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Existence and Stability of Supersonic Euler Flows Past Lipschitz Wedges

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Abstract

It is well known that, when the vertex angle of a straight wedge is less than the critical angle, there exists a shock-front emanating from the wedge vertex so that the constant states on both sides of the shock-front are supersonic. Since the shock-front at the vertex is usually strong, especially when the vertex angle of the wedge is large, then a global flow is physically required to be governed by the isentropic or adiabatic Euler equations. In this paper, we systematically study two-dimensional steady supersonic Euler (i.e. nonpotential) flows past Lipschitz wedges and establish the existence and stability of supersonic Euler flows when the total variation of the tangent angle functions along the wedge boundaries is suitably small. We develop a modified Glimm difference scheme and identify a Glimm-type functional, by naturally incorporating the Lipschitz wedge boundary and the strong shock-front and by tracing the interaction not only between the boundary and weak waves, but also between the strong shock-front and weak waves, to obtain the required BV estimates. These estimates are then employed to establish the convergence of both approximate solutions to a global entropy solution and corresponding approximate strong shock-fronts emanating from the vertex to the strong shock-front of the entropy solution. The regularity of strong shock-fronts emanating from the wedge vertex and the asymptotic stability of entropy solutions in the flow direction are also established.

1. Introduction

We are concerned with the existence and behavior of two-dimensional steady supersonic Euler flows past Lipschitz wedges with arbitrary vertex angles that are less than the critical angle so that there is a supersonic shock-front emanating from the wedge vertex. The two-dimensional steady supersonic Euler flows are generally governed by

$$
\begin{aligned}
(\rho u)_x + (\rho v)_y &= 0,\\
(\rho u^2 + p)_x + (\rho u v)_y &= 0,\\
(\rho u v)_x + (\rho v^2 + p)_y &= 0,\\
(u(E+p))_x + (v(E+p))_y &= 0,
\end{aligned}
\tag{1.1}
$$

where (u, v) is the velocity, ρ the density, p the scalar pressure, and

$$E = \frac{1}{2}\rho(u^2 + v^2) + \rho e(\rho, p)$$

is the total energy with e the internal energy (a given function of (ρ, p) defined through thermodynamic relationships). The other two thermodynamic variables are the temperature T and the entropy S. If ρ and S are chosen as the independent variables, then we have the constitutive relations:

$$(e, p, T) = (e(\rho, S), p(\rho, S), T(\rho, S)), \tag{1.2}$$

governed by

$$T dS = de - \frac{p}{\rho^2} d\rho. \tag{1.3}$$

For an ideal gas,

$$p = R\rho T, \qquad e = c_v T, \qquad \gamma = 1 + \frac{R}{c_v} > 1, \tag{1.4}$$

and

$$p = p(\rho, S) = \kappa \rho^\gamma e^{S/c_v}, \qquad e = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1} e^{S/c_v} = \frac{RT}{\gamma - 1}, \tag{1.5}$$

where R, κ, and c_v are all positive constants.

If the flow is isentropic, i.e. S is constant, then p is a function of ρ, $p = p(\rho)$, and the flow is governed by the following, simpler, isentropic Euler equations:

$$
\begin{aligned}
(\rho u)_x + (\rho v)_y &= 0,\\
(\rho u^2 + p)_x + (\rho u v)_y &= 0,\\
(\rho u v)_x + (\rho v^2 + p)_y &= 0.
\end{aligned}
\tag{1.6}
$$

For polytropic isentropic gases, by scaling, the pressure–density relationship can be expressed as

$$p(\rho) = \rho^\gamma/\gamma, \qquad \gamma > 1. \tag{1.7}$$

For the isothermal flow, $\gamma = 1$. The quantity

$$c = \sqrt{p_\rho(\rho, S)}$$

is defined as the sonic speed and, for polytropic gases, $c = \sqrt{\gamma p/\rho}$.

System (1.1), or (1.6), governing a supersonic flow (i.e. $u^2 + v^2 > c^2$) has all real eigenvalues and is hyperbolic, whilst system (1.1), or (1.6), governing a subsonic flow (i.e. $u^2 + v^2 < c^2$) has complex eigenvalues and is elliptic-hyperbolic mixed and composite.

The study of two-dimensional steady supersonic flows past wedges can date back to the 1940s (cf. COURANT & FRIEDRICHS [8]). Local solutions around the wedge vertex were first constructed by GU [12], LI [16], SCHAEFFER [20] (and references cited therein). Global potential solutions are constructed in [4–8, 25, 26] when the wedge has certain convexity or the wedge is a small perturbation of the straight wedge with fast decay in the flow direction, with a vertex angle less than the critical angle. In particular, in ZHANG [26], the existence of two-dimensional steady supersonic potential flows past piecewise smooth curved wedges, which are a small perturbation of the straight wedge, was established.

As is well known, the potential flow equation is an excellent model for flow containing only weak shocks, since it approximates to the isentropic Euler equations up to third-order in shock strength. For a flow containing shocks of large strength, the isentropic or adiabatic Euler equations are required to govern the physical flow. For the wedge problem, when the vertex angle is large, the flow contains a strong shock-front emanating from the wedge vertex and, for this case, the Euler equations should be used to describe the physical flow. Hence, it is important to study the two-dimensional steady supersonic flows governed by the Euler equations, rather than the potential flow equation, for the wedge problem with a large vertex angle. When a wedge is straight, and the wedge vertex angle is less than the critical angle, there exists a supersonic shock-front emanating from the wedge vertex such that the constant states on both sides of the shock are supersonic; the critical-angle condition is necessary, and sufficient, for the existence of the supersonic shock. These facts can be seen through the shock polar in Fig. 3, Section 2 (cf. COURANT & FRIEDRICHS [8]; see also CHANG & HSIAO [1] and CHEN [2]).

In this paper, we analyze the two-dimensional steady supersonic Euler flows past two-dimensional Lipschitz wedges, with vertex angles less than the critical angle (along which the total variation of the tangent angle function is suitably small); and we establish the existence and behavior of such global supersonic Euler flows, especially the nonlinear stability of the strong shock-front emanating from the wedge vertex under the BV perturbation.

For clarity, we will analyze the problem in the region below the lower side Γ of the wedge for the Euler flows for $U = (u, v, p, \rho)$ governed by system (1.1) and $U = (u, v, \rho)$ by (1.6); the case above the wedge can be handled in the same fashion. We then have

(i) there exists a Lipschitz function $g \in Lip(\mathbb{R}_+)$ with $g' \in BV(\mathbb{R}_+)$, $g'(0+) = 0$, and $g(0) = 0$ such that

$$\Omega := \{(x, y) \ : \ y < g(x),\ x \geqq 0\}, \quad \Gamma := \{(x, y) \ : \ y = g(x),\ x \geqq 0\},$$

and $\mathbf{n}(x\pm) = \frac{(-g'(x\pm), 1)}{\sqrt{(g'(x\pm))^2+1}}$ is the outer normal vector to Γ at the point $x\pm$ (see Fig. 1);

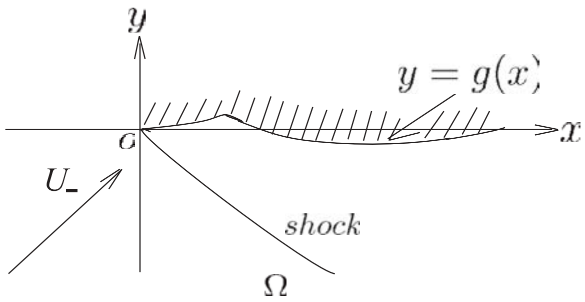

Fig. 1. Supersonic flow past a curved wedge.

(ii) the upstream flow is a constant state U_- satisfying

$$u_- > 0, \quad v_- > 0, \quad u_-^2 + v_-^2 > c_-^2 := \frac{\gamma p_-}{\rho_-},$$

and

$$0 < \arctan(v_-/u_-) < \omega_{crit},$$

so that there is a supersonic shock-front emanating from the wedge vertex, where ω_{crit} is the critical vertex angle (cf. Fig. 3).

With this setup, the wedge problem can be formulated into the following problem of initial-boundary value type for system (1.1) or (1.6):

Cauchy Condition:

$$U|_{x=0} = U_-; \tag{1.8}$$

Boundary Condition:

$$(u, v) \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma. \tag{1.9}$$

The main theorem of this paper is the following:

Main theorem (Existence and stability)**.** There exist $\varepsilon > 0$ and $C > 0$ such that, if

$$TV(g'(\cdot)) < \varepsilon, \tag{1.10}$$

then there exists a pair of functions

$$U \in BV_{loc}(\mathbb{R}_+^2), \qquad \sigma \in BV(\mathbb{R}_+),$$

with $\chi = \int_0^x \sigma(t)dt \in Lip(\mathbb{R}_+)$ such that

(i) U is a global entropy solution of problem (1.1), or (1.6), and (1.8)–(1.9) in Ω with

$$TV\{U(x,\cdot) : (-\infty, g(x)]\} \leqq C\, TV(g'(\cdot)) \quad \text{for every } x \in \mathbb{R}_+, \tag{1.11}$$

$$(u, v) \cdot \mathbf{n}|_{y=g(x)} = 0 \quad \text{in the trace sense}; \tag{1.12}$$

(ii) the curve $y = \chi(x)$ is a strong shock-front with $\chi(x) < g(x)$ for any $x > 0$ and

$$U|_{\{y<\chi(x)\}} = U_-, \qquad \sqrt{u^2 + v^2}|_{\{\chi(x)<y<g(x)\}} < u_-, \tag{1.13}$$

that is, $y = \chi(x)$ is the strong shock next to the constant state U_-;

(iii) there exist constants p_∞ and σ_∞ such that

$$\lim_{x\to\infty} \sup\{|p(x, y) - p_\infty| \,:\, \chi(x) < y < g(x)\} = 0,$$
$$\lim_{x\to\infty} |\sigma(x) - \sigma_\infty| = 0,$$

and

$$\lim_{x\to\infty} \sup\{|\arctan(v(x, y)/u(x, y)) - \omega_\infty| \,:\, \chi(x) < y < g(x)\} = 0,$$

where $\omega_\infty = \lim_{x\to\infty} \arctan(g'(x+))$.

This theorem indicates that the strong shock-front emanating from the wedge vertex is nonlinearly stable in structure, although there may be many weak waves and vortex sheets between the wedge boundary and the strong shock-front, under the BV perturbation of the wedge boundary so long as the wedge vertex angle is less than the critical angle. This asserts that any supersonic shock for the wedge problem is nonlinearly stable.

In order to establish this theorem, we develop a modified Glimm scheme and identify a Glimm-type functional by naturally incorporating the curved wedge boundary and the strong shock-front and by tracing the interactions not only between the wedge boundary and weak waves but also the interaction between the strong shock-front and weak waves. Some detailed interaction estimates are carefully made to ensure that the Glimm-type functional monotonically decreases in the flow direction. In particular, one of the essential estimates is on the strengths of the reflected 4-waves for (1.1), or 3-waves for (1.6), in the interaction between the strong shock-front and weak waves. The second essential estimate is the interaction estimate between the wedge boundary and the weak waves. Another essential estimate is made by tracing the approximate strong shocks, in order to establish the nonlinear stability and asymptotic behavior of the strong shock-front emanating from the wedge vertex under the wedge perturbation.

We remark that, in Lien & Liu [17], the nonlinear stability of a self-similar three-dimensional gas flow past an infinite cone (with small vertex angle) was established upon the perturbation of the obstacle. It would be interesting to combine the analysis in this paper with the argument in [17] to study the nonlinear stability of a self-similar three-dimensional gas flow past an infinite cone with arbitrary vertex angle. We also remark, in passing, that condition (1.10) can be relaxed by combining the analysis in this paper with the argument in [22, 23].

In this paper, we first focus on the isentropic Euler flows in Sections 2–5 and then extend to the adiabatic (full) Euler flows in Section 6.

In Section 2, we study the lateral Riemann problem and the classical Riemann problem, and analyze the properties of the Riemann solutions of the isentropic

Euler equations (1.6), which are essential for the interaction estimates among the nonlinear waves and the wedge boundary in Section 3, and for the existence and behavior of entropy solutions of the wedge problem in Sections 4–5. In Section 3, we make estimates on the wave interactions and reflections on the wedge and the strong shock. In Section 4, we develop a modified Glimm scheme to construct approximate solutions and establish necessary estimates for them in the approximate domains. In Section 5, we establish the convergence of approximate solutions to a global entropy solution and prove the nonlinear stability and asymptotic behavior of the strong shock-front emanating from the wedge vertex under the wedge perturbation. We extend the analysis in Section 6 to establish the existence and behavior of two-dimensional steady supersonic flows past the Lipschitz wedges for the adiabatic Euler equations.

2. Riemann problems and Riemann solutions

In this section, we study the lateral Riemann problem and the classical Riemann problem, and analyze the properties of the Riemann solutions to the isentropic Euler equations (1.6), which are essential not only for the interaction estimates among the nonlinear waves and the wedge boundary but also for the existence and behavior of solutions for the wedge problem in Sections 3–5.

2.1. Euler equations

The Euler system can be written in the following conservation form:

$$W(U)_x + H(U)_y = 0, \tag{2.1}$$

where

$$U = (u, v, \rho), \;\; W(U) = (\rho u, \rho u^2 + p, \rho uv), \;\; H(U) = (\rho v, \rho uv, \rho v^2 + p).$$

For a smooth solution $U(x, y)$, (2.1) is equivalent to

$$\nabla_U W(U) U_x + \nabla_U H(U) U_y = 0. \tag{2.2}$$

Then the eigenvalues of (2.1) are the roots of the third-order polynomial:

$$det(\lambda \nabla_U W(U) - \nabla_U H(U)) \tag{2.3}$$

and are thus the solutions of the cubic equation:

$$(v-\lambda)\left((v-\lambda u)^2 - c^2(1+\lambda^2)\right) = 0,$$

where $c = \sqrt{p'(\rho)}$ is the sonic speed. If the flow is supersonic (i.e., $u^2 + v^2 > c^2$), then we have three eigenvalues λ_j, $j = 1, 2, 3$:

$$\lambda_2 = v/u, \qquad \lambda_j = \frac{uv + (-1)^{\frac{j+1}{2}} c\sqrt{u^2+v^2-c^2}}{u^2-c^2}, \quad j = 1, 3, \tag{2.4}$$

which implies that the flow in the system is always hyperbolic. When $u \neq 0$, the corresponding eigenvectors are

$$\mathbf{r}_2 = (1, v/u, 0)^\top, \qquad \mathbf{r}_j = \kappa_j(-\lambda_j, 1, \rho(\lambda_j u - v)/c^2)^\top, \quad j = 1, 3, \tag{2.5}$$

where κ_j are chosen so that $\mathbf{r}_j \cdot \nabla\lambda_j = 1$ because of genuine nonlinearity of the j^{th}-characteristic fields, $j = 1, 3$. Note that the second characteristic field is always linearly degenerate: $\mathbf{r}_2 \cdot \nabla\lambda_2 = 0$.

Definition 2.1. (Entropy Solutions)**.** A function $U = U(x, y) \in BV_{loc}(\Omega)$ is called an entropy solution of problem (1.6) and (1.8)–(1.9) provided that

(i) U is a weak solution of (1.6) and satisfies

$$(u, v) \cdot \mathbf{n}|_{y=g(x)} = 0 \qquad \text{in the trace sense;}$$

(ii) U satisfies the following entropy inequality:

$$(u(E + p(\rho)))_x + (v(E + p(\rho)))_y \leqq 0 \tag{2.6}$$

in the sense of distributions in Ω including the boundary.

2.2. Basic properties of nonlinear waves

In this subsection, we analyze some basic properties of nonlinear waves, especially the global behavior of shock curves and rarefaction wave curves in the phase space.

We first seek the self-similar solution of (1.6):

$$(u, v, \rho)(x, y) = (u(\xi), v(\xi), \rho(\xi)), \qquad \xi = y/x,$$

which connects to a state $U_0 = (u_0, v_0, \rho_0)$. We then have

$$det\,(\xi\nabla_U W(U) - \nabla_U H(U)) = 0. \tag{2.7}$$

Hence,

$$\xi = \lambda_2 = v/u, \quad \text{or} \quad \xi = \lambda_j = \frac{uv + (-1)^{\frac{j+1}{2}} c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}, \quad j = 1, 3. \tag{2.8}$$

Substituting $\xi = \lambda_2$ into (2.7), we obtain

$$d\rho = 0, \quad v du - u dv = 0.$$

Then, the contact discontinuity curve $C_2(U_0)$ in the phase space is:

$$C_2(U_0): \quad \rho = \rho_0, \quad w = v/u = v_0/u_0, \tag{2.9}$$

which describes compressible vortex sheets.

Substituting $\xi = \lambda_j$ into (2.7), we get the j^{th}-rarefaction wave curve $R_j(U_0)$ in the phase space through U_0:

$$R_j(U_0): \quad du = -\lambda_j dv, \quad \rho(\lambda_j u - v)dv = dp, \qquad j = 1, 3. \tag{2.10}$$

We now compute $\frac{d\lambda_j}{d\rho}$ along $R_j(U_0)$, $j = 1, 3$. Since $(v - \xi u)^2 = c^2(1 + \xi^2)$ along $R_j(U_0)$, we differentiate the equation to obtain

$$\left(c^2\lambda_j + u(v - \lambda_j u)\right)\frac{d\lambda_j}{d\rho} = -(1 + \lambda_j^2)\left(\frac{1}{\rho}\frac{dp}{d\rho} + c\frac{dc}{d\rho}\right) < 0.$$

Noting that $c^2\lambda_j + u(v - \lambda_j u) = (-1)^{\frac{j-1}{2}} c\sqrt{u^2 + v^2 - c^2}$, we conclude that

$$\frac{d\lambda_1}{d\rho}|_{R_1(U_0)} < 0, \qquad \frac{d\lambda_3}{d\rho}|_{R_3(U_0)} > 0. \tag{2.11}$$

Now we consider discontinuous solutions, so that (1.6) is satisfied in the distributional sense, which implies that the following Rankine-Hugoniot conditions hold along the discontinuity with speed σ, which connects to a state $U_0 = (u_0, v_0, \rho_0)$:

$$\sigma[\rho u] = [\rho v], \tag{2.12}$$
$$\sigma[\rho u^2 + p] = [\rho uv], \tag{2.13}$$
$$\sigma[\rho uv] = [\rho v^2 + p], \tag{2.14}$$

where the jump symbol $[\cdot]$ stands for the value of the quantity of the front state minus that of the back state, which can be rewritten as

$$\begin{pmatrix} -\sigma\rho & \rho & v_0 - \sigma u_0 \\ \rho(v_0 - \sigma u_0) & 0 & -\sigma\bar{c}_0^2 \\ 0 & \rho(v_0 - \sigma u_0) & \bar{c}_0^2 \end{pmatrix}\begin{pmatrix} [u] \\ [v] \\ [\rho] \end{pmatrix} = 0 \tag{2.15}$$

with $\bar{c}_0^2 = \frac{\rho}{\rho_0}\frac{[p]}{[\rho]}$. We then have

$$\sigma = \sigma_2 := v_0/u_0, \quad \sigma = \sigma_j := \frac{u_0 v_0 + (-1)^{\frac{j+1}{2}}\bar{c}_0\sqrt{u_0^2 + v_0^2 - \bar{c}_0^2}}{u_0^2 - \bar{c}_0^2}, \quad j = 1, 3. \tag{2.16}$$

Substituting σ_2 into (2.15), we get the same $C_2(U_0)$ as defined in (2.9). Substituting σ_j into (2.15), we obtain the j^{th}-shock curve $S_j(U_0)$ in the phase space through U_0:

$$S_j(U_0): \quad [u] = -\sigma[v], \quad \rho_0(\sigma_j u_0 - v_0)[v] = [p], \qquad j = 1, 3. \tag{2.17}$$

It is straightforward to see that the shock curve $S_j(U_0)$ contacts with $R_j(U_0)$ at U_0 up to second-order and, along $S_j(U_0)$, $j = 1, 3$,

$$\frac{d\sigma_1}{d\rho}|_{S_1(U_0)} < 0, \qquad \frac{d\sigma_3}{d\rho}|_{S_3(U_0)} > 0. \tag{2.18}$$

Lemma 2.1. *If U is a piecewise smooth solution, then, on the shock wave, the entropy inequality* (2.6) *in* Definition 2.1 *is equivalent to any of the following:*

(i) the physical entropy condition – the density increases across the shock in the flow direction:

$$\rho_{front} < \rho_{back}; \tag{2.19}$$

(ii) the Lax entropy condition – on the j^{th}-shock with the shock speed σ_j:

$$\lambda_j(back) < \sigma_j < \lambda_j(front), \qquad j = 1, 3, \tag{2.20}$$

$$\sigma_1 < \lambda_2(back), \qquad \lambda_2(front) < \sigma_3. \tag{2.21}$$

Proof. Since the system is Galilean invariant, we may assume that the back state of the shock is $U_+ = (u_+, 0, \rho_+)$ with $u_+ > 0$.

