Arch. Rational Mech. Anal. 181 (2006) 43–96
Digital Object Identifier (DOI) 10.1007/s00205-005-0406-1

The Concept of a Minimal State in Viscoelasticity: New Free Energies and Applications to PDEs

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Communicated by C.M. DAFERMOS

Abstract

We show here the impact on the initial-boundary value problem, and on the evolution of viscoelastic systems of the use of a new definition of state based on the stress-response (see, e.g., [48, 16, 41]). Comparisons are made between this new approach and the traditional one, which is based on the identification of histories and states. We shall refer to a stress-response definition of state as the *minimal state* [29]. Materials with memory and with relaxation are discussed.

The energetics of linear viscoelastic materials is revisited and new free energies, expressed in terms of the minimal state descriptor, are derived together with the related dissipations. Furthermore, both the minimum and the maximum free energy are recast in terms of the minimal state variable and the current strain.

The initial-boundary value problem governing the motion of a linear viscoelastic body is re-stated in terms of the minimal state and the velocity field through the principle of virtual power. The advantages are (i) the elimination of the need to know the past-strain history at each point of the body, and (ii) the fact that initial and boundary data can now be prescribed on a broader space than resulting from the classical approach based on histories. These advantages are shown to lead to natural results about well-posedness and stability of the motion.

Finally, we show how the evolution of a linear viscoelastic system can be described through a strongly continuous semigroup of (linear) contraction operators on an appropriate Hilbert space. The family of all solutions of the evolutionary system, obtained by varying the initial data in such a space, is shown to have exponentially decaying energy.

1. Introduction

In the classical approach to materials with memory, the state is identified with the history of variables carrying information about the input processes. We show in this paper how the notion of state given in [16, 41] (for the linear case), based on

Noll's definition of equivalent histories [48], is more convenient for application to such materials. Indeed, Noll's approach takes the material response as the basis for such an equivalence, leading to the definition of state: if an arbitrary continuation of different given histories leads to the same response of the material, then the given histories are equivalent and the state is represented as the class of all the equivalent histories. We shall refer to this definition of state as the *minimal state* [29].*

The idea of minimal state was developed and applied in [41] to the case of linear viscoelasticity with scalar relaxation functions given by a sum of exponentials. A subsequent paper [16] presents a more comprehensive treatment in three dimensions, and in the more general context of thermodynamically compatible (tensor-valued) relaxation functions, taking into account the weak regularity of histories and processes.

For such materials, the past strain history $\mathbf{E}^t(s)$ $\left(s \in \mathbb{R}^{++}\right)$ and the current value of the strain $\mathbf{E}(t)$ determine the stress response. The state is, however, identifiable through the variable

$$\mathbf{I}^t(\tau) = -\mathbb{G}(\tau)\mathbf{E}(t) - \int_0^\infty \dot{\mathbb{G}}(\tau + s)\mathbf{E}^t(s)ds, \tag{1.1}$$

which is the negative of the stress response to the constant process $\mathbf{E}(r) = \mathbf{E}(t),\ r \in [0, \tau)$. The negative signs on the right are to maintain consistency with developments in Section 4. The relaxation tensor $\mathbb{G}$ is discussed in Section 4. The regularity assumptions made in [16] on histories, processes and the relaxation function yield

$$\lim_{\tau \to \infty} \mathbf{I}^t(\tau) = -\mathbf{T}_r(t), \tag{1.2}$$

with $|\mathbf{T}_r(t)| < +\infty$; in fact:

$$\mathbf{T}_r(t) = \mathbb{G}_\infty \mathbf{E}(t), \tag{1.3}$$

which is the constitutive equation of an elastic (in fact hyperelastic) material with elastic modulus $\mathbb{G}_\infty$. Equation (1.2) is a *fading memory* property for the stress functional in linear viscoelasticity.

This property enters directly into the definition of state, which we shall consider below.

Because of equations (1.2) and (1.3), it seems more appropriate to refer to such materials as *materials with relaxation* instead of materials with fading memory. In order to better understand this distinction, we see that the state is now characterized by the future response of the material when, ideally, the null process is imposed for

* Surprisingly, the first contribution in the direction of applying Noll's approach to linear viscoelasticity was presented in [1]. As far as the authors are concerned, being aware of this paper might have helped to develop the subject one decade ahead of time.

a finite time interval on a material element (i.e., a "small" neighborhood of a fixed and arbitrary point of the body). This characterization of the state is then alternative to the usual one based on knowledge of the deformation history.

A fading memory property of the response functional [5] is usually required, as opposed to the case in which the minimal state is adopted, where indeed the relaxation property of the response functional suffices. Obviously, whenever the stress response functional is such that the knowledge of the minimal state turns is equivalent to the knowledge of the past history, the property of relaxation of the stress response implies the fading memory of the related functional. In this sense, the class of materials with relaxation is larger than that described by constitutive equations with fading memory.

The first characterization of such a definition of state for linear viscoelastic materials in the frequency domain was provided in [20], whereas an extension of such a characterization was given in [35] in terms of equivalent variables.

As was already pointed out in [16], the main advantage of a response-based definition of state relates to the physical features of the state itself. Indeed, the "future stress", $\mathbf{I}^t(\tau)$, is detectable through measurements and does not require any knowledge of the past history.

Another advantage that we shall highlight is the fact that the response-based definition of state is useful for both the study of Initial-Boundary Value Problems (IBVP) (see Section 9) on one hand, and for the evolution of linear viscoelastic systems on the other (Section 10).

To this end, we shall show that $\mathbf{I}^t(\tau)$ leads to the minimal information required to identify the state of the material. Indeed, from (1.1) we see from any given equivalent history, and from the knowledge of the current strain, that the state variable $\mathbf{I}^t(\tau)$ is completely, and uniquely, determined, although the converse is not true. Indeed, if $\mathbf{I}^t(\tau)$ is given instead, the current value of strain can be determined but not, in general, the history. The inability to determine the history is due to the fact that $\mathbf{I}^t(\tau)$ represents all the equivalent histories leading to the same future stress, that is to say, the new state variable represents a family of histories. Examples in which the set of equivalent histories is not a singleton are well known [41, 16]. These are cases in which the relaxation functions a finite sum of exponentials, in which the state is described by a finite list of quantities (see, e.g., [52]). This topic is addressed in Section 8.

In Sections 2 and 3, definitions and properties relating to a general theory of materials, including nonlinear viscoelasticity, are presented in terms of the abstract formulation of thermodynamics [8, 9, 29]. In particular, the minimal state and a notion of fading memory via relaxation are established, and theorems characterizing the minimal and maximal free energies are recalled.

In Section 4, linear viscoelasticity is considered and various formulae, required in subsequent sections (including an explicit expression for the minimum free energy), are presented. The functional $\mathbf{I}^t$, given by (1.1), and related quantities are introduced and discussed.

In Section 5 the space of processes is defined.

As is well known, there are two different definitions of free energy in viscoelasticity (see, e.g., [16]), and according to one of these definitions, a free energy has to be a function of state, i.e., of $\mathbf{I}^t(\tau)$.[1] In this paper, we provide new thermodynamic potentials for the stress, i.e., new free energies, which are also functions of state. Indeed, a new class of *single-integral type* free energy is introduced in Section 6 as a quadratic form of the time derivative of the state variable (see, e.g., [43, 44] for discussions and analysis of single integral-type free energies that are quadratic forms of histories). For exponentially decaying relaxation functions, it can be shown that the dissipation associated with such energies is bounded from below by a time-decay coefficient multiplied by the purely viscoelastic part of the free energy. This property turns out to be crucial in the analysis of *PDE*s developed in Section 9.

An analogous property holds for a family of multiple-integral free-energy functionals which are the generalization of the previous single-integral type free energy. We may refer to such a family as the *n-family*. Incidentally, for $n = 0$, the free energy for single-exponential relaxation functions is recovered, whereas for $n = 1$, one recovers the single-integral type free energy discussed above.

An explicit formula is derived in Section 7 for the *minimal free energy* starting from the original formula derived in [20]. This is a quadratic functional depending on the second derivative of $\mathbf{I}^t(\tau)$ with respect to τ. The use of such derivatives of $\mathbf{I}^t$ is connected with convergence issues of certain inverse Fourier transforms. The dissipation associated with the minimal free energy is also given as a similar quadratic form.

A natural question may be how can the *maximum free energy* be represented in terms of the minimal state descriptor. It is noteworthy that a new way to link the minimum and maximum free energy, together with many other interesting results has been recently established [15] in a very general context; in particular, the maximum free energy as a function of the minimal state has been characterized, although no explicit formula for it has been provided.

In our paper (see Section 8), the Euler-Lagrange equation associated with the problem of determining the minimum work expended going from the natural state to a prescribed state is solved for a fairly large class of relaxation functions. First, the issue of whether the set of equivalent histories corresponding to a given $\mathbf{I}^t$ is a singleton, or not, is addressed in a fairly general context (the argument is restricted to materials for which the eigenspaces of $\dot{\mathbb{G}}$ are time-independent [20, 26], which almost certainly must be assumed in any case if the formulae giving the minimum free energy are used to get explicit results). It is shown that the set is a singleton, unless all the singularities of the Fourier transform of $\dot{\mathbb{G}}$ are isolated. This corresponds to $\dot{\mathbb{G}}$ given by sums (not integrals) of decaying exponentials multiplying trigonometric functions and polynomials or convergent-power series.

In this case, an explicit formula is given for the maximum free energy, generalizing results given previously in [29, 26]. It is similar in structure to the formula for the minimum free energy in [20]. This formula can also be expressed as a quadratic functional of the second derivative of a quantity closely related to $\mathbf{I}^t$ by steps

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[1] The phrase "function of state" shall, in general, be taken to mean "function of the minimal state".

entirely analogous to those for the minimum free energy. An explicit expression is also given for the dissipation associated with the maximum free energy, as a similar quadratic functional. If the set of equivalent histories is a singleton, then the minimal state is the current value and history of strain, and the maximum free energy is the work function.

