mathbf{v}_{e1}) - \frac{1}{\theta_e}\mathbf{v}_{i1}\\ -\partial_t\mathbf{v}_{e0} - \theta_e(\nabla\mathbf{n}^\sharp_{e1} - (\mathbf{v}_{e0}\cdot\nabla)\mathbf{v}_{e0}) - (\mathbf{v}_{e0}\times\mathbf{B}_1))\end{pmatrix}_p.$$

In the above right-hand side, the nonlinear terms are

$$(\mathbf{n}^\sharp_{e1}\mathbf{v}_{e0})_p = n^\sharp_{e1,0}v_{e0,p},\quad (\mathbf{v}_{e0}\times\mathbf{B}_1)_p = v_{e0,p}\times B_{1,0},\quad p=-1,1,$$

and

$$(\mathbf{v}_{e0} \cdot \nabla)\mathbf{v}_{e0})_p = 0, \quad p = -1, 1,$$

a transparency relation for the convective term. The compatibilty relation is the Schrödinger equation for the electric field,

$$-2ip\omega\partial_t E_{0,p} + \Delta_e E_{0,p} - \frac{1}{\theta_e^2} E_{0,p} - n^\sharp_{e1,0} E_{0,p} + \frac{\theta_e}{ip\omega} E_{0p} \times B_{1,0} = 0, \quad (86)$$

where

$$\Delta_e z := \theta_e^2 \nabla(\nabla \cdot z) - \nabla \times (\nabla \times z). \quad (87)$$

The nonoscillating terms satisfy

$$\theta_e\big((\mathbf{v}_{e0} \cdot \nabla)\mathbf{v}_{e0}\big)_0 = -\theta_e \nabla n^\sharp_{e1,0} - (E_{2,0} + \theta_e(\mathbf{v}_{e0} \times \mathbf{B}_1)_0). \quad (88)$$

The above relation is the crucial equation that couples the Schrödinger equation (86) with the evolution equation for $n^\sharp_{e1,0}$, the latter being made explicit below. The equations for the terms in $O(1)$ also contain the relation,

$$\nabla \times B_{1,0} = v_{e2,0} - \frac{1}{\theta_e} v_{i1,0} + (\mathbf{n}^\sharp_{e1}\mathbf{v}_{e0})_0, \quad (89)$$

and a linear wave equation for $v_{i0,0}$ and $n^\sharp_{i1,0}$. As the initial data for $v_{i0,0}$ and $n^\sharp_{i1,0}$ are null, $n^\sharp_{i0,0} = 0$ and $v_{i0,0} = 0$. With (85), and because $B_{0,0} = 0$, this implies $v_{e1,0} = 0$.

Equations for the terms in $O(\varepsilon)$.

$$\begin{aligned}
&\omega\partial_\theta \mathbf{B}_3 + \partial_t \mathbf{B}_1 + \nabla \times \mathbf{E}_2 = 0,\\
&\omega\partial_\theta \mathbf{E}_3 + \partial_t \mathbf{E}_1 - \nabla \times \mathbf{B}_2 = \mathbf{v}_{e3} + \mathbf{n}^\sharp_{e1}\mathbf{v}_{e1} + \mathbf{n}^\sharp_{e0}\mathbf{v}_{e2} + \mathbf{n}^\sharp_{e2}\mathbf{v}_{e0}\\
&\quad -\frac{1}{\theta_e}\left(\mathbf{v}_{i1} + \mathbf{n}^\sharp_{i1}\mathbf{v}_{i0} + \mathbf{n}^\sharp_{i0}\mathbf{v}_{i1}\right),\\
&\omega\partial_\theta \mathbf{v}_{e3} + \partial_t \mathbf{v}_{e1} + \theta_e((\mathbf{v}_{e1} \cdot \nabla)\mathbf{v}_{e0} + (\mathbf{v}_{e0} \cdot \nabla)\mathbf{v}_{e1}) = -\theta_e \nabla \mathbf{n}^\sharp_{e2}\\
&\quad +\theta_e(\mathbf{n}^\sharp_{e1}\nabla\mathbf{n}^\sharp_{e0} + \mathbf{n}^\sharp_{e1}\nabla\mathbf{n}^\sharp_{e1}) - (\mathbf{E}_3 + \theta_e(\mathbf{v}_{e2} \times \mathbf{B}_0 + \mathbf{v}_{e1} \times \mathbf{B}_1 + \mathbf{v}_{e0} \times \mathbf{B}_2)),\\
&\omega\partial_\theta \mathbf{v}_{i3} + \partial_t \mathbf{v}_{i1} + (\mathbf{v}_{i0} \cdot \nabla)\mathbf{v}_{i0} = -\alpha^2 \nabla \mathbf{n}^\sharp_{i1} + \alpha^2 \mathbf{n}^\sharp_{i0}\nabla\mathbf{n}^\sharp_{i0}\\
&\quad +\frac{1}{\theta_e}(\mathbf{E}_2 + \mathbf{v}_{i1} \times \mathbf{B}_0 + \mathbf{v}_{i0} \times \mathbf{B}_1),\\
&\omega\partial_\theta \mathbf{n}^\sharp_{e3} + \partial_t \mathbf{n}^\sharp_{e1} + \theta_e \nabla \cdot (\mathbf{v}_{e2} + \mathbf{n}^\sharp_{e1}\mathbf{v}_{e0} + \mathbf{n}^\sharp_{e0}\mathbf{v}_{e1}) = 0,\\
&\omega\partial_\theta \mathbf{n}^\sharp_{i3} + \partial_t \mathbf{n}^\sharp_{i1} + \nabla \cdot (\mathbf{v}_{i1} + \mathbf{n}^\sharp_{i0}\mathbf{v}_{i0}) = 0.
\end{aligned}$$

In particular,

$$\begin{aligned}
\partial_t v_{i1,0} + \alpha^2 \nabla n^\sharp_{i1,0} &= \frac{1}{\theta_e} E_{2,0},\\
\partial_t n^\sharp_{i1,0} + \nabla \cdot v_{i1,0} &= 0,\\
\partial_t n^\sharp_{e1,0} + \theta_e \nabla \cdot (v_{e2,0} + (\mathbf{n}^\sharp_{e1}\mathbf{v}_{e0})_0) &= 0.
\end{aligned}$$

