Arch. Rational Mech. Anal. 183 (2007) 187–214
Digital Object Identifier (DOI) 10.1007/s00205-006-0003-y

A Constructive Existence Proof for the Extreme Stokes Wave

L. E. FRAENKEL

Communicated by C.A. STUART

Abstract

Stokes conjectured in 1880 that an extreme gravity wave on water (or "wave of greatest height") exists, has sharp crests of included angle $2\pi/3$ and has a boundary that is convex between successive crests. These three conjectures have all been proved recently, but by diverse methods that are not conspicuously direct. The present paper proceeds from a first approximate solution, of the extreme form of the integral equation due to Nekrasov, to a contraction mapping for a related integral equation that governs a new dependent variable in the space $L_2(0, \pi)$. This method provides: (a) a constructive approach to an extreme wave with the sharp crests predicted by Stokes; and (b) a rather accurate second approximation. However, the method has not led (so far, at least) to the convexity.

1. Introduction

The STOKES conjecture [11] for the "highest" gravity wave on water may be said to consist of three parts:

(i) existence of an extreme wave (an end-point of a family of "smaller" waves, described fully in the Introductions to [2] and [10]);
(ii) that this wave is distinguished by sharp crests of included angle $2\pi/3$;
(iii) convexity of the upper boundary of the water between successive crests.

All three parts have now been proved, but not easily and not by a unified method. A preliminary step was the existence proof in [5] for smooth waves (not extreme) of large amplitude. Part (i) of the conjecture was proved in [12] and in [6] by means of sequences of smooth waves tending to an extreme one; the passage to the limit was treated quite differently in the two papers.

Part (ii) was proved in [2] by means of real-variable methods and the limiting form (equation (2.1) below) of the integral equation due to Nekrasov; at the same time, it was proved in [9] by means of complex-variable methods and an extension of a certain function beyond its domain in the plane of the complex potential. (This was an inspired sharpening for a particular case of a general construction due to H. Lewy.)

Part (iii) has been proved only in [10]; the proof uses complex-variable methods of the kind initiated in [9]. The paper is formidable, both as an achievement by the authors and as a task for the reader.

The present paper deals jointly with parts (i) and (ii) of the conjecture. We solve a nonlinear integral equation $u = Au$, which is a variant of the limiting Nekrasov equation, in three steps as follows.

(a) Explicit formulae define an approximation u_a and the corresponding function Au_a such that $Au_a - u_a$ is small.

(b) Posing $u = u_a + h$ for the exact solution, we write the equation $u = Au$ as

$$u_a + h = Au_a + A'(u_a)\,h + R_a h, \tag{1.1}$$

and prove that the linear equation

$$\mathcal{L}_a w := w - A'(u_a)\,w = f \tag{1.2}$$

has a unique, bounded solution $w \in L_2(0, \pi)$ for each $f \in L_2(0, \pi)$.

(c) Equation (1.1) may now be written

$$h = \mathcal{L}_a^{-1}(Au_a - u_a + R_a h) =: S_a h \quad \text{(say)}, \tag{1.3}$$

and estimates of the nonlinear term $R_a h$ show that S_a is a contraction map of a small, closed ball in $L_2(0, \pi)$. (Accordingly, there is exactly one solution h of (1.3) in that ball.)

It might be hoped that this construction would yield a simple proof of part (iii) of the conjecture, but I have been unable to show this.

Throughout the paper, results depend on the numerical evaluation and numerical integration of functions defined by explicit formulae. (These calculations were all done with a Texas Instruments TI-92 calculator.) Therefore, purists may believe that the theorems in the paper have not been proved. I have much sympathy with this point of view, but it seems unlikely that, without numerical evaluation of known functions, a construction as direct as that in this paper could be obtained.

2. Preliminaries

The Nekrasov equation governing the extreme Stokes wave, for the case of periodic waves on water of infinite depth, is

$$\varphi(s) = \frac{1}{3}\int_0^\pi K(s,t)\,\frac{\sin\varphi(t)}{\int_0^t \sin\varphi}\,\mathrm{d}t, \quad 0 < s < \pi, \tag{2.1}$$

where

$$K(s,t) := \frac{1}{\pi}\log\frac{\tan\frac{1}{2}s + \tan\frac{1}{2}t}{\left|\tan\frac{1}{2}s - \tan\frac{1}{2}t\right|} = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin ks\ \sin kt}{k}. \tag{2.2}$$

The physical meanings of s, φ and the u mentioned in Section 1 are as follows. In the physical plane we take axes Ox, Oy moving with the wave (axes fixed relative to a crest) with Oy pointing vertically upwards. The wavelength is λ, the complex potential is $\Phi + \mathrm{i}\Psi$, and the complex velocity

$$\frac{\mathrm{d}(\Phi + \mathrm{i}\Psi)}{\mathrm{d}(x + \mathrm{i}y)} = q\mathrm{e}^{-\mathrm{i}\vartheta} \to c \quad \text{as} \quad y \to -\infty \quad (c = \text{const.} > 0)\,.$$

Then the co-ordinate s in (2.1) is defined by $s := -(2\pi/c\lambda)\,\Phi$, with $s = 0$ at a crest and $s = \pi$ at the adjacent trough to the left of that crest. The angle φ is such that $\tan\varphi$ is the slope of the upper boundary $\{\Psi = 0\}$ of the water, φ is also the restriction of ϑ to that upper boundary, and

$$u(s) := 3\frac{\partial}{\partial s}\left(\log\frac{q}{c}\right)\bigg|_{\Psi=0},$$

in which $\log(q/c)$ is a harmonic conjugate of $-\vartheta$.

As we shall use a variant of (2.1), an elaborate definition of admissible function is required. The symbol $C^{0,\alpha}[0,\pi]$ denotes the set of functions that are uniformly Hölder continuous on $[0,\pi]$ with Hölder exponent α.

Definition 2.1.

(i) The set X of *admissible functions* is the set of functions $\psi : [0,\pi] \to \mathbb{R}$ such that

(a) $\psi \in C^{0,\alpha}[0,\pi]$ for some $\alpha \in (0,1)$; (2.3a)

(b) $\psi(0) = \pi/6$ and $\psi(\pi) = 0$; (2.3b)

(c) $-\pi/12 \leqq \psi(s) \leqq \pi/3$ for $s \in (0,\pi)$; (2.3c)

(d) $\displaystyle\int_0^s \sin\psi \geqq ks$ on $[0,\pi]$ for some constant $k > 0$. (2.3d)

(ii) By a *solution of* (2.1) we mean a function $\varphi \in X$ that satisfies (2.1) pointwise.

This definition of a solution is wider than those in [2] and [1] in that it allows negative values of φ; it is also narrower than the definition in [2] in that it implies that a solution represents a wave having a sharp crest of included angle $2\pi/3$.

We shall write, at least for $0 < s < \pi$,

$$(\mathcal{N}\psi)(s) := \frac{\sin\psi(s)}{\int_0^s \sin\psi} = \frac{\mathrm{d}}{\mathrm{d}s}\log\int_0^s \sin\psi, \tag{2.4}$$

$$(\mathcal{K}f)(s) := \int_0^\pi K(s,t)\,f(t)\,\mathrm{d}t, \tag{2.5}$$

whenever $\psi \in X$ and f is such that the integral in (2.5) exists. Obviously (2.4) holds also at $s = \pi$. Limiting values at $s = 0$ and at $s = \pi$ will be adopted wherever they exist. Equation (2.1) becomes

$$\varphi = T\varphi, \quad \text{where} \quad T\psi := \frac{1}{3}\mathcal{K} \circ \mathcal{N}\psi \quad \text{for all } \psi \in X. \tag{2.6}$$

We shall invert the nonlinear operator $\mathcal{N}$; to that end, we define a suitable co-domain.

Definition 2.2. The set Y is the set of functions $v : (0, \pi] \to \mathbb{R}$ such that

(a) the function $s \mapsto sv(s)$ is in $C^{0,\alpha}[0, \pi]$ for some $\alpha \in (0, 1)$; (2.7a)

(b) $sv(s) = 1 + O\left(s^{\alpha}\right)$ near $s = 0$ and $v(\pi) = 0$. (2.7b)

First we show that $\mathcal{N} : X \to Y$ is injective. For ψ and θ in X,

$$\mathcal{N}\psi = \mathcal{N}\theta \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}s} \log \frac{\int_0^s \sin\psi}{\int_0^s \sin\theta} = 0 \Longrightarrow \log \frac{\int_0^s \sin\psi}{\int_0^s \sin\theta} = 0 \tag{2.8}$$

because

$$\frac{\int_0^s \sin\psi}{\int_0^s \sin\theta} = 1 + O\left(s^{\alpha}\right), \quad \alpha > 0, \text{ near } s = 0.$$

Thus,

$$\int_0^s \sin\psi = \int_0^s \sin\theta \quad \text{on } [0, \pi];$$

we differentiate, recall that $\psi(s)$ and $\theta(s)$ are in $[-\pi/12, \pi/3]$ and conclude that $\psi = \theta$.

Accordingly, $\mathcal{N}$ has an inverse $\mathcal{N}^{-1} : \mathcal{N}(X) \to X$; we write it explicitly as follows. By (2.4),

$$\mathcal{N}\psi = v \Longleftrightarrow \int_0^s \sin\psi = \left(\int_0^s \sin\psi_*\right) \exp \int_0^s (v - v_*) \tag{2.9}$$

for any *comparison function* $\psi_* \in X$ with $v_* := \mathcal{N}\psi_*$. (Note that v is not integrable at 0, but $v(t) - v_*(t) = O\left(t^{\alpha-1}\right)$ for some $\alpha > 0$. Possible comparison functions ψ_* are φ_0 and φ_1, defined by

$$\sin\varphi_0(s) := \frac{1}{2}\cos\frac{s}{2}, \qquad (\mathcal{N}\varphi_0)(s) = \frac{1}{2}\cot\frac{s}{2}, \tag{2.10}$$

$$\left.\begin{aligned} \varphi_1(s) &:= (T\varphi_0)(s) = \tfrac{\pi - s}{6}, \\ (\mathcal{N}\varphi_1)(s) &= \tfrac{1}{6}\sin\tfrac{\pi - s}{6} \Big/ \left(\cos\tfrac{\pi - s}{6} - \cos\tfrac{\pi}{6}\right). \end{aligned}\right\} \tag{2.11}$$

The functions ψ_c and $T\psi_c$ in Section 3 will be perturbations of φ_0 and φ_1, respectively.) It follows from (2.9) by differentiation that

$$\sin\psi(s) = \sin\left(\mathcal{N}^{-1}v\right)(s) = \left(\int_0^s \sin\psi_*\right) v(s)\, \exp\int_0^s (v - v_*)\,. \quad (2.12)$$

That this is independent of the choice of comparison function follows from our derivation and is confirmed by the proof of Remark 2.3 below.