The new approach outlined above to the theory of viscoelasticity, and the new free energies, lead to applications to the *PDE*s governing the motion of a suitable class of viscoelastic bodies. In particular the use of the new quadratic forms of the minimal state variables yield results relating to well-posedeness and stability for the IBVP. This formulation allows for initial data belonging to broader functional spaces than those usually considered in the literature, which are based on histories.

Furthermore, we present an application of semi-group theory to the class of materials discussed above. Here, besides having the system of equations in a more general form than in the classical approach, results on asymptotic stability are obtained again for initial data belonging to a space broader than that usually employed when states and histories are identified.

Various notations and assumptions used in later sections are defined in the Appendix.

2. Fading memory and thermodynamics

The mechanical properties of any material are based on the concepts of *state* and *process* [48, 8, 34]. We consider a body occupying the region $\mathcal{B}$. For any material point $\mathbf{X} \in \mathcal{B}$ and time t, we define the *configuration* $C(\mathbf{X}, t)$ given by the *deformation gradient* $\mathbf{F}(\mathbf{X}, t)$. Writing $C(t)$ or $\mathbf{F}(t)$ means that the dependence on t is examined, while $\mathbf{X}$ is kept fixed.

A *mechanical process* P, of duration $d_p > 0$, is a piece-wise continuous function on $\left[0, d_p\right)$, with values in $Lin(\mathbb{R}^3)$, given by

$$P(\tau) = \mathbf{L}^P(\tau), \tag{2.1}$$

where $\mathbf{L} = \nabla \mathbf{v}$ is the *velocity gradient* and $\mathbf{L}^P$ is the specified segment of values of this quantity. The assignment $P_{[t_1,t_2)}$ is the restriction of P to $[t_1, t_2) \subset \left[0, d_p\right)$. In particular we denote by P_t the restriction of P to $[0, t)$, $t < d_p$.

Given two processes P_1, P_2 of duration d_{P_1}, d_{P_2}, the composition $P_1 * P_2$ of P_1 with P_2 is defined as

$$P_1 * P_2(\tau) = \begin{cases} P_1(\tau) & \text{if } \tau \in \left[0, d_{p_1}\right), \\ P_2(\tau - d_{p_1}) & \text{if } \tau \in \left[d_{p_1}, d_{p_1} + d_{p_2}\right). \end{cases}$$

Definition 2.1. A simple material element, at any $\mathbf{X} \in \mathcal{B}$, is a set $\left\{\Pi, \Upsilon, \Sigma, \hat{\rho}, \tilde{T}\right\}$ such that

1. Π is the space of mechanical processes P satisfying the following properties:

(i) if $P \in \Pi$, then $P_{[t_1,t_2)} \in \Pi$ for every $[t_1, t_2) \subset [0, d_p)$,
(ii) if $P_1, P_2 \in \Pi$, then $P_1 * P_2 \in \Pi$;

2. the set Υ is the space of all the stress tensor processes $\mathbf{T}^P$, defined by Property 5 below;
3. Σ is a set, the *state space,* whose elements σ are the possible *states* of the system;
4. the map $\hat{\rho} : \Sigma \times \Pi \to \Sigma$ is the *state transition function* with the property

$$\hat{\rho}(\sigma, P_1 * P_2) = \hat{\rho}(\hat{\rho}(\sigma, P_1), P_2)$$

for all $P_1, P_2 \in \Pi$, $\sigma \in \Sigma$;

5. the map $\tilde{\mathbf{T}} : \Sigma \times \Pi \to \Upsilon$ is the *response function* which, to any state σ and process P, assigns the stress tensor process $\mathbf{T}^P$ over the time interval $[0, d_p)$:

$$\mathbf{T}^P(\tau) = \tilde{\mathbf{T}}(\sigma, P)(\tau), \quad \tau \in [0, d_p) .$$

Definition 2.2. The system is said to be *causal*, if property 5 can be replaced by the new condition:

5′. the map $\hat{\mathbf{T}} : \Sigma \times Lin \to Sym$ is such that

$$\mathbf{T}^P(\tau) = \hat{\mathbf{T}}(\sigma(\tau), P(\tau)), \quad \sigma(\tau) = \hat{\rho}(\sigma, P_\tau).$$

Remark 2.3. In this paper we consider only causal systems. For these systems, the funtion $\tilde{\mathbf{T}}$ is connected to $\hat{\mathbf{T}}$ by

$$\hat{\mathbf{T}}(\hat{\rho}(\sigma, P_\tau), P(\tau)) = \tilde{\mathbf{T}}(\sigma, P)(\tau). \tag{2.2}$$

In this framework, following [48], we can introduce a concept of equivalence in the state space Σ.

Definition 2.4. Two states $\sigma_1, \sigma_2 \in \Sigma$ are said to be *equivalent,* if they satisfy the identity

$$\tilde{\mathbf{T}}(\sigma_1, P) = \tilde{\mathbf{T}}(\sigma_2, P) \tag{2.3}$$

for all $P \in \Pi$.

Moreover, we introduce the concept of a minimal state σ_R as the equivalence class of the states according to Definition 2.4. In the following we denote by Σ_R the set of all σ_R.

A material with fading memory is defined by a constitutive equation which relates the stress tensor $\mathbf{T}$ and the deformation gradient $\mathbf{F}$ by a functional of the type

$$\mathbf{T}(\mathbf{X}, t) = \hat{\mathbf{T}}(\mathbf{F}^t(\mathbf{X})), \tag{2.4}$$

where $\mathbf{F}^t(\mathbf{X}, s) = \mathbf{F}(\mathbf{X}, t - s), s \in \mathbb{R}^+$, is the history of $\mathbf{F}$. The fading memory property is characterized in detail below. Usually, for these materials, the state is represented by the history $\mathbf{F}^t$.

If the initial state is $\sigma_0 = \mathbf{F}^{t_0}$ and $P(t) = \mathbf{L}^P(t),\ t \in \left[t_0, t_0 + d_p\right)$ is a process in Π, then the transition function is defined by

$$\hat{\rho}(\sigma_0, P) := \mathbf{F}^{t_0+d_p}(s) = \begin{cases} \mathbf{F}^{t_0}(s - d_p) & \text{if } s \in \left[d_p, \infty\right), \\ \mathbf{F}^P(t_0 + d_p - s) & \text{if } s \in \left[0, d_p\right), \end{cases} \tag{2.5}$$

where $\mathbf{F}^P$ is the solution of the Cauchy problem on the interval $\left[t_0, t_0 + d_p\right)$:

$$\begin{aligned} \frac{d}{dt}\mathbf{F}(s) &= \mathbf{L}^P(s)\mathbf{F}(s), \\ \mathbf{F}(t_0) &= \mathbf{F}^{t_0}(0). \end{aligned} \tag{2.6}$$

The response function is represented by the constitutive equation (2.4).

Materials with fading memory, described by (2.4)–(2.6), are simple material elements in the sense of Definition 2.1.

A history $\mathbf{F}^t$ will be viewed as the pair $(\mathbf{F}(t), \mathbf{F}^t(s))$ of the present value $\mathbf{F}(t)$ and the past history $\mathbf{F}^t(s),\ s > 0$. One way to intrepret these two pieces of information as one item can be found in [15], where a history $\mathbf{F}^t$ is defined as follows: past histories are taken to be (i) of total bounded variation on $\mathbb{R}^+$, and (ii) continuous from the right. Thus, $\mathbf{F}^t(s) = \mathbf{F}^t(s^+)$ for all $s \in (0, \infty)$.

We define the *static* and *null continuations* of duration $\tau > 0$, to be the histories $\mathbf{F}^t_\tau$ and ${}_\tau\mathbf{F}^t$ defined by

$$\begin{aligned} \mathbf{F}^t_\tau(s) &:= \begin{cases} \mathbf{F}(t) & \text{if } s \leqq \tau \\ \mathbf{F}^t(s-\tau) & \text{if } s > \tau \end{cases}; \\ {}_\tau\mathbf{F}^t(s) &:= \begin{cases} 0 & \text{if } s \leqq \tau \\ \mathbf{F}^t(s-\tau) & \text{if } s > \tau. \end{cases} \end{aligned} \tag{2.7}$$

Now we are in a position to characterize the fading memory property of the constitutive equation (2.4).

Definition 2.5. A material with memory represented by (2.4) satisfies the fading memory property if the function $\hat{\mathbf{T}}(\mathbf{F}^t_\tau)$ is bounded for any $\mathbf{F}^t \in \Sigma$ and for all $\tau > 0$; and there is an elastic material $\hat{\mathbf{T}}_{el}(\mathbf{F}(t))$ such that

$$\lim_{\tau\to\infty} \hat{\mathbf{T}}(\mathbf{F}^t_\tau) = \hat{\mathbf{T}}_{el}(\mathbf{F}(t));$$

moreover

$$\lim_{\tau\to\infty} \hat{\mathbf{T}}({}_\tau\mathbf{F}^t) = \mathbf{0}.$$

3. The dissipation principle and maximum recoverable work

Thermodynamics plays a central role in the characterization of natural states in which to consider physical problems, and in the determination of the most suitable norms for such studies. We show later how, for linear problems, explicit expressions for various free energies allow us to decide on suitable norms and function spaces.

We begin by recalling the traditional Clausius-Duhem inequality and the definition of a *cycle*, which is a pair (σ, P) such that $\hat{\rho}(\sigma, P) = \sigma$. For any state σ and process P, the function $\hat{\rho}$ determines the one-parameter family of states $\sigma(t) = \hat{\rho}(\sigma, P_t)$, $t \in \left[0, d_p\right)$.