The last two equations in the above system, together with (89), imply the quasi-neutrality relation

$$n^{\sharp}_{e1,0} = n^{\sharp}_{i1,0}. \tag{90}$$

The first two equations in the above system, together with (88) and (90), give

$$\begin{cases} \partial_t v_{i1,0} + (\alpha^2+1)\nabla n^{\sharp}_{i1,0} = -(\mathbf{v}_{e0}\cdot\nabla)\mathbf{v}_{e0} + \mathbf{v}_{e0}\times\mathbf{B}_1)_0, \\ \qquad \partial_t n^{\sharp}_{i1,0} + \nabla\cdot v_{i1,0} = 0, \end{cases}$$

where the nonlinear term can be computed with the above polarization conditions

$$\big((\mathbf{v}_{e0}\cdot\nabla)\mathbf{v}_{e0} + \mathbf{v}_{e0}\times\mathbf{B}_1\big)_0 = \nabla|E_{0,p}|^2.$$

The equations at order $O(1)$ also yield

$$\partial_t B_{1,0} + \nabla\times E_{2,0} = 0,$$

which, together with (88), implies that $E_{2,0}$ is a gradient. Hence the term $B_{1,0}$ in (86) vanishes.

Finally, $E_{0,p}$ and $n_{i1,0}$ satisfy the vector Zakharov system

$$\begin{cases} -2ip\partial_t E_{0,p} + \Delta_e E_{0,p} - \dfrac{1}{\theta_e^2}E_{0,p} = \quad n^{\sharp}_{i1,0}E_{0,p} \\ \qquad (\partial_t^2 - (\alpha^2+1)\Delta)n^{\sharp}_{i1,0} = \quad -\Delta|E_{0,p}|^2. \end{cases} \tag{91}$$

In (91), $p = 1$ or $p = -1$. The Laplace-type operator Δ_e was introduced in (87). With the initial condition $E_{0,1} = E$, $n_{i1,0} = 0$, and $\partial_t n_{i1,0} = 0$, OZAWA and TSUTSUMI's result [18] guarantees the existence and uniqueness of a solution $(E_{0,1}, n_{i1,0})$ to (91), over a time interval $[0, t^*)$, with the same Sobolev regularity as the initial condition. Note that the crucial coupling term $\Delta|E_{0,p}|^2$ comes from the convective term and from the Lorentz force term. In the above Schrödinger equation, the term $\theta_e^{-2}E_{0,p}$ means a small shift in frequency. This term was not present in the (Z) system given in the introduction; it accounts for the contribution of the ions to the fundamental frequency. Indeed, by letting $\varepsilon = (\omega_{pe}t_0)^{-1}$, we took as a reference the electronic plasma frequency ω_{pe}, which is only an approximation of the plasma frequency

$$\omega_p = \sqrt{4\pi e^2 n_0\left(\frac{1}{m_e} + \frac{1}{m_i}\right)}.$$

As the ratio m_e/m_i is equal to ε^2/θ_e^2 (a consequence of (7)), at first order in ε^2

$$\omega_p = \omega_{pe}\left(1 + \frac{1}{2}\frac{\varepsilon^2}{\theta_e^2}\right),$$

and thus the shift in frequency in (91) means that the electric field actually oscillates at the plasma frequency ω_p.

Higher-order terms. The WKB expansion can be carried out up to any order. For $m \geqq 2$, the terms $E_{m,p}, n_{i,m+1,0}, \ p = -1, 1$, are seen to satisfy a linearized Zakharov system of the form

$$\begin{cases} \left(-2ip\partial_t + \Delta_e - \dfrac{1}{\theta_e^2}\right) E_{m,p} = n^\sharp_{im+1,0} E_{0,p} + n^\sharp_{i1,0} E_{m,p} + r_{m,p}, \\ (\partial_t^2 - (\alpha^2+1)\Delta) n^\sharp_{im+1,0} = -\Delta(E_{m,p}E_{0,-p} + E_{0,p}E_{m,-p}) + r_{m,p}, \end{cases} \tag{92}$$

where $r_{m,p}$ represent the p^{th} harmonics of smooth functions of profiles $\partial_x^k \mathbf{u}_{k'}$, for $k, k' \leqq m - 1$. The system (92) is the linearization of (91) around $(E_{0,p}, n^\sharp_{i1,0})$. The initial data for $E_{m,p}$ and $n^\sharp_{im+1,0}$ are null. Ozawa and Tsutsumi's method for the Zakharov equations [18] allows us to solve the initial value problem for (92). It provides an existence time which *is a priori* smaller than the existence time of the data $r_{m,p}, \ E_{0,p}$ and $n^\sharp_{i1,0}$. The solution has the same Sobolev regularity as the data.

Thus, by induction, we can construct a family of profiles $\mathbf{u}_m$ that determines an approximate solution u^ε_{app}, as follows. If we assume the $\mathbf{u}_k$, for $k \leqq m-1$, to be known, with enough Sobolev regularity, then

(i) $E_{m,p}$ and $n_{im+1,0}$ are defined as the unique solution of $(Z)_m$,
(ii) the terms $u_{m,p}$, for $p \in \mathcal{R}^*$, are deduced from $E_{m,p}$ and $n_{im+1,0}$ by polarization conditions similar to the ones found in the first terms of the expansion;
(iii) the terms $u_{m,p}$, for $p \neq \mathcal{R}^*$, are computed by elliptic inversions; that is, in terms of $(ip\omega + L_0)^{-1} r_{m-1,p}$.

We obtain a profile $\mathbf{u}_m$, whose Sobolev regularity is smaller than the regularity of $\mathbf{u}_{m-1}$ by one, and the induction is complete. The profile

$$\mathbf{u}^\varepsilon_{app} = (\mathbf{B}^\varepsilon, \mathbf{E}^\varepsilon, \mathbf{v}^\varepsilon_e, \mathbf{n}^{\varepsilon,\sharp}_e, \mathbf{v}^\varepsilon_i, \mathbf{n}^{\varepsilon,\sharp}_i)$$

provides, via (82), an approximate solution to $(\text{EM})^\sharp$. Then

$$u^\varepsilon_a = \left[\mathbf{B}^\varepsilon, \mathbf{E}^\varepsilon, \mathbf{v}^\varepsilon_e, \frac{1}{\varepsilon}\log(1+\varepsilon\mathbf{n}^{\varepsilon,\sharp}_e), \mathbf{v}^\varepsilon_i, \frac{1}{\varepsilon}\frac{1}{\alpha}\log(1+\varepsilon\mathbf{n}^{\varepsilon,\sharp}_i)\right]_{\theta=\omega t/\varepsilon^2}$$

is an approximate solution to (80), in the sense of (39). The approximate solution can be made arbitrarily precise, in a Sobolev norm, provided the initial data have enough Sobolev regularity. Then, by construction, u^ε_a satisfies all the conditions stated in Assumption 4.

