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Tractions, Balances, and Boundary Conditions for Nonsimple Materials with Application to Liquid Flow at Small-Length Scales

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Abstract

Using a nonstandard version of the principle of virtual power, we develop general balance equations and boundary conditions for second-grade materials. Our results apply to both solids and fluids as they are independent of constitutive equations. As an application of our results, we discuss flows of incompressible fluids at small-length scales. In addition to giving a generalization of the Navier–Stokes equations involving higher-order spatial derivatives, our theory provides conditions on free and fixed boundaries. The free boundary conditions involve the curvature of the free surface; among the conditions for a fixed boundary are generalized adherence and slip conditions, each of which involves a material length scale. We reconsider the classical problem of plane Poiseuille flow for generalized adherence and slip conditions.

Contents

1. Introduction

1.1. Toupin's results for a second-grade elastic material

In two monumental works, TOUPIN [40, 41] derived general balance equations and associated traction boundary conditions for an *elastic* body whose strain energy depends on *first and second gradients of the deformation*. Toupin's derivation is based on a virtual work principle[1] asserting that the variation of the total elastic energy be equal to the virtual work exerted on the body by tractions and body forces.[2] A central consequence of Toupin's work is the observation that Cauchy's hypothesis — that the surface traction at a point $\mathbf{x}$ on a surface $\mathcal{S}$ depend on $\mathcal{S}$ through its normal field at $\mathbf{x}$ — is not valid in a theory involving second gradients of the deformation, because in Toupin's theory the traction depends also on the curvature of $\mathcal{S}$ at $\mathbf{x}$.

[1] We use the term virtual *work* when the principle describes equilibrium, and virtual *power* when the principle describes an evolving body, allowing for inertia.

[2] Toupin's results were applied by MINDLIN [23] and MINDLIN & ESHEL [24] to linear elastostatics and were partially generalized by PODIO-GUIDUGLI [32] to include third deformation gradients. Precursors to Toupin's work trace back to the introduction of couple stresses by VOIGT [45] (cf. Section 1) and E. & F. COSSERAT [5] (cf. Section 53).

Unfortunately, because it assumes that the material is elastic and the body is in equilibrium, Toupin's *derivation* of the balance equations and associated traction boundary conditions cannot be applied to a general dynamical framework that includes dissipation.[3]

1.2. The general virtual power principle for a second-grade material

Our goal here is to derive Toupin's results within a framework that is *independent of constitutive equations*. To do so we use a *nonstandard* form of the principle of virtual power (Gurtin [12]).[4] Conventional versions of this principle are formulated for the body B as a whole rather than for control volumes and as such generally involve particular boundary conditions applied to the boundary ∂B of B. Such formulations allow for a weak statement of the basic force balances and when combined with constitutive equations result in weak statements of the resulting boundary-value problems. *Here the principle of virtual power is used instead as a basic tool to determine the structure of the tractions and of the local force balances*. As such, conditions on ∂B play a role no different from those on the boundary of any control volume. Basic to this view is the premise, central to all of continuum mechanics, that any basic law for the body should hold also for all subregions of the body.

Classically, the power expended *within* an arbitrary control volume R in the region of space occupied by the deformed body has the simple form

$$\mathcal{W}_{\text{int}}(R) = \int_R \mathbf{T} : \operatorname{grad}\mathbf{v}\, \mathrm{d}v = \int_R T_{ij} v_{i,j}\, \mathrm{d}v \tag{1}$$

with $\mathbf{T}$ the Cauchy stress and $\mathbf{T} : \operatorname{grad}\mathbf{v}$ the stress power. We generalize the classical theory by including an analogous $\mathbb{G} \vdots \operatorname{grad}^2\mathbf{v}$ linear in the second gradient $\operatorname{grad}^2\mathbf{v}$, with $\mathbb{G}$ a third-order *hyperstress*, and therefore rewrite the power expended within R in the form

$$\mathcal{W}_{\text{int}}(R) = \int_R \left(\mathbf{T} : \operatorname{grad}\mathbf{v} + \mathbb{G} \vdots \operatorname{grad}^2\mathbf{v}\right) \mathrm{d}v = \int_R \left(T_{ij} v_{i,j} + G_{ijk} v_{i,jk}\right) \mathrm{d}v. \tag{2}$$

Within our framework *the grade of the material is defined to be the order* (here 2) *of the highest velocity gradient in the internal power*.

In conjunction with the internal power expenditure (2), we introduce a corresponding external power expenditure

$$\mathcal{W}_{\text{ext}}(R) = \int_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}} \cdot \mathbf{v} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \mathbf{v}}{\partial n}\right) \mathrm{d}a + \int_R \mathbf{b} \cdot \mathbf{v}\, \mathrm{d}v, \tag{3}$$

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[3] In this regard, Toupin [40] gave an alternative derivation of the basic balance that allows for inertia, again for an elastic material, using an argument based on balance of energy.

[4] Cf. Antman & Osborn [1] for a rigorous treatment of the classical virtual-work principle for forces and a similar treatment of a corresponding principle for torques. The central result of Antman and Osborn — the equivalence of these principles to the classical balances for linear and angular momentum — is independent of constitutive equations.

in which $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$ represent tractions on the bounding surface $\mathcal{S} = \partial R$ of R, while $\mathbf{b}$ *represents the net inertial and noninertial body force* acting within the body.[5] Here the term

$$\mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \mathbf{v}}{\partial n},$$

which is not present in classical theories, is needed to balance the effects of the second-grade term $\mathbb{G}\,\vdots\,\mathrm{grad}^2\mathbf{v}$ in the internal power.

The *principle of virtual power* we use is based on the requirement that

$$\mathcal{W}_{\mathrm{ext}}(R) = \mathcal{W}_{\mathrm{int}}(R)$$

for all control volumes R and any choice of the velocity field $\mathbf{v}$, here considered as *virtual*. Consequences of the virtual power principle and the requirement that the internal power expenditure be frame indifferent are that:

(i) the classical macroscopic balance $\mathrm{div}\,\mathbf{T} + \mathbf{b}^{\mathrm{ni}} = \rho\dot{\mathbf{v}}$, with ρ the density and $\mathbf{b}^{\mathrm{ni}}$ the noninertial body force, need to be replaced by the balance

$$\mathrm{div}\,(\mathbf{T} - \mathrm{div}\,\mathbb{G}) + \mathbf{b}^{\mathrm{ni}} = \rho\dot{\mathbf{v}} \quad \left(T_{ij,j} - G_{ijk,jk} + b_i^{\mathrm{ni}} = \rho\dot{v}_i\right) \tag{4}$$

with $\mathbf{T}$ symmetric as in the classical theory;

(ii) Cauchy's classical condition $\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n}$ for the traction across a surface $\mathcal{S}$ with unit normal $\mathbf{n}$ need to be replaced by the conditions

$$\mathbf{t}_{\mathcal{S}} = \mathbf{T}\mathbf{n} - (\mathrm{div}\,\mathbb{G})\mathbf{n} - \mathrm{div}_{\mathcal{S}}(\mathbb{G}\mathbf{n}) - 2K(\mathbb{G}\mathbf{n})\mathbf{n}, \quad \mathbf{m}_{\mathcal{S}} = (\mathbb{G}\mathbf{n})\mathbf{n}, \tag{5}$$

in which K is the mean curvature of $\mathcal{S}$; equivalently, in components, for $\mathbf{K}$ the curvature tensor of $\mathcal{S}$,

$$\left.\begin{aligned} (\mathbf{t}_{\mathcal{S}})_i &= T_{ij}n_j - 2G_{ijk,k}n_j + G_{ijk,l}n_jn_kn_l + G_{ijk}(K_{jk} - 2Kn_jn_k), \\ (\mathbf{m}_{\mathcal{S}})_i &= G_{ijk}n_jn_k. \end{aligned}\right\} \tag{6}$$

The results (4) and (5) (applied to the boundary of the body) are the same as those given in equations (7.8) and (7.9) of TOUPIN [40], but *our derivation is independent of constitutive relations and hence valid for both solids and fluids.*

1.3. Application of the theory to liquid flow at small-length scales

We discuss in some detail applications of the general field theory defined by (4) and (5), a discussion which we now summarize.

We consider a purely mechanical theory with an underlying "second law" – a free-energy imbalance asserting that the free energy of an arbitrary region

[5] Specifically, $\mathbf{b}$ accounts not only for the usual body forces like that due to gavitation but also for accelerations terms treated, in the manner of D'ALEMBERT [6], as reversed forces.

that convects with the body increases at a rate not greater than the rate at which work is performed.[6]

Consistent with the goal of developing a theory for liquids at small-length scales, we restrict attention to incompressible materials, so that: (i) the density and specific free energy are constant; (ii) the stress has the form

$$\mathbf{T} = \mathbf{T}_0 - P\mathbf{1},$$

with $\mathbf{T}_0$ the traceless extra stress and P an indeterminate pressure; and (iii) the hyperstress has the form

$$\mathbb{G} = \mathbb{G}_0 - \mathbf{1} \otimes \boldsymbol{\pi} \quad \left(G_{ijk} = G_{0ijk} - \delta_{ij}\pi_k\right),$$

with extra hyperstress $\mathbb{G}_0$ traceless in its first two indices and $\boldsymbol{\pi}$ an indeterminate vectorial hyperpressure. More importantly, the free energy imbalance reduces to a dissipation inequality

$$\mathbf{T}_0 : \mathbf{D} + \mathbb{G}_0 \vdots \operatorname{grad}^2\mathbf{v} \geqq 0. \tag{7}$$

We restrict attention to linear isotropic constitutive relations which, in components, have the form[7]

$$\left.\begin{aligned} T_{0ij} &= 2\mu D_{ij} = \mu(v_{i,j} + v_{j,i}), \\ G_{0ijk} &= \eta_1 v_{i,jk} + \eta_2(v_{k,ij} + v_{j,ik} - v_{i,rr}\delta_{jk}), \end{aligned}\right\} \tag{8}$$

where isotropy rules out coupling between T_{0ij} and G_{0ijk}. Most importantly, (8) involves only two constitutive moduli η_1 and η_2 in addition to the classical viscosity μ.

With the constitutive relations (8) and noninertial body forces neglected, the force balance (4) yields the flow equation

$$\rho\dot{\mathbf{v}} = -\operatorname{grad} p + \mu\triangle\mathbf{v} - \zeta\triangle\triangle\mathbf{v} \quad \left(\rho\dot{v}_i = -p_{,i} + \mu v_{i,jj} - \zeta v_{i,jjkk}\right), \tag{9}$$

where $\triangle$ represents the Laplace operator, while

$$p = P - \operatorname{div}\boldsymbol{\pi} \quad \text{and} \quad \zeta = \eta_1 - \eta_2. \tag{10}$$

The flow equation (9) differs from the conventional Navier–Stokes equation only by the term with coefficient ζ.

As a consequence of the dissipation inequality (7), we find that $\mu \geqq 0$ and $\zeta \geqq 0$. Granted that $\mu > 0$ or $\zeta > 0$ our flow equation (9) is — like the Navier–Stokes equation — parabolic. Further, a dimensional argument yields a material length

$$L = \sqrt{\frac{\zeta}{\mu}} \tag{11}$$

relevant to the description of liquid flows at small scales.

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[6] If the theory was based on continuum formulations of balance of energy and growth of entropy (the Clausius–Duhem inequality), then this free-energy imbalance would be satisfied in isothermal processes.

[7] A future paper — based on constitutive equations more general than (8) — will discuss *turbulence*.

In addition to the flow equation, the theory also provides boundary conditions. The classical no-slip boundary condition is replaced by what we refer to as the *generalized adherence conditions*

$$\mathbf{v} = \mathbf{0} \quad \text{and} \quad \mathbf{m}_{\mathcal{S}} = -\mu l \frac{\partial \mathbf{v}}{\partial n}, \tag{12}$$

in which the constitutive modulus $l \geqq 0$, a material length, measures the strength of the fluid's adherence to the boundary. Alternatively, the theory allows also for slip and for a free surface, each of which involves the introduction of an additional constitutive parameter.

To exhibit some features of the theory, we revisit the classical problem of plane Poiseuille flow for generalized adherence and slip conditions. For the generalized adherence conditions (12), the theory predicts a flow rate lower than that obtained classically. When slip is allowed, the flow rate exceeds that obtained under weak adherence but also lies below that obtained for the Navier–Stokes equations subject to the Navier slip condition. These lower flow rates trace directly to the additional sources of dissipation associated with the hyperstress and the boundary conditions. As would be expected, these effects are important only for channels with sufficiently small gaps.

1.4. Kinetic energy dependent upon the velocity gradient

The theory discussed thus far is based on a generalization of the classical virtual-power principle (for a continuum) to include higher-order velocity gradients and concomitant higher-order stresses and tractions. In accordance with this, we conclude the paper by modifying the kinetic energy to account for dependence on the velocity gradient. Specifically, assuming that the kinetic energy per unit volume is of the form

$$\tfrac{1}{2}\rho|\mathbf{v}|^2 + \tfrac{1}{2}\beta\,|\mathrm{grad}\,\mathbf{v}|^2,$$

where $\beta = (\text{constant})\rho$, we seek to determine appropriate *inertial and noninertial* components $\mathbf{b}^{\mathrm{in}}$ and $\mathbf{b}^{\mathrm{ni}}$ of the body force $\mathbf{b}$, and $\mathbf{t}_{\mathcal{S}}^{\mathrm{in}}$ and $\mathbf{t}_{\mathcal{S}}^{\mathrm{ni}}$ of the traction $\mathbf{t}_{\mathcal{S}}$. To achieve this, we adapt the virtual power analysis used previously to the *inertial power balance*

$$\int_R (\mathbf{p}\cdot\mathbf{v} + \mathbf{M} \colon \mathrm{grad}\,\mathbf{v})\; \mathrm{d}v = -\int_R \mathbf{b}^{\mathrm{in}}\cdot\mathbf{v}\,\mathrm{d}v - \int_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}^{\mathrm{in}}\cdot\mathbf{v}\,\mathrm{d}a, \tag{13}$$

where $\mathbf{p}$ and $\mathbf{M}$ are (vectorial and tensorial) *momentum-rate forces* defined via

$$\mathbf{p} = \rho\dot{\mathbf{v}} \quad \text{and} \quad \mathbf{M} = \beta\,\overset{\boldsymbol{\cdot}}{\overline{\mathrm{grad}\,\mathbf{v}}}. \tag{14}$$

We find that for an incompressible material,

$$\left.\begin{aligned} \mathbf{b} &= -\mathrm{grad}\,\kappa - (\rho - \beta\triangle)\dot{\mathbf{v}} - \beta[(\mathrm{grad}\,\mathbf{v})\triangle\mathbf{v} + (\mathrm{grad}^2\mathbf{v})\mathrm{grad}\,\mathbf{v}] + \mathbf{b}^{\mathrm{ni}}, \\ \mathbf{t}_{\mathcal{S}} &= [\kappa\mathbf{1} - \beta(\mathrm{grad}\,\dot{\mathbf{v}} - (\mathrm{grad}\,\mathbf{v})\mathrm{grad}\,\mathbf{v})]\mathbf{n} + \mathbf{t}_{\mathcal{S}}^{\mathrm{ni}} \end{aligned}\right\} \tag{15}$$

and when noninertial body forces are neglected, the force balance (4) and the constitutive equations (8) yield the flow equation

$$\rho\dot{\mathbf{v}} - \beta\left[\triangle\dot{\mathbf{v}} - (\operatorname{grad}\mathbf{v})\triangle\mathbf{v} - (\operatorname{grad}^2\mathbf{v})\operatorname{grad}\mathbf{v}\right] = -\operatorname{grad}p + \mu\triangle\mathbf{v} - \zeta\triangle\triangle\mathbf{v}, \tag{16}$$

where $p = P + \kappa - \operatorname{div}\boldsymbol{\pi}$. In components, (16) reads

$$\rho\dot{v}_i - \beta\left(\dot{v}_{i,jj} - v_{i,j}v_{j,kk} - v_{i,jk}v_{j,k}\right) = -p_{,i} + \mu v_{i,jj} - \zeta v_{i,jjkk}.$$

The generalized adherence conditions (12) are unaffected by the inclusion of gradient inertia, but, interestingly, the conditions at a fixed surface that allows for slip and at a free surface contain inertial terms.

2. Preliminaries

To simplify our calculations, we use direct notation. However, for clarity, we also present key definitions and results in component form.