We first show the equivalence between (2.6) and (2.19). Along the discontinuity with speed σ and back state U_+, (2.6) is equivalent to

$$\sigma[\,u(E+p)] \geqq [\,v(E+p)]\,. \tag{2.22}$$

Substituting (2.12) into (2.22) and using $v_+ = 0$, we have

$$\frac{\rho^{\gamma-1}}{(\gamma-1)} + \frac{(u^2+v^2)}{2} \geqq \frac{\rho_+^{\gamma-1}}{(\gamma-1)} + \frac{u_+^2}{2}. \tag{2.23}$$

Using (2.12) and (2.13), we have

$$u = u_+ - \frac{p-p_+}{\rho_+ u_+}. \tag{2.24}$$

Using (2.12) and (2.14), we obtain

$$p - p_+ = \sigma \rho_+ u_+ v. \tag{2.25}$$

Combining (2.12) with (2.25) yields

$$v^2 = \frac{(\rho u - \rho_+ u_+)(p - p_+)}{\rho \rho_+ u_+}. \tag{2.26}$$

Then substituting (2.24) and (2.26) into (2.23) yields that (2.6) is equivalent to

$$H(\rho) := 2\gamma(p\rho_+ - p_+\rho) - (\gamma-1)(p-p_+)(\rho+\rho_+) \geqq 0,$$

with $H(\rho_+) = 0$, which implies $\rho < \rho_+$, that is (2.19).

Now we show the equivalence between (2.19) and (2.20)–(2.21).

Case (2.19)⇒ (2.20)–(2.21). We now prove that the 1-shock (the shock-front corresponding to the first characteristic field) with the physical entropy condition satisfies the Lax entropy condition; the 3-shock (the shock-front corresponding to the third characteristic field) can be proved in the same way.

First, since $\lambda_1(U_+) = \frac{-c_+}{\sqrt{u_+^2 - c_+^2}}$ and $\sigma_1 = \frac{-\bar{c}_+}{\sqrt{u_+^2 - \bar{c}_+^2}}$, then $\sigma_1 < \lambda_2(U_+)$ is direct, and $\lambda_1(U_+) < \sigma_1$ is equivalent to the inequality:

$$c_+/\sqrt{u_+^2 - c_+^2} > \bar{c}_+/\sqrt{u_+^2 - \bar{c}_+^2}. \tag{2.27}$$

Since the function $f(x) = \frac{x}{\sqrt{u_+^2 - x^2}}$ is strictly increasing in $x \in [0, u_+)$, inequality (2.27) holds if and only if $c_+ > \bar{c}_+$, which is equivalent to

$$p'(\rho_+) > \frac{\rho_-}{\rho_+}\frac{p_- - p_+}{\rho_- - \rho_+} = \frac{\rho_-}{\rho_+} p'(\theta\rho_+ + (1-\theta)\rho_-) \tag{2.28}$$

for some $\theta \in (0, 1)$, where $U_- = (u_-, v_-, \rho_-)$ is the front state of the 1-shock.

By the entropy condition, $\rho_- < \rho_+$, so that $\theta\rho_+ + (1-\theta)\rho_- < \rho_+$. Then the convexity of $p(\rho)$ implies $p'(\theta\rho_+ + (1-\theta)\rho_-) \leqq p'(\rho_+)$. Then (2.28) follows.

Secondly, we set $\omega_- = \arctan(v_-/u_-)$ and

$$\omega_{ma} = \arctan\left(\frac{c_-}{\sqrt{u_-^2 + v_-^2 - c_-^2}}\right), \quad \bar{\omega}_{ma} = \arctan\left(\frac{\bar{c}_-}{\sqrt{u_-^2 + v_-^2 - \bar{c}_-^2}}\right).$$

A direct calculation shows that $\sigma_1 < \lambda_1(U_-) \Leftrightarrow \omega_{ma} < \bar{\omega}_{ma} \Leftrightarrow c_- < \bar{c}_-$, while $c_- < \bar{c}_-$ is equivalent to

$$p'(\rho_-) < \frac{\rho_+}{\rho_-}\frac{p_+ - p_-}{\rho_+ - \rho_-} = \frac{\rho_+}{\rho_-} p'(\theta\rho_+ + (1-\theta)\rho_-)$$

for some $\theta \in (0, 1)$, which is a corollary of the convexity of $p(\rho)$ and $\rho_+ > \rho_-$.

Case (2.20)–(2.21)⇒ (2.19). We prove this case by contradiction. Instead, if $\rho_+ < \rho_-$, similarly to the previous case, we find that $\lambda_1(U_+) < \sigma_1$ is equivalent to $c_+ > \bar{c}_+$, i.e.,

$$p'(\rho_+) > \frac{\rho_-}{\rho_+}\frac{p_- - p_+}{\rho_- - \rho_+} = \frac{\rho_-}{\rho_+} p'(\theta\rho_+ + (1-\theta)\rho_-)$$

for some $\theta \in (0, 1)$, which is impossible since $p''(\rho) > 0$ and $\rho_+ < \rho_-$. □

In view of (2.16) and (2.19), for a shock wave, $u_0^2 + v_0^2 \geqq \bar{c}_0^2 > c_0^2$, which indicates that the front state of a shock must be supersonic. Choosing a coordinate system so that $u_0 > 0$ and $v_0 = 0$, we then have $\frac{y}{x} = \sigma_j = (-1)^{\frac{j+1}{2}} \bar{c}_0/\sqrt{u_0^2 - \bar{c}_0^2}$. Thus, 1-shocks and 3-shocks must be as shown in Fig. 2.

We define the angle of the flow direction as

$$\omega = \arctan(v/u).$$

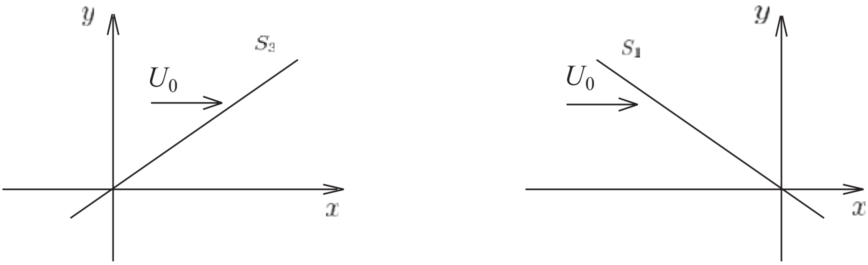

Fig. 2. Shock waves in the physical plane.

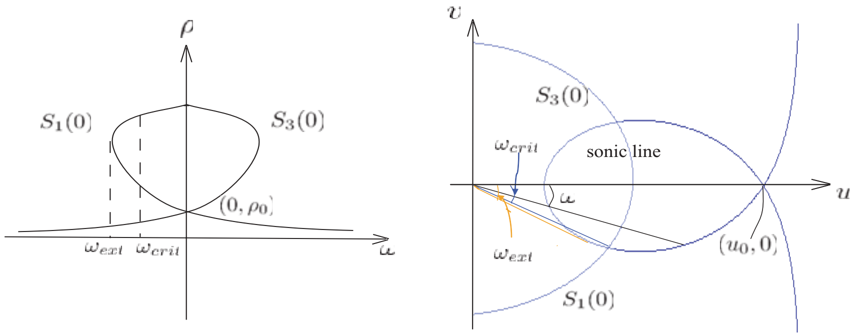

Fig. 3. Shock polar and critical angle.

Then the shock curves $S_j(U_0)$, $j = 1, 3$, in the (ω, ρ)-plane and (u, v)-plane form shock polars, as shown in Fig. 3. In general,

$$\omega_{crit} = \sup\{|\omega(u, v) - \omega(u_0, v_0)| \ : \ (u, v, \rho) \in S(U_0),\ c_*^2 < u^2 + v^2 < u_0^2 + v_0^2\},$$

where $S(U_0) = S_1(U_0) \cup S_3(U_0)$ is the shock polar associated with U_0, similar to that shown above, and $c_* > 0$ is a constant such that $u^2 + v^2 \geqq c^2(\rho)$ is equivalent to $u^2 + v^2 > c_*^2$ on $S(U_0)$.

2.3. Lateral Riemann problem

The simplest case of problem (1.6) and (1.8)–(1.9) is $g \equiv 0$. It has been shown in [8] that if $g \equiv 0$, then problem (1.6) yields an entropy solution that consists of the constant states U_- and U_+, with $U_+ = (u_+, 0, \rho_+)$ and $u_+ > c_+ > 0$ in the subdomain of Ω, separated by a straight shock-front emanating from the vertex. That is to say that the state ahead of the shock-front is U_-, whilst the state behind the shock-front is U_+ (see Fig. 4). When the angle between the flow direction of the front state and the wedge boundary at a boundary vertex is larger than π, the entropy solution contains a rarefaction wave that separates the front state from the back state (see Fig. 5).

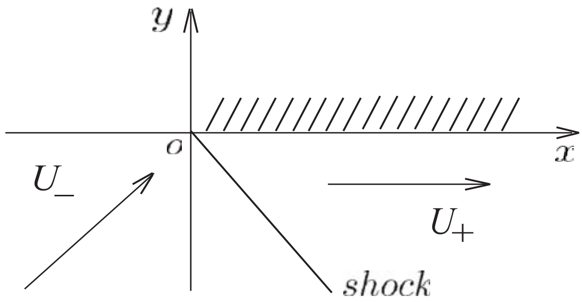

Fig. 4. Unperturbed case when $g \equiv 0$.

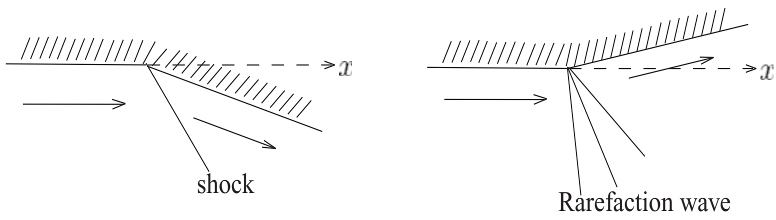

Fig. 5. Lateral Riemann solutions.

2.4. Riemann problem involving only weak waves

Consider the Riemann problem for (2.1):

$$U|_{x=x_0} = \underline{U} = \begin{cases} U_a, \ y > y_0, \\ U_b, \ y < y_0, \end{cases} \tag{2.29}$$

where U_a and U_b are constant states which are regarded as the *above* state and *below* state with respect to the line $y = y_0$.

Following LAX [14], we can parametrize any physically admissible wave curve in a neighborhood of a constant state U_+, $O_\varepsilon(U_+)$, by $\alpha_j \mapsto \Phi_j(\alpha_j; U_b)$, with $\Phi \in C^2$, $\Phi_j|_{\alpha_j=0} = U_b$, and $\frac{\partial \Phi_j}{\partial \alpha_j}|_{\alpha_j=0} = \mathbf{r}_j(U_b)$. Set

$$\Phi(\alpha_3, \alpha_2, \alpha_1; U_b) := \Phi_3(\alpha_3; \Phi_2(\alpha_2; \Phi_1(\alpha_1; U_b))).$$

From this point forward, we denote by $O_\varepsilon(W)$ a universal ball with radius $M\varepsilon > 0$ and center W, where $M > 0$ is a universal constant depending only on the parameters in the system and possibly on the boundary function $g(x)$ (see Section 4.2), which may be different for each occurrence. Then we have

Lemma 2.2. *There exits $\varepsilon > 0$ such that, for any states $U_a, U_b \in O_\varepsilon(U_+)$, the Riemann problem* (2.29) *yields a unique admissible solution consisting of three elementary waves. In addition, state U_a can be represented by*

$$U_a = \Phi(\alpha_3, \alpha_2, \alpha_1; U_b),$$

with $\Phi|_{\alpha_1=\alpha_2=\alpha_3=0} = U_b$ and $\frac{\partial \Phi}{\partial \alpha_i}|_{\alpha_1=\alpha_2=\alpha_3=0} = \mathbf{r}_i(U_b)$, $i = 1, 2, 3$.

Furthermore, we find that the renormalization factors $\kappa_j(U)$, $j = 1, 3$, in (2.5) are positive in a neighborhood $O_\varepsilon(U_0)$ of any state $U_0 = (u_0, 0, \rho_0)$ with $u_0 > 0$.

Lemma 2.3. *At any state $U_0 = (u_0, 0, \rho_0)$ with $u_0 > 0$,*

$$\kappa_1(U_0) = \kappa_3(U_0) > 0,$$

which implies $\kappa_j(U) > 0$ for any $U \in O_\varepsilon(U_0)$, since $\kappa_j(U)$ are continuous for $j = 1, 3$.

At state $U_0 = (u_0, 0, \rho_0)$, it is straightforward to see that

$$\nabla_U \lambda_{10} \cdot (-\lambda_{10}, 1, \rho_0 u_0 \lambda_{10}/c_0^2) = \nabla_U \lambda_{30} \cdot (-\lambda_{30}, 1, \rho_0 u_0 \lambda_{30}/c_0^2) > 0,$$

where $\lambda_{j0} = \lambda_j(U_0)$, $j = 1, 3$. Therefore, we have $\kappa_1(U_0) = \kappa_3(U_0) > 0$.

2.5. Riemann problem involving a strong 1-shock

For simplicity, we use the notation $\{U_b, U_a\} = (\alpha_1, \alpha_2, \alpha_3)$ to denote that $U_a = \Phi(\alpha_3, \alpha_2, \alpha_1; U_b)$ throughout the paper. For any $U \in S_1(U_-)$, we also use $\{U_-, U\} = (\sigma, 0, 0)$ to denote the 1-shock that connects U_- and U with speed σ. We then have

Lemma 2.4. *Let $\{U_-, U_+\} = (\sigma_0, 0, 0)$ with $U_+ = (u_+, 0, \rho_+)$, $\rho_+ > \rho_-$, and $\gamma \geqq 1$. Then*

$$\sigma_0 < 0, \qquad u_+ < u_- < (1 + 1/\gamma)u_+.$$

Proof. From the Rankine-Hugoniot conditions (2.12)–(2.14), we have

$$\sigma_0(\rho_- u_- - \rho_+ u_+) = \rho_- v_-, \tag{2.30}$$

$$\sigma_0(\rho_- u_-^2 - \rho_+ u_+^2 + p_- - p_+) = \rho_- v_- u_-, \tag{2.31}$$

which implies $\sigma_0 = -\bar{c}_0/\sqrt{u_+^2 - \bar{c}_0^2} < 0$. Substituting (2.30) into (2.31), we have $p_+ - p_- = \rho_+ u_+(u_- - u_+)$. By the entropy condition: $\rho_+ > \rho_-$, which implies $p_+ > p_-$. Thus $u_- > u_+$. Furthermore, since $p_- > 0$, we have $p_+ > \rho_+ u_+(u_- - u_+)$, which implies

$$\rho_+ c_+^2/\gamma > \rho_+ u_+(u_- - u_+).$$

Using $u_+ > c_+$, we have $u_+/\gamma > u_- - u_+$ and thus $u_- < (1 + 1/\gamma)u_+$. □

Moreover, it is direct to conclude

Lemma 2.5. *There exists a neighborhood $O_\varepsilon(U_+)$ of U_+ such that the shock polar $S_1(U_-) \cap O_\varepsilon(U_+)$ can be parametrized by the shock speed σ as*

$$\sigma \to G(\sigma)$$

with $G \in C^2$ near σ_0 and $G(\sigma_0) = U_+$.

The following lemma is essential to estimate the strengths of reflected weak waves in the interaction between the strong 1-shock and weak waves (see the proofs for Propositions 3.1–3.4).

Lemma 2.6. *Set* $A = \nabla_U H(U_+) - \sigma_0 \nabla_U W(U_+)$. *Then*

$$\det A < 0, \quad \det(A\mathbf{r}_3, A\mathbf{r}_2, A\mathbf{r}_1)|_{U=U_+} < 0,$$
$$\det(A\mathbf{r}_3, A\mathbf{r}_2, AG_\sigma(\sigma_0))|_{U=U_+} < 0.$$

Proof. A direct calculation shows that

$$A = \begin{pmatrix} -\sigma_0\rho_+ & \rho_+ & -\sigma_0 u_+ \\ -2\sigma_0\rho_+ u_+ & \rho_+ u_+ & -\sigma_0(u_+^2 + c_+^2) \\ 0 & -\sigma_0\rho_+ u_+ & c_+^2 \end{pmatrix},$$

and

$$\mathbf{r}_2(U_+) = (1, 0, 0)^\top,$$
$$\mathbf{r}_j(U_+) = \kappa_j(U_+)(-\lambda_{j+}, 1, \rho_+ u_+ \lambda_{j+}/c_+^2)^\top, \ j = 1, 3.$$

We then have

$$A\mathbf{r}_2(U_+) = -\sigma_0\rho_+(1, 2u_+, 0)^\top,$$
$$A\mathbf{r}_j(U_+) = \frac{\kappa_j(U_+)\rho_+(\lambda_{j+} - \sigma_0)}{\lambda_{j+}}(1, u_+, u_+\lambda_{j+})^\top, \ \ j = 1, 3,$$

and

$$AG_\sigma(\sigma_0) = W(U_+) - W(U_-) = -\frac{\rho_- v_-}{\sigma_0}(1, u_-, \sigma_0 u_-)^\top.$$

Using Lemma 2.4, we can directly identify the signs of the following determinants:

$$\det A = \sigma_0 \rho_+^2 u_+ (\lambda_{1+}^2 - \sigma_0^2)(u_+^2 - c_+^2) < 0,$$
$$\det(A\mathbf{r}_3, A\mathbf{r}_2, A\mathbf{r}_1)|_{U=U_+}$$
$$= \frac{(\kappa_3(U_+))^2 \sigma_0 \rho_+^3 u_+^2}{\lambda_{1+}\lambda_{3+}}(\lambda_{3+} - \sigma_0)(\sigma_0 - \lambda_{1+})(\lambda_{1+} - \lambda_{3+}) < 0,$$

and

$$\det(A\mathbf{r}_3, A\mathbf{r}_2, AG_\sigma(\sigma_0))|_{U=U_+}$$
$$= \frac{\kappa_3(U_+)\rho_- v_- \rho_+^2 u_+}{\lambda_{3+}}(\lambda_{3+} - \sigma_0)\left((2u_+ - u_-)\lambda_{3+} - \sigma_0 u_-\right) < 0. \quad \square$$

3. Estimates on wave interactions and reflections

We now make estimates on wave interactions and reflections on the wedge and the strong 1-shock.