3.2. Stability of the approximate solution

Consider now a perturbation of the initial condition a introduced in (81):

$$a^\varepsilon := a + \varepsilon^{k_0}\varphi^\varepsilon, \tag{93}$$

where φ^ε is a bounded family in $H_\varepsilon^{\sigma-l_0-2}$, and $k_0+3 \leqq l_0 < \sigma-2-\frac{d}{2}$. We need to assume that (4) is satisfied so that a^ε is a proper initial datum for the Euler–Maxwell system. This amounts to the conditions

$$\nabla \cdot B_{\varphi^\varepsilon} = 0, \quad \nabla \cdot E_{\varphi^\varepsilon} = \frac{1}{\varepsilon\theta_e}(n_{e,\varphi^\varepsilon} - n_{i,\varphi^\varepsilon})$$

for the coordinates of the perturbation φ^ε.

Consider the approximate solution u_a^ε to (80) associated with the initial datum a at order l_0.

Theorem 9. *If $k_0 > 3 + \frac{3}{2}$, the system* (80)*, together with the initial datum* (93) *has a unique solution u^ε defined over* $[0, t^*)$*, independent of ε. Moreover for all* $0 < t_0 < t^*$*, there holds for ε small enough*

$$\sup_{0 \leqq t \leqq t_0} \|u^\varepsilon - u_a^\varepsilon\|_{\varepsilon,s} \leqq C\varepsilon^{k_0-1}, \tag{94}$$

with a constant C independent of ε, and a Sobolev index $s > 1 + \frac{d}{2}$.

The proof shows that the existence time t^* is bounded from below by the existence time of the approximate solution u_a^ε. Note that the estimate (94), the condition $k_0 > 3 + \frac{3}{2}$, and the description of the approximate solution given in the above section, imply

$$\sup_{0 \leqq t \leqq t_0} \sup_{x \in \mathbb{R}^3} \left| E^\varepsilon - (E_{0,1}e^{i\omega t/\varepsilon^2} + (E_{0,1})^* e^{-i\omega t/\varepsilon^2} \right| + \left| n^\varepsilon - \varepsilon n_{i1,0} \right| \leqq C\varepsilon^2, \tag{95}$$

where $E^\varepsilon, n^\varepsilon$ are coordinates of the solution u^ε of (80), where $n_{i1,0} = \frac{1}{\varepsilon}\log(1 + \varepsilon n_{i1,0}^\sharp)$, and $E_{0,1}$ and $n_{i1,0}^\sharp$ solve the Zakharov system (91), with the initial condition $E_{0,1}(t=0) = E^0$, $n_{i1,0}^\sharp(t=0) = 0$, $\partial_t n_{i1,0}^\sharp(t=0) = 0$.

The asymptotic estimate (95) is the estimate that validates the Zakharov model, as it actually gives a description of the electric field E^ε and the fluctuation of density n^ε in (EM) by means of the solution $(E_{0,1}, n_{i1,0}^\sharp)$ of (91).

Theorem 9 follows as a corollary of Theorem 8 if we can prove that the (EM) system satisfies the assumptions of Theorem 8. An approximate solution satisfying Assumption 4 was constructed in the above section. The next sections are devoted to the verification of Assumption 3 (hyperbolic structure, regularity of the eigenvalues and eigenprojectors), Assumption 5 (localization of the resonances) and Assumption 6 (transparency).

3.2.1. Eigenvalues and eigenvectors. We check in this section that the (EM) system satisfies Assumption 3. The operator $\mathcal{A}$ defined at the beginning of Section 3 obviously belongs to $C^\infty\mathcal{M}^1$. For all ε, u, ξ, $\mathcal{A}(\varepsilon, u, \xi)$ is a Hermitian matrix, and satisfies a decomposition of the form (36). It remains to check hypotheses (i) to (iii) in Assumption 3.

The eigenvalues that appear in the spectral decomposition (36) are the solutions

$$\omega = \lambda(\varepsilon, u, \xi), \quad \varepsilon > 0, u \in \mathbb{R}^{14}, \xi \in \mathbb{R}^3,$$

of the polynomial equation in ω,

$$\det(i\omega + \mathcal{A}(\varepsilon, u, i\xi)) = 0. \tag{96}$$

A look at the definition of $\mathcal{A}_1$ shows that the eigenvalues depend on u only through the scalar terms

$$\mathrm{x} := \varepsilon\theta_e v_e \cdot \xi, \quad \text{and} \quad \mathrm{y} := \varepsilon^2 v_i \cdot \xi,$$

representing the electronic and ionic convections (v_e and v_i are coordinates of u). Equation (96) factorizes into a transverse, degree four equation

$$(\omega - \mathrm{x})(\omega - \mathrm{y})\left(\omega^2 - 1 - |\xi|^2 - \frac{\varepsilon^2}{\theta_e^2}\right) = \mathrm{x}(\omega - \mathrm{y}) + \frac{\varepsilon^2}{\theta_e^2}\mathrm{y}(\omega - \mathrm{x}), \tag{97}$$

and a longitudinal, degree five equation

$$\begin{aligned} &\omega((\omega - \mathrm{y})^2 - \varepsilon^2\alpha^2|\xi|^2)((\omega - \mathrm{x})^2 - 1 - \theta_e^2|\xi|^2) \\ &\quad = -\mathrm{x}((\omega - \mathrm{y})^2 - \varepsilon^2\alpha^2|\xi|^2) \\ &\qquad + \frac{\varepsilon^2}{\theta_e^2}(\omega - \mathrm{y})((\omega - \mathrm{x})^2 - \theta_e^2|\xi|^2). \end{aligned} \tag{98}$$

For all $\varepsilon, \mathrm{x}, \mathrm{y}, \xi$, the Kernel of $\mathcal{A}(\varepsilon, u, \xi)$ has dimension one. It is generated by

$$e_0 := \left(\frac{\xi}{|\xi|}, 0, 0, 0, 0, 0\right).$$

The solutions of (97) have multiplicity two, while the solutions of (98) have multiplicity one. The solutions of (97) and (98) are algebraic functions of $\varepsilon, \xi, \mathrm{x}, \mathrm{y}$. Evaluations of these functions $\omega = \omega(\varepsilon, (\mathrm{x}, \mathrm{y}), \xi)$ at $(\mathrm{x}, \mathrm{y}) = (\varepsilon\theta_e v_e \cdot \xi, \varepsilon^2 v_i \cdot \xi)$ give the nonzero eigenvalues of the (EM) system.