Moreover, we define the space

$$\Sigma_\sigma = \left\{\sigma' \in \Sigma\,;\, \exists\ P \in \Pi,\ \text{such that } \sigma' = \hat{\rho}(\sigma, P)\right\}. \tag{3.1}$$

Let us define the work done by a process P acting on a given state σ as follows:

$$W(\sigma, P) = \int_0^{d_P} \hat{\mathbf{T}}\left(\sigma\left(\tau\right), P\left(\tau\right)\right) \cdot \mathbf{L}^P\left(\tau\right)\, d\tau. \tag{3.2}$$

Clausius-Duhem inequality (for isothermal processes)
For every cycle $(\sigma, P) \in \Sigma \times \Pi$, *the inequality*

$$\oint_0^{d_P} \hat{\mathbf{T}}\left(\sigma\left(\tau\right), P\left(\tau\right)\right) \cdot \mathbf{L}^P\left(\tau\right)\, d\tau \geqq 0 \tag{3.3}$$

holds, where $\mathbf{L}^P$ *determines* P *as in* (2.1).

For materials with fading memory, cycles are quite rare, because usually the material reaches a state, which is different from the initial state, although it may be "close" to it. For this reason it is more convenient to use the following approach [27].

Strong dissipation principle
The set

$$\mathcal{W}\left(\sigma\right) := \left\{W\left(\sigma, P\right);\ \ P \in \Pi\right\} \tag{3.4}$$

of the work done in passing from a given state σ *to any state* $\sigma' \in \Sigma_\sigma$, *is bounded below. There exists a state* $\sigma^\dagger$, *called zero state, such that*

$$\inf \mathcal{W}\left(\sigma^\dagger\right) = 0, \ \textit{and} \quad W\left(\sigma^\dagger, P\right) > 0\,,\ \forall\ P \neq 0$$

For materials with fading memory, the zero state is given by the history $\mathbf{F}^\dagger(s)$ equal to the unit tensor for all $s \in [0, \infty)$.

Definition 3.1. A function $\psi : \mathcal{S}_\psi \to \mathbf{R}^+$ is a free energy if

(i) the domain $\mathcal{S}_\psi \subset \Sigma$ is invariant under ρ, namely, for every $\sigma_1 \in \mathcal{S}_\psi$ and $P \in \Pi$, the state $\sigma = \hat{\rho}\left(\sigma_1, P\right) \in \mathcal{S}_\psi$. Also $\sigma^\dagger \in \mathcal{S}_\psi$, and $\psi\left(\sigma^\dagger\right) = 0$;
(ii) for any pair σ_1, $\sigma_2 \in \mathcal{S}_\psi$ and $P \in \Pi$ such that $\hat{\rho}\left(\sigma_1, P\right) = \sigma_2$ we have:

$$\psi\left(\sigma_2\right) - \psi\left(\sigma_1\right) \leqq W\left(\sigma_1, P\right). \tag{3.5}$$

It is well known that for materials with memory there are many free energies (see, e.g., [8] for general dissipative materials and [28, 16] and many others), no matter what the definition of the energy. Depending on the definition of the free energy, the family $\mathcal{F}$ of the free energies might be a convex set. A general property of the resulting sets is that they are bounded [16] and their minimum and maximum elements are denoted here by ψ_m and ψ_M.

Definition 3.2. A free energy ψ_m is called the *minimum free energy* if:

(i) ψ_m is a non-negative function with domain $\mathcal{S} = \Sigma$,
(ii) the zero state $\sigma^\dagger \in \Sigma$ is such that $\psi_m(\sigma^\dagger) = 0$,
(iii) for any free energy $\psi : \mathcal{S} \to I\!R^+$ such that $\sigma^\dagger \in \mathcal{S}$ and $\psi(\sigma^\dagger) = 0$, we have

$$\psi(\sigma) \geqq \psi_m(\sigma) \ , \quad \forall\, \sigma \in \mathcal{S}.$$

The following theorem is proved in [27].

Theorem 3.3. *The functional*

$$\psi_m(\sigma) := -\inf \mathcal{W}(\sigma) \tag{3.6}$$

is the minimum free energy.

Equivalent characterizations for the minimum free energy have been proved in [35].

Let us recall definition (3.1) given above for the set Σ_σ. For any pair $\sigma_0, \sigma \in \Sigma$, such that $\sigma \in \Sigma_{\sigma_0}$, we can consider the set:

$$N(\sigma_0, \sigma) = \left\{ W(\sigma_0, P), \quad \forall \ P \in \Pi; \ \hat{\rho}(\sigma_0, P) = \sigma \right\}. \tag{3.7}$$

From the Strong Dissipation Principle, this set is bounded below.

The following theorem is proved in [29].

Theorem 3.4. *For any fixed σ^i, the functional $\psi_M^{\sigma^i} : \Sigma_{\sigma^i} \to \mathbb{R}^+$, defined by*

$$\psi_M^{\sigma^i}(\sigma) = \inf N(\sigma^i; \sigma) + \psi_m(\sigma^i), \tag{3.8}$$

is a free energy, called a maximum free energy. *For any free energy $\psi : \mathcal{S}_\psi \to \mathbb{R}^+$, such that $\mathcal{S}_\psi \supset \Sigma_{\sigma^i}$, and $\psi(\sigma^i) = \psi_m(\sigma^i)$,*

$$\psi(\sigma) \leqq \psi_M^{\sigma^i}(\sigma), \qquad \forall \quad \sigma \in \Sigma_{\sigma^i}. \tag{3.9}$$

Remark 3.5. Of course, for any $\sigma^i \in \Sigma$ we may obtain a different free energy. Moreover, for a fixed $\sigma^i \in \Sigma$, the definition of maximum free energy may depend on the definition of state. We can, however, construct a maximum free energy that is defined on the space of minimal states. In other words, if we consider the definition of minimal state, then (3.7) is replaced by

$$N(\sigma_{0R}, \sigma_R) = \left\{ W(\sigma_{0R}, P), \quad \forall \quad P \in \Pi; \ \hat{\rho}(\sigma_{0R}, P) = \sigma_R \right\}.$$

This set is generally larger than $N(\sigma_0, \sigma)$, if $\sigma_0 \in \sigma_{0R}$ and $\sigma \in \sigma_R$. For this reason the maximum free energy, defined on Σ_R as

$$\psi_M^{\sigma_R^i}(\sigma_R) = \inf N(\sigma_R^i; \sigma_R) + \psi_m(\sigma_R^i), \tag{3.10}$$

satisfies the inequality

$$\psi_M^{\sigma_R^i}(\sigma_R) \leqq \psi_M^{\sigma^i}(\sigma), \quad \sigma^i \in \sigma_R^i, \quad \sigma \in \sigma_R.$$

Relation (3.9) will apply to any free energy $\psi(\sigma_R)$ defined on Σ_R, provided $\psi(\sigma_R^i) = \psi_m(\sigma_R^i)$.

It follows from (3.2) and (3.5) that, for any time t where the process is continuous, we have

$$\dot{\psi}(\sigma(t)) \leqq \hat{\mathbf{T}}(\sigma(t), P(t)) \cdot \mathbf{L}^P(t). \tag{3.11}$$

4. Linear viscoelasticity

Let us now consider the linear theory of viscoelasticity. For this case the tensor $\mathbf{F}$ is replaced by the infinitesimal deformation gradient

$$\mathbf{E} = \frac{\nabla\mathbf{u} + (\nabla\mathbf{u})^T}{2} \in Sym, \tag{4.1}$$

where $\mathbf{u}$ denotes the displacement field, such that $\mathbf{F} = \nabla\mathbf{u} + \mathbf{I}$ (where $\mathbf{I}$ is the second-order identity tensor), while a process is now given by a specified set of values of $\dot{\mathbf{E}}(t),\ t \in [0, d_p)$. We denote position by $\mathbf{x}$ which, in the linear approximation, can be taken to be either the current or reference position. Then the constitutive equation (2.4) becomes

$$\begin{aligned}\tilde{\mathbf{T}}(\mathbf{E}^t) = \mathbf{T}(\mathbf{x}, t) &= \mathbb{G}_0(\mathbf{x})\mathbf{E}(\mathbf{x}, t) + \int_0^\infty \dot{\mathbb{G}}(\mathbf{x}, s)\mathbf{E}^t(\mathbf{x}, s)\, ds \\ &= \mathbb{G}_\infty(\mathbf{x})\mathbf{E}(\mathbf{x}, t) + \int_0^\infty \dot{\mathbb{G}}(\mathbf{x}, s)\mathbf{E}_r^t(\mathbf{x}, s) ds,\end{aligned}$$

$$\begin{aligned}&\mathbf{E}^t(\mathbf{x}, s) := \mathbf{E}(\mathbf{x}, t - s), \quad \mathbf{E}_r^t(\mathbf{x}, s) := \mathbf{E}^t(\mathbf{x}, s) - \mathbf{E}(\mathbf{x}, t), \quad s \in \mathbb{R}^{++}, \\ &\mathbb{G}_0(\mathbf{x}) := \mathbb{G}(\mathbf{x}, 0), \qquad \mathbb{G}_\infty(\mathbf{x}) := \mathbb{G}(\mathbf{x}, \infty).\end{aligned} \tag{4.2}$$

The quantity $\mathbf{E}(\mathbf{x}, t) \in Sym$ is the *instantaneous* or *present value* of the strain and $\mathbf{E}^t : \mathbb{R}^{++} \to Sym$ denotes the *past history*. We refer to $\mathbf{E}_r^t$ as the relative strain history. The quantity $\mathbb{G}_0(\mathbf{x}) \in Lin(Sym)$, the instantaneous modulus, is a symmetric tensor as is the equilibrium modulus, given by [34],

$$\mathbb{G}_\infty(\mathbf{x}) = \mathbb{G}_0(\mathbf{x}) + \int_0^\infty \dot{\mathbb{G}}(\mathbf{x}, s) ds. \tag{4.3}$$

Both $\mathbb{G}_0(\mathbf{x})$ *and* $\mathbb{G}_\infty(\mathbf{x})$ are positive tensors. The function $\dot{\mathbb{G}}$ represents the time derivative of the *relaxation function*. We will assume that $\dot{\mathbb{G}}(\mathbf{x}, \cdot) \in L^1(\mathbb{R}^{++},$

$Lin(Sym)$) and is symmetric. The first assumption is simply a weak "fading" memory requirement on the response functional $\tilde{\mathbf{T}}$ (see, *e.g.,* [17, 16]) whereas the second is introduced here for convenience. Some results for the case where $\dot{\mathbb{G}}$ is not symmetric are presented in [34]. Henceforth, it is understood that the statements are relative to any fixed point $\mathbf{x} \in \mathcal{B}$, if this variable is omitted.