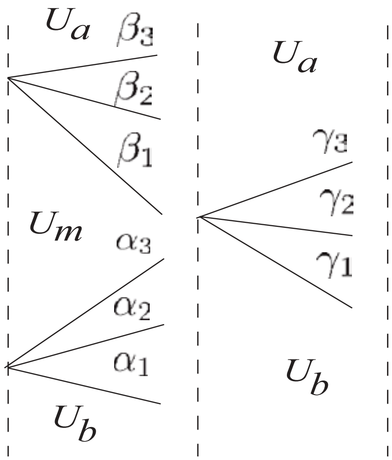

Fig. 6. Weak wave interactions.

3.1. Estimates on weak wave interactions

We first estimate the interactions among weak waves. We will use the following elementary identities, the proofs of which are straightforward.

Lemma 3.1.

(i) *If* $f \in C^1(\mathbb{R})$, *then for any* $x \in \mathbb{R}$,

$$f(x) - f(0) = x \int_0^1 f_x(rx)dr. \tag{3.1}$$

(ii) *If* $f \in C^2(\mathbb{R}^2)$, *then, for any* $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned} &f(x, y) - f(x, 0) - f(0, y) + f(0, 0) \\ &\quad = xy \int_0^1 \int_0^1 f_{xy}(rx, sy)drds. \end{aligned} \tag{3.2}$$

Proposition 3.1. *Suppose that* U_b, U_m, *and* U_a *are three states in a small neighborhood* $O_\varepsilon(U_+)$ *with*

$$\{U_b, U_m\} = (\alpha_1, \alpha_2, \alpha_3), \quad \{U_m, U_a\} = (\beta_1, \beta_2, \beta_3), \quad \{U_b, U_a\} = (\gamma_1, \gamma_2, \gamma_3),$$

then

$$\gamma_i = \alpha_i + \beta_i + O(1)\triangle(\alpha, \beta), \tag{3.3}$$

where $\triangle(\alpha, \beta) = |\alpha_3||\beta_1| + |\alpha_2||\beta_1| + |\alpha_3||\beta_2| + \sum_{j=1,3} \triangle_j(\alpha, \beta)$ *with*

$$\triangle_j(\alpha, \beta) = \begin{cases} 0, & \alpha_j \geqq 0 \text{ and } \beta_j \geqq 0, \\ |\alpha_j||\beta_j|, & \text{otherwise.} \end{cases}$$

Proof. First, Lemma 2.6 yields

$$\det\left(\frac{\partial\Phi(\gamma_3,\gamma_2,\gamma_1;U_b)}{\partial(\gamma_3,\gamma_2,\gamma_1)}\right)|_{\gamma_1=\gamma_2=\gamma_3=0}=\frac{1}{\det A}\det(A\mathbf{r}_3,A\mathbf{r}_2,A\mathbf{r}_1)|_{U=U_+}>0.$$

Then, by the implicit function theorem, there exists $(\gamma_3,\gamma_2,\gamma_1)$ as a C^2 function of $(\beta_1,\beta_2,\beta_3,\alpha_3,\alpha_2,\alpha_1;U_b)$ so that

$$\Phi(\beta_3,\beta_2,\beta_1;\Phi(\alpha_3,\alpha_2,\alpha_1;U_b))=\Phi(\gamma_3,\gamma_2,\gamma_1;U_b).$$

We omit U_b now, for simplicity, and will only compute γ_3 since the estimates for γ_1 and γ_2 can be carried out in the same way. We can rewrite

$$\gamma_3=I_1+I_2+I_3+I_4,$$

where

$$\begin{aligned}
I_1&=\gamma_3(\beta_3,\beta_2,\beta_1,\alpha_3,\alpha_2,\alpha_1)-\gamma_3^a(\beta_3,\beta_2,\beta_1,0,\alpha_2,\alpha_1)\\
&\quad-\gamma_3^b(\beta_3,\beta_2,0,\alpha_3,\alpha_2,\alpha_1)+\gamma_3(\beta_3,\beta_2,0,0,\alpha_2,\alpha_1),\\
I_2&=\gamma_3^a(\beta_3,\beta_2,\beta_1,0,\alpha_2,\alpha_3)-\gamma_3^c(\beta_3,\beta_2,\beta_1,0,0,\alpha_1)\\
&\quad-\gamma_3(\beta_3,\beta_2,0,0,\alpha_2,\alpha_1)+\gamma_3(\beta_3,\beta_2,0,0,0,\alpha_1),\\
I_3&=\gamma_3^b(\beta_3,\beta_2,0,\alpha_3,\alpha_2,\alpha_1)-\gamma_3(\beta_3,\beta_2,0,0,\alpha_2,\alpha_1)\\
&\quad-\gamma_3^d(\beta_3,0,0,\alpha_3,\alpha_2,\alpha_1)+\gamma_3(\beta_3,0,0,0,\alpha_2,\alpha_1),\\
I_4&=\gamma_3^c(\beta_3,\beta_2,\beta_1,0,0,\alpha_1)+\gamma_3^d(\beta_3,0,0,\alpha_3,\alpha_2,\alpha_1)\\
&\quad+\gamma_3(\beta_3,\beta_2,0,0,\alpha_2,\alpha_1)-\gamma_3(\beta_3,\beta_2,0,0,0,\alpha_1)\\
&\quad-\gamma_3(\beta_3,0,0,0,\alpha_2,\alpha_1).
\end{aligned}$$

Note that we add the superscripts a, b, c, and d only to trace the terms.

From (3.2), we have

$$I_1=O(1)|\beta_1||\alpha_3|,\qquad I_2=O(1)|\beta_1||\alpha_2|,\qquad I_3=O(1)|\beta_2||\alpha_3|.$$

Now we estimate I_4. First we have to rely on the implicit function theorem to obtain the uniqueness of the solution in a small neighborhood of U_+, $O_\varepsilon(U_+)$. Then we obtain the following facts, which make the above decomposition possible:

$$\begin{aligned}
&\gamma_3(\beta_3,\beta_2,\beta_1,0,0,0)=\gamma_3(\beta_3,\beta_2,0,0,0,0)=\beta_3,\\
&\gamma_3(0,0,0,\alpha_3,\alpha_2,\alpha_1)=\alpha_3,\ \gamma_3(0,0,0,0,\alpha_2,\alpha_1)=0,\\
&\gamma_3(\beta_3,\beta_2,0,0,0,\alpha_1)=\gamma_3(\beta_3,0,0,0,\alpha_2,\alpha_1)=\beta_3.
\end{aligned}$$

The implicit function theorem also implies $\gamma_3(\beta_3,\beta_2,0,0,\alpha_2,\alpha_1)=\beta_3$. Therefore,

$$I_4=\beta_3+\alpha_3+O(1)(|\alpha_1||\beta_1|+|\alpha_3||\beta_3|).$$

Summing up I_1, I_2, I_3, and I_4, we obtain (3.3). □

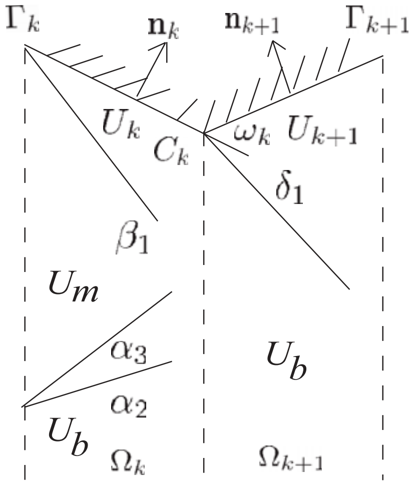

Fig. 7. Weak wave reflections on the boundary.

3.2. Estimates of the weak wave reflections on the boundary (see Fig. 7)

Denote $\{C_k(a_k, b_k)\}_{k=0}^{\infty}$ by the points $\{(a_k, b_k)\}_{k=0}^{\infty}$ in the xy-plane with $a_{k+1} > a_k > 0$. Set

$$\begin{aligned} &\omega_{k,k+1} = \arctan\left(\frac{b_{k+1}-b_k}{a_{k+1}-a_k}\right), \quad \omega_k = \omega_{k,k+1} - \omega_{k-1,k}, \quad \omega_{-1,0} = 0, \\ &\Omega_{k+1} = \{(x, y) : x \in [a_k, a_{k+1}),\ y < b_k + (x - a_k)\tan(\omega_{k,k+1})\}, \\ &\Gamma_{k+1} = \{(x, y) : x \in [a_k, a_{k+1}),\ y = b_k + (x - a_k)\tan(\omega_{k,k+1})\}, \end{aligned} \tag{3.4}$$

and the outer normal vector to Γ_k:

$$\mathbf{n}_{k+1} = \frac{-(b_{k+1}-b_k, a_{k+1}-a_k)}{\sqrt{(b_{k+1}-b_k)^2 + (a_{k+1}-a_k)^2}} = (-\sin(\omega_{k,k+1}), \cos(\omega_{k,k+1})). \tag{3.5}$$

We then consider the initial-boundary value problem:

$$\begin{cases} (2.1) & \text{in } \Omega_{k+1}, \\ U|_{x=a_k} = \underline{U}, & \\ (u, v)\cdot \mathbf{n}_{k+1} = 0 & \text{on } \Gamma_{k+1}, \end{cases}$$

where $\underline{U}$ is a constant state.

Proposition 3.2. *Let* $\{U_b, U_m\} = (\alpha_3, \alpha_2, 0)$ *and* $\{U_m, U_k\} = (0, 0, \beta_1)$ *with*

$$(u_k, v_k)\cdot \mathbf{n}_k = 0.$$

Then there exists U_{k+1} *such that*

$$\{U_b, U_{k+1}\} = (0, 0, \delta_1) \quad \text{with } (u_{k+1}, v_{k+1})\cdot \mathbf{n}_{k+1} = 0.$$

Furthermore,

$$\delta_1 = \beta_1 + K_{b3}\alpha_3 + K_{b2}\alpha_2 + K_{b0}\omega_k,$$

where K_{b3}, K_{b2}, and K_{b0} are C^2 functions of $(\alpha_3, \alpha_2, \beta_1, \omega_k; U_b)$ satisfying

$$K_{b3}|_{\{\omega_k=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}} = 1, \quad K_{b2}|_{\{\omega_k=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}} = 0,$$

and K_{b0} is bounded.

Proof. Since

$$\begin{aligned}
&\frac{\partial}{\partial \delta_1}(\Phi(0,0,\delta_1; U_b)\cdot(\mathbf{n}_{k+1},0))|_{\{\delta_1=0, U_b=U_+, \omega_{k,k+1}=0\}}\\
&\quad = \kappa_1(U_+)(-\lambda_{1+}, 1, \rho_+ u_+ \lambda_{1+}/c_+^2)\cdot(0,1,0)\\
&\quad > 0,
\end{aligned} \tag{3.6}$$

we know from the implicit function theorem that δ_1 can be solved as a C^2 function of $(\alpha_3, \alpha_2, \beta_1, \omega_{k-1,k}, \omega_k; U_b)$ such that

$$\Phi(0,0,\beta_1; \Phi(\alpha_3,\alpha_2,0;U_b))\cdot(\mathbf{n}_k,0) = \Phi(0,0,\delta_1;U_b)\cdot(\mathbf{n}_{k+1},0). \tag{3.7}$$

Since $\omega_{k-1,k}$ and U_b are constant, we can also omit U_b and $\omega_{k-1,k}$ and for simplicity can write $\delta_1 = \delta_1(\omega_k, \alpha_2, \alpha_3, \beta_1)$. Again, from (3.1), we can obtain

$$\begin{aligned}
\delta_1(\omega_k,\alpha_2,\alpha_3,\beta_1) &= \delta_1(\omega_k,\alpha_2,\alpha_3,\beta_1) - \delta_1(0,\alpha_2,\alpha_3,\beta_1)\\
&\quad + \delta_1(0,\alpha_2,\alpha_3,\beta_1) - \delta_1(0,0,\alpha_3,\beta_1)\\
&\quad + \delta_1(0,0,\alpha_3,\beta_1) - \delta_1(0,0,0,\beta_1) + \delta_1(0,0,0,\beta_1)\\
&= K_{b0}\omega_k + K_{b2}\alpha_2 + K_{b3}\alpha_3 + \beta_1.
\end{aligned}$$

Differentiating (3.7) with respect to α_3 and α_2, and letting $\omega_k = \alpha_3 = \alpha_2 = \beta_1 = 0$ and $U_b = U_+$, yields

$$\begin{aligned}
\mathbf{r}_3(U_+)\cdot(0,1,0) &= \frac{\partial \delta_1}{\partial \alpha_3}\mathbf{r}_1(U_+)\cdot(0,1,0),\\
\mathbf{r}_2(U_+)\cdot(0,1,0) &= \frac{\partial \delta_1}{\partial \alpha_2}\mathbf{r}_1(U_+)\cdot(0,1,0).
\end{aligned}$$

Hence, $K_{b3}|_{\{\omega_k=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}} = 1$ and $K_{b2}|_{\{\omega_k=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}} = 0$. It is clear that $K_{b0} = \frac{\partial \delta_1}{\partial \omega_k}$ is bounded. This completes the proof. □

3.3. Estimate on the boundary perturbation of the strong shock

Proposition 3.3. *For $\varepsilon > 0$, there exists $\hat{\varepsilon} = \hat{\varepsilon}(\varepsilon) < \varepsilon$ so that $G(O_{\hat{\varepsilon}}(\sigma_0)) \subset O_\varepsilon(U_+)$ and, when $|\omega_k| < \varepsilon$, the following equation*

$$G(\sigma) \cdot (\mathbf{n}_k, 0) = 0 \tag{3.8}$$

yields a unique solution $\sigma_k \in O_{\hat{\varepsilon}}(\sigma_0)$. Moreover, we have

$$\sigma_{k+1} = \sigma_k + K_{bs}\omega_k + O(1)|\omega_k|^2, \tag{3.9}$$

where K_{bs} is bounded.

Proof. We first show that there exists a solution $\sigma = \sigma(h)$ to the following equation:

$$G(\sigma) \cdot (-\sin h, \cos h, 0) = 0. \tag{3.10}$$

This solution may be proved as follows. By differentiating the quantity $G(\sigma) \cdot (-\sin h, \cos h, 0)$ with respect to σ (following the calculations of Lemma 2.6) and denoting by (A^*_{ij}) the cofactor matrix of (a_{ij}), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\sigma}(G(\sigma) \cdot (-\sin h, \cos h, 0))|_{\{\sigma=\sigma_0, h=0\}} \\
&\quad = (A^{-1} \cdot AG_\sigma(\sigma_0)) \cdot (0, 1, 0) \\
&\quad = \frac{1}{\det A}(A^*_{12}, A^*_{22}, A^*_{32}) \cdot AG_\sigma(\sigma_0) \\
&\quad = \frac{\rho_- v_- \rho_+}{\det A}\Big((u_- - 2u_+)c_+^2 + \sigma_0^2 u_-(c_+^2 - u_+^2)\Big) \\
&\quad > 0.
\end{aligned}$$

Hence, by the implicit function theorem, there exists a unique C^2 function $\sigma = \sigma(h)$ with $\sigma(0) = \sigma_0$, which solves (3.10) in some neighborhood of $(\sigma, h) = (\sigma_0, 0)$. Then

$$\sigma(\omega_j) = \sigma_j, \qquad j = k, k+1,$$

and, by the Taylor expansion formula, we have the desired estimates (3.9). □

3.4. Estimates on the interactions between the strong shock and weak waves

Proposition 3.4. *Let $U_m, U_a \in O_\varepsilon(U_+)$ with*

$$\{G(\sigma), U_m\} = (0, \alpha_2, \alpha_3), \qquad \{U_m, U_a\} = (\beta_1, \beta_2, 0).$$

Then there exists a unique $(\sigma', \delta_2, \delta_3)$ such that the Riemann problem (2.29) *with $U_b = U_-$ yields an admissible solution that consists of a strong* 1*-shock of strength σ', a contact discontinuity of strength δ_2, and a weak* 3*-wave of strength δ_3:*

$$\{U_-, U_a\} = (\sigma', \delta_2, \delta_3).$$

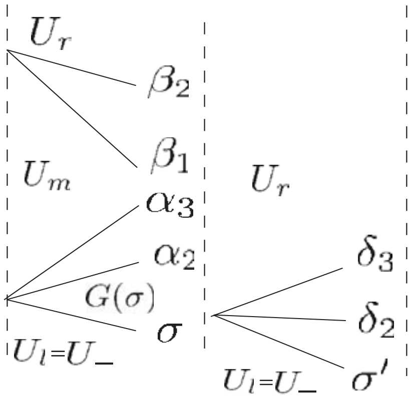

Fig. 8. Interactions between the strong shock and weak waves.

Moreover,

$$\begin{aligned}\delta_3 &= \alpha_3 + K_{s3}\beta_1 + O(1)\Delta, \\ \delta_2 &= \alpha_2 + \beta_2 + K_{s2}\beta_1 + O(1)\Delta, \\ \sigma' &= \sigma + K_{s1}\beta_1 + O(1)\Delta,\end{aligned} \tag{3.11}$$

where

$$|K_{s3}| < 1, \tag{3.12}$$

$|K_{s2}|$ *and* $|K_{s3}|$ *are bounded, and* $\Delta = |\alpha_3||\beta_1| + |\alpha_2||\beta_1| + |\alpha_3||\beta_2|$*. Furthermore, we can write the estimates in a more precise fashion:*

$$\begin{aligned}\sigma' &= \sigma + \widetilde{K_{s1}}\beta_1 + O(1)|\alpha_3||\beta_2|, \\ \delta_2 &= \alpha_2 + \beta_2 + \widetilde{K_{s2}}\beta_1 + O(1)|\alpha_3||\beta_2|, \\ \delta_3 &= \alpha_3 + \widetilde{K_{s3}}\beta_1 + O(1)|\alpha_3||\beta_2|,\end{aligned} \tag{3.13}$$

where

$$|\widetilde{K_{s3}}| < 1, \qquad |\widetilde{K_{s1}}| + |\widetilde{K_{s2}}| \leqq M \qquad \textit{for some constant } M > 0. \tag{3.14}$$

Proof. We first show that there exists a unique solution

$$(\sigma', \delta_2, \delta_3) = (\sigma'(\sigma, \alpha_2, \alpha_3, \beta_1, \beta_2), \delta_2(\sigma, \alpha_2, \alpha_3, \beta_1, \beta_2), \delta_3(\sigma, \alpha_2, \alpha_3, \beta_1, \beta_2))$$

to

$$\Phi(0, \beta_2, \beta_1; \Phi(\alpha_3, \alpha_2, 0; G(\sigma))) = \Phi(\delta_3, \delta_2, 0; G(\sigma')). \tag{3.15}$$

By Proposition 3.1, there exists $(\gamma_3, \gamma_2, \gamma_1)$ such that

$$\Phi(0, \beta_2, \beta_1; \Phi(\alpha_3, \alpha_2, 0; G(\sigma))) = \Phi(\gamma_3, \gamma_2, \gamma_1; G(\sigma)), \tag{3.16}$$

with $\gamma_1 = \beta_1 + O(1)\Delta$, $\gamma_2 = \beta_2 + \alpha_2 + O(1)\Delta$, and $\gamma_3 = \alpha_3 + O(1)\Delta$. Thus, (3.15) can be reduced to

$$\Phi(\gamma_3, \gamma_2, \gamma_1; G(\sigma)) = \Phi(\delta_3, \delta_2, 0; G(\sigma')). \tag{3.17}$$

Furthermore, Lemma 2.6 implies

$$\begin{aligned}
&\det\left(\frac{\partial\Phi(\delta_3, \delta_2, 0; G(\sigma'))}{\partial(\delta_3, \delta_2, \sigma')}\right)|_{\{\delta_3=\delta_2=0, \sigma'=\sigma_0\}} \\
&= \frac{1}{\det A}\det(A\mathbf{r}_3(U_+), A\mathbf{r}_2(U_+), AG_\sigma(\sigma_0)) \\
&> 0.
\end{aligned}$$

Therefore, the implicit function theorem implies that $(\delta_3, \delta_2, \sigma')$ can be solved uniquely as a C^2 function of $(\gamma_1, \gamma_2, \gamma_3, \sigma)$:

$$\delta_3 = \delta_3(\gamma_3, \gamma_2, \gamma_1, \sigma), \quad \delta_2 = \delta_2(\gamma_3, \gamma_2, \gamma_1, \sigma), \quad \sigma' = \sigma'(\gamma_3, \gamma_2, \gamma_1, \sigma).$$

Using (3.1), we find that, for $i = 2, 3$,

$$\begin{aligned}
\delta_i &= \delta_i(\gamma_3, \gamma_2, \gamma_1, \sigma) - \delta_i(\gamma_i, \gamma_2, 0, \sigma) + \delta_i(\gamma_3, \gamma_2, 0, \sigma) = K_{si}\gamma_1 + \gamma_i, \\
\sigma' &= \sigma'(\gamma_3, \gamma_2, \gamma_1, \sigma) - \sigma'(\gamma_3, \gamma_2, 0, \sigma) + \sigma'(\gamma_3, \gamma_2, 0, \sigma) = K_{s1}\gamma_1 + \sigma,
\end{aligned}$$

where $K_{s1} = \int_0^1 \partial_{\gamma_1}\sigma'(\gamma_3, \gamma_2, \lambda\gamma_1, \sigma)d\lambda$ and $K_{si} = \int_0^1 \partial_{\gamma_1}\delta_i(\gamma_3, \gamma_2, \lambda\gamma_1, \sigma)d\lambda$.