For $(\mathrm{x}, \mathrm{y}) = (0, 0)$, equation (97) and (98) simplify to

$$\omega^2\left(\omega^2 - 1 - |\xi|^2 - \frac{\varepsilon^2}{\theta_e^2}\right) = 0, \tag{99}$$

and

$$\omega((\omega^2 - \varepsilon^2\alpha^2|\xi|^2)(\omega^2 - 1 - \theta_e^2|\xi|^2) - \frac{\varepsilon^2}{\theta_e^2}(\omega^2 - \theta_e^2|\xi|^2)) = 0. \tag{100}$$

The solutions $\omega = \omega(\varepsilon, \xi)$ of (99)–(100) are represented in Figure 2.

The transverse modes at $(\varepsilon, 0, \xi)$ are

$$\lambda_s(e_-) = 0, \quad \lambda_s(e_+) = 0, \quad \lambda_\pm = \pm\sqrt{1 + |\xi|^2 + \frac{\varepsilon^2}{\theta_e^2}}. \tag{101}$$

The Klein–Gordon longitudinal modes at $(\varepsilon, 0, \xi)$ are

$$\mu_\pm = \pm\sqrt{1 + \theta_e^2|\xi|^2} + \frac{\varepsilon}{2\theta_e^2(1 + \theta_e^2|\xi|^2)} + O(\varepsilon^2),$$

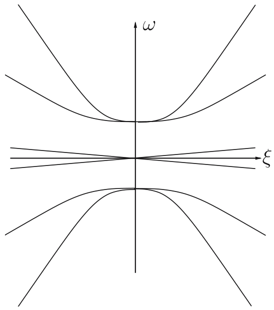

Fig. 2. The characteristic variety for the operator linearized around 0

and the acoustic longitudinal modes at $(\varepsilon, 0, \xi)$ are

$$\mu_s(e_-) = 0, \quad \mu_s(e_+)_\pm = O(\varepsilon), \quad \mu_\pm = \pm\sqrt{1 + \theta_e^2|\xi|^2} + O(\varepsilon),$$

locally uniform in ξ. The Klein–Gordon modes $\mu_\pm$ have constant multiplicity, and hence are analytical in ε, ξ. For the acoustic modes, crossing of eigenvalues occurs only at the zero frequency. Thus away from the zero frequency, the acoustic modes are analytical as well. It is easy to check that

$$0 \leqq |\mu_s(e^+)_\pm| \leqq C\varepsilon|\xi|, \tag{102}$$

uniformly in ε, ξ, and that

$$\mu_s(e^+)_\pm = \pm\varepsilon|\xi|\big(\sqrt{1+\alpha^2} + O(\varepsilon^2 + |\xi|^2)\big), \tag{103}$$

uniformly in ε, $|\xi| \leqq c_l$. Regularity at infinity can be directly checked using an exact description of the solutions of (100): we find that the longitudinal eigenvalues have the form

$$\begin{aligned} \mu_s(e_+)_\pm &= \pm\varepsilon\alpha|\xi| + \tfrac{1}{|\xi|}F_\pm\left(\tfrac{1}{|\xi|}, \tfrac{\xi}{|\xi|}, \varepsilon\right), \\ \mu_\pm &= \theta_e|\xi| + \tfrac{1}{|\xi|}G_\pm\left(\tfrac{1}{|\xi|}, \tfrac{\xi}{|\xi|}, \varepsilon\right), \end{aligned} \tag{104}$$

where $F_\pm$ and $G_\pm$ are analytical in $(|\xi_0|, \infty) \times \mathbb{S}^2 \times \mathbb{R}_+$, for $|\xi_0| > 0$.

Thus the eigenvalues at $(\mathtt{x}, \mathtt{y}) = (0, 0)$ satisfy (16) (regularity in ε, ξ for ξ away from zero, decay at infinity) with $m = 1$. That is, condition (ii) in Assumption 3 is satisfied by the eigenvalues.

For ξ in a compact subset of $\mathbb{R}^3$, the eigenvalues evaluated at $(\mathtt{x}, \mathtt{y}) = (\varepsilon\theta_e v_e \cdot \xi, \varepsilon^2 v_i \cdot \xi)$ are small perturbations of the eigenvalues evaluated at $(\mathtt{x}, \mathtt{y}) = (0, 0)$. This implies in particular that the Klein–Gordon modes are separated from the acoustic modes. For large ξ, the contribution of the convective terms to the eigenvalues is not negligible, but the acoustic modes are all $O(\varepsilon|\xi|)$, while the Klein–Gordon modes are $O(|\xi|)$. Thus condition (i) in Assumption 3 is satisfied.

The Klein–Gordon modes are single eigenvalues of (97) and (98), for all ξ. Hence the eigenvalues and the eigenprojectors corresponding to the Klein–Gordon modes are analytical.

For $\xi \neq 0$, let $\{\xi_1, \xi_2\}$ be an orthonormal basis of $\{\xi\}^\perp$.

At $(\varepsilon, 0, \xi)$, $\xi \neq 0$, the eigenvectors associated with the transverse eigenvalues are

$$e_\pm := \left(\frac{|\xi|\xi_2}{\lambda_\pm}\,,\xi_1\,,\frac{-i\xi_1}{\lambda_\pm}\,,0\,,\frac{i\varepsilon}{\theta_e}\frac{\xi_1}{\lambda_\pm}\,,0\right),$$
$$e'_\pm := \left(\frac{-|\xi|\xi_1}{\lambda_\pm}\,,\xi_2\,,\frac{-i\xi_2}{\lambda_\pm}\,,0\,,\frac{i\varepsilon}{\theta_e}\frac{\xi_2}{\lambda_\pm}\,,0\right),$$
$$e_s(e^-) := \frac{1}{\sqrt{1+|\xi|^2}}\,\big(\xi_1, 0, i|\xi|\xi_2, 0, 0, 0\big),$$
$$e'_s(e^-) := \frac{1}{\sqrt{1+|\xi|^2}}\,\big(\xi_2, 0, i|\xi|\xi_1, 0, 0, 0\big),$$
$$e_s(e^+) := \frac{|\xi|}{|\xi|+\varepsilon}\left(\frac{-i\varepsilon\xi_2}{\theta_e|\xi|}, 0, 0, 0, \xi_1, 0\right),$$
$$e'_s(e^+) := \frac{|\xi|}{|\xi|+\varepsilon}\left(\frac{i\varepsilon\xi_1}{\theta_e|\xi|}, 0, 0, 0, \xi_2, 0\right).$$

