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# *Surfactants in Foam Stability: A Phase-Field Model*

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**Abstract**

The role of surfactants in stabilizing the formation of bubbles in foams is studied using a phase-field model. The analysis is centered on a van der Walls–Cahn–Hilliard-type energy with an added term which accounts for the interplay between the presence of a surfactant density and the creation of interfaces. In particular, it is concluded that the surfactant segregates to the interfaces, and that the prescription of the distribution of surfactant will dictate the locus of interfaces, which is in agreement with the experimental results.

## 1. Introduction

In this paper we use a phase-field model in an attempt to explain the role of surfactants in stabilizing, and possibly encouraging, the formation of bubbles in foams. Ultimately, the goal is to examine solid foams (e.g. oxides such as $AL_2O_3$) and metallic foams which have important applications in industry such as the manufacturing of lightweight sandwich structures in automotive engineering, and in biotechnology, for example in the making of highly-porous scaffolds for bone tissue engineering. Most research has focused on aqueous foams (shampoo, dishwasher detergent, beer and soap froth etc.), with some incursions into polymeric foams, but the realm of solid foams has been virtually untouched by a rigorous mathematical treatment. In solid foams anisotropy plays a very important role in determining the polyhedral shapes in cellular packing, and an important analytical and geometrical challenge is to explain the different sizes of clusters in cellular packing.

The applicability of foams depends in a crucial way on their wetness. It is well known that very little liquid is contained on the faces of the bubbles and that most of it migrates to the edges of the lattice, i.e. regions between three touching bubbles (*plateau borders* in liquid foams, *struts* in solid foams), and the junctions of four channels (*nodes* in liquid foams, *joints* in solid foams). Therefore, it is of the utmost interest to understand the mechanism by which surfactants (such as soap) induce the formation of interfaces.

Here we adopt the (commonly agreed) viewpoint that formation of bubbles is intrinsically related to phase transitions phenomena, and that solid foams and liquid foams share many topological and geometrical properties, due in part to the fact that solid foams typically evolve in the fluid state as gas bubbles, expanding and deforming under the influence of viscous forces, surface tension and surfactants etc. (see [18]). This conforms to the model proposed by PERKINS, SEKERKA, WARREN and LANGER [23] which is a modification of van der Waals–Cahn–Hilliard model for fluid–fluid phase transitions, with an added term that accounts for the influence of the surfactant in preventing the coalescence of bubbles and in encouraging the formation of interfaces. Precisely, let $\Omega \subset \mathbb{R}^N$ be a domain, and let $u : \Omega \to \mathbb{R}$ be a phase (order) parameter, where $u = 1$ corresponds to the liquid (water) phase and $u = 0$ to the gas (argon) phase. Another parameter of the model is the density $\rho : \Omega \to [0, +\infty)$ of the surfactant. The volume of the surfactant is given *a priori* and fixed, and the total amount of bulk material is preserved, i.e.

$$\int_\Omega \rho\, dx = \alpha \quad \text{and} \quad \int_\Omega u\, dx = \beta \tag{1.1}$$

for some $\alpha, \beta > 0$ with $\beta < |\Omega|$. The energy of the system is given by

$$G_\varepsilon(u, \rho) := \frac{1}{\varepsilon}\int_\Omega f(u)\, dx + \varepsilon \int_\Omega |\nabla u|^2\, dx + \alpha(\varepsilon)\int_\Omega (\rho - |\nabla u|)^2\, dx, \tag{1.2}$$

where $\varepsilon, \alpha(\varepsilon) > 0$ are small parameters, and the double-well potential $f : \mathbb{R} \to [0, +\infty)$, with $\{f = 0\} = \{0, 1\}$, drives the system to the two phases.

We want to study the stable configurations of the physical system, which correspond to (local) minimizers of the energy. Since $\varepsilon$ is a small parameter, it is a usual procedure to study the problem as $\varepsilon \to 0$, investigate the properties of the limiting system and then transfer those back to the original system with $\varepsilon$ small enough. The right mathematical tool for this is De Giorgi's $\Gamma$-convergence (see [11]).

The asymptotic behavior of the model depends on the parameter $\alpha(\varepsilon)$ and we expect the physically relevant case to emerge when $\alpha(\varepsilon) = O(\varepsilon)$. Indeed, if $\alpha(\varepsilon) \ll \varepsilon$ then the surfactant energy term $\alpha(\varepsilon)\int_\Omega (\rho - |\nabla u|)^2\, dx$ does not have any influence on the limiting problem, and as $\varepsilon \to 0$ we obtain the well known Cahn–Hilliard model which leads to perimeter minimization. Therefore, the influence of the surfactant is absent, contrary to what we seek with this model. If $\varepsilon \ll \alpha(\varepsilon)$ then it may be shown that the energy becomes degenerate if the volume of the surfactant is smaller than the jump in the order parameter, i.e. the amount of the surfactant is not enough to promote the creation of interfaces and the energy to create a jump is infinite. Again, this goes against our aim, as we expect that even a small amount of the surfactant should influence the interfacial energy. This is exactly what happens when $\alpha(\varepsilon) = \varepsilon$.

In this case we may split the energy into two terms: the Cahn–Hilliard energy

$$\frac{1}{\varepsilon}\int_\Omega f(u)\, dx + \varepsilon \int_\Omega |\nabla u|^2\, dx, \quad \int_\Omega u\, dx = \beta,$$

and the surfactant energy

$$\varepsilon \int_\Omega (\rho - |\nabla u|)^2 \, dx, \quad \int_\Omega \rho(x) \, dx = \alpha.$$

The Cahn–Hilliard energy is responsible for the formation of the interfaces, and the surfactant energy term "promotes" them in that it favors the creation of interfaces where the surfactant is present. Indeed, if we had only the Cahn–Hilliard energy term, then it is a well-known result (see [21]) that as $\varepsilon \to 0$ the problem reduces to the minimization of the perimeter of the jump set of $u$, and minimizers locally have a hyperbolic tangent profile. Combining this with the surfactant energy to obtain the total energy of the system, leads to a compromise as the Cahn–Hilliard energy term penalizes the formation of multiple interfaces while the surfactant term favors the occurrence of interfaces there where $\rho$ is present, or, better, $\rho = |\nabla u|$.

To recall briefly this history, the analysis of the asymptotic behavior of singularly perturbed energies

$$I_\varepsilon(u) := \int_\Omega \frac{1}{\varepsilon} f(u) + \varepsilon |\nabla u|^2 \, dx,$$

where $f$ is a non-negative potential with $\{f = 0\} = \{0, 1\}$ was first studied by Modica and Mortola [20], and subsequently it was applied by Modica [21] to the van der Waals–Cahn–Hilliard theory of fluid–fluid phase transitions to solve an optimal design problem proposed by Gurtin [15]. It was shown in [20, 21] that $\{I_\varepsilon\}$ $\Gamma$ converges (with respect to $L^1$) to

$$I(u) := \begin{cases} \left(2 \int_0^1 \sqrt{f}(s) \, ds\right) \mathcal{H}^{N-1}(S_u) & \text{if } u \in BV(\Omega; \{0, 1\}), \\ +\infty & \text{otherwise,} \end{cases}$$

and thus in the limit as $\varepsilon \to 0$ partitions with minimal interfacial area and given volume fraction $\beta$ are selected. Generalizations were obtained by Bouchitté [6] and by Owen and Sternberg [22] for the coupled problem, and Kohn and Sternberg [19] undertook the study of local minimizers. The vector-valued setting, where $u : \Omega \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}^N$, $d, N > 1$, was considered independently by Sternberg [25] and by Fonseca and Tartar [13], where the latter addressed the two-well setting (see also Barroso and Fonseca [5] and Sternberg [26]).

This was generalized to the case of multiple wells by Baldo [4]. Higher-order variants were addressed by Fonseca and Mategazza [12], Conti, Fonseca and Leoni [9], and current work by Conti and Schweizer [10] extends the latter to the case where $f$ vanishes on two rank-one connected copies of the set of rotations in $\mathbb{R}^N$ and exploits notions related to geometric rigidity (for related issues within the realm of the Eikonal equation we refer to [2, 3, 17] etc).

The main analytical goal of this paper is to identify the asymptotic behavior of equilibria. Precisely, if $(u_\varepsilon, \rho_\varepsilon)$ minimizes $G_\varepsilon$ then can we establish that $\{(u_\varepsilon, \rho_\varepsilon)\}$ converges to some macroscopic state $(u, \rho)$, and, if so, what characterizes $(u, \rho)$, e.g. does $(u, \rho)$ minimize a new, macroscopic (relaxed) energy?

Relaxing the ambiance space of $\rho$ to include non-negative bounded Radon measures $\mu$ (and identifying every integrable surfactant energy density $\rho$ with the measure $\mu = \rho dx$), in Theorem 2.1 we show that when $\alpha(\varepsilon) = \varepsilon$ the asymptotic problem (1.1), (1.2) in the limit as $\varepsilon \to 0$ becomes

$$F(u,\mu) := \begin{cases} \displaystyle\int_{S_u} \phi\left(\frac{d\mu}{d(\mathcal{H}^{N-1}\lfloor S_u)}\right) d\mathcal{H}^{N-1}, & \text{if } u \in BV(\Omega;\{0,1\}) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.3}$$

where $\phi$ is a suitable non-increasing convex surface energy density. Moreover, the function $\phi$ can be characterized through an optimal profile formula (see (2.5)) and it turns out that it is strictly decreasing in $(0, 1)$ and satisfies

$$\phi(0) = 2\sqrt{2}\int_0^1 \sqrt{f(s)}\,ds, \qquad \phi(\gamma) = 2\int_0^1 \sqrt{f(s)}\,ds \quad \text{for } \gamma \geqq 1. \tag{1.4}$$

Based on results from $\Gamma$-convergence theory we conclude that minimizers $(u_\varepsilon, \rho_\varepsilon)$ of $G_\varepsilon$, subject to (1.1), converge (up to a subsequence) to a minimizer $(u, \mu)$ of $F$, subject to the constraints (see the discussion at the end of Section 4)

$$\mu(\Omega) = \alpha \quad \text{and} \quad \int_\Omega u\,dx = \beta. \tag{1.5}$$

A direct inspection of the energy (1.3) allows us to conclude that:

(i) the macroscopic energy $F$ is only sensitive to the restriction of $\mu$ to the interface $S_u$, and we interpret this fact by saying that the surfactant segregates to the interface;
(ii) if we have a prescribed distribution of the surfactant (say, in one-dimension $\mu = \Sigma a_i \delta_{x_i}$) then the interfaces will be created exactly on the support of $\mu$ (resp. at the concentration points $x_i$);
(iii) the macroscopic energy $F$ will remain unchanged if the density of the surfactant density $\mu$ on the interface $S_u$, $d\mu/d\mathcal{H}^{N-1}$, exceeds 1. Indeed, in view of (1.4) the energy is impervious to adding more surfactant and the system reaches saturation;
(iv) the decreasing character of the surface energy density $\phi$ in the interval $(0, 1)$ shows that below the saturation threshold the addition of an arbitrarily small amount of surfactant lowers the surface tension, in agreement with experimentation.

We expect the model to also explain how the presence of surfactant influences the metastability of multiple interface configurations. This expectation is supported by the observation that the lower and the upper bounds for the persistence time of metastable configurations for the Cahn–Hilliard energy depend on the surface tension constant $\sigma := \left(2\int_0^1 \sqrt{f}(s)\,ds\right)$ in a monotone way (see [8, 14, 7] etc.). Although we expect that the $\Gamma$-convergence result obtained in this paper will play a crucial role in the analysis of metastability (as in [7] and [14]), we leave the dynamical aspects of the theory for future investigation.

The paper is organized as follows. In Section 2 we state the central theorem of this work, Theorem 2.1. Section 3 is dedicated to the case where $N = 1$—for expository reasons we start with the proof of Theorem 2.1 in the one-dimensional case where it is possible to present the key ideas in a more transparent way without invoking the heavier technical machinery required in the multidimensional setting. Here we highlight the detailed description of the optimal profile, and Theorem 3.6 where we prove that $\phi$ is convex. This is by no means a necessary condition for lower-semicontinuity of $F$ in the scalar case, but it turns out to be an important ingredient in the proof of the higher-dimensional case. We also mention here Corollary 3.10. which shows that, at least in some important cases, the function $\phi$ is in fact strictly convex in the interval (0,1). This implies that for limiting stable configuration the surfactant is uniformly distributed along the interfaces, in accordance with the so-called "Marangoni effect". Section 4 is devoted to the $N$-dimensional setting, and, finally, in Section 5 we offer a discussion on the role of the surfactant in the stability of the system.

## 2. Statement of the main result

Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz domain and consider the family of functionals

$$G_\varepsilon(u,\rho) := \frac{1}{\varepsilon}\int_\Omega f(u)\,dx + \varepsilon\int_\Omega |\nabla u|^2\,dx + \alpha(\varepsilon)\int_\Omega (\rho - |\nabla u|)^2\,dx \tag{2.1}$$

where $\varepsilon > 0$, $\alpha : [0,+\infty) \to [0,+\infty)$ is continuous at 0 and strictly positive in $(0,+\infty)$, and $f$ is a *double-well potential*, that is $f \in C(\mathbb{R};[0,\infty))$ and $f$ vanishes only at 0 and 1. We will work in the ambiance space $X(\Omega) := L^1(\Omega) \times \mathcal{M}_+(\Omega)$ endowed with the convergence $\tau_1 \times \tau_2$ where $\tau_1$ denotes the strong convergence in $L^1(\Omega)$, while $\tau_2$ denotes the weak*-convergence in the space of non-negative bounded Radon measures $\mathcal{M}_+(\Omega)$. Extend the functionals $G_\varepsilon$ to the whole space $X$ by setting for every $(u,\mu) \in X$

$$F_\varepsilon(u,\mu) := \begin{cases} G_\varepsilon(u,\rho) & \text{if } u \in H^1(\Omega) \text{ and } \mu = \rho dx, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

When $\mu = \rho dx$, with an obvious abuse of notation we will often write $F_\varepsilon(u,\rho)$ instead of $F_\varepsilon(u,\mu)$. We will need to localize the family $G_\varepsilon(u,\rho)$ (and in turn $F_\varepsilon(u,\rho)$): for every open subset $A \subset \Omega$ the functional $G_\varepsilon(u,\rho;A)$ is defined as in (2.1), with $\Omega$ replaced by $A$.

We are interested in the asymptotic behavior of the functionals (2.2) as $\varepsilon \to 0^+$. Our main result is stated in the following theorem.

**Theorem 2.1.** *Let $\varepsilon_n \searrow 0$ and assume that*

$$\lim_{n\to\infty} \frac{\alpha(\varepsilon_n)}{\varepsilon_n} =: c \in [0,+\infty]. \tag{2.3}$$

*Then there exists a non-increasing convex function $\phi : [0,+\infty) \to [0,+\infty]$ such that the family $\{F_{\varepsilon_n}\}$ $\Gamma$-converges with respect to the $\tau_1 \times \tau_2$ convergence in $X(\Omega)$*

*to a functional $F$ of the form*

$$F(u,\mu) := \begin{cases} \displaystyle\int_{S_u} \phi\left(\frac{d\mu}{d(\mathcal{H}^{N-1}\lfloor S_u)}\right) d\mathcal{H}^{N-1}, & \textit{if } u \in BV(\Omega;\{0,1\}) \\ +\infty & \textit{otherwise,} \end{cases} \tag{2.4}$$

*for every $(u,\mu) \in X(\Omega)$. Moreover, the energy density $\phi$ depends on the asymptotic behavior of the sequence $\alpha(\varepsilon_n)$; i.e. on the constant $c$ in* (2.3) *and on $\alpha(0)$, according to the following formulae:*

(i) *if $0 < c < +\infty$, then for every $\gamma \geqq 0$*

$$\phi(\gamma) := \inf\left\{\int_{-\infty}^{+\infty} f(u)\,dx + \int_{-\infty}^{+\infty} \min\{c\lambda^2 + |u'|^2, (1+c)|u'|^2\}\,dx : (u,\lambda) \in \mathcal{A}(\gamma)\right\}, \tag{2.5}$$

*where $\mathcal{A}(\gamma)$ is the class of admissible pairs for $\gamma$ defined as*

$$\mathcal{A}(\gamma) := \left\{(u,\lambda) \in H^1_{\mathrm{loc}}(\mathbb{R})\times(-\infty,0] : \lim_{t\to-\infty} u(t) = 0,\ \lim_{t\to+\infty} u(t) = 1, \int_{-\infty}^{+\infty} \max\{\lambda + |u'|, 0\}\,dx \leqq \gamma\right\}; \tag{2.6}$$

(ii) *if $c = 0$, then*

$$\phi(\gamma) \equiv 2\int_0^1 \sqrt{f(s)}\,ds;$$

(iii) *if $c = +\infty$ and $\alpha(0) = 0$, then*

$$\phi(\gamma) = \begin{cases} 2\int_0^1 \sqrt{f(s)}\,ds & \textit{if } \gamma \geqq 1, \\ +\infty & \textit{otherwise;} \end{cases}$$

(iv) *if $\alpha(0) > 0$; i.e., $\alpha(\varepsilon_n)$ is bounded away from* 0, *then*

$$\phi(\gamma) = \begin{cases} 2\int_0^1 \sqrt{f(s)}\,ds & \textit{if } \gamma = 1, \\ +\infty & \textit{otherwise.} \end{cases}$$

**Remark 2.2.** (Compactness)**.** Assume that the double-well potential $f$ satisfies the following growth condition

$$f(s) \geqq C|s| - \frac{1}{C}$$

for some $C > 0$. Then a comparison with the well-known Modica–Mortola functional shows immediately that every sequence $\{(u_n, \rho_n)\}$ such that

$$\sup_n \int_\Omega \rho_n\,dx < \infty \qquad \text{and} \qquad \sup_n F_{\varepsilon_n}(u_n,\rho_n) < \infty$$

is relatively compact with respect to the $\tau_1 \times \tau_2$-convergence of $X(\Omega)$ (see [13]).

We will mostly focus on the case (i) of Theorem 2.1, which is the most interesting from both the physical and the mathematical viewpoints.

## 3. The one-dimensional case

As we already mentioned in the previous section, we will focus on case (i) of Theorem 2.1 and leave the (easier) proofs for the other regimes to the interested reader.