When $\gamma_3 = \gamma_2 = \gamma_1 = 0$, it is clear that $|\frac{\partial\delta_3}{\partial\gamma_1}|$, $|\frac{\partial\delta_2}{\partial\gamma_1}|$, and $|\frac{\partial\sigma'}{\partial\gamma_1}|$ are bounded. We can further claim the important feature that

$$|\frac{\partial\delta_3}{\partial\gamma_1}| < 1 \qquad \text{when } \gamma_3 = \gamma_2 = \gamma_1 = 0.$$

This can be achieved as follows: differentiating (3.17) with respect to γ_1 and letting $\gamma_3 = \gamma_2 = \gamma_1 = 0$ yields

$$\mathbf{r}_1(U_+) = \mathbf{r}_3(U_+)\frac{\partial\delta_3}{\partial\gamma_1} + \mathbf{r}_2(U_+)\frac{\partial\delta_2}{\partial\gamma_1} + G_\sigma(\sigma_0)\frac{\partial\sigma'}{\partial\gamma_1}. \tag{3.18}$$

Multiplying both sides by A as defined in Lemma 2.6, we have

$$\begin{aligned}
|\frac{\partial\delta_3}{\partial\gamma_1}| &= |\frac{\det(A\mathbf{r}_1(U_+), A\mathbf{r}_2(U_+), AG_\sigma(\sigma_0))}{\det(A\mathbf{r}_3(U_+), A\mathbf{r}_2(U_+), AG_\sigma(\sigma_0))}| \\
&= |\frac{\lambda_{3+} + \sigma_0}{\lambda_{3+} - \sigma_0}||\frac{(2u_+ - u_-)\lambda_{3+} + \sigma_0 u_-}{(2u_+ - u_-)\lambda_{3+} - \sigma_0 u_-}| < 1.
\end{aligned}$$

Combining these with the estimates we had for γ_1, γ_2, and γ_3, we conclude the proof. □

4. Approximate solutions

In this section, we develop a modified Glimm difference scheme to construct a family of approximate solutions and establish their necessary estimates for the initial-boundary value problem (1.1) and (1.8)–(1.9) in the corresponding approximate domains $\Omega_{\Delta x}$.

4.1. A modified Glimm scheme

To define the scheme more clearly, we first use the fact that the boundary is a perturbation of the straight wedge:

$$\sup_{x \geqq 0} |g'(x)| < \varepsilon \qquad \text{for sufficiently small } \varepsilon > 0.$$

For any $\Delta x \geqq 0$, set $a_k := k\Delta x$ and $b_k := y_k = g(k\Delta x)$ in (3.4) and (3.5), and follow the notations in Section 3.2 (also see Fig. 7). Then

$$m := \sup_{k>0}\{\frac{y_k - y_{k-1}}{\Delta x}\} < \varepsilon. \tag{4.1}$$

Define

$$\Omega_{\Delta x} = \bigcup_{k \geqq 0} \Omega_{\Delta x,k},$$

where $\Omega_{\Delta x,k} = \{(x, y) : (k-1)\Delta x \leqq x < k\Delta x, y \leqq y_{k-1} + (x - (k-1)\Delta x)\tan(\omega_{k-1,k})\}$. We also need the Courant–Friedrichs–Lewy type condition:

$$\frac{\Delta y - m\Delta x}{\Delta x} < |\sigma_0| + \max_{j=1,3}\left(\sup_{U \in O_\varepsilon(U_+)} |\lambda_j(U)|\right).$$

Define

$$a_{k,n} = (2n + 1 + \theta_k)\Delta y + y_k,$$

where θ_k is randomly chosen in $(-1, 1)$. Then we choose

$$p_{k,n} = (k\Delta x, a_{k,n}), \qquad k \geqq 0, n = 0, -1, -2, ...,$$

to be the mesh points and define the approximate solutions $U_{\Delta x,\theta}$ in $\Omega_{\Delta x}$ for any $\theta = (\theta_0, \theta_1, \theta_2, ...)$ in an inductive way:

For $k = 0$, we define $U_{\Delta x,\theta}$ in $\{0 \leqq x < \Delta x\} \cap \Omega_{\Delta x}$ to be the strong 1-shock solution starting from $U_{\Delta x,\theta}|_{\{x=0,y<0\}} = U_-$.

We assume that $U_{\Delta x,\theta}$ has been constructed for $\{0 \leqq x < k\Delta x\}$. Denoting, for $n \leqq -1$,

$$U_k^0 := U_{\Delta x,\theta}(k\Delta x-, a_{k,n}) \qquad \text{if } y \in (y_k + 2n\Delta y, y_k + (2n+2)\Delta y),$$

then we define $U_{\Delta x,\theta}$ in $\{k\Delta x \leqq x < (k+1)\Delta x\}$ as follows: First we solve the following lateral Riemann problem in diamond $T_{k,0}$, whose vertices are $((k+1)\Delta x, y_{k+1})$, $((k+1)\Delta x, -\Delta y + y_{k+1})$, $(k\Delta x, y_k)$, and $(k\Delta x, -\Delta y + y_k)$:

$$\begin{aligned} W(U_k)_x + H(U_k)_y = 0 &\qquad \text{in } T_{k,0},\\ U_k|_{x=k\Delta x} = U_k^0&,\\ (u_k, v_k) \cdot \mathbf{n}_k = 0 &\qquad \text{on } \Gamma_k,\end{aligned}$$

to obtain the lateral Riemann solution U_k in $T_{k,0}$ as constructed in Section 2.3 and define

$$U_{\Delta x,\theta} = U_k \qquad \text{in } T_{k,0}.$$

Then we solve the following Riemann problem in each diamond $T_{k,n}$ for $n \leqq -1$, whose vertices are $((k+1)\Delta x, (2n-1)\Delta y + y_{k+1})$, $((k+1)\Delta x, (2n+1)\Delta y + y_{k+1})$, $(k\Delta x, (2n-1)\Delta x + y_k)$, and $(k\Delta x, (2n+1)\Delta y + y_k)$:

$$W(U_k)_x + H(U_k)_y = 0 \quad \text{in } T_{k,n},$$
$$U_k|_{x=k\Delta x} = U_k^0,$$

to obtain the Riemann solution $U_k(x, y)$ in $T_{k,n}$ as constructed in Sections 2.4–2.5, and define

$$U_{\Delta x,\theta} = U_k \qquad \text{in } T_{k,n},\ n \leqq -1.$$

In this way, we have constructed the approximate solutions $U_{\Delta x,\theta}(x, y)$ globally, provided that we can obtain a uniform bound of the approximate solutions.

4.2. Glimm-type functional and its bounds

In this section, we prove that the approximate solutions can indeed be well defined in $\Omega_{\Delta x}$ via the steps in Section 4.1 by providing a uniform bound for them. First, we introduce the following lemma.

Lemma 4.1.

(i) *If* $\{U_b, U_a\} = (\alpha_1, \alpha_2, \alpha_3)$ *with* $U_b, U_a \in O_\varepsilon(U_+)$, *then*

$$|U_b - U_a| \leqq s_1(|\alpha_1| + |\alpha_2| + |\alpha_3|),$$

with $s_1 = \max_{1\leqq i\leqq 3}\left(\sup_{U\in O_\varepsilon(U_+)} |\partial_{\alpha_i}\Phi(\alpha_3, \alpha_2, \alpha_1; U)|\right)$;

(ii) *for any* $\sigma \in O_{\hat\varepsilon}(\sigma_0) \subset O_\varepsilon(U_+)$,

$$|G(\sigma) - G(\sigma_0)| \leqq s_2|\sigma - \sigma_0|$$

with $s_2 = \max_{\sigma\in O_{\hat\varepsilon}(\sigma_0)}\{G'_\sigma(\sigma)\}$.

Next, we show that $U_{\Delta x,\theta}$ can be globally defined. Assume that $U_{\Delta x,\theta}$ has been defined in $\{x < k\Delta x\} \cap \Omega_{\Delta x}$ by the steps in Section 4.1 and assume that the following conditions are satisfied:

$C_1(k-1)$: in each $\Omega_{\Delta x,j}$ for $0 \leqq j \leqq k-1$, there is a strong 1-shock $S_*(\sigma_{(j)})$ in $U_{\Delta x,\theta}$ with speed $\sigma_{(j)} \in O_{\hat\varepsilon}(\sigma_0)$, which divides $\Omega_{\Delta x,j}$ into two parts: $\Omega^+_{\Delta x,j}$ and $\Omega^-_{\Delta x,j}$, where $\Omega^+_{\Delta x,j}$ is the part bounded by $S_*(\sigma_{(j)})$ and $\Gamma_j = \{y = g(x, j, \Delta x)\}$;

$C_2(k-1)$: $U_{\Delta x,\theta}|_{\Omega^+_{\Delta x,j}} \in O_\varepsilon(U_+)$ and $U_{\Delta x,\theta}|_{\Omega^-_{\Delta x,j}} = U_-$ for $0 \leqq j \leqq k-1$;

$C_3(k-1)$: $\{S_*(\sigma_{(j)})\}_{j=0}^{k-1}$ form an approximate 1-characteristic $\chi_{\Delta x,\theta}$: $y = \chi_{\Delta x,\theta}(x)$, emanating from the origin.

Here and from this point forth, we use $S_*(\sigma_{(j)})$ to denote the strong 1-shock with speed $\sigma_{(j)}$. Then we prove that $U_{\Delta x,\theta}$ can be defined in $\Omega_{\Delta x,k}$ and satisfies $C_1(k)$, $C_2(k)$, and $C_3(k)$.

From the construction steps in Section 4.1, we first define $U_{\Delta x,\theta}$ and the strong 1-shock $S_*(\sigma_{(k)})$ in $\Omega_{\Delta x,k}$. Then there exists a diamond $\Lambda_{k,n(k)}$ such that $S_*(\sigma_{(k-1)})$ enters $\Lambda_{k,n(k)}$ and $S_*(\sigma_{(k)})$ emanates from the center of $\Lambda_{k,n(k)}$. We extend $\chi_{\Delta x,\theta}$ to $\Omega_{\Delta x,k}$ such that $\chi_{\Delta x,\theta} = S_*(\sigma_{(k)})$ in $\Omega_{\Delta x,k}$ and define $\Omega^-_{\Delta x,j}$ and $\Omega^+_{\Delta x,j}$ in the same way as in $C_1(k-1)$. It then suffices to impose some conditions so that $C_2(k-1)$ holds and $\sigma_{(k)} \in O_{\hat{\varepsilon}}(\sigma_0)$.

To achieve this, we establish the bound on the total variation of $U_{\Delta x,\theta}$ on a class of space-like curves. Denote by

$$N(\theta_{k+1}, n) = \begin{cases} P_{k+1,n} & \text{if } \theta_{k+1} \leqq 0, \\ P_{k+1,n-1} & \text{if } \theta_{k+1} > 0, \end{cases} \qquad S(\theta_k, n) = \begin{cases} P_{k-1,n-1} & \text{if } \theta_k \leqq 0, \\ P_{k-1,n} & \text{if } \theta_k > 0. \end{cases}$$

Then we introduce

Definition 4.1. A j-mesh curve J is defined to be an unbounded space-like curve lying in the strip $\{(j-1)\Delta x \leqq x \leqq (j+1)\Delta x\}$ and consisting of the segments of the form $P_{k,n-1}N(\theta_{k+1}, n)$, $P_{k,n-1}S(\theta_k, n)$, $S(\theta_k, n)P_{k,n}$, and $N(\theta_{k+1}, n)P_{k,n}$.

This definition means that we can connect the mesh point $P_{k,n}$ by two line segments to the two mesh points $P_{k-1,n-1}$ and $P_{k-1,n}$ if $\theta_k \leqq 0$, or we can connect the mesh point $P_{k,n}$ by two line segments to the two mesh points $P_{k-1,n}$ and $P_{k-1,n+1}$ if $\theta_k > 0$ (see Fig. 9).

Clearly, for any $k > 0$, each k-mesh curve I divides the plane $\mathbb{R}^2$ into an I^+ part and an I^- part, where I^- is the part containing the set $\{x < 0\}$. As in GLIMM [10], we also partially order these mesh curves by saying $J > I$ if every point of the mesh curve J is either on I or contained in I^+. Thus we call J an immediate successor to I if $J > I$ and every mesh point of J, except one, is on I.

With these mesh curves J, we associate the Glimm-type functional $F_s(J)$:

Definition 4.2. We define

$$F_s(J) = C^*|\sigma^J - \sigma_0| + F(J),$$

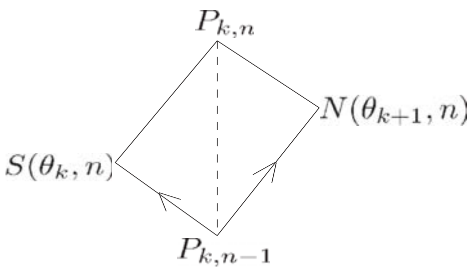

Fig. 9. Interaction diamond $\Lambda_{k,n}$ and orientation of the segments.

with

$$F(J) = L(J) + KQ(J),$$
$$L(J) = K_0^* L_0(J) + L_1(J) + K_2^* L_2(J) + K_3^* L_3(J),$$
$$Q(J) = \sum \{|\alpha_i||\beta_j| \ : \ \text{both } \alpha_i \text{ and } \beta_j \text{ cross } J \text{ and approach}\},$$

and

$$L_0(J) = \sum \{|\omega(C_k)| \ : \ C_k \in \Omega_J\},$$
$$L_j(J) = \sum \{|\alpha_j| \ : \ \alpha_j \text{ crosses } J\}, \ \ j = 1, 2, 3,$$

where K and C^* will be defined later; Ω_J is the set of the corner points C_k lying in J^+:

$$\Omega_J = \{C_k \in J^+ \cap \partial\Omega_{\Delta x} \ : \ k \geqq 0\};$$

σ^J stands for the speed of the strong shock crossing J; K_0^*, K_2^* and K_3^* are the constants that satisfy the following conditions:

$$K_0^* > |K_{b0}|, \qquad |K_{b2}| < K_2^* < \frac{1 - |K_{s3}|K_3^*}{|K_{s2}|}, \qquad |K_{b3}| < K_3^* < \frac{1}{K_{s3}},$$

which can be achieved from our discussions of the properties of K_{bi} and K_{si}, $i = 1, 2, 3$, as in Propositions 3.1–3.3 in Section 3.

As indicated in Section 2.4, from now on, we denote $M > 0$ a universal constant, depending only the parameters in the system and the boundary function $g(x)$, which may be different at each occurrence. We now prove the decreasing property of our functional F_s. We first have

Proposition 4.1. *Suppose that the wedge function $g(x)$ satisfies* (4.1)*, and I and J are two k-mesh curves such that J is an immediate successor of I. Suppose that*

$$|U_{\Delta x,\theta}|_{I \cap (\Omega^+_{\Delta x,k-1} \cup \Omega^+_{\Delta x,k})} - U_+| < \varepsilon, \qquad |\sigma^I - \sigma_0| < \hat{\varepsilon}(\varepsilon),$$

where $\hat{\varepsilon}(\varepsilon)$ is determined in Proposition 3.3 *and* Lemma 4.1. *Then there exist constants $\tilde{\varepsilon} > 0$, $K > 0$, and $C^* > 1$, depending only on the system in* (1.6) *and states U_- and U_+, such that, if $F_s(I) \leqq \tilde{\varepsilon}$, then*

$$F_s(J) \leqq F_s(I),$$

and hence

$$|U_{\Delta x,\theta}|_{J \cap (\Omega^+_{\Delta x,k-1} \cup \Omega^+_{\Delta x,k})} - U_+| < \varepsilon, \qquad |\sigma^J - \sigma_0| < \hat{\varepsilon}(\varepsilon).$$

Proof. Let Λ be the diamond that is formed by I and J. We can always assume that $I = I_0 \cup I'$ and $J = J_0 \cup J'$ such that $\partial\Lambda = I' \cup J'$. We divide our proof into four cases depending on the location of the diamond.

Case 1 (interior weak-weak interaction). Λ *lies in the interior of* $\Omega_{\Delta x}$ *and does not touch* $\chi_{\Delta x,\theta}$. *Then only weak waves enter* Λ. *Denote* $Q(\Lambda) = \Delta(\alpha, \beta)$ *as defined as in Proposition 3.1. Then, for some constant* $M > 0$,

$$L(J) - L(I) \leqq (1 + K_2^* + K_3^*) M Q(\Lambda).$$

Since $L(I_0) < \tilde{\varepsilon}$ from $F_s(I) < \tilde{\varepsilon}$, we have

$$\begin{aligned} Q(J) - Q(I) = {} & Q(I_0) + Q(\gamma_1, I_0) + Q(\gamma_2, I_0) + Q(\gamma_3, I_0) \\ & -(Q(I_0) + Q(\Lambda) + Q(\alpha_1, I_0) + Q(\alpha_2, I_0) + Q(\alpha_3, I_0) \\ & \quad + Q(\beta_1, I_0) + Q(\beta_2, I_0) + Q(\beta_3, I_0)) \\ \leqq {} & Q(MQ(\Lambda), I_0) - Q(\Lambda) = (ML(I_0) - 1) Q(\Lambda) \leqq -\frac{1}{2} Q(\Lambda). \end{aligned}$$

Hence, by choosing suitably large K, we obtain

$$F(J) - F(I) \leqq \left((1 + K_2^* + K_3^*) M - K/2\right) Q(\Lambda) \leqq -\frac{1}{4} Q(\Lambda).$$

Case 2 (near the boundary). Λ *touches the approximate boundary* $\partial\Omega_{\Delta x}$ *and is away from the strong shock* $\chi_{\Delta x,\theta}$. *Then* $\Omega_J = \Omega_I \backslash \{C_k\}$ *for certain k and* $\sigma^I = \sigma^J$.