At $(\varepsilon, 0, \xi)$, $\xi \neq 0$, the eigenvectors associated with the longitudinal eigenvalues are

$$f_\pm := \left(0\,,\ \frac{\mu_\pm^2-\theta_e^2|\xi|^2}{\mu_\pm}\frac{\xi}{|\xi|}\,,\ -i\frac{\xi}{|\xi|}\,,\ \frac{-i\theta_e|\xi|}{\mu_\pm}\,,\frac{i\varepsilon}{\theta_e}\frac{\mu_\pm^2-\theta_e^2|\xi|^2}{\mu_\pm^2-\varepsilon^2\alpha^2|\xi|^2}\frac{\xi}{|\xi|}\frac{\mu_\pm}{\mu_\pm}\,,\right.$$
$$\left.\frac{i\alpha\varepsilon^2}{\theta_e}\frac{\mu_\pm^2-\theta_e^2|\xi|^2}{\mu_\pm^2-\varepsilon^2\alpha^2|\eta|^2}\frac{|\xi|}{\mu_\pm}\right),$$

$$f_s(e^-) := \frac{1}{\sqrt{1+\theta_e^2|\xi|^2}}\left(0, -i\theta_e\xi, 0, 1, 0, -\frac{1}{\alpha}\right),$$

and

$$f_s(e^+)_\pm := \left(0\,,\ \frac{\theta_e}{\varepsilon}\tilde{\mu}_s(e^+)_\pm\frac{\xi}{|\xi|}\,,\ \frac{-i\theta_e}{\varepsilon}\frac{\tilde{\mu}_s(e^+)_\pm\mu_s(e^+)_\pm}{\mu_s(e^+)_\pm^2-\theta_e^2|\xi|^2}\frac{\xi}{|\xi|}\,,\right.$$
$$\left.\frac{-i\theta_e^2}{\varepsilon}\frac{|\xi|\tilde{\mu}_s(e^+)}{\mu_s(e^+)_\pm^2-\theta_e^2|\xi|^2)}\frac{\xi}{|\xi|}\,,\ \frac{i\xi}{|\xi|}\,,\ \frac{i\alpha\varepsilon|\xi|}{\mu_s(e^+)_\pm}\right),$$

where

$$\tilde{\mu}_s(e^+)_\pm := \frac{\mu_s(e^+)_\pm^2-\varepsilon^2\alpha^2|\xi|^2}{\mu_s(e^+)_\pm}.$$

From (103), we see that $\tilde{\mu}_s(e^+)_\pm = O(\varepsilon|\xi|)$. This yields a description of $f_s(e^+)_\pm$ for small frequencies:

$$f_s(e^+)_\pm = \big(0, O(|\xi|), O(\varepsilon), O(1), O(1), O(1)\big).$$

The corresponding orthogonal eigenprojectors (Π_j, $1 \leqq j \leqq n$ with the notation introduced in Assumption 3) are defined by

$$\underline{e} \otimes \underline{e} + \underline{e}' \otimes \underline{e}', \quad \underline{f} \otimes \underline{f},$$

where $e = e_\pm$, $e_s(e^-)$ or $e_s(e^+)$, $f = f_\pm$, $f_s(e^-)$ or $f_s(e^+)_\pm$, and $\underline{e}, \underline{f} := \frac{e}{|e|}, \frac{f}{|f|}$. With the above description of the eigenvalues for small and large $|\xi|$, it is straightforward to check that all the eigenprojectors, evaluated at $(\varepsilon, 0)$, are Fourier multipliers in the class $C^\infty\mathcal{M}^0$. Thus condition (ii) in Assumption 3 is satisfied.

Condition (iii) follows from the above description of the acoustic modes, as they all have a prefactor ε.

We now turn to condition (38). The total eigenprojectors, defined in (37), are

$$\Pi_0 = \sum_\pm \underline{e}_\pm \otimes \underline{e}_\pm + \underline{e}'_\pm \otimes \underline{e}'_\pm + \underline{f}_\pm \otimes f_\pm,$$

and

$$\begin{aligned}\Pi_s = &\sum_\pm \underline{e}_s(e^\pm) \otimes \underline{e}_s(e^\pm) + \underline{e}'_s(e^\pm) \otimes \underline{e}'_s(e^\pm) \\ &+\underline{f}_s(e^-) \otimes \underline{f}_s(e^-) + e_0 \otimes e_0 + \sum_\pm f_s(e^+)_\pm \otimes \underline{f}_s(e^+)_\pm.\end{aligned}$$

Given $(\varepsilon_0, u_0, \xi_0) \in (0, 1] \times \mathbb{R}^{14} \times \mathbb{R}^3$, for (ε, u, ξ) in a neighbourhood of $(\varepsilon_0, u_0, \xi_0)$,

$$\begin{aligned}\Pi_0(\varepsilon, u, \xi) = &\frac{1}{2i\pi} \int_{\Gamma_{0+}} (z - \mathcal{A}(\varepsilon, u, \xi))^{-1} dz \\ &+ \frac{1}{2i\pi} \int_{\Gamma_{0-}} (z - \mathcal{A}(\varepsilon, u, \xi))^{-1} dz,\end{aligned} \tag{105}$$

and

$$\Pi_s(\varepsilon, u, \xi) = \frac{1}{2i\pi} \int_{\Gamma_s} (z - \mathcal{A}(\varepsilon, u, \xi))^{-1} dz, \tag{106}$$

where Γ_{0+} is a contour enclosing all the positive Klein–Gordon eigenvalues at $(\varepsilon_0, u_0, \xi_0)$, and no other eigenvalues of $\mathcal{A}$, and where Γ_{0-} is a contour enclosing all the negative Klein–Gordon eigenvalues at $(\varepsilon_0, u_0, \xi_0)$, and no other eigenvalues of $\mathcal{A}$, and Γ_s is a contour enclosing all the acoustic eigenvalues at $(\varepsilon_0, u_0, \xi_0)$, and no other eigenvalues of $\mathcal{A}$. The above description of the eigenvalues shows in particular that such contours do exist. It follows from these representations that Π_0 and Π_s are analytical in ε, u, ξ. To investigate the behaviour for large ξ, notice that $\mathcal{A}$ can be written, for $|\xi| \neq 0$,