Although $\mathcal{N}$ is not monotonic under the usual partial ordering of functions in $C[0,\pi]$, the formula (2.12) shows that $\mathcal{N}^{-1}v$ is an increasing function of v. The formula also enables us to distinguish functions in $\mathcal{N}(X)$ from those merely in Y, as follows. If $v \in Y$ and also

$$-\sin\frac{\pi}{12} \leqq \left(\int_0^s \sin\psi_*\right) v(s) \exp\int_0^s (v - v_*) \leqq \sin\frac{\pi}{3} \quad \text{for } s \in (0,\pi), \quad (2.13)$$

then (2.12) shows that (2.3a) to (2.3c) hold, while (2.9) implies (2.3d).

Here is a variant (which seems to be new) of the Nekrasov equation (2.1). Instead of pursuing φ by means of $\varphi = T\varphi$, we pursue $u := \mathcal{N}\varphi$ by means of

$$u = Au, \quad (2.14)$$

where

$$(Av)(s) := \frac{\sin\left(\frac{1}{3}\mathcal{K}v\right)(s)}{\int_0^s \sin\psi_*} \exp\int_0^s (v_* - v) \quad (2.15)$$

for all $v \in \mathcal{N}(X)$ and for any comparison function $\psi_* \in X$ with $v_* := \mathcal{N}\psi_*$.

Remark 2.3. The operator A is independent of the choice of comparison function ψ_*.

Proof. Let A_* and A_0 be the operator A with comparison functions ψ_* and ψ_0, respectively. At a zero, say $s_0 \in (0,\pi]$, of $\sin\left(\frac{1}{3}\mathcal{K}v\right)$ we have $(A_*v)(s_0) = (A_0v)(s_0)$. At all other points $s \in (0,\pi)$,

$$\begin{aligned}\frac{(A_*v)(s)}{(A_0v)(s)} &= \frac{\int_0^s \sin\psi_0}{\int_0^s \sin\psi_*} \exp\int_0^s (v_* - v_0)\\ &= \frac{\int_0^s \sin\psi_0}{\int_0^s \sin\psi_*} \exp\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\int_0^t \sin\psi_*}{\int_0^t \sin\psi_0}\,\mathrm{d}t\\ &= 1. \qquad \square\end{aligned}$$

If we choose $\psi_* = \psi := \mathcal{N}^{-1}v$, then (2.15) becomes

$$(Av)(s) = \frac{\sin(T\psi)(s)}{\int_0^s \sin\psi} \qquad \left(\psi = \mathcal{N}^{-1}v\right), \quad (2.16)$$

where T is as in (2.6). This shows (since $v(s) = \sin\psi(s)\big/\int_0^s \sin\psi$) that the equations $\varphi = T\varphi$ and $u = Au$ are equivalent when $\varphi \in X$ and $u \in \mathcal{N}(X)$.

3. A formal approximation to the solution

We define the *Grant number* β to be the smallest positive zero of $\sqrt{3}(1+x) - \tan(\pi x/2)$, whence $\beta = 0.80268\ldots$ If φ is a solution (perhaps not quite in the present sense) of the Nekrasov equation (2.1), then

$$\varphi(s) = \frac{\pi}{6} + C_1 s^{\beta} + C_2 s^{2\beta} + O\left(s^{3\beta}\right) \quad \text{as } s \downarrow 0, \tag{3.1}$$

where $C_1 < 0$ and $C_2 > 0$. The exponents β, 2β and many others were found and applied heuristically in [4] and [8]; the form of the whole asymptotic expansion for $s \downarrow 0$ (with known exponents and unknown coefficients) was established rigorously in [1]; that $C_1 < 0$ was proved in [7]; and that $C_2 > 0$ is a result of [1]. The whole asymptotic expansion contains infinitely many exponents *other than* $n\beta$, $n \in \mathbb{N}$, where $\mathbb{N} := \{1, 2, 3, \ldots\}$.

The precise hypotheses in [1] and [7] need not concern us here, because we shall use (3.1) only as a guide (and not in any proof).

Lemma A.3 of [2] provides a method of evaluating $T\psi$ explicitly for certain functions ψ. Guided by (3.1), we apply the lemma to construct functions ψ_c and $T\psi_c$ as follows. The holomorphic function w in the lemma is first chosen to be cw_1, where c =const.$\in (0, 1)$ and

$$w_1(\zeta_0) := -\int_1^{\zeta_0} \frac{1}{\zeta}\left\{(1-\zeta)^{\beta-1} - 1\right\} \mathrm{d}\zeta, \tag{3.2}$$

with ζ_0 and ζ in $\mathcal{D}_1 := \left\{\zeta \in \mathbb{C} \,\middle|\, |\zeta| \leqq 1, \quad \zeta \neq 1\right\}$; a cut outside $\mathcal{D}_1$ restricts $\arg(1-\zeta)$ to $(-\pi/2, \pi/2)$ when $\zeta \in \mathcal{D}_1$. The real and imaginary parts of $w_1\left(\mathrm{e}^{\mathrm{i}s_0}\right)$, $0 \leqq s_0 \leqq \pi$, are respectively

$$a_1(s_0) = \int_0^{s_0} \left(2\sin\frac{s}{2}\right)^{\beta-1} \sin\frac{(1-\beta)(\pi-s)}{2}\,\mathrm{d}s, \tag{3.3}$$

and

$$b_1(s_0) = -\int_0^{s_0} \left(2\sin\frac{s}{2}\right)^{\beta-1} \cos\frac{(1-\beta)(\pi-s)}{2}\,\mathrm{d}s \quad + \quad s_0; \tag{3.4}$$

the lemma then states that

$$\int_0^s \sin\psi_c = \sin\frac{s}{2}\exp\{-ca_1(s)\}, \tag{3.5a}$$

$$\sin\psi_c(s) = \left\{\frac{1}{2}\cos\frac{s}{2} - ca_1'(s)\sin\frac{s}{2}\right\}\exp\{-ca_1(s)\}, \tag{3.5b}$$

$$(\mathcal{N}\psi_c)(s) = \frac{1}{2}\cot\frac{s}{2} - ca_1'(s), \tag{3.5c}$$

$$(T\psi_c)(s) = \frac{\pi - s}{6} + \frac{1}{3}cb_1(s). \tag{3.6}$$

Here $b_1(\pi) = 0$ because $w_1(\zeta_0)$ is real for real $\zeta_0 \in [-1, 1]$. It is laborious but not difficult to check that $\psi_c \in X$ and $T\psi_c \in X$ when $c \in (0, 1)$. In particular, $\psi_c'(s) < 0$ on $(0, \pi]$, $b_1''(s) > 0$ in $(0, \pi)$ and, for $s \in (0, \pi)$,

$$0 < \sin \psi_c(s) < \frac{1}{2} \cos \frac{s}{2}, \tag{3.7a}$$

$$\frac{\pi - s}{9} < (T\psi_c)(s) < \frac{\pi - s}{6}. \tag{3.7b}$$

Contour integration shows that

$$a_1(\pi) = \int_{-1}^{1} \frac{1}{\xi} \left\{ (1-\xi)^{\beta-1} - 1 \right\} d\xi \tag{3.8a}$$

$$= 0.53826\ldots \tag{3.8b}$$

by numerical evaluation of (3.8a); equally, by numerical evaluation of (3.3) with $s_0 = \pi$. Of course, over short intervals near awkward points (such as $s = 0$ for (3.3), or $\xi = 0$ and $\xi = 1$ for (3.8a)) approximations to the integrand have been integrated analytically, here and elsewhere.

We wish to choose c so that $T\psi_c - \psi_c$ is numerically small. Now, as $s \downarrow 0$,

$$\sin \psi_c(s) = \frac{1}{2} - cBs^\beta + O\left(s^{2\beta}\right) \qquad \left(B := \frac{\beta+1}{2\beta} \cos \frac{\beta\pi}{2}\right),$$

$$\sin (T\psi_c)(s) = \frac{1}{2} - cBs^\beta + \frac{2c-1}{4\sqrt{3}} s + O\left(s^{2\beta}\right),$$

so that the choice $c = 1/2$ makes $T\psi_c$ close to ψ_c for small s. On the other hand, $(T\psi_c)'(\pi) = \psi_c'(\pi)$ if and only if $c = 0.62633\ldots$, so that this value of c makes $T\psi_c$ close to ψ_c for s near π. We shall use

$$g(s, c) := \frac{\sin (T\psi_c)(s) - \sin \psi_c(s)}{\int_0^s \sin \psi_c} \tag{3.9}$$

as a measure of error; note that, by (2.16) and (2.4),

$$g(\cdot, c) = Av_c - v_c \quad \text{if} \quad v_c := \mathcal{N}\psi_c, \tag{3.10}$$

and that $\sin T\psi_c - \sin \psi_c$ is much smaller than $g(\cdot, c)$ near $s = 0$.