We may assume without loss of generality that $c = 1$ and $\alpha(\varepsilon) = \varepsilon$. For any $J \subset I$ open subset of $I$, with $I \subset \mathbb{R}$ bounded open interval, and for all $(u, \rho) \in H^1(\Omega)\times L^1_+(\Omega)$ we set

$$G_\varepsilon(u, \rho; J) := \frac{1}{\varepsilon}\int_J f(u)\,dx + \varepsilon\int_J |u'|^2\,dx + \varepsilon\int_J (\rho - |u'|)^2\,dx.$$

The family of functionals we are going to study takes the form

$$F_\varepsilon(u, \mu; J) := \begin{cases} G_\varepsilon(u, \rho; J) & \text{if } u \in H^1(I) \text{ and } \mu = \rho dx, \\ +\infty & \text{otherwise} \end{cases}$$

for all $(u, \mu) \in X(I)$. When $J = I$ we write $F_\varepsilon(u, \mu)$ instead of $F_\varepsilon(u, \mu; J)$. We will show that $\Gamma(X(I))\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon = F$, with $F$ defined as

$$F(u, \mu) := \begin{cases} \displaystyle\sum_{x\in S_u} \phi(\mu(\{x\})) & \text{if } u \in BV(I; \{0, 1\}), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\phi(\gamma) := \inf\left\{\int_{-\infty}^{+\infty} f(u)\,dx + \int_{-\infty}^{+\infty} \min\{\lambda^2 + |u'|^2, 2|u'|^2\}\,dx : (u, \lambda) \in \mathcal{A}(\gamma)\right\}, \tag{3.1}$$

and $\mathcal{A}(\gamma)$ is as in (2.6).

### *3.1. Preliminary lemmas*

If $I \subset \mathbb{R}$ is an open interval then for every $(u, \lambda) \in H^1_{\mathrm{loc}}(I) \times \mathbb{R}$ we denote

$$E(u, \lambda; I) := \int_I f(u)\,dx + \int_I \min\{\lambda^2 + |u'|^2, 2|u'|^2\}\,dx. \tag{3.2}$$

By (3.1) we clearly have

$$\phi(\gamma) := \inf\{E(u, \lambda; \mathbb{R}) : (u, \lambda) \in \mathcal{A}(\gamma)\}. \tag{3.3}$$

As it will be shown in Theorem 3.5, the infimum in the previous formula is attained.

We start by collecting some simple facts which will be used repeatedly in the sequel.

**Remark 3.1.**

(i) If $\gamma \geqq 1$ then

$$\phi(\gamma) = 2\int_0^1 \sqrt{f(s)}\, ds. \tag{3.4}$$

Indeed, if $u \in H^1_{\mathrm{loc}}(\mathbb{R})$ is non-decreasing and satisfies $\lim_{t\to-\infty} u(t) = 0$, $\lim_{t\to+\infty} u(t) = 1$, then $(u, 0) \in \mathcal{A}(\gamma)$. Therefore

$$\begin{aligned}\phi(\gamma) &\leqq \inf\Bigg\{\int_{-\infty}^{+\infty} f(u)\,dx + \int_{-\infty}^{+\infty} |u'|^2\,dx : u \in H^1_{\mathrm{loc}}(\mathbb{R}),\\ &\qquad u \text{ non-decreasing}, \lim_{t\to-\infty} u(t) = 0 \text{ and } \lim_{t\to+\infty} u(t) = 1\Bigg\}\\ &= 2\int_0^1 \sqrt{f(s)}\,ds,\end{aligned}$$

where the last equality is well known, and follows from the solution of the standard Modica–Mortola optimal profile problem (see [20, 21]). Since the opposite inequality is trivial, (3.4) follows.
Similarly if $\gamma = 0$ then $(u, \lambda) \in \mathcal{A}(0)$ entails $|\lambda| \geqq |u'|$ a.e., and thus

$$\begin{aligned}\phi(0) &= \inf\Bigg\{\int_{-\infty}^{+\infty} f(u)\,dx + \int_{-\infty}^{+\infty} 2|u'|^2\,dx : u \in H^1_{\mathrm{loc}}(\mathbb{R}),\\ &\qquad u' \in L^\infty(\mathbb{R}), \lim_{t\to-\infty} u(t) = 0 \text{ and } \lim_{t\to+\infty} u(t) = 1\Bigg\}\\ &= 2\sqrt{2}\int_0^1 \sqrt{f(s)}\,ds,\end{aligned}$$

where again the last equality follows from the solution of the standard Modica–Mortola optimal profile problem.

(ii) The minimization problem in (3.3) is equivalent to

$$\phi(\gamma) = \min\{E(u, \lambda; \mathbb{R}) : (u, \lambda) \in \tilde{\mathcal{A}}(\gamma)\},$$

where

$$\begin{aligned}\tilde{\mathcal{A}}(\gamma) := \Bigg\{&(u, \lambda) \in H^1_{\mathrm{loc}}(\mathbb{R}) \times (-\infty, 0] : \lim_{t\to-\infty} u(t) = 0,\\ &\lim_{t\to+\infty} u(t) = 1, \int_{-\infty}^{+\infty} \max\{\lambda + |u'|, 0\}\,dx = \min\{\gamma, 1\}\Bigg\}.\end{aligned}$$

Indeed, if $\gamma < 1$, $(u, \lambda') \in \mathcal{A}(\gamma)$, and $\int_{-\infty}^{+\infty} \max\{\lambda' + |u'|, 0\}\,dx < \gamma$, then necessarily $\lambda' < 0$. Moreover, assuming without loss of generality that $\int_{-\infty}^{+\infty} |u'|^2\,dx < +\infty$, by Hölder's and Chebyshev's inequalities it is easy to see that $\int_{-\infty}^{+\infty} \max\{\lambda + |u'|, 0\}\, dx < +\infty$ for every $\lambda < 0$. It follows from the dominated convergence theorem that the function

$$\lambda \mapsto \int_{-\infty}^{+\infty} \max\{\lambda + |u'|, 0\}\, dx$$

is continuous in $(-\infty, 0)$, and by Lebesgue monotone convergence theorem it converges to the value $\int_{-\infty}^{+\infty} |u'|\, dx \geqq 1 > \gamma$ as $\lambda \to 0^-$. Thus, in particular, we may find $\lambda'' \in (\lambda', 0)$ such that $\int_{-\infty}^{+\infty} \max\{\lambda'' + |u'|, 0\}\, dx = \gamma$ and $E(u, \lambda''; \mathbb{R}) < E(u, \lambda'; \mathbb{R})$. If $\gamma \geqq 1$ then, as shown in (i), the unique minimizing pair is given by $(u, 0)$ where $u$ is the solution of the Modica–Mortola optimal profile problem, and clearly satisfies $\int_{-\infty}^{+\infty} |u'|\, dx = 1$.

(iii) For all $\lambda \leqq 0$ and $w \geqq 0$ the following identities hold:

$$\min\{\lambda^2 + w^2, 2w^2\} = w^2 + \min\{\lambda^2, w^2\} = w^2 + (\max\{\lambda + w, 0\} - w)^2. \tag{3.5}$$

Thus for every $(u, \lambda) \in \mathcal{A}(\gamma)$ we have

$$E(u, \lambda; I) = \int_I f(u)\, dx + \int_I |u'|^2\, dx + \int_I (\max\{\lambda + |u'|, 0\} - |u'|)^2\, dx. \tag{3.6}$$

We now state and prove some auxiliary results which will be needed in the proof of the Theorem 2.1.

**Lemma 3.2.** *Let $(Y, \mu)$ be a measure space, with $\mu$ a non-atomic and positive measure, and let $g : Y \to [0, +\infty)$ be a non-zero function belonging to $L^1(Y, \mu) \cap L^2(Y, \mu)$. Then for any fixed $0 < \gamma \leqq \int_Y g\, d\mu$ the problem*

$$\min\left\{\int_Y (v - g)^2\, d\mu : v \geqq 0, \int_Y v\, d\mu = \gamma\right\} \tag{3.7}$$

*admits a unique (modulo $\mu$-a.e. equivalence) solution $u$ given by*

$$u := \max\{\lambda + g, 0\},$$

*where $\lambda$ is the unique (non-positive) constant such that*

$$\int_Y \max\{\lambda + g, 0\}\, d\mu = \gamma. \tag{3.8}$$

**Proof.** If $\gamma = \int_Y g\, d\mu$ then trivially the function $g$ itself is the unique minimizer and $\lambda = 0$.

Assume now that $\gamma < \int_Y g\, d\mu$ and consider the "relaxed" problem

$$\min\left\{\int_Y (v - g)^2\, d\mu : v \geqq 0, \int_Y v\, d\mu \leqq \gamma\right\}. \tag{3.9}$$

**Step 1.** We show that if $u$ minimizes (3.9) then $u = \max\{\lambda + g, 0\}$ with $\lambda \leqq 0$.

The existence and uniqueness of the solution to (3.9) are an immediate consequence of the projection theorem in Hilbert spaces after observing that the problem can be recast as

$$\min\{\|v - g\|_{L^2(Y,\mu)} : v \in V\},$$

where $V$ is the closed convex set

$$V := \left\{v \in L^2(Y, \mu) : v \geqq 0, \int_Y v \, d\mu \leqq \gamma\right\}.$$

The solution $u$ is the (unique) projection of $g$ onto $V$, and using the variational characterization of projections, we have

$$\int_Y (v - u)(g - u) \, d\mu \geqq 0 \tag{3.10}$$

for every $v \in V$.

For every $n \in \mathbb{N}$ consider the set $J_n := \{x \in Y : u(x) > 1/n\}$ and let $J := \cup_n J_n = \{x \in Y : u(x) > 0\}$. If $\varphi \in L^\infty(Y, \mu)$ and $\int_{J_n} \varphi \, d\mu = 0$, then the function $v := u \pm \varepsilon\varphi\chi_{J_n}$ belongs to $V$ if $\varepsilon$ is sufficiently small, and in view of (3.10) it follows that $\int_{J_n} (u - g)\varphi \, d\mu = 0$. Since $\mu$ is non-atomic, this in turn implies that $u - g$ is constant $\mu$-a.e. in $J_n$ for every $n$. We conclude that

$$u = \lambda + g \quad \text{a. e. in } J \tag{3.11}$$

for a suitable constant $\lambda$, and $u = 0$ in $Y \setminus J$.

We claim that $u = \max\{\lambda + g, 0\}$, i.e. $\lambda + g \leqq 0$ a.e. on $Y \setminus J$. In order to show this, we assume by contradiction the existence of $\varepsilon > 0$, $J' \subset J$, with $0 < \mu(J') < +\infty$, and of $H \subset Y \setminus J$, with $0 < \mu(H) < +\infty$, such that

$$\lambda + g \geqq \varepsilon \quad \text{in } J' \cup H. \tag{3.12}$$

Here we used the fact that $\mu(J) > 0$, or else $u = 0$ and $u$ would not be a minimizer (take as a competitor $v := \frac{\gamma}{\int_Y g \, d\mu} g$). In particular, $\mu(\{\lambda + g > 0\}) > 0$. Setting $\lambda_t := \lambda - t$ we also have

$$\lambda_t + g > 0 \quad \text{in } J' \cup H \text{ for all } t \in (0, \varepsilon). \tag{3.13}$$

By (3.11) and (3.12), $\lim_{t\to 0} \int_{J'\cup H} (\lambda_t + g) \, d\mu = \int_{J'\cup H} (\lambda + g) \, d\mu > \tilde{\gamma}$, where

$$\tilde{\gamma} := \int_{J'} (\lambda + g) \, d\mu = \int_{J'} u \, d\mu, \tag{3.14}$$

and so we can find $\bar{t} \in (0, \varepsilon)$ such that $\int_{J'\cup H} (\lambda_{\bar{t}} + g) \, d\mu > \tilde{\gamma}$. Moreover, as $\int_{J'} (\lambda_{\bar{t}} + g) \, d\mu < \tilde{\gamma}$ and $\mu$ is non-atomic, there exists $H' \subset H$ such that

$$\int_{J'\cup H'} (\lambda_{\bar{t}} + g) \, d\mu = \tilde{\gamma}. \tag{3.15}$$

Let

$$\bar{u}(x) := \begin{cases} u(x) & \text{if } x \in Y \setminus (J' \cup H'), \\ \lambda_{\bar{t}} + g(x) & \text{if } x \in J' \cup H'. \end{cases}$$

In view of (3.13), (3.14), and (3.15) we have that $\bar{u} \in V$. By Jensen's inequality for every $v$ such that $\int_{J' \cup H'} v \, d\mu = \tilde{\gamma}$ it follows that

$$\begin{aligned} \int_{J' \cup H'} (v - g)^2 \, d\mu &\geqq \frac{1}{\mu(J' \cup H')} \left( \int_{J' \cup H'} (v - g) \, d\mu \right)^2 = \\ &= \frac{1}{\mu(J' \cup H')} \left( \tilde{\gamma} - \int_{J' \cup H'} g \, d\mu \right)^2 = \\ &= \int_{J' \cup H'} (\bar{u} - g)^2 d\mu, \end{aligned} \tag{3.16}$$

where in the last equality we used (3.15). The equality in (3.16) holds if and only if $v - g = \lambda_{\bar{t}}$ a.e. in $J' \cup H'$, that is if and only if $v = \bar{u}$ a.e. in $J' \cup H'$. Therefore, since $u \neq \bar{u}$ in $J' \cup H'$, we have by (3.16)

$$\int_{J' \cup H'} (\bar{u} - g)^2 \, d\mu < \int_{J' \cup H'} (u - g)^2 \, d\mu$$

and, in turn,

$$\int_Y (\bar{u} - g)^2 \, d\mu < \int_Y (u - g)^2 \, d\mu$$

which contradicts the minimality of $u$. We have established the claim, i.e. $u = \max\{\lambda + g, 0\}$. Moreover, since

$$\int_Y u \, d\mu \leqq \int_Y g \, d\mu$$

we conclude that $\lambda \leqq 0$.

**Step 2.** Here we prove that $\lambda$, found in Step 1 satisfies (3.8). If not, then an argument entirely similar to that used in Remark 3.1 (ii) would yield the existence of $\lambda' \in (\lambda, 0)$ such that $v := \max\{\lambda' + g, 0\} \in V$, and by (3.5),

$$\int_Y (v - g)^2 \, d\mu = \int_Y \min\{|\lambda'|^2, g^2\} \, d\mu < \int_Y \min\{|\lambda|^2, g^2\} \, d\mu = \int_Y (u - g)^2 \, d\mu,$$

which violates the minimality of $u$.

**Step 3.** To show the uniqueness of $\lambda$ it suffices to observe that if $\lambda' \neq \lambda$, say $\lambda' < \lambda$, and if

$$\gamma = \int_{A'} (\lambda' + g) \, d\mu = \int_A (\lambda + g) \, d\mu, \tag{3.17}$$

where $A := \{\lambda + g > 0\}$ and $A' := \{\lambda' + g > 0\}$, then clearly $A' \subset A$ and thus

$$\int_A (\lambda + g)\, d\mu = \int_{A'} (\lambda' + g)\, d\mu + \mu(A')(\lambda - \lambda') + \int_{A \setminus A'} (\lambda + g)\, d\mu.$$

The last identity is incompatible with (3.17) unless $\mu(A') = 0$, i.e. $\gamma = 0$, and this contradicts the hypotheses on $\gamma$. □

For every $\gamma \geqq 0$ and for $\delta \in [0, \frac{1}{2})$ define

$$\phi_\delta(\gamma) := \inf_{t>0} \inf \left\{ E(u, \lambda; (-t, t)) \,:\, (u, \lambda) \in \mathcal{A}_{\delta,t}(\gamma) \right\}, \tag{3.18}$$

where

$$\mathcal{A}_{\delta,t}(\gamma) := \Bigg\{ (u, \lambda) \in H^1(-t, t) \times (-\infty, 0] : u(-t) = \delta,\ u(t) = 1 - \delta, \\ \int_{-t}^{t} \max\{\lambda + |u'|, 0\}\, dx \leqq \gamma + \delta \Bigg\}. \tag{3.19}$$

**Remark 3.3.**

(i) It can easily be shown that

$$\phi_\delta(\gamma) = \inf_{t>0} \inf \left\{ E(u, \lambda; (-t, t)) \,:\, (u, \lambda) \in \tilde{\mathcal{A}}_{\delta,t}(\gamma) \right\},$$

where

$$\tilde{\mathcal{A}}_{\delta,t}(\gamma) := \Bigg\{ (u, \lambda) \in H^1(-t, t) \times (-\infty, 0] : u(-t) = \delta,\ u(t) = 1 - \delta, \\ \int_{-t}^{t} \max\{\lambda + |u'|, 0\}\, dx = \min\{\gamma + \delta,\ 1 - 2\delta\} \Bigg\}.$$

Also, $\phi_\delta(0) = 2\sqrt{2} \int_\delta^{1-\delta} f(s)\, ds$ and

$$\phi_\delta(\gamma) = 2 \int_\delta^{1-\delta} f(s)\, ds \qquad \text{for } \gamma \geqq 1 - 3\delta. \tag{3.20}$$

(ii) Let $(u, \lambda) \in \mathcal{A}_{0,t}(\gamma)$, and for $t' > t$ let $\bar{u}$ be the function which is zero in $(-t', -t)$, coincides with $u$ in $(-t, t)$, and equals 1 in $(t, t')$. Then $(\bar{u}, \lambda) \in \mathcal{A}_{0,t'}(\gamma)$ and $E(u, \lambda; (-t, t)) = E(\bar{u}, \lambda; (-t', t'))$. Hence,

$$\phi_0(\gamma) = \lim_{t\to\infty} \inf \left\{ E(u, \lambda; (-t, t)) \,:\, (u, \lambda) \in \mathcal{A}_{0,t}(\gamma) \right\}. \tag{3.21}$$

Note that $\mathcal{A}_{0,t}(\gamma + \delta) \subset \mathcal{A}_{0,t}(\gamma + \delta')$ if $\delta < \delta'$, and so we can write

$$\tilde{\phi}_0(\gamma) := \sup_{\delta>0} \phi_0(\gamma + \delta) = \lim_{\delta\to 0^+} \phi_0(\gamma + \delta).$$

In particular we have

$$\tilde{\phi}_0(\gamma) = 2 \int_0^1 f(s)\, ds \quad \text{if } \gamma \geqq 1. \tag{3.22}$$

**Lemma 3.4.** *For every $\gamma \geqq 0$ there holds*

$$\phi_0(\gamma) = \phi(\gamma) \quad \text{and} \quad \phi_\delta(\gamma) \nearrow \tilde{\phi}_0(\gamma) \text{ as } \delta \to 0^+. \tag{3.23}$$

**Proof.**

**Step 1.** We show that $\phi_\delta(\gamma) \nearrow \tilde{\phi}_0(\gamma)$ as $\delta \to 0^+$. If $\gamma \geqq 1$ by (3.20) and (3.22) we have

$$\phi_\delta(\gamma) = 2\int_\delta^{1-\delta} f(s)\, ds,$$

and thus $\phi_\delta(\gamma) \to 2\int_0^1 f(s)\, ds = \tilde{\phi}_0(\gamma)$ as $\delta \to 0^+$.