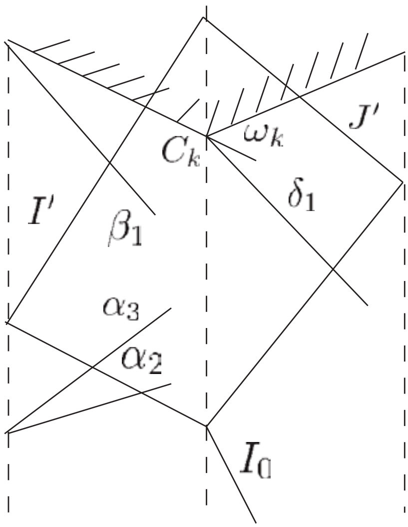

Fig. 10. Case 2: near the boundary.

Let δ_1 be the weak 1-wave going out of Λ through J', and let β_1, α_2, and α_3 be the weak waves entering Λ through I', as shown in Fig. 10. Then

$$\begin{aligned}
L_0(J) - L_0(I) &= -|\omega_k|,\\
L_2(J) - L_2(I) &= \sum_{\gamma_2 \text{ crosses } I_0} |\gamma_2| - (|\alpha_2| + \sum_{\gamma_2 \text{ crosses } I_0} |\gamma_2|) = -|\alpha_2|,\\
L_3(J) - L_3(I) &= \sum_{\gamma_3 \text{ crosses } I_0} |\gamma_3| - (|\alpha_3| + \sum_{\gamma_3 \text{ crosses } I_0} |\gamma_3|) = -|\alpha_3|,\\
L_1(J) - L_1(I) &= (|\delta_1| + \sum_{\gamma_1 \text{ crosses } I_0} |\gamma_1|) - (|\beta_1| + \sum_{\gamma_1 \text{ crosses } I_0} |\gamma_1|)\\
&= |\delta_1| - |\beta_1| \leqq |K_{b3}||\alpha_3| + |K_{b2}||\alpha_2| + |K_{b0}||\omega_k|,
\end{aligned}$$

where the last step is from Proposition 3.2. Thus,

$$L(J) - L(I) \leqq (|K_{b0}| - K_0^*)|\omega_k| + (|K_{b2}| - K_2^*)|\alpha_2| + (|K_{b3}| - K_3^*)|\alpha_3|.$$

From our requirement in Definition 4.2, we find $L(J) - L(I) \leqq 0$. Since $F_s(I) \leqq \tilde{\varepsilon}$ implies $L(I) \leqq \tilde{\varepsilon}$, the higher order term $Q(I)$ can always be bounded by the linear term $L(I)$. We can then easily conclude that $F(J) \leqq F(I)$.

Case 3 (near the wedge vertex)**.** *Λ covers a part of $\partial\Omega_{\Delta x}$, and $S_*(\sigma_{(k-1)})$ emanates from $\{C_{k-1}\}$ and enters Λ. Then, from our construction, we find $\Omega_J = \Omega_I \backslash \{C_k\}$, $S_*(\sigma_{(k)})$ emanates from $\{C_k\}$ and crosses J, $\sigma^I = \sigma_{(k-1)}$, and $\sigma^J = \sigma_{(k)}$. Moreover, there is no weak wave crossing I' or J'. Then we have*

$$F(J) - F(I) \leqq -K_0^*|\omega_k|.$$

Since

$$|\sigma^J - \sigma_0| - |\sigma^I - \sigma_0| \leqq |\sigma^J - \sigma^I| \leqq |K_{bs}||\omega_k| + M|\omega_k|^2,$$

we can further choose C^* suitably small and $\tau > 0$ such that

$$F_s(J) - F_s(I) \leqq C^*|\sigma^J - \sigma^I| + F(J) - F(I) \leqq -\tau|\omega_k|.$$

Case 4 (near the strong 1-shock)**.** *Λ lies in the interior of $\Omega_{\Delta x}$, and the strong 1-shock $S_*(\sigma_{(k-1)})$ enters Λ. Then $S_*(\sigma_{(k)})$ is generated from the inside of Λ, $\sigma^I = \sigma_{(k-1)}$, and $\sigma^J = \sigma_{(k)}$.*

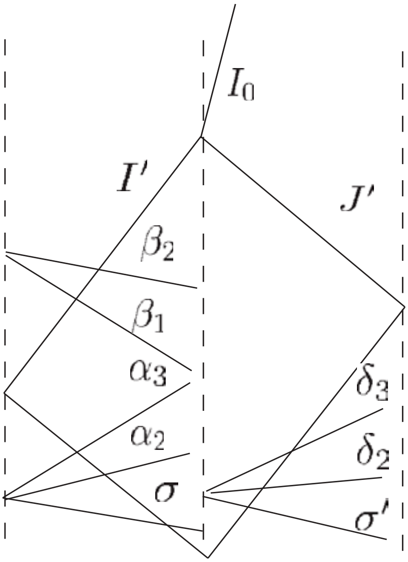

Fig. 11. Case 4: near the strong 1-shock.

Let δ_3 and δ_2 be the weak waves going out of Λ through J', and let α_3, α_2, β_1, and β_2 be the weak waves entering Λ through I', as shown in Fig. 11. Then

$$L_1(J) - L_1(I) = \sum_{\gamma_1 \text{ crosses } I_0} |\gamma_1| - (|\beta_1| + \sum_{\gamma_1 \text{ crosses } I_0} |\gamma_1|) = -|\beta_1|,$$

$$\begin{aligned} L_2(J) - L_2(I) &= (|\delta_2| + \sum_{\gamma_2 \text{ crosses } I_0} |\gamma_2|) - (|\alpha_2| + |\beta_2| + \sum_{\gamma_2 \text{ crosses } I_0} |\gamma_2|) \\ &\leqq |K_{s2}||\beta_1| + M|\alpha_3||\beta_2|, \end{aligned}$$

$$\begin{aligned} L_3(J) - L_3(I) &= (|\delta_3| + \sum_{\gamma_3 \text{ crosses } I_0} |\gamma_3|) - (|\alpha_3| + \sum_{\gamma_3 \text{ crosses } I_0} |\gamma_3|) \\ &\leqq |K_{s3}||\beta_1| + M|\alpha_3||\beta_2|, \end{aligned}$$

where we have used the estimates in Proposition 3.4.

This case is more complicated, which requires a careful calculation of $Q(J) - Q(I)$. For simplicity, for any weak wave γ, we denote

$$Q(\gamma, I_0) = |\gamma| \sum \{|\gamma_j| \,:\, \gamma_j \text{ and } \gamma \text{ approach}\}.$$

Then

$$\begin{aligned}Q(J)-Q(I) &= Q(I_0)+Q(\delta_3,I_0)+Q(\delta_2,I_0)-(Q(I_0)+|\beta_1||\alpha_2|+|\alpha_3||\beta_1|\\ &\quad +|\alpha_3||\beta_2|+Q(\alpha_2,I_0)+Q(\alpha_3,I_0)+Q(\beta_1,I_0)+Q(\beta_2,I_0))\\ &\leqq -(|\beta_1||\alpha_2|+|\alpha_3||\beta_1|+|\alpha_3||\beta_2|)\\ &\quad +\left(|\widetilde{K_{s3}}|+|\widetilde{K_{s2}}|-1\right)Q(\beta_1,I_0)+Q(M|\alpha_3||\beta_2|,I_0)\\ &\leqq (-1+ML(I_0))\,|\alpha_3||\beta_2|+(-|\alpha_2|-|\alpha_3|+ML(I_0))\,|\beta_1|.\end{aligned}$$

Again, since $L(I_0)\leqq\tilde{\varepsilon}$ sufficiently small, then

$$Q(J)-Q(I)\leqq -\frac{1}{2}|\alpha_3||\beta_2|+ML(I_0)|\beta_1|.$$

Therefore, we have

$$\begin{aligned}F(J)-F(I) &\leqq \left(-1+K_2^*|K_{s2}|+K_3^*|K_{s3}|\right)|\beta_1|+M|\alpha_3||\beta_2|\\ &\quad +K\left(-\frac{1}{2}|\alpha_3||\beta_2|+ML(I_0)|\beta_1|\right)\\ &\leqq -\frac{1}{4}|\beta_1|+ML(I_0)|\beta_1|-\frac{1}{8}|\alpha_3||\beta_2|\\ &\leqq -\frac{1}{8}(|\beta_1|+|\alpha_3||\beta_2|),\end{aligned}$$

where we have chosen suitably large K and used the fact that $L(I_0)\leqq\tilde{\varepsilon}$. Furthermore, since $|\sigma^J-\sigma^I|\leqq|K_{s1}||\beta_1|+M|\alpha_3||\beta_2|$, we can further choose C^* suitably small such that

$$F_s(J)-F_s(I)\leqq C^*|\sigma^J-\sigma^I|+F(J)-F(I)\leqq -\frac{1}{16}|\beta_1|-\frac{1}{16}|\alpha_3||\beta_2|.$$

Again, we have $F(J)\leqq F(I)$.

Then, from Lemma 4.1, there exists $\tilde{\epsilon}>0$ such that, when $F(I)<\tilde{\epsilon}$, we have $|U-U_+|<\epsilon$. □

Let I_k be the k-mesh curve lying in $\{(j-1)\Delta x\leqq x\leqq j\Delta x\}$. From Proposition 4.1, we obtain the following theorem for any $k\geqq 1$.

Theorem 4.1. *Suppose that the function $g(x)$ satisfies* (4.1). *Let ε, $\tilde{\varepsilon}$, $\hat{\varepsilon}(\varepsilon)$, K, and C^* be the constants specified in* Proposition 4.3. *If the induction hypotheses $C_1(k-1)$, $C_2(k-1)$, and $C_3(k-1)$ hold and if $F_s(I_{k-1})\leqq\tilde{\varepsilon}$, then*

$$|U_{\Delta x,\theta}|_{\Omega^+_{\Delta x,k}}-U_+|<\varepsilon,\quad U_{\Delta x,\theta}|_{\Omega^-_{\Delta x,k}}=U_-,\quad |\sigma_k-\sigma_0|<\hat{\varepsilon}(\varepsilon),$$

and

$$F_s(I_k)\leqq F_s(I_{k-1}). \tag{4.2}$$

Moreover, we obtain

Theorem 4.2. *There exists $\varepsilon > 0$ such that, if $TV(g'(\cdot)) < \varepsilon$, then, for any $\theta \in \prod_{k=0}^{\infty}(-1, 1)$ and every $\Delta x > 0$, the modified Glimm scheme defines a family of global approximate solutions $U_{\Delta x,\theta}$ and the corresponding family of approximate strong 1-shocks $\chi_{\Delta x,\theta}$ in $\Omega_{\Delta x,\theta}$ which satisfy $C_1(k-1)$, $C_2(k-1)$, $C_3(k-1)$, and (4.2) for any $k \geqq 1$. In addition,*

$$TV\{U_{\Delta x,\theta}(k\Delta x-, \cdot) : (-\infty, y_k]\} \leqq C\, TV(g'(\cdot))$$

for any $k \geqq 0$ and

$$|\chi_{\Delta x,\theta}(x+h) - \chi_{\Delta x,\theta}(x)| \leqq (|\sigma_0| + M)|h| + 2\Delta x$$

for any $x \geqq 0$ and $h > 0$, where the constant C depends only on the bound M, K, C^*, and K_i^*, $i = 0, 2, 3$.

4.3. Estimates on the approximate shock-fronts

We use the notations and estimates in the previous section and define

$$\sigma_{\Delta x,\theta}(x) = \sigma_{(k)} \qquad \text{if } x \in (k\Delta x, (k+1)\Delta x]\,.$$

From Proposition 3.4, we have

Proposition 4.2. *There exists a constant M, independent of Δx, θ, and $U_{\Delta x,\theta}$, such that*

$$TV\{\sigma_{\Delta x,\theta} : [0, \infty)\} = \sum_{k=0}^{\infty} |\sigma_{(k+1)} - \sigma_{(k)}| \leqq M.$$

Proof. For any $k \geqq 1$, and any interaction diamond $\Lambda \subset \{(k-1)\Delta x \leqq x \leqq (k+1)\Delta x\}$, define

$$E_{\Delta x,\theta}(\Lambda) = \begin{cases} 0 & \text{for Case 1,} \\ |\omega_k| + |\alpha_2| + |\alpha_3| & \text{for Case 2,} \\ |\omega_k| & \text{for Case 3,} \\ |\beta_1| & \text{for Case 4;} \end{cases}$$

and

$$Q_{\Delta x,\theta}(\Lambda) = \begin{cases} Q(\Lambda) & \text{for Case 1,} \\ 0 & \text{for Case 2,} \\ |\omega|^2 & \text{for Case 3,} \\ |\alpha_3||\beta_2| & \text{for Case 4.} \end{cases}$$

Then

$$\sum_{\Lambda} E_{\Delta x,\theta}(\Lambda) \leqq \sum_{\Lambda} \frac{1}{\varepsilon'}(F(I) - F(J)) \leqq \frac{1}{\varepsilon'} F(0) := \tilde{M},$$

and

$$\sum_{\Lambda} Q_{\Delta x,\theta}(\Lambda) \leqq \tilde{M},$$

where $\varepsilon' = \sup_\Lambda \max\{K_0^* - |K_{b0}|, K_2^* - |K_{b2}|, K_3^* - |K_{b3}|, K_0^*, K_{s2}, K_{s3}\}$.

From Proposition 3.4, we know that, for some $M > 0$,

$$\sum_{k=0}^{\infty} |\sigma_{(k+1)} - \sigma_{(k)}| \leqq \sum_{\Lambda} (\widetilde{K_{s1}} E_{\Delta x,\theta}(\Lambda) + M Q_{\Delta x,\theta}) \leqq (\widetilde{K_{s1}} + 1) M \leqq M,$$

where $\widetilde{K_{s1}}$ is the constant in (3.13). □

5. Global entropy solutions

In this section we establish the convergence of approximate solutions to a global entropy solution. We also show the nonlinear stability and asymptotic behavior of the strong shock emanating from the wedge vertex under the BV wedge perturbation.

5.1. Convergence of approximate solutions

Following the discussions in Section 4, we can extend $U_{\Delta x,\theta}$ by the constant $U_{k,0}$ continuously across the approximate shock-front to the whole strip $\{k\Delta x < x < (k+1)\Delta x\}$ for each $k \leqq 0$.

Let the line $x = a > 0$ intersect $\partial\Omega_{\Delta x} = \cup\{C_{k-1}C_k, k \geqq 1\}$ at the point $(a, p_a^{\Delta x})$. Similar to [26], by Theorem 4.2, we can prove

Lemma 5.1. *For any $h > 0$ and $x \geqq 0$, there exists a constant $M > 0$ independent of $\Delta x, \theta$, and h such that*

$$\int_{-\infty}^{0} |U_{\Delta x,\theta}(x+h, y + p_{x+h}^{\Delta x}) - U_{\Delta x,\theta}(x, y + p_x^{\Delta x})| dy \leqq M|h|.$$

Denote

$$J(\theta, \Delta x, \phi) = \sum_{k=1}^{\infty} \int_{-\infty}^{0} \phi(k\Delta x, y + y_k) \cdot (U_{k\Delta x+,\theta} - U_{k\Delta x-,\theta}) dy$$

for $\phi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^3)$. Following the steps in [10] (see also [3, 9, 17, 20]), we have

Lemma 5.2. *There exists a null set $N \subset \Pi_{k=0}^\infty(-1,1)$ and a subsequence $\{\Delta x_j\}_{j=1}^\infty \subset \{\Delta x\}$, which tends to 0, such that*

$$J(\theta, \Delta x_j, \phi) \longrightarrow 0 \qquad \textit{when } \Delta x_j \to 0$$

for any $\theta \in \Pi_{k=0}^\infty(-1,1)\backslash N$ and $\phi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^3)$.

To establish the main theorem, we need to estimate the jumps of the approximate strong shock-fronts. Let

$$d_k = \frac{\sigma_{(k-1)}\Delta x - (y_k - y_{k-1}) + \Delta y}{\Delta y}.$$

Then, by the choice of Δx and $\{y_k\}$, we find that $d_k \in (0, 1)$ and depends only on $\{\theta_l\}_{l=1}^{k-1}$. Thus, we define

$$I(x, \Delta x, \theta) = \sum_{k=1}^{[x/\Delta x]} I_k(\Delta x, \theta),$$

where $I_k(\Delta x, \theta) = \mathbf{1}_{(-1,d_k)}(\theta_k)(d_k - 1)\Delta y + \mathbf{1}_{(d_k,1)}(\theta_k)(d_k + 1)\Delta y$, in which $\mathbf{1}_A$ denotes the characteristic function of the set A, and $[x/\Delta x]$ denotes the largest integer less than, or equal to, $x/\Delta x$. Notice that $I_k(\Delta x, \theta)$ is the jump of the function $y = \chi_{\Delta x,\theta}(x)$ at $x = k\Delta x$ and is a measurable function of $(\Delta x, \theta)$, which depends only on $U_{\Delta x,\theta}|_{\{0\leqq x\leqq k\Delta x\}}$ and $\{\theta_l\}_{l=0}^{k}$.

Lemma 5.3.

(i) *For any* $x \geqq 0$, $\Delta x > 0$, *and* $\theta \in \Pi_{k=0}^{\infty}(-1, 1)$,

$$\chi_{\Delta x,\theta}(x) = I(x, \Delta x, \theta) + \int_0^x \sigma_{\Delta x,\theta}(s)ds;$$

(ii) *there exist a null set* N_1 *and a subsequence* $\{\Delta_l\}_{l=1}^{\infty} \subset \{\Delta x_j\}_{j=1}^{\infty}$ *such that*

$$\int_0^{\infty} e^{-x}|I(x, \Delta_l, \theta)|^2 dx \longrightarrow 0 \qquad when\ \ \Delta_l \to 0$$

for any $\theta \in \Pi_{k=0}^{\infty}(-1, 1)\backslash N_1$.

Proof. Part (i) can be obtained by a direct calculation. We thus focus only on part (ii). As in [10], let $d\theta = \Pi_{k=0}^{\infty}(d\theta_k/2)$. Then, for any $l > j$, we have

$$\int I_l I_j d\theta = \int \Pi_{i=1}^{l-1} d\theta_i \left(I_j \int I_l d\theta_l \right) = 0.$$

Therefore, we can deduce

$$\int |I(x, \Delta x, \theta)|^2 d\theta = \sum_{k=1}^{[x/\Delta x]} \int |I_k(\Delta x, \theta)|^2 d\theta \leqq 4|\frac{\Delta y}{\Delta x}|^2 x\Delta x.$$

Then, by choosing a subsequence $\{\Delta_l\}_{l=1}^{\infty} \subset \{\Delta x_j\}_{j=1}^{\infty}$ with $\sum_{l=0}^{\infty} \Delta_l < \infty$ as in Lemma 5.2, we arrive at (ii). □

By Theorem 4.2, Proposition 4.2, and Lemmas 5.1–5.2, we have

Theorem 5.1 (Existence and stability)**.** *There exist* $\varepsilon > 0$ *and* $C > 0$ *such that, if* (1.10) *holds, then, for each* $\theta \in (\Pi_{k=0}^{\infty}(-1, 1))\backslash(N \cup N_1)$, *there exist a subsequence* $\{\Delta_l\}_{l=1}^{\infty}$ *of mesh sizes with* $\Delta_l \to 0$ *as* $l \to \infty$ *and a pair of functions* $U_\theta \in L^\infty(\Omega; O_\varepsilon(U_+))$ *and* $\chi_\theta \in Lip([0, \infty))$ *with* $\chi_\theta(0) = 0$ *such that*

(i) $U_{\Delta_l,\theta}(x,\cdot)$ *converges to* $U_\theta(x,\cdot)$ *in* $L^1(-\infty, g(x))$ *for every* $x > 0$*, and* U_θ *is a global entropy solution of problem* (1.6) *and* (1.8)–(1.9) *in* Ω *and satisfies* (1.11)–(1.12)*;*

(ii) $\chi_{\Delta_l,\theta}$ *converges to* χ_θ *uniformly in any bounded* x*-interval;*

(iii) $\sigma_{\Delta_l,\theta}$ *converges to* $\sigma_\theta \in BV([0,\infty))$ *a.e. with* $|\sigma_\theta - \sigma_0| \leqq \hat{\varepsilon} < \epsilon$ *and*

$$\chi_\theta(x) = \int_0^x \sigma_\theta(t)dt.$$

In addition, if θ *is equidistributed, then* $\chi_\theta(x) < g(x)$ *for any* $x > 0$ *with* (1.13) *and the Rankine-Hugoniot conditions a.e. along the curve* $\{y = \chi_\theta(x)\}$.