$$\mathcal{A}(\varepsilon, u, \xi) = \underline{\mathcal{A}}_0(\varepsilon) + \underline{\mathcal{A}}_1(\varepsilon, u, \xi),$$

where $\mathcal{A}_1$ is linear in ξ, so that, using polar coordinates $\xi = (|\xi|, \theta) \in \mathbb{R}_+^* \times \mathbb{S}^2$,

$$\mathcal{A}(\varepsilon, u, \xi) = |\xi| \left(\frac{1}{|\xi|} \underline{\mathcal{A}}_0(\varepsilon) + \underline{\mathcal{A}}_1(\varepsilon, u, \theta) \right).$$

Thus an eigenvector $e(\varepsilon, u, |\xi|, \theta)$ of $\mathcal{A}(\varepsilon, u, \xi)$, associated with an eigenvalue $\lambda(\varepsilon, u, |\xi|, \theta)$, is also an eigenvector $\tilde{e}(\varepsilon, u, \frac{1}{|\xi|}, \theta)$ of

$$\tilde{\mathcal{A}}\left(\varepsilon, u, \frac{1}{|\xi|}, \theta\right) := \frac{1}{|\xi|}\underline{\mathcal{A}}_0(\varepsilon) + \underline{\mathcal{A}}_1(\varepsilon, u, \theta), \tag{107}$$

associated with the eigenvalue $\frac{1}{|\xi|}\lambda(\varepsilon, u, \frac{1}{|\xi|}, \theta)$. The operator $\tilde{\mathcal{A}}$ is the long-wave operator associated with $\mathcal{A}$, introduced in [26] in the study of the short-wave limit.

For small $|\xi|$, the eigenvalues of $\tilde{\mathcal{A}}$ split into "acoustic" eigenvalues, of size $O(\varepsilon)$, and "Klein–Gordon" eigenvalues, of size $O(1)$. The total eigenprojectors

$$\tilde{\Pi}_0 = \frac{1}{2i\pi}\int_{\tilde{\Gamma}_{0+}} (z - \tilde{\mathcal{A}}(\varepsilon, u, |\xi|, \theta))^{-1} dz + \frac{1}{2i\pi}\int_{\Gamma_{0-}} (z - \tilde{\mathcal{A}}(\varepsilon, u, |\xi|, \theta))^{-1} dz,$$

and

$$\tilde{\Pi}_s = \frac{1}{2i\pi}\int_{\Gamma_s} (z - \tilde{\mathcal{A}}(\varepsilon, u, |\xi|, \theta))^{-1} dz,$$

are analytical in $\varepsilon, u, |\xi|, \theta$, for $|\xi|$ in a neighbourhood of 0. There holds

$$\Pi_j(\varepsilon, u, \xi) = \tilde{\Pi}_j\left(\varepsilon, u, \frac{1}{|\xi|}, \frac{\xi}{|\xi|}\right), \quad j = 0, s,$$

from which we deduce that the total eigenprojectors Π_0, Π_s of $\mathcal{A}$ are analytical in $\varepsilon, u, \frac{1}{|\xi|}, \frac{\xi}{|\xi|}$, for ξ in a neighbourhood of ∞. In particular, they satisfy decay estimates of the form (16), and (38) is proved.

To conclude this section, we now indicate how the Klein–Gordon eigenvalues and eigenvectors depend on $\mathtt{x}$ and $\mathtt{y}$. These descriptions are needed in the evaluation of the interaction coefficients that enter Assumption 6.

The eigenvalues satisfy

$$\partial_{\mathtt{x}}\lambda_\pm = \frac{1}{2(1 + |\xi|^2)} + O(\varepsilon^2), \quad \partial_{\mathtt{x}}\mu_\pm = \frac{\theta_e^2|\xi|^2}{2(1 + \theta_e^2|\xi|^2)} + O(\varepsilon),$$

at $(\mathtt{x}, \mathtt{y}) = (0, 0)$, locally uniformly in ξ.

The eigenvectors are

$$\underline{e}_\pm := \left(\frac{|\eta|\eta_2}{\lambda_\pm}, \eta_1, \frac{-i\eta_1}{\lambda_\pm - \mathtt{x}}, 0, \frac{i\varepsilon}{\theta_e}\frac{\eta_1}{\lambda_\pm - \mathtt{y}}, 0\right),$$
$$\underline{e}'_\pm := \left(\frac{-|\eta|\eta_1}{\lambda_\pm}, \eta_2, \frac{-i\eta_2}{\lambda_\pm - \mathtt{x}}, 0, \frac{i\varepsilon}{\theta_e}\frac{\eta_2}{\lambda_\pm - \mathtt{y}}, 0\right),$$

and

$$\begin{aligned}\underline{f}_\pm := \Big(0, &\frac{(\mu_\pm - \mathtt{x})^2 - \theta_e^2|\eta|^2}{\mu_\pm - \mathtt{x}}\frac{\eta}{|\eta|}, -i\frac{\eta}{|\eta|}, \frac{-i\theta_e|\eta|}{\mu_\pm - \mathtt{x}},\\ &\frac{i\varepsilon}{\theta_e}\frac{(\mu_\pm - \mathtt{x})^2 - \theta_e^2|\eta|^2}{(\mu_\pm - \mathtt{y})^2 - \varepsilon^2\alpha^2|\eta|^2}\frac{\eta}{|\eta|}\frac{\mu_\pm - \mathtt{y}}{\mu_\pm - \mathtt{x}},\\ &\frac{i\varepsilon^2}{\theta_e}\frac{(\mu_\pm - \mathtt{x})^2 - \theta_e^2|\eta|^2}{(\mu_\pm - \mathtt{y})^2 - \varepsilon^2\alpha^2|\eta|^2}\frac{|\eta|}{\mu_\pm - \mathtt{x}}\Big).\end{aligned}$$