2.1. Notation

We find it most convenient to work spatially i.e. to use what is commonly called an Eulerian description. We write $\rho(\mathbf{x}, t)$ for the *mass density*, $\mathbf{v}(\mathbf{x}, t)$ for the velocity, and

$$\mathbf{D} = \tfrac{1}{2}\left(\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^{\top}\right) \quad \text{and} \quad \mathbf{W} = \tfrac{1}{2}\left(\operatorname{grad}\mathbf{v} - (\operatorname{grad}\mathbf{v})^{\top}\right) \tag{17}$$

for the *stretching* and *spin*. As is usual, we use a superposed dot for the *material time-derivative* e.g. for $\varphi(\mathbf{x}, t)$ a scalar field

$$\dot{\varphi} = \frac{\partial\varphi}{\partial t} + \mathbf{v}\cdot\operatorname{grad}\varphi.$$

Balance of mass is then the requirement that

$$\dot{\rho} + \rho\operatorname{div}\mathbf{v} = 0. \tag{18}$$

2.2. Control volume R. Differential geometry on ∂R

We denote by R an arbitrary region, fixed in time, that is contained in the region of space occupied by the body over some time interval. We refer to R as a *control volume* and write

$$\mathcal{S} = \partial R$$

for the boundary of R and $\mathbf{n}$ for the outward unit normal on $\mathcal{S}$. Unless stated to the contrary, we assume that $\mathcal{S}$ is *smooth*, cf. Section 4.4. We let $\mathbf{P} = \mathbf{P}(\mathbf{n})$ denote the projection onto the plane normal to $\mathbf{n}$:

$$\mathbf{P} = \mathbf{1} - \mathbf{n}\otimes\mathbf{n} \quad \left(P_{ij} = \delta_{ij} - n_i n_j\right).$$

The operator $\operatorname{grad}_{\mathcal{S}}$ defined on any vector field $\mathbf{g}$ by

$$\operatorname{grad}_{\mathcal{S}}\mathbf{g} = (\operatorname{grad}\mathbf{g})\mathbf{P} \quad \big((\operatorname{grad}_{\mathcal{S}}\mathbf{g})_{ij} = g_{i,j} - g_{i,k}n_k n_j\big)$$

is the *surface gradient*;[8]

$$\operatorname{div}_{\mathcal{S}}\mathbf{g} = \operatorname{tr}(\operatorname{grad}_{\mathcal{S}}\mathbf{g}) = \mathbf{P}:\operatorname{grad}\mathbf{g} = \operatorname{div}\mathbf{g} - \mathbf{n}\cdot(\operatorname{grad}\mathbf{g})\mathbf{n} = g_{i,i} - g_{i,k}n_i n_k$$

defines the *surface divergence*; $\partial/\partial n$ defined by

$$\frac{\partial \mathbf{g}}{\partial n} = (\operatorname{grad}\mathbf{g})\mathbf{n}$$

is the *normal derivative*. Then

$$\operatorname{grad}\mathbf{g} = \operatorname{grad}_{\mathcal{S}}\mathbf{g} + \frac{\partial \mathbf{g}}{\partial n}\otimes\mathbf{n}, \quad \operatorname{div}\mathbf{g} = \operatorname{div}_{\mathcal{S}}\mathbf{g} + \frac{\partial \mathbf{g}}{\partial n}\cdot\mathbf{n}. \tag{19}$$

The *surface divergence* of a tensor field $\mathbf{A}$ is the vector field defined by

$$(\operatorname{div}_{\mathcal{S}}\mathbf{A})_i = A_{ij,k}P_{kj}. \tag{20}$$

Using a smooth extension of the unit normal $\mathbf{n}$ to a neighborhood of $\mathcal{S}$, $\operatorname{grad}\mathbf{n}$ is defined and the field

$$\mathbf{K} = -\operatorname{grad}_{\mathcal{S}}\mathbf{n} = -(\operatorname{grad}\mathbf{n})\mathbf{P},$$

which is independent of the particular extension used, is the *curvature tensor* of $\mathcal{S}$; as is well known, $\mathbf{K}$ is symmetric and satisfies

$$\mathbf{K}\mathbf{n} = \mathbf{0}.$$

The scalar field

$$K = \tfrac{1}{2}\operatorname{tr}\mathbf{K} = -\tfrac{1}{2}\operatorname{div}_{\mathcal{S}}\mathbf{n}$$

is the *mean curvature* of $\mathcal{S}$.

Let $\mathbf{A}$ be a (second-order) tensor field and let $\mathbf{g}$ be a vector field. We make considerable use of the identities

$$\left.\begin{aligned} \operatorname{div}_{\mathcal{S}}(\mathbf{A}\mathbf{P}) &= \operatorname{div}_{\mathcal{S}}\mathbf{A} + 2K\mathbf{A}\mathbf{n}, \\ \operatorname{div}_{\mathcal{S}}(\mathbf{A}^{\top}\mathbf{g}) &= \mathbf{g}\cdot\operatorname{div}_{\mathcal{S}}\mathbf{A} + \mathbf{A}:\operatorname{grad}_{\mathcal{S}}\mathbf{g} \end{aligned}\right\} \tag{21}$$

and, in particular, their specializations

$$\operatorname{div}_{\mathcal{S}}\mathbf{P} = 2K\mathbf{n}, \quad \operatorname{div}_{\mathcal{S}}\left(\mathbf{A}^{\top}\mathbf{n}\right) = \mathbf{n}\cdot\operatorname{div}_{\mathcal{S}}\mathbf{A} - \mathbf{A}:\mathbf{K}, \tag{22}$$

which arise, respectively, on choosing $\mathbf{A} = \mathbf{1}$ in $(21)_1$ and $\mathbf{g} = \mathbf{n}$ in $(21)_2$.

Given a third-order tensor field $\mathbb{B}$ and a vector field $\mathbf{g}$, the product $\mathbb{B}\mathbf{g}$ is a (second-order) tensor field defined by

$$(\mathbb{B}\mathbf{g})_{ij} = B_{ijk}g_k. \tag{23}$$

In view of the definition (20) of the surface divergence, it then follows that $\operatorname{div}_{\mathcal{S}}(\mathbb{B}\mathbf{n})$ is the vector field with components

$$[\operatorname{div}_{\mathcal{S}}(\mathbb{B}\mathbf{n})]_i = B_{ijk,j}n_k - B_{ijk,l}n_l n_j n_k - B_{ijk}K_{jk}. \tag{24}$$

[8] The domain of $\operatorname{grad}_{\mathcal{S}}\mathbf{g}$ is then restricted to the surface $\mathcal{S}$, and similarly for $\operatorname{div}_{\mathcal{S}}\mathbf{g}$, $\partial\mathbf{g}/\partial n$, and $\operatorname{div}_{\mathcal{S}}\mathbf{A}$.

3. Power expenditures

Throughout this section R — with boundary $\mathcal{S}$ and outward unit normal $\mathbf{n}$ — is an arbitrary control volume.

3.1. Internal power

In discussing the manner in which power is expended internally, bear in mind that our goal is a theory that accounts explicitly for first and second velocity gradients. To accomplish this we generalize the classical theory — which has internal power of the form (1) with $\mathbf{T}$ the *stress* and $\mathbf{T} \colon \mathrm{grad}\,\mathbf{v}$ the stress power — by introducing a third-order *hyperstress* $\mathbb{G}$ with associated hyperstress power $\mathbb{G} \vdots \mathrm{grad}^2\mathbf{v}$, and therefore write the *internal power* in the form

$$\mathcal{W}_{\mathrm{int}}(R) = \int_R (\mathbf{T} \colon \mathrm{grad}\,\mathbf{v} + \mathbb{G} \vdots \mathrm{grad}^2\mathbf{v})\,\mathrm{d}v = \int_R (T_{ij}v_{i,j} + G_{ijk}v_{i,jk})\,\mathrm{d}v. \quad (25)$$

The fields $\mathbf{T}$ and $\mathbb{G}$ are defined over the deformed body for all time. Since $\mathrm{grad}^2\mathbf{v}$ is symmetric in its last two subscripts, we may, without loss in generality, require that $\mathbb{G}$ be *symmetric in its last two subscripts*:

$$G_{ijk} = G_{ikj}. \quad (26)$$

3.2. External power

Conventionally, power is expended on a control volume R by surface tractions acting on $\mathcal{S} = \partial R$ and body forces acting over R, and each of these force fields expends power (pointwise) over the velocity $\mathbf{v}$. Conventional continuum mechanics is based on a classical hypothesis of Cauchy asserting that the surface traction at a point $\mathbf{x}$ on $\mathcal{S}$ and time t be a function $\mathbf{t_n}(\mathbf{x}, t)$ of the normal $\mathbf{n}(\mathbf{x}, t)$. Here, as we shall see, *it is necessary to abandon this hypothesis* and assume instead that for each control volume R and each time t there is a *surface-traction* field $\mathbf{t}_{\mathcal{S}}$ defined over $\mathcal{S} = \partial R$ such that $\mathbf{t}_{\mathcal{S}}$ gives the surface force, per unit area, on $\mathcal{S}$.

As is classical, we assume that the body force is given by a field $\mathbf{b}$, and that both $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{b}$ are power conjugate to the velocity $\mathbf{v}$. Further, we stipulate that $\mathbf{b}$ *accounts for inertia*.

The external power expended on the boundary of the body sets the stage for the formulation of boundary conditions; this power should therefore be based on kinematical fields that — when restricted to the boundary — may be specified independently. Further, since the *internal* power depends on $\mathrm{grad}^2\mathbf{v}$, the *external power* should include a boundary expenditure involving $\mathrm{grad}\,\mathbf{v}$. However, the fields $\mathbf{v}$ and $\mathrm{grad}\,\mathbf{v}$ are kinematically coupled on $\mathcal{S}$, since a knowledge of $\mathbf{v}$ on $\mathcal{S}$ implies a knowledge of the tangential derivatives of $\mathbf{v}$ on $\mathcal{S}$; thus the *tangential* part of $\mathrm{grad}\,\mathbf{v}$ cannot be specified independently of $\mathbf{v}$. Bearing this in mind, we consider a (vectorial) *hypertraction* $\mathbf{m}_{\mathcal{S}}(\mathbf{x}, t)$ that expends power over the *normal* part $\partial\mathbf{v}/\partial n$ of the velocity gradient. Based on this discussion, we assume that the power expended externally on an arbitrary control volume R has the form

$$\mathcal{W}_{\text{ext}}(R) = \int\limits_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}} \cdot \mathbf{v} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \mathbf{v}}{\partial n} \right) \mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}v. \tag{27}$$

Since $\partial \mathbf{v}/\partial n = (\operatorname{grad} \mathbf{v})\mathbf{n}$, we may write

$$\mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \mathbf{v}}{\partial n} = (\mathbf{m}_{\mathcal{S}} \otimes \mathbf{n}) \,\vdots\, \operatorname{grad} \mathbf{v}; \tag{28}$$

we refer to $\mathbf{m}_{\mathcal{S}} \otimes \mathbf{n}$ as the *true hypertraction*, because its conjugate, $\operatorname{grad} \mathbf{v}$, carries no information associated with the surface $\mathcal{S}$.

4. Principle of virtual power

Most commonly, the principle of virtual power is used to generate weak formulations of boundary-value problems. In this form, the principle is stated for the body B as a whole and is contingent upon the provision of particular conditions on the boundary ∂B of B. Here, we use the principle of virtual power to *determine the structure of the tractions and of the local force balances*. This involves a *nonstandard* formulation in which the principle is stated for an arbitrary control volume R as opposed to the body as a whole. As such, conditions on ∂B play a role no different from those on the boundary ∂R of any control volume R. Basic to this view is the premise, central to all continuum mechanics, that any basic law for the body should hold also for all subregions of the body.

4.1. Statement of the principle of virtual power

To state this principal, assume that at some arbitrarily chosen but *fixed time*, the region occupied by the body is known, as are the tractions $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$, the body force $\mathbf{b}$, and the stresses $\mathbf{T}$ and $\mathbb{G}$, and consider the velocity field $\mathbf{v}$ as a virtual field $\tilde{\mathbf{v}}$ that may be specified *independently of the actual evolution of the body*. Then, writing

$$\left. \begin{aligned} \mathcal{W}_{\text{ext}}(R, \tilde{\mathbf{v}}) &= \int\limits_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}} \cdot \tilde{\mathbf{v}} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \tilde{\mathbf{v}}}{\partial n} \right) \mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \tilde{\mathbf{v}} \, \mathrm{d}v, \\ \mathcal{W}_{\text{int}}(R, \tilde{\mathbf{v}}) &= \int\limits_{R} (\mathbf{T} \,\vdots\, \operatorname{grad} \tilde{\mathbf{v}} + \mathbb{G} \,\vdots\, \operatorname{grad}\operatorname{grad} \tilde{\mathbf{v}}) \, \mathrm{d}v, \end{aligned} \right\} \tag{29}$$

respectively, for the external and internal expenditures of *virtual power*, the *principle of virtual power* is the requirement that *the external and internal powers be balanced*: given any control volume R,

$$\mathcal{W}_{\text{ext}}(R, \tilde{\mathbf{v}}) = \mathcal{W}_{\text{int}}(R, \tilde{\mathbf{v}}) \quad \text{for all virtual velocities } \tilde{\mathbf{v}}. \tag{30}$$

4.2. Consequences of the principle of virtual power

To determine the consequence of this principle, we first consider the individual terms in the internal power. Using the divergence theorem, we obtain

$$\int_R \mathbf{T} : \operatorname{grad}\tilde{\mathbf{v}}\,\mathrm{d}v = -\int_R \operatorname{div}\mathbf{T}\cdot\tilde{\mathbf{v}}\,\mathrm{d}v + \int_{\mathcal{S}} \mathbf{T}\mathbf{n}\cdot\tilde{\mathbf{v}}\,\mathrm{d}a. \tag{31}$$

Similarly, the divergence theorem applied twice yields

$$\begin{aligned}\int_R \mathbb{G} \vdots \operatorname{grad}\operatorname{grad}\tilde{\mathbf{v}}\,\mathrm{d}v &= -\int_R \operatorname{div}\mathbb{G} : \operatorname{grad}\tilde{\mathbf{v}}\,\mathrm{d}v + \int_{\mathcal{S}} \mathbb{G}\mathbf{n} : \operatorname{grad}\tilde{\mathbf{v}}\,\mathrm{d}a \\ &= \int_R (\operatorname{div}\operatorname{div}\mathbb{G})\cdot\tilde{\mathbf{v}}\,\mathrm{d}v + \int_{\mathcal{S}} \left(\mathbb{G}\mathbf{n} : \operatorname{grad}\tilde{\mathbf{v}} - [(\operatorname{div}\mathbb{G})\,\mathbf{n}]\cdot\tilde{\mathbf{v}}\right)\,\mathrm{d}a.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{W}_{\mathrm{int}}(R,\tilde{\mathbf{v}}) = &\int_R (\operatorname{div}\operatorname{div}\mathbb{G} - \operatorname{div}\mathbf{T})\cdot\tilde{\mathbf{v}}\,\mathrm{d}v \\ &+ \int_{\mathcal{S}} \left(\mathbb{G}\mathbf{n} : \operatorname{grad}\tilde{\mathbf{v}} + (\mathbf{T}\mathbf{n} - (\operatorname{div}\mathbb{G})\mathbf{n})\cdot\tilde{\mathbf{v}}\right)\,\mathrm{d}a.\end{aligned} \tag{32}$$

Further, by $(19)_1$,

$$\operatorname{grad}\tilde{\mathbf{v}} = \operatorname{grad}_{\mathcal{S}}\tilde{\mathbf{v}} + \frac{\partial\tilde{\mathbf{v}}}{\partial n}\otimes\mathbf{n};$$

therefore

$$\mathbb{G}\mathbf{n} : \operatorname{grad}\tilde{\mathbf{v}} = \mathbb{G}\mathbf{n} : \operatorname{grad}_{\mathcal{S}}\tilde{\mathbf{v}} + [(\mathbb{G}\mathbf{n})\,\mathbf{n}]\cdot\frac{\partial\tilde{\mathbf{v}}}{\partial n}$$

and (32) becomes

$$\begin{aligned}\mathcal{W}_{\mathrm{int}}(R,\tilde{\mathbf{v}}) = &\int_R (\operatorname{div}\operatorname{div}\mathbb{G} - \operatorname{div}\mathbf{T})\cdot\tilde{\mathbf{v}}\,\mathrm{d}v \\ &+ \int_{\mathcal{S}} \left(\underline{\mathbb{G}\mathbf{n} : \operatorname{grad}_{\mathcal{S}}\tilde{\mathbf{v}}} + [(\mathbb{G}\mathbf{n})\,\mathbf{n}]\cdot\frac{\partial\tilde{\mathbf{v}}}{\partial n} + (\mathbf{T}\mathbf{n} - (\operatorname{div}\mathbb{G})\mathbf{n})\cdot\tilde{\mathbf{v}}\right)\,\mathrm{d}a.\end{aligned} \tag{33}$$