We also assume that $\dot{\mathbb{G}}(\mathbf{x}, \cdot) \in L^2(\mathbb{R}^{++}, Lin(Sym))$ (see Appendix (A.6) and after). From now on we shall drop the dependence upon the space variable $\mathbf{x}$. The Fourier transform of $\dot{\mathbb{G}}(t)$, namely $\dot{\mathbb{G}}_F(\omega) = \dot{\mathbb{G}}_c(\omega) - i\,\dot{\mathbb{G}}_s(\omega)$, for real ω, belongs to $L^2(\mathbb{R}, Lin(Sym))$. It is clear that $\dot{\mathbb{G}}_c(\omega)$ is even, as a function of ω, and that $\dot{\mathbb{G}}_s(\omega)$ is odd. The quantity $\dot{\mathbb{G}}_s(\omega)$ therefore vanishes at the origin. In fact, a consequence of our assumption of analyticity of Fourier-transformed quantities on the real axis of Ω, is that it vanishes at least linearly at the origin. It is assumed that it vanishes no more stronger than linearly.

From [33, 34] we have, on the basis of thermodynamical arguments

$$\dot{\mathbb{G}}_s(\omega) < \mathbf{0}_4 \quad \forall\, \omega \in \mathbb{R}^{++}, \tag{4.4}$$

where $\mathbf{0}_4$ is the zero in Lin(Sym), and from [34], we also have

$$\mathbb{G}_\infty - \mathbb{G}_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{\dot{\mathbb{G}}_s(\omega)}{\omega}. \tag{4.5}$$

By virtue of (4.4), this yields

$$\mathbb{G}_0 > \mathbb{G}_\infty. \tag{4.6}$$

It is shown in [34] that (4.4) implies the inequality

$$\mathbb{G}_0 - \mathbb{G}(s) > \mathbf{0}_4. \tag{4.7}$$

It follows that $\dot{\mathbb{G}}(0) \leqq \mathbf{0}_4$. For $\dot{\mathbb{G}}(0) < \mathbf{0}_4$, we have (see (A.12))

$$\dot{\mathbb{G}}_c(\omega) \sim -\frac{\ddot{\mathbb{G}}(0)}{\omega^2}, \qquad \dot{\mathbb{G}}_s(\omega) \sim \frac{\dot{\mathbb{G}}(0)}{\omega}, \tag{4.8}$$

at large ω, where $\sim$ stands for "behaves as".

From the conclusions and assumptions of the Appendix, $\dot{\mathbb{G}}_F$ is analytic on Ω^-, which includes the real axis. The analyticity of $\dot{\mathbb{G}}_F$ implies that any singularities are at least slightly removed into $\Omega^{(+)}$, which turn means that $\dot{\mathbb{G}}$ decays exponentially at large positive times. However, formulae of physical interest will generally be continuous with respect to taking the limit to non-exponential behavior.

Applying Plancherel's theorem to (4.2), we obtain

$$\begin{aligned} \mathbf{T}(t) &= \mathbb{G}_0\mathbf{E}(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\dot{\mathbb{G}}}_F(\omega)\mathbf{E}^t_+(\omega)\, d\omega \\ &= \mathbb{G}_\infty\mathbf{E}(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\dot{\mathbb{G}}}_F(\omega)\mathbf{E}^t_{r+}(\omega)\, d\omega. \end{aligned} \tag{4.9}$$

It is worth noting that if we replace $\overline{\dot{\mathbb{G}}}_F(\omega)$ by $\left[\overline{\dot{\mathbb{G}}}_F(\omega) + \mathbb{F}(\omega)\right]$, where $\mathbb{F} \in Lin(Sym)$ is analytic on Ω^- and goes to zero at large frequencies at least as ω^{-1}, the relationship still holds. This follows by a simple application of Cauchy's theorem. In particular, we have

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\overline{\dot{\mathbb{G}}_F(\omega)} + \lambda\dot{\mathbb{G}}_F(\omega)\right]\mathbf{E}_+^t(\omega)\,d\omega, \tag{4.10}$$

where λ is any complex constant. Choosing $\lambda = -1$ yields

$$\begin{aligned}\mathbf{T}(t) &= \mathbb{G}_0\mathbf{E}(t) + \frac{i}{\pi}\int_{-\infty}^{\infty}\dot{\mathbb{G}}_s(\omega)\mathbf{E}_+^t(\omega)d\omega \\ &= \mathbb{G}_\infty\mathbf{E}(t) + \frac{i}{\pi}\int_{-\infty}^{\infty}\dot{\mathbb{G}}_s(\omega)\mathbf{E}_{r+}^t(\omega)d\omega, \\ \mathbf{E}_{r+}^t(\omega) &= \mathbf{E}_+^t(\omega) - \frac{\mathbf{E}(t)}{i\omega^-},\end{aligned} \tag{4.11}$$

where $\mathbf{E}_{r+}^t$ is the Fourier transform of $\mathbf{E}_r^t$, defined in $(4.2)_3$, as can be seen from (A.9). The equivalence of the two expressions for $\mathbf{T}$ can be seen with the aid of (4.5). For $\lambda = 1$,

$$\begin{aligned}\mathbf{T}(t) &= \mathbb{G}_0\mathbf{E}(t) + \frac{1}{\pi}\int_{-\infty}^{\infty}\dot{\mathbb{G}}_c(\omega)\mathbf{E}_+^t(\omega)d\omega \\ &= \mathbb{G}_\infty\mathbf{E}(t) + \frac{1}{\pi}\int_{-\infty}^{\infty}\dot{\mathbb{G}}_c(\omega)\mathbf{E}_{r+}^t(\omega)d\omega.\end{aligned} \tag{4.12}$$

In fact, (4.11) corresponds to taking the even extension of $\mathbb{G}$ to $\mathbb{R}$, namely $\mathbb{G}(s) = \mathbb{G}(|s|),\ \ s \in \mathbb{R}$ which yields the odd extension of $\dot{\mathbb{G}}$. This amounts to writing (4.2) as

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \int_{-\infty}^{\infty}\frac{d}{ds}\mathbb{G}(|s|)\mathbf{E}^t(s)\,ds, \tag{4.13}$$

noting that $\mathbf{E}^t$ is taken to be zero on $\mathbb{R}^-$. We can then derive $(4.11)_1$ by observing that

$$\int_{-\infty}^{\infty}\frac{d}{ds}\mathbb{G}(|s|)e^{-i\omega s}\,ds = -2i\,\dot{\mathbb{G}}_s(\omega). \tag{4.14}$$

Also, (4.12) corresponds to taking the even extension of $\dot{\mathbb{G}}$, which we denote by $\dot{\mathbb{G}}_e$. We see that (4.2) can be written as

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \int_{-\infty}^{\infty}\dot{\mathbb{G}}_e(s)\mathbf{E}^t(s)\,ds. \tag{4.15}$$

Then (4.12) follows from

$$\int_{-\infty}^{\infty}\dot{\mathbb{G}}_e(s)e^{-i\omega s}ds = 2\dot{\mathbb{G}}_c(\omega). \tag{4.16}$$

We define $\breve{\mathbb{G}} : \mathbb{R}^+ \mapsto Lin(Sym)$ as

$$\breve{\mathbb{G}}(s) := \mathbb{G}(s) - \mathbb{G}(\infty), \tag{4.17}$$

and note the relation

$$\dot{\mathbb{G}}_F(\omega) = i\omega\breve{\mathbb{G}}_+(\omega) - \breve{\mathbb{G}}(0), \tag{4.18}$$

or

$$\dot{\mathbb{G}}_s(\omega) = -\omega\breve{\mathbb{G}}_c(\omega), \qquad \dot{\mathbb{G}}_c(\omega) = \omega\breve{\mathbb{G}}_s(\omega) - \breve{\mathbb{G}}(0). \tag{4.19}$$

The relation

$$\frac{d\mathbf{E}^t_+(\omega)}{dt} = -i\omega\mathbf{E}^t_+(\omega) + \mathbf{E}(t), \tag{4.20}$$

obtained by differentiating the integral definition of $\mathbf{E}^t_+(\omega)$ and carrying out a partial integration, is required for manipulations relating to the minimum and maximum free energies. Generally, it is sufficient that it holds almost everywhere. For this to be so, it is sufficient to assume that $\mathbf{E}^t \in BV(\mathbb{R}^+) \cap \mathcal{D}^{temp}(\mathbb{R}^+)$, where $BV(\mathbb{R}^+)$ is the space of functions of bounded variation on $\mathbb{R}^+$ and $\mathcal{D}^{temp}(\mathbb{R}^+)$ is the space of tempered distributions on $\mathbb{R}^+$ (see, e.g., [20]). For the sake of simplicity, it will be assumed here that, as well as belonging to $L^2(\mathbb{R}^+)$, $\mathbf{E}^t \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$ and that its derivative also belongs to $L^1(\mathbb{R}^+)$ [50].