Figure 3.1 shows graphs of $g(\cdot, c)$ for $c = 1/2$, $c = 0.62633$ and $c = 0.56$. We adopt the value $c = 0.56$ and *define a formal approximation* φ_a *to the desired solution* φ *by* $\varphi_a := \psi_{0.56}$, with error

$$g_a(s) := g(s, 0.56) = \frac{\sin (T\varphi_a)(s) - \sin \varphi_a(s)}{\int_0^s \sin \varphi_a}. \tag{3.11}$$

The explicit formula for g_a and numerical integration (done in two different ways) yield

$$\left\| g_a \middle| L_2(0, \pi) \right\| := \left(\int_0^\pi g_a^2 \right)^{1/2} < \frac{1}{60}. \tag{3.12}$$

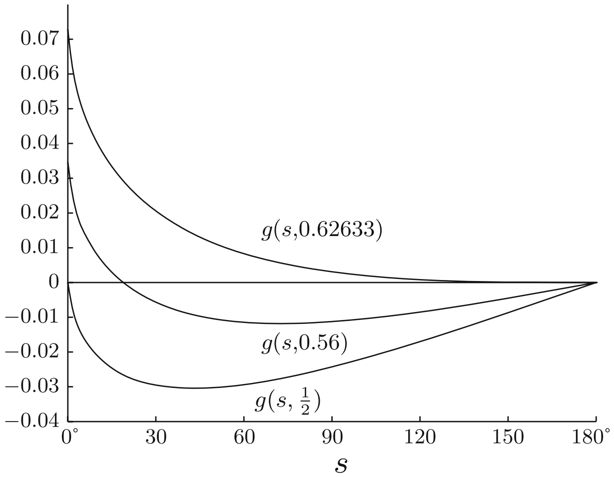

Fig. 3.1.

Table 3.1.

s	$\sin \varphi_a(s)$	$\sin(T\varphi_a)(s) - \sin \varphi_a(s)$	$\lambda(s)$
0°	0.5	0	0
3	0.48231	0.00051	−0.00074
6	0.46944	0.00069	−0.00129
9	0.45798	0.00069	−0.00182
12	0.44736	0.00057	−0.00237
15	0.43732	0.00036	−0.00296
30	0.39193	−0.00130	−0.00662
60	0.31082	−0.00474	−0.01752
90	0.23319	−0.00626	−0.03117
120	0.15586	−0.00560	−0.04425
150	0.07810	−0.00325	−0.05358
180°	0	0	−0.05696

Table 3.1 shows values of $\sin \varphi_a$ and of $\sin T\varphi_a - \sin \varphi_a$; Figure 3.2 shows the graph of $\sin \varphi_a$.

Appendix B presents an improved approximation, based on the first of the successive approximations that are implied by the existence proof in Section 6.

4. Inversion of a linear operator

We seek a solution u of $u = Au$ in the form

$$u = u_a + h, \quad u_a := \mathcal{N}\varphi_a, \quad h \in L_2(0, \pi), \tag{4.1}$$

where φ_a is the approximation constructed in Section 3. Whether $u_a + h$ will be in $\mathcal{N}(X)$ remains to be seen. With

$$R_a h := A(u_a + h) - Au_a - A'(u_a)\, h \tag{4.2}$$

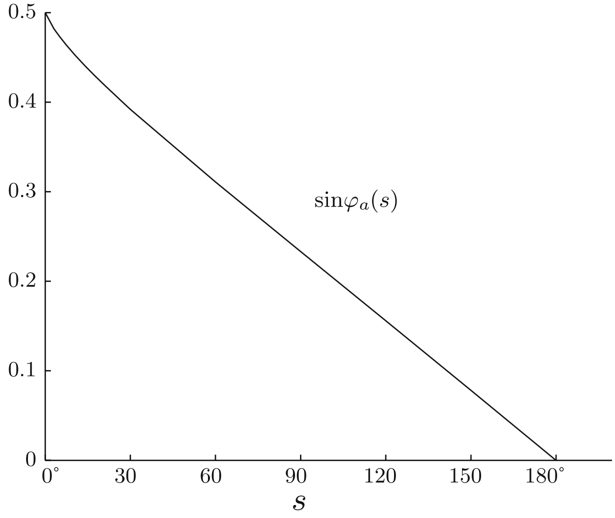

Fig. 3.2.

denoting a nonlinear remainder, the equation $u = Au$ becomes

$$u_a + h = Au_a + A'(u_a)\, h + R_a h,$$

equivalently,

$$\mathcal{L}_a h := h - A'(u_a)\, h = g_a + R_a h, \tag{4.3}$$

where

$$g_a := Au_a - u_a. \tag{4.4}$$

We have encountered g_a in (3.10) to (3.12); the function $R_a h$ will be the subject of Section 5. Section 4 concerns the linear operator $\mathcal{L}_a$. Let the comparison function, in the definition (2.15) of the operator A, be $\psi_* = \varphi_a$. Then $v_* = u_a$ and the concrete form of $\mathcal{L}_a h$ is given by

$$(\mathcal{L}_a h)(s) = h(s) - \frac{1}{3}\frac{\cos\varphi_b(s)}{\int_0^s \sin\varphi_a}\,(\mathcal{K}h)(s) + \frac{\sin\varphi_b(s)}{\int_0^s \sin\varphi_a}\int_0^s h\,, \tag{4.5a}$$

where

$$\varphi_b := T\varphi_a. \tag{4.5b}$$

Notation. $\mathbb{N} := \{1, 2, 3, \dots\}$, $L_2 := L_2(0, \pi)$ and $\|\cdot\| := \big\|\cdot\big|\, L_2\big\|$.

Lemma 4.1. *Given $w \in L_2$, define*

$$f(s) := \frac{1}{\sin\frac{s}{2}}\,(\mathcal{K}w)(s) \quad \textit{and} \quad g(s) := \frac{1}{\sin\frac{s}{2}}\int_0^s w. \tag{4.6}$$

Then $f \in L_2$, $g \in L_2$ with $\|f\| \leqq 4\|w\|$, $\|g\| \leqq 4\|w\|$ and $\|f\| = \|g\|$.

Proof.
(i) We may write (for $0 < s < \pi$)

$$w(s) = \sum_{k=1}^{\infty} c_k \sin ks, \quad (c_k) \in l_2,$$

the convergence being in the norm of L_2. Let w_n denote the partial sum from $k = 1$ to $k = n$, and define

$$u_n(s) := (\mathcal{K}w_n)(s) = \sum_{k=1}^{n} \frac{c_k \sin ks}{k}, \tag{4.7a}$$

$$f_n(s) := \frac{1}{\sin\frac{s}{2}} u_n(s), \qquad \phi := u_n - u_m, \quad m < n. \tag{4.7b}$$

Since $\cot\frac{\pi}{2} = 0$ and $\phi(s) = O(s)$ as $s \downarrow 0$, integration by parts yields

$$\begin{aligned}\int_0^{\pi} \left(\frac{\phi}{\sin\frac{s}{2}}\right)^2 \mathrm{d}s &= \int_0^{\pi} 2\phi\phi'.2\cot\frac{s}{2}.\,\mathrm{d}s \\ &\leqq 4\left\{\int_0^{\pi}\left(\frac{\phi}{\sin\frac{s}{2}}\right)^2\right\}^{1/2}\left\{\int_0^{\pi}\left(\phi'\cos\frac{s}{2}\right)^2\right\}^{1/2}\end{aligned}$$

by the Schwarz inequality. Dividing through by the first of these square roots, we have

$$\|f_n - f_m\| = \left\{\int_0^{\pi}\left(\frac{\phi}{\sin\frac{s}{2}}\right)^2\right\}^{1/2} \leqq 4\left\{\int_0^{\pi}\phi'^2\,\mathrm{d}s\right\}^{1/2} = 4\left\{\frac{\pi}{2}\sum_{k=m+1}^{n} c_k^2\right\}^{1/2}, \tag{4.8}$$

so that (f_n) is a Cauchy sequence in the complete space L_2; its limit is certainly f because $u_n \to \mathcal{K}w$ in $C[0, \pi]$. That $\|f\| \leqq 4\|w\|$ follows from (4.8) if f_m, u_m and m are replaced by 0 there.
If we now define

$$v_n(s) := \int_0^s w_n = \sum_{k=1}^{n} \frac{c_k(1 - \cos ks)}{k}, \tag{4.9a}$$

$$g_n(s) := \frac{1}{\sin\frac{s}{2}} v_n(s), \tag{4.9b}$$

then the proof that $g \in L_2$, with $\|g\| \leqq 4\|w\|$, is similar to the foregoing one.
(ii) For an elementary proof that $\|f\| = \|g\|$, we write $f_n(s)$ and $g_n(s)$ in terms of $\theta := s/2$. First,

$$\frac{\sin 2k\theta}{\sin\theta} = 2\{\cos\theta + \cos 3\theta + \cdots + \cos(2k-1)\theta\},$$

$$\frac{1 - \cos 2k\theta}{\sin\theta} = 2\{\sin\theta + \sin 3\theta + \cdots + \sin(2k-1)\theta\}.$$

(To verify these identities, multiply each by $\sin\theta$ and use $2\sin\theta\cos 3\theta = \sin 4\theta - \sin 2\theta$, etc.) Accordingly,

$$f_n(s) = b_1\cos\theta + b_3\cos 3\theta + \cdots + b_{2n-1}\cos(2n-1)\theta,$$
$$g_n(s) = b_1\sin\theta + b_3\sin 3\theta + \cdots + b_{2n-1}\sin(2n-1)\theta,$$

where

$$b_{2m-1} = 2\sum_{k=m}^{n}\frac{c_k}{k}, \quad m \leqq n.$$

The functions $\cos(2m-1)\theta$, $m \in \mathbb{N}$, are mutually orthogonal on $(0, \pi/2)$, and so are the functions $\sin(2m-1)\theta$, $m \in \mathbb{N}$. Hence, (for all $n \in \mathbb{N}$)

$$\|f_n\|^2 = \frac{\pi}{2}\sum_{m=1}^{n} b_{2m-1}^2 = \|g_n\|^2, \tag{4.10}$$

and we know from step (i) that $f_n \to f$ and $g_n \to g$ in the norm of L_2.
(iii) A more revealing proof that $\|f\| = \|g\|$, suggested by J.B. McLeod and by J.F. Toland, proceeds from a transformation of conjugate functions on the unit circle to conjugate functions on the real line. If $u(s) := (\mathcal{K}w)(s)$ and $v(s) := \int_0^s w$ are extended from $[0,\pi]$ to $[0, 2\pi)$ as odd and even functions, respectively, of $s-\pi$, then u and v are conjugate functions on the unit circle (that is, boundary-value functions of harmonic conjugates in the unit disc).
The conformal transformation

$$z = 2\mathrm{i}\,\frac{1+\zeta}{1-\zeta}, \quad z = x + \mathrm{i}y, \quad \zeta = \rho e^{\mathrm{i}s}, \tag{4.11}$$

maps $\mathcal{D}_1 = \{\zeta \in \mathbb{C} \mid |\zeta| \leqq 1,\ \zeta \neq 1\}$ onto $\{z \in \mathbb{C} \mid \operatorname{Im} z \geqq 0\}$. For $\zeta = e^{\mathrm{i}s}$, $0 < s < 2\pi$, we have

$$x = 2\tan\frac{s-\pi}{2}, \quad \mathrm{d}x = \frac{1}{\left(\sin\frac{s}{2}\right)^2}\,\mathrm{d}s. \tag{4.12}$$