Suppose now $\gamma < 1$. If $t > 0$ and $(u, \lambda) \in \mathcal{A}_{0,t}(\gamma + \delta)$ then, by continuity, we can find $-t < t_1 < t_2 < t$ such that $u(t_1) = \delta$ and $u(t_2) = 1 - \delta$. Setting $\bar{u}(\cdot) := u(\cdot + t_1 + t_2/2)$, the pair $(\bar{u}, \lambda) \in \mathcal{A}_{\delta, t_2 - t_1/2}(\gamma)$ and, due to the translation invariance of $E$,

$$\phi_\delta(\gamma) \leqq E\left(\bar{u}, \lambda; \left(-\tfrac{t_2-t_1}{2}, \tfrac{t_2-t_1}{2}\right)\right) = E(u, \lambda; (t_1, t_2)) \leqq E(u, \lambda; (-t, t)),$$

which yields $\phi_\delta(\gamma) \leqq \phi_0(\gamma + \delta)$ for every $\gamma \in [0, 1)$, and thus

$$\limsup \phi_\delta(\gamma) \leqq \tilde{\phi}_0(\gamma). \tag{3.24}$$

In order to prove the opposite inequality, let $(u_n, \lambda_n) \in \mathcal{A}_{\delta_n, t_n}(\gamma)$ be such that $t_n > 0$, $\delta_n \to 0^+$, and

$$\lim_{n \to +\infty} E(u_n, \lambda_n; (-t_n, t_n)) = \lim_{n \to \infty} \phi_{\delta_n}(\gamma) = \liminf_{\delta \to 0} \phi_\delta(\gamma). \tag{3.25}$$

We claim that there exists a constant $c > 0$ such that

$$\lambda_n < -c \text{ for all } n \in \mathbb{N}. \tag{3.26}$$

Indeed, assume by contradiction that up to a subsequence (not relabeled), $\lambda_n \to 0$. By continuity, for any fixed $\delta$ satisfying (recall that $\gamma < 1$)

$$1 - 2\delta > \gamma \tag{3.27}$$

and for $n$ large enough, we can find an interval $I_n := (x_{1,n}, x_{2,n}) \subset (-t_n, t_n)$ such that $\delta \leqq u_n \leqq 1 - \delta$ in $I_n$, $u_n(x_{1,n}) = \delta$, and $u_n(x_{2,n}) = 1 - \delta$. Since by (3.25)

$$|I_n| \min_{u \in [\delta, 1-\delta]} f(u) \leqq \int_{I_n} f(u_n)\, dx \leqq \sup_{m \in \mathbb{N}} E(u_m, \lambda_m; I_m) < \infty,$$

we deduce that $\sup_n |I_n| < +\infty$. Therefore, since $(u_n \lambda_n) \in A_{\delta_n, t_n}(\gamma)$ we have

$$\begin{aligned} \gamma = \lim_{n \to \infty} (\gamma + \delta_n) &\geqq \limsup_{n \to \infty} \int_{I_n} \max\{\lambda_n + |u_n'|, 0\}\, dx \\ &\geqq \limsup_{n \to \infty} \int_{I_n} (\lambda_n + |u_n'|)\, dx \geqq \lim_{n \to \infty} (1 - 2\delta + |I_n|\lambda_n) = 1 - 2\delta, \end{aligned}$$

which contradicts (3.27). This establishes claim (3.26).

Set

$$T_{1,n} := \frac{\delta_n}{\sqrt{\int_0^1 f(\delta_n y)\,dy}}, \quad T_{2,n} := \frac{\delta_n}{\sqrt{\int_0^1 f(\delta_n y + 1 - \delta_n)\,dy}}, \tag{3.28}$$

and define

$$\bar{u}_n(x) := \begin{cases} 0 & \text{if } x \leqq -t_n - T_{1,n}, \\ \dfrac{\delta_n}{T_{1,n}}\big(x + t_n + T_{1,n}\big) & \text{if } -t_n - T_{1,n} \leqq x \leqq -t_n, \\ u_n(x) & \text{if } -t_n \leqq x \leqq t_n, \\ \dfrac{\delta_n}{T_{2,n}}(x - t_n) + 1 - \delta_n & \text{if } t_n \leqq x \leqq t_n + T_{2,n}, \\ 1 & \text{if } x \geqq t_n + T_{2,n}. \end{cases}$$

Note that

$$\lim_{n\to\infty} \frac{\delta_n}{T_{1,n}} = \lim_{n\to\infty} \sqrt{\int_0^1 f(\delta_n y)\,dy} = 0$$

and

$$\lim_{n\to\infty} \frac{\delta_n}{T_{2,n}} = \lim_{n\to\infty} \sqrt{\int_0^1 f(\delta_n y + 1 - \delta_n)\,dy} = 0$$

since $f(0) = f(1) = 0$. Therefore, recalling that $\lambda_n$ is bounded away from 0 (see (3.26)), for $n$ sufficiently large we have

$$\lambda_n + \bar{u}_n' < 0 \text{ in } (-\infty, -t_n) \cup (t_n, \infty),$$

which yields for any $T > t_n$

$$\int_{-T}^{T} \max\{\lambda_n + |\bar{u}_n'|, 0\}\,dx = \int_{-t_n}^{t_n} \max\{\lambda_n + |u_n'|, 0\}\,dx \leqq \gamma + \delta_n$$

since $(u_n, \lambda_n) \in \mathcal{A}_{\delta_n, t_n}(\gamma)$. This shows that for $n$ large enough $(\bar{u}_n, \lambda_n) \in \mathcal{A}_{0,T}(\gamma + \delta_n)$ for every $T > t_n + \max\{T_{1,n}, T_{2,n}\}$. Choosing any such $T$ and using (3.5) we can estimate

$$
\begin{aligned}
\phi_0(\gamma+\delta_n) &\leqq E(\bar{u}_n,\lambda_n;(-T,T))\\
&\leqq E(\bar{u}_n,\lambda_n;(-t_n,t_n)) + \int_{-t_n-T_{1,n}}^{-t_n} f\left(\tfrac{\delta_n}{T_{1,n}}\left(x+t_n+T_{1,n}\right)\right)dx\\
&\quad+\int_{t_n}^{t_n+T_{2,n}} f\left(\tfrac{\delta_n}{T_{2,n}}(x-t_n)+1-\delta_n\right)dx + \frac{2\delta_n^2}{T_{1,n}} + \frac{2\delta_n^2}{T_{2,n}}\\
&= E(u_n,\lambda_n;(-t_n,t_n)) + T_{1,n}\int_0^1 f(\delta_n y)\,dy\\
&\quad+T_{2,n}\int_0^1 f(\delta_n y+1-\delta_n)\,dy + \frac{2\delta_n^2}{T_{1,n}} + \frac{2\delta_n^2}{T_{2,n}}\\
&= E(u_n,\lambda_n;(-t_n,t_n)) + 3\delta_n\sqrt{\int_0^1 f(\delta_n y)\,dy}\\
&\quad+3\delta_n\sqrt{\int_0^1 f(\delta_n y+1-\delta_n)\,dy},
\end{aligned}
\tag{3.29}
$$

where in the last equality we used (3.28). Letting $n \to \infty$ in (3.29), and recalling (3.25), we deduce that $\tilde{\phi}_0(\gamma) \leqq \liminf_{\delta\to 0}\phi_\delta(\gamma)$ which, together with (3.24), yields that $\lim_{\delta\to 0^+}\phi_\delta(\gamma) = \tilde{\phi}_0(\gamma)$.

**Step 2.** We show that $\phi_0(\gamma) = \phi(\gamma)$. We remark that if $\gamma \geqq 1$ then clearly $\phi_0(\gamma) = \phi(\gamma)$. If $\gamma < 1$ then $\lambda < 0$, and trivially $\phi \leqq \phi_0$. To establish the opposite inequality it is enough to show that for any $(u,\lambda) \in \mathcal{A}(\gamma)$ we can construct a sequence $\{(u_n,\lambda)\}$, with $(u_n,\lambda) \in \mathcal{A}_{0,n+1}(\gamma)$, such that $E(u_n,\lambda;(-n-1,n+1)) \to E(u,\lambda;\mathbb{R})$. This can be done by considering the restriction of $u$ to the interval $(-n,n)$ and then by connecting $u(n)$ to 1 on $[n,n+1]$ and $u(-n)$ to $-1$ on $[-(n+1),-n]$ with affine functions. Then, since $\lambda < 0$ for $n$ large enough $(u_n,\lambda)$ is admissible for $A_{0,n+1}(\gamma)$ and satisfies the required approximation property. We leave the details to the reader. □

The next theorem deals with the existence of an optimal profile, that is, of a minimizing pair for the problem (3.1). Although the direct method of the calculus of variations cannot be applied due to the lack of convexity of $E(\cdot,\cdot;\mathbb{R})$, and therefore possible failure of lower-semicontinuity, the proof will be achieved by showing that lower-semicontinuity is ensured along minimizing sequences.

**Theorem 3.5** (Existence of an optimal profile)**.** *For every $\gamma \geqq 0$ the optimal profile problem* (3.1) *admits a solution $(u,\lambda) \in \mathcal{A}(\gamma)$, with $u$ a non-decreasing function. Moreover, for every optimal pair $(u,\lambda) \in \mathcal{A}(\gamma)$ the function $u$ is non-decreasing and strictly increasing in the set* $\{0 < u < 1\}$.

**Proof.** We only consider the case $\gamma \in (0,1)$ since for $\gamma \notin (0,1)$ the problem reduces to the standard Modica–Mortola optimal profile problem (see Remark 3.1 (ii)). We split the proof in two steps.

**Step 1.** With the notation introduced in (3.18), (3.19), for any fixed $T > 0$ we consider the problem

$$\min\{E(u,\lambda;(-T,T)) : (u,\lambda) \in \mathcal{A}_{0,T}(\gamma)\}$$

and we show that it admits a non-decreasing solution.

Let $\{(u_n, \lambda_n)\}$ be a minimizing sequence, extract a subsequence (not relabeled) such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(-T, T). \tag{3.30}$$

By Remark 3.3 (i) we can assume that $\int_{-T}^{T} \max\{\lambda_n + |u_n'|, 0\}\, dx = \gamma$. We claim that

$$\sup_n |\lambda_n| < +\infty. \tag{3.31}$$

Indeed, denoting $I_n := \{x \in (-T, T) : |u_n'| > |\lambda_n|\}$, by Hölder's and Chebyshev's Inequalities, we have

$$\begin{aligned}
\int_{-T}^{T} |u_n'|^2\, dx &\geqq \frac{1}{|I_n|}\left(\int_{I_n} |u_n'|\, dx\right)^2 \\
&\geqq \frac{1}{|I_n|}\left(\int_{-T}^{T} \max\{\lambda_n + |u_n'|, 0\}\, dx\right)^2 \\
&= \frac{\gamma^2}{|I_n|} \geqq \gamma^2 \frac{|\lambda_n|^2}{\int_{-T}^{T} |u_n'|^2\, dx},
\end{aligned}$$

and this implies (3.31) since $\sup_n \int_{-T}^{T} |u_n'|^2\, dx < +\infty$. By (3.30) and (3.31) we can extract a further subsequence such that

$$|u_n'| \rightharpoonup w \quad \text{and} \quad \max\{\lambda_n + |u_n'|, 0\} \rightharpoonup z \quad \text{weakly in } L^2(-T, T). \tag{3.32}$$

Using (3.5) and noticing that the function

$$(x, y) \mapsto (x - y)^2$$

is convex, by lower-semicontinuity and Fatou's lemma we obtain

$$\begin{aligned}
&\lim_{n\to\infty} E(u_n, \lambda_n; (-T, T)) \\
&= \lim_{n\to\infty} \left(\int_{-T}^{T} f(u_n)\, dx + \int_{-T}^{T} |u_n'|^2\, dx + \right. \\
&\qquad\qquad \left. + \int_{-T}^{T} (\max\{\lambda_n + |u_n'|, 0\} - |u_n'|)^2\, dx\right) \\
&\geqq \int_{-T}^{T} f(u)\, dx + \int_{-T}^{T} |u'|^2\, dx + \int_{-T}^{T} (z - w)^2\, dx.
\end{aligned} \tag{3.33}$$

By Lemma 3.2, identity (3.5), and the fact that $w \geqq |u'|$ (which follows from (3.30) and (3.32)), we also have

$$\begin{aligned}
\int_{-T}^{T} (z - w)^2\, dx &\geqq \int_{-T}^{T} (\max\{\lambda + w, 0\} - w)^2\, dx \\
&= \int_{-T}^{T} \min\{\lambda^2, w^2\}\, dx \geqq \int_{-T}^{T} \min\{\lambda^2, |u'|^2\}\, dx,
\end{aligned} \tag{3.34}$$

where $\lambda < 0$ is uniquely determined by

$$\int_{-T}^{T} \max\{\lambda + w, 0\}\, dx = \int_{-T}^{T} z\, dx = \gamma.$$

Since

$$\int_{-T}^{T} \max\{\lambda + |u'|, 0\}\, dx \leqq \int_{-T}^{T} \max\{\lambda + w, 0\}\, dx = \gamma$$

we deduce that $(u, \lambda) \in \mathcal{A}_{0,T}(\gamma)$, and from (3.33), (3.34), (3.5) we obtain

$$\begin{aligned}\lim_{n\to\infty} E(u_n, \lambda_n; (-T, T)) &\geqq \int_{-T}^{T} f(u)\, dx + \int_{-T}^{T} \min\{\lambda^2 + |u'|^2, 2|u'|^2\}\, dx \\ &= E(u, \lambda; (-T, T)),\end{aligned}$$

and this implies that $(u, \lambda)$ is an optimal pair.

In order to show that $u$ is non-decreasing, we first observe that a truncation argument yields $0 \leqq u \leqq 1$ in $(-T, T)$. Now suppose by contradiction that there exist $0 < x_1 < x_2$ such that $u(x_1) > u(x_2)$. We can also assume without loss of generality that

$$u(x_1) = \max_{x\in[-T,x_2]} u(x), \quad u(x_2) = \min_{x\in[x_1,T]} u(x). \tag{3.35}$$

Let $\bar{v}$ be such that

$$f(\bar{v}) = \min_{v\in[u(x_2),u(x_1)]} f(v). \tag{3.36}$$

If $\bar{v} = u(x_1)$ then we consider the first point $x_3 \in (x_2, T]$ for which $u(x_3) = u(x_1)$ (such a point exists since $u(T) = 1 \geqq u(x_1)$), and define a new function $\bar{u}$ as

$$\bar{u}(x) := \begin{cases} u(x) & \text{for } x \in (-T, T) \setminus [x_1, x_3], \\ \bar{v} & \text{for } x \in (x_1, x_3). \end{cases}$$

If $\bar{v} < u(x_1)$ then we consider the last point $x_0 \in [-T, x_1)$ and the first point $x_3 \in (x_1, x_2]$ such that $u(x_0) = u(x_3) = \bar{v}$, and define

$$\bar{u}(x) := \begin{cases} u(x) & \text{for } x \in (-T, T) \setminus [x_0, x_3], \\ \bar{v} & \text{for } x \in (x_0, x_3). \end{cases}$$

In both cases, using (3.35) and (3.36) it follows that that $(\bar{u}, \lambda) \in \mathcal{A}_{0,T}(\gamma)$ and $E(\bar{u}, \lambda; (-T, T)) < E(u, \lambda; (-T, T))$, and this contradicts the minimality of $(u, \lambda)$.

**Step 2.** Given a sequence $T_n \uparrow +\infty$, let $(u_n, \lambda_n)$ be a solution of the problem considered in Step 1 with $T = T_n$. By Lemma 3.4 and (3.21) we have $\lim_{n\to\infty} E(u_n, \lambda_n; (-T_n, T_n)) = \phi(\gamma)$. Extending $u_n$ to $\mathbb{R}$ as $u_n := \chi_{(0,+\infty)}$ in

$\mathbb{R} \setminus (-T_n, T_n)$, using the translation invariance of $E$ and monotonicity of $u_n$ we may assume without loss of generality that

$$u_n \leqq \frac{1}{2} \text{ in } (-\infty, 0] \quad \text{and} \quad u_n \geqq \frac{1}{2} \text{ in } [0, +\infty). \tag{3.37}$$

Arguing as in the previous step, up to the extraction of a subsequence we have $u_n \rightharpoonup u$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R})$, for some function $u \in H^1_{\mathrm{loc}}(\mathbb{R})$, and we can find $\lambda < 0$ such that

$$\int_{-\infty}^{+\infty} \max\{\lambda + |u'|, 0\}\, dx \leqq \gamma \quad \text{and} \quad E(u, \lambda; \mathbb{R}) \leqq \lim_{n\to\infty} E(u_n, \lambda_n; \mathbb{R}) = \phi(\gamma).$$

As each $u_n$ is non-increasing, $u$ is also non-decreasing, and thus there exist $\lim_{x\to+\infty} u(x) =: \alpha$ and $\lim_{x\to-\infty} u(x) =: \beta$. Since by (3.37) $u \leqq \frac{1}{2}$ in $(-\infty, 0]$ and $u \geqq \frac{1}{2}$ in $[0, +\infty)$, and taking into account that $\int_{\mathbb{R}} f(u)\, dx < +\infty$, we must have $\alpha = 1$ and $\beta = 0$. We can now conclude that $(u, \lambda)$ belongs to $\mathcal{A}(\gamma)$ and minimizes $E$. Finally, the monotonicity of any optimal function $u$ can be proved exactly as in Step 1, while the strict monotonicity in the set $\{0 < u < 1\}$ follows from the observation that the energy can be strictly decreased by removing the intervals where $u \equiv c$, with $c \in (0, 1)$. $\square$

We now show that the surface energy density $\phi$ is convex (see Fig 3.1). This fact will play a crucial role in the $N$-dimensional estimates.