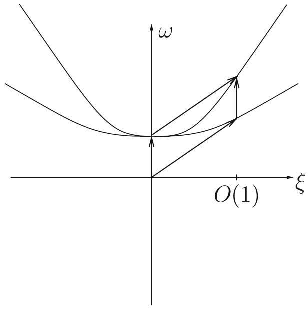

Fig. 3. (0-0) resonances

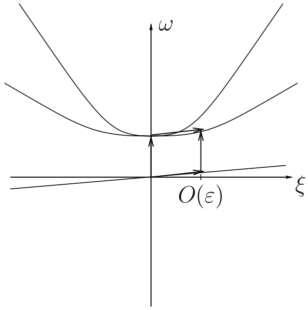

Fig. 4. (0-s) resonances

3.2.2. Resonances. We use here the description of the eigenvalues given in Section 3.2.1. Equation (40) for $j, k \leqq n_0$ (resonances between Klein–Gordon modes) and $\varepsilon = 0$ is

$$\sqrt{1 + |\xi|^2} - \sqrt{1 + \theta_e^2 |\xi|^2} - 1 = 0. \tag{108}$$

If θ_e is small enough, we choose $c_m = 1$ and $C_m = 2$. The left-hand side in (108) is bounded away from zero for $|\xi| \notin [1]$, and hypothesis (0-0) in Assumption 5 is satisfied.

Equation (40) for $j \leqq n_0 < k$ (resonances between Klein–Gordon and acoustic modes) is

$$\sqrt{1 + \kappa^2 |\xi^2|} - 1 - \mu = 0, \tag{109}$$

where $\kappa = 1$ or $\kappa = \theta_e$, and $\mu = 0$ or $\mu = \mu_s(e^+)_\pm$. If θ_e and ε are small enough, we can choose $c_l = 1/2$. Then the left-hand side in (109) is bounded away from zero for $|\xi| \geqq 1/2$.

Equation (41) (second-order resonances between Klein–Gordon and acoustic modes) is

$$\sqrt{1 + \kappa^2 |\xi|^2} - (p + p') = 0, \tag{110}$$

where $p, p \in \{-1, 1\}$, $\kappa = 1$ or $\kappa = \theta_e$. The left-hand side in (110) is bounded away from 0 for $|\xi| \leqq 1$, and hypothesis (0-0-s) in Assumption 5 is satisfied.

The resonances are pictured in Figures 3–5.

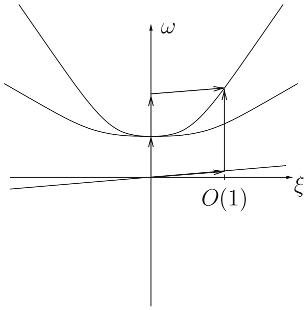

Fig. 5. (0-0-s) resonances

3.2.3. Transparency. We check in this section that Assumptions 6 and 7 are satisfied.

The definition of $\mathcal{A}$ given at the beginning of Section 3 implies that $\mathcal{A}^{(2)} = 0$. It is straightforward to check that $\Pi_s(0,0)\mathcal{D}(u_a)\Pi_0(0,0)$, where $\mathcal{D}$ is defined by (43), has size $O(\varepsilon)$, and (47) is satisfied.

Up to terms of size $O(\varepsilon^2)$, the source term defined in (42) is

$$\mathcal{B}(u_a)u = \begin{pmatrix} 0 \\ n_e\mathbf{v}_e + \varepsilon\mathbf{n}_{e1}v_e \\ -\theta_e(\mathbf{v}_e \times B + (\mathbf{v}_e\cdot\xi)v_e + \varepsilon\mathbf{B}_1 \times v_e + \varepsilon v_e\cdot\nabla\mathbf{v}_e) \\ -\theta_e(\mathbf{v}_e\cdot\xi)n_e \\ 0 \\ 0 \end{pmatrix}.$$

It depends on the coordinates $\mathbf{B}$, $\mathbf{v}_e$, $\mathbf{n}_e$, of the approximate solution u_a, evaluated at $\theta = \omega t/\varepsilon^2$.

We compute

$$e_+^*\mathcal{B}(u_a)e_s(e^-) = \frac{i\theta_e}{\lambda_+^2}\frac{\mathbf{v}_{e0}\cdot\xi}{|\xi|} + O(\varepsilon),$$

and also

$$f_+^*\mathcal{B}(u_a)f_s(e^-) = \frac{\mu_+^2 - \theta_e^2|\xi|^2}{\mu_+^2}\frac{\mathbf{v}_{e0}\cdot\xi}{|\xi|} + O(\varepsilon).$$

In particular, these interaction coefficients are not small for small frequencies. This shows that the nontransparency condition (44) is satisfied.

We now check that the transparency condition of Assumption 6 is satisfied.

The symbol ρ defined in (45) is $\rho = \rho^{(0)} + O(\varepsilon)$, where

$$\rho^{(0)} := \Pi_s \sum_{|\alpha|=1}\Big(\partial_\xi^\alpha(\Pi_s\mathcal{A}^{(0)})\partial_v\Pi_0\cdot\partial_x^\alpha u_a + \partial_\xi^\alpha\Pi_s\mathcal{A}^{(1)}(\partial_x^\alpha u_a)$$
$$- \Pi_s\mathcal{A}^{(0)}\partial_\xi^\alpha\Pi_0\partial_v\Pi_0\cdot\partial_x^\alpha u_a - \partial_\xi^\alpha\Pi_s\mathcal{B}(0,\partial_x^\alpha u_a)\Big)\Pi_0,$$

where, unless otherwise noted, the symbols and their derivatives are evaluated at $(\varepsilon, v) = (0, 0)$.

Assumption 6 is a transparency assumption for the interaction coefficient $\partial_u B^{\mathrm{r}}(\varepsilon, 0) \cdot u_a$. Direct computations, using the description of the eigenvalues and the eigenvectors given above, yield the bound

$$|\Pi_k(\varepsilon, 0)(\partial_u B^{\mathrm{r}}(\varepsilon, 0) \cdot u_a)\Pi_j(\varepsilon, 0)| \leqq C_B(|\xi|^2 + \varepsilon|\xi|), \tag{111}$$

for $j \leqq n_0 < k$, uniformly in ε, t, x and $|\xi| \leqq c_l$. Now (111) and the above description of the phases imply (48), as follows. Let $\delta > 0$ be given.