The work done on the material by the strain history $\mathbf{E}^t$ is

$$\begin{aligned}\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) &:= \int_{-\infty}^{t} \mathbf{T}(\tau)\cdot\dot{\mathbf{E}}(\tau)d\tau \\ &= \frac{1}{2}\mathbb{G}_0\mathbf{E}(t)\cdot\mathbf{E}(t) + \int_{-\infty}^{t}\int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^\tau(s)\cdot\dot{\mathbf{E}}(\tau)ds d\tau. \end{aligned}\tag{4.21}$$

It will be clear from the representation of $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$ in the frequency domain, given below, that it is a non-negative quantity. We will restrict our considerations to histories such that $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) < \infty$ (see Section 5). The quantity $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$ is, in some circumstances, the maximum free energy (see [18, 16] and Section 8). It will be denoted by $W(t)$. We can show that

$$\begin{aligned} W(t) &= \phi(t) + \frac{1}{2}\int_0^\infty\int_0^\infty \mathbf{E}^t_r(s_1)\cdot\mathbb{G}_{12}(|s_1-s_2|)\mathbf{E}^t_r(s_2)ds_1 ds_2 \\ &= S(t) + \frac{1}{2}\int_0^\infty\int_0^\infty \mathbf{E}^t(s_1)\cdot\mathbb{G}_{12}(|s_1-s_2|)\mathbf{E}^t(s_2)ds_1 ds_2; \\ \mathbb{G}_{12}(|s_1-s_2|) &= \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\mathbb{G}(|s_1-s_2|); \\ \phi(t) &:= \frac{1}{2}\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t); \\ S(t) &:= \mathbf{T}(t)\cdot\mathbf{E}(t) - \frac{1}{2}\mathbb{G}_0\mathbf{E}(t)\cdot\mathbf{E}(t). \end{aligned}\tag{4.22}$$

A representation of the work $W(t)$, given by (4.22), in the frequency domain, has been obtained in [27] (see also [24]) in terms of integrals over $\mathbb{R}^+$. Using symmetry arguments, this representation can be expressed as an integral over $\mathbb{R}$ of the form

$$
\begin{aligned}
W(t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{H}(\omega)\mathbf{E}^t_{r+}(\omega)\cdot\overline{\mathbf{E}}^t_{r+}(\omega)d\omega \\
&= S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{H}(\omega)\mathbf{E}^t_{+}(\omega)\cdot\overline{\mathbf{E}}^t_{+}(\omega)d\omega,
\end{aligned}
\tag{4.23}
$$

where, for each given $\omega \in \mathbb{R}$, the fourth-order tensor $\mathbb{H}(\omega) \in Lin(Sym)$ is defined as

$$
\mathbb{H}(\omega) := -\,\omega\dot{\mathbb{G}}_s(\omega). \tag{4.24}
$$

The equivalence of the two forms of (4.23) follows from (4.5) and (4.11). These reduce to the relations of GOLDEN [37] in the scalar case. It follows from $(4.8)_2$ that

$$
\mathbb{H}(\infty) = -\dot{\mathbb{G}}(0). \tag{4.25}
$$

As discussed at the beginning of the Appendix, $\mathbb{H}(\omega)$ (or indeed any tensor in $Lin(Sym)$) can be represented as a matrix acting on $\mathbb{R}^6$. From a result in [20], based on a theorem of GOHBERG & KREĬN [36], $\mathbb{H}(\omega)$ can be factorized as:

$$
\mathbb{H}(\omega) = \mathbb{H}_+(\omega)\,\mathbb{H}_-(\omega), \tag{4.26}
$$

with

$$
\mathbb{H}_+(\omega) = \mathbb{H}^*_-(\omega), \tag{4.27}
$$

where $\mathbb{H}^*_-$ is the hermitean conjugate of $\mathbb{H}_-$ (see (A.4)). The matrix functions $\mathbb{H}_\pm$ admit analytic continuations which are analytic in the interior and continuous up to the boundary of the complex half planes $\Omega^\mp$, and are such that

$$
\det\,\mathbb{H}_\pm(\zeta) \neq 0 \quad , \quad \zeta \in \Omega^\mp. \tag{4.28}
$$

Similarly, $\mathbb{H}$ has a right factorization with corresponding properties [20]. The factorization is unique up to a multiplication on the left of $\mathbb{H}_-$ by a constant, unitary matrix, and multiplication of $\mathbb{H}_+$ on the right by the inverse of this matrix. From (4.25), $\mathbb{H}_\pm(\infty)$ are non-zero and

$$
\mathbb{H}_+(\infty)\,\mathbb{H}_-(\infty) = -\dot{\mathbb{G}}(0). \tag{4.29}
$$

The notation for $\mathbb{H}_+(\omega)$ and $\mathbb{H}_-(\omega)$ follows the convention used in [37], i.e., the sign indicates the half plane where any singularities of the tensor, and any zeros in the determinant of the corresponding matrix, occur.

Consider now the second-order symmetric tensors $\mathbb{H}_-(\omega)\mathbf{E}^t_{r+}(\omega)$ and $\mathbb{H}_-(\omega)$ $\mathbf{E}^t_+(\omega)$, whose components are continuous by virtue of the properties of $\mathbb{H}_-(\omega)$ and $\mathbf{E}^t_+(\omega)$. The Plemelj formulae [47]

$$
\begin{aligned}
\mathbf{P}^t(\omega) &:= \mathbb{H}_-(\omega)\mathbf{E}^t_{r+}(\omega) = \mathbf{p}^t_-(\omega) - \mathbf{p}^t_+(\omega), \\
\mathbf{Q}^t(\omega) &:= \mathbb{H}_-(\omega)\mathbf{E}^t_{+}(\omega) = \mathbf{q}^t_-(\omega) - \mathbf{q}^t_+(\omega),
\end{aligned}
\tag{4.30}
$$

where

$$\begin{aligned} \mathbf{p}^t(z) &:= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{P}^t(\omega)}{\omega - z}d\omega\ , \quad \mathbf{p}^t_{\pm}(\omega) := \lim_{\alpha\to 0^{\mp}}\mathbf{p}^t(\omega + i\alpha), \\ \mathbf{q}^t(z) &:= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{Q}^t(\omega)}{\omega - z}d\omega\ , \quad \mathbf{q}^t_{\pm}(\omega) := \lim_{\alpha\to 0^{\mp}}\mathbf{q}^t(\omega + i\alpha). \end{aligned} \tag{4.31}$$

Moreover, $\mathbf{p}^t(z) = \mathbf{p}^t_+(z)$ is analytic in $z \in \Omega^{(-)}$ and $\mathbf{p}^t(z) = \mathbf{p}^t_-(z)$ is analytic in $z \in \Omega^{(+)}$. Both are analytic on the real axis (as indeed is $\mathbf{P}^t$) by virtue of the assumption in Section 10 on the analyticity of Fourier-transformed quantities on the real axis, and by virtue of an argument given in [37]. Similar statements apply to $\mathbf{q}^t$ and $\mathbf{Q}^t$. It can be shown that

$$\mathbf{q}^t_+(\omega) = \mathbf{p}^t_+(\omega). \tag{4.32}$$

The minimum free energy has the form [20, 21]:

$$\begin{aligned} \psi_m(t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}^t_-(\omega)|^2 d\omega \\ &= S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{q}^t_-(\omega)|^2 d\omega, \end{aligned} \tag{4.33}$$

while the work function $W(t)$, defined by (4.22), is given by

$$\begin{aligned} W(t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[|\mathbf{p}^t_-(\omega)|^2 + |\mathbf{p}^t_+(\omega)|^2\right]d\omega \\ &= S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[|\mathbf{q}^t_-(\omega)|^2 + |\mathbf{q}^t_+(\omega)|^2\right]d\omega \geqq \psi_m(t). \end{aligned} \tag{4.34}$$

Using the static continuation defined by (2.7) and (4.2) we have

$$\hat{\mathbf{T}}(\mathbf{E}^t_\tau) = \mathbb{G}(\tau)\mathbf{E}(t) + \int_0^{\infty}\dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)ds. \tag{4.35}$$

From Definition 2.5

$$\begin{aligned} &\left|\hat{\mathbf{T}}(\mathbf{E}^t_\tau)\right| < \infty, \qquad \forall\ \tau \geqq 0 \\ &\lim_{\tau\to\infty}\textstyle\int_0^{\infty}\dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)ds = 0, \end{aligned} \tag{4.36}$$

giving

$$\lim_{\tau\to\infty}\hat{\mathbf{T}}(\mathbf{E}^t_\tau) = \mathbb{G}_\infty\mathbf{E}(t). \tag{4.37}$$

Relation (4.37) follows from (4.35) and $(4.36)_2$. Since the right-hand side is the constitutive equation for an elastic solid, we must have

$$\mathbb{G}_\infty > 0. \tag{4.38}$$

It follows from (4.6) that

$$\mathbb{G}_0 > 0. \tag{4.39}$$

In [16, 18], it is observed that the linear constitutive equation (4.2) allows us to rewrite the equivalence relation among states specified by Definition 2.4 as an equivalence relation among histories (see also [41]). It is easy to prove the following statement.