Our next step is to establish an important identity for the underlined term in (33); specifically, letting $\mathbf{A} = \mathbb{G}\mathbf{n}$, we now show that

$$\int_{\mathcal{S}} \mathbf{A} : \operatorname{grad}_{\mathcal{S}}\tilde{\mathbf{v}}\,\mathrm{d}a = -\int_{\mathcal{S}} (\operatorname{div}_{\mathcal{S}}\mathbf{A} + 2K\mathbf{A}\mathbf{n})\cdot\tilde{\mathbf{v}}\,\mathrm{d}a. \tag{34}$$

The verification of (34) is based on the *surface divergence-theorem*: let $\boldsymbol{\tau}$ be a *tangential* vector field on $\mathcal{S}$ and let $\mathcal{T}$ be a subsurface of $\mathcal{S}$ with $\boldsymbol{\nu}$ the outward unit normal to the boundary curve $\partial\mathcal{T}$;[9] then

$$\int_{\partial\mathcal{T}} \boldsymbol{\tau}\cdot\boldsymbol{\nu}\, ds = \int_{\mathcal{T}} \mathrm{div}_{\mathcal{S}}\boldsymbol{\tau}\, \mathrm{d}a. \tag{35}$$

To establish (34), note that

$$\boldsymbol{\tau} \overset{\text{def}}{=} \mathbf{P}\mathbf{A}^{\top}\tilde{\mathbf{v}}$$

represents a tangential vector field, so that, by (35),

$$\int_{\partial\mathcal{T}} \boldsymbol{\tau}\cdot\boldsymbol{\nu}\, ds = \int_{\mathcal{T}} \mathrm{div}_{\mathcal{S}}\left(\mathbf{P}\mathbf{A}^{\top}\tilde{\mathbf{v}}\right)\, \mathrm{d}a. \tag{36}$$

Further, by $(21)_2$ — with $\mathbf{A}$ replaced by $\mathbf{AP}$ — and $(21)_1$,

$$\begin{aligned}\mathrm{div}_{\mathcal{S}}\left(\mathbf{P}\mathbf{A}^{\top}\tilde{\mathbf{v}}\right) &= \tilde{\mathbf{v}}\cdot\mathrm{div}_{\mathcal{S}}\,(\mathbf{AP}) + (\mathbf{AP}):\mathrm{grad}_{\mathcal{S}}\tilde{\mathbf{v}}\\ &= \tilde{\mathbf{v}}\cdot(\mathrm{div}_{\mathcal{S}}\mathbf{A} + 2K\mathbf{An}) + \mathbf{A}:\mathrm{grad}_{\mathcal{S}}\tilde{\mathbf{v}};\end{aligned}$$

hence (36) takes the form

$$\int_{\partial\mathcal{T}} \boldsymbol{\tau}\cdot\boldsymbol{\nu}\, \mathrm{d}s = \int_{\mathcal{T}} ((\mathrm{div}_{\mathcal{S}}\mathbf{A} + 2K\mathbf{An})\cdot\tilde{\mathbf{v}} + \mathbf{A}:\mathrm{grad}_{\mathcal{S}}\tilde{\mathbf{v}})\, \mathrm{d}a. \tag{37}$$

Finally, if we take $\mathcal{T} = \mathcal{S}$, then $\mathcal{T}$ is empty and the left side of (37) vanishes; thus (34) is satisfied.

Combining (33) and (34), we obtain

$$\begin{aligned}\mathcal{W}_{\mathrm{int}}(R,\tilde{\mathbf{v}}) = &\int_R (\mathrm{div}\,\mathrm{div}\,\mathbb{G} - \mathrm{div}\,\mathbf{T})\cdot\tilde{\mathbf{v}}\,\mathrm{d}v\\ &+\int_{\mathcal{S}}\left((\mathbf{Tn} - (\mathrm{div}\,\mathbb{G})\mathbf{n} - \mathrm{div}_{\mathcal{S}}(\mathbb{G}\mathbf{n}) - 2K(\mathbb{G}\mathbf{n})\mathbf{n})\cdot\tilde{\mathbf{v}} + [(\mathbb{G}\mathbf{n})\mathbf{n}]\cdot\frac{\partial\tilde{\mathbf{v}}}{\partial n}\right)\mathrm{d}a.\end{aligned} \tag{38}$$

We are now in a position to apply the virtual-power balance (30): by $(29)_1$ and (38),

$$\begin{aligned}\int_{\mathcal{S}}\left(\mathbf{t}_{\mathcal{S}}\cdot\tilde{\mathbf{v}} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\tilde{\mathbf{v}}}{\partial n}\right)\mathrm{d}a + \int_R \mathbf{b}\cdot\tilde{\mathbf{v}}\,\mathrm{d}v = &\int_R (\mathrm{div}\,\mathrm{div}\,\mathbb{G} - \mathrm{div}\,\mathbf{T})\cdot\tilde{\mathbf{v}}\,\mathrm{d}v\\ &+\int_{\mathcal{S}}\left((\mathbf{Tn} - (\mathrm{div}\,\mathbb{G})\mathbf{n} - \mathrm{div}_{\mathcal{S}}(\mathbb{G}\mathbf{n}) - 2K(\mathbb{G}\mathbf{n})\mathbf{n})\cdot\tilde{\mathbf{v}} + [(\mathbb{G}\mathbf{n})\mathbf{n}]\cdot\frac{\partial\tilde{\mathbf{v}}}{\partial n}\right)\mathrm{d}a,\end{aligned} \tag{39}$$

and rearranging (39), we have the “only if" implication in the next result.

[9] So that $\boldsymbol{\nu}$ is tangent to $\mathcal{S}$, normal to $\partial\mathcal{T}$, and directed outward from $\mathcal{T}$.

(#) *Given any virtual velocity* $\tilde{\mathbf{v}}$ *and any control volume R, the virtual balance*

$$\underbrace{\int_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}} \cdot \tilde{\mathbf{v}} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \tilde{\mathbf{v}}}{\partial n} \right) \mathrm{d}a + \int_R \mathbf{b} \cdot \tilde{\mathbf{v}} \, \mathrm{d}v}_{\mathcal{W}_{\mathrm{ext}}(R,\tilde{\mathbf{v}})} = \underbrace{\int_R (\mathbf{T} \colon \operatorname{grad} \tilde{\mathbf{v}} + \mathbb{G} \vdots \operatorname{grad}\operatorname{grad} \tilde{\mathbf{v}}) \, \mathrm{d}v}_{\mathcal{W}_{\mathrm{int}}(R,\tilde{\mathbf{v}})} . \tag{40}$$

is satisfied if and only if

$$\int_{\mathcal{S}} (\mathbf{t}_{\mathcal{S}} - (\mathbf{T}\mathbf{n} - (\operatorname{div} \mathbb{G})\mathbf{n} - \operatorname{div}_{\mathcal{S}}(\mathbb{G}\mathbf{n}) - 2K(\mathbb{G}\mathbf{n})\mathbf{n})) \cdot \tilde{\mathbf{v}} \, \mathrm{d}a$$
$$+ \int_{\mathcal{S}} (\mathbf{m}_{\mathcal{S}} - (\mathbb{G}\mathbf{n})\mathbf{n}) \cdot \frac{\partial \tilde{\mathbf{v}}}{\partial n} \, \mathrm{d}a = \int_R (\operatorname{div}\operatorname{div} \mathbb{G} - \operatorname{div} \mathbf{T} - \mathbf{b}) \cdot \tilde{\mathbf{v}} \, \mathrm{d}v. \tag{41}$$

The reverse implication, that (41) implies (40), follows upon reversing the argument leading to (40).

4.3. Local force balance and traction conditions

Since the control volume R and the virtual field $\tilde{\mathbf{v}}$ in (41) *may be arbitrarily chosen*, we may appeal to the the fundamental lemma of the calculus of variations and arrive at the *local force balance*

$$\operatorname{div}(\mathbf{T} - \operatorname{div} \mathbb{G}) + \mathbf{b} = \mathbf{0} \quad \big(T_{ij,j} - G_{ijk,jk} + b_i = 0\big) \tag{42}$$

and the *traction conditions*[10]

$$\mathbf{t}_{\mathcal{S}} = \mathbf{T}\mathbf{n} - (\operatorname{div} \mathbb{G})\mathbf{n} - \operatorname{div}_{\mathcal{S}}(\mathbb{G}\mathbf{n}) - 2K(\mathbb{G}\mathbf{n})\mathbf{n}, \quad \mathbf{m}_{\mathcal{S}} = (\mathbb{G}\mathbf{n})\mathbf{n}, \tag{43}$$

which, in components, have the form

$$\left.\begin{aligned} (\mathbf{t}_{\mathcal{S}})_i &= T_{ij}n_j - 2G_{ijk,k}n_j + G_{ijk,l}n_j n_k n_l + G_{ijk}(K_{jk} - 2Kn_j n_k), \\ (\mathbf{m}_{\mathcal{S}})_i &= G_{ijk}n_j n_k. \end{aligned}\right\} \tag{44}$$

Next, recall our agreement that $\mathbf{b}$ includes inertial body forces. Thus, if the underlying frame is inertial, then

$$\mathbf{b} = -\rho\dot{\mathbf{v}} + \mathbf{b}^{\mathrm{ni}} \tag{45}$$

with $\mathbf{b}^{\mathrm{ni}}$ the noninertial body force and the local force balance becomes

$$\rho\dot{\mathbf{v}} = \operatorname{div} \mathbf{T} - \operatorname{div}\operatorname{div} \mathbb{G} + \mathbf{b}^{\mathrm{ni}} \quad \big(\rho\dot{v}_i = T_{ij,j} - G_{ijk,jk} + b_i^{\mathrm{ni}}\big). \tag{46}$$

[10] Since $\tilde{\mathbf{v}}$ is arbitrary, $\tilde{\mathbf{v}}$ and $\partial\tilde{\mathbf{v}}/\partial n$ may be arbitrarily chosen independent of one another on $\mathcal{S}$ (cf. the paragraph containing (27)).

4.4. Traction conditions when the boundary of the control volume is not smooth

Thus far the control volumes R under consideration were assumed to have smooth boundaries. We now consider a control volume whose boundary $\mathcal{S}$ is piecewise smooth: specifically, $\mathcal{S}$ is the union of a finite number of smooth surfaces, smooth curves, and points, with the smooth curves referred to as *edges*. When this is the case — writing $\mathcal{C}$ for the union of the edges — we allow for the possibility of an *edge traction* $\mathbf{f}_{\mathcal{C}}$ measured per unit length along $\mathcal{C}$ and add the terms

$$\int_{\mathcal{C}} \mathbf{f}_{\mathcal{C}} \cdot \mathbf{v}\, ds \quad \text{and} \quad \int_{\mathcal{C}} \mathbf{f}_{\mathcal{C}} \cdot \tilde{\mathbf{v}}\, ds, \tag{47}$$

respectively, to the right sides of the external power (27) and its virtual counterpart $(29)_2$. The corresponding internal power expenditure remains (25), but the analysis used in finding the consequences of the virtual-power principle must be altered. Specifically, the identity (34) — which followed from the surface divergence theorem (35) with $\mathcal{T} = \mathcal{S}$ — is no longer valid: we must now apply (35) on each of the smooth surfaces $\mathcal{S}_k$ comprising $\mathcal{S}$ and then sum over all k. If we do this we find that the term

$$\int_{\mathcal{C}} \{\!\{(\mathbb{G}\mathbf{n})\boldsymbol{\nu}\}\!\} \cdot \tilde{\mathbf{v}}\, ds \tag{48}$$

must be added to the right side of (38), where $\{\!\{\dots\}\!\}$, at each point $\mathbf{x}$ on $\mathcal{C}$, denotes the sum of the values of the enclosed quantity as $\mathbf{x}$ is approached from the surfaces on either side of $\mathcal{C}$. (If $\mathcal{S}$ is smooth, then the values of $\boldsymbol{\nu}$ on the two sides of each edge are equal and opposite, while the values of $\mathbf{n}$ are equal; thus, granted that $\mathbb{G}$ is smooth, (48) vanishes.) Continuing the analysis with the extra term (48), we find that the *edge condition*

$$\mathbf{f}_{\mathcal{C}} = \{\!\{(\mathbb{G}\mathbf{n})\boldsymbol{\nu}\}\!\} \tag{49}$$

on $\mathcal{C}$ must be added to the traction conditions (43) (cf. equation (7.10) TOUPIN) [40].

4.5. Consequences of frame indifference

Frame indifference requires that the theory be invariant under all changes in frame. In accordance with this principle we require that the internal power be invariant under transformations of the form

$$\tilde{\mathbf{v}}(\mathbf{x}, t) \mapsto \tilde{\mathbf{v}}(\mathbf{x}, t) + \underbrace{\boldsymbol{\alpha}(t) + \boldsymbol{\Omega}(t)\mathbf{x}}_{\mathbf{w}(\mathbf{x},t)}, \tag{50}$$

where $\boldsymbol{\alpha}(t)$ is an arbitrary scalar and $\boldsymbol{\Omega}(t)$ is an arbitrary skew tensor, at each t. It then follows, as a consequence of the virtual balance (30), that *the external power is automatically consistent with frame indifference*.

Consider the internal power. By (50), the velocity gradient transforms according to $\operatorname{grad}\mathbf{v}(\mathbf{x},t) \mapsto \operatorname{grad}\mathbf{v}(\mathbf{x},t) + \mathbf{\Omega}$. We may therefore conclude from (25) that for the internal power to be frame indifferent we must have

$$\int_R \mathbf{T}:\mathbf{\Omega}\,\mathrm{d}v = 0$$

for all skew tensors $\mathbf{\Omega}$ and all control volumes R; hence the stress $\mathbf{T}$ is *symmetric*:

$$\mathbf{T} = \mathbf{T}^{\top}. \tag{51}$$

A consequence of (51) is that the *stress power* $\mathbf{T}:\operatorname{grad}\mathbf{v}$ in the internal power (25) may equally well be written in the form $\mathbf{T}:\mathbf{D}$, with $\mathbf{D}$ the stretching defined in $(17)_1$.

We now turn to the external power (27), which is automatically frame indifferent: invariance under (50) implies that[11]

$$\left.\begin{aligned} \int_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}\,\mathrm{d}a + \int_R \mathbf{b}\,\mathrm{d}v = \mathbf{0}, \\ \int_{\mathcal{S}} (\mathbf{x}\times\mathbf{t}_{\mathcal{S}} + \mathbf{n}\times\mathbf{m}_{\mathcal{S}})\,\mathrm{d}a + \int_R \mathbf{x}\times\mathbf{b}\,\mathrm{d}v = \mathbf{0}, \end{aligned}\right\} \tag{52}$$

which bears similarity to its classical conterparts in which $\mathbf{t}_{\mathcal{S}} = \mathbf{T}\mathbf{n}$ and $\mathbf{m}_{\mathcal{S}} = \mathbf{0}$. Thus, bearing in mind the paragraph containing (28), the skew part of the true hypertraction $\mathbf{m}_{\mathcal{S}}\otimes\mathbf{n}$ represents a distribution of moments on $\mathcal{S}$.

Our formulation of the virtual-power principle ensures that the classical balances (52) are satisfied automatically.

4.6. Locality of the tractions. Action-reaction principle

A consequence of (44) is that the *tractions are local*: at any point $\mathbf{x}$ on $\mathcal{S}$, $\mathbf{t}_{\mathcal{S}}(\mathbf{x})$ depends on $\mathcal{S}$ through a dependence on the normal $\mathbf{n}(\mathbf{x})$ and curvature $\mathbf{K}(\mathbf{x})$ at $\mathbf{x}$, while $\mathbf{m}_{\mathcal{S}}(\mathbf{x})$ depends on $\mathcal{S}$ through $\mathbf{n}(\mathbf{x})$ (where for convenience we have suppressed the argument t). Thus, writing $\mathbf{t}_{(\mathbf{K},\mathbf{n})}$ and $\mathbf{m}_{\mathbf{n}}$ for the corresponding functions, we obtain

$$\mathbf{t}_{\mathcal{S}} = \mathbf{t}_{(\mathbf{K},\mathbf{n})}, \quad \mathbf{m}_{\mathcal{S}} = \mathbf{m}_{\mathbf{n}} \tag{53}$$

i.e. $\mathbf{t}_{\mathcal{S}}(\mathbf{x}) = \mathbf{t}_{(\mathbf{K}(\mathbf{x}),\mathbf{n}(\mathbf{x}))}(\mathbf{x})$. Then, letting $-\mathcal{S}$ denote for the surface $\mathcal{S}$ oriented by $-\mathbf{n}$ (which has curvature $-\mathbf{K}$), we see that by (44), $\mathbf{m}_{\mathcal{S}} = \mathbf{m}_{-\mathcal{S}}$ and

$$\mathbf{t}_{\mathcal{S}} = -\mathbf{t}_{-\mathcal{S}}, \quad \mathbf{m}_{\mathcal{S}}\otimes\mathbf{n} = -\mathbf{m}_{-\mathcal{S}}\otimes(-\mathbf{n}), \tag{54}$$

or equivalently, $\mathbf{m}_{\mathbf{n}} = \mathbf{m}_{-\mathbf{n}}$ and

$$\mathbf{t}_{(\mathbf{K},\mathbf{n})} = -\mathbf{t}_{(-\mathbf{K},-\mathbf{n})}, \quad \mathbf{m}_{\mathbf{n}}\otimes\mathbf{n} = -\mathbf{m}_{-\mathbf{n}}\otimes(-\mathbf{n}); \tag{55}$$

[11] When combined with (45), (52) represent balances for linear and angular momentum.