**Theorem 3.6.** *The function $\phi$ defined in* (3.1) *is convex.*

**Proof.** The proof will be split in several steps.

**Step 1.** We start by considering the following auxiliary energy density:

$$\psi(M, a, b; \gamma) := \inf_{\mu>0} \inf \left\{ M\mu + \int_0^{\mu} \min\{\lambda^2 + |u'|^2, 2|u'|^2\}\, dx : \right.$$
$$\left. (u, \lambda) \in \mathcal{A}(\mu, a, b; \gamma) \right\},$$

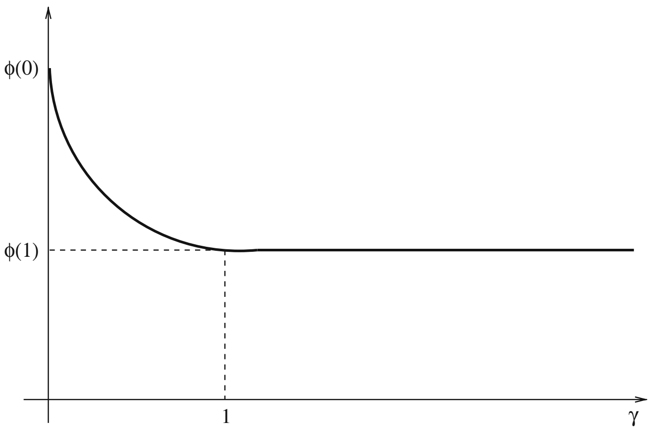


**Fig. 3.1.** The surface density $\phi$.

where $M > 0$, $0 < a < b$, $\gamma \geqq 0$, and

$$\mathcal{A}(\mu, a, b; \gamma) := \left\{ (u, \lambda) \in H^1(0, \mu) \times (-\infty, 0] : u(0) = a,\ u(\mu) = b, \right.$$
$$\left. \int_0^\mu \max\{\lambda + |u'|, 0\}\, dx \leqq \gamma \right\}. \tag{3.38}$$

Note that $\psi(M, a, b; \cdot)$ is non-increasing and we prove that it is convex. Arguing as in Remark 3.1, it can be shown that the class $\mathcal{A}(\mu, a, b; \gamma)$ can be replaced by

$$\tilde{\mathcal{A}}(\mu, a, b; \gamma) := \left\{ (u, \lambda) \in H^1(0, \mu) \times (-\infty, 0] : u(0) = a,\ u(\mu) = b, \right.$$
$$\left. \int_0^\mu \max\{\lambda + |u'|, 0\}\, dx = \min\{\gamma, b - a\} \right\},$$

without affecting the definition of $\psi$. Fix $\mu > 0$ and let $(u, \lambda) \in \tilde{\mathcal{A}}(\mu, a, b; \gamma)$ be a minimizer of

$$M\mu + \int_0^\mu \min\{\lambda^2 + |u'|^2, 2|u'|^2\}\, dx.$$

As in the proof of Theorem 3.5, we have that $u$ is increasing and, using Lemma 3.2 and identity (3.5), we deduce that the pair $(u, \max\{u' + \lambda, 0\})$ minimizes

$$M\mu + \int_0^\mu |v'|^2\, dx + \int_0^\mu (\rho - |v'|)^2\, dx, \tag{3.39}$$

among all pairs $(v, \rho)$ such that $v \in H^1(0, \mu)$ with $v(0) = a$ and $v(\mu) = b$, and $\rho \geqq 0$ satisfies

$$\int_0^\mu \rho\, dx = \min\{\gamma, b - a\}. \tag{3.40}$$

Indeed, if $(v, \rho)$ is one such pair, then

$$\begin{aligned}
&\int_0^\mu |v'|^2\, dx + \int_0^\mu (\rho - |v'|)^2\, dx \\
&\quad \geqq \int_0^\mu |v'|^2\, dx + \int_0^\mu (\max\{|v'| + \bar{\lambda}, 0\} - |v'|)^2\, dx \\
&\quad = \int_0^\mu \min\{|v'|^2 + \bar{\lambda}^2, 2|v'|^2\}\, dx \geqq \int_0^\mu \min\{|u'|^2 + \lambda^2, 2|u'|^2\}\, dx \\
&\quad = \int_0^\mu |u'|^2\, dx + \int_0^\mu (\max\{u' + \lambda, 0\} - u')^2\, dx,
\end{aligned}$$

where $\bar{\lambda}$ is such that

$$\int_0^\mu \max\{\bar{\lambda} + |v'|, 0\}\, dx = \min\{\gamma, b - a\}.$$

Therefore, variations of the form $(u + \varepsilon\varphi, \max\{u' + \lambda, 0\})$ in the functional (3.39) yield the Euler–Lagrange equation

$$2u' - \max\{u' + \lambda, 0\} = C \quad \text{a.e. in } (0, \mu) \tag{3.41}$$

for a suitable constant $C$. From (3.41) it immediately follows that

$$u' = C/2 \quad \text{a.e. on } A := \{x \in (0, \mu) : u' < -\lambda\} \tag{3.42}$$

and

$$u' = C + \lambda \quad \text{a.e. on } B := \{x \in (0, \mu) : u' \geqq -\lambda\}. \tag{3.43}$$

We claim that $u'$ is constant almost everywhere. If $\gamma = 0$ then by (3.40) $\max\{u' + \lambda, 0\} = 0$ almost everywhere and so the claim follows immediately from (3.41). If $\gamma > 0$ then we show that $|A| = 0$. Indeed, suppose by contradiction that $A$ has positive measure. Since $B$ must also have positive measure (otherwise $\rho = \max\{u' + \lambda, 0\} = 0$ a.e. and (3.40) would be violated), from (3.42) we deduce $C < -2\lambda$, whereas from (3.43) we get $C \geqq -2\lambda$ and thus we have a contradiction. We conclude that $u$ is affine in the interval $(0, \mu)$.

We are now in a position to compute explicitly the value of $\psi(M, a, b; \gamma)$. From the preceding discussion we know that there is a unique optimal pair $(u, \rho)$ for (3.39) given by

$$u(x) = a + \frac{b-a}{\mu}x \quad \text{and} \quad \rho = u' + \lambda,$$

where $\lambda$ is determined by (3.40) and reads

$$\lambda = \frac{\min\{(\gamma - (b-a)), 0\}}{\mu}.$$

We conclude that

$$\begin{aligned}\psi(M, a, b; \gamma) &= \min_{\mu>0} \left\{ M\mu + \frac{(b-a)^2}{\mu} + \frac{[\min\{(\gamma - (b-a)), 0\}]^2}{\mu} \right\} \\ &= \begin{cases} 2\sqrt{M}\sqrt{(b-a)^2 + (\gamma - (b-a))^2} & \text{if } \gamma \leqq b-a, \\ 2\sqrt{M}(b-a) & \text{otherwise,} \end{cases}\end{aligned}$$

and thus $\psi(M, a, b; \cdot)$ is convex.

**Step 2.** We now assume that the double-well potential $f$ is lower semicontinuous and piecewise constant in $[0, 1]$, i.e. there exists a finite subdivision

$$0 = a_0 < a_1 < \cdots < a_{m-1} < a_m = 1,$$

and $M_i > 0$, $i = 1, \ldots, m$, such that $f \equiv M_i$ in $(a_{i-1}, a_i)$. We claim that

$$\begin{aligned}\phi(\gamma) &= \min\left\{ \sum_{i=1}^{m} \psi(M_i, a_{i-1}, a_i; \gamma_i) : \gamma_i \geqq 0, \sum_{i=1}^{m} \gamma_i = \min\{\gamma, 1\} \right\} \\ &= \begin{cases} \big[\, \psi(M_1, a_0, a_1; \cdot) \,\square \ldots \square\, \psi(M_m, a_{m-1}, a_m; \cdot) \big](\gamma) & \text{if } \gamma \leqq 1, \\[2ex] \big[\, \psi(M_1, a_0, a_1; \cdot) \,\square \ldots \square\, \psi(M_m, a_{m-1}, a_m; \cdot) \big](1) & \text{if } \gamma > 1, \end{cases}\end{aligned}$$

where the symbol "$\square$" denotes the *infimal convolution* (see ROCKAFELLAR [24]). Once (3.44) is established the convexity of $\phi$ follows from Step 1 and the fact that the infimal convolution of convex non-increasing functions is still convex and non-increasing (see again [24]).

In order to prove the claim, we first observe that there exists an optimal pair $(u, \lambda)$ for the infimum problem defining $\phi(\gamma)$. Indeed, the same argument used in the proof of Theorem 3.5 works without changes if we assume the double-well potential to be just lower semicontinuous. We also recall that $u$ must be strictly increasing in the set $u^{-1}(0, 1)$. Since $\min_{u\in(0,1)} f(u) > 0$ the set $u^{-1}(0, 1)$ is a finite interval, and taking into account the strict monotonicity of $u$,

$$u^{-1}(0, 1) = \bigcup_{i=1}^{m} \left(u^{-1}(a_{i-1}), u^{-1}(a_i)\right) \bigcup_{i=1}^{m-1} \{a_i\},$$

where we set

$$u^{-1}(a_0) = u^{-1}(0) := \max\{x \in \mathbb{R} : u(x) = 0\}$$

and

$$u^{-1}(a_m) = u^{-1}(1) := \min\{x \in \mathbb{R} : u(x) = 1\}.$$

Writing $I_i := \left(u^{-1}(a_{i-1}), u^{-1}(a_i)\right)$, $\mu_i := |I_i|$,

$$\tilde{\gamma}_i := \int_{I_i} \max\{u' + \lambda, 0\}\, dx \quad \text{and} \quad v_i(x) := u\left(x + u^{-1}(a_{i-1})\right),$$

we see that $(v_i, \lambda) \in \mathcal{A}(\mu_i, a_{i-1}, a_i; \tilde{\gamma}_i)$ (recall (3.38)),

$$\sum_{i=1}^{m} \tilde{\gamma}_i = \int_{u^{-1}(0,1)} \max\{u' + \lambda, 0\}\, dx = \min\{\gamma, 1\},$$

and thus, by the translation invariance of the energy $E$, we have

$$\begin{aligned}
\phi(\gamma) &\geqq \sum_{i=1}^{m} E(u, \lambda; I_i) = \sum_{i=1}^{m} E(v_i, \lambda; (0, \mu_i)) \\
&\geqq \sum_{i=1}^{m} \psi(M_i, a_{i-1}, a_i; \tilde{\gamma}_i) \\
&\geqq \min\left\{\sum_{i=1}^{m} \psi(M_i, a_{i-1}, a_i; \gamma_i) : \gamma_i \geqq 0, \sum_{i=1}^{m} \gamma_i = \min\{\gamma, 1\}\right\}. \qquad (3.44)
\end{aligned}$$

For the opposite inequality we choose $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m$ such that

$$\sum_{i=1}^{m} \tilde{\gamma}_i = \min\{\gamma, 1\} \qquad (3.45)$$

and

$$\sum_{i=1}^{m} \psi(M_i, a_{i-1}, a_i; \tilde{\gamma}_i)$$
$$= \min\left\{\sum_{i=1}^{m} \psi(M_i, a_{i-1}, a_i; \gamma_i) : \gamma_i \geqq 0, \sum_{i=1}^{m} \gamma_i = \min\{\gamma, 1\}\right\}. \tag{3.46}$$

Correspondingly, we can find $\mu_i$ and $(v_i, \lambda_i) \in \mathcal{A}(\mu_i, a_{i-1}, a_i; \tilde{\gamma}_i)$ such that

$$E(v_i, \lambda_i; (0, \mu_i)) = \psi(M_i, a_{i-1}, a_i; \tilde{\gamma}_i). \tag{3.47}$$

We now set $t_0 := 0$, $t_i := \sum_{j=1}^{i} \mu_j$, for $i = 1, \ldots, m$,

$$u(x) := \begin{cases} 0 & \text{if } x < 0, \\ v_i(x - t_{i-1}) & \text{if } x \in (t_{i-1}, t_i) \text{ for } i = 1, \ldots, m, \\ 1 & \text{if } x > t_m, \end{cases}$$

and

$$\rho(x) := \begin{cases} 0 & \text{if } x < 0, \\ \max\{u' + \lambda_i, 0\} & \text{if } x \in (t_{i-1}, t_i) \text{ for } i = 1, \ldots, m, \\ 0 & \text{if } x > t_m. \end{cases}$$

Finally, take $\lambda$ such that $\int_{-\infty}^{+\infty} \max\{u' + \lambda, 0\}\, dx = \min\{\gamma, 1\}$. Clearly $(u, \lambda)$ belongs to $\mathcal{A}(\gamma)$, i.e. it is admissible for the minimum problem defining $\phi(\gamma)$. Moreover, by (3.45),

$$\int_{-\infty}^{+\infty} \rho\, dx = \sum_{i=1}^{m} \int_0^{\mu_i} \max\{v_i' + \lambda_i, 0\}\, dx$$
$$= \sum_{i=1}^{m} \min\{\gamma_i, a_i - a_{i-1}\} \leqq \min\{\gamma, 1\}.$$

Therefore, by Lemma 3.2 and by (3.5) we deduce that

$$\sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \min\{\lambda_i^2, |u'|^2\}\, dx = \int_{-\infty}^{+\infty} (\rho - u')^2\, dx \geqq$$
$$\geqq \int_{-\infty}^{+\infty} (\max\{u' + \lambda, 0\} - u')^2\, dx = \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \min\{\lambda^2, |u'|^2\}\, dx. \tag{3.48}$$

Using (3.46), (3.47), (3.48), and the translation invariance of $E$, we obtain

$$\begin{aligned}\min\left\{\sum_{i=1}^{m}\psi(M_i, a_{i-1}, a_i; \gamma_i) : \gamma_i \geqq 0,\ \sum_{i=1}^{m}\gamma_i = \min\{\gamma, 1\}\right\}\\ = \sum_{i=1}^{m} E(v_i, \lambda_i; (0, \mu_i)) = \sum_{i=1}^{m} E(u, \lambda_i; (t_{i-1}, t_i))\\ \geqq \sum_{i=1}^{m} E(u, \lambda; (t_{i-1}, t_i)) = E(u, \lambda; \mathbb{R}) \geqq \phi(\gamma),\end{aligned}$$

which, together with (3.44), concludes the proof of the claim.

**Step 3.** Let $f$ be any continuous double-well potential. We conclude by an approximation procedure. Indeed, construct a sequence $f_n$ of lower-semicontinuous double-well potentials satisfying the hypotheses of Step 2, coinciding with $f$ in $\mathbb{R} \setminus (0, 1)$, and decreasing uniformly to $f$. If we call $\phi_n$ the surface energy density associated with $f_n$ according to formula (3.1), then by Step 2 we have that each $\phi_n$ is convex. In order to conclude it is enough to show that $\phi_n \to \phi$ pointwise.

Clearly we have $\liminf_{n\to\infty} \phi_n \geqq \phi$. For the opposite inequality, fix $\gamma \geqq 0$, $\varepsilon > 0$, and choose $t > 0$ and $(u, \lambda) \in \mathcal{A}_{0,t}(\gamma)$ (see (3.19)) such that $u$ is non-decreasing and $E(u, \lambda; (-t, t)) \leqq \phi(\gamma) + \varepsilon$. This is possible thanks to Lemma 3.4 and Theorem 3.5. Denoting by $E_n$ the energy associated with the potential $f_n$, it is easy to see that $E_n(u, \lambda; (-t, t)) \to E(u, \lambda; (-t, t))$. Hence,

$$\limsup_{n\to\infty} \phi_n(\gamma) \leqq \lim_{n\to\infty} E_n(u, \lambda; (0, t)) = E(u, \lambda; (0, t)) \leqq \phi(\gamma) + \varepsilon.$$

The conclusion follows from the arbitrariness of $\varepsilon$. □

**Corollary 3.7.** *Let $\phi_\delta$ be the function defined in* (3.18). *Then $\phi_\delta \nearrow \phi$ as $\delta \to 0^+$ and the convergence is uniform on the compact subsets of* $[0, +\infty)$.

**Proof.** This corollary follows immediately from Lemma 3.4 provided we show that $\phi = \tilde{\phi}_0$. By Lemma 3.4 we have $\phi = \phi_0$, and thus $\phi_0$ is continuous thanks to Theorem 3.6. This, in turn, implies that $\phi_0 = \tilde{\phi}_0$. □

### *3.2. Proof of Theorem 2.1: The case $N = 1$*

**Step 1** ($\Gamma$-liminf inequality)**.** Let $\varepsilon_n \to 0$, $u_n \to u$ in $L^1(I)$, and $\rho_n \stackrel{*}{\rightharpoonup} \mu$ weakly* in $\mathcal{M}_+(I)$. Without loss of generality we may assume that

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n, \rho_n) = \lim_{n\to\infty} F_{\varepsilon_n}(u_n, \rho_n) < +\infty.$$

Then, since

$$F_{\varepsilon_n}(u_{\varepsilon_n}, \rho_{\varepsilon_n}) \geqq \frac{1}{\varepsilon_n}\int_I f(u_{\varepsilon_n})dx + \varepsilon_n \int_I |u'_{\varepsilon_n}|^2 dx,$$

by the well-known Modica–Mortola estimate we get that $u \in BV(I; \{0, 1\})$ and $\sharp S_u < \infty$. Extracting a subsequence, if necessary, we may assume that $u_n \to u$ almost everywhere.