Proposition 4.1. *Two histories* $\mathbf{E}_1^t$, $\mathbf{E}_2^t$ *represent equivalent states, if*

$$\mathbf{E}_1(t) = \mathbf{E}_2(t), \tag{4.40}$$

and the past histories satisfy:

$$\int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}_1^t(s)ds = \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}_2^t(s)ds\,, \quad \forall\, \tau \in \mathbb{R}^+. \tag{4.41}$$

The equality (4.41) allows us to define the minimal state by means of the function

$$\breve{\mathbf{I}}^t(\tau, \mathbf{E}^t) := -\int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}_r^t(s)ds, \tag{4.42}$$

instead of the past history $\mathbf{E}^t$. The state is given by the pair $(\mathbf{E}(t), \breve{\mathbf{I}}^t(\tau))$, or equivalently by the function

$$\mathbf{I}^t(\tau, \mathbf{E}^t) := -\mathbb{G}(\tau)\mathbf{E}(t) - \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)ds = -\mathbb{G}_\infty\mathbf{E}(t) + \breve{\mathbf{I}}^t(\tau, \mathbf{E}^t), \tag{4.43}$$

which is of course the negative of the stress associated with the static continuation in the interval $[0, \tau)$, namely $-\hat{\mathbf{T}}(\mathbf{E}_\tau^t)$, given by (4.35). From $\mathbf{I}^t(\tau, \mathbf{E}^t)$, we can obtain both $\mathbf{E}(t)$ and $\breve{\mathbf{I}}^t(\tau, \mathbf{E}^t)$, since

$$\lim_{\tau\to\infty} \mathbf{I}^t(\tau, \mathbf{E}^t) = -\mathbb{G}_\infty\mathbf{E}(t),$$

and

$$\breve{\mathbf{I}}^t(\tau, \mathbf{E}^t) = \mathbf{I}^t(\tau, \mathbf{E}^t) - \lim_{\tau\to\infty} \mathbf{I}^t(\tau, \mathbf{E}^t).$$

Thus, an equivalent form of the statement of Proposition 4.1 is: we say that $\mathbf{E}_1^t \sim \mathbf{E}_2^t$ if

$$\mathbf{I}^t(\tau, \mathbf{E}_1^t) = \mathbf{I}^t(\tau, \mathbf{E}_2^t), \quad \forall\, \tau \in \mathbb{R}^+, \tag{4.44}$$

Hence, the function $\mathbf{I}^t(\tau, \mathbf{E}^t)$ represents the equivalence class of histories, because any history $\mathbf{E}^t$ which belongs to this class yields the same value of $\mathbf{I}^t(\tau, \mathbf{E}^t)$ or $(\mathbf{E}(t), \breve{\mathbf{I}}^t(\tau))$. For this reason, $\mathbf{I}^t(\tau, \mathbf{E}^t)$ will be referred to as the minimal state.

We also define

$$\tilde{\mathbf{I}}^t(\tau, \mathbf{E}^t) := -\int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)ds \\ = \mathbb{G}(\tau)\mathbf{E}(t) + \mathbf{I}^t(\tau, \mathbf{E}^t) = \breve{\mathbb{G}}(\tau)\mathbf{E}(t) + \breve{\mathbf{I}}^t(\tau, \mathbf{E}^t). \tag{4.45}$$

In the following, for the sake of simplicity, we will usually denote by $\breve{\mathbf{I}}^t(\tau)$, $\mathbf{I}^t(\tau)$ and $\tilde{\mathbf{I}}^t(\tau)$ the functions $\breve{\mathbf{I}}^t(\tau, \mathbf{E}^t)$, $\mathbf{I}^t(\tau, \mathbf{E}^t)$ and $\tilde{\mathbf{I}}^t(\tau, \mathbf{E}^t)$. The quantities $\breve{\mathbf{I}}^t$ and $\tilde{\mathbf{I}}^t$ are functions of the minimal state $\mathbf{I}^t$ in the sense that they have the same value for different histories in the same minimal state.

We will require later the result

$$\frac{d}{dt}\breve{\mathbf{I}}^t(\tau) = -\breve{\mathbb{G}}(\tau)\dot{\mathbf{E}}(t) + \dot{\breve{\mathbf{I}}}^t(\tau), \tag{4.46}$$

where

$$\dot{\breve{\mathbf{I}}}^t(\tau) := \frac{d}{d\tau}\breve{\mathbf{I}}^t(\tau) = -\int_0^\infty \ddot{\mathbb{G}}(\tau+s)\mathbf{E}^t_r(s)ds. \tag{4.47}$$

Plancherel's theorem yields

$$\breve{\mathbf{I}}^t(\tau) = -\frac{1}{2\pi}\int_{-\infty}^\infty \overline{\dot{\mathbb{G}}_F(\omega)}\mathbf{E}^t_{r+}(\omega)e^{-i\omega\tau}\, d\omega, \qquad \tau \geqq 0. \tag{4.48}$$

Just as in (4.10):

$$\breve{\mathbf{I}}^t(\tau) = -\frac{1}{2\pi}\int_{-\infty}^\infty \left[\overline{\dot{\mathbb{G}}_F(\omega)} + \lambda\dot{\mathbb{G}}_F(\omega)\right]\mathbf{E}^t_{r+}(\omega)e^{-i\omega\tau}\, d\omega, \qquad \tau \geqq 0, \tag{4.49}$$

since the added term gives zero, which can be seen by integrating over a contour around Ω^- (noting that the exponential goes to zero as $Im\omega \to -\infty$). Let

$$\mathbf{J}^t(\tau) := \breve{\mathbf{I}}^t(-\tau), \qquad \tau \leqq 0. \tag{4.50}$$

We now have

$$\mathbf{J}^t_-(\omega) := \int_{-\infty}^0 \mathbf{J}^t(\tau)e^{-i\omega\tau}d\tau \\ = -\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{\left[\overline{\dot{\mathbb{G}}_F(\omega')} + \lambda\dot{\mathbb{G}}_F(\omega')\right]\mathbf{E}^t_{r+}(\omega')d\omega'}{\omega' - \omega^+}. \tag{4.51}$$

Similarly, let $\mathbf{J}^t$ be defined by (4.49), (4.50) for $\tau > 0$. In this case, it depends on λ. Thus

$$\mathbf{J}^t_+(\omega, \lambda) = \int_0^\infty \mathbf{J}^t(\tau, \lambda)e^{-i\omega\tau}d\tau, \\ = \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{\left[\overline{\dot{\mathbb{G}}_F(\omega')} + \lambda\dot{\mathbb{G}}_F(\omega')\right]\mathbf{E}^t_{r+}(\omega')d\omega'}{\omega' - \omega^-} \tag{4.52}$$

and

$$\mathbf{J}_F^t(\omega,\lambda) = \mathbf{J}_+^t(\omega,\lambda) + \mathbf{J}_-^t(\omega,\lambda) = -\left[\overline{\dot{\mathbb{G}}_F(\omega)} + \lambda\dot{\mathbb{G}}_F(\omega)\right]\mathbf{E}_{r+}^t(\omega), \tag{4.53}$$

by the Plemelj formulae. Referring to (4.8), we see that

$$\mathbf{J}_F^t(\omega,\lambda) \sim \omega^{-3}, \quad \lambda \neq 1, \qquad \mathbf{J}_F^t(\omega,1) \sim \omega^{-4}, \quad \lambda = 1, \tag{4.54}$$

at large ω, since $\mathbf{E}_{r+}^t(\omega) \sim \omega^{-2}$ by (A.11).

For $\lambda = -1$, we obtain

$$\begin{aligned} \mathbf{J}_-^t(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathbb{H}(\omega')\mathbf{E}_{r+}^t(\omega')d\omega'}{\omega'(\omega'-\omega^+)} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega')\left[\mathbf{p}_-^t(\omega') - \mathbf{p}_+^t(\omega')\right]d\omega'}{\omega'(\omega'-\omega^+)} \end{aligned} \tag{4.55}$$

by virtue of (4.24) and $(4.30)_1$. The $\mathbf{p}_+^t(\omega')$ term vanishes on integrating over Ω^-, and we obtain

$$\mathbf{J}_-^t(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega')\mathbf{p}_-^t(\omega')dw'}{\omega'(\omega'-\omega^+)}, \tag{4.56}$$

which is the frequency space version of a result given in [20] (for histories rather than relative histories). Since $\mathbf{p}_-^t$ is a function of the minimal state [20], it follows that $\mathbf{J}_-^t$ is also—which we have already observed. There is of course no corresponding result for $\mathbf{J}_+^t$.

Note that $\mathbf{J}_F^t(\omega,-1)$ (or $\mathbf{J}^t(s,-1)$ in the time domain) corresponds to $\breve{\mathbf{I}}^t(-\tau)$, defined by the odd extension of $\dot{\mathbb{G}}$ to $\mathbb{R}$, or

$$\breve{\mathbf{I}}^t(-\tau) = \int_0^{\infty} \frac{\partial}{\partial\tau}\mathbb{G}(|\tau-u|)\mathbf{E}_r^t(u)du, \qquad \tau\in\mathbb{R}. \tag{4.57}$$

The integral can be extended over $\mathbb{R}$, if $\mathbf{E}_r^t$ is understood to vanish on $\mathbb{R}^-$. Taking the Fourier transform immediately yields (4.53) (for $\lambda = -1$) with the aid of (4.14). Also, $\mathbf{J}_F^t(\omega,1)$ corresponds to the even extension of $\dot{\mathbb{G}}$ to $\mathbb{R}$:

$$\breve{\mathbf{I}}^t(-\tau) = -\int_0^{\infty} \dot{\mathbb{G}}_e(\tau-u)\mathbf{E}_r^t(u)du, \qquad \tau\in\mathbb{R}, \tag{4.58}$$

where $\mathbb{G}_e$ was introduced in (4.15). Again, we can extend the integration to $\mathbb{R}$ and, with the aid of (4.16), obtain (4.53) (for $\lambda = 1$), by taking the Fourier transforms.