The proof of (i) and (ii), and the convergence proof of $\sigma_{\Delta_k,\theta}$ in (iii), can be carried out in the same way as in the standard cases (see [3, 10, 11, 25]) by using the structure of the approximate solutions. In particular, for any $\varphi \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$,

$$\begin{aligned}&\int_{\Omega_{\Delta x,\theta}} (\rho^{\Delta x,\theta}u^{\Delta x,\theta}\varphi_x + \rho^{\Delta x,\theta}v^{\Delta x,\theta}\varphi_y)dxdy\\ &\quad = \int_\Omega \chi_{\Omega_{\Delta x,\theta}}(\rho^{\Delta x,\theta}u^{\Delta x,\theta}\varphi_x + \rho^{\Delta x,\theta}v^{\Delta x,\theta}\varphi_y)dxdy\end{aligned}$$

weak-star converges, hence the initial condition is satisfied by the trace theorem for BV functions (cf. [24]). Similarly, the boundary condition can be shown to be satisfied. The equality in (iii) can be deduced from Lemma 5.3 and the result on the convergence of $\{\chi_{\Delta_l,\theta}\}$ and $\{\sigma_{\Delta_l,\theta}\}$.

5.2. Asymptotic behavior of the strong shock

As in Theorem 5.1, let $\theta \in (\Pi_{k=0}^\infty(-1,1))\backslash(N \cup N_1)$ be equidistributed, and let U_θ be the solution and χ_θ its shock-front, respectively. By Theorem 5.1, we conclude that the solution U_θ contains, at most, countable shock-fronts and countable points of wave interactions. Moreover, we can modify the solution U_θ such that U_θ is continuous except along the shock curves and the points of wave interactions (see [9, 11, 19]). Then we have

Lemma 5.4. *The solution* U_θ *and its strong shock-front* χ_θ *satisfy*

$$\lim_{x\to\infty} TV\{\arctan(v_\theta(x,\cdot)/u_\theta(x,\cdot)) \;:\; (\chi_\theta(x), g(x))\} = 0$$

and

$$\lim_{x\to\infty} TV\{\rho_\theta(x,\cdot) \;:\; (\chi_\theta(x), g(x))\} = 0.$$

Proof. Let $\{\Delta_l\}$ be the sequence given as in Theorem 5.1, and let $E_{\Delta_l,\theta}(\Lambda)$ and $Q_{\Delta_l,\theta}(\Lambda)$ be the quantities defined in Proposition 4.2. As in [11], we denote by $dE_{\Delta_l,\theta}$ and $dQ_{\Delta_l,\theta}$ the measures assigned to $E_{\Delta_l,\theta}(\Lambda)$ and $Q_{\Delta_l,\theta}(\Lambda)$ to the center of Λ.

The boundedness of $E_{\Delta_l,\theta}(\Lambda)$ and $Q_{\Delta_l,\theta}(\Lambda)$ in Proposition 4.2 implies the compactness of $\{dE_{\Delta_l,\theta}\}$ and $\{dQ_{\Delta_l,\theta}\}$. We can then select their subsequences (still denoted by themselves) so that $\Delta_l \to 0$ and the limits

$$dE_{\Delta_l,\theta} \to dE_\theta \quad \text{and} \quad dQ_{\Delta_l,\theta} \to dQ_\theta$$

exist in the weak-star topology in the measure space, and the limits are finite on Ω. Therefore, for any $\delta > 0$, we can choose $x_\delta > 0$ independent of $\{U_{\Delta_l,\theta}\}$ and $\{\Delta_l\}$ such that, for any $l > 0$,

$$\sum_{k \geqq [x_\delta/\Delta_l]} E_{\lambda,\theta}(\Lambda_{k,n}) < \delta, \qquad \sum_{k \geqq [x_\delta/\Delta_l]} Q_{\lambda,\theta}(\Lambda_{k,n}) < \delta.$$

Moreover, let $X_\delta^1 = (x_\delta, y_\delta^1)(\,or\ X_\delta^3 = (x_\delta, y_\delta^3))$ be the point lying in $\chi_{\Delta_l,\theta}$ (or $\partial\Omega_{\Delta_k}$). Let $\chi^3_{\Delta_l,\theta}$ be the minimum approximate 3-characteristics in $U_{\Delta_l,\theta}$ emanating from the point X_δ^1, and $\chi^1_{\Delta_l,\theta}$ the maximum approximate 1-characteristic in $U_{\Delta_l,\theta}$ emanating from the point X_δ^3. From the construction of the approximate solutions, we have

$$|\chi^j_{\Delta_l,\theta}(x+h) - \chi^j_{\Delta_l,\theta}(x)| \leqq M(|h| + \Delta_l), \qquad j = 1, 3,$$

for some constant $M > 0$ which is independent of Δx and θ. Then, for $\theta \in (\prod_{k=0}^{\infty}(-1,1))\backslash(N \cup N_1)$, we can select a subsequence (still denoted by $\{\Delta_l\}_{l=1}^{\infty}$) such that

$$\chi^j_{\Delta_l,\theta} \to \chi^j_\theta \qquad \text{uniformly on every bounded interval as } \Delta_l \to 0$$

for some $\chi^j_\theta \in Lip$ with $(\chi^j_\theta)'$ bounded.

Let the characteristics $y = \chi^3_\theta(x)$ and $y = \chi^1_\theta(x)$ intersect $\partial\Omega$ and $y = \chi_\theta(x)$, respectively, at $(t_\delta^3, \chi_\theta(t_\delta^3))$ and $(t_\delta^1, \chi_\theta(t_\delta^1))$ for some t_δ^3 and t_δ^1. Then, since the flow angle $\arctan(v/u)$ and the density ρ are invariant across the contact discontinuities, by the approximate conservation laws for the weak j-waves, $j = 1, 3$, we can deduce (in the same way as in [11]) that

$$TV\{\arctan\big(v_{\Delta_l,\theta}(x-,\cdot)/u_{\Delta_l,\theta}(x-,\cdot)\big) \ :\ (\chi_{\Delta_l,\theta}(x), g_l(x))\} \leqq C\,\delta$$

and

$$TV\{\rho_{\Delta_l,\theta}(x-,\cdot) \ :\ (\chi_{\Delta_l,\theta}(x), g_l(x))\} \leqq C\delta$$

for $x > 2(t_\delta^1 + t_\delta^3)$, where $C > 0$ is independent of δ, x, $U_{\Delta_l,\theta}$, and Δ_l.

Thus, taking the limit as $\Delta_l \to 0$ and using Theorem 5.1 and the regularity of U_θ yield, for $x > 2(t_\delta^1 + t_\delta^3)$,

$$TV\{\arctan(v_\theta(x-,\cdot)/u_\theta(x-,\cdot)) \ :\ (\chi_\theta(x), g_l(x))\} \leqq C\,\delta$$

and

$$TV\{\rho_\theta(x-,\cdot) \ :\ (\chi_\theta(x), g_l(x))\} \leqq C\,\delta. \quad \square$$

Theorem 5.2.

(i) *Let* $\omega_\infty = \lim_{x\to\infty} \arctan(g'(x+))$. *Then*

$$\lim_{x\to\infty} \sup\{|\arctan(v_\theta(x,y)/u_\theta(x,y)) - \omega_\infty| \;:\; \chi_\theta(x) < y < g(x)\} = 0.$$

(ii) *There exist constants* ρ_∞ *and* σ_∞ *such that*

$$\lim_{x\to\infty} \sup\{|\rho_\theta(x,y) - \rho_\infty| \;:\; \chi_\theta(x) < y < g(x)\} = 0,$$
$$\lim_{x\to\infty} \sup|\sigma_\theta(x) - \sigma_\infty| = 0.$$

Proof. Let $U_{l,\theta} = U_{\Delta_l,\theta}$, $\sigma_{l,\theta} = \sigma_{\Delta_l,\theta}$, and $\chi_{l,\theta} = \chi_{\Delta_l,\theta}$, where Δ_l is chosen as in the proof of Lemma 5.4. Following the construction of the approximate solutions, we conclude that, for every $x > 0$,

$$\arctan(v_{l,\theta}/u_{l,\theta})|_{\Gamma_k} = \arctan\left(\frac{y_{k+1}-y_k}{\Delta x}\right) = \arctan(g'(\eta_k))$$

for some $\eta_k \in [k\Delta x, (k+1)\Delta x)$. Then, choosing x_δ so that $|g'(x+) - g'(\infty)| < \delta$ for $x > x_\delta$, we have

$$\begin{aligned}&\sup\{|\arctan(v_{l,\theta}(x,y)/u_{l,\theta}(x,y)) - \omega_\infty| \;:\; \chi_\theta(x) < y < g(x)\}\\ &\quad\leqq TV\{\arctan\left(v_{l,\theta}(x,\cdot)/u_{l,\theta}(x,\cdot)\right) \;:\; (\chi_\theta(x), g(x))\} + M\delta \qquad \text{for } x > 2x_\delta.\end{aligned}$$

Therefore, taking the limit as $\Delta_l \to 0$, by Theorem 5.1 and Lemma 5.4, and by the regularity of U_θ, we can deduce part (i).

Moreover, from Theorem 5.1, we also have

$$\sigma_\theta \in BV(\mathbb{R}_+),\;\; |\sigma_\theta - \sigma_0| \leqq \hat{\varepsilon} < \varepsilon, \qquad G(\sigma_\theta) \in BV(\mathbb{R}_+; O_\varepsilon(U_+)).$$

Let $\sigma_\infty = \lim_{x\to\infty} \sigma_\theta(x+)$ and $U_\infty = \lim_{x\to\infty} G(\sigma_\theta(x))$. Then part (ii) follows from Lemma 5.4. □

6. Extension to the adiabatic Euler flows past Lipschitz wedges

In this section, we turn to the adiabatic Euler equations (1.1) for steady supersonic flows, which can be written in the following conservation form:

$$W(U)_x + H(U)_y = 0, \qquad U = (u, v, p, \rho) \tag{6.1}$$

with

$$W(U) = \left(\rho u, \rho u^2 + p, \rho u v, \rho u\left(h + \frac{u^2+v^2}{2}\right)\right),$$
$$H(U) = \left(\rho v, \rho u v, \rho v^2 + p, \rho v\left(h + \frac{u^2+v^2}{2}\right)\right),$$

and $h = \frac{\gamma p}{(\gamma-1)\rho}$. As in Section 1, the problem of supersonic Euler flows governed by (6.1) past Lipschitz wedges can be formulated as problem (1.8)–(1.9) for system (6.1) in the region below the lower edge Γ of the wedge.

Definition 6.1. (Entropy Solutions). A BV function $U = U(x, y)$ is called an entropy solution of problem (6.1) and (1.8)–(1.9) provided that

(i) U is a weak solution of (6.1) and satisfies

$$(u, v) \cdot \mathbf{n}|_{y=g(x)} = 0 \qquad \text{in the trace sense;}$$

(ii) U satisfies the entropy inequality:

$$(\rho u S)_x + (\rho v S)_y \leqq 0 \tag{6.2}$$

in the sense of distributions in Ω including the boundary.

6.1. Riemann problems and Riemann solutions

The eigenvalues of system (6.1) are the solutions of the fourth-order polynomial equation:

$$(v - \lambda u)^2 \Big((v - \lambda u)^2 - c^2(1 + \lambda^2)\Big) = 0,$$

where $c^2 = \gamma p/\rho$. We then have

$$\lambda_j = \frac{uv + (-1)^j c\sqrt{u^2 + v^2 - c^2}}{u^2 - c^2}, \quad j = 1, 4, \qquad \lambda_i = v/u, \; i = 2, 3. \tag{6.3}$$

When the flow is supersonic (i.e., $u^2 + v^2 > c^2$), system (6.1) is hyperbolic and the corresponding eigenvectors for $u \neq 0$ are

$$\begin{aligned}
&\mathbf{r}_j = \kappa_j(-\lambda_j, 1, \rho(\lambda_j u - v), \rho(\lambda_j u - v)/c^2)^\top, \; j = 1, 4,\\
&\mathbf{r}_2 = (1, v/u, 0, 0)^\top,\\
&\mathbf{r}_3 = (0, 0, 0, 1)^\top,
\end{aligned}$$

where κ_j are chosen so that $\mathbf{r}_j \cdot \nabla\lambda_j = 1$, since the j-characteristic fields are genuinely nonlinear, $j = 1, 4$. Note that the second and third characteristic fields are always linearly degenerate: $\mathbf{r}_j \cdot \nabla\lambda_j = 0$, $j = 2, 3$.

6.1.1. Wave curves in the phase space. Similarly to Section 2, the contact discontinuity curves $C_i(U_0)$ through U_0 are

$$C_i(U_0): \quad p = p_0, \quad w = v/u = v_0/u_0, \quad i = 2, 3, \tag{6.4}$$

which describe compressible vortex sheets. Moreover, the rarefaction wave curves $R_j(U_0)$ in the phase space through U_0 are

$$R_j(U_0): \quad dp = c^2 d\rho, \quad du = -\lambda_j dv, \quad \rho(\lambda_j u - v)dv = dp, \qquad j = 1, 4. \tag{6.5}$$

It is easy to check that $\frac{d\lambda_j}{d\rho}$ along $R_j(U_0)$, $j = 1, 4$, satisfies

$$\frac{d\lambda_1}{d\rho}|_{R_1(U_0)} < 0, \qquad \frac{d\lambda_4}{d\rho}|_{R_4(U_0)} > 0.$$

Similarly, the Rankine-Hugoniot conditions for (6.1) are

$$\sigma[\rho u] = [\rho v], \tag{6.6}$$
$$\sigma[\rho u^2 + p] = [\rho uv], \tag{6.7}$$
$$\sigma[\rho uv] = [\rho v^2 + p], \tag{6.8}$$
$$\sigma\left[\rho u\left(h + \frac{u^2+v^2}{2}\right)\right] = \left[\rho v\left(h + \frac{u^2+v^2}{2}\right)\right]. \tag{6.9}$$

We then have

$$(v_0 - \sigma u_0)^2\left((v_0 - \sigma u_0)^2 - \bar{c}_0^2(1+\sigma^2)\right) = 0,$$

where $\bar{c}_0^2 = \frac{c_0^2}{b_0}\frac{\rho}{\rho_0}$ and $b_0 = \frac{\gamma+1}{2} - \frac{\gamma-1}{2}\frac{\rho}{\rho_0}$. This implies

$$\sigma = \sigma_j := \frac{u_0 v_0 + (-1)^j \bar{c}_0\sqrt{u_0^2 + v_0^2 - \bar{c}_0^2}}{u_0^2 - \bar{c}_0^2},\ j = 1, 4, \quad \sigma = \sigma_i = v_0/u_0,\ i = 2, 3.$$

Substituting $\sigma_i, i = 2, 3$, into (6.6)–(6.9), we get the same $C_i(U_0), i = 2, 3$, as defined in (6.4); whilest substituting $\sigma_j, j = 1, 4$, into (6.6)–(6.9), we get the j^{th}-shock wave curve $S_j(U_0)$ through U_0:

$$S_j(U_0): \quad [p] = \frac{c_0^2}{b_0}[\rho], \quad [u] = -\sigma_j[v], \quad \rho_0(\sigma_j u_0 - v_0)[v] = [p], \quad \text{j} = 1, 4. \tag{6.10}$$

Notice that $S_j(U_0)$ contacts with $R_j(U_0)$ at U_0 up to second-order and

$$\frac{d\sigma_1}{d\rho}|_{S_1(U_0)} < 0, \qquad \frac{d\sigma_4}{d\rho}|_{S_4(U_0)} > 0. \tag{6.11}$$

Similarly, Lemma 2.1 still holds for system (6.1), which implies that the entropy inequality (6.2) is equivalent to (2.19) or

$$\begin{aligned} \lambda_j(back) &< \sigma_j < \lambda_j(front), \qquad j = 1, 4, \\ \sigma_1 &< \lambda_{2,3}(back), \\ \lambda_{2,3}(front) &< \sigma_4. \end{aligned}$$

Here we only show the equivalence between (6.2) and (2.19) when the back state $U_+ = (u_+, 0, p_+, \rho_+)$ with $u_+ > 0$ so that $\sigma < 0$. First, the entropy condition (6.2) is equivalent to

$$\sigma[\rho u(\ln p - \gamma \ln \rho)] < [\rho v(\ln p - \gamma \ln \rho)]. \tag{6.12}$$

From (6.6), we know that

$$\sigma\rho u = \sigma\rho_+ u_+ + \rho v. \tag{6.13}$$

Substituting (6.13) into (6.12), we get $\ln(p/p_+) - \gamma\ln(\rho/\rho_+) > 0$. That is,

$$p/p_+ > (\rho/\rho_+)^\gamma. \tag{6.14}$$

On the other hand, from (6.6) and (6.9), we have the Bernoulli law:

$$h + \frac{(u^2+v^2)}{2} = h_+ + \frac{u_+^2}{2}, \tag{6.15}$$

which implies that

$$\frac{p}{p_+} = H(t) := \frac{(\gamma-1)-(\gamma+1)t}{(\gamma-1)t-(\gamma+1)} \quad \text{with } t = \rho/\rho_+.$$

Then the function $G(t) := H(t)/t^\gamma$ is strictly decreasing in t, since

$$G'(t) = \frac{\gamma(1-\gamma^2)(t+1)^2}{t^{\gamma+1}((\gamma-1)t-(\gamma+1))^2} < 0.$$

Because $G(1) = 1$ and $G(t) > G(1)$ from (6.14), we conclude $t < 1$, which implies that $\rho < \rho_+$.

6.1.2. Lateral Riemann problem. Again, the simplest case of problem (6.1) and (1.8)–(1.9) occurs when $g \equiv 0$. It can be shown that, if $g \equiv 0$, then problem (6.1) yields an entropy solution that consists of a constant state U_- and a constant state U_+, with $U_+ = (u_+, 0, p_+, \rho_+)$ and $u_+ > c_+ > 0$ in the subdomain of Ω separated by a straight shock emanating from the vertex. That is to say that the state ahead of the shock-front is U_-, while the state behind the shock-front is U_+ (see Fig. 4). When the angle between the flow direction of the front state and the wedge boundary at a boundary vertex is larger than π, an entropy solution will contain a rarefaction wave that separates the front state from the back state (see Fig. 5).