Let $x_0 \in S_u$ be a jump point of $u$ and suppose that $\mu(\{x_0\}) < 1$. Without loss of generality we can also assume

$$\lim_{x \to x_0^-} u(x) = 0 \quad \text{and} \quad \lim_{x \to x_0^+} u(x) = 1. \tag{3.49}$$

Fix $\delta \in (0, \frac{1}{2})$ such that

$$\mu(\{x_0\}) < 1 - 3\delta \tag{3.50}$$

and let $k \in \mathbb{N}$. By the pointwise convergence of $u_n$ and by (3.49), up to a translation we can find two sequences $x_{1,n} \to x_0^-$ and $x_{2,n} \to x_0^+$ such that $u_n(x_{1,n}) = \delta$ and $u_n(x_{2,n}) = 1 - \delta$, and $I_n := (x_{1,n}, x_{2,n}) \subset \left(-\frac{1}{k} + x_0, x_0 + \frac{1}{k}\right)$ for $n \geqq n(k)$. For every $k \in \mathbb{N}$ let $\psi_k$ be a cut-off function such that

$$\operatorname{supp}\psi_k \subseteq \left(-\frac{2}{k} + x_0, x_0 + \frac{2}{k}\right) \quad \text{and} \quad \psi_k \equiv 1 \text{ in } \left(-\frac{1}{k} + x_0, x_0 + \frac{1}{k}\right).$$

We have

$$\limsup_{n \to \infty} \int_{I_n} \rho_n \, dx \leqq \lim_{k \to \infty} \lim_{n \to \infty} \int_I \psi_k \rho_n \, dx = \lim_{k \to \infty} \int_I \psi_k \, d\mu = \mu(\{x_0\}),$$

and thus

$$\int_{I_n} \rho_n \, dx \leqq \mu(\{x_0\}) + \delta \tag{3.51}$$

for $n$ large enough. Setting $v_n(x) := u_n(x_0 + \varepsilon_n x)$ and $\sigma_n := \varepsilon_n \rho_n(x_0 + \varepsilon_n x)$ we can estimate:

$$\begin{aligned}
F_{\varepsilon_n}(u_n, \rho_n; I_n) &= \frac{1}{\varepsilon_n} \int_{I_n} f\left(v_n\left(\tfrac{x - x_0}{\varepsilon_n}\right)\right) dx + \frac{1}{\varepsilon_n} \int_{I_n} \left|v_n'\left(\tfrac{x - x_0}{\varepsilon_n}\right)\right|^2 dx + \\
&\quad + \frac{1}{\varepsilon_n} \int_{I_n} \left(\sigma_n\left(\tfrac{x - x_0}{\varepsilon_n}\right) - \left|v_n'\left(\tfrac{x - x_0}{\varepsilon_n}\right)\right|\right)^2 dx = \\
&= \int_{\varepsilon_n^{-1}(I_n - x_0)} f(v_n) \, dy + \int_{\varepsilon_n^{-1}(I_n - x_0)} |v_n'|^2 \, dy + \\
&\quad + \int_{\varepsilon_n^{-1}(I_n - x_0)} (\sigma_n - |v_n'|)^2 \, dy \geqq \\
&\geqq \int_{\varepsilon_n^{-1}(I_n - x_0)} f(v_n) \, dy + \int_{\varepsilon_n^{-1}(I_n - x_0)} |v_n'|^2 \, dy + \\
&\quad + \int_{\varepsilon_n^{-1}(I_n - x_0)} (\max\{\lambda_n + |v_n'|, 0\} - |v_n'|)^2 \, dy,
\end{aligned} \tag{3.52}$$

where in the last inequality we applied Lemma 3.2, and $\lambda_n$ is determined by

$$\int_{\varepsilon_n^{-1}(I_n-x_0)} \max\{\lambda_n + |v_n'|, 0\}\, dx = \int_{\varepsilon_n^{-1}(I_n-x_0)} \sigma_n\, dx$$
$$= \int_{I_n} \rho_n\, dx \leqq \mu(\{x_0\}) + \delta \tag{3.53}$$

(in the last inequality we have used (3.51)). Note that (3.50) and (3.53) imply $\lambda_n < 0$. Setting $\bar{v}_n(\cdot) := v_n(\cdot - t_n)$, where $t_n$ is chosen in such a way that $t_n + \varepsilon_n^{-1}(I_n - x_0)$ is a symmetric interval centered at the 0, it follows that $(\bar{v}_n, \lambda_n) \in \mathcal{A}_{\delta, \frac{\varepsilon_n^{-1}|I_n|}{2}}(\mu(\{x_0\}))$ and thus, from (3.52), (3.5), and the translation invariance of $E$ we obtain

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n, \rho_n; I_n) \geqq \liminf_{n\to\infty} E(\bar{v}_n, \lambda_n; t_n + \varepsilon_n^{-1}(I_n - x_0)) \geqq \phi_\delta(\mu(\{x_0\})).$$

By Corollary 3.7, letting $\delta \to 0$, we get

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n, \rho_n; I_n) \geqq \phi(\mu(\{x_0\})).$$

If $\mu(\{x_0\}) \geqq 1$ then the above inequality is an immediate consequence of Remark 3.1 (ii) and the classical Modica–Mortola $\Gamma$-convergence result. Repeating the same procedure in a neighborhood of each point $x \in S_u$ we finally obtain

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n, \rho_n; I) \geqq \sum_{x\in S_u} \phi(\mu(\{x\})).$$

**Step 2** ($\Gamma$-limsup inequality)**.** For every $M > 0$ we consider the subset $X_M(I) := \{(u, \mu) \in X(I) : \mu(I) \leqq M\}$ endowed with the convergence inherited from $X(I)$. Setting for every $(u, \mu) \in X_M(I)$

$$\Gamma(X_M(I))\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u, \mu)$$
$$:= \inf\left\{\limsup_{n\to\infty} F_{\varepsilon_n}(u_n, \rho_n) : (u_n, \rho_n) \to (u, \mu) \text{ in } X_M(I)\right\},$$

we can consider the following functional defined for every $(u, \mu) \in X(I)$:

$$\overline{F}_M(u, \mu) := \begin{cases} \Gamma(X_M(I))\text{-}\limsup\limits_{n\to\infty} F_{\varepsilon_n}(u, \mu) & \text{if } (u, \mu) \in X_M(I), \\ +\infty & \text{otherwise.} \end{cases}$$

It is clear that

$$\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n} \leqq \overline{F}_M$$

for every $M > 0$, and thus it will be enough to show that

$$\overline{F}_M(u, \mu) \leqq F(u, \mu) \tag{3.54}$$

for every $M > 0$ and for every $(u, \mu) \in X(I)$ with $\mu(I) \leqq M$. The advantage of considering $\overline{F}_M$ lies in the fact that $\overline{F}_M$ is sequentially lower semicontinuous with respect to the $\tau_1 \times \tau_2$ convergence in $X(I)$. This is an easy consequence of the metrizability of the subset of $X_M(I)$. Note that, on the other hand, the lower-semicontinuity of $\Gamma$- $\limsup_{n\to\infty} F_{\varepsilon_n}$ is not *a priori* clear.

We start by constructing a recovering sequence $\{(u_n, \rho_n)\}$ for a pair $(u, \mu)$ such that $u \in BV(I; \{0, 1\})$ with $\sharp S_u$ finite and $\mu \in \mathcal{S}$, where $\mathcal{S}$ is the class of all positive finite linear combinations of Dirac measures. Write $\mu$ as

$$\mu = \sum_{i=1}^{N_1} \gamma_i \delta_{x_i} + \sum_{i=1}^{N_2} \beta_i \delta_{y_i},$$

where $\gamma_i, \beta_i \geqq 0$, $\cup_{i=1}^{N_1}\{x_i\} = S_u$, and $y_i \in I \setminus S_u$ for $i = 1, \ldots, N_2$. Since the construction of the recovering sequence will be localized near each atom of $\mu$, and $u_n$ will match $u$ on the boundary of disjoint intervals centered at $x_i$ and $y_i$, it suffices to consider the particular case where $\mu = \gamma\delta_{x_1} + \beta\delta_{y_1}$, with $x_1 \in S_u$ and $y_1 \in I \setminus S_u$. By the same reason it is not restrictive to assume that $u(x) = 0$ for $x < x_1$ and $u(x) = 1$ for $x > x_1$. For any fixed $\eta > 0$, by Lemma 3.4 and Remark 3.3 (ii), we can find $t > 0$ and $(v, \lambda) \in \mathcal{A}_{0,t}(\gamma)$ such that

$$\int_{-t}^{t} f(v)\, dx + \int_{-t}^{t} \min\{\lambda^2 + |v'|^2, 2|v'|^2\}\, dx \leqq \phi(\gamma) + \eta, \tag{3.55}$$

where $\lambda$ satisfies

$$\int_{-t}^{t} \max\{\lambda + |v'|, 0\} = \min\{\gamma, 1\}. \tag{3.56}$$

Define $u_n$ as

$$u_n(x) := \begin{cases} 0 & \text{if } x \in I \cap \{y : y < x_1\}, \\ v\left(\frac{2(x-x_1)-\varepsilon_n t}{\varepsilon_n}\right) & \text{if } x \in (x_1, x_1 + t\varepsilon_n), \\ 1 & \text{otherwise in } I, \end{cases}$$

and $\rho_n$ by

$$\rho_n(x) := \begin{cases} \max\left\{|u_n'| + \frac{\lambda}{\varepsilon_n}, 0\right\} & \text{for } x \in (x_1, x_1 + t\varepsilon_n), \\ \dfrac{\max\{\gamma - 1, 0\}}{\sqrt{\varepsilon_n}} & \text{for } x \in (x_1 + t\varepsilon_n, x_1 + t\varepsilon_n + \sqrt{\varepsilon_n}), \\ \dfrac{\beta}{\sqrt{\varepsilon_n}} & \text{for } x \in (y_1, y_1 + \sqrt{\varepsilon_n}), \\ 0 & \text{otherwise in } I. \end{cases}$$

Note that $\rho_n$ is well defined when $n$ is large enough. Clearly $u_n \to u$ in $L^1(I)$, and using (3.56) it is also easy to see that $\int_I \rho_n\, dx = \mu(I) = \gamma + \beta$ for every $n$ and

$\rho_n \overset{*}{\rightharpoonup} \mu$. Moreover, we have

$$
\begin{aligned}
F_{\varepsilon_n}(u_n, \rho_n) = & \frac{1}{\varepsilon_n} \int_{x_1}^{x_1+t\varepsilon_n} f(u_{\varepsilon_n})\, dx + \varepsilon_n \int_{x_1}^{x_1+t\varepsilon_n} |u_n'|^2\, dx + \\
& + \varepsilon_n \int_{x_1}^{x_1+t\varepsilon_n} \left(\max\left\{|u_n'| + \tfrac{\lambda}{\varepsilon_n}, 0\right\} - |u_n'|\right)^2 dx + \\
& + \varepsilon_n \int_{x_1+t\varepsilon_n}^{x_1+t\varepsilon_n+\sqrt{\varepsilon_n}} \left(\tfrac{\max\{\gamma-1,0\}}{\sqrt{\varepsilon_n}}\right)^2 dx + \varepsilon_n \int_{y_1}^{y_1+\sqrt{\varepsilon_n}} \left(\tfrac{\beta}{\sqrt{\varepsilon_n}}\right)^2 dx = \\
= & \int_{-t}^{t} f(v)\, dx + \int_{-t}^{t} \min\{\lambda^2 + |v'|^2, 2|v'|^2\}\, dx + \\
& + (\max\{\gamma - 1, 0\})^2 \sqrt{\varepsilon_n} + \beta^2 \sqrt{\varepsilon_n},
\end{aligned}
$$

where the second equality is obtained by a change of variables and by (3.5). Therefore, recalling (3.55), we deduce that

$$\limsup_{n\to\infty} F_{\varepsilon_n}(u_n, \rho_n) \leqq \phi(\gamma) + \eta.$$

The arbitrariness of $\eta$ yields

$$\overline{F}_M(u, \mu) \leqq F(u, \mu) \tag{3.57}$$

for all $M > 0$ and for all $(u, \mu) \in BV(I; \{0, 1\}) \times \mathcal{S}$ with $\mu(I) \leqq M$.

In order to remove the restriction on $\mu$ (i.e. $\mu \in \mathcal{S}$), we decompose $\mu$ as $\mu = \mu\lfloor S_u + \mu\lfloor(I \setminus S_u)$ and construct a sequence of purely atomic measures $\nu_k \in \mathcal{S}$ such that $\nu_k\lfloor S_u = 0$ for every $k$, $\nu_k(I \setminus S_u) = \mu(I \setminus S_u)$, and $\nu_k \overset{*}{\rightharpoonup} \mu\lfloor(I \setminus S_u)$. Setting $\mu_k := \mu\lfloor S_u + \nu_k$ it follows that $\mu_k \in \mathcal{S}$, $\mu_k \overset{*}{\rightharpoonup} \mu$, and $F(u, \mu_k) = F(u, \mu)$ for every $k$. Therefore, by the lower-semicontinuity of $\overline{F}_M$ (with $M \geqq \mu(I)$) and by (3.57) we have

$$
\begin{aligned}
\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u, \mu) & \leqq \overline{F}_M(u, \mu) \\
& \leqq \liminf_{k\to\infty} \overline{F}_M(u, \mu_k) \leqq \lim_{k\to\infty} F(u, \mu_k) = F(u, \mu)
\end{aligned}
$$

and this concludes the proof. □

### *3.3. The optimal profile*

In this subsection we show that for a large class of double-well potentials the optimal profile problem admits a unique (up to translation of the function $u$) minimizing pair $(u, \lambda)$, and we provide an explicit construction. The additional assumptions on the double-well potential $f$ are the following:

(H1) the restriction $f_{|[0,1]}$ is of class $C^1$;
(H2) there exists $u_0 \in (0, 1)$ such that $f' > 0$ in $(0, u_0)$ and $f' < 0$ in $(u_0, 1)$.

For simplicity, in the sequel we will assume in addition

(H3) $u_0 = 1/2$ and $f$ is symmetric with respect to $1/2$, that is $f(u) = f(1-u)$ for every $u \in (0, 1)$.

The symmetry condition stated in (H3) will allow us to simplify some arguments, but it will be clear that all the analysis below can be extended with minor changes to the case where just (H1) and (H2) hold. A typical example of a potential f satisfying our hypotheses is given by $f(u) = u^2(1-u)^2$.

Fix $\gamma \in (0, 1)$ and let $(u, \lambda) \in \mathcal{A}(\gamma)$ be a minimizer for the problem defining $\phi(\gamma)$. In view of Theorem 3.5, we already know that $u$ must be non-decreasing and in fact strictly increasing in the set $u^{-1}(0, 1)$. We claim that $u$ is of class $C^1$. Indeed, setting $\rho := \max\{u' + \lambda, 0\}$, by Lemma 3.2 $u$ minimizes the functional

$$v \mapsto \int_{-\infty}^{+\infty} f(v)\,dx + \int_{-\infty}^{+\infty} |v'|^2\,dx + \int_{-\infty}^{+\infty} (\rho - |v'|)^2\,dx$$

among the functions $v$ satisfying the same conditions at infinity as $u$ and thus, using the Euler–Lagrange equation, we deduce the existence of a $C^1$ function $g$, with $g'(x) = f'(u(x))/2$, and of a suitable representative of $u'$ (still denoted by $u'$) such that

$$2u'(x) - \max\{u'(x) + \lambda, 0\} = g(x) \qquad \text{for all } x \in \mathbb{R}. \tag{3.58}$$

In order to show that $u'$ is continuous, let $x_n \to x$ and assume first that $u'(x) > -\lambda$. Then from (3.58) we get $g(x) = u'(x) - \lambda > -2\lambda$ and so, by the continuity of $g$ and again by (3.58), $2u'(x_n) - \max\{u'(x_n) + \lambda, 0\} > -2\lambda$ for $n$ large. It follows that necessarily $u'(x_n) > -\lambda$ for $n$ large and thus

$$u'(x_n) - \lambda = g(x_n) \to g(x) = u'(x) - \lambda,$$

that is, $u'(x_n) \to u'(x)$. A similar argument shows that if $u'(x) \leqq -\lambda$ then $\max\{u'(x_n) + \lambda, 0\} \to 0$ and thus, from (3.58),

$$\begin{aligned}\lim_{n\to\infty} 2u'(x_n) &= \lim_{n\to\infty} (2u'(x_n) + \max\{u'(x_n) + \lambda, 0\}) \\ &= \lim_{n\to\infty} g(x_n) = g(x) = 2u'(x),\end{aligned}$$

which concludes the proof of the continuity of $u'$.

Now taking into account condition (H2) and the strict monotonicity of $u$ in $u^{-1}(0, 1)$, an elementary study of the differential equation (3.58) yields the following conclusion: after translation of the function $u$, there exists $t > 0$ such that

$$\{x \in \mathbb{R} : u'(x) < -\lambda\} = (-\infty, 0) \cup (t, +\infty), \tag{3.59}$$

$$\{x \in \mathbb{R} : u'(x) > -\lambda\} = (0, t), \tag{3.60}$$

and $u$ satisfies

$$4u'' = f'(u) \quad \text{in } (-\infty, 0) \cup (t, +\infty), \tag{3.61}$$

and

$$2u'' = f'(u) \quad \text{in } (0, t). \tag{3.62}$$

Moreover,

$$0 < u(0) < \frac{1}{2} \qquad \text{and} \qquad u'(0) = u'(t) = -\lambda. \tag{3.63}$$

Next we show that all the conditions listed above, together with the volume constraint

$$\int_{-\infty}^{+\infty} \max\{u' + \lambda, 0\}\, dx = \gamma, \tag{3.64}$$

determine uniquely $u$ and $\lambda$. First we observe that equation (3.61) integrates in $(-\infty, 0)$ to

$$u' = \sqrt{\frac{f(u)}{2} + C},$$

for a suitable constant $C$. Since $\lim_{x\to-\infty} u(x) = 0$ and $\liminf_{x\to-\infty} u'(x) = 0$, we deduce that $C = 0$. The same conclusion holds in $(t, +\infty)$, and thus

$$u' = \sqrt{\frac{f(u)}{2}} \qquad \text{in } (-\infty, 0) \cup (t, +\infty). \tag{3.65}$$

In particular, using (3.63) we get $2\lambda^2 = f(u(0)) = f(u(t))$, or equivalently,

$$u(0) = h(2\lambda^2) \qquad \text{and} \qquad u(t) = 1 - h(2\lambda^2), \tag{3.66}$$

where $h$ denotes the inverse of $f_{|(0,1/2]}$. Note that in the second equality in (3.66) we used the symmetry of $f$ (see condition (H3)). Arguing as before and using now (3.62), we have that $u' = \sqrt{f(u) + C}$ in $(0, t)$, where, by (3.63) and (3.66), $C := (u'(0))^2 - f(u(0)) = \lambda^2 - f(h(2\lambda^2)) = -\lambda^2$. It follows that $u_{|(0,t)}$ coincides with the solution $u_\lambda$ of the Cauchy problem

$$\begin{cases} u'_\lambda = \sqrt{f(u_\lambda) - \lambda^2}, \\ u_\lambda(0) = h(2\lambda^2). \end{cases} \tag{3.67}$$

From (3.66) and (3.67) we get

$$t = t(\lambda) = \int_{h(2\lambda^2)}^{1-h(2\lambda^2)} \frac{1}{\sqrt{f(u) - \lambda^2}}\, du. \tag{3.68}$$

Also, the volume constraint (3.64), (3.59), (3.60), (3.66), and (3.68) yield

$$\begin{aligned} \gamma &= \int_0^t u' + \lambda\, dx = u(t) - u(0) + \lambda t \\ &= 1 - 2h(2\lambda^2) + \int_{h(2\lambda^2)}^{1-h(2\lambda^2)} \frac{\lambda}{\sqrt{f(u) - \lambda^2}}\, du. \end{aligned}$$

Setting

$$F(\lambda) := 1 - 2h(2\lambda^2) + \int_{h(2\lambda^2)}^{1-h(2\lambda^2)} \frac{\lambda}{\sqrt{f(u) - \lambda^2}}\, du, \tag{3.69}$$

we have

$$F'(\lambda) = \int_{h(2\lambda^2)}^{1-h(2\lambda^2)} \frac{f(u)}{(f(u) - \lambda^2)^{3/2}}\, du > 0,$$

that is, $F$ is invertible and thus $\lambda$ is uniquely determined as $\lambda = \lambda(\gamma) = F^{-1}(\gamma)$.