These formulae are generalizations of (4.13) and (4.15).

A central aim of this work is to express various free energies in terms of $(\mathbf{E}(t), \breve{\mathbf{I}}^t)$, or equivalently in terms of $\mathbf{I}^t$. This ensures, in particular, that they are functions of the minimal state. New functionals, expressible in terms of $\mathbf{I}^t$, which are free energies for a fairly wide class of materials, are introduced in Section 6. Also, the minimum free energy, which can be shown to be a function of the minimal state in all cases where it exists [20], is expressed in terms of $(\mathbf{E}(t), \breve{\mathbf{I}}^t)$ in Section 7. Materials for which the maximum free energy is a function of the minimal state are characterized in Section 8. For these materials, the maximum free energy is expressed as a function of $\breve{\mathbf{I}}^t$.

5. The space of processes

We define any process $\tilde{P} \in \Pi$ over $\mathbb{R}^+$, by means of the trivial extension

$$\tilde{P}(\tau) = \begin{cases} P(\tau)\ , & \tau \in [0, d_p)\,, \\ 0, & \tau \in [d_p, \infty)\,. \end{cases} \tag{5.1}$$

This new space will be denoted by $\overset{\circ}{\Pi}$. Now let us consider the work $W(\sigma_0, \tilde{P})$, where $\sigma_0 = \mathbf{E}^0$ is the history at $t = 0$, and $\tilde{P} \in \overset{\circ}{\Pi}$ is a process such that $P(\tau) = \dot{\mathbf{E}}(\tau),\ \ \tau \in [0, d_p)$.

We have $\mathbf{E}^t = \hat{\rho}(\mathbf{E}^0, P_t)$; also the stress is given by

$$\begin{aligned} \mathbf{T}(\mathbf{E}^t) &= \mathbb{G}_0\mathbf{E}(t) + \int_0^t \dot{\mathbb{G}}(s)\mathbf{E}^t(s)\,ds + \int_t^\infty \dot{\mathbb{G}}(s)\mathbf{E}^t(s)\,ds \\ &= \mathbb{G}_0\mathbf{E}(t) + \int_0^t \dot{\mathbb{G}}(s)\mathbf{E}^t(s)\,ds - \tilde{\mathbf{I}}^0(t), \end{aligned} \tag{5.2}$$

where $\tilde{\mathbf{I}}^0$ is defined by (4.45) with $t = 0$. From (5.1), we conclude that the limit $\mathbf{E}(\infty) = \lim_{t\to+\infty} \mathbf{E}(t)$ exists. Then,

$$\begin{aligned} W(\sigma, P) &= \int_0^\infty \left\{ \mathbb{G}_0\mathbf{E}(t) + \int_0^t \dot{\mathbb{G}}(s)\mathbf{E}^t(s)\,ds \right\} \cdot \dot{\mathbf{E}}(t)\,dt - \int_0^\infty \tilde{\mathbf{I}}^0(t) \cdot \dot{\mathbf{E}}(t)\,dt \\ &= \int_0^\infty (\mathbb{G}_0\mathbf{E}(t) \cdot \dot{\mathbf{E}}(t)dt \\ &\quad + \int_0^\infty \left\{ \mathbb{G}(s)\mathbf{E}(t-s)\ |_0^t + \int_0^t \mathbb{G}(s)\dot{\mathbf{E}}^t(s)\,ds) \right\} \cdot \dot{\mathbf{E}}(t)\,dt \\ &\quad - \int_0^\infty \tilde{\mathbf{I}}^0(t) \cdot \dot{\mathbf{E}}(t)\,dt \\ &= \frac{1}{2}\int_0^\infty \int_0^\infty \mathbb{G}(|t-\tau|)\dot{\mathbf{E}}(t) \cdot \dot{\mathbf{E}}(\tau)d\tau dt - \int_0^\infty \mathbf{I}^0(t) \cdot \dot{\mathbf{E}}(t)\,dt, \end{aligned} \tag{5.3}$$

where (see (4.45))

$$\mathbf{I}^0(t) = -\mathbb{G}(t)\mathbf{E}(0) + \tilde{\mathbf{I}}^0(t)\,. \tag{5.4}$$

As noted after (4.1), we interpret $\dot{\mathbf{E}}(t)(t > 0)$ as a process, denoting it by $\dot{\mathbf{E}}^P(t)$. Let us consider the following definition of a finite work process, given by Gentili in [35].

Definition 5.1. A process $\dot{\mathbf{E}}^P : [0, \infty) \to Sym$ is said to be a *finite work process* if

$$W(0^\dagger, \dot{\mathbf{E}}^P) = \int_0^{d_P} \hat{\mathbf{T}}(0^\dagger, \dot{\mathbf{E}}^P_{[0,\tau)}) \cdot \dot{\mathbf{E}}^P(\tau)d\tau < \infty.$$

Here, the notation of (2.2) has been used. It follows from the Strong Dissipation Principle that for any $\dot{\mathbf{E}}^P \neq 0$,

$$W(0^\dagger, \dot{\mathbf{E}}^P) > 0.$$

Moreover, by virtue of (5.3), the work $W(0^\dagger, \dot{\mathbf{E}}^P)$ can be written as

$$\begin{aligned} W(0^\dagger, \dot{\mathbf{E}}^P) &= \frac{1}{2}\mathbb{G}_\infty \mathbf{E}(d_P)\cdot\mathbf{E}(d_P) \\ &\quad +\frac{1}{2}\int_0^\infty\int_0^\infty \breve{\mathbb{G}}(|\tau-s|)\dot{\mathbf{E}}^P(\tau)\cdot\dot{\mathbf{E}}^P(s)d\tau ds \\ &= \frac{1}{2}\mathbb{G}_\infty \mathbf{E}(d_P)\cdot\mathbf{E}(d_P) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\breve{\mathbb{G}}_c(\omega)\dot{\mathbf{E}}^P_+(\omega)\cdot\overline{\dot{\mathbf{E}}^P_+(\omega)}d\omega, \end{aligned} \tag{5.5}$$

where $\breve{\mathbb{G}}_c(\omega)$ is defined by (4.19).

Then, GENTILI [35] defines the *process space* as the set of finite work processes:

$$\begin{aligned} \mathcal{H}_G(\mathbb{R}^+) = \Bigg\{ &\dot{\mathbf{E}}^P \in Sym;\ \mathbf{E}(d_p)=\mathbf{0};\ \frac{1}{2}\int_0^\infty\int_0^\infty \breve{\mathbb{G}}(|\tau-s|)\dot{\mathbf{E}}^P(\tau)\cdot\dot{\mathbf{E}}^P(s)d\tau ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\breve{\mathbb{G}}_c(\omega)\dot{\mathbf{E}}^P_+(\omega)\cdot\overline{\dot{\mathbf{E}}^P_+(\omega)}d\omega < \infty\Bigg\}. \end{aligned} \tag{5.6}$$

Let us return to (5.3), referred to an arbitrary time t, denoting (σ, P) by $(\mathbf{I}^t, \dot{\mathbf{E}}^P)$. We can write:

$$\begin{aligned} W(\mathbf{I}^t, \dot{\mathbf{E}}^P) &= \frac{1}{2}\int_0^\infty\int_0^\infty \mathbb{G}\left(\left|\tau-\tau'\right|\right)\dot{\mathbf{E}}^P(\tau')\cdot\dot{\mathbf{E}}^P(\tau)d\tau d\tau' \\ &\quad -\int_0^\infty \mathbf{I}^t(\tau', \mathbf{E}^t)\cdot\dot{\mathbf{E}}^P(\tau')\,d\tau' \\ &= \frac{1}{2}\mathbb{G}_\infty\mathbf{E}_0(\infty)\cdot\mathbf{E}_0(\infty) \\ &\quad +\frac{1}{2}\int_0^\infty\int_0^\infty \breve{\mathbb{G}}(\left|\tau-\tau'\right|)\dot{\mathbf{E}}^P(\tau')\cdot\dot{\mathbf{E}}^P(\tau)d\tau d\tau' \\ &\quad +\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}_0(\infty) - \int_0^\infty \breve{\mathbf{I}}^t(\tau', \mathbf{E}^t)\cdot\dot{\mathbf{E}}^P(\tau')\,d\tau', \end{aligned} \tag{5.7}$$

where $\mathbf{E}_t(\infty) = \int_t^\infty \dot{\mathbf{E}}^P(\tau)d\tau$, and $\breve{\mathbf{I}}^t$ is defined by (4.42). Thus,

$$\begin{aligned} W(\mathbf{I}^t, \dot{\mathbf{E}}^P) = \frac{1}{2}&\mathbb{G}_\infty\mathbf{E}_0(\infty)\cdot\mathbf{E}_0(\infty) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\breve{\mathbb{G}}_c(\omega)\dot{\mathbf{E}}^P_+(\omega)\cdot\overline{\dot{\mathbf{E}}^P_+(\omega)}d\omega \\ &+\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}_0(\infty) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\breve{\mathbf{I}}^t_+(\omega)\cdot\overline{\dot{\mathbf{E}}^P_+(\omega)}d\omega < \infty, \end{aligned} \tag{5.8}$$

where

$$\breve{\mathbf{I}}^t_+(\omega) = \int_0^\infty \breve{\mathbf{I}}^t(\tau)e^{-i\omega\tau}d\tau,$$

which is in fact equal to $\mathbf{J}^t_-(-\omega)$ defined by (4.51).