(55) represents an *action-reaction principle* for oppositely oriented surfaces that touch and are tangent at a point.

Consider an *arbitrary surface* $\mathcal{S}$ with orientation $\mathbf{n}$ and define the plus side of $\mathcal{S}$ as the side into which $\mathbf{n}$ points and the minus side as the other side. In the definition (27) of the external power the quantity $\mathcal{W}_{\rm surf}(\mathcal{S})$ defined by

$$\begin{aligned}\mathcal{W}_{\rm surf}(\mathcal{S}) &= \int\limits_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\right)\,{\rm d}a \\ &= \int\limits_{\mathcal{S}} (\mathbf{t}_{(\mathbf{K},\mathbf{n})}\cdot\mathbf{v} + (\mathbf{m}_{\mathbf{n}}\otimes\mathbf{n}):\operatorname{grad}\mathbf{v})\,{\rm d}a\end{aligned} \tag{56}$$

represents the power expended on the boundary of a control volume. However, because the tractions are local, this definition is also meaningful for an arbitrary surface $\mathcal{S}$ with orientation $\mathbf{n}$. In this instance, $\mathcal{W}_{\rm surf}(\mathcal{S})$ represents the power expended *by* the material on the plus side of $\mathcal{S}$ *on* the material on the minus side of $\mathcal{S}$, so that, by (55), we have the *power balance*

$$\mathcal{W}_{\rm surf}(\mathcal{S}) = -\mathcal{W}_{\rm surf}(-\mathcal{S}). \tag{57}$$

The balance (57) written in the form $\mathcal{W}_{\rm surf}(\mathcal{S}) + \mathcal{W}_{\rm surf}(-\mathcal{S}) = 0$ constitutes a balance for the "pillbox" represented by the infinitesimally thin region bounded by the oriented surfaces $\mathcal{S}$ and $-\mathcal{S}$, a notion discussed in Section 8.2.

5. Free-energy imbalance. Dissipation inequality

Let $\mathcal{R}(t)$ be an arbitrary region that convects with the body. We restrict attention to a purely mechanical theory based on the requirement that *the temporal increase in free energy of* $\mathcal{R}(t)$ *be less than or equal to the power expended on* $\mathcal{R}(t)$. Precisely, letting ψ denote the *specific free energy*, this requirement takes the form of a free-energy imbalance

$$\frac{\rm d}{{\rm d}t}\int\limits_{\mathcal{R}(t)} \rho\psi\,{\rm d}v \leqq \mathcal{W}_{\rm ext}(\mathcal{R}(t)) = \mathcal{W}_{\rm int}(\mathcal{R}(t)). \tag{58}$$

Balance of mass implies that $({\rm d}/{\rm d}t)\int_{\mathcal{R}(t)}\rho\psi\,{\rm d}v = \int_{R(t)}\rho\dot{\psi}\,{\rm d}v$. Therefore, using the expression (25) for the internal power $\mathcal{W}_{\rm int}(\mathcal{R}(t))$ in conjunction with the symmetry of $\mathbf{T}$, we may localize (58) to yield the *local free-energy imbalance*

$$\rho\dot{\psi} - \mathbf{T}:\mathbf{D} - \mathbb{G}\vdots\operatorname{grad}^2\mathbf{v} \leqq 0 \quad \left(\rho\dot{\psi} - T_{ij}D_{ij} - G_{ijk}v_{i,jk} \leqq 0\right), \tag{59}$$

where $\mathbf{D}$ is the stretching defined in $(17)_1$. The difference

$$\Gamma \stackrel{\rm def}{=} \mathbf{T}:\mathbf{D} + \mathbb{G}\vdots\operatorname{grad}^2\mathbf{v} - \rho\dot{\psi} \geqq 0 \tag{60}$$

represents the *dissipation* and allows us to rewrite (58) in the form

$$\frac{\rm d}{{\rm d}t}\int\limits_{\mathcal{R}(t)} \rho\psi\,{\rm d}v - \mathcal{W}_{\rm ext}(\mathcal{R}(t)) = -\int\limits_{\mathcal{R}(t)} \Gamma\,{\rm d}v \leqq 0. \tag{61}$$

6. Explicit form of the inertial body force. Kinetic energy

6.1. Inertia and kinetic energy

We assume that the underlying frame is *inertial* and, for convenience, neglect noninertial body forces, so that

$$\mathbf{b} = -\rho\dot{\mathbf{v}}. \tag{62}$$

Granted this, the local force balance (42) reduces to a local *momentum balance*:

$$\operatorname{div}(\mathbf{T} - \operatorname{div}\mathbb{G}) = \rho\dot{\mathbf{v}}. \tag{63}$$

The power expended by the body force has the form

$$\mathbf{b}\cdot\mathbf{v} = -\tfrac{1}{2}\rho\dot{\overline{|\mathbf{v}|^2}},$$

and we may rewrite the external power expenditure as the sum of a noninertial expenditure minus a kinetic-energy rate:

$$\mathcal{W}_{\text{ext}}(\mathcal{R}(t)) = \underbrace{\int\limits_{\partial\mathcal{R}(t)} \left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\right) \mathrm{d}a}_{\text{noninertial power expenditure}} - \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int\limits_{\mathcal{R}(t)} \tfrac{1}{2}\rho|\mathbf{v}|^2\,\mathrm{d}v}_{\text{kinetic energy}}. \tag{64}$$

6.2. Imbalance of free and kinetic energy

By (64), the free-energy imbalance (61) — for a control volume $\mathcal{R}(t)$ that convects with the fluid — takes the form of an imbalance of free and kinetic energy

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{R}(t)} \rho(\psi + \tfrac{1}{2}|\mathbf{v}|^2)\,\mathrm{d}v}_{\text{net energy rate}} - \int\limits_{\partial\mathcal{R}(t)} \left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\right) \mathrm{d}a = -\underbrace{\int\limits_{\mathcal{R}(t)} \Gamma\,\mathrm{d}v}_{\text{net dissipation}} \leqq 0. \tag{65}$$

Further, appealing to (#) on page 525, we may rewrite (65) as an imbalance for a control volume R:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{R} \rho\big(\psi + \tfrac{1}{2}|\mathbf{v}|^2\big)\,\mathrm{d}v + \int\limits_{\mathcal{S}} \rho\big(\psi + \tfrac{1}{2}|\mathbf{v}|^2\big)\,\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a \\ - \int\limits_{\mathcal{S}} \left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\right) \mathrm{d}a = -\int\limits_{R} \Gamma\,\mathrm{d}v \leqq 0.\end{aligned} \tag{66}$$

7. Application of the theory to the flow of an incompressible fluid at small-length scales

7.1. Constitutive equations for an incompressible fluid

We assume that the fluid is incompressible, so that

$$\operatorname{div}\mathbf{v} = \operatorname{tr}\mathbf{D} = 0, \quad \rho = \text{constant}, \quad \psi = \text{constant}. \tag{67}$$

Without loss in generality, we may then suppose that

$$T_{ij} = T_{0ij} - P\delta_{ij}, \quad T_{0kk} = 0 \tag{68}$$

and

$$G_{ijk} = G_{0ijk} - \delta_{ij}\pi_k, \quad G_{0iik} = 0, \tag{69}$$

where the *pressure* P and the *hyperpressure* $\boldsymbol{\pi}$ are constitutively indeterminate fields that do not affect the internal power (25)[12] and where the fields $\mathbf{T}_0$ and $\mathbb{G}_0$, respectively, represent the *extra stress* and the *extra hyperstress*. Then

$$\mathbf{T}:\mathbf{D} = \mathbf{T}_0:\mathbf{D}, \quad \mathbb{G}\vdots\operatorname{grad}^2\mathbf{v} = \mathbb{G}_0\vdots\operatorname{grad}^2\mathbf{v}, \tag{70}$$

and the local free-energy imbalance (59) reduces to the *dissipation inequality*

$$\Gamma \stackrel{\text{def}}{=} \mathbf{T}_0:\mathbf{D} + \mathbb{G}_0\vdots\operatorname{grad}^2\mathbf{v} \geqq 0. \tag{71}$$

The field Γ represents the *bulk dissipation*, measured per unit volume.

We consider constitutive equations giving $\mathbf{T}_0$ and $\mathbb{G}_0$ as linear *isotropic* functions of $\mathbf{D}$ and $\operatorname{grad}^2\mathbf{v}$. Specifically, since coupling between $\mathbf{T}_0$ and $\mathbb{G}_0$ is ruled out by isotropy, we consider constitutive relations of the form

$$\left.\begin{aligned} T_{0ij} &= 2\mu D_{ij} = \mu(v_{i,j} + v_{j,i}), \\ G_{0ijk} &= \eta_1 v_{i,jk} + \eta_2(v_{k,ij} + v_{j,ik} - v_{i,rr}\delta_{jk}). \end{aligned}\right\} \tag{72}$$

The relation $(72)_1$ for $\mathbf{T}_0$ is of the form $\mathbf{T}_0 = 2\mu\mathbf{D}$, familiar from the theory of incompressible, linearly viscous fluids.[13]

Using (17), we can rewrite the constitutive relation $(72)_2$ for $\mathbb{G}_0$ in terms of the gradients of the stretching and the spin and the Laplacian of the velocity field:

$$\begin{aligned} G_{0ijk} = (\eta_1 + \eta_2)\, D_{ij,k} + (\eta_1 - \eta_2)\, W_{ij,k} \\ +\eta_2\left(D_{ki,j} + W_{ki,j}\right) - \eta_2 v_{i,rr}\delta_{jk}. \end{aligned} \tag{73}$$

---

[12] As we shall see, the pressure relevant to the local balance of linear momentum is the *effective pressure* $p = P - \operatorname{div}\boldsymbol{\pi}$.

[13] We conjecture that $(72)_2$ is the most general linear, isotropic relation possible between $\mathbb{G}_0$ and $\operatorname{grad}^2\mathbf{v}$ (cf. (2.4) and (2.18) of MINDLIN & ESHEL [24]).

Thus the dissipation (71) has the form

$$\Gamma = 2\mu D_{ij} D_{ij} + (\eta_1 + 2\eta_2)\, D_{ij,k} D_{ij,k} \\ + (\eta_1 - 2\eta_2)\, W_{ij,k} W_{ij,k} - \eta_2 v_{i,jj} v_{i,kk} \geqq 0. \tag{74}$$

Further, by expanding $D_{ij,k}$ and $W_{ij,k}$ into deviatoric and spherical parts with respect to the indices i and k (j fixed), it can be shown that the following conditions are both necessary and sufficient for the dissipation to be nonnegative:

$$\mu \geqq 0, \quad \eta_1 + 2\eta_2 \geqq 0, \quad \eta_1 - 6\eta_2 \geqq 0. \tag{75}$$

Important consequences of (75) are the inequalities

$$\eta_1 \geqq 0, \quad \eta_1 - \eta_2 \geqq 0, \quad \eta_1 - 2\eta_2 \geqq 0. \tag{76}$$

Conversely, by (71) and (72), the dissipation has the form

$$\Gamma = 2\mu|\mathbf{D}|^2 + \eta_1|\mathrm{grad}\,\mathbf{v}|^2 + \eta_2(2|\mathrm{grad}\,\mathbf{D}|^2 - 2|\mathrm{grad}\,\mathbf{W}|^2 - |\triangle\mathbf{v}|^2) \tag{77}$$

and is nonnegative as long as (75) are satisfied.

Finally, using (68) and (69), we may rewrite the constitutive relations (72) taking into account the pressure P and hyperpressure $\boldsymbol{\pi}$:

$$\left.\begin{aligned} T_{ij} &= 2\mu D_{ij} - P\delta_{ij}, \\ G_{ijk} &= \eta_1 v_{i,jk} + \eta_2(v_{k,ij} + v_{j,ik} - v_{i,rr}\delta_{jk}) - \pi_k\delta_{ij}. \end{aligned}\right\} \tag{78}$$

7.2. The flow equation

Bearing in mind $(67)_1$, we may use (78) to yield

$$\left.\begin{aligned} T_{ij,j} &= \mu v_{i,jj} - P_{,i}, \\ G_{ijk,k} &= \eta_1 v_{i,jkk} + \eta_2(v_{j,ikk} - v_{i,jkk}) - \delta_{ij}\pi_{k,k}, \\ G_{ijk,kj} &= (\eta_1 - \eta_2) v_{i,jjkk} - \pi_{j,ji}. \end{aligned}\right\} \tag{79}$$

Thus, letting

$$\zeta = \eta_1 - \eta_2 \geqq 0, \tag{80}$$

we may conclude from (79) that

$$\mathrm{div}\,(\mathbf{T} - \mathrm{div}\,\mathbf{G}) = -\mathrm{grad}\,(P - \mathrm{div}\,\boldsymbol{\pi}) + \mu\triangle\mathbf{v} - \zeta\triangle\triangle\mathbf{v}, \tag{81}$$

and — neglecting noninertial body forces as in Section 6 — the local momentum balance (63) takes the form

$$\rho\dot{\mathbf{v}} = -\mathrm{grad}\,p + \mu\triangle\mathbf{v} - \zeta\triangle\triangle\mathbf{v} \tag{82}$$

or, in components,

$$\rho\dot{v}_i = -p_{,i} + \mu v_{i,jj} - \zeta v_{i,jjkk}, \tag{83}$$

where

$$p = P - \operatorname{div} \boldsymbol{\pi} \tag{84}$$

is the *effective pressure*. We refer to (82) as the *flow equation*. Provided that $\zeta > 0$, the flow equation is *parabolic*. Moreover, for the choice $\eta = 0$, the flow equation reduces to the conventional Navier–Stokes equation $\rho\dot{\mathbf{v}} = -\operatorname{grad} p + \mu\triangle\mathbf{v}$ for an incompressible fluid.

Consistent with the expectation that gradient effects should be important only at small-length scales, we introduce the *gradient length*

$$L = \sqrt{\frac{\zeta}{\mu}}. \tag{85}$$

7.3. Spin and vorticity equations

Assume that the noninertial body force $\mathbf{b}^{\mathrm{ni}}$ vanishes. Then the flow equation leads to an interesting equation for the spin $\mathbf{W}$. A well-known kinematical relation for the spin has the form (for example cf. GURTIN [11] pp. 80)

$$\dot{\mathbf{W}} + \mathbf{DW} + \mathbf{WD} = \operatorname{skw}(\operatorname{grad}\dot{\mathbf{v}}), \tag{86}$$

where $\operatorname{skw}\mathbf{A} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^{\top}\right)$ for any tensor $\mathbf{A}$. Next, since the operators grad and $\triangle$ commute, (67) and (82) imply that

$$\rho\operatorname{grad}\dot{\mathbf{v}} = -\operatorname{grad}\operatorname{grad} p + \mu\triangle\operatorname{grad}\mathbf{v} - \zeta\triangle\triangle\operatorname{grad}\mathbf{v}.$$

Taking the skew part of this relation we find that since $\operatorname{grad}\operatorname{grad} p$ is symmetric,

$$\rho\operatorname{skw}(\operatorname{grad}\dot{\mathbf{v}}) = \mu\triangle\mathbf{W} - \zeta\triangle\triangle\mathbf{W};$$

thus, by (86), we have the *spin equation*

$$\rho(\dot{\mathbf{W}} + \mathbf{DW} + \mathbf{WD}) = \mu\triangle\mathbf{W} - \zeta\triangle\triangle\mathbf{W}, \tag{87}$$

or, in components,

$$\rho\dot{W}_{ij} + D_{ik}W_{kj} + W_{ik}D_{kj} = \mu W_{ij,kk} - \zeta W_{ij,kkll}, \tag{88}$$

which, when $\zeta = 0$, reduces to the classical spin equation (for example cf. GURTIN [11] pp. 152).