Since all the solutions of (3.65) taking values in $(0, 1)$ are obtained by translating the particular solution $u_0$ which satisfies $u_0(0) = 1/2$, there exist $\tau_1$ and $\tau_2$ such that

$$u_{|(-\infty,0)}(\,\cdot\,) = u_0(\,\cdot + \tau_1) \qquad \text{and} \qquad u_{|(t,+\infty)}(\,\cdot\,) = u_0(\,\cdot + \tau_2).$$

In order to identify $\tau_1$, observe that

$$\begin{aligned} h(2\lambda^2) = u(0) &= \int_{-\infty}^{0} u'\, dx \\ &= \int_{-\infty}^{0} \sqrt{\frac{f(u_0(\tau_1 + x))}{2}}\, dx = \int_{-\infty}^{\tau_1} \sqrt{\frac{f(u_0)}{2}}\, dx, \end{aligned} \tag{3.70}$$

and so $\tau_1$ is uniquely determined as a smooth function of $\lambda$. The symmetry of $f$ yields

$$u_0(x) = 1 - u_0(-x) \quad \text{for } x > 0 \qquad \text{and} \quad \tau_2 = -t - \tau_1. \tag{3.71}$$

Finally, note that $\lambda$ is a $C^2$ function of $\gamma$, while $t$, $\tau_1$, and $\tau_2$ are $C^1$ functions of $\lambda$. This means that the dependence of the solution $(u, \lambda)$ on $\gamma$ is at least of class $C^1$, and, in turn, the function $\phi$ is of class $C^1$ in $(0, 1)$. In fact, all functions $\lambda$, $t$, $\tau_1$, $\tau_2$, and therefore $\phi$, inherit at least the same regularity as $f_{|[0,1]}$. In particular, if $f_{|[0,1]}$ is analytic then $\phi_{|(0,1)}$ is also analytic. We summarize what we proved so far in the following proposition (see Fig. 3.2).

**Proposition 3.8.** *Let $f$ be a double-well potential satisfying the conditions* (H1), (H2), and (H3) *listed above. Then for every $\gamma \geqq 0$ the optimal profile problem* (3.1) *admits a unique (up to translations of the function $u$) minimizing pair* $(u, \lambda)$ *given by:*

$$\lambda = \lambda(\gamma) = F^{-1}(\gamma),$$

*where $F$ is the function defined in* (3.69)*, and*

$$u(x) = \begin{cases} u_0(\tau_1(\lambda) + x) & \text{in } (-\infty, 0), \\ u_\lambda(x) & \text{in } (0, t(\lambda)), \\ u_0(\tau_2(\lambda) + x) & \text{in } (t(\lambda), +\infty), \end{cases} \tag{3.72}$$

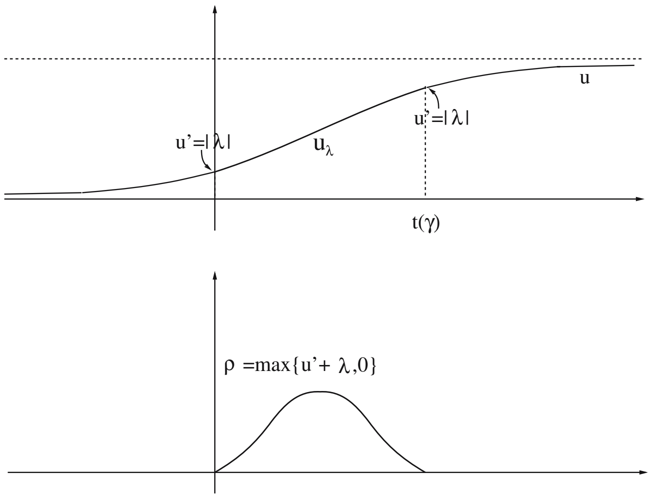


**Fig. 3.2.** The solution $u$ to the optimal profile problem and the corresponding $\rho$.

*where $t(\lambda)$ is given by* (3.68)*, $\tau_1(\lambda)$ is implicitly defined in* (3.70)*, $\tau_2(\lambda) = -t(\lambda) - \tau_1(\lambda)$, and $u_0$ is the solution of the equation* (3.65) *satisfying $u_0(0) = 1/2$. The dependence of the solution on $\gamma$ is as smooth as $f_{|[0,1]}$, and, in turn, $\phi_{|(0,1)}$ has at least the same regularity as $f_{|[0,1]}$. Moreover, $\phi$ is continuously differentiable in* $(0, +\infty)$.

**Proof of Proposition 3.8.** In view of what was established prior to the statement of Proposition 3.8, all that remains is to prove is the global $C^1$ regularity of $\phi$. To this end, it is enough to show that $\lim_{\gamma\to 1^-} \phi'(\gamma) = 0$.

Fix $\gamma \in (0, 1)$ and consider the minimizing pair $(u, \lambda)$ constructed above. We will write $u = u(\gamma, x)$ to highlight the $(C^1)$ dependence of $u$ on $\gamma$. Clearly, from the definition (3.72) of $u$, we have

$$\begin{aligned}\phi(\gamma) = E(u, \lambda; \mathbb{R}) &= E\big(u_0(\tau_1 + \cdot), \lambda; (-\infty, 0)\big)\\ &+E\big(u_\lambda, \lambda; (0, t)\big) + E\big(u_0(\tau_2 + \cdot), \lambda; (t, +\infty)\big).\end{aligned} \tag{3.73}$$

Using the identity $2|u_0'|^2 = f(u_0)$ and (3.71), we easily get

$$\begin{aligned}&E\big(u_0(\tau_1 + \cdot), \lambda; (-\infty, 0)\big) + E\big(u_0(\tau_2 + \cdot), \lambda; (t, +\infty)\big)\\ &= 4\int_{-\infty}^{\tau_1(\lambda)} f(u_0)\, dx.\end{aligned} \tag{3.74}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\left(4\int_{-\infty}^{\tau_1(\lambda)} f(u_0)\, dx\right) = 4f(u_0(\tau_1(\lambda)))\tau_1'(\lambda)\lambda' = 8\lambda^2\tau'\lambda', \tag{3.75}$$

where we used that $f(u_0(\tau_1)) = f(u(0)) = f(h(2\lambda^2)) = 2\lambda^2$. Using (3.70) we also have

$$\tau_1'(\lambda) = \frac{4\sqrt{2}\lambda h'(2\lambda^2)}{\sqrt{f(u_0(\tau_1))}} = -4h'(2\lambda^2) \tag{3.76}$$

and from (3.69)

$$\lambda' = (F^{-1})' = \left( \int_{h(2\lambda^2)}^{1-h(2\lambda^2)} \frac{f(u)}{(f(u) - \lambda^2)^{3/2}}\, du \right)^{-1}.$$

By (3.73), (3.75), and (3.76) we conclude

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\gamma} \left( E\big(u_0(\tau_1 + \cdot), \lambda; (-\infty, 0)\big) + E\big(u_0(\tau_2 + \cdot), \lambda; (t, +\infty)\big) \right)(\gamma_n) \\ &\quad = -32\lambda^2 h'(2\lambda^2)\lambda'. \end{aligned} \tag{3.77}$$

Next, since $u' > -\lambda$ in $(0, t(\lambda))$, we have

$$E(u_\lambda, \lambda; (0, t(\lambda))) = \int_0^{t(\lambda)} \left( f(u_\lambda) + \left| \frac{\partial u_\lambda}{\partial x} \right|^2 \right) dx + \lambda^2 t(\lambda).$$

In order to determine its derivative we first assume that $f_{|[0,1]}$ is of class $C^2$. This implies that $(\gamma, x) \mapsto u_\lambda(\gamma, x)$ is of class $C^2$ as well, and so:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\gamma} (E(u_\lambda, \lambda; (0, t(\lambda)))) &= \left( f(u_\lambda(\gamma, t(\lambda))) + \left| \frac{\partial}{\partial x} u_\lambda(\gamma, t(\lambda)) \right|^2 \right) t'(\lambda)\lambda' \\ &+ \int_0^{t(\lambda)} \left( f'(u_\lambda) \frac{\partial u_\lambda}{\partial \gamma} + 2 \frac{\partial u_\lambda}{\partial x} \frac{\partial^2 u_\lambda}{\partial x \partial \gamma} \right) dx + \lambda^2 t'(\lambda)\lambda' + 2\lambda\lambda' t(\lambda). \end{aligned}$$

We recall that $u_\lambda(\gamma, 0) = h(2\lambda^2)$, $u_\lambda(\gamma, t(\lambda)) = 1 - h(2\lambda^2)$, $f(u_\lambda(\gamma, t(\lambda))) = 2\lambda^2$, $\frac{\partial u_\lambda}{\partial x}(\gamma, 0) = \frac{\partial u_\lambda}{\partial x}(\gamma, t(\lambda)) = -\lambda$, and $u_\lambda$ solves (3.62). In particular,

$$\begin{aligned} \frac{\partial u_\lambda}{\partial \gamma}(\gamma, t(\lambda)) &= \frac{\mathrm{d}u_\lambda}{\mathrm{d}\gamma}(\gamma, t(\lambda)) - \frac{\partial u_\lambda}{\partial x}(\gamma, t(\lambda)) t'(\lambda)\lambda' \\ &= -4h'(2\lambda^2)\lambda\lambda' + \lambda t'(\lambda)\lambda'. \end{aligned}$$

Therefore, after integration by parts we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\gamma} (E(u_\lambda, \lambda; (0, t(\lambda)))) &= 4\lambda^2 t'(\lambda)\lambda' + \int_0^{t(\lambda)} \frac{\partial u_\lambda}{\partial \gamma} \left( f'(u_\lambda) - 2\frac{\partial^2 u_\lambda}{\partial x^2} \right) dx \\ &\qquad + \left[ 2 \frac{\partial u_\lambda}{\partial \gamma} \frac{\partial u_\lambda}{\partial x} \right]_{(\gamma,0)}^{(\gamma, t(\lambda))} + 2\lambda\lambda' t(\lambda) \\ &= 2\lambda^2 t'(\lambda)\lambda' + 16\lambda^2 h'(2\lambda^2)\lambda' + 2\lambda\lambda' t(\lambda). \end{aligned} \tag{3.78}$$

Moreover, by (3.68)

$$2\lambda^2 t'(\lambda)\lambda' = 16\lambda^2 h'(2\lambda^2)\lambda' + \int_{h(2\lambda^2)}^{1-h(2\lambda^2)} \frac{2\lambda^3\lambda'}{(f(u) - \lambda^2)^{3/2}}\, du. \tag{3.79}$$

Summing up, by (3.68), (3.77), (3.78), and (3.79), we finally get

$$\phi'(\gamma) = \int_{h(2\lambda^2)}^{1-h(2\lambda^2)} \frac{2\lambda\lambda'}{\sqrt{f(u) - \lambda^2}}\, du + \int_{h(2\lambda^2)}^{1-h(2\lambda^2)} \frac{2\lambda^3\lambda'}{(f(u) - \lambda^2)^{3/2}}\, du. \tag{3.80}$$

If $f$ is simply of class $C^1$, then we proceed by approximation, i.e. we construct a sequence of $C^2$ potentials $f_n$ satisfying the assumptions (H1), (H2), and (H3) and converging uniformly to $f$ in $[0, 1]$. Then the corresponding sequence $\phi_n$ converges to $\phi$, and by the above arguments we obtain

$$\phi_n'(\gamma) = \int_{h_n(2\lambda_n^2)}^{1-h_n(2\lambda_n^2)} \frac{2\lambda_n\lambda_n'}{\sqrt{f_n(u) - \lambda_n^2}}\, du + \int_{h_n(2\lambda_n^2)}^{1-h_n(2\lambda_n^2)} \frac{2\lambda_n^3\lambda_n'}{(f_n(u) - \lambda_n^2)^{3/2}}\, du, \tag{3.81}$$

where $\lambda_n$, $\lambda_n'$, and $h_n$ are defined exactly in the same way with $f$ replaced by $f_n$. Note that all these quantities depend on $\gamma$ and $f_n$ only, and do not depend on $f_n'$ or on $f_n''$. It is then easy to verify that $\lambda_n$, $\lambda_n'$, and $h_n$ converge to the corresponding quantities $\lambda$, $\lambda'$, $h$. In particular, the right-hand side of (3.81) converges to the right-hand side of (3.80) which must then coincide with $\phi'$. We leave the details to the reader. We deduce that (3.80) holds also if $f$ is of class $C^1$. Moreover, since by construction $f_n \geqq 2\lambda_n^2$ in the interval $(h_n(2\lambda_n^2), 1 - h_n(2\lambda_n^2))$, we also have that $f(u) \geqq 2\lambda^2$ in $(h(2\lambda^2), 1 - h(2\lambda^2))$, and thus both integrands in (3.80) are dominated by $2\lambda'$. Since $\lambda$ vanishes as $\gamma \to 1^-$, the dominated convergence theorem implies that both integrals vanish as well, that is, $\phi'(\gamma) \to 0$ as $\gamma \to 1^-$, which concludes the proof. □

From the proof of the proposition it is clear that the same approximation procedure holds if $f$ is simply continuous and has the same increasing–decreasing structure we assumed before. This is made precise in the following corollary.

**Corollary 3.9.** *Let $f$ be a continuous double well-potential such that $f(u) = f(1-u)$ for every $u \in (0, 1)$ and $f$ is strictly increasing in $(0, 1/2)$ and strictly decreasing in $(1/2, 1)$. Then $\phi$ is of class $C^1$.*

**Proof.** As in the last part of the previous proof, we can approximate $f$ by a sequence of regular potentials satisfying the assumptions of Proposition 3.8. Since the expression of $\phi_n'$ does not involve any derivative of $f_n$, we can pass to the limit and deduce that (3.80) still holds. We then argue as before. □

We conclude this section with the following:

**Corollary 3.10.** *Under the assumptions of* Proposition 3.8*, in addition if $f_{|[0,1]}$ is analytic, then $\phi$ is strictly convex in* $(0, 1)$.

**Proof of Corollary 3.10.** From Proposition 3.8 $\phi$ is analytic in $(0, 1)$, and by Theorem 3.6 it is convex. Thus, $\phi$ is either strictly convex or affine in $(0, 1)$, but the latter possibility is ruled out by the fact that $\phi'(1) = 0$. □

## 4. The N-dimensional case

Here we prove Theorem 2.1 in the case where $\Omega \subset \mathbb{R}^N$ and $N \geqq 2$. As in the one-dimensional framework we consider only the case (i), and assume without loss of generality that $\alpha(\varepsilon) = \varepsilon$.

### *4.1. The* $\Gamma$-$\liminf$ *inequality*

Let $\varepsilon_n \searrow 0$, $u_n \to u$ in $L^1(\Omega)$, and $\rho_n \overset{*}{\rightharpoonup} \mu$ weakly* in $\mathcal{M}_+(\Omega)$ be such that

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n, \rho_n) < +\infty.$$

Extracting a subsequence (not relabeled), if necessary, we may assume that

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n, \rho_n) = \lim_{n\to\infty} F_{\varepsilon_n}(u_n, \rho_n),$$

$u_n \to u$ a.e. in $\Omega$, and

$$\frac{1}{\varepsilon_n} f(u_n) + \varepsilon_n \left( |\nabla u_n|^2 + (\rho_n - |\nabla u_n|)^2 \right) \overset{*}{\rightharpoonup} \sigma$$

weakly* in $\mathcal{M}_+(\Omega)$. Set $\rho(x) := \frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_u}(x)$. In order to proof the $\Gamma$-liminf inequality it is enough to show that

$$\frac{d\sigma}{d\mathcal{H}^{N-1}\lfloor S_u}(x) \geqq \phi(\rho(x)) \tag{4.1}$$

for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$. For every point $x \in S_u$ where the generalized normal vector $\nu(x)$ is defined we denote by $Q_{x,\delta}$ the cube of side-length $\delta$ centered at $x$ and with two of its faces orthogonal to $\nu(x)$. Let $x_0 \in S_u$ satisfy

(a) $\lim_{\delta\to 0^+} \delta^{1-N} \mathcal{H}^{N-1}(Q_{x_0,\delta} \cap S_u) = 1$;

(b) $\displaystyle\lim_{\delta\to 0^+} \frac{\mu(Q_{x_0,\delta})}{\mathcal{H}^{N-1}(Q_{x_0,\delta} \cap S_u)} = \lim_{\delta\to 0^+} \delta^{1-N} \mu(Q_{x_0,\delta}) = \rho(x_0)$;

(c) $\displaystyle\lim_{\delta\to 0^+} \frac{\sigma(Q_{x_0,\delta})}{\mathcal{H}^{N-1}(Q_{x_0,\delta} \cap S_u)} = \lim_{\delta\to 0^+} \delta^{1-N} \sigma(Q_{x_0,\delta}) = \frac{d\sigma}{d\mathcal{H}^{N-1}\lfloor S_u}(x_0)$;

(d) $\displaystyle\lim_{\delta\to 0^+} \delta^{-N} \int_{Q^+_{x_0,\delta}} |u(x) - 1|\, dx = \lim_{\delta\to 0^+} \delta^{-N} \int_{Q^-_{x_0,\delta}} |u(x)|\, dx = 0$,

where we define

$$Q^{\pm}_{x_0,\delta} := \{x \in Q_{x_0,\delta} : \pm(x - x_0) \cdot \nu(x_0) \geqq 0\}.$$

Note that condition (d) simply states that 1 and 0 are the upper and lower traces, respectively, of $u$ on $S_u$ at $x_0$. It can be restated as

(d)' $\displaystyle\lim_{\delta\to 0^+} \delta^{-N} \mathcal{L}^N\{x \in Q^+_{x_0,\delta} : u(x) \neq 1\} = \lim_{\delta\to 0^+} \delta^{-N} \mathcal{L}^N\{x \in Q^-_{x_0,\delta} : u(x) \neq 0\} = 0$.