6.1.3. Riemann problem involving only weak waves. Consider the following initial value problem:

$$\begin{aligned} &W(U)_x + H(U)_y = 0,\\ &U|_{x=x_0} = \underline{U} = \begin{cases} U_a, & y > y_0,\\ U_b, & y < y_0, \end{cases} \end{aligned} \tag{6.16}$$

where U_b and U_a are constant states. As before, we can parametrize the physically admissible elementary solution curve in a neighborhood of U_+, $O_\varepsilon(U_+)$, by $\alpha_j \mapsto \Phi_j(\alpha_j; U_b)$, with $\Phi \in C^2$, $\Phi_j|_{\alpha_j=0} = U_b$, and $\frac{\partial\Phi_j}{\partial\alpha_j}|_{\alpha_j=0} = \mathbf{r}_j(U_b)$.

Denote $\Phi(\alpha_4,\alpha_3,\alpha_2,\alpha_1; U_b) = \Phi_4(\alpha_4, \Phi_3(\alpha_3, \Phi_2(\alpha_2, \Phi_1(\alpha_1; U_b))))$. Then we have

Lemma 6.1. *There exists $\varepsilon > 0$ such that, for any states $U_a, U_b \in O_\varepsilon(U_+)$, problem* (6.16) *yields a unique admissible solution consisting of four elementary waves. In addition, state U_a can be represented by $U_a = \Phi(\alpha_4,\alpha_3,\alpha_2,\alpha_1; U_b)$ with $\Phi|_{\alpha_1=\alpha_2=\alpha_3=\alpha_4=0} = U_b$ and $\frac{\partial\Phi}{\partial\alpha_i}|_{\alpha_1=\alpha_2=\alpha_3=\alpha_4=0} = \mathbf{r}_i(U_b)$, $i = 1,2,3,4$.*

Similar to the argument for Lemma 2.3, we have

Lemma 6.2. *It can be shown that $U_+ = (u_+, 0, p_+, \rho_+)$ with $u_+ > 0$, $\kappa_1(U_+) = \kappa_4(U_+) > 0$, which implies $\kappa_j(U) > 0$, $j = 1,4$, for any state $U \in O_\varepsilon(U_+)$ since κ_j are continuous, $j = 1,4$.*

6.1.4. Riemann problem involving a strong 1-shock. For simplicity, we use the notation $\{U_b, U_a\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ to denote that $U_a = \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U_b)$ throughout this section. For any $U \in S_1(U_-)$, we also use $\{U_-, U\} = (\sigma, 0, 0, 0)$ to denote the 1-shock that connects U_- and U with speed σ. Then we have

Lemma 6.3. *Let $\{U_-, U_+\} = (\sigma_0, 0, 0, 0)$, $\rho_+ > \rho_-$, and $\gamma > 1$. Then*

$$\sigma_0 < 0, \qquad u_+ < u_- < (1 + 1/\gamma)u_+.$$

The slight difference in the proof of this lemma from that of Lemma 2.4 lies in the fact that $\bar{c}_0$ and $\bar{c}_-$ are different in system (6.1). However, we still have $c_0 > \bar{c}_0$ and $c_- < \bar{c}_-$, and hence there is no problem carrying out the same steps.

Lemma 6.4. *There exists a neighborhood $O_\varepsilon(U_+)$ of U_+ such that the shock polar $S_1(U_-) \cap O_\varepsilon(U_+)$ can be parametrized by the shock speed σ as*

$$\sigma \to G(\sigma)$$

with $G \in C^2$ near σ_0 and $G(\sigma_0) = U_+$.

Lemma 6.5. *For states U_- and U_+ in the unperturbed solution,*

$$P := u_+ \left(h_- + \frac{u_-^2 + v_-^2}{2} - \frac{u_+^2}{2} \right) + \left(\frac{c_+^2}{\gamma - 1} + u_+^2 \right) (u_+ - u_-) > 0.$$

Proof. From the Bernoulli law (6.15), we have

$$P = \frac{c_+^2}{\gamma - 1}(2u_+ - u_-) + u_+^2(u_+ - u_-). \tag{6.17}$$

Following the same steps as in Lemma 2.4, we have the following facts from Lemma 6.3: $u_+(u_- - u_+) < c_+^2/\gamma$, $u_- < (1 + 1/\gamma)u_+$, which implies

$$P > \frac{\gamma u_+(u_- - u_+)}{\gamma - 1} ((1 + 1/\gamma)u_+ - u_-) > 0. \quad \square$$

Lemma 6.6. *Let $A = \nabla_U H(U_+) - \sigma_0 \nabla_U W(U_+)$. Then*

$$\det A > 0, \ \det(A\mathbf{r}_4, A\mathbf{r}_3, A\mathbf{r}_2, A\mathbf{r}_1)|_{U=U_+} > 0,$$
$$\det(A\mathbf{r}_4, A\mathbf{r}_3, A\mathbf{r}_2, AG_\sigma(\sigma_0))|_{U=U_+} > 0.$$

Proof. A direct calculation shows that

$$A = \begin{pmatrix} -\sigma_0\rho_+ & \rho_+ & 0 & -\sigma_0 u_+ \\ -2\sigma_0\rho_+ u_+ & \rho_+ u_+ & -\sigma_0 & -\sigma_0 u_+^2 \\ 0 & -\sigma_0\rho_+ u_+ & 1 & 0 \\ -\sigma_0\rho_+(\frac{c_+^2}{\gamma-1} + \frac{3}{2}u_+^2) & \rho_+(\frac{c_+^2}{\gamma-1} + \frac{u_+^2}{2}) & -\frac{\gamma}{\gamma-1}\sigma_0 u_+ & -\frac{1}{2}\sigma_0 u_+^3 \end{pmatrix},$$

and

$$\begin{aligned}
\mathbf{r}_j(U_+) &= \frac{(-1)^{j-1}\kappa_j(U_+)}{c_+\sqrt{u_+^2 - c_+^2}} \\
&\quad \times \left(c_+^2, (-1)^{j-1}c_+\sqrt{u_+^2 - c_+^2}, \rho_+ u_+ c_+, \rho_+ u_+\right)^\top, \ \ j = 1, 4, \\
\mathbf{r}_2(U_0) &= (1, 0, 0, 0)^\top, \\
\mathbf{r}_3(U_0) &= (0, 0, 0, 1)^\top.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
A\mathbf{r}_j(U_+) &= \frac{\kappa_j(U_+)\rho_+(\lambda_{j+} - \sigma_0)}{\lambda_{j+}} \\
&\quad \times \left(1, u_+, u_+\lambda_{j+}, \left(\frac{c_+^2}{\gamma - 1} + \frac{u_+^2}{2}\right)\right)^\top \quad \text{for } j = 1, 4,
\end{aligned}$$

$$\begin{aligned}
A\mathbf{r}_2(U_+) &= -\sigma_0\rho_+\left(1, 2u_+, 0, \frac{c_+^2}{\gamma - 1} + \frac{3}{2}u_+^2\right)^\top, \\
A\mathbf{r}_3(U_+) &= -\sigma_0 u_+\left(1, u_+, 0, \frac{u_+^2}{2}\right)^\top,
\end{aligned}$$

and

$$\begin{aligned}
AG_\sigma(\sigma_0) &= W(U_+) - W(U_-) \\
&= -\frac{\rho_- v_-}{\sigma_0}\left(1, u_-, \sigma_0 u_-, h_- + \frac{u_-^2 + v_-^2}{2}\right)^\top,
\end{aligned}$$

where $\lambda_{j+} = \lambda_j(U_+)$, $j = 1, 4$.

By Lemmas 6.3 and 6.5, we find

$$\begin{aligned}
\det A &= \frac{\sigma_0^2\rho_+^2 u_+^2}{\gamma - 1}(\lambda_{1+}^2 - \sigma_0^2)(u_+^2 - c_+^2) > 0, \\
\det(A\mathbf{r}_4, A\mathbf{r}_3, A\mathbf{r}_2, A\mathbf{r}_1)|_{U=U_+} &= \frac{(\kappa_4(U_+))^2\sigma_0^2\rho_+^3 u_+^3 c_+^2}{(\gamma - 1)\lambda_{1+}\lambda_{4+}} \\
&\quad \times(\sigma_0 - \lambda_{4+})(\sigma_0 - \lambda_{1+})(\lambda_{4+} - \lambda_{1+}) > 0,
\end{aligned}$$

and

$$\begin{aligned}
&\det(A\mathbf{r}_4, A\mathbf{r}_3, A\mathbf{r}_2, AG_\sigma(\sigma_0))|_{U=U_+} \\
&\quad = \frac{\kappa_4(U_+)\sigma_0\rho_- v_-\rho_+^2 u_+}{\lambda_{4+}}(\sigma_0 - \lambda_{4+})(u_+\lambda_{4+}P + \sigma_0 u_- Q) > 0,
\end{aligned}$$

since $P > 0$ and $Q = -c_+^2/(\gamma - 1) < 0$. □

6.2. Estimates on wave interactions and reflections

We now make essential estimates as in Section 3. The interaction estimates are similar and the corresponding Figs. 5–7 are the same except that 2-contact discontinuities and 3-waves in Section 3 are now replaced by 2, 3-contact discontinuities and 4-waves, respectively.

6.2.1. Estimates on weak wave interactions. First we have

Proposition 6.1. *Suppose that* $U_b, U_m, U_a \in O_\varepsilon(U_+)$ *are three states with* $\{U_b, U_m\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\{U_m, U_a\} = (\beta_1, \beta_2, \beta_3, \beta_4)$, *and* $\{U_b, U_a\} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ (cf. Fig. 5). *Then*

$$\gamma_i = \alpha_i + \beta_i + O(1)\triangle(\alpha, \beta),$$

where $\triangle(\alpha, \beta) = (|\alpha_4| + |\alpha_3| + |\alpha_2|)|\beta_1| + |\alpha_4|(|\beta_2| + |\beta_3|) + \sum_{j=1,4} \triangle_j(\alpha, \beta)$ *with*

$$\triangle_j(\alpha, \beta) = \begin{cases} 0, & \alpha_j \geqq 0,\ \beta_j \geqq 0, \\ |\alpha_j||\beta_j|, & \text{otherwise.} \end{cases}$$

Since, by Lemma 6.6,

$$\det\left(\frac{\partial \Phi(\gamma_4, \gamma_3, \gamma_2, \gamma_1; U_l)}{\partial(\gamma_4, \gamma_3, \gamma_2, \gamma_1)}\right)|_{\gamma_1=\gamma_2=\gamma_3=\gamma_4=0}$$
$$= \det(A\mathbf{r}_4, A\mathbf{r}_3, A\mathbf{r}_2, A\mathbf{r}_1)|_{U=U_+} > 0,$$

then, by the implicit function theorem, there exists $(\gamma_4, \gamma_3, \gamma_2, \gamma_1)$ as a C^2 function of $(\beta_4, \beta_3, \beta_2, \beta_1, \alpha_4, \alpha_3, \alpha_2, \alpha_1; U_b)$ so that

$$\Phi(\beta_4, \beta_3, \beta_2, \beta_1; \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U_b)) = \Phi(\gamma_4, \gamma_3, \gamma_2, \gamma_1; U_b).$$

Then we follow the proof of Proposition 3.1 to arrive at the result.

6.2.2. Estimates on the weak wave reflections on the boundary. We use the same notation as in Section 3.2 for $C_k(a_k, b_k)$ with $a_{k+1} > a_k > 0$, $\omega_{k,k+1}$, ω_k, Ω_k, Γ_k, and the outer normal vector $\mathbf{n}_k$ to Γ_k (cf. Fig. 7). Then we consider the initial-boundary value problem with $\underline{U}$ a constant state:

$$\begin{cases} (6.1) & \text{in } \Omega_{k+1}, \\ U|_{x=a_k} = \underline{U}, & \\ (u, v) \cdot \mathbf{n}_{k+1} = 0 & \text{on } \Gamma_{k+1}. \end{cases}$$

Proposition 6.2. *Let* $\{U_b, U_m\} = (\alpha_4, \alpha_3, \alpha_2, 0)$ *and* $\{U_m, U_k\} = (0, 0, 0, \beta_1)$ *with*

$$(u_k, v_k) \cdot \mathbf{n}_k = 0.$$

Then there exists U_{k+1} *such that*

$$\{U_b, U_{k+1}\} = (0, 0, 0, \delta_1) \qquad \text{and } (u_{k+1}, v_{k+1}) \cdot \mathbf{n}_{k+1} = 0.$$

Furthermore,

$$\delta_1 = \beta_1 + K_{b4}\alpha_4 + K_{b3}\alpha_3 + K_{b2}\alpha_2 + K_{b0}\omega_k,$$

where K_{b4}, K_{b3}, K_{b2}*, and* K_{b0} *are* C^2 *functions of* $(\alpha_4, \alpha_3, \alpha_2, \beta_1, \omega_k; U_b)$ *satisfying*

$$K_{b4}|_{\{\omega_k=\alpha_4=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}} = 1,$$
$$K_{b2}|_{\{\omega_k=\alpha_4=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}} = K_{b3}|_{\{\omega_k=\alpha_4=\alpha_3=\alpha_2=\beta_1=0, U_b=U_+\}} = 0,$$

and K_{b0} *is bounded.*

Since

$$\begin{aligned}&\frac{\partial}{\partial\delta_1}(\Phi(0,0,0,\delta_1;U_b)\cdot(\mathbf{n}_{k+1},0,0))|_{\{\delta_1=0,U_b=U_+,\omega_{k,k+1}=0\}}\\ &\quad= \kappa_1(U_+)(-\lambda_{1+}, 1, \rho_+u_+\lambda_{1+}, \frac{\rho_+u_+\lambda_{1+}}{c_+^2})\cdot(0,1,0,0) > 0,\end{aligned}$$

we know from the implicit function theorem that δ_1 can be solved as a C^2 function of $(\alpha_4, \alpha_3, \alpha_2, \beta_1, \omega_{k-1,k}, \omega_k; U_b)$ such that

$$\begin{aligned}&\Phi(0,0,0,\beta_1;\Phi(\alpha_4,\alpha_3,\alpha_2,0;U_b))\cdot(\mathbf{n}_k,0,0)\\ &\quad= \Phi(0,0,0,\delta_1;U_b)\cdot(\mathbf{n}_{k+1},0,0).\end{aligned} \tag{6.18}$$

Then, following the argument in the proof of Proposition 3.2 yields the results.

6.2.3. Estimate on the boundary perturbation of the strong shock. We have

Proposition 6.3. *For* $\varepsilon > 0$ *sufficiently small, there exists* $\hat{\varepsilon} = \hat{\varepsilon}(\varepsilon) < \varepsilon$ *so that* $G(O_{\hat{\varepsilon}}(\sigma_0)) \subset O_\varepsilon(U_+)$ *and, when* $|\omega_k| < \varepsilon$*, the following equation*

$$G(\sigma)\cdot(\mathbf{n}_k, 0, 0) = 0 \tag{6.19}$$

yields a unique solution $\sigma_k \in O_{\hat{\varepsilon}}(\sigma_0)$*. Moreover, we have*

$$\sigma_{k+1} = \sigma_k + K_{bs}\omega_k + O(1)|\omega_k|^2, \tag{6.20}$$

where $|K_{bs}|$ *is bounded.*

Proof. It suffices to find a solution $\sigma = \sigma(h)$ to the following equation:

$$G(\sigma)\cdot(-\sin h, \cos h, 0, 0) = 0. \tag{6.21}$$

Differentiating both sides of (6.21) in σ, following the part of the calculation we had in Lemma 6.6, and denoting by (A_{ij}^*) the co-factor matrix of (a_{ij}), we obtain

$$\begin{aligned}&\frac{\partial}{\partial\sigma}(G(\sigma)\cdot(-\sin h, \cos h, 0, 0))|_{\{\sigma=\sigma_0, h=0\}}\\ &\quad= \frac{1}{\det A}(A_{12}^*, A_{22}^*, A_{32}^*, A_{42}^*)\cdot AG_\sigma(\sigma_0) > 0.\end{aligned}$$

By the implicit function theorem, we can find a unique C^2 function $\sigma = \sigma(h)$ with $\sigma(0) = \sigma_0$, which solves (6.21) in some neighborhood of $(\sigma, h) = (\sigma_0, 0)$. Then $\sigma(\omega_j) = \sigma_j$, $j = k, k+1$, and by Taylor's expansion formula, we have the desired estimates (6.20). □

6.2.4. Estimates on the interaction between the strong shock and weak waves.

Proposition 6.4. *Let* $U_m, U_a \in O_\varepsilon(U_+)$ *with*

$$\{G(\sigma), U_m\} = (0, \alpha_2, \alpha_3, \alpha_4), \qquad \{U_m, U_a\} = (\beta_1, \beta_2, \beta_3, 0).$$

Then there exists a unique $(\sigma', \delta_2, \delta_3, \delta_4)$ *such that the Riemann problem* (6.16) *with* $U_b = U_-$ *yields an admissible solution consisting of a strong* 1*-shock, two contact discontinuities of strengths* δ_2 *and* δ_3*, and a weak* 4*-wave of strength* δ_4*:*

$$\{U_-, U_a\} = (\sigma', \delta_2, \delta_3, \delta_4).$$

Moreover,

$$\sigma' = \sigma_0 + K_{s1}\beta_1 + O(1)\Delta, \quad \delta_2 = \alpha_2 + \beta_2 + K_{s2}\beta_1 + O(1)\Delta,$$

$$\delta_3 = \alpha_3 + \beta_3 + K_{s3}\beta_1 + O(1)\Delta, \quad \delta_4 = \alpha_4 + K_{s4}\beta_1 + O(1)\Delta,$$

where

$$|K_{s4}| < 1 \quad \text{and} \quad |K_{s3}|, |K_{s2}|, \text{ and } |K_{s1}| \text{ are bounded},$$

and $\Delta = |\alpha_3||\beta_1| + |\alpha_2||\beta_1| + |\alpha_4||\beta_1| + |\alpha_4||\beta_2| + |\alpha_4||\beta_3|$*. Furthermore, we can write the estimates in a more precise fashion:*

$$\delta_4 = \alpha_4 + \widetilde{K_{s4}}\beta_1 + O(1)\tilde{\Delta}, \qquad \delta_3 = \alpha_3 + \widetilde{K_{s3}}\beta_1 + O(1)\tilde{\Delta},$$
$$\delta_2 = \alpha_2 + \widetilde{K_{s2}}\beta_1 + O(1)\tilde{\Delta}, \qquad \sigma' = \sigma + \widetilde{K_{s1}}\beta_1 + O(1)\tilde{\Delta},$$

where

$$|\widetilde{K_{s4}}| < 1, \qquad |\widetilde{K_{s3}}| + |\widetilde{K_{s2}}| + |\widetilde{K_{s1}}| \leqq M,$$

for some $M > 0$ *and* $\tilde{\Delta} = |\alpha_4||\beta_3| + |\alpha_4||\beta_2|$.