Let $\hat{u}(x) := u(s)$ and $\hat{v}(x) := v(s)$ under this transformation; then $\hat{u}$ and $\hat{v}$ are conjugate functions on the real line (are Hilbert transforms of each other), from which it follows that

$$\int_0^{2\pi}\left(\frac{u(s)}{\sin\frac{s}{2}}\right)^2 \mathrm{d}s = \int_{-\infty}^{\infty}\hat{u}(x)^2\,\mathrm{d}x = \int_{-\infty}^{\infty}\hat{v}(x)^2\,\mathrm{d}x = \int_0^{2\pi}\left(\frac{v(s)}{\sin\frac{s}{2}}\right)^2 \mathrm{d}s,$$

and the integrals over $(0, 2\pi)$ are twice those over $(0, \pi)$. □

Lemma 4.2. *If $w \in L_2$ and $\mathcal{L}_a$ is as in* (4.5)*, then*

$$\int_0^{\pi} w\mathcal{L}_a w \geqq 0.43\,\|w\|^2, \tag{4.13}$$

with equality only if $\|w\| = 0$.

Proof.
(i) The coefficients in $\mathcal{L}_a$ may be written, in view of (3.5a),

$$p_a(s) := \frac{1}{3}\frac{\mathrm{e}^{a(s)}\cos\varphi_b(s)}{\sin\frac{s}{2}}, \quad q_a(s) := \frac{\mathrm{e}^{a(s)}\sin\varphi_b(s)}{\sin\frac{s}{2}}, \tag{4.14}$$

where $a(s) := 0.56\, a_1(s)$. Consequently,

$$\int_0^\pi w\mathcal{L}_a w = \int_0^\pi \left\{ w^2 - p_a w\mathcal{K}w + q_a w W \right\}, \tag{4.15}$$

where $W(s) := \int_0^s w$. We shall show that the adverse second term is dominated by the favourable first and third terms. First,

$$p_a(s) \leqq \frac{\kappa}{\sin\frac{s}{2}}, \quad \kappa := \frac{1}{3}\mathrm{e}^{a(\pi)} = 0.45059\ldots, \tag{4.16}$$

by (3.8b). Then, for every constant $\alpha > 0$,

$$\begin{aligned}
\left|\int_0^\pi p_a w\mathcal{K}w\right| &\leqq \kappa \int_0^\pi \frac{|w\mathcal{K}w|}{\sin\frac{s}{2}}\,\mathrm{d}s \\
&\leqq \frac{\kappa}{2}\int_0^\pi \left\{ \alpha w^2 + \frac{1}{\alpha}\left(\frac{\mathcal{K}w}{\sin\frac{s}{2}}\right)^2 \right\} \mathrm{d}s \\
&= \frac{\kappa}{2}\int_0^\pi \left\{ \alpha w^2 + \frac{1}{\alpha}\left(\frac{W}{\sin\frac{s}{2}}\right)^2 \right\} \mathrm{d}s,
\end{aligned} \tag{4.17}$$

by Lemma 4.1.
(ii) Define

$$\begin{aligned}
r_a(s) &:= \frac{1}{2}\cot\frac{s}{2} - q_a(s) && (4.18\text{a}) \\
&= \left\{ \frac{1}{2}\cot\frac{s}{2} - \frac{\sin\varphi_a(s)}{\int_0^s \sin\varphi_a} \right\} - \frac{\sin\varphi_b(s) - \sin\varphi_a(s)}{\int_0^s \sin\varphi_a} \\
&= a'(s) - g_a(s), && (4.18\text{b})
\end{aligned}$$

by (3.5) and (3.11), since $\varphi_b := T\varphi_a$. Then

$$\int_0^\pi q_a w W = \int_0^\pi \left\{ \frac{1}{2}\cot\frac{s}{2} - r_a(s) \right\} wW\,\mathrm{d}s. \tag{4.19}$$

Now, because $\cot(\pi/2) = 0$ and because the Schwarz inequality implies that

$$|W(s)| \leqq s^{1/2}\left\{ \int_0^s w^2 \right\}^{1/2} = o\left(s^{1/2}\right) \quad \text{as} \quad s \downarrow 0, \tag{4.20}$$

integration by parts gives

$$\int_0^\pi \frac{1}{2}\cot\frac{s}{2}.wW.\,\mathrm{d}s = \int_0^\pi \frac{1}{4\left(\sin\frac{s}{2}\right)^2}.\frac{W^2}{2}.\,\mathrm{d}s. \tag{4.21}$$

Use of (4.17), (4.19) and (4.21) in (4.15) now shows that (for every constant $\alpha > 0$)

$$\int_0^\pi w\mathfrak{L}_a w \geqq \int_0^\pi \left\{ \left(1 - \frac{\kappa\alpha}{2}\right) w^2 + \left(\frac{1}{8} - \frac{\kappa}{2\alpha}\right)\left(\frac{W}{\sin\frac{s}{2}}\right)^2 - r_a(s) w W \right\} \mathrm{d}s. \tag{4.22}$$

We choose $\alpha = 4\kappa$ in order to make the middle term vanish.
(iii) It remains to consider the last term in (4.22). Since $r_a(s) = O\left(s^{\beta-1}\right)$ as $s \downarrow 0$ and $r_a(\pi) = 0$, integration by parts and two applications of (4.20) give

$$\int_0^\pi r_a w W = -\int_0^\pi r_a' \frac{W^2}{2} \leqq \int_0^\pi |r_a'(s)| \frac{s\|w\|^2}{2}\,\mathrm{d}s. \tag{4.23}$$

We claim that

$$r_a'(s) < 0 \quad \text{on} \quad (0, \pi]. \tag{4.24}$$

For, by (4.18a),

$$r_a'(s) = \frac{\lambda(s)}{\left(\sin\frac{s}{2}\right)^2},$$

where, with the notation $\sigma(s) := \sin\frac{s}{2}$,

$$\lambda := \mathrm{e}^{2a}\sin\varphi_a \sin\varphi_b + \mathrm{e}^a \sigma \cos\varphi_b \left(-\varphi_b'\right) - \frac{1}{4}. \tag{4.25}$$

It follows that

$$\lambda(s) = -0.00842\, s^\beta + O\left(s^{2\beta}\right) \qquad \text{as} \quad s \downarrow 0,$$
$$\lambda(s) = -0.05696 + O\left((\pi - s)^2\right) \quad \text{as} \quad s \uparrow \pi,$$

and that λ has numerical values as in Table 3.1.
The inequalities (4.23) and (4.24) imply that

$$\int_0^\pi r_a w W \leqq -\frac{1}{2}\|w\|^2 \int_0^\pi r_a'(s)\, s\,\mathrm{d}s =: \mu\|w\|^2, \tag{4.26a}$$

where, by a final integration by parts and by (4.18b),

$$\mu = \frac{1}{2}\int_0^\pi r_a = \frac{1}{2}\left\{a(\pi) - \int_0^\pi g_a\right\} < 0.161. \tag{4.26b}$$

(iv) With $\alpha = 4\kappa$, the inequality (4.22) becomes

$$\int_0^\pi w\mathfrak{L}_a w \geqq \left(1 - 2\kappa^2 - \mu\right)\|w\|^2$$

and

$$1 - 2\kappa^2 - \mu > 0.43. \quad \square$$

Theorem 4.3. *The equation $\mathcal{L}_a w = f$, with $f \in L_2$, has a unique solution $w \in L_2$ and*

$$\|w\| \leqq \frac{1}{0.43} \|f\|. \tag{4.27}$$

Proof.
(i) Granted existence of a solution, we have the estimate (4.27) immediately:

$$0.43 \|w\|^2 \leqq \int_0^\pi w\mathcal{L}_a w = \int_0^\pi wf \leqq \|w\| \|f\|,$$

and this inequality also implies uniqueness.
(ii) To prove existence of a solution, we apply the Lax–Milgram lemma [3] (p. 207) to the bilinear form B defined by

$$B(v, w) := \int_0^\pi v\mathcal{L}_a w \quad \text{for all } v \text{ and } w \text{ in } L_2.$$

In order to bound B, we recall that

$$(\mathcal{L}_a w)(s) := w(s) - \frac{1}{3} \mathrm{e}^{a(s)} \cos\varphi_b(s) \frac{(\mathcal{K}w)(s)}{\sin\frac{s}{2}} + \mathrm{e}^{a(s)} \sin\varphi_b(s) \frac{\int_0^s w}{\sin\frac{s}{2}}.$$

Lemma 4.1 and (4.16) imply that

$$\|\mathcal{L}_a w\| \leqq \|w\| + 4\kappa \|w\| + 6\kappa \|w\|,$$

whence

$$|B(v, w)| \leqq (1 + 10\kappa)\|v\| \|w\| \tag{4.28}$$

for all v and w in L_2. This inequality and Lemma 4.2 show that B satisfies the hypotheses of the Lax–Milgram lemma. □

5. Estimates of the remainder operator R_a

In view of Theorem 4.3, the version (4.3) of the equation $u = Au$ may be written

$$h = \mathcal{L}_a^{-1} \left(g_a + R_a h\right) =: S_a h \quad \text{(say)}. \tag{5.1}$$

Abandoning the condition $u_a + h \in \mathcal{N}(X)$ for the moment, we seek a solution h that is merely in L_2 in the first instance. Therefore, we consider

$$R_a w := A\left(u_a + w\right) - Au_a - A'\left(u_a\right) w \quad \text{for all } w \in L_2, \tag{5.2}$$

and seek estimates of $R_a w$ and of $R_a v - R_a w$ that will make the operator S_a in (5.1) a contraction map of a small closed ball in L_2.