Equivalently, writing $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$ for the *vorticity* and making use of the relations $\operatorname{div}\mathbf{v} = 0$, $\operatorname{div}\boldsymbol{\omega} = \operatorname{div}(\operatorname{curl}\mathbf{v}) = 0$, $\mathbf{Lv} = \operatorname{grad}\left(\frac{1}{2}|\mathbf{v}|^2\right) + \boldsymbol{\omega}\times\mathbf{v}$, and $\mathbf{W}\boldsymbol{\omega} = \mathbf{0}$, we have the *vorticity equation*

$$\rho\dot{\boldsymbol{\omega}} - \mathbf{D}\boldsymbol{\omega} = \mu\triangle\boldsymbol{\omega} - \zeta\triangle\triangle\boldsymbol{\omega}. \tag{89}$$

7.4. Free-energy imbalance

Next, provided that the noninertial body force $\mathbf{b}^{\mathrm{ni}}$ vanishes, we may use (60) to write the free-energy imbalance (66) (for a control volume R) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_R \tfrac{1}{2}\rho|\mathbf{v}|^2\,\mathrm{d}v + \int\limits_{\mathcal{S}} \tfrac{1}{2}\rho|\mathbf{v}^2|\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a - \int\limits_{\mathcal{S}}\left(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\right)\mathrm{d}a$$
$$= -\int\limits_R \big(2\mu|\mathbf{D}|^2 + \eta_1|\mathrm{grad}\,\mathbf{v}|^2$$
$$+\eta_2\big(2|\mathrm{grad}\,\mathbf{D}|^2 - 2|\mathrm{grad}\,\mathbf{W}|^2 - |\triangle\mathbf{v}|^2\big)\big)\,\mathrm{d}v \leqq 0. \tag{90}$$

8. Formulation of boundary conditions

This section is independent of constitutive equations and is valid for both compressible and incompressible materials. Even so, the boundary conditions we consider are most relevant to liquids and, for that reason, we find it convenient to refer to the body as a liquid.

Let $B(t)$ denote the region of space occupied by the liquid at an arbitrarily chosen time and let $\mathbf{n}(\mathbf{x}, t)$ denote the outward unit normal to $\partial B(t)$. Further, let $V(\mathbf{x}, t)$ denote the scalar normal velocity of $\partial B(t)$, so that

$$V = \mathbf{v}\cdot\mathbf{n}. \tag{91}$$

Unless mentioned to the contrary, we assume that $\partial B(t)$ is smooth. In Section 8.2.5 we discuss boundary conditions for situations in which $\partial B(t)$ is not smooth.

8.1. Free energy imbalance for a boundary pillbox

Consider an *arbitrary* evolving subsurface $\mathcal{S}(t)$ of $\partial B(t)$ and let $V_{\partial\mathcal{S}}(\mathbf{x}, t)$ denote the normal velocity of the boundary curve $\partial\mathcal{S}(t)$ in the direction of its outward unit normal $\boldsymbol{\nu}(\mathbf{x}, t)$ (cf. Footnote 9, pp. 524). We view $\mathcal{S}$ as a *boundary pillbox* of infinitesimal thickness containing a portion of the boundary — a view that allows us to isolate the physical processes in the material on the two sides of the boundary. The geometric boundary of $\mathcal{S}$ consists of its boundary curve $\partial\mathcal{S}$. However, $\mathcal{S}$ viewed as pillbox has a *pillbox boundary* consisting of [14]:

(i) a surface $\mathcal{S}$ with unit normal $\mathbf{n}$; $\mathcal{S}$ is viewed as lying in the *environment*;
(ii) a surface $-\mathcal{S}$ with unit normal $-\mathbf{n}$; $-\mathcal{S}$ is viewed as lying in the fluid adjacent to the boundary;
(iii) a "lateral face" represented by $\partial\mathcal{S}$ (see Figure 1).

[14] In a forthcoming paper on turbulence, we find it more convenient to use a pillbox with surface $\mathcal{S}$ lying at the interface between the fluid and the environment.

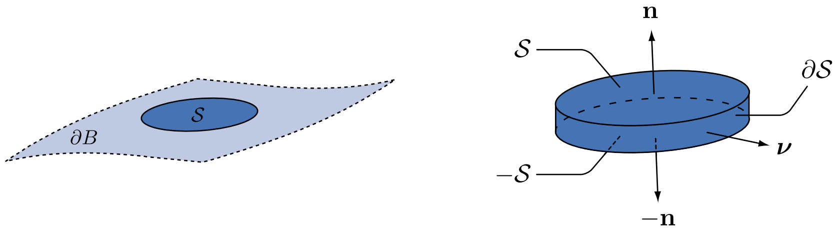

Fig. 1. Pillbox corresponding to a subsurface $\mathcal{S}$ of the boundary ∂B of the region B of space occupied by the body. Only a portion of ∂B is depicted. Whereas $\mathbf{n}$ is oriented into the environment, $-\mathbf{n}$ is oriented into the fluid. The outward unit normal on the lateral face $\partial\mathcal{S}$ of the pillbox is denoted by $\boldsymbol{\nu}$.

We base our discussion on a free-energy imbalance for the pillbox $\mathcal{S}(t)$ requiring that *the temporal increase in free energy of* $\mathcal{S}(t)$ *be less than or equal to the power expended on* $\mathcal{S}(t)$ (cf. (58)). We let ψ^{x} denote the *excess free energy* and σ the *surface tension* of the fluid at the boundary, so that $\int_{\mathcal{S}} \psi^{\mathrm{x}}\,\mathrm{d}a$ represents the net free energy of the pillbox, while $\int_{\partial\mathcal{S}} \sigma V_{\partial\mathcal{S}}\,\mathrm{d}s$ represents the power expended by the fluid on the lateral face of the pillbox by surface tension. Further, in view of the paragraph containing (56), $\mathcal{W}_{\mathrm{surf}}(-\mathcal{S})$ represents the power expended by the fluid on the pillbox surface $-\mathcal{S}$. Finally, denoting by $\mathcal{W}_{\mathrm{env}}(\mathcal{S})$ the power expended by the environment on the pillbox surface $\mathcal{S}$, we express the *free-energy imbalance* for the pillbox as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}(t)} \psi^{\mathrm{x}}\,\mathrm{d}a \leqq \mathcal{W}_{\mathrm{surf}}(-\mathcal{S}(t)) + \mathcal{W}_{\mathrm{env}}(\mathcal{S}(t)) + \int_{\partial\mathcal{S}(t)} \sigma V_{\partial\mathcal{S}}\,\mathrm{d}s. \tag{92}$$

Assuming that ψ^{x} is constant, we use a standard transport theorem to yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}(t)} \psi^{\mathrm{x}}\,\mathrm{d}a = -\int_{\mathcal{S}(t)} 2\psi^{\mathrm{x}} K V\,\mathrm{d}a + \int_{\partial\mathcal{S}(t)} \psi^{\mathrm{x}} V_{\partial\mathcal{S}}\,\mathrm{d}s$$

(e.g. CERMELLI, FRIED & GURTIN [4] equation (3.10)), so that (92) becomes

$$\mathcal{W}_{\mathrm{surf}}(-\mathcal{S}) + \int_{\mathcal{S}} 2\psi^{\mathrm{x}} K V\,\mathrm{d}a + \mathcal{W}_{\mathrm{env}}(\mathcal{S}) + \int_{\partial\mathcal{S}} (\sigma - \psi^{\mathrm{x}}) V_{\partial\mathcal{S}}\,\mathrm{d}s \geqq 0. \tag{93}$$

Thus, since $V_{\partial\mathcal{S}}$ may be arbitrarily specified at any time without affecting the other quantities in (93) at that time, we must have the classical result

$$\sigma = \psi^{\mathrm{x}}$$

and, using (56), (57), and (91), we find that the inequality (93) reduces to

$$-\int_{\mathcal{S}} \left((\mathbf{t}_{\mathcal{S}} - 2\sigma K \mathbf{n})\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial \mathbf{v}}{\partial n} \right) \mathrm{d}a + \mathcal{W}_{\mathrm{env}}(\mathcal{S}) \geqq 0. \tag{94}$$

8.2. Boundary conditions for a passive environment

We restrict attention to situations in which the environment expends no power on the body. Such an environment, termed *passive*, is defined by the requirement that, for any motion of the fluid and any evolving subsurface $\mathcal{S}(t)$ of $\partial B(t)$[15],

$$\mathcal{W}_{\text{env}}(\mathcal{S}) = 0.$$

Granted this, the free-energy imbalance (94) becomes

$$\int_{\mathcal{S}} \left((\mathbf{t}_{\mathcal{S}} - 2\sigma K\mathbf{n})\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n} \right) \mathrm{d}a \leqq 0, \tag{95}$$

and, since the choice of $\mathcal{S}$ is arbitrary, (95) is equivalent to the *boundary dissipation inequality*

$$\gamma \overset{\text{def}}{=} -(\mathbf{t}_{\mathcal{S}} - 2\sigma K\mathbf{n})\cdot\mathbf{v} - \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n} \geqq 0, \tag{96}$$

a condition basic to the formulation of passive boundary conditions for the fluid. The field γ represents the *boundary dissipation*, measured per unit area.

We consider three classes of passive boundary conditions.

8.2.1. Boundary subsurfaces that are free. A simple set of boundary conditions, trivially consistent with (96), are the *free surface conditions*

$$\mathbf{t}_{\mathcal{S}} = 2\sigma K\mathbf{n} \quad \text{and} \quad \mathbf{m}_{\mathcal{S}} = \mathbf{0} \quad \text{on } \mathcal{S}. \tag{97}$$

These conditions place no constraint on the velocity or its normal derivative. In the terminology of the calculus of variations, (97) represent *natural boundary conditions*.

By (44), at a free surface

$$\begin{gathered} T_{ij}n_j - 2G_{ijk,k}n_j + G_{ijk,l}n_jn_kn_l + G_{ijk}K_{jk} = 2\sigma Kn_i, \\ G_{ijk}n_jn_k = 0, \end{gathered} \tag{98}$$

where we have used $(98)_2$ to simplify $(98)_1$. Thus the conventional traction $\mathbf{Tn}$ generally does not vanish; in fact,

$$T_{ij}n_j = 2G_{ijk,k}n_j - G_{ijk,l}n_jn_kn_l - G_{ijk}K_{jk} + 2\sigma Kn_i,$$

so that, interestingly, surface tension is not the sole source of curvature dependence.

[15] An example of a nonpassive environment is that associated with Couette flow.

8.2.2. Boundary subsurfaces that are fixed and that do not allow for slip. A boundary subsurface $\mathcal{S}(t)$ is *fixed* if the normal velocity of the fluid vanishes on $\mathcal{S}(t)$[16]. Classically, a fixed boundary without slip is characterized by the requirement that $\mathbf{v} = \mathbf{0}$. Here such a boundary condition is not sufficient to ensure satisfaction of the boundary dissipation inequality, which may be satisfied by taking $\partial\mathbf{v}/\partial n = \mathbf{0}$, or by taking $\mathbf{m}_{\mathcal{S}} = \mathbf{0}$, or by allowing for an appropriately signed relation between $\partial\mathbf{v}/\partial n$ and $\mathbf{m}_{\mathcal{S}}$. Since the third of these includes the first two, we consider the *generalized adherence conditions*

$$\mathbf{v} = \mathbf{0} \quad \text{and} \quad \mathbf{m}_{\mathcal{S}} = -\mu l \frac{\partial\mathbf{v}}{\partial n} \quad \text{on } \mathcal{S}, \tag{99}$$

in which the constitutive modulus $l \geqq 0$ represents a material length scale characterizing the strength of the adherency. We refer to l as the *adherence length*. We consider two potentially important special cases of (99): *weak adherence* ($l = 0$), for which

$$\mathbf{v} = \mathbf{0} \quad \text{and} \quad \mathbf{m}_{\mathcal{S}} = \mathbf{0} \quad \text{on } \mathcal{S}; \tag{100}$$

and *strong adherence* ($l = \infty$), for which

$$\mathbf{v} = \mathbf{0} \quad \text{and} \quad \frac{\partial\mathbf{v}}{\partial n} = \mathbf{0} \quad \text{on } \mathcal{S}. \tag{101}$$

Thus — in contrast with the classical no-slip condition — our more general theory allows for a one-parameter family of no-slip conditions.

8.2.3. Boundary subsurfaces that are fixed and that allow for slip. The classical condition of Navier for such a boundary has the form

$$\mathbf{PTn} = -\frac{\mu}{\lambda}\mathbf{v} \tag{102}$$

in which the constant constitutive modulus $\lambda > 0$ represents the *slip length*; since $\mathbf{PTn}$ is the *tangential traction*, this condition implies that

$$\mathbf{v}\cdot\mathbf{n} = 0.$$

Since $K\mathbf{Pn} = \mathbf{0}$, within our theory the natural counterpart of (102) is $\lambda\mathbf{Pt}_{\mathcal{S}} = -\mu\mathbf{v}$, a condition we combine with $\mathbf{m}_{\mathcal{S}} = \mathbf{0}$, which places no constraint on the normal derivative of the velocity. We therefore consider the *slip conditions*

$$\mathbf{Pt}_{\mathcal{S}} = -\frac{\mu}{\lambda}\mathbf{v} \quad \text{and} \quad \mathbf{m}_{\mathcal{S}} = \mathbf{0} \quad \text{on } \mathcal{S}, \tag{103}$$

with $\lambda > 0$.

[16] This condition does not require that $\mathcal{S}(t)$ be independent of t, only that $\mathcal{S}(t)$ depend on t at most through a dependence of its boundary curve $\partial\mathcal{S}(t)$ on t. Note that, since $\mathbf{v}\cdot\mathbf{n} = 0$, it follows that $(\operatorname{grad}\mathbf{v})^{\top}\mathbf{n} = \mathbf{Kv}$ and hence that $(\operatorname{grad}\mathbf{v})\mathbf{n} = 2\mathbf{Wn} = 2\mathbf{Dn}$. In particular, $\partial\mathbf{v}/\partial n = 2\boldsymbol{\omega} \times \mathbf{n}$ and hence $\mathbf{m}_{\mathcal{S}}\cdot(\partial\mathbf{v}/\partial n) = (2\mathbf{n} \times \mathbf{m}_{\mathcal{S}})\cdot\boldsymbol{\omega}$ represents a torque $2\mathbf{n} \times \mathbf{m}_{\mathcal{S}}$ that expends power over the vorticity $\boldsymbol{\omega}$.

8.2.4. A more general condition for a boundary subsurface that is fixed. The adherence and slip conditions may be generalized by the condition[17]

$$\left.\begin{aligned} \mathbf{P}\mathbf{t}_{\mathcal{S}} &= -\tfrac{\mu}{\lambda}\mathbf{v} - \mu\alpha\frac{\partial \mathbf{v}}{\partial n}, \quad \mathbf{v}\cdot\mathbf{n} = 0, \\ \mathbf{m}_{\mathcal{S}} &= -\mu\beta\mathbf{v} - \mu l\frac{\partial \mathbf{v}}{\partial n}, \end{aligned}\right\} \tag{104}$$

with

$$\lambda > 0 \quad \text{and} \quad l \geqq \tfrac{1}{4}(\alpha+\beta)^2\lambda. \tag{105}$$

For $\beta = \lambda = 0$, (104) reduces to the generalized adherence conditions (99); for $\alpha = \beta = l = 0$, (104) reduces to the slip conditions (103).

8.2.5. Conditions when ∂B is not smooth. In this case, the discussion in Section 4.4 and, in particular, (47) lead us to conclude that

$$\int_{\mathcal{S}} \left((\mathbf{t}_{\mathcal{S}} - 2\sigma K\mathbf{n})\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial \mathbf{v}}{\partial n} \right) \mathrm{d}a + \int_{\mathcal{C}\cap\mathcal{S}} \mathbf{f}_{\mathcal{C}}\cdot\mathbf{v}\,\mathrm{d}s \leqq 0 \tag{106}$$

for every subsurface $\mathcal{S}$ of ∂B, and hence that the *edge dissipation inequality*

$$\mathbf{f}_{\mathcal{C}}\cdot\mathbf{v} \leqq 0 \tag{107}$$

should join the boundary dissipation inequality (96) as a condition basic to the formulation of passive boundary conditions.

For $\mathcal{S}$ a free surface we would supplement (97) with the boundary condition

$$\mathbf{f}_{\mathcal{C}} = \mathbf{0}$$

on $\mathcal{C}\cap\mathcal{S}$, so that, by (49),

$$\{\!\{(\mathbb{G}\mathbf{n})\boldsymbol{\nu}\}\!\} = \mathbf{0}.$$

For $\mathcal{S}$ a fixed surface the adherence conditions (99)–(101) are appropriate on $\mathcal{S}$ away from $\mathcal{C}$, and continuity would require that $\mathbf{v} = \mathbf{0}$ on $\mathcal{C}\cap\mathcal{S}$, so an additional condition would not be needed.

A discussion of slip conditions in the presence of an edge is delicate and beyond the scope of this study.

[17] Choosing $\alpha = \beta$ in (104) gives conditions consistent with the reciprocity relations of ONSAGER [27].