Proof. First we show that there exists a unique solution $(\sigma', \delta_2, \delta_3, \delta_4)$, as a function of $(\sigma, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3)$, to

$$\Phi(0, \beta_3, \beta_2, \beta_1; \Phi(\alpha_4, \alpha_3, \alpha_2, 0; G(\sigma))) = \Phi(\delta_4, \delta_3, \delta_2, 0; G(\sigma')). \tag{6.22}$$

By Proposition 6.1, there exists $(\gamma_4, \gamma_3, \gamma_2, \gamma_1)$ such that

$$\Phi(0, \beta_3, \beta_2, \beta_1; \Phi(\alpha_4, \alpha_3, \alpha_2, 0; G(\sigma))) = \Phi(\gamma_4, \gamma_3, \gamma_2, \gamma_1; G(\sigma)) \tag{6.23}$$

with

$$\gamma_1 = \beta_1 + O(1)\Delta, \qquad \gamma_2 = \beta_2 + \alpha_2 + O(1)\Delta,$$
$$\gamma_3 = \alpha_3 + \beta_3 + O(1)\Delta, \quad \gamma_4 = \alpha_4 + O(1)\Delta.$$

Thus, (6.22) can be reduced to

$$\Phi(\gamma_4, \gamma_3, \gamma_2, \gamma_1; G(\sigma)) = \Phi(\delta_4, \delta_3, \delta_2, 0; G(\sigma')). \tag{6.24}$$

Furthermore, Lemma 6.6 implies

$$\det\left(\frac{\partial\Phi(\delta_4,\delta_3,\delta_2,0;G(\sigma'))}{\partial(\delta_4,\delta_3,\delta_2,\sigma')}\right)\Big|_{\{\delta_4=\delta_3=\delta_2=0,\sigma'=\sigma_0\}}$$
$$=\frac{1}{\det A}\det(A\mathbf{r}_4(U_+),A\mathbf{r}_3(U_+),A\mathbf{r}_2(U_+),AG_\sigma(\sigma_0))>0.$$

Therefore, the implicit function theorem implies that $(\delta_4,\delta_3,\delta_2,\sigma')$ can be uniquely solved as a C^2 function of $(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\sigma)$:

$$\sigma'=\sigma'(\gamma_4,\gamma_3,\gamma_2,\gamma_1,\sigma),\qquad \delta_i=\delta_i(\gamma_4,\gamma_3,\gamma_2,\gamma_1,\sigma),\quad i=2,3,4.$$

Using identity (3.1) in Lemma 3.1, we find

$$\sigma'=K_{s1}\gamma_1+\sigma,\qquad \delta_i=K_{si}\gamma_1+\gamma_i,\qquad i=2,3,4,$$

where $K_{s1}=\int_0^1\partial_{\gamma_1}\sigma'(\gamma_3,\gamma_2,\lambda\gamma_1,\sigma)d\lambda$ and $K_{si}=\int_0^1\partial_{\gamma_1}\delta_i(\gamma_4,\gamma_3,\gamma_2,\lambda\gamma_1,\sigma)d\lambda$.

When $\gamma_4=\gamma_3=\gamma_2=\gamma_1=0$, it is clear that $|\frac{\partial\sigma'}{\partial\gamma_1}|$ and $|\frac{\partial\delta_i}{\partial\gamma_1}|$, $i=2,3,4$, are bounded. We can further claim the important fact that

$$\left|\frac{\partial\delta_4}{\partial\gamma_1}\right|<1\quad\text{when } \gamma_4=\gamma_3=\gamma_2=\gamma_1=0.$$

This can be shown by differentiating (6.24) with respect to γ_1 and letting $\gamma_4=\gamma_3=\gamma_2=\gamma_1=0$. We then have

$$\mathbf{r}_1(U_+)=\mathbf{r}_4(U_+)\frac{\partial\delta_4}{\partial\gamma_1}+\mathbf{r}_3(U_+)\frac{\partial\delta_3}{\partial\gamma_1}+\mathbf{r}_2(U_+)\frac{\partial\delta_2}{\partial\gamma_1}+G_\sigma(\sigma_0)\frac{\partial\sigma'}{\partial\gamma_1}.$$

Multiplying both sides by A (defined in Lemma 6.6), we obtain

$$\left|\frac{\partial\delta_4}{\partial\gamma_1}\right|=\left|\frac{\det(A\mathbf{r}_1(U_+),A\mathbf{r}_2(U_+),A\mathbf{r}_3(U_+),AG_\sigma(\sigma_0))}{\det(A\mathbf{r}_4(U_+),A\mathbf{r}_2(U_+),A\mathbf{r}_3(U_+),AG_\sigma(\sigma_0))}\right|$$
$$=\left|\frac{\lambda_{4+}+\sigma_0}{\lambda_{4+}-\sigma_0}\right|\cdot\left|\frac{\sigma_0u_-Q-u_+\lambda_{4+}P}{\sigma_0u_-Q+u_+\lambda_{4+}P}\right|<1,$$

since $P>0, Q<0, \sigma_0<0$, and $\lambda_{4+}=\lambda_4(U_+)>0$. Combining with the estimates we had on $\gamma_1,\gamma_2,\gamma_3$, and γ_4, we complete the proof. □

6.3. Approximate solutions

Similarly to Section 4, we can construct the globally defined, modified Glimm approximate solutions $U_{\Delta x,\theta}$ in the approximate domains (see Fig. 7):

$$\Omega_{\Delta x}=\bigcup_{k\geqq 0}\Omega_{\Delta x,k}$$

with

$$\Omega_{\Delta x,k}=\{(x,y)\ :\ (k-1)\Delta x<x\leqq k\Delta x,\ y=y_{k-1}+(x-(k-1)\Delta x)\tan(\omega_{k-1,k})\}$$

under the Courant–Friedrichs–Lewy type condition:

$$\frac{\Delta y - m\Delta x}{\Delta x} < |\sigma_0| + \max_{j=1,4}\left(\sup_{O_\varepsilon(U_+)} |\lambda_j(U)|\right) \qquad \text{with } m \text{ as defined in (4.1).}$$

Lemma 6.7.
(i) *If* $\{U_b, U_a\} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *with* $U_b, U_a \in O_\varepsilon(U_+)$*, then*

$$|U_b - U_a| \leqq s_1(|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4|),$$

where $s_1 = \max_{1\leqq i\leqq 4}(\sup_{U\in O_\varepsilon(U_+)} |\partial_{\alpha_i}\Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U)|)$;
(ii) *for any* $\sigma \in O_{\hat\varepsilon}(\sigma_0)$ *with* $G(O_{\hat\varepsilon}(\sigma_0)) \subset O_\varepsilon(U_+)$ *for* $\hat\varepsilon = \hat\varepsilon(\varepsilon)$,

$$|G(\sigma) - G(\sigma_0)| \leqq s_2|\sigma - \sigma_0|,$$

where $s_2 = \sup_{\sigma\in O_{\hat\varepsilon(\sigma_0)}} |G'_\sigma(\sigma)|$.

We need to establish the estimates on $U_{\Delta x,\theta}$ on a class of space-like curves, and j-mesh curves J as introduced in Definition 4.1. To achieve this, we now define the Glimm-type functional.

Definition 6.2. We define

$$F_s(J) = C^*|\sigma^J - \sigma_0| + F(J),$$

with

$$F(J) = L(J) + KQ(J),$$
$$L(J) = K_0^*L_0(J) + L_1(J) + K_2^*L_2(J) + K_3^*L_3(J) + K_4^*L_4(J),$$
$$Q(J) = \sum\{|\alpha_i||\beta_j| : \text{ both } \alpha_i \text{ and } \beta_j \text{ cross } J \text{ and approach}\},$$

and

$$L_0(J) = \sum\{|\omega(C_k)| : C_k \in \Omega_J\},$$
$$L_j(J) = \sum\{|\alpha_j| : \alpha_j \text{ crosses } J\},\ 1 \leqq j \leqq 4,$$

where K and C^* will be defined later, while $\Omega_J = \{C_k \in J^+ \cap \partial\Omega_{\Delta x} : k \geqq 0\}$ is the set of the corner points C_k lying in J^+, σ^J stands for the speed of the strong shock crossing J, and $K_0^*, K_2^*, K_3^*, K_4^*$ are the constants that satisfy the following conditions:

$$K_0^* > |K_{b0}|, \quad |K_{b4}| < K_4^* < \frac{1}{|K_{s4}|}, \quad |K_{b3}| < K_3^* < \frac{1 - |K_{s4}|K_4^*}{|K_{s3}|},$$

and

$$|K_{b2}| < K_2^* < \frac{1 - |K_{s3}|K_3^* - |K_{s4}|K_4^*}{|K_{s2}|},$$

which can be achieved from our discussions of the properties of K_{bi} and K_{si}, $0 \leqq i \leqq 4$, as in the propositions in Section 6.2.

Now we prove the decreasing property of our functional F_s. We have

Proposition 6.5. *Suppose that the wedge boundary function* $g(x)$ *satisfies* (4.1), *and* I *and* J *are two* k*-mesh curves such that* J *is an immediate successor of* I. *Suppose that*

$$\left|U_{\Delta x,\theta|I\cap(\Omega^+_{\Delta x,k-1}\cup\Omega^+_{\Delta x,k})} - U_+\right| < \varepsilon, \quad \left|\sigma^I - \sigma_0\right| < \hat{\varepsilon},$$

where $\hat{\varepsilon} = \hat{\varepsilon}(\varepsilon)$ *is defined in* Proposition 6.3 *and* Lemma 6.7. *Then there exist constants* $\tilde{\varepsilon} > 0$, $K > 0$, *and* $C^* > 1$, *depending only on the system in* (1.1) *and states* U_- *and* U_+, *such that, if* $F_s(I) < \tilde{\varepsilon}$, *then*

$$F_s(J) \leqq F_s(I),$$

and hence

$$\left|U_{\Delta x,\theta|J\cap(\Omega^+_{\Delta x,k-1}\cup\Omega^+_{\Delta x,k})} - U_+\right| < \varepsilon, \quad \left|\sigma^J - \sigma_0\right| < \hat{\varepsilon}.$$

Proof. Let Λ be the diamond that is formed by I and J. We can always assume that $I = I_0 \cup I'$ and $J = J_0 \cup J'$ such that $\partial\Lambda = I' \cup J'$. As in the proof of Proposition 4.1, we divide our proof in four cases depending on the location of the diamond.

Case 1 (interior weak-weak interaction)**.** *Denote* $Q(\Lambda) = \Delta(\alpha, \beta)$ *as defined in* Proposition 6.1. *Then, for some constant* $M > 0$,

$$L(J) - L(I) = (1 + K_2^* + K_3^* + K_4^*)MQ(\Lambda),$$

and, since $L(I_0) < \tilde{\varepsilon}$ *from* $F_s(I) < \tilde{\varepsilon}$,

$$Q(J) - Q(I) = (ML(I_0) - 1)Q(\Lambda) \leqq -\frac{1}{2}Q(\Lambda).$$

Hence, we have

$$F(J) - F(I) = \big((1 + K_2^* + K_3^* + K_4^*)M - K/2\big)\, Q(\Lambda) \leqq -\frac{1}{4}Q(\Lambda),$$

by choosing suitably large K.

Case 2 (near the boundary)**.** *Then* $\Omega_J = \Omega_I \backslash \{C_k\}$ *for certain* k *and* $\sigma^I = \sigma^J$. *Let* δ_1 *be the weak 1-wave going out of* Λ *through* J', *and* β_1, α_2, *and* α_3 *be the weak waves entering* Λ *through* I', *as shown in* Fig. 10. *Then*

$$L_0(J) - L_0(I) = -|\omega_k|,$$

$$L_i(J) - L_i(I) = \sum_{\gamma_i \text{ crosses } I_0} |\gamma_i| - (|\alpha_i| + \sum_{\gamma_i \text{ crosses } I_0} |\gamma_i|) = -|\alpha_i|, \quad i = 2, 3, 4,$$

$$L_1(J) - L_1(I) = |\delta_1| - |\beta_1| \leqq |K_{b4}||\alpha_4| + |K_{b3}||\alpha_3| + |K_{b2}||\alpha_2| + |K_{b0}||\omega_k|,$$

where the last step is from Proposition 6.2. *Thus,*

$$\begin{aligned} &L(J) - L(I) \\ &\leqq (|K_{b0}| - K_0^*)|\omega_k| + (|K_{b2}| - K_2^*)|\alpha_2| + (|K_{b3}| - K_3^*)|\alpha_3| \\ &\quad + (|K_{b4}| - K_4^*)|\alpha_4|. \end{aligned}$$

From our requirement in Definition 6.2, we get $L(J)-L(I) \leqq 0$. Since $F_s(I) \leqq \tilde{\varepsilon}$ implies $L(I) \leqq \tilde{\varepsilon}$, the higher-order term $Q(I)$ can always be bounded by the linear term $L(I)$. Then we can easily conclude that $F(J) \leqq F(I)$.

Case 3 (near the wedge vertex)**.** *From our construction, we find that $\Omega_J = \Omega_I \backslash \{C_k\}$, and $S_*(\sigma_{(k)})$ emanates from C_k and crosses J, $\sigma^I = \sigma_{(k-1)}$, and $\sigma^J = \sigma_{(k)}$. Moreover, there is no weak wave crossing I' or J'. We have*

$$F(J) - F(I) \leqq -K_0^*|\omega_k|.$$

Since $|\sigma^J - \sigma_0| - |\sigma^I - \sigma_0| \leqq |\sigma^J - \sigma^I| \leqq |K_{bs}||\omega_k| + M|\omega_k|^2$ and $|K_{bs}|$ is bounded, we can further choose suitably small C^* and $\tau > 0$ such that

$$F_s(J) - F_s(I) \leqq C^*|\sigma^J - \sigma^I| + F(J) - F(I) \leqq -\tau|\omega_k|.$$

Case 4 (near the strong 1-shock)**.** *The shock $S_*(\sigma_{(k)})$ is generated from the inside of Λ, $\sigma^I = \sigma_{(k-1)}$, and $\sigma^J = \sigma_{(k)}$. Let δ_4, δ_3, and δ_2 be the weak waves going out of Λ through J', and let $\alpha_4, \alpha_3, \alpha_2$, β_1, β_2, and β_3 be the weak waves entering Λ through I', as shown in* Fig. 11.

Then

$$\begin{aligned} L_1(J) - L_1(I) &= \sum_{\gamma_1 \text{ crosses } I_0} |\gamma_1| - (|\beta_1| + \sum_{\gamma_1 \text{ crosses } I_0} |\gamma_1|) = -|\beta_1|, \\ L_i(J) - L_i(I) &\leqq |K_{si}||\beta_1| + M|\alpha_4||\beta_2| + |\alpha_4||\beta_3|, \quad i = 2, 3, \\ L_4(J) - L_4(I) &\leqq |K_{s4}||\beta_1| + M(|\alpha_4||\beta_2| + |\alpha_4||\beta_3|), \end{aligned}$$

where we have used the estimates from Proposition 6.4.

Again, this case is much more complicated and requires careful calculation of $Q(J) - Q(I)$. For simplicity, for any weak wave γ, we denote

$$Q(\gamma, I_0) = |\gamma| \sum \{|\gamma_j| \ : \ \gamma_j \text{ and } \gamma \text{ approach}, \gamma_j \text{ crosses } I_0\}.$$

Then

$$\begin{aligned} Q(J) - Q(I) \leqq& -(|\alpha_4||\beta_1| + |\alpha_4||\beta_2| + |\alpha_4||\beta_3| + |\beta_1||\alpha_2| + |\beta_1||\alpha_3|) \\ &+(\sum_{i=2}^{4} |\widetilde{K_{si}}| - 1)Q(\beta_1, I_0) \\ &+ Q(M(|\alpha_4||\beta_2| + |\alpha_4||\beta_3|), I_0) \\ =& (-1 + ML(I_0))(|\alpha_4||\beta_2| + |\alpha_4||\beta_3|) \\ &+(ML(I_0) - \sum_{i=2}^{4} |\alpha_i|)|\beta_1|. \end{aligned}$$

Since $L(I_0) < \tilde{\varepsilon}$ from $F_s(I) < \tilde{\varepsilon}$, then

$$Q(J) - Q(I) \leqq -\frac{1}{2}(|\alpha_4||\beta_2| + |\alpha_4||\beta_3|) + ML(I_0)|\beta_1|.$$

Therefore, we have

$$F(J) - F(I) \leqq \left(-1 + \sum_{i=2}^{4} K_i^* |K_{si}|\right) |\beta_1| + M(|\alpha_4||\beta_2| + |\alpha_4||\beta_3|)$$
$$+K(-\frac{1}{2}(|\alpha_4||\beta_2| + |\alpha_4||\beta_3|) + ML(I_0)|\beta_1|)$$
$$\leqq -\frac{1}{8}(|\beta_1| + |\alpha_4||\beta_2| + |\alpha_4||\beta_3|),$$

where we have chosen suitably large K and used the fact that $L(I_0) < \tilde{\varepsilon}$.

Furthermore, since

$$|\sigma^J - \sigma^I| \leqq |K_{s1}||\beta_1| + M(|\alpha_4||\beta_2| + |\alpha_4||\beta_3|),$$

we can further choose suitably small C^* such that

$$F_s(J) - F_s(I) \leqq C^* |\sigma^J - \sigma^I| + F(J) - F(I)$$
$$\leqq -\frac{1}{16}|\beta_1| - \frac{1}{16}(|\alpha_4||\beta_2| + |\alpha_4||\beta_3|).$$

Again we have $F(J) \leqq F(I)$. Then, from Lemma 6.7, there exists $\tilde{\epsilon} > 0$ such that, when $F(I) < \tilde{\epsilon}$, we have $|U - U_+| < \epsilon$. □

Then the same argument as in Section 4 yields the following theorem.

Theorem 6.1 (Existence and stability)**.** *There exist $\varepsilon > 0$ and $C > 0$ such that, if* (1.10) *holds, then, for each $\theta \in (\Pi_{k=0}^{\infty}(-1, 1))\backslash(N \cup N_1)$, there exist a sequence $\{\Delta_l\}_{l=1}^{\infty}$ of mesh sizes with $\Delta_l \to 0$ as $l \to \infty$ and a pair of functions $U_\theta \in O_\varepsilon(U_+)$ and $\chi_\theta \in Lip(\mathbb{R}_+)$ with $\chi_\theta(0) = 0$ such that*

- (i) *$U_{\Delta_l,\theta}(x, \cdot)$ converges to $U_\theta(x, \cdot)$ in $L^1(-\infty, g(x))$ for every $x > 0$, and U_θ is a global entropy solution of problem* (1.1) *and* (1.8)–(1.9) *in Ω and satisfies* (1.11)–(1.12)*;*
- (ii) *$\chi_{\Delta_l,\theta}$ converges to χ_θ uniformly in any bounded x-interval;*
- (iii) *$\sigma_{\Delta_l,\theta}$ converges a.e. to $\sigma_\theta \in BV(\mathbb{R}_+)$ with $|\sigma_\theta - \sigma_0| < \hat{\varepsilon}$ and $\chi_\theta(x) = \int_0^x \sigma_\theta(t)dt$.*

In addition, if θ is equidistributed, then $\chi_\theta(x) < g(x)$ for any $x > 0$ with (1.13) *and the Rankine-Hugoniot conditions a.e. along the curve $\{y = \chi_\theta(x)\}$.*

Furthermore, let $\theta \in (\Pi_{k=0}^{\infty}(-1, 1))\backslash(N \cup N_1)$ be equidistributed, and let U_θ be the solution and χ_θ its shock-front, respectively. By Theorem 6.1, we find that the solution U_θ contains at most countable shock fronts and countable points of wave interactions. Moreover, we can modify the solution U_θ such that U_θ is continuous except on the shock curves and the points of wave interactions (cf. [9]). Then we have

Theorem 6.2.

(i) *Let $\omega_\infty = \lim_{x\to\infty} \arctan(g'(x+))$. Then*

$$\lim_{x\to\infty} \sup\{|\arctan(v_\theta(x, y)/u_\theta(x, y)) - \omega_\infty| \ : \ \chi_\theta(x) < y < g(x)\} = 0.$$

(ii) *There exist constants p_∞ and σ_∞ such that*

$$\lim_{x\to\infty} \sup\{|p_\theta(x, y) - p_\infty| \,:\, \chi_\theta(x) < y < g(x)\} = 0$$

and

$$\lim_{x\to\infty} |\sigma_\theta(x) - \sigma_\infty| = 0.$$

Theorem 6.1 can be proved in the same way as Theorem 5.2.

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