According to the definition (2.15) of the operator A, with comparison function $\psi_* = \varphi_a$, and hence with $v_* = u_a := \mathcal{N}\varphi_a$,

$$A\left(u_a + w\right)(s) = \frac{\sin\left\{\left(T\varphi_a\right)(s) + \frac{1}{3}\left(\mathcal{K}w\right)(s)\right\} \exp\left(-\int_0^s w\right)}{\int_0^s \sin\varphi_a}. \tag{5.3}$$

Lemma 5.1. *Let* $W(s) := \int_0^s w$. *If* $w \in L_2$, *then* W *and* $\mathcal{K}w$ *are in* $C^{0,1/2}[0,\pi]$ *and*

$$|W(s)| \leqq \|w\| s^{1/2}, \tag{5.4a}$$

$$\left|\left(\mathcal{K}w\right)(s)\right| \leqq \|w\| s^{1/2} \left(1 - \frac{s}{\pi}\right)^{1/2}, \tag{5.4b}$$

$$\int_0^s |\mathcal{K}w| \leqq \|w\| \left(\frac{s^3}{3} - \frac{s^4}{4\pi}\right)^{1/2}. \tag{5.4c}$$

Proof.
(i) The identity (4.7a) implies that $\|\left(\mathcal{K}w\right)'\| = \|w\|$. Hence the Schwarz inequality yields

$$|(\mathcal{K}w)(s) - (\mathcal{K}w)(\sigma)| = \left|\int_\sigma^s (\mathcal{K}w)'\right| \leqq \|w\|\, |s - \sigma|^{1/2} \tag{5.5}$$

for all s and σ in $[0,\pi]$, and similarly for W.
(ii) The bounds in (5.4a,b,c) also result from the Schwarz inequality. The first is a coarse version of (4.20). For the second, we have

$$|(\mathcal{K}w)(s)| \leqq \|K(s,\cdot)\|\, \|w\|, \qquad 0 \leqq s \leqq \pi,$$

where $K(s,\cdot) \in L_2$ for fixed s, with Fourier sine coefficients as in (2.2), so that

$$\begin{aligned}\|K(s,\cdot)\|^2 &= \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{\sin ks}{k}\right)^2 = \frac{1}{\pi}\left\{\frac{\pi^2}{6} - \sum_{k=1}^{\infty}\frac{\cos 2ks}{k^2}\right\} \\ &= \frac{1}{\pi} s\,(\pi - s)\,.\end{aligned}$$

For the inequality (5.4c), let

$$Q(s,t) := \int_0^s K(\sigma,t)\, \mathrm{d}\sigma = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(1-\cos ks)\sin kt}{k^2}. \tag{5.6}$$

Then

$$\begin{aligned}\|Q(s,\cdot)\|^2 &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(1-\cos ks)^2}{k^4}, \quad 0 \leqq s \leqq \pi,\\ &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^4}\left(\frac{3}{2}-2\cos ks+\frac{1}{2}\cos 2ks\right)\\ &= \frac{s^3}{3}-\frac{s^4}{4\pi},\end{aligned}$$

and it follows from

$$|(\mathcal{K}w)(\sigma)| \leqq \int_0^{\pi} K(\sigma,t)\,|w(t)|\,\mathrm{d}t,$$

that

$$\int_0^s |\mathcal{K}w| \leqq \int_0^{\pi} Q(s,t)|w(t)|\,\mathrm{d}t \leqq \|Q(s,\cdot)\|\,\|w\|. \quad \square$$

Lemma 5.2.
(i) *With the notation*

$$\varphi_b := T\varphi_a, \quad l_w := \frac{1}{3}\mathcal{K}w, \quad W(s) := \int_0^s w, \tag{5.7}$$

we have

$$(R_a w)(s) = \frac{1}{\int_0^s \sin\varphi_a}\int_0^1 (1-\theta)\{P(\theta,s)+Q(\theta,s)\}\,\mathrm{d}\theta, \tag{5.8a}$$

where

$$P(\theta,\cdot) := e^{-\theta W}\left\{W^2 - l_w^2\right\}\sin(\varphi_b+\theta l_w), \tag{5.8b}$$

$$Q(\theta,\cdot) := e^{-\theta W}\{-2Wl_w\}\cos(\varphi_b+\theta l_w). \tag{5.8c}$$

(ii) *With the further notation*

$$z := v - w, \quad l_z := \frac{1}{3}\mathcal{K}z, \quad Z(s) := \int_0^s z, \tag{5.9}$$

we have

$$(R_a v)(s) - (R_a w)(s) = \frac{1}{\int_0^s \sin\varphi_a}\int_0^1 \{S(\theta,s)l_z(s) - T(\theta,s)Z(s)\}\,\mathrm{d}\theta, \tag{5.10a}$$

where

$$S(\theta,\cdot) := e^{-W-\theta Z}\cos(\varphi_b+l_w+\theta l_z) - \cos\varphi_b, \tag{5.10b}$$

$$T(\theta,\cdot) := e^{-W-\theta Z}\sin(\varphi_b+l_w+\theta l_z) - \sin\varphi_b. \tag{5.10c}$$

Proof. It suffices to consider the numerator in (5.3). Let

$$B(u_a + w) := \sin(\varphi_b + l_w)\mathrm{e}^{-W},$$

and

$$R_B w := B(u_a + w) - Bu_a - B'(u_a)w.$$

(i) To prove (5.8) we define $F : [0, 1] \to C[0, \pi]$ by

$$F(\theta) := B(u_a + \theta w) = \sin(\varphi_b + \theta l_w)\mathrm{e}^{-\theta W}, \quad 0 \leqq \theta \leqq 1.$$

Then

$$\begin{aligned} R_B w &= F(1) - F(0) - F'(0).1 \\ &= \int_0^1 (1-\theta)F''(\theta)\,\mathrm{d}\theta, \end{aligned}$$

and (5.8) follows from two differentiations of $F(\theta)$.
(ii) To prove (5.10) we define $G : [0, 1] \to C[0, \pi]$ by

$$G(\theta) := B(u_a + w + \theta z) = \sin(\varphi_b + l_w + \theta l_z)\mathrm{e}^{-W-\theta Z}, \quad 0 \leqq \theta \leqq 1.$$

Then,

$$\begin{aligned} R_B v - R_B w &= B(u_a + v) - Bu_a - B'(u_a)v - \left\{B(u_a + w) - Bu_a - B'(u_a)w\right\} \\ &= \int_0^1 G'(\theta)\,\mathrm{d}\theta - B'(u_a)z, \end{aligned}$$

and $B'(u_a)z$ is $F'(0)$ with w replaced by z. Differentiation of $G(\theta)$ now yields (5.10). □

Now we begin estimates for v and w in a small closed ball

$$\overline{\mathcal{B}_\rho} := \left\{w \in L_2 \middle| \, \|w\| \leqq \rho\right\}, \quad \rho > 0;$$

the radius ρ will be chosen later.

Theorem 5.3. *If* $\|w\| \leqq \rho$*, then*

$$\|R_a w\| \leqq \exp(\rho\pi^{1/2})\,\rho^2\,(1.773 + 0.654\rho). \tag{5.11}$$

Proof.
(i) We begin with pointwise estimates; that is, with repeated application of Lemma 5.1 to the functions in (5.8). Let

$$\eta(s) := \left(1 - \frac{s}{\pi}\right)^{1/2}, \quad \zeta(s) := s^{1/2}\left(1 - \frac{s}{\pi}\right)^{1/2}, \quad \varphi_1(s) := \frac{\pi - s}{6}. \tag{5.12}$$

Then (for $\|w\| \leqq \rho$ and $0 \leqq s \leqq \pi$)

$$|W(s)| \leqq \rho s^{1/2} \leqq \rho \pi^{1/2},$$
$$-\frac{1}{9}\rho^2 s\left(1-\frac{s}{\pi}\right) \leqq W(s)^2 - l_w(s)^2 \leqq \rho^2 s,$$
$$-\frac{1}{3}\rho\zeta(s) \leqq \sin\{\varphi_b(s)+\theta l_w(s)\} \leqq \sin\varphi_1(s) + \frac{1}{3}\rho\zeta(s),$$

because $0 \leqq \varphi_b \leqq \varphi_1$ by (3.7b). Accordingly,

$$|P(\theta,s)| \leqq \exp\left(\rho\pi^{1/2}\right)\,\rho^2 s\left\{\sin\varphi_1(s) + \frac{1}{3}\rho\zeta(s)\right\}.$$

Similarly,

$$|Q(\theta,s)| \leqq \exp\left(\rho\pi^{1/2}\right)\,\frac{2}{3}\rho^2 s\eta(s).$$

These bounds are independent of θ, so that we may use $\int_0^1 (1-\theta)\,\mathrm{d}\theta = 1/2$ to obtain

$$|(R_a w)(s)| \leqq \frac{1}{2}\exp\left(\rho\pi^{1/2}\right)\,\rho^2\frac{s}{\int_0^s \sin\varphi_a}\left\{\sin\varphi_1(s) + \frac{2}{3}\eta(s) + \frac{1}{3}\rho\zeta(s)\right\}. \tag{5.13}$$