(i) If $|\xi| \leqq \delta\varepsilon$, then (111) directly implies (48), with $C = C_B\delta(1+\delta)$.
(ii) If $|\xi| > \delta\varepsilon$, then for $0 < \varepsilon < \varepsilon_0$, $|\Phi_{j,k,p}(\varepsilon)| > C_0(\delta)|\xi|^2$, with

$$C_0(\delta) := \frac{3\theta_e^2}{8} - \frac{C(\alpha, \varepsilon_0)}{\delta},$$

for some nondecreasing function $C(\alpha, \varepsilon_0)$. If δ is chosen to be large enough, then $C_0(\delta) > 0$, and (48) is satisfied with

$$C = C_B\left(\frac{1}{\delta} + 1\right) C_0(\delta)^{-1}.$$

Assumption 7 is a symmetrizability condition for the interaction coefficients

$$B_0 = \Pi_0 \mathcal{B}(u_a)\Pi_0, \quad B_s = \Pi_s \mathcal{B} \Pi_s.$$

As these interaction coefficients enter the equation with a prefactor $\frac{1}{\varepsilon}$, it is sufficient to consider the leading term in ε in $\mathcal{B}(u_a)$. In particular, the contribution of $\mathcal{A}^{(1)}$ in B_0 and B_s is only $O(\varepsilon)$. We can indeed write

$$\mathrm{op}_\varepsilon(\mathcal{A}^{(1)}(u))u_a = \varepsilon \mathrm{op}_1(\mathcal{A}^{(1)}(u))u_a, \tag{112}$$

because $\mathcal{A}^{(1)}$ is linear in ξ, and the H^s_ε norm of (112) is bounded by

$$\varepsilon C(\|u_a\|_{1,s+d_0})\|u\|_{\varepsilon,s}.$$

The auto-interaction coefficients are all purely imaginary:

$$e^*\mathcal{B}(u_a)e \in i\mathbb{R},$$

where e is any eigenvector of the Euler–Maxwell equations. The other interaction coefficients between the Klein–Gordon modes are

$$e_\pm^*\mathcal{B}(u_a)e'_\pm = 0, \quad e_\pm^*\mathcal{B}(u_a)f_\pm = -i\theta_e\frac{|\xi|}{\mu_\pm}\mathbf{v}_e\cdot\xi_1,$$
$$(e'_\pm)^*\mathcal{B}(u_a)e_\pm = 0, \quad (e'_\pm)^*\mathcal{B}(u_a)f_\pm = -i\theta_e\frac{|\xi|}{\mu_\pm}\mathbf{v}_e\cdot\xi_2,$$
$$f_\pm^*\mathcal{B}(u_a)e_\pm = -i\theta_e\frac{|\xi|}{\lambda_\pm}\mathbf{v}_e\cdot\xi_1, \quad f_\pm^*\mathcal{B}(u_a)e'_+ = -i\theta_e\frac{|\xi|}{\lambda_\pm}\mathbf{v}_e\cdot\xi_2,$$

where for all $\xi \neq 0$, $\{\xi_1, \xi_2\}$ is an orthonormal basis of $\{\xi\}^{\perp}$. Let $S_0 = S_0^{(+)} + S_0^{(-)}$, where

$$S_0^{(\pm)} := e_{\pm} \otimes e_{\pm} + e'_{\pm} \otimes e'_{\pm} + \frac{\lambda_{\pm}}{\mu_{\pm}} f_{\pm} \otimes f_{\pm}.$$

Then S_0 is a symmetrizer, in the sense that

$$\frac{1}{\varepsilon}(S_0 B_0 + (S_0 B_0)^*) \in C^{\infty}\mathcal{M}^0.$$

Finally, we turn to the interaction coefficients between acoustic modes. Remark first that the divergence-free condition for the magnetic field

$$\nabla \cdot B = 0 \tag{113}$$

is equivalent to

$$\mathrm{op}_{\varepsilon}(e_0) u = 0.$$

Condition (113) is propagated by the equations, that is, the solution belongs to the orthogonal of the image of $\mathrm{op}_{\varepsilon}(e_0)$ in $L^2(\mathbb{R}^3)$ if the initial datum does. Thus we can overlook the interaction coefficient involving e_0. At first order in ε, the interaction coefficients with the other acoustic modes all vanish, except for

$$\begin{aligned} f_s(e_-)^* \mathcal{B}(u_a) f_s(e_+)_{\pm} &= \frac{-\mathbf{v}_e \cdot \xi}{\sqrt{1 + \theta_e^2 |\xi|^2}} \frac{\mu_s(e_+)_{\pm}}{\varepsilon \theta_e} \frac{-\theta_e^2 |\xi|^2}{\mu_s(e_+)_{\pm}^2 - \theta_e^2 |\xi|^2}, \\ (f_s(e_+)_{\pm})^* \mathcal{B}(u_a) f_s(e_-) &= \frac{\mathbf{v}_e \cdot \xi}{\sqrt{1 + \theta_e^2 |\xi|^2}} \frac{\mu_s(e_+)_{\pm}}{\varepsilon \theta_e}. \end{aligned}$$

It follows from the description of the eigenvalues given in Section 3.2.1 that, for ε small enough,

$$\frac{1}{2} \leqq \frac{-\theta_e^2 |\xi|^2}{\mu_s(e_+)_{\pm}^2 - \theta_e^2 |\xi|^2} \leqq 2,$$

uniformly in $\xi \in \mathbb{R}^3$. Thus we can take $\gamma = 2$ in Assumption 7. Let

$$\begin{aligned} S_s := & e_0 \otimes e_0 + e_s(e_-) \otimes e_s(e_-) + e_s(e_+) \otimes e_s(e_+) + f_s(e_-) \otimes f_s(e_-) \\ & + \frac{-\theta_e^2 |\xi|^2}{\mu_s(e_+)_+^2 - \theta_e^2 |\xi|^2} f_s(e_+)_+ \otimes f_s(e_+)_+ \\ & + \frac{-\theta_e^2 |\xi|^2}{\mu_s(e_+)_-^2 - \theta_e^2 |\xi|^2} f_s(e_-)_+ \otimes f_s(e_+)_-. \end{aligned}$$

Then,

$$\frac{1}{\varepsilon}(S_s B_s + (S_s B_s)^*) \in C^{\infty}\mathcal{M}^0.$$

As S_0 and S_s are diagonal matrices in a basis of eigenvectors of $\mathcal{A}$, the matrices $S_0 A_0$ and $S_s A_s$ are Hermitian. Finally,

$$S := \begin{pmatrix} S_0 & 0 \\ 0 & S_s \end{pmatrix}$$

defines a symmetrizer that satisfies Assumption 6.

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