9. Weak formulation of the flow equation and boundary conditions at a prescribed time

Because we work within a framework based on the principle of virtual power, it is fairly straightforward to derive a weak (variational) formulation of the flow equation and the boundary conditions (113). Here rather than work with an arbitrary control volume R, *we work with the region B occupied by the body at a fixed time t*.

We consider *boundary conditions* in which a portion $\mathcal{S}_{\mathrm{free}}$ of ∂B is free and the remainder $\mathcal{S}_{\mathrm{fxd}}$ is fixed:

$$\left.\begin{array}{l}\left.\begin{array}{c}\mathbf{Tn} - (\operatorname{div}\mathbb{G})\mathbf{n} - \operatorname{div}_{\mathcal{S}}(\mathbb{G}\mathbf{n}) = 2\sigma K\mathbf{n} \\ \text{and} \quad (\mathbb{G}\mathbf{n})\mathbf{n} = \mathbf{0}\end{array}\right\} \quad \text{on } \mathcal{S}_{\mathrm{free}}, \\ \mathbf{v} = \mathbf{0} \quad \text{and} \quad (\mathbb{G}\mathbf{n})\mathbf{n} = -\mu l \dfrac{\partial \mathbf{v}}{\partial n} \quad \text{on } \mathcal{S}_{\mathrm{fxd}};\end{array}\right\} \tag{108}$$

cf. (98) and (99).

We refer to an arbitrary virtual field $\tilde{\mathbf{v}}$ as *kinematically admissible* if

$$\tilde{\mathbf{v}} = \mathbf{0} \text{ on } \mathcal{S}_{\mathrm{fxd}}. \tag{109}$$

Given such a field, the virtual-power balance (30) applied with $R = B$, with $\mathbf{b}$ given by (45), and with the replacements indicated by

$$\mathbf{t}_{\mathcal{S}} \to 2\sigma K\mathbf{n} \text{ and } \mathbf{m}_{\mathcal{S}} \to \mathbf{0} \text{ on } \mathcal{S}_{\mathrm{free}}, \quad \mathbf{m}_{\mathcal{S}} \to -\mu l \frac{\partial \mathbf{v}}{\partial n} \text{ on } \mathcal{S}_{\mathrm{fxd}} \tag{110}$$

yields the *virtual balance*:

$$\int_{\mathcal{S}_{\mathrm{free}}} 2\sigma K\mathbf{n}\cdot\tilde{\mathbf{v}}\,\mathrm{d}a - \int_{\mathcal{S}_{\mathrm{fxd}}} \mu l \frac{\partial \mathbf{v}}{\partial n}\cdot\frac{\partial \tilde{\mathbf{v}}}{\partial n}\,\mathrm{d}a - \int_B \rho\dot{\mathbf{v}}\cdot\tilde{\mathbf{v}}\,\mathrm{d}v + \int_R \mathbf{b}_0\cdot\tilde{\mathbf{v}}\,\mathrm{d}v$$
$$= \int_B (\mathbf{T}\colon \operatorname{grad}\tilde{\mathbf{v}} + \mathbb{G}\,\vdots\,\operatorname{grad}\operatorname{grad}\tilde{\mathbf{v}})\,\mathrm{d}v. \tag{111}$$

The result (#) on page 525 then implies that given any kinematically admissible $\tilde{\mathbf{v}}$, (111) *is equivalent to* (41), also with the replacements (110):

$$\int_{\mathcal{S}_{\mathrm{free}}} (2\sigma K\mathbf{n} - (\mathbf{Tn} - (\operatorname{div}\mathbb{G})\mathbf{n} - \operatorname{div}_{\mathcal{S}}(\mathbb{G}\mathbf{n}) - 2K(\mathbb{G}\mathbf{n})\mathbf{n}))\cdot\tilde{\mathbf{v}}\,\mathrm{d}a$$
$$- \int_{\mathcal{S}_{\mathrm{free}}} (\mathbb{G}\mathbf{n})\mathbf{n}\cdot\frac{\partial \tilde{\mathbf{v}}}{\partial n}\,\mathrm{d}a - \int_{\mathcal{S}_{\mathrm{fxd}}} \left(\mu l \frac{\partial \mathbf{v}}{\partial n} + (\mathbb{G}\mathbf{n})\mathbf{n}\right)\cdot\frac{\partial \tilde{\mathbf{v}}}{\partial n}\,\mathrm{d}a$$
$$= \int_B (\operatorname{div}\operatorname{div}\mathbb{G} - \operatorname{div}\mathbf{T} + \rho\dot{\mathbf{v}} - \mathbf{b}^{\mathrm{ni}})\cdot\tilde{\mathbf{v}}\,\mathrm{d}v. \tag{112}$$

Thus, arguing as in the steps leading to (42) and (43), we see that the momentum balance (63) is satisfied in B, while

$$\left.\begin{array}{l}\left.\begin{array}{r}\mathbf{T}\mathbf{n}-(\operatorname{div}\mathbb{G})\mathbf{n}-\operatorname{div}_{\mathcal{S}}(\mathbb{G}\mathbf{n})-2K(\mathbb{G}\mathbf{n})\mathbf{n}=2\sigma K\mathbf{n}\\ \text{and}\quad(\mathbb{G}\mathbf{n})\mathbf{n}=\mathbf{0}\end{array}\right\}\quad\text{on } \mathcal{S}_{\text{free}},\\ \qquad(\mathbb{G}\mathbf{n})\mathbf{n}=-\mu l\dfrac{\partial\mathbf{v}}{\partial n}\quad\text{on } \mathcal{S}_{\text{fxd}}.\end{array}\right\}\tag{113}$$

Conversely, (63) and (113) imply that (112) and (hence) (111) are satisfied for all kinematically admissible $\tilde{\mathbf{v}}$. Finally, as is clear from the discussion in Section 7.2, granted the constitutive equations (78), the momentum balance is equivalent to the flow equation (82). We have therefore established a *weak formulation of the flow equation and traction boundary conditions*: *granted the constitutive equations* (78), *the virtual balance* (111) *is satisfied for all kinematically admissible virtual fields* $\tilde{\mathbf{v}}$ *if and only if*:

(i) *the flow equation* (82) *is satisfied within the fluid*;
(ii) *the boundary conditions* (113) *are satisfied on the boundary of the fluid.*

10. Plane Poiseuille flow

We now reconsider the classical problem of steady, laminar flow through an infinite, rectangular channel formed by two parallel surfaces separated by a gap h. Specifically, using the notation of Figure 2, we assume that the fluid velocity $\mathbf{v}$ has the form

$$\mathbf{v}(\mathbf{x})=v(y)\mathbf{e}_x\tag{114}$$

and is hence consistent with $\operatorname{div}\mathbf{v}=0$ and obeys $\dot{\mathbf{v}}=\mathbf{0}$. In view of (114), the flow equation (82) yields

$$-\zeta\frac{\partial^4 v}{\partial y^4}+\mu\frac{\partial^2 v}{\partial y^2}=\frac{\partial p}{\partial x},\qquad\frac{\partial p}{\partial y}=\frac{\partial p}{\partial z}=0.\tag{115}$$

Since v depends only on y, (115) implies that

$$\operatorname{grad}p=-\beta\mathbf{e}_x\quad\text{with}\quad\beta=\text{constant};\tag{116}$$

without loss of generality we assume that the effective pressure decreases with increasing x, so that

$$\beta>0.\tag{117}$$

Using a prime to denote differentiation with respect to y, we express the flow equation (118) as

$$-L^2v''''+v''=-\frac{\beta}{\mu}.\tag{118}$$

We next discuss the behavior, according to (118), of the fluid subject to the generalized adherence conditions (99) and slip conditions (103) at the points $y=0$

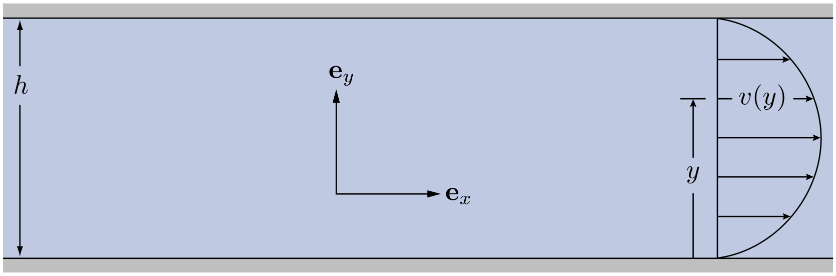

Fig. 2. Schematic of the channel for the problem of plane Poiseuille flow. The coordinates in the directions downstream and out of the plane are x and z.

and $y = h$ corresponding to the floor and ceiling of the channel. To facilitate this discussion, we record here the general solution of (118), which has the form

$$v(y) = c_0 + c_1 y - \frac{\beta y^2}{2\mu} + c_2 \sinh\frac{y}{L} + c_3 \cosh\frac{y}{L}, \tag{119}$$

with c_0, c_1, c_2, and c_3 constants.

10.1. Generalized adherence conditions

For a channel of the assumed geometry and a velocity field of the form (114), the generalized adherence conditions (99) become

$$v(0) = 0, \quad v(h) = 0, \quad lv'(0) = L^2 v''(0), \quad lv'(h) = -L^2 v''(h). \tag{120}$$

Applying these conditions to (119), we find that the velocity profile has the form

$$v(y) = \frac{\beta h^2}{2\mu}\left\{\frac{y}{h}\left(1 - \frac{y}{h}\right) - \frac{b_l}{\frac{h}{L}\sinh\frac{h}{L}}\left(\sinh\frac{h}{L} - \sinh\frac{y}{L} - \sinh\frac{h-y}{L}\right)\right\}, \tag{121}$$

with

$$b_l = \frac{\frac{2L}{h} + \frac{l}{L}}{1 + \frac{l}{L}\tanh\frac{h}{2L}} \tag{122}$$

a nonnegative dimensionless measure of the effective adhesion length. As might be expected in view of the geometry of the problem and the boundary conditions (120), v as defined by (121)–(122) obeys

$$v(y) = v(h - y), \quad 0 \leqq y \leqq h, \tag{123}$$

and, thus, is symmetric about the midplane of the channel.

To facilitate the discussion of (121), we note that

$$v(y) = v_c(y) + v_g(y), \tag{124}$$

where

$$v_c(y) = \frac{\beta h^2}{2\mu}\frac{y}{h}\left(1 - \frac{y}{h}\right) \tag{125}$$

is the *classical solution* of the analogous problem for a Navier–Stokes fluid and

$$v_g(y) = -\frac{\beta h^2}{2\mu}\frac{b_l L}{h \sinh \frac{h}{L}}\left(\sinh\frac{h}{L} - \sinh\frac{y}{L} - \sinh\frac{h-y}{L}\right), \tag{126}$$

arises from higher-order terms characterized by the gradient length L.

Importantly, since

$$b_l \to 0 \quad \text{as} \quad \frac{L}{h} \to 0 \quad \text{and} \quad \frac{l}{L} \to 0$$

it follows from (121) and (126) that

$$v(y) \to v_c(y) \quad \text{as} \quad \frac{L}{h} \to 0 \quad \text{and} \quad \frac{l}{L} \to 0, \quad 0 \leqq y \leqq h. \tag{127}$$

The theory therefore yields the correct classical limit when gradient length is negligibly small in comparison to the channel gap and the adherence length is negligibly small in comparison to the gradient length. Although this result is confined to the problem of plane Poiseuille flow subject to weak adherence conditions, it is not unexpected and we anticipate that it will carry over to general flows. When the channel gap h is large compared to the gradient length L, so that $L \ll h$, $v_g = \mathrm{O}(L^2/h^2)$. Hence, gradient effects are important only for channels with sufficiently small gaps.

Consistent with (100) and (101), the specialized conditions of weak and strong adherence arise, respectively, on setting $l = 0$ and $l = \infty$ in (120). The corresponding expressions for v follow on taking appropriate limits in (122). In particular, since

$$b_l \to \frac{2L}{h} \quad \text{as } l \to 0,$$

the specialization of (126) to weak adherence conditions is

$$v_w(y) = \frac{\beta h^2}{2\mu}\left\{\frac{y}{h}\left(1 - \frac{y}{h}\right) - \frac{2L^2}{h^2 \sinh\frac{h}{L}}\left(\sinh\frac{h}{L} - \sinh\frac{y}{L} - \sinh\frac{h-y}{L}\right)\right\}. \tag{128}$$

Further, since

$$b_l \to \coth\frac{h}{2L} \quad \text{as} \quad \frac{l}{L} \to \infty \quad \text{with} \quad 0 < L < \infty,$$

the identities $\sinh 2A = (1+\cosh 2A)\tanh A$ and $(\cosh A + \cosh B)\sinh(A+B) = (\sinh A + \sinh B)(1 + \cosh(A+B))$ allow us to express the specialization of (126) to the strong adherence conditions as

$$v_s(y) = \frac{\beta h^2}{2\mu}\left\{\frac{y}{h}\left(1 - \frac{y}{h}\right) - \frac{L}{h \sinh\frac{h}{L}}\left(1 + \cosh\frac{h}{L} - \cosh\frac{y}{L} - \cosh\frac{h-y}{L}\right)\right\}. \tag{129}$$

For $0 < L < \infty$, b_l is a monotonically increasing function of l/L and obeys

$$\frac{2L}{h} \leqq b_l \leqq \coth \frac{h}{2L}. \tag{130}$$

Thus, since

$$\sinh \frac{y}{L} + \sinh \frac{h-y}{L} \leqq \sinh \frac{h}{L}, \quad 0 \leqq y \leqq h, \tag{131}$$

a straightforward argument shows that

$$v_w(y) \geqq v(y) \geqq v_s(y) \quad 0 \leqq y \leqq h. \tag{132}$$

Hence, the solution (121) to the generalized adherence problem is bounded above by the solution (128) to the weak adherence problem and below by the solution (129) to the strong adherence problem. Furthermore, it can be shown that

$$v_c(y) \geqq v_w(y) \quad \text{and} \quad v_s(y) \geqq 0, \quad 0 \leqq y \leqq h, \tag{133}$$

and, thus, that the solution (125) to the corresponding problem for the Navier–Stokes equations provides an upper bound for the weak adherence solution and that solution (129) to the strong adherence problem is never negative.

For all choices of the characteristic lengths $L \geqq 0$ and $l \geqq 0$, it can be shown that v behaves much like the classical solution v_c, increasing monotonically from its value of zero at the channel floor $y = 0$ to a maximum at the channel midplane $y = h/2$ and then decreases monotonically to its value of zero at the channel ceiling $y = h$.

Plots of v, v_w, v_s, and v_c, normalized by $2\mu/\beta h^2$, as functions of y/h are provided in Figure 3. For each of these plots, we take $L = h/5$. Also, for the plot of v, we take $4l = 3\coth(5/2)$. These plots display the general features of v as determined by (121) and discussed above.

To gain further insight concerning the influence of gradient effects, we integrate v over the channel gap and *divide by the flow rate* $\int_0^h v_c(y)\,\mathrm{d}y = \beta h^3/12\mu$ *for the classical velocity* v_c to obtain the dimensionless *flow rate*

$$Q = \frac{12\mu}{\beta h^3} \int_0^h v(y)\,\mathrm{d}y = 1 - \frac{12L^2 b_l}{h^2}\left(\frac{h}{2L} - \tanh \frac{h}{2L}\right). \tag{134}$$

Since $b_l > 0$ and $A \geqq \tanh A$ for $A \geqq 0$, it follows that $Q < 1$ for all values of $L/h > 0$. The flow rates corresponding to the solutions v_w and v_s of the problems for strong and weak adherence are given by

$$Q_w = 1 - \frac{12L^2}{h^2}\left(1 - \frac{2L}{h} \tanh \frac{h}{2L}\right) \tag{135}$$

and

$$Q_s = 1 + \frac{12L^2}{h^2}\left(1 - \frac{h}{2L} \coth \frac{h}{2L}\right), \tag{136}$$

respectively. Thus, in view of (130),

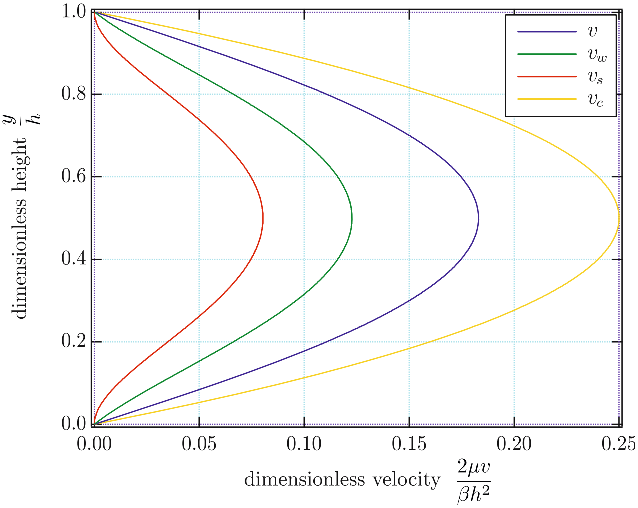

Fig. 3. Velocity profiles, normalized by $\beta h^2/2\mu$ and plotted versus the dimensionless height y/h, arising from solutions of the flow equation (118) subject to weak (right), generalized (left), and strong (far left) adherence conditions. For each of these plots, $L = h/5$. Also, $4l = 3L\coth(5/2)$ for generalized adherence. Shown for comparison is a plot of the classical solution (far right) normalized by $\beta h^2/2\mu$ and plotted versus the dimensionless height y/h.

$$0 \leqq Q_s \leqq Q \leqq Q_w \leqq 1. \tag{137}$$

Consistent with our discussion of the velocity field, the flow rate for generalized adherence is always less than or equal to its classical counterpart, while the flow rates corresponding to strong and weak adherence provide lower and upper bounds for all flow rates possible under generalized adherence conditions. Importantly, since $Q = 1 + \mathrm{O}(L^2/h^2)$, the impact of gradient effects on the flow rate is significant only for channels with sufficiently small gaps. This is consistent with our discussion of the velocity field v. Plots of Q (with $4l = 3L\coth(h/2L)$), Q_w, and Q_s as functions of the ratio L/h of the gradient length L to the channel gap h are provided in Figure 4.