(ii) Writing $a(s) := 0.56a_1(s)$ and

$$\frac{s}{\int_0^s \sin\varphi_a} = \frac{s\mathrm{e}^{a(s)}}{\sin\frac{s}{2}} \leqq \frac{s\left(1+0.214s^{\beta}\right)}{\sin\frac{s}{2}} =: f_1(s), \tag{5.14a}$$

we have equality only at $s=0$. Let

$$f_1(s)\sin\varphi_1(s) =: f_2(s), \quad f_1(s)\eta(s) =: f_3(s), \quad f_1(s)\zeta(s) =: f_4(s); \tag{5.14b}$$

then

$$\|R_a w\| < \exp\left(\rho\pi^{1/2}\right)\rho^2\left\{\frac{1}{2}\|f_2\| + \frac{1}{3}\|f_3\| + \frac{1}{6}\rho\|f_4\|\right\}.$$

These $\|f_j\|$ have been evaluated numerically (and rounded upwards) to yield (5.11). Of course, our cavalier treatment of the dependence on θ, and our final use of the triangle inequality, have led to an overestimate. □

Theorem 5.4. *If $\|v\| \leqq \rho$ and $\|w\| \leqq \rho$, then*

$$\|R_a v - R_a w\| \leqq \rho\|v-w\|\left\{2.423c_\rho + 1.227 + \rho\left(1.308c_\rho + 0.069\right)\right\}, \tag{5.15a}$$

where

$$c_\rho := \frac{\exp\left(\rho\pi^{1/2}\right)-1}{\rho\pi^{1/2}}. \tag{5.15b}$$

Proof.
(i) The significance of c_ρ is that, because of the convexity of the function $x \mapsto \mathrm{e}^x$,

$$|\mathrm{e}^x - 1| \leqq c_\rho |x| \qquad \text{if} \quad |x| \leqq \rho\pi^{1/2}. \tag{5.16}$$

(ii) Let $V(s) := \int_0^s v$; we continue to use the notation in Lemma 5.2 and in (5.12). First,

$$\begin{aligned} |W(s) + \theta Z(s)| &= |\theta V(s) + (1-\theta)\, W(s)| \leqq \theta\rho s^{1/2} + (1-\theta)\rho s^{1/2} \\ &= \rho s^{1/2}; \end{aligned} \tag{5.17}$$

similarly,

$$|l_w(s) + \theta l_z(s)| \leqq \frac{1}{3}\rho\zeta(s). \tag{5.18}$$

Now, (5.10b) may be written

$$S(\theta,\cdot) = \left\{\mathrm{e}^{-W-\theta Z} - 1\right\} \cos(\varphi_b + l_w + \theta l_z) + \{\cos(\varphi_b + l_w + \theta l_z) - \cos\varphi_b\},$$

whence, in view of (5.16) to (5.18),

$$\begin{aligned} |S(\theta, s)| &\leqq c_\rho \rho s^{1/2} + \sin\left\{\varphi_1(s) + \frac{1}{3}\rho\zeta(s)\right\} \frac{1}{3}\rho\zeta(s) \\ &\leqq \rho s^{1/2}\left\{c_\rho + \frac{1}{3}\eta(s)\left[\sin\varphi_1(s) + \frac{1}{3}\rho\zeta(s)\right]\right\}; \end{aligned}$$

also,

$$|l_z(s)| \leqq \frac{1}{3}\|z\| s^{1/2}\eta(s).$$

Next, (5.10c) may be written

$$T(\theta,\cdot) = \left\{\mathrm{e}^{-W-\theta Z} - 1\right\} \sin(\varphi_b + l_w + \theta l_z) + \{\sin(\varphi_b + l_w + \theta l_z) - \sin\varphi_b\},$$

whence, by (5.16) to (5.18) again,

$$|T(\theta, s)| \leqq c_\rho \rho s^{1/2}\left[\sin\varphi_1(s) + \frac{1}{3}\rho\zeta(s)\right] + \frac{1}{3}\rho\zeta(s);$$

also,

$$|Z(s)| \leqq \|z\| s^{1/2}.$$

The last four inequalities are independent of θ; it follows from (5.10a) that

$$\begin{aligned} &|\,(R_a v)(s) - (R_a w)(s)| \\ &\quad \leqq \rho\|z\| \frac{s}{\int_0^s \sin\varphi_a} \left\{\frac{1}{3}c_\rho\eta(s) + \frac{1}{9}\eta(s)^2 \sin\varphi_1(s) + \frac{1}{27}\rho\eta(s)^2\zeta(s)\right. \\ &\qquad\qquad \left. + c_\rho \sin\varphi_1(s) + \frac{1}{3}c_\rho\rho\zeta(s) + \frac{1}{3}\eta(s)\right\}. \end{aligned} \tag{5.19}$$

(iii) We add to the list (5.14) the definitions

$$f_1(s)\eta(s)^2 \sin\varphi_1(s) =: f_5(s), \tag{5.20a}$$
$$f_1(s)\eta(s)^2\zeta(s) =: f_6(s). \tag{5.20b}$$

Then

$$\|R_a v - R_a w\| \leqq \rho\|z\| \left\{c_\rho\|f_2\| + \frac{1}{3}\left(c_\rho+1\right)\|f_3\| + \frac{1}{3}c_\rho\rho\|f_4\| + \frac{1}{9}\|f_5\| + \frac{1}{27}\rho\|f_6\|\right\}.$$

Numerical evaluation of these $\|f_j\|$ yields (5.15). □

6. Existence and positivity of a solution

Notation. As before, $L_2 := L_2(0,\pi)$, $\|\cdot\| = \|\cdot\,|\,L_2\|$, $u_a := \mathcal{N}\varphi_a$ and $\varphi_b := T\varphi_a$, where φ_a is the approximation in Section 3.

Theorem 6.1.
(a) *The equation* $u = Au$ *has exactly one solution* $u = u_a + h$ *such that* $h \in L_2$ *and* $\|h\| \leqq 0.054$.
(b) *It follows that there is a solution* $\varphi = \varphi_b + l_h$ *of* $\varphi = T\varphi$ *such that*

$$l_h := \frac{1}{3}\mathcal{K}h \in C^{0,1/2}[0,\pi] \tag{6.1}$$

and

$$|l_h(s)| \leqq 0.018\,\zeta(s), \quad \textit{where} \quad \zeta(s) := s^{1/2}\left(1-\frac{s}{\pi}\right)^{1/2}. \tag{6.2}$$

(c) *This solution* $\varphi_b + l_h$ *is the only solution* $\tilde{\varphi}$ *of* $\tilde{\varphi} = T\tilde{\varphi}$ *such that* $\|\mathcal{N}\tilde{\varphi} - \mathcal{N}\varphi_a\| \leqq 0.054$.

Proof.
(i) Recalling the definition (3.11) of g_a and the version (4.3) of the equation $u = Au$, together with the invertibility of $\mathcal{L}_a$ (Theorem 4.3), we consider

$$S_a w := \mathcal{L}_a^{-1}\left(g_a + R_a w\right) \tag{6.3}$$

$$\text{for all} \quad w \in \overline{\mathcal{B}_\rho} := \left\{ v \in L_2 \middle| \, \|v\| \leqq \rho \right\}. \tag{6.4}$$

In step (ii) we shall show that the choice $\rho = 0.054$ makes S_a a contraction map of $\overline{\mathcal{B}_\rho}$. Hence, S_a has exactly one fixed point h in $\overline{\mathcal{B}_\rho}$ (that is, $h = S_a h$) for $\rho = 0.054$. It follows from (4.3) and (5.1) that $u_a + h = A\,(u_a + h)$.
(ii) With $\rho = 0.054$, we have (to the number of figures shown)

$$\exp\left(\rho\pi^{1/2}\right) = 1.10044, \quad c_\rho = 1.04942,$$

and, for $w \in \overline{\mathcal{B}_\rho}$,

$$\|R_a w\| < 0.00581 \quad \text{by Theorem 5.3,}$$
$$\|S_a w\| < \frac{1}{0.43}\left(\frac{1}{60} + 0.00581\right) < 0.053$$

by Theorem 4.3 and (3.12). Thus, S_a maps $\overline{\mathcal{B}_\rho}$ into itself.

Next, for v and w in $\overline{\mathcal{B}_\rho}$ with $\rho = 0.054$,

$$\|R_a v - R_a w\| \leqq 0.208\|v - w\| \quad \text{by Theorem 5.4,}$$
$$\|S_a v - S_a w\| = \|\mathcal{L}_a^{-1}\left(R_a v - R_a w\right)\| \leqq 0.49\|v - w\|$$

by Theorem 4.3 once more. Thus, S_a is a contraction of $\overline{\mathcal{B}_\rho}$.
(iii) Now we show that $\varphi_b + l_h \in X$.

(a) Equations (3.4) and (3.6) show that $\varphi_b \in C^{0,\beta}[0,\pi]$, while $l_h \in C^{0,1/2}[0,\pi]$ by Lemma 5.1. Therefore, $\varphi \in C^{0,1/2}[0,\pi]$.

(b) Equations (3.4) and (3.6), in which $b_1(\pi) = 0$, show that $\varphi_b(0) = \pi/6$ and $\varphi_b(\pi) = 0$. The inequality (6.2), which follows from Lemma 5.1, shows that $l_h(0) = 0$ and $l_h(\pi) = 0$.

(c) The very coarse estimates

$$0 < \varphi_b(s) < \frac{\pi}{6} \quad \text{in} \quad (0,\pi), \quad |l_h(s)| \leqq 0.018\left(\frac{\pi}{4}\right)^{1/2}$$

are sufficient to show that $-\pi/12 \leqq \varphi(s) \leqq \pi/3$.