We claim that (4.1) holds for $x_0$. We treat only the case $\rho(x_0) < 1$, as the other case reduces to the standard Modica–Mortola estimate. Fix $r > 0$ such that

$$(1 - 4r) > (1 + r)\rho(x_0), \tag{4.2}$$

and choose $\delta \ll 1$ such that $\sigma(\partial Q_{x_0,\delta}) = \mu(\partial Q_{x_0,\delta}) = 0$,

$$\delta^{1-N}\mu(Q_{x_0,\delta}) \leqq (1+r)\rho(x_0), \tag{4.3}$$

$$\delta^{1-N}\sigma(Q_{x_0,\delta}) \leqq (1+r)\frac{d\sigma}{d\mathcal{H}^{N-1}\lfloor S_u}(x_0), \tag{4.4}$$

$$\mathcal{L}^N\{x \in Q^+_{x_0,\delta} : u(x) = 1\} \geqq \left(\frac{1-r}{2}\right)\delta^N,$$

and

$$\mathcal{L}^N\{x \in Q^-_{x_0,\delta} : u(x) = 0\} \geqq \left(\frac{1-r}{2}\right)\delta^N.$$

By Severini–Egoroff's Theorem, we can find two closed sets $C^+ \subset \{x \in Q^+_{x_0,\delta} : u(x) = 1\}$ and $C^- \subset \{x \in Q^-_{x_0,\delta} : u(x) = 0\}$ such that

$$\mathcal{L}^N\left(C^\pm\right) \geqq \frac{(1-r)^2}{2}\delta^N$$

and $u_n \to u$ uniformly in $C^+ \cup C^-$. In particular, we have that the orthogonal projection $K^\pm$ of $C^\pm$ onto $Q^0_{x_0,\delta} := Q^+_{x_0,\delta} \cap Q^-_{x_0,\delta}$, that is, the set

$$K^\pm := \left\{y \in Q^0_{x_0,\delta} : \exists t \in \left(0, \frac{\delta}{2}\right) \text{ such that } y \pm t\nu(x_0) \in C^\pm\right\}$$

satisfies

$$\mathcal{H}^{N-1}(K^\pm) \geqq (1-r)^2\delta^{N-1}$$

and thus, setting $K := K^+ \cap K^-$, we have

$$\mathcal{H}^{N-1}(K) \geqq (1-4r)\delta^{N-1}. \tag{4.5}$$

For every $y \in Q^0_{x_0,\delta}$ and $t \in (-\frac{\delta}{2}, \frac{\delta}{2})$ set

$$u_{n,y}(t) := u_n(y + t\nu(x_0)).$$

Now let $\lambda_n \in \mathbb{R}$ be such that

$$\int_{Q_{x_0,\delta}} \max\{\lambda_n + |\nabla u_n|, 0\}\,dx = \int_{Q_{x_0,\delta}} \rho_n\,dx. \tag{4.6}$$

We claim that $\lambda_n < 0$ for $n$ large enough. Indeed, assume by contradiction that $\lambda_{n_k} \geqq 0$ for a subsequence $n_k \to \infty$ and fix $\eta \ll 1$. By the uniform convergence of $u_n$ to $u$ in $C^+ \cup C^-$ we deduce that for $k$ large enough the total variation of $u_{n_k,y}$ is bigger than $1-2\eta$ for almost every $y \in K$, and thus

$$\begin{aligned}\mu(Q_{x_0,\delta}) &= \lim_{k\to\infty} \int_{Q_{x_0,\delta}} \rho_{n_k}\,dx = \lim_{k\to\infty}\int_{Q_{x_0,\delta}} \max\{\lambda_{n_k} + |\nabla u_{n_k}|, 0\}\,dx \\ &\geqq \limsup_{k\to\infty} \int_K \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \max\{\lambda_{n_k} + |(u_{n_k,y})'(t)|, 0\}\,dt\,dy \\ &\geqq \limsup_{k\to\infty} \int_K \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |(u_{n_k,y})'(t)|\,dt\,dy \geqq (1-4r)\delta^{N-1}(1-2\eta),\end{aligned}$$

where in the last inequality we used (4.5). Dividing by $\delta^{N-1}$ and recalling (4.3) we deduce

$$(1+r)\rho(x_0) \geqq (1-4r)(1-2\eta),$$

which contradicts (4.2) if $\eta$ is small enough. Hence $\lambda_n < 0$ for $n$ large enough, and thus using (4.6), Lemma 3.2, and (3.5), we can estimate

$$\begin{aligned}&F_{\varepsilon_n}(u_n, \rho_n; Q_{x_0,\delta}) \\ &\quad\geqq \frac{1}{\varepsilon_n}\int_{Q_{\delta,x_0}} f(u_n)\,dx + \varepsilon_n \int_{Q_{x_0,\delta}} \min\{\lambda_n^2 + |\nabla u^n_{\varepsilon_n}|^2, 2|\nabla u^n_{\varepsilon_n}|^2\}\,dx \\ &\quad\geqq \int_K \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{1}{\varepsilon_n} f(u_{n,y}) + \varepsilon_n \min\{\lambda_n^2 + |(u_{n,y})'(t)|^2, 2|(u_{n,y})'(t)|^2\}\,dt\,dy \\ &\quad\geqq \int_K \int_{-\frac{\delta}{2\varepsilon_n}}^{\frac{\delta}{2\varepsilon_n}} f(v_{n,y}) + \min\{\mu_n^2 + |(v_{n,y})'(t)|^2, 2|(v_{n,y})'(t)|^2\}\,dt\,dy, \qquad (4.7)\end{aligned}$$

where we set $v_{n,y}(t) := u_{n,y}(\varepsilon_n t)$ and $\mu_n := \varepsilon_n \lambda_n$, and, without loss of generality, we assume that $x_0 = 0$. For every $y \in Q^0_{x_0,\delta}$ define

$$g_n(y) := \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \max\{\lambda_n + |\nabla u_n(y + t\nu)|, 0\}\,dt,$$

and note that

$$\int_{-\frac{\delta}{2\varepsilon_n}}^{\frac{\delta}{2\varepsilon_n}} \max\{\mu_n + |(v_{n,y})'(t)|, 0\}\,dt = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \max\{\lambda_n + |(u_{n,y})'(t)|, 0\}\,dt \leqq g_n(y) \qquad (4.8)$$

and that, recalling (4.3),

$$\int_{Q^0_{x_0,\delta}} g_n(y)\,dy = \int_{Q_{x_0,\delta}} \rho_n\,dx \leqq (1+2r)\rho(x_0)\delta^{N-1} \qquad (4.9)$$

if $n$ is large enough. By Chebyshev's inequality and by (4.5) it is possible to find $M > 0$ so large that

$$\mathcal{H}^{N-1}\left(\{y \in K : g_n(y) \leqq M\}\right) \geqq (1-r)\mathcal{H}^{N-1}(K) \geqq (1-r)(1-4r)\delta^{N-1} \tag{4.10}$$

for all $n$. Moreover, by Corollary 3.7 we can find $\eta_0 \ll 1$ such that for all $0 < \eta < \eta_0$

$$\phi_\eta(\gamma) \geqq (1-r)\phi(\gamma) \qquad \text{for } 0 \leqq \gamma \leqq M. \tag{4.11}$$

Now by the uniform convergence of $u_n$ to $u$ in $C^+ \cup C^-$, we deduce that for $n$ large and for all almost every $y \in K$ there exist $(s_n(y), t_n(y)) \subseteq (-\frac{\delta}{2\varepsilon_n}, \frac{\delta}{2\varepsilon_n})$ such that $v_{n,y}(s_n(y)) = \eta$ and $v_{n,y}(t_n(y)) = 1-\eta$ and therefore, by (4.8) and the very definition of $\phi_\eta$, we get

$$\int_{-\frac{\delta}{2\varepsilon_n}}^{\frac{\delta}{2\varepsilon_n}} f(v_{n,y}) + \min\{\mu_n^2 + |(v_{n,y})'(t)|^2, 2|(v_{n,y})'(t)|^2\}\,dtdy$$
$$\geqq \int_{s_n(y)}^{t_n(y)} f(v_{n,y}) + \min\{\mu_n^2 + |(v_{n,y})'(t)|^2, 2|(v_{n,y})'(t)|^2\}\,dtdy \geqq \phi_\eta(g_n(y)).$$

Hence, from (4.7) and (4.11) we obtain

$$F_{\varepsilon_n}(u_n, \rho_n; Q_{x_0,\delta}) \geqq (1-r)\int_{\{y\in K:\, g_n(y)\leqq M\}} \phi(g_n(y))\,dy, \tag{4.12}$$

for $n$ large enough. Since by (4.9) and (4.10)

$$⨍_{\{y\in K:\, g_n(y)\leqq M\}} g_n(y)\,dy \leqq \frac{1+2r}{(1-r)(1-4r)}\rho(x_0),$$

and recalling that $\phi$ is convex and non-increasing, from (4.12) we deduce that

$$\begin{aligned} &F_{\varepsilon_n}(u_n, \rho_n; Q_{x_0,\delta}) \geqq \\ &\geqq (1-r)\mathcal{H}^{N-1}\left(\{y \in K : g_n(y) \leqq M\}\right)\phi\left(\tfrac{1+2r}{(1-r)(1-4r)}\rho(x_0)\right) \\ &\geqq (1-r)(1-4r)\delta^{N-1}\phi\left(\tfrac{1+2r}{(1-r)(1-4r)}\rho(x_0)\right), \end{aligned}$$

for $n$ large enough, where we have used (4.10) again. Dividing this inequality by $\delta^{N-1}$, and using the fact that

$$\lim_n F_{\varepsilon_n}(u_n, \rho_n; Q_{x_0,\delta}) = \sigma(Q_{x_0,\delta}),$$

and (4.4), we finally obtain

$$(1+r)\frac{d\sigma}{d\mathcal{H}^{N-1}\lfloor S_u}(x_0) \geqq (1-r)(1-4r)\phi\left(\frac{1+2r}{(1-r)(1-4r)}\rho(x_0)\right).$$

Owing to the arbitrariness of $r$ and the continuity of $\phi$, we conclude that (4.1) holds for any point $x_0$ satisfying conditions (a)–(d), that is, for $\mathcal{H}^{N-1}$-a.e. point $x_0 \in S_u$. This completes the proof of the $\Gamma$-liminf inequality.

*4.2. The* $\Gamma$-lim sup *inequality*

The proof of the $\Gamma$-lim sup inequality will be split in several steps.

**Step 1.** Assume firstly that $u = \chi_{A\cap\Omega}$, where $A$ is an open set with $\partial A$ a smooth $(N-1)$ manifold, and that $\mu = g\mathcal{H}^{N-1}\lfloor S_u + \sum_{i=1}^{j} c_i\delta_{x_i}$, where $g$ is piecewise constant on $S_u$ and the atoms $x_i$ are in $\Omega \setminus S_u$. More precisely, there exists a finite collection of pairwise disjoint compact subsets, $K_1, \dots, K_m \subset S_u$, and positive constants $\gamma_1, \dots, \gamma_m$ such that $g_{|K_i} \equiv \gamma_i$ and $g \equiv 0$ on $S_u \setminus \cup_i^m K_i$. We also assume that $K_i = \overline{B(y_i, r_i)} \cap S_u$ for some $r_i > 0$ and $y_i \in S_u$. We claim that there exist $v_n \to u$ in $L^1$ and $\rho_n \overset{*}{\rightharpoonup} \mu$ with $\int_\Omega \rho_n\,dx = \mu(\Omega)$ such that

$$\limsup_{n\to\infty} F_n(v_n, \rho_n) \leqq \int_{S_u} \phi\left(\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_u}\right) d\mathcal{H}^{N-1}. \tag{4.13}$$

Since the construction of the recovering sequence can be localized near each set $K_i$ and each atom $x_i$, it suffices to consider the special case where

$$\mu = \gamma\chi_K\mathcal{H}^{N-1}\lfloor S_u + \beta\delta_{x_0},$$

with $\gamma, \beta > 0$, $K = \overline{B(y_0, r)} \cap S_u$ for some $r > 0$ and $y_0 \in S_u$, and $x_0 \in \Omega \setminus S_u$. We fix $\eta \ll 1$ and choose $t > 0$ and $(u_1, \lambda_1) \in \mathcal{A}_{0,t}(\gamma)$ such that

$$\int_{-t}^{t} \max\{\lambda_1 + |u_1'|, 0\}\,dx = \min\{\gamma, 1\} \tag{4.14}$$

and

$$E(u_1, \lambda_1; (-t, t)) \leqq \phi(\gamma) + \eta, \tag{4.15}$$

and let $u_2 \in H^1_{\text{loc}}(\mathbb{R})$ with $u_2 = \chi_{(0,+\infty)}$ in $\mathbb{R}\setminus(-t, t)$ be such that (see Remark 3.1)

$$\int_{-t}^{t} \left(f(u_2) + 2|u_2'|^2\right) dx < \phi(0) + \eta. \tag{4.16}$$

We extend $u_1$ to the whole real line by $\chi_{(0,+\infty)}$ in $\mathbb{R} \setminus (-t, t)$. For $\delta > 0$ we denote $K_\delta := B(y_0, r+\delta) \cap S_u$, and we choose a cut-off function $\varphi \in C_0^\infty(S_u; [0, 1])$ such that $\varphi \equiv 1$ in $K$, $\phi \equiv 0$ in $S_u \setminus K_\delta$, and $\|\nabla\varphi\|_\infty \leqq C/\delta$ with $C > 0$ independent of $\delta$. Finally, since $S_u$ is smooth we know that the signed distance function $d$ from $S_u$, and the projection $\pi$ on $S_u$ are well defined and smooth in the $\eta$-neighborhood $(S_u)_\eta$ of $S_u$, provided $\eta$ is small enough. Moreover, without loss of generality, we may assume that $u(x) = 1$ if $d(x) > 0$ and $u(x) = 0$ if $d(x) < 0$.

We can now define (for $t\varepsilon_n < \eta$)

$$v_n(x) := \begin{cases} \varphi(\pi(x))u_1\left(\frac{d(x)}{\varepsilon_n}\right) + (1-\varphi(\pi(x)))u_2\left(\frac{d(x)}{\varepsilon_n}\right) & \text{if } x \in (S_u)_{t\varepsilon_n}\cap\Omega, \\ u & \text{otherwise,} \end{cases}$$

and

$$\rho_n(x) := c_n \cdot \begin{cases} \dfrac{\max\left\{u_1'\left(\frac{d(x)}{\varepsilon_n}\right) + \lambda_1, 0\right\}}{\varepsilon_n} & \text{if } x \in (S_u)_{t\varepsilon_n} \text{ and } \pi(x) \in K, \\[2ex] \dfrac{\max\{\gamma - 1, 0\}}{2\sqrt{\varepsilon_n}} & \text{if } x \in \left((S_u)_{t\varepsilon_n + \sqrt{\varepsilon_n}} \setminus (S_u)_{t\varepsilon_n}\right) \\ & \text{and } \pi(x) \in K, \\[2ex] \dfrac{\beta}{\alpha_N \sqrt{\varepsilon_n}} & \text{if } x \in B\left(x_0, \varepsilon_n^{\frac{1}{2N}}\right), \\[2ex] 0 & \text{otherwise,} \end{cases}$$

where $\alpha_N$ denotes the measure of the $N$-dimensional unit ball and $c_n$ is a normalization constant chosen in such a way that $\int_\Omega \rho_n \, dx = \mu(\Omega)$. Note that $\rho_n$ is well defined provided that $n$ is large enough. We claim that $c_n \to 1$ as $n \to \infty$. Indeed, using the Coarea formula (see [1]), we have

$$\begin{aligned} &\frac{1}{c_n} \int_\Omega \rho_n \, dx \\ &= \frac{1}{\varepsilon_n} \int_{-t\varepsilon_n}^{t\varepsilon_n} \max\{u_1'(\tfrac{s}{\varepsilon_n}) + \lambda_1, 0\} \mathcal{H}^{N-1}(\{x \in \Omega : \pi(x) \in K,\ d(x) = s\}) \, ds \\ &\quad + \tfrac{\max\{\gamma-1,0\}}{2\sqrt{\varepsilon_n}} \int_{t\varepsilon_n}^{t\varepsilon_n + \sqrt{\varepsilon_n}} \mathcal{H}^{N-1}(\{x \in \Omega : \pi(x) \in K,\ d(x) = s\}) \, ds \\ &\quad + \tfrac{\max\{\gamma-1,0\}}{2\sqrt{\varepsilon_n}} \int_{-t\varepsilon_n - \sqrt{\varepsilon_n}}^{t\varepsilon_n} \mathcal{H}^{N-1}(\{x \in \Omega : \pi(x) \in K,\ d(x) = s\}) \, ds + \beta \\ &=: I_n^1 + I_n^2 + I_n^3 + \beta. \end{aligned}$$

Since

$$\lim_{s \to 0} \mathcal{H}^{N-1}(\{x \in \Omega : \pi(x) \in K,\ d(x) = s\}) = \mathcal{H}^{N-1}(K), \tag{4.17}$$

and

$$\begin{aligned} &\lim_{n \to \infty} (\sqrt{\varepsilon_n})^{-1} \int_{t\varepsilon_n}^{t\varepsilon_n + \sqrt{\varepsilon_n}} \mathcal{H}^{N-1}(\{x \in \Omega : \pi(x) \in K,\ d(x) = s\}) \, ds \\ &= \lim_{n \to \infty} (\sqrt{\varepsilon_n})^{-1} \int_{-t\varepsilon_n - \sqrt{\varepsilon_n}}^{-t\varepsilon_n} \mathcal{H}^{N-1}(\{x \in \Omega : \pi(x) \in K,\ d(x) = s\}) \, ds \\ &= \mathcal{H}^{N-1}(K), \end{aligned}$$

we have

$$\begin{aligned}
&\lim_{n\to\infty} I_n^1 \\
&\quad = \lim_{n\to\infty} \int_{-t}^{t} \max\{\lambda + u_1', 0\}\mathcal{H}^{N-1}(\{x \in \Omega : \pi(x) \in K,\ d(x) = \varepsilon_n z\})\, dz \\
&\quad = \mathcal{H}^{N-1}(K) \int_{-t}^{t} \max\{\lambda + u_1', 0\}\, dx = \min\{\gamma, 1\}\mathcal{H}^{N-1}(K),
\end{aligned}$$

where the last equality follows from (4.14), and, similarly,

$$\lim_{n\to\infty} I_n^2 + I_n^3 = \max\{\gamma - 1, 0\}\mathcal{H}^{N-1}(K).$$

We conclude that

$$\lim_{n\to\infty} \frac{1}{c_n} \int_\Omega \rho_n\, dx = \gamma\mathcal{H}^{N-1}(K) + \beta = \mu(\Omega),$$

and thus $c_n \to 1$. Using this fact and again the Coarea formula it is now easy to show that $\rho_n \overset{*}{\rightharpoonup} \mu$. The convergence of $v_n$ to $u$ is clear.