Under generalized adherence, the net dissipation includes contributions from both the bulk and the boundary. For $\mathbf{v}$ of the assumed form (114), the bulk dissipation Γ per unit volume, as defined by (77), simplfies to $\Gamma = \mu\left(|v'|^2 + L^2|v''|^2\right)$. Integrating this expression over the channel gap and *dividing by the dissipation* $\mu\int_0^h |v_c'(y)|^2\,\mathrm{d}y = \beta^2h^3/12\mu$ *per unit channel height for the classical velocity* v_c, we obtain a dimensionless measure

$$\begin{aligned}\Gamma^* &\overset{\text{def}}{=} \frac{12\mu^2}{\beta^2h^3}\int\limits_0^h \left(|v'(y)|^2 + L^2|v''(y)|^2\right)\mathrm{d}y,\\ &= 1 + \frac{12L^2}{h^2}\left(1 - \frac{b_lh}{L} + \frac{b_l^2h}{L}\tanh\frac{h}{2L}\right)\end{aligned}$$

of the net bulk dissipation. Using the definition (122) of b_l, we have

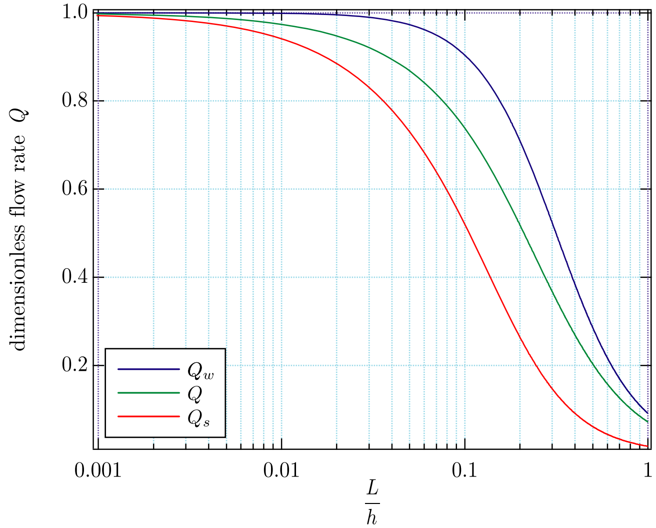

Fig. 4. Dimensionless flow rates, normalized by $\beta h^3/12\mu$ and plotted versus the dimensionless ratio L/h of the gradient length L to the channel gap h, for weak (right), generalized (middle), and strong (left) adherence conditions. For generalized adherence, $4l = 3L\coth(h/2L)$.

$$
\begin{aligned}
1-\frac{b_l h}{L}+\frac{b_l^2 h}{L}\tanh\frac{h}{2L} &= 1-\frac{b_l h}{2L}-\frac{b_l h}{2L}\left(1-b_l\tanh\frac{h}{2L}\right)\\
&= -\frac{\frac{l}{L}\left(\frac{h}{2L}-\tanh\frac{h}{2L}\right)}{1+\frac{l}{L}\tanh\frac{h}{2L}}-\frac{b_l\left(\frac{h}{2L}-\tanh\frac{h}{2L}\right)}{1+\frac{l}{L}\tanh\frac{h}{2L}}\\
&= -\frac{b_l+\frac{l}{L}}{\frac{2L}{h}+\frac{l}{L}}\left(\frac{h}{2L}-\tanh\frac{h}{2L}\right)
\end{aligned}
$$

and, on recalling the expression (134) for the flow rate Q, we find that

$$
\begin{aligned}
\Gamma^* &= 1-\frac{b_l+\frac{l}{L}}{\frac{2L}{h}+\frac{l}{L}}\frac{12L^2 b_l}{h^2}\left(\frac{h}{2L}-\tanh\frac{h}{2L}\right)\\
&= 1-\frac{b_l+\frac{l}{L}}{\frac{2L}{h}+\frac{l}{L}}(1-Q).
\end{aligned}
\tag{138}
$$

Simple calculations show that

$$
Q_s \leqq \Gamma^* \leqq Q_w, \tag{139}
$$

with the upper and lower bounds being attained for the respective cases of weak and strong adherence. Next, by (99) and (114), for $\mathbf{v}$ of the assumed form (114) the boundary dissipation γ per unit area, as defined by (96), simplifies to $\gamma =$

$\mu l\left(|v'(0)|^2+|v'(h)|^2\right)=2\mu l|v'(0)|^2$. Normalizing as in the case of the bulk dissipation, we obtain a dimensionless measure

$$\begin{aligned}\gamma^* &= \frac{24\mu^2 l}{\beta^2 h^3}|v'(0)|^2\\ &= \frac{6L}{h}\frac{l}{L}\left(1-b_l\tanh\frac{h}{2L}\right)^2\\ &= \frac{l}{3L}\frac{(1-Q)^2}{\frac{2L}{h}\left(\frac{2L}{h}+\frac{l}{L}\right)^2},\end{aligned} \tag{140}$$

of the net boundary dissipation. It can be shown that γ^* vanishes for both weak and strong adherence and achieves a maximum when the adherence length takes the value $l=L\coth\frac{h}{2L}$.

The net dimensionless dissipation $\Gamma^*+\gamma^*$ can be shown to differ from the classical amount $\beta^2h^3/12\mu$ by terms of $\mathrm{O}(L^2/h^2)$. Hence, as with the velocity and the flow rate, the impact of gradient effects on the net dissipation is significant only for channels with sufficiently small gaps.

10.2. Slip conditions

For a channel of the assumed geometry and a velocity field of the form (114), the slip conditions (103) become

$$\left.\begin{aligned}\mu v(0) &= \lambda[\mu v'(0)-\zeta v'''(0)], \qquad v''(0)=0,\\ \mu v(h) &= -\lambda[\mu v'(h)-\zeta v'''(h)], \quad v''(h)=0.\end{aligned}\right\} \tag{141}$$

Using (141) in (119) and noting from (85) that $\eta_2/\mu L^2=\eta_2/\zeta$, we find that

$$v(y)=\frac{\beta h\lambda}{2\mu}+v_w(y), \tag{142}$$

which differs from the expression (128) obtained for weak adherence conditions by the classical value $\beta\lambda h/2\mu$ of the effective slip-length for the analogous problem for a Navier–Stokes fluid.

In view of (142), the dimensionless flow rate for slip conditions is simply

$$Q=\frac{12\mu}{\beta h^3}\int_0^h v(y)\,\mathrm{d}y=Q_w+\frac{6\lambda}{h}, \tag{143}$$

which exceeds the flow rate (135) obtained for weak adherence by the amount $6\lambda/h$. As in classical theory, an allowance for slip is therefore accompanied by an increased flow rate.

Finally, the net dimensionless dissipation for slip coincides with the dimensionless flow rate (143). This is consistent with what occurs for weak adherence and observations analogous to those made in connection with that problem apply here as well.

10.3. Role of the pressure

An interesting nuance of the generalized adherence and slip problems is that a detailed characterization of the pressure P and hyperpressure $\boldsymbol{\pi}$ is not necessary to determine solutions to these problems consistent with the constraint $\operatorname{div}\mathbf{v} = 0$. Rather, it suffices only to determine the effective pressure $p = P - \operatorname{div}\boldsymbol{\pi}$. By (116), the hyperpressure $\boldsymbol{\pi}$ must satisfy the partial differential equation $\operatorname{div}\boldsymbol{\pi}(\mathbf{x}) = P(\mathbf{x}) + \beta x$. In the generalized adherence problem, the condition involving $\mathbf{m}_{\mathcal{S}}$ requires in addition to $(120)_{3,4}$ that $\boldsymbol{\pi}\cdot\mathbf{n} = 0$ on the channel walls. Hence, for the generalized adherence problem, $\boldsymbol{\pi}$ and P must satisfy $\operatorname{div}\boldsymbol{\pi}(\mathbf{x}) = P(\mathbf{x}) + \beta x$ in the interior of the channel along with $\boldsymbol{\pi}\cdot\mathbf{n} = 0$ on the channel walls. There exists an infinity of choices of $\boldsymbol{\pi}$ and P satisfying this boundary-value problem. For the slip problem, the boundary conditions involve neither P nor $\boldsymbol{\pi}$. Hence, these fields may be chosen as arbitrary solutions of $\operatorname{div}\boldsymbol{\pi}(\mathbf{x}) = P(\mathbf{x}) + \beta x$.

10.4. Brief summary of results

For the generalized adherence conditions (12), the theory predicts a flow rate lower than that obtained classically. When slip is allowed, the flow rate exceeds that obtained under weak adherence but also lies below that obtained for the Navier–Stokes equations subject to the Navier slip condition. These lower flow rates trace directly to the additional sources of dissipation associated with the hyperstress and the boundary conditions. Importantly, the reduced flow rates predicted by the theory are important only for channels with sufficiently small gaps. Specifically, for a channel with gap h, the flow rate decreases by terms of $\mathrm{O}(L^2/h^2)$ where L is the gradient length.

11. Applicability of the theory to liquid flow at small-length scales

An understanding of the flow of liquids at small-length scales is of central importance in a wide variety of disciplines, including biology [2, 25, 39], chemistry [8, 10, 16], and the rapidly expanding fields of micro- and nanotechnology [9, 37, 38, 44]. Recent experiments indicate that the Navier–Stokes equations and their classical boundary conditions accurately describe the flow of incompressible, Newtonian liquids through smooth channels with cross-sectional dimensions as small as 50 μm [14, 35], a conclusion supported by atomistic simulations. What is more important, such simulations indicate that *the classical theory breaks down below* 10 *molecular diameters* [3, 18, 19, 22, 26, 42, 43], and therefore it could be asked: can a continuum theory such as ours be applicable at such small-length scales? In fact, materials science is replete with examples where extensions of classical continuum theories have been exploited to capture effects at nanometer length scales. As a striking illustration of this, RASTELLI, VON KÄNEL, SPENCER & TERSOFF [34] use a continuum theory to characterize the formation of faceted islands of nanometer size during molecular beam epitaxy[18]; their results show that theory agrees well with experiments for quantum dots of 2 nm high and 80 nm wide (cf. [34] Figure 2b). A similar approach is used by SIEGEL, MIKSIS & VOORHEES [36] to study the formation of wrinkles on the surface bounding a void of radius 10 nm.

Aside from possibly describing flows in nanometer scale devices and to problems of nanolubrication, our theory might also apply to flows in micron scale devices with rough walls. For such applications, the gradient length L would be small compared to the characteristic linear dimension of the region of flow. However, to model wall roughness, the adherence length l would exceed L. With such a choice of scales, differences between our theory and classical theory should be confined to regions close to fixed boundaries. In particular, reduced flow rates of the sort predicted in our analysis of plane Poiseuille flow are consistent with experiments performed in microchannels with rough walls [15, 17, 20, 21, 28–30, 33].

12. Kinetic energy dependent on the velocity gradient

The theory discussed thus far is based on a generalization of the classical virtual-power principle to include higher-order velocity gradients and associated higher-order stresses and tractions. In accordance with this, it would seem reasonable to modify the kinetic energy to account for dependence on the velocity gradient. We consider that modification here.

12.1. Gradient kinetic energy. Inertial power balance

We now consider

$$\tfrac{1}{2}\rho|\mathbf{v}|^2 + \tfrac{1}{2}\beta\,|\mathrm{grad}\,\mathbf{v}|^2$$

as the a kinetic energy, per unit volume, so that the kinetic energy of any region $\mathcal{R}(t)$ that *convects with the fluid* is given by

$$\mathcal{K}(\mathcal{R}(t)) = \int\limits_{\mathcal{R}(t)} \tfrac{1}{2}\left(\rho|\mathbf{v}|^2 + \beta\,|\mathrm{grad}\,\mathbf{v}|^2\right)\,\mathrm{d}v. \tag{144}$$

To simplify the analysis, we assume that $\beta = (\text{constant})\rho$ throughout, so that, given any field φ and any region $\mathcal{R}(t)$ that convects with the fluid,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{R}(t)} \beta\varphi\,\mathrm{d}v = \int\limits_{\mathcal{R}(t)} \beta\dot{\varphi}\,\mathrm{d}v,$$

and the kinetic-energy *rate* takes the form

[18] Rastelli, von Känel, Spencer & Tersoff allow the surface energy of the film to depend on surface curvature (HERRING [13]; DICARLO, GURTIN & PODIO-GUIDUGLI [7]); in all other respects their theory is conventional.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(\mathcal{R}(t)) = \int\limits_{\mathcal{R}(t)} \Big(\rho\dot{\mathbf{v}}\cdot\mathbf{v} + \beta\, \dot{\overline{\mathrm{grad}\,\mathbf{v}}} : \mathrm{grad}\,\mathbf{v}\Big)\,\mathrm{d}v. \tag{145}$$

Our goal is to determine inertial components of the body force $\mathbf{b}$ and the traction $\mathbf{t}_{\mathcal{S}}$ appropriate to the kinetic-energy rate (145). Bearing in mind that the argument leading to the virtual balance (40) is independent of what specific form the inertia might take, we assume that

(i) $\mathbf{t}_{\mathcal{S}}$, $\mathbf{m}_{\mathcal{S}}$, and $\mathbf{b}$ continue to satisfy the virtual balance (40), so that (42) and (43) are satisfied.
(ii) There are *inertial and noninertial tractions* $\mathbf{t}_{\mathcal{S}}^{\mathrm{in}}$ and $\mathbf{t}_{\mathcal{S}}^{\mathrm{ni}}$ as well as *inertial and noninertial body forces* $\mathbf{b}^{\mathrm{in}}$ and $\mathbf{b}^{\mathrm{ni}}$ such that

$$\mathbf{b} = +\mathbf{b}^{\mathrm{ni}}, \quad \mathbf{t}_{\mathcal{S}} = \mathbf{t}_{\mathcal{S}}^{\mathrm{in}} + \mathbf{t}_{\mathcal{S}}^{\mathrm{ni}}. \tag{146}$$

(iii) The kinetic-energy rate (145) is balanced by the *negative* of the power expended by $\mathbf{b}^{\mathrm{in}}$ and $\mathbf{t}_{\mathcal{S}}^{\mathrm{in}}$ in the sense of the *inertial power balance* [19]:

$$\int\limits_{R} \big(\rho\dot{\mathbf{v}}\cdot\mathbf{v} + \beta\, \dot{\overline{\mathrm{grad}\,\mathbf{v}}} : \mathrm{grad}\,\mathbf{v}\big)\,\mathrm{d}v = -\int\limits_{R} \cdot\mathbf{v}\,\mathrm{d}v - \int\limits_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}^{\mathrm{in}}\cdot\mathbf{v}\,\mathrm{d}a, \tag{147}$$

where in writing (147) we have, without loss in generality, replaced $\mathcal{R}$ by an arbitrary control volume R and $\partial\mathcal{R}$ by $\partial R = \mathcal{S}$.