(d) From the result $b_1''(s) > 0$ in $(0,\pi)$ and the value

$$-\varphi_b'(\pi) = \frac{1}{6} - \frac{0.56}{3}\left(1 - 2^{\beta-1}\right) =: k_1 = 0.14280\ldots,$$

it can be shown without difficulty that

$$\int_0^s \sin\varphi_b \geqq \frac{3k_1}{\pi} s\left(\pi - \frac{s}{2}\right) \geqq \frac{3}{2} k_1 s.$$

Also, by Lemma 5.1,

$$\int_0^s |l_h| \leqq 0.018\left(\frac{s^3}{3} - \frac{s^4}{4\pi}\right)^{1/2} \leqq 0.006\,\pi^{1/2} s,$$

whence

$$\int_0^s \sin\varphi \geqq \frac{1}{5}s.$$

(iv) It is to be expected from (2.16) and the remark following it that the fixed point u of A can be written

$$u = \mathcal{N}\varphi, \quad \text{where} \quad u = u_a + h, \quad \varphi = \varphi_b + l_h \in X, \tag{6.5}$$

but it seems worth while to prove this *ab initio*.

By the definition (2.15) of A, with comparison function $\psi_* = \varphi_a$ and with $v_* = u_a$,

$$u(s) = Au(s) := \frac{\sin\left(\frac{1}{3}\mathcal{K}u\right)(s)\exp\int_0^s (u_a - u)}{\int_0^s \sin\varphi_a}, \tag{6.6}$$

where

$$\frac{1}{3}\mathcal{K}u = \frac{1}{3}\mathcal{K}\left(u_a + h\right) = \varphi_b + l_h. \tag{6.7}$$

Hence,

$$\begin{aligned}\sin\left\{\varphi_b(s) + l_h(s)\right\} &= u(s)\left(\int_0^s \sin\varphi_a\right)\exp\int_0^s (u - u_a) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left\{\left(\int_0^s \sin\varphi_a\right)\exp\int_0^s (u - u_a)\right\},\end{aligned} \tag{6.8}$$

so that

$$\int_0^s \sin\left(\varphi_b + l_h\right) = \left(\int_0^s \sin\varphi_a\right)\exp\int_0^s (u - u_a)\,. \tag{6.9}$$

By (6.8) and (6.9),

$$\mathcal{N}\left(\varphi_b + l_h\right) = u, \tag{6.10a}$$

as desired, and then

$$T\left(\varphi_b + l_h\right) := \frac{1}{3}\mathcal{K}\circ\mathcal{N}\left(\varphi_b + l_h\right) = \frac{1}{3}\mathcal{K}u = \varphi_b + l_h \tag{6.10b}$$

by (6.7) once more. Thus, $T\varphi = \varphi$. We proved in (iii) that $\varphi_b + l_h \in X$ and mentioned there that (6.1) and (6.2) are implied by Lemma 5.1.

(v) The uniqueness statement for φ in part (c) of the theorem is merely a repetition of that for h in part (a). □

As far as we know at present, the solution φ in Theorem 6.1 may be negative in a small interval near π. In fact, we know only that

$$\varphi(s) \geqq \varphi_b(s) - 0.018\,\zeta(s),$$

and a calculation shows that

$$\sin\varphi_b < 0.018\,\zeta(s) \quad \text{in} \quad (\pi-\delta,\pi) \quad \text{for} \quad \delta = 0.0158\ldots. \tag{6.11}$$

(The reason that $\sin\varphi_b$ rather than φ_b is used in (6.11) will appear in the proof of Lemma 6.2.) However, this is a weakness of Theorem 6.1 that we can repair, as follows.

Lemma 6.2. *Numerical evaluation of explicit formulae yields functions h_1 and h_2* (*shown in* Table 6.1) *such that*

$$h_1(s) < h(s) < h_2(s), \quad 0 < s < \pi, \tag{6.12}$$

and

$$\textit{for} \quad s \downarrow 0, \quad u_a(s) = s^{-1} - 0.17081\,s^{\beta-1} + O\left(s^{\beta}\right), \tag{6.13a}$$

$$h_1(s), h_2(s) = \mp 0.05678\,s^{-1/2} + O(1), \quad \textit{respectively,} \tag{6.13b}$$

and

$$\textit{for} \quad r := \pi - s \downarrow 0, \quad u_a(s) = 0.20181\,r + O\left(r^3\right), \tag{6.14a}$$

$$h_1(s), h_2(s) = \mp 0.02572\,r^{1/2} + O(r), \quad \textit{respectively.} \tag{6.14b}$$

Proof. Let

$$\xi(s) := \left(\frac{s^3}{3} - \frac{s^4}{4\pi}\right)^{1/2}, \tag{6.15}$$

so that

$$\int_0^s |l_h| \leqq 0.018\,\xi(s), \tag{6.16}$$

by Lemma 5.1. Since

$$u(s) := (\mathcal{N}\varphi)(s) = \frac{\sin\{\varphi_b(s) + l_h(s)\}}{\int_0^s \sin(\varphi_b + l_h)},$$

we have

$$u(s) \geqq u_1(s) := \begin{cases} \dfrac{\sin\varphi_b(s) - 0.018\,\zeta(s)}{\int_0^s \sin\varphi_b + 0.018\,\xi(s)} & \text{in } (0, \pi-\delta], \\ \dfrac{\sin\varphi_b(s) - 0.018\,\zeta(s)}{\int_0^s \sin\varphi_b - 0.018\,\xi(s)} & \text{in } [\pi-\delta, \pi], \end{cases}$$

$$u(s) \leqq u_2(s) := \frac{\sin\varphi_b(s) + 0.018\,\zeta(s)}{\int_0^s \sin\varphi_b - 0.018\,\xi(s)} \quad \text{in} \quad (0,\pi].$$

Defining $h_j := u_j - u_a$ for $j = 1, 2$, where

$$u_a(s) := (\mathcal{N}\varphi_a)(s) = \frac{1}{2}\cot\frac{s}{2} - 0.56\, a_1'(s),$$

we obtain the results of the lemma. The calculations have been checked by means of

(a) five-term expansions of $h_1(s)$ and $h_2(s)$ for $s \downarrow 0$, with error $O\left(s^{\beta}\right)$;
(b) four-term expansions of $h_1(s)$ and $h_2(s)$ for $r := \pi - s \downarrow 0$, with error $O\left(r^3\right)$. □

Remark 6.3. Define, for $0 < s \leqq \pi$,

$$h_3(s) := -0.05678\sqrt{2}\, a_2'(s) - 0.02572\sqrt{2}\,\left(a_3'\right)(\pi - s) - 0.01050\, E(s), \quad (6.17)$$

$$h_4(s) := 0.05678\sqrt{2}\, a_2'(s) + 0.02572\sqrt{2}\,\left(a_3'\right)(\pi - s) + 0.01850\left(1 - \frac{s}{\pi}\right) - 0.01590\sin s, \quad (6.18)$$

where a_2', a_3' and Clausen's integral E are defined in Appendix A. Then, according to tabulation of these functions (Table 6.1), $h_3 \leqq h_1$ and $h_4 \geqq h_2$.

Explanation. The functions h_3 and h_4 have the advantage, displayed in Appendix A, of explicit formulae for $\mathcal{K}h_3$ and $\mathcal{K}h_4$. Note that h_3 and h_4 have the same leading terms as h_1 and h_2, respectively, for $s \downarrow 0$ and for $s \uparrow \pi$.

Theorem 6.4. *The solution* $\varphi = \varphi_b + l_h$ *in* Theorem 6.1 *is (strictly) positive on* $[0, \pi)$. *In particular, for* $r := \pi - s \downarrow 0$,

$$\varphi_b(s) = 0.14280\, r + O\left(r^3\right),$$

$$l_h(s) \geqq l_{h,3}(s) = -0.02284\, r + 0.00571\, r^{3/2} + O\left(r^{5/2}\right),$$

where $l_{h,j} := \frac{1}{3}\mathcal{K}h_j$ *for* $j = 3, 4$. *Graphs of* $-l_{h,3}$ *and* $l_{h,4}$ *are compared with that of* $0.018\,\zeta$ *in* Figure 6.1.

Table 6.1.

s	$h_3(s)$	$h_1(s)$	$h_2(s)$	$h_4(s)$
0°	$-\infty$	$-\infty$	∞	∞
3	−0.2477	−0.2400	0.2594	0.2629
6	−0.1758	−0.1709	0.1830	0.1884
9	−0.1443	−0.1412	0.1482	0.1547
12	−0.1258	−0.1238	0.1269	0.1342
15	−0.1134	−0.1121	0.1121	0.1198
30	−0.0836	−0.0835	0.0742	0.0820
60	−0.0632	−0.0612	0.0466	0.0511
90	−0.0520	−0.0477	0.0343	0.0357
120	−0.0411	−0.0358	0.0264	0.0264
150	−0.0272	−0.0230	0.0186	0.0186
180°	0	0	0	0

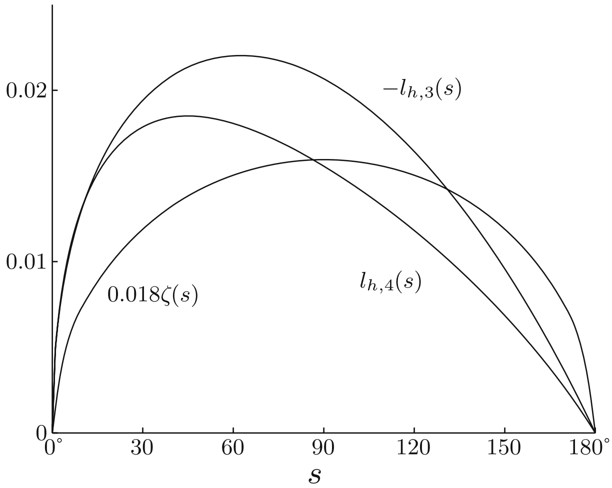

Fig. 6.1.