It remains to estimate $F_{\varepsilon_n}(v_n, \rho_n)$. We can write

$$\begin{aligned}
&F_{\varepsilon_n}(v_n, \rho_n) = \\
&\quad = \int_{\{x\in(S_u)_{t\varepsilon_n} : \pi(x)\in K\}} \frac{1}{\varepsilon_n} f(v_n) + \varepsilon_n|\nabla v_n|^2 + \varepsilon_n\left(\rho_n - |\nabla v_n|\right)^2\, dx \\
&\qquad + \int_{\{x\in(S_u)_{t\varepsilon_n} : \pi(x)\notin K_\delta\}} \frac{1}{\varepsilon_n} f(v_n) + \varepsilon_n|\nabla v_n|^2 + \varepsilon_n\left(\rho_n - |\nabla v_n|\right)^2\, dx \\
&\qquad + \int_{\{x\in(S_u)_{t\varepsilon_n} : \pi(x)\in K_\delta\setminus K\}} \frac{1}{\varepsilon_n} f(v_n) + \varepsilon_n|\nabla v_n|^2 + \varepsilon_n\left(\rho_n - |\nabla v_n|\right)^2\, dx \\
&\qquad + c_n \tfrac{(\max\{\gamma-1,0\})^2}{4} \mathcal{L}^N\Big(\{x \in (S_u)_{t\varepsilon_n + \sqrt{\varepsilon_n}} \setminus (S_u)_{t\varepsilon_n} : \pi(x) \in K\}\Big) \\
&\qquad\qquad + \tfrac{\beta^2}{\alpha_N}\sqrt{\varepsilon_n} \\
&\quad =: I_n^1 + I_n^2 + I_n^3 + O(\sqrt{\varepsilon_n}).
\end{aligned} \tag{4.18}$$

Using the Coarea formula and changing variables as before, we easily get

$$I_n^1 = \int_{-t}^{t} \Big(f(u_1) + \min\{\lambda^2 + |u_1'|^2, 2|u_1'|^2\}\Big)\, h_n\, ds$$

where $h_n(s) := \mathcal{H}^{N-1}(\{x \in \Omega : \pi(x) \in K,\ d(x) = \varepsilon_n s\})$. By (4.15) and (4.17) we conclude that

$$\limsup_{n\to\infty} I_n^1 \leqq (\phi(\gamma) + \eta)\mathcal{H}^{N-1}(K). \tag{4.19}$$

Similarly, it can be shown that

$$\limsup_{n\to\infty} I_n^2 \leqq (\phi(0) + \eta)\mathcal{H}^{N-1}(\Omega \cap \partial A \setminus K). \tag{4.20}$$

Finally we have the estimate

$$I_n^3 \leqq O(\delta). \tag{4.21}$$

Combining (4.18), (4.19), (4.20), and (4.21), due to the arbitrariness of $\eta$ and $\delta$ we can conclude

$$\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u,\mu) \leqq F(u,\mu) = \phi(\gamma)\mathcal{H}^{N-1}(K) + \phi(0)\mathcal{H}^{N-1}(\partial A \cap (\Omega|K).$$

As in the one-dimensional case it is convenient to consider for every $M > 0$ the subset $X_M(\Omega) := \{(u,\mu) \in X(\Omega) : \mu(\Omega) \leqq M\}$ endowed with the convergence inherited from $X(\Omega)$. Setting for every $(u,\mu) \in X_M(\Omega)$

$$\begin{aligned}&\Gamma(X_M(\Omega))\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u,\mu)\\ &\quad := \inf\left\{\limsup_{n\to\infty} F_{\varepsilon_n}(u_n,\rho_n) : (u_n,\rho_n) \to (u,\mu) \text{ in } X_M(\Omega)\right\},\end{aligned}$$

we can consider the functional

$$\overline{F}_M(u,\mu) := \begin{cases}\Gamma(X_M(\Omega))\text{-}\limsup\limits_{n\to\infty} F_{\varepsilon_n}(u,\mu) & \text{if } (u,\mu) \in X_M(\Omega),\\ +\infty & \text{otherwise,}\end{cases}$$

defined for every $(u,\mu) \in X(\Omega)$. What we have proved so far can be restated in the following way:

$$\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u,\mu) \leqq \overline{F}_M(u,\mu) \leqq F(u,\mu)$$

for every pair $(u,\mu)$ satisfying the assumptions of Step 1 and with $\mu(\Omega) \leqq M$. As we already observed the advantage of considering $\overline{F}_M$ lies in the fact that $\overline{F}_M$ is sequentially lower semicontinuous with respect to the $\tau_1 \times \tau_2$ convergence in $X(\Omega)$.

**Step 2.** Let $u = \chi_{(A\cap\Omega)}$ with $\partial A$ a smooth $(N-1)$ manifold and $\mu = g\mathcal{H}^{N-1}\lfloor S_u + \sum_{j=1}^n c_i\delta_{x_i}$ where $g : \Omega \to \mathbb{R}$ is a continuous function. We may find a sequence $g_k$ of piecewise constant functions satisfying the assumptions of the previous step and converging to $g$ in $L^p(S_u;\mathcal{H}^{N-1})$ for every $p > 1$. We may also assume that $\int_{S_u} g_k\, d\mathcal{H}^{N-1} = \int_{S_u} g\, d\mathcal{H}^{N-1}$ for every $k$. Then, setting $\mu_k := g_k\mathcal{H}^{N-1}\lfloor S_u + \sum_{j=1}^n c_i\delta_{x_i}$, we clearly have $\mu_k(\Omega) = \mu(\Omega)$ for every $k$ and $\mu_k \overset{*}{\rightharpoonup} \mu$. Let $M > \mu(\Omega)$. By the lower-semicontinuity of $\overline{F}_M$ and from Step 1 we have

$$\begin{aligned}\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u,\mu) &\leqq \overline{F}_M(u,\mu) \leqq \liminf_{k\to\infty} \overline{F}_M(u,\mu_k)\\ &\leqq \lim_{k\to\infty}\int_{S_u}\phi(g_k)\, d\mathcal{H}^{N-1} = \int_{S_u}\phi(g)\, d\mathcal{H}^{N-1} = F(u,\mu).\end{aligned}$$

**Step 3.** Let $u = \chi_{(A\cap\Omega)}$ with $A$ an arbitrary set of finite perimeter, and let $\mu = g\mathcal{H}^{N-1}\lfloor S_u + \sum_{j=1}^n c_i\delta_{x_i}$ where $g : \Omega \to \mathbb{R}$ is a continuous function. By a well-known approximation result (see [21]), we may find a sequence $\{A_k\}$ of open sets such that $\partial A_k$ is a smooth manifold and

$$\chi_{A_k} \to \chi_A \text{ in } L^1(\mathbb{R}^N) \quad \text{and} \quad \mathrm{Per}\,(A_k, \Omega) \to \mathrm{Per}\,(A, \Omega).$$

We define $\mu_k := g\mathcal{H}^{N-1}\lfloor\partial A_k + t_k\sum_{j=1}^n c_i\delta_{x_i}$, where $t_k$ is chosen so that $\mu_k(\Omega) = \mu(\Omega)$. Since by Reshetnyak's theorem (see [1])

$$\int_\Omega \psi g\, d(\mathcal{H}^{N-1}\lfloor\partial A_k) \to \int_\Omega \psi g\, d(\mathcal{H}^{N-1}\lfloor\partial^* A)$$

for any $\psi \in C(\Omega)$, we have that $t_k \to 1$ and $\mu_k \overset{*}{\rightharpoonup} \mu$. Set $u_k := \chi_{A_k\cap\Omega}$. From the previous step and again by Reshetnyak's theorem, we get

$$\begin{aligned}
\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u,\mu) &\leqq \overline{F}_M(u,\mu) \leqq \liminf_{k\to\infty} \overline{F}_M(u_k,\mu_k) \\
&\leqq \lim_{k\to\infty}\int_\Omega \phi(g)\, d(\mathcal{H}^{N-1}\lfloor\partial A_k) = \int_\Omega \phi(g)\, d(\mathcal{H}^{N-1}\lfloor\partial^* A) \\
&= \int_{S_u} \phi(g)\, d\mathcal{H}^{N-1} = F(u,\mu).
\end{aligned}$$

**Step 4.** Let $u = \chi_{(A\cap\Omega)}$ with $A$ an arbitrary set of finite perimeter, and let $\mu$ be an arbitrary positive finite Radon measure. We can construct a sequence $\{\mu_k\}$ of the form

$$\mu_k = g_k\mathcal{H}^{N-1}\lfloor S_u + \sum_{j=1}^{n_k} c_i^k\delta_{x_i^k},$$

where each $g_k : \Omega \to \mathbb{R}$ is continuous, $g_k \to \frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_u}$ in $L^1(S_u; \mathcal{H}^{N-1})$, and $\sum_{j=1}^{n_k} c_i^k\delta_{x_i^k} \overset{*}{\rightharpoonup} \mu - \frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_u}\mathcal{H}^{N-1}\lfloor\partial^* A$. Clearly $\mu_k \overset{*}{\rightharpoonup} \mu$, and from Step 3 we conclude that

$$\begin{aligned}
\Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u,\mu) &\leqq \overline{F}_M(u,\mu) \leqq \liminf_{k\to\infty} \overline{F}_M(u,\mu_k) \\
&\leqq \lim_{k\to\infty}\int_{S_u} \phi(g_k)\, d\mathcal{H}^{N-1} = \int_{S_u} \phi\left(\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_u}\right) d\mathcal{H}^{N-1} = F(u,\mu).
\end{aligned}$$

The theorem is proved. □

From the preceding proof it is clear that given $(u,\mu) \in X(\Omega)$ the recovering sequence $\{(u_k,\mu_k)\}$ can be constructed in such a way that $\mu_k(\Omega) = \mu(\Omega)$. Moreover, if $f$ grows at least quadratically near the two wells we can argue as in [13] to show that the constraint $\int_\Omega u_k\, dx = \int_\Omega u\, dx$ can be imposed. In other words, the same $\Gamma$-convergence result remains true if we fix the volume of both $u$ and $\mu$. In order to state this precisely, assume that $f$ is a continuous double-well potential with wells at 0 and 1 and that it satisfies the following additional growth assumption: there exist $\delta > 0$ and $C > 0$ such that

$$f(u) \geqq C|u|^2 \qquad \text{and} \qquad f(1-u) \geqq C|u|^2$$

for $|u| \leqq \delta$. For $\alpha \in (0, \mathcal{L}^N(\Omega))$ and $\beta > 0$ consider the space

$$X^{\alpha,\beta}(\Omega) := \{(u,\mu) \in X(\Omega) : \int_\Omega u\,dx = \alpha \text{ and } \mu(\Omega) = \beta\}$$

and define

$$F_\varepsilon^{\alpha,\beta}(u,\mu) := \begin{cases} G_\varepsilon(u,\rho) & \text{if } (u,\mu) \in X^{\alpha,\beta}(\Omega) \text{ and } \mu = \rho dx, \\ +\infty & \text{otherwise,} \end{cases}$$

where $G_\varepsilon$ is the functional defined in (2.1) with $\alpha(\varepsilon) = \varepsilon$, and

$$F^{\alpha,\beta}(u,\mu) := \begin{cases} \displaystyle\int_{S_u} \phi\left(\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S_u}(x)\right) d\mathcal{H}^{N-1} & \begin{array}{l}\text{if } (u,\mu) \in X^{\alpha,\beta}(\Omega) \\ \text{and } u \in BV(\Omega;\{0,1\}),\end{array} \\ +\infty & \text{otherwise,} \end{cases}$$

where $\phi$ is the function defined in (3.1). Then we have

**Theorem 4.1.** *Under the above assumptions the family* $\{F_\varepsilon^{\alpha,\beta}\}\Gamma$*-converges to* $F^{\alpha,\beta}$ *with respect to the* $(\tau_1 \times \tau_2)$*-convergence of* $X(\Omega)$.

## 5. Remarks on stability

The following theorem deals with the existence of local minimizers for the approximating functionals $F_\varepsilon$ (see (2.2)) near a stable configuration of the limit energy $F$ (see (2.4)), in the spirit of Kohn & Sternberg (see [19]).

**Definition 1.** Let $F : L^1(\Omega) \to \mathbb{R}$ be a functional. We say that $u \in L^1(\Omega)$ is a *local minimizer* for $F$ if there exists $\delta > 0$ such that

$$F(u) \leqq F(v) \tag{5.1}$$

whenever $0 < \|u - v\|_{L^1} \leqq \delta$ with $v$ satisfying the same volume constraint as $u$, that is, $\int_\Omega u\,dx = \int_\Omega v\,dx$. We say that $u_0$ is an *isolated local minimizer* for $F$ if (5.1) holds with the strict inequality.

**Theorem 5.1.** *Let* $(u_0, \mu_0) \in BV(\Omega;\{0,1\}) \times \mathcal{M}_+(\Omega)$ *be such that* $u_0$ *is an isolated local minimizer for the functional* $F(\cdot, \mu_0)$. *Then there exists a sequence* $(u_\varepsilon, \rho_\varepsilon)$ *with*

$$u_\varepsilon \to u_0 \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad \rho_\varepsilon \overset{*}{\rightharpoonup} \mu_0 \quad \text{in } \mathcal{M}_+(\Omega)$$

*such that for* $\varepsilon$ *small enough* $u_\varepsilon$ *is a local minimizer for the functional* $F_\varepsilon(\cdot, \rho_\varepsilon)$.

**Proof.** Let $(v_\varepsilon, \rho_\varepsilon)$ such that

$$v_\varepsilon \to u_0 \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad \rho_\varepsilon \overset{*}{\rightharpoonup} \mu_0 \quad \text{in } \mathcal{M}_+(\Omega)$$

and

$$F_\varepsilon(v_\varepsilon, \rho_\varepsilon) \to F(u_0, \mu_0). \tag{5.2}$$

By assumption there exists $\delta > 0$ such that $F(u_0, \mu_0) < F(v, \mu_0)$ whenever $0 < \|u_0 - v\|_{L^1} \leqq \delta$ and $\int_\Omega u_0\, dx = \int_\Omega v\, dx$. We choose $u_\varepsilon$ solution to the problem

$$\min\left\{F_\varepsilon(v, \rho_\varepsilon) : \ \|v - u_0\|_{L^1} \leqq \delta \ \int_\Omega v\, dx = \int_\Omega u_0\, dx\right\}. \tag{5.3}$$

The existence of such $u_\varepsilon$ is easily deduced by applying the direct method of the calculus of variations. We claim that $u_\varepsilon \to u_0$. Indeed, suppose by contradiction that (up to a subsequence) $0 < \delta_1 \leqq \|u_\varepsilon - u_0\| \leqq \delta$. Since

$$\sup_\varepsilon F_\varepsilon(u_\varepsilon, \rho_\varepsilon) < +\infty,$$

by compactness we may assume that $u_\varepsilon \to u^*$ in $L^1(\Omega)$ for some $u^* \in BV(\Omega;\{0, 1\})$. Clearly we still have $\delta_1 \leqq \|u_0 - u^*\| \leqq \delta$ and $\int_\Omega u^*\, dx = \int_\Omega u_0\, dx$. In light of the minimality of $u_\varepsilon$ we know $F_\varepsilon(u_\varepsilon, \rho_\varepsilon) \leqq F_\varepsilon(v_\varepsilon, \rho_\varepsilon)$ from which we deduce

$$\begin{aligned} F(u^*, \mu_0) &\leqq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \rho_\varepsilon) \\ &\leqq \lim_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon, \rho_\varepsilon) = F(u_0, \mu_0), \end{aligned} \tag{5.4}$$

where the first inequality is a consequence of the $\Gamma$-convergence of $F_\varepsilon$ to $F$ while the last equality follows from (5.2). The inequalities in (5.4) are in contradiction with the fact that $u_0$ is an isolated local minimizer. Therefore, $u_\varepsilon \to u_0$, and this concludes the proof of the theorem. □

We now use the previous theorem to show that the presence of surfactant may influence the structure of local minimizers. Let $\Omega$ be the two-dimensional cube $(0, 1) \times (0, 1)$ and let $u_0 \in BV(\Omega; \{0, 1\})$ be a characteristic function with a jump set made of a finite collection of line segments parallel to the $x$-axis. We start by assuming that no surfactant is present in the system, that is, $\mu_0 = 0$. In this situation $u_0$ corresponds to a non-isolated stable configuration for the functional $F(\cdot, 0)$. Indeed we can obtain energetically equivalent configurations by sliding the interfaces a little bit. As consequence we cannot apply the previous theorem and in fact by a result from GURTIN and MATANO ([16]) we know that for every $\varepsilon > 0$ all local minimizers for $F_\varepsilon(\cdot, 0)$ are monotone in the $y$-direction and therefore they cannot be close to a multiple interface configuration like $u_0$. In other words for $\varepsilon$ finite the configuration given by $u_0$ is unstable when there is no surfactant. The situation changes as soon as we add surfactant. Indeed if $\mu_0$ is a positive measure whose support coincides with the jump set of $u_0$, then it is easy to see that $u_0$ is an isolated local minimizer for $F(\cdot, \mu_0)$ and thus, by Theorem 5.1 we can find a sequence $\{\rho_\varepsilon\}$ of surfactant densities approaching the limit distribution $\mu_0$ and a sequence $\{u_\varepsilon\}$ of local minimizers for $F_\varepsilon(\cdot, \rho_\varepsilon)$ converging to $u_0$. This shows that the presence of surfactant makes it possible to have stable configurations for the functionals $F_\varepsilon$ close to a multiple interface configuration.

We conclude by observing that so far we considered only configurations which are only stable with respect to variations of the phase variable $u$. It would be interesting from the physical point of view to investigate to prove the existence of multiple interface configurations which are stable with respect to variations in the pair $(u, \rho)$.

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