12.2. Inertial virtual-power balance. Inertial body force and surface traction

With a view toward making use of experience gained in the virtual-power analysis of Sections 3–4, we define (vectorial and tensorial) *momentum-rate forces* $\mathbf{p}$ and $\mathbf{M}$ through

$$\mathbf{p} = \rho\dot{\mathbf{v}} \quad \text{and} \quad \mathbf{M} = \beta\, \dot{\overline{\mathrm{grad}\,\mathbf{v}}}, \tag{148}$$

in which case we may rewrite (147) in the form

$$\int\limits_{R} (\mathbf{p}\cdot\mathbf{v} + \mathbf{M} : \mathrm{grad}\,\mathbf{v})\,\mathrm{d}v = -\int\limits_{R} \mathbf{b}^{\mathrm{in}}\cdot\mathbf{v}\,\mathrm{d}v - \int\limits_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}^{\mathrm{in}}\cdot\mathbf{v}\,\mathrm{d}a. \tag{149}$$

Guided by our discussion of virtual power in Section 4 (in particular, in the paragraph containing (29)) and comparing (149) to the virtual power relation defined by (29) and (30), we assume that at some arbitrarily chosen but *fixed time*, the region occupied by the body is known, as are the inertial traction $\mathbf{t}_{\mathcal{S}}^{\mathrm{in}}$, the inertial body force $\mathbf{b}^{\mathrm{in}}$, and the momentum-rate forces $\mathbf{p}$ and $\mathbf{M}$, and consider the velocity field

[19] Cf. PODIO-GUIDUGLI [31], who bases his discussion of classical kinetic energy on a balance between the rate of kinetic energy and the power expended by an inertial body force.

$\mathbf{v}$ as a virtual field $\tilde{\mathbf{v}}$ that may be specified *independently of the actual evolution of the body*[20]:

$$\int_R (\mathbf{p}\cdot\tilde{\mathbf{v}} + \mathbf{M}:\operatorname{grad}\tilde{\mathbf{v}})\,\mathrm{d}v = -\int_R \mathbf{b}^{\text{in}}\cdot\tilde{\mathbf{v}}\,\mathrm{d}v - \int_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}^{\text{in}}\cdot\tilde{\mathbf{v}}\,\mathrm{d}a. \tag{150}$$

Writing $\mathbf{n}$ for the outward unit normal to $\mathcal{S} = \partial R$, if we integrate the term in (150) involving $\mathbf{M}:\operatorname{grad}\tilde{\mathbf{v}}$ by parts we find that

$$\int_R \big(\mathbf{b}^{\text{in}} + \mathbf{p} - \operatorname{div}\mathbf{M}\big)\cdot\tilde{\mathbf{v}}\,\mathrm{d}v + \int_{\mathcal{S}} \big(\mathbf{t}^{\text{in}}{}_{\partial\mathcal{R}} + \mathbf{M}\mathbf{n}\big)\cdot\tilde{\mathbf{v}}\,\mathrm{d}a = 0; \tag{151}$$

since this relation is to hold for all virtual fields $\tilde{\mathbf{v}}$, we arrive at explicit expressions for $\mathbf{b}^{\text{in}}$ and $\mathbf{t}_{\mathcal{S}}^{\text{in}}$:

$$\left.\begin{aligned} \mathbf{b}^{\text{in}} &= -\mathbf{p} + \operatorname{div}\mathbf{M}, \\ \mathbf{t}_{\mathcal{S}}^{\text{in}} &= -\mathbf{M}\mathbf{n}. \end{aligned}\right\} \tag{152}$$

Next, using (146), (148), and (152), we may express the net body force $\mathbf{b}$ and the net surface traction $\mathbf{t}_{\mathcal{S}}$ in the forms

$$\left.\begin{aligned} \mathbf{b} &= -\rho\dot{\mathbf{v}} + \beta\operatorname{div}\Big(\overline{\operatorname{grad}\mathbf{v}}^{\boldsymbol{\cdot}}\Big) + \mathbf{b}^{\text{ni}}, \\ \mathbf{t}_{\mathcal{S}} &= -\beta\Big(\overline{\operatorname{grad}\mathbf{v}}^{\boldsymbol{\cdot}}\Big)\,\mathbf{n} + \mathbf{t}_{\mathcal{S}}^{\text{ni}}. \end{aligned}\right\} \tag{153}$$

Since, by (i), $\mathbf{b}$ and $\mathbf{t}_{\mathcal{S}}$ are given by (42) and (43), the relations (153) may be viewed as expressions for the the noninertial fields $\mathbf{b}^{\text{ni}}$ and $\mathbf{t}_{\mathcal{S}}^{\text{ni}}$.

12.3. The flow equation and free-energy imbalance for an incompressible, linearly viscous fluid

We now return to the topic of incompressible fluids. For an imcompressible body, it follows from the constraint $\operatorname{div}\mathbf{v} = 0$ that the tensorial momentum-rate force $\mathbf{M}$ is determined only up to an additive spherical factor. In analogy to the decomposition (68) of the Cauchy stress, we may therefore replace $(148)_1$ by

$$\mathbf{M} = -\kappa\mathbf{1} + \beta\,\overline{\operatorname{grad}\mathbf{v}}^{\boldsymbol{\cdot}}, \tag{154}$$

where κ is an *indeterminate kinetic pressure*. Consider the argument leading to the representations (153) for $\mathbf{b}^{\text{in}}$ and $\mathbf{t}_{\mathcal{S}}^{\text{in}}$. When $\mathbf{v}$ satisfies $\operatorname{div}\mathbf{v} = 0$, it is not necessary to require that the $\tilde{\mathbf{v}}$ entering the balance (150) be divergence free. Indeed, since the kinetic pressure κ is indeterminate, we may, without loss in generality, consider

---

[20] This paradigm represents an intrinsic method of decomposing the rate of the kinetic energy into the negative of a power expenditure by a body force field $\mathbf{b}^{\text{in}}$ and a traction field $\mathbf{t}_{\mathcal{S}}^{\text{in}}$, with each of these fields uniquely determined. As a bonus, this treatment of inertia guarantees a variational framework for the resulting partial differential equation.

κ as a *Lagrange multiplier* and *require that the balance* (150) *holds for all virtual fields* $\tilde{\mathbf{v}}$ *on* R. It then follows that

$$\left.\begin{aligned}\mathbf{b} &= -\operatorname{grad}\kappa - \rho\dot{\mathbf{v}} + \beta\operatorname{div}\left(\overline{\operatorname{grad}\mathbf{v}}^{\boldsymbol{\cdot}}\right) + \mathbf{b}^{\mathrm{ni}},\\ \mathbf{t}_{\mathcal{S}} &= \left(\kappa\mathbf{1} - \beta\,\overline{\operatorname{grad}\mathbf{v}}^{\boldsymbol{\cdot}}\right)\mathbf{n} + \mathbf{t}_{\mathcal{S}}^{\mathrm{ni}}.\end{aligned}\right\} \tag{155}$$

With a view toward writing (155) in terms of the acceleration $\dot{\mathbf{v}}$, we note that

$$\begin{aligned}\left(\overline{\operatorname{grad}\mathbf{v}}^{\boldsymbol{\cdot}}\right)_{ij} &= \frac{\partial v_{i,j}}{\partial t} + v_{i,jk}v_k,\\ &= \left(\frac{\partial v_i}{\partial t}\right)_{,j} + (v_{i,k}v_k)_{,j} - v_{i,k}v_{k,j},\\ &= (\dot{v}_i)_{,j} - v_{i,k}v_{k,j},\end{aligned} \tag{156}$$

and hence that

$$\left[\operatorname{div}\left(\overline{\operatorname{grad}\mathbf{v}}^{\boldsymbol{\cdot}}\right)\right]_i = \triangle\dot{v}_i - v_{i,k}\triangle v_k - v_{i,kj}v_{k,j}. \tag{157}$$

Thus, on defining

$$\left[\left(\operatorname{grad}^2\mathbf{v}\right)\operatorname{grad}\mathbf{v}\right]_i \stackrel{\text{def}}{=} v_{i,kj}v_{k,j},$$

we may write the identities (156) and (157) in direct notation as follows:

$$\begin{aligned}\overline{\operatorname{grad}\mathbf{v}}^{\boldsymbol{\cdot}} &= \operatorname{grad}\dot{\mathbf{v}} - (\operatorname{grad}\mathbf{v})\operatorname{grad}\mathbf{v},\\ \operatorname{div}\left(\overline{\operatorname{grad}\mathbf{v}}^{\boldsymbol{\cdot}}\right) &= \triangle\dot{\mathbf{v}} - (\operatorname{grad}\mathbf{v})\triangle\mathbf{v} - \left(\operatorname{grad}^2\mathbf{v}\right)\operatorname{grad}\mathbf{v}.\end{aligned} \tag{158}$$

Thus, by (155),

$$\left.\begin{aligned}\mathbf{b} &= -\operatorname{grad}\kappa - (\rho - \beta\triangle)\dot{\mathbf{v}} - \beta[(\operatorname{grad}\mathbf{v})\triangle\mathbf{v} + \left(\operatorname{grad}^2\mathbf{v}\right)\operatorname{grad}\mathbf{v}] + \mathbf{b}^{\mathrm{ni}},\\ \mathbf{t}_{\mathcal{S}} &= [\kappa\mathbf{1} - \beta(\operatorname{grad}\dot{\mathbf{v}} + (\operatorname{grad}\mathbf{v})\operatorname{grad}\mathbf{v})]\mathbf{n} + \mathbf{t}_{\mathcal{S}}^{\mathrm{ni}}.\end{aligned}\right\} \tag{159}$$

We continue to work within the constitutive framework set out in Section 7. With this provision and $(159)_1$, the local force balance (42) specializes to yield the *flow equation*

$$\begin{aligned}\rho\dot{\mathbf{v}} - \beta[\triangle\dot{\mathbf{v}} - (\operatorname{grad}\mathbf{v})\triangle\mathbf{v} - \left(\operatorname{grad}^2\mathbf{v}\right)\operatorname{grad}\mathbf{v}]&\\ = -\operatorname{grad}p + \mu\triangle\mathbf{v} - \zeta\triangle\triangle\mathbf{v} + \mathbf{b}^{\mathrm{ni}},&\end{aligned} \tag{160}$$

with *effective pressure* now given by

$$p = P + \kappa - \operatorname{div}\boldsymbol{\pi}. \tag{161}$$

Assume now that $\mathbf{b}^{\mathrm{ni}} = \mathbf{0}$. Then, arguing as in the steps leading up to (90) we find, using (147), that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_R \tfrac{1}{2}\big(\rho|\mathbf{v}|^2 + \beta\,|\mathrm{grad}\,\mathbf{v}|^2\big)\,\mathrm{d}v \\
&\qquad + \int_{\mathcal{S}} \tfrac{1}{2}\big(\rho|\mathbf{v}|^2 + \beta\,|\mathrm{grad}\,\mathbf{v}|^2\big)\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a - \int_{\mathcal{S}}\bigg(\mathbf{t}_{\mathcal{S}}^{\mathrm{ni}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\bigg)\,\mathrm{d}a \\
&= -\int_R \big(2\mu|\mathbf{D}|^2 + \eta_1|\mathrm{grad}\,\mathbf{v}|^2 \\
&\qquad + \eta_2\big(2|\mathrm{grad}\,\mathbf{D}|^2 - 2|\mathrm{grad}\,\mathbf{W}|^2 - |\triangle\mathbf{v}|^2\big)\big)\,\mathrm{d}v \leqq 0.
\end{aligned} \tag{162}$$

12.4. Boundary conditions for a passive environment

We now generalize the boundary conditions derived in Section 8.2 to account for the gradient kinetic energy. In this more general theory the inequality (94) remains valid, but the discussion of a passive environment is delicate, as we must account for the inertial power expended by the environment. With this in mind, we assume that $\mathcal{W}_{\mathrm{env}}(\mathcal{S})$, the environmental power expended on the pillbox $\mathcal{S}$ of Figure 1, admits a decomposition

$$\mathcal{W}_{\mathrm{env}}(\mathcal{S}) = \mathcal{W}_{\mathrm{env}}^{\mathrm{in}}(\mathcal{S}) + \mathcal{W}_{\mathrm{env}}^{\mathrm{ni}}(\mathcal{S}) \tag{163}$$

into inertial and noninertial power expenditures $\mathcal{W}_{\mathrm{env}}^{\mathrm{in}}(\mathcal{S})$ and $\mathcal{W}_{\mathrm{env}}^{\mathrm{ni}}(\mathcal{S})$. Granted this, we refer to the boundary as *passive* if for any motion of the fluid and any evolving subsurface $\mathcal{S}(t)$ of $\partial B(t)$:

(i) the *noninertial* power expended by the environment at the boundary vanishes,

$$\mathcal{W}_{\mathrm{env}}^{\mathrm{ni}}(\mathcal{S}) = 0;$$

(ii) the *inertial* power expended by the environment at the boundary is *balanced* by the *inertial* power expended in the body at the boundary,

$$\mathcal{W}_{\mathrm{env}}^{\mathrm{in}}(\mathcal{S}) = \int_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}^{\mathrm{in}}\cdot\mathbf{v}\,\mathrm{d}a,$$

cf. (94). Then, for a passive environment, (94), (146), and (163) imply that

$$\int_{\mathcal{S}}\bigg(\big(\mathbf{t}_{\mathcal{S}}^{\mathrm{ni}} - 2\sigma K\mathbf{n}\big)\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\bigg)\,\mathrm{d}a \leqq 0, \tag{164}$$

and since the choice of $\mathcal{S}$ is arbitrary, (164) is equivalent to the boundary dissipation inequality

$$-\big(\mathbf{t}_{\mathcal{S}}^{\mathrm{ni}} - 2\sigma K\mathbf{n}\big)\cdot\mathbf{v} - \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n} \geqq 0, \tag{165}$$

where, by $(43)_1$ and $(155)_2$,

$$\mathbf{t}_{\mathcal{S}}^{\rm ni} = \mathbf{T}\mathbf{n} - (\operatorname{div}\mathbb{G})\mathbf{n} - \operatorname{div}_{\mathcal{S}}(\mathbb{G}\mathbf{n}) - 2K(\mathbb{G}\mathbf{n})\mathbf{n} - \left(\kappa\mathbf{1} - \beta\,\overline{\operatorname{grad}\mathbf{v}}^{\bullet}\right)\mathbf{n}. \tag{166}$$

As in Section 8.2 the dissipation inequality (165) may be used to motivate boundary conditions. The only difference between this dissipation inequality and the inequality (96) of Section 8.2 is that the relevant traction here is $\mathbf{t}_{\mathcal{S}}^{\rm ni}$ rather than $\mathbf{t}_{\mathcal{S}}$. Consequently, *each of the boundary conditions specified in* Section 8.2 *has a direct counterpart for the theory discussed here: we simply replace* $\mathbf{t}_{\mathcal{S}}$ *by* $\mathbf{t}_{\mathcal{S}}^{\rm ni}$. Thus, because they do not involve the surface traction $\mathbf{t}_{\mathcal{S}}$, *the generalized adherence conditions* (99) *as well as its weak and strong counterparts* (100) *and* (101) *are unchanged.*

On the other hand, the boundary conditions for a *free surface* are $\mathbf{m}_{\mathcal{S}} = \mathbf{0}$ (as in Section 8.2) and $\mathbf{t}_{\mathcal{S}}^{\rm ni} = 2\sigma K\mathbf{n}$, or equivalently, by (44) and (166), $G_{ijk}n_jn_k = 0$ and

$$T_{ij}n_j - 2G_{ijk,k}n_j + G_{ijk,l}n_jn_kn_l + G_{ijk}K_{jk} + \beta\,\overline{(v_{i,j})}^{\bullet}n_j = 2\sigma Kn_i. \tag{167}$$

Thus, interestingly, in this more general theory the boundary condition at a free surface involves the inertial term

$$\beta\left(\overline{\operatorname{grad}\mathbf{v}}^{\bullet}\right)\mathbf{n}.$$

Similarly, the boundary conditions for a surface that is *fixed but allows for slip* are $\mathbf{m}_{\mathcal{S}} = \mathbf{0}$ and $\mathbf{P}\mathbf{t}_{\mathcal{S}}^{\rm ni} = -(\mu/\lambda)\mathbf{v}$, the second of which, by (166), includes a term of the form

$$\beta\mathbf{P}\left(\overline{\operatorname{grad}\mathbf{v}}^{\bullet}\right)\mathbf{n}.$$

12.5. Plane Poiseuille flow revisited

Granted that the fluid velocity is of the form (114) assumed for the problem of plane Poiseuille flow, kinematics alone yields the conclusion that

$$\overline{\operatorname{grad}\mathbf{v}}^{\bullet} = \mathbf{0}. \tag{168}$$

Thus, when gradient kinetic energy is accounted for and inertial body forces are neglected, it follows from (158) that the flow equation (160) reduces to the form (82) of the theory without gradient kinetic energy. In addition, as noted in the paragraph following (166), the generalized adherence condition (99) is unaltered. As a further consequence of (168), it follows from (166) that the conditions at a surface that allows for slip reduce to the form (103) in the theory without gradient kinetic energy. The problem of plane Poiseuille flow for the theory with gradient kinetic energy therefore reduces to the problem considered in Section 10 and the resulting solutions for generalized adherence and slip conditions are therefore unchanged.

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