Proof. This is merely a matter of evaluating $\frac{1}{3}\mathcal{K}h_3$ and $\frac{1}{3}\mathcal{K}h_4$ by means of the formulae in Appendix A. The results have been checked by means of

(a) two-term expansions of $l_{h,3}(s)$ and $l_{h,4}(s)$ for $s \downarrow 0$, with error $O\left(s^{3/2}\right)$;
(b) two-term expansions of $l_{h,3}(s)$ and $l_{h,4}(s)$ for $r := \pi - s \downarrow 0$, with error $O\left(r^{5/2}\right)$. □

Appendx A. Some formulae for $\mathcal{K}f$

Here we give seven applications of Lemma A.1 in [2]. In all cases ($j = 1, 2, \ldots, 7$)

$$\zeta \in \mathcal{D}_1 := \left\{\zeta \in \mathbb{C} \,\middle|\, |\zeta| \leqq 1,\ \zeta \neq 1\right\},$$
$$\arg(1-\zeta) \in (-\pi/2, \pi/2), \quad w_j\left(\mathrm{e}^{\mathrm{i}s}\right) = a_j(s) + \mathrm{i}b_j(s)$$

and $\mathcal{K}\left(a_j'\right) = -b_j$ in $(0, \pi]$.

(i) $$w_1(\zeta_0) = -\int_1^{\zeta_0} \frac{1}{\zeta}\left\{(1-\zeta)^{\beta-1} - 1\right\} \mathrm{d}\zeta,$$
$$a_1'(s) = \left(2\sin\frac{s}{2}\right)^{\beta-1} \sin\frac{(1-\beta)(\pi-s)}{2},$$
$$-b_1(s_0) = \int_0^{s_0} \left(2\sin\frac{s}{2}\right)^{\beta-1} \cos\frac{(1-\beta)(\pi-s)}{2}\, \mathrm{d}s - s_0.$$

$$\text{(ii)}\quad w_2(\zeta_0) = -\int_1^{\zeta_0} \frac{1}{\zeta}\left\{(1-\zeta)^{-1/2} - 1\right\} \mathrm{d}\zeta,$$

$$a_2'(s) = \left(2\sin\frac{s}{2}\right)^{-1/2} \sin\frac{\pi - s}{4},$$

$$-b_2(s_0) = \int_0^{s_0} \left(2\sin\frac{s}{2}\right)^{-1/2} \cos\frac{\pi - s}{4}\,\mathrm{d}s - s_0$$

$$= \pi - 2\sin^{-1}\left(\sqrt{2}\sin\frac{\pi - s_o}{4}\right) - s_0.$$

$$\text{(iii)}\quad w_3(\zeta_0) = \int_1^{\zeta_0} \frac{1}{\zeta}\left\{(1-\zeta)^{1/2} - 1\right\} \mathrm{d}\zeta,$$

$$a_3'(s) = \left(2\sin\frac{s}{2}\right)^{1/2} \sin\frac{\pi - s}{4},$$

$$b_3(s_0) = \int_0^{s_0} \left(2\sin\frac{s}{2}\right)^{1/2} \cos\frac{\pi - s}{4}\,\mathrm{d}s - s_0$$

$$= -4\sin\frac{\pi - s_0}{4}\left\{\frac{1}{2} - \left(\sin\frac{\pi - s_0}{4}\right)^2\right\}^{1/2} - b_2(s_0).$$

$$\text{(iv)}\quad w_4(\zeta_0) = -\int_1^{\zeta_0} \frac{1}{\zeta}\log\frac{1}{1-\zeta}\,\mathrm{d}\zeta,$$

$$a_4'(s) = \frac{\pi - s}{2},$$

$$-b_4(s_0) = \int_0^{s_0} \log\frac{1}{2\sin\frac{s}{2}}\,\mathrm{d}s = \sum_{k=1}^{\infty}\frac{\sin ks_0}{k^2} =: E(s_0),$$

which is Clausen's integral.

$$\text{(v)}\quad w_5(\zeta) = \sum_{k=1}^{\infty}\frac{1-\zeta^k}{k^3}$$

$$a_5'(s) = \sum_{k=1}^{\infty}\frac{\sin ks}{k^2} = E(s),$$

$$-b_5(s) = \sum_{k=1}^{\infty}\frac{\sin ks}{k^3} = \frac{\pi^2 s}{6} - \frac{\pi s^2}{4} + \frac{s^3}{12}.$$

$$\text{(vi)}\quad w_6(\zeta) = \frac{1-\zeta^n}{n}, \qquad n \in \mathbb{N},$$

$$a_6'(s) = \sin ns,$$

$$-b_6(s) = \frac{\sin ns}{n}.$$

(vii) $$w_7(\zeta_0) = \int_1^{\zeta_0} \frac{1}{\zeta} \left\{ (1-\zeta)^{2\beta-1} - 1 \right\} \mathrm{d}\zeta,$$

$$a_7'(s) = \left(2 \sin \frac{s}{2}\right)^{2\beta-1} \sin \frac{(2\beta-1)(\pi-s)}{2},$$

$$b_7(s_0) = \int_0^{s_0} \left(2 \sin \frac{s}{2}\right)^{2\beta-1} \cos \frac{(2\beta-1)(\pi-s)}{2} \mathrm{d}s - s_0.$$

Appendix B. A second approximation to the solution

The solution φ in this paper is characterized by $\mathcal{N}\varphi = \mathcal{N}\varphi_a + h$. The existence proof in Section 6 ensures convergence to h, in the space $L_2(0, \pi)$, of successive approximations

$$h_1 = \mathcal{L}_a^{-1} g_a, \quad h_{n+1} = S_a h_n \quad (n \in \mathbb{N}), \tag{B.1}$$

where g_a, $\mathcal{L}_a$ and S_a are defined by (3.11), (4.5) and (5.1) respectively.

We construct an approximation h_* to h_1 as follows. Let

$$h_*(s) := k_1 a_1'(s) + k_2 a_7'(s) + k_3 \sin s, \tag{B.2a}$$

where a_1' and a_7' are defined in Appendix A. A calculation, which includes use of (i), (vii) and (vi) of Appendix A, shows that

$$(\mathcal{L}_a h_*)(s) = g_a(s) \quad \text{at} \quad s = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \tag{B.2b}$$

if

$$k_1 = 0.02141, \quad k_2 = -0.05334, \quad k_3 = 0.01475 \tag{B.2c}$$

to five decimal places. In addition, $(\mathcal{L}_a h_*)(\pi) = g_a(\pi) = 0$.

Accordingly, we characterize a second approximation φ_* by

$$(\mathcal{N}\varphi_*)(s) = \left\{ \frac{1}{2} \cot \frac{s}{2} - 0.56\, a_1'(s) \right\} + h_*(s) = \frac{1}{2} \cot \frac{s}{2} - a_*'(s), \tag{B.3}$$

where

$$a_*(s) = (0.56 - k_1)\, a_1(s) - k_2 a_7(s) - k_3(1 - \cos s), \tag{B.4}$$

$$b_*(s) = (0.56 - k_1)\, b_1(s) - k_2 b_7(s) + k_3 \sin s, \tag{B.5}$$

and where $a_j(0) = 0$ for all j. Then, by Lemma A.3 of [2],

$$\sin \varphi_*(s) = \left\{ \frac{1}{2} \cos \frac{s}{2} - a_*'(s) \sin \frac{s}{2} \right\} \exp\{-a_*(s)\}, \tag{B.6}$$

$$(T\varphi_*)(s) = \frac{\pi - s}{6} + \frac{1}{3} b_*(s). \tag{B.7}$$

Table B.1 shows values of $\sin \varphi_*$ and of $\sin T\varphi_* - \sin \varphi_*$.

Table B.1.

s	$\sin\varphi_*(s)$	$\sin(T\varphi_*)(s) - \sin\varphi_*(s)$
0°	0.5	0
3	0.48270	−0.00005
6	0.46979	−0.00013
9	0.45809	−0.00020
12	0.44710	−0.00026
15	0.43660	−0.00032
30	0.38827	−0.00036
60	0.30195	−0.00003
90	0.22262	0.00010
120	0.14702	−0.00003
150	0.07320	−0.00010
180°	0	0

References

1. AMICK, C.J., FRAENKEL, L.E.: On the behavior near the crest of waves of extreme form. *Trans. Amer. Math. Soc.* **299**, 273–298 (1987)
2. AMICK, C.J., FRAENKEL, L.E., TOLAND, J.F.: On the Stokes conjecture for the wave of extreme form. *Acta Math.* **148**, 193–214 (1982)
3. FRIEDMAN, A.: *Foundations of Modern Analysis*. Holt, Rinehart and Winston, 1970
4. GRANT, M.A.: The singularity at the crest of a finite amplitude progressive Stokes wave. *J. Fluid Mech.* **59**, 257–262 (1973)
5. KEADY, G., NORBURY, J.: On the existence theory for irrotational water waves. *Math. Proc. Cambridge Philos. Soc.* **83**, 137–157 (1978)
6. MCLEOD, J.B.: The Stokes and Krasovskii conjectures for the wave of greatest height. *Math. Research Center Report No.* 2041, *University of Wisconsin, Madison* (1979). Also, *Stud. Appl. Math.* **98**, 311–333 (1997)
7. MCLEOD, J.B.: The asymptotic behavior near the crest of waves of extreme form. *Trans. Amer. Math. Soc.* **299**, 299–302 (1987)
8. NORMAN, A.C.: Expansions for the shape of maximum amplitude Stokes waves. *J. Fluid Mech.* **66**, 261–265 (1974)
9. PLOTNIKOV, P.I.: A proof of the Stokes conjecture in the theory of surface waves. Russian version: *Dinamika Sploshnoi Sredy*, **57**, 41–76 (1982). English translation: *Stud. Appl. Math.* **108**, 217–244 (2002)
10. PLOTNIKOV, P.I., TOLAND, J.F.: Convexity of Stokes waves of extreme form. *Arch. Ration. Mech. Anal.* **171**, 349–416 (2004)
11. STOKES, G.G.: Considerations relative to the greatest height of oscillatory irrotational waves which can be propagated without change of form. *Mathematical and Physical Papers*, Vol. I, 225–228, Cambridge, 1880
12. TOLAND, J.F.: On the existence of a wave of greatest height and Stokes's conjecture. *Proc. R. Soc. Lond.* **363**, 469–485 (1978)

Department of Mathematical Sciences
University of Bath
Bath, BA2 7AY
United Kingdom
e-mail: lef@maths.bath.ac.uk

(*Received June 8, 2005 / Accepted December 12, 2005*)
Published online May 31, 2006 –