Therefore, the set of admissible states $\mathbf{I}^t$ or $(\mathbf{E}(t), \check{\mathbf{I}}^t(\cdot))$ belongs to the set $Sym \times \mathcal{H}'_G(\mathbb{R}^+)$, where $\mathcal{H}'_G(\mathbb{R}^+)$ is the dual of $\mathcal{H}_G(\mathbb{R}^+)$, namely

$$\mathcal{H}'_G(\mathbb{R}^+) = \left\{ \check{\mathbf{I}}^t ; \int_0^\infty \check{\mathbf{I}}^t(\tau) \cdot \dot{\mathbf{E}}^P(\tau) d\tau < \infty, \forall\, \dot{\mathbf{E}}^P \in \mathcal{H}_G(\mathbb{R}^+) \right\}. \tag{5.9}$$

6. New free energies expressed as functionals of $\check{\mathbf{I}}^t$

Let us first present the Volterra-Graffi functional [39, 40, 16, 18]

$$\psi_G\left(\mathbf{E}^t\right) = \phi(t) - \frac{1}{2}\int_0^\infty \dot{\mathbb{G}}(s)\, \mathbf{E}^t_r(s) \cdot \mathbf{E}^t_r(s) ds, \tag{6.1}$$

which is a free energy if $\dot{\mathbb{G}}(s) \leqq \mathbf{0}_4$, $\ddot{\mathbb{G}}(s) \geqq \mathbf{0}_4$, $\forall s \in \mathbb{R}^+$. This is the Graffi-Volterra free energy and is frequently used in applications.

Remark 6.1. Note that the assumption $\ddot{\mathbb{G}}(u) \geqq \mathbf{0}_4$, $u \geqq s$ implies $\dot{\mathbb{G}}(s) \geqq \mathbf{0}_4$. It implies $\dot{\mathbb{G}}(s) < \mathbf{0}_4$ if $\ddot{\mathbb{G}}$ is non-zero on a set of finite measure with elements $u > s$, which further implies that $\dot{\mathbb{G}}(0) < \mathbf{0}_4$, except in trivial cases.

We now introduce a functional which is a free energy for a certain class of materials, in fact the same class for which the Volterra-Graffi functional, given by (6.1), is a free energy. We also present a family of related functionals which are free energies for more restrictive conditions on the material.

Let us first write down a general representation of a free energy, noting some of its properties. These were presented for the scalar case in [37]. Consider the quantity

$$\psi(t) = \phi(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}^t_r(s) \cdot \mathbb{G}_{12}(s,u)\mathbf{E}^t_r(u) ds du, \tag{6.2}$$

where $\phi(t)$ is given by $(4.22)_4$ and a numerical subscript on $\mathbb{G}$ indicates differentiation with respect to the corresponding argument, so that

$$\mathbb{G}_{12}(s,u) := \frac{\partial}{\partial s}\frac{\partial}{\partial u}\mathbb{G}(s,u). \tag{6.3}$$

We obtain

$$\mathbb{G}_{12}(s,u) = \mathbb{G}_{12}^\top(u,s) \tag{6.4}$$

without loss of generality. Since the integral in (6.2) must exist for finite relative histories, we have $\mathbb{G}_{12}(\infty, s), \mathbb{G}_{12}(s,\infty) = 0$.

The requirement that $\psi(t) \geqq \phi(t)$ for all t [5, 20] implies that the kernel $\mathbb{G}_{12}$ must be such that the integral term in (6.2) is non-negative for all histories. Furthermore, we must have

$$\dot{\psi}(t) + D(t) = \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t), \tag{6.5}$$

where $D(t)$ must be non-negative according to the second law. Equation (6.5) leads us to impose the properties

$$\mathbb{G}_1(s,0) = \mathbb{G}_2(0,s) = \dot{\mathbb{G}}(s), \qquad \mathbb{G}_2(\infty,s) = \mathbb{G}_1(s,\infty) = 0 \tag{6.6}$$

where $\dot{\mathbb{G}}(s)$ is the derivative of the relaxation function and

$$\begin{aligned} D(t) &= -\frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}_r^t(s)\cdot \mathbb{K}_{12}(s,u)\mathbf{E}_r^t(u)dsdu \geqq 0, \\ \mathbb{K}(s,u) &= \mathbb{G}_1(s,u) + \mathbb{G}_2(s,u). \end{aligned} \tag{6.7}$$

Thus, the kernel $\mathbb{K}$ must be such that the non-negativity of D is ensured. If we assume that

$$\mathbb{G}(s,\infty) = \mathbb{G}(\infty,s) = \mathbb{G}_\infty, \quad s \in \mathbb{R}^+, \tag{6.8}$$

where $\mathbb{G}_0$, $\mathbb{G}_\infty$ are defined by $(4.2)_4$, then $(6.6)_1$ yields

$$\mathbb{G}(s,0) = \mathbb{G}(0,s) = \mathbb{G}(s), \qquad \mathbb{G}(0,0) = \mathbb{G}_0, \tag{6.9}$$

and

$$\mathbb{G}(s,u) = \mathbb{G}_\infty + \int_s^\infty \int_u^\infty \mathbb{G}_{12}(s',u')ds'du', \tag{6.10}$$

which clearly obeys (6.3). It follows that $\psi(t)$ can be expressed as

$$\begin{aligned} \psi(t) &= \phi_r(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \dot{\mathbf{E}}^t(s)\cdot \mathbb{G}(s,u)\dot{\mathbf{E}}^t(u)dsdu, \\ \phi_r(t) &= \phi(t) - \frac{1}{2}\mathbb{G}_\infty \mathbf{E}_r^t(\infty)\cdot \mathbf{E}_r^t(\infty), \qquad \dot{\mathbf{E}}^t(s) = -\frac{d}{ds}\mathbf{E}^t(s) = \frac{d}{dt}\mathbf{E}^t(s) \end{aligned} \tag{6.11}$$

if $\mathbf{E}(-\infty)$ is finite.

It is worth noting that $\mathbb{G}(\cdot,\cdot)$, given by (6.10), is an absolutely continuous function whenever $\mathbb{G}_{12} \in L^1(\mathbb{R}^+ \times \mathbb{R}^+)$; in this case, (6.2) is well defined even if $\mathbf{E}_r^t \in BV(\mathbb{R}^+)$. The corresponding expression (6.11) is also well defined since the kernel is absolutely continuous and $\dot{\mathbf{E}}^t \in L^1(\mathbb{R}^+)$.

Consider the functional

$$\Psi_F(\mathbf{I}^t) = \phi(t) - \frac{1}{2}\int_0^\infty \dot{\mathbb{G}}^{-1}(\tau)\dot{\check{\mathbf{I}}}^t(\tau)\cdot \dot{\check{\mathbf{I}}}^t(\tau)d\tau, \tag{6.12}$$

where $\dot{\mathbb{G}}^{-1}(\tau)$ is the inverse of the tensor $\dot{\mathbb{G}}(\tau)$ and $\dot{\check{\mathbf{I}}}^t$ is defined by (4.47). It is assumed that $\ddot{\mathbb{G}}(\tau)$ is positive semidefinite for all $\tau \in \mathbb{R}^+$, from which it follows that $\dot{\mathbb{G}}(\tau)$ is negative semidefinite for all $\tau \in \mathbb{R}^+$. Thus, the integral term in (6.12) is non-negative. The tensor $\dot{\mathbb{G}}^{-1}$ becomes singular at large τ, but it is clear from the representation (6.13) below that the integral exists.

The domain of definition of the functional Ψ_F will be denoted by $\mathcal{H}_F(\mathbb{R}^+) = Sym \times \mathcal{H}_F^*(\mathbb{R}^+)$, where

$$\mathcal{H}_F^*(\mathbb{R}^+) = \left\{ \breve{\mathbf{I}}^t\,;\ \int_0^\infty \dot{\mathbb{G}}^{-1}(\tau\dot{\breve{\mathbf{I}}}^t(\tau)\cdot\dot{\breve{\mathbf{I}}}^t(\tau)d\tau < \infty \right\}.$$

This space is very much larger than the domain of definition of the Graffi-Volterra free energy, as we see for a kernel given by an exponential or a sum of exponentials.

As a function of the relative history $\mathbf{E}_r^t$, the functional Ψ_F can be written

$$\begin{aligned}\Psi_F(\mathbf{I}^t) &= \phi(t)\\ &\quad -\frac{1}{2}\int_0^\infty\int_0^\infty\int_0^\infty \ddot{\mathbb{G}}(\tau+s_2)\dot{\mathbb{G}}^{-1}(\tau)\\ &\quad \ddot{\mathbb{G}}(\tau+s_1)\mathbf{E}_r^t(s_1)\cdot\mathbf{E}_r^t(s_2)ds_1ds_2d\tau\\ &= \phi(t)+\frac{1}{2}\int_0^\infty\int_0^\infty \mathbb{G}_{12}(s_1,s_2)\mathbf{E}_r^t(s_1)\cdot\mathbf{E}_r^t(s_2)ds_1ds_2,\end{aligned} \tag{6.13}$$

where

$$\mathbb{G}(s_1,s_2) = -\int_0^\infty \dot{\mathbb{G}}(\tau+s_2)\dot{\mathbb{G}}^{-1}(\tau)\dot{\mathbb{G}}(\tau+s_1)d\tau + \mathbb{G}_\infty = \mathbb{G}(s_2,s_1), \tag{6.14}$$

and $\mathbb{G}_{12}$ is given by (6.3). We see that $\mathbb{G}(s_1,s_2)$ obeys (6.4) and (6.9). Also, the quantity $\mathbb{K}$ in (6.7) is given by

$$\mathbb{K}(s_1,s_2) = -\int_0^\infty (\ddot{\mathbb{G}}(\tau+s_1)\dot{\mathbb{G}}^{-1}(\tau)\dot{\mathbb{G}}(\tau+s_2)+\dot{\mathbb{G}}(\tau+s_1)\dot{\mathbb{G}}^{-1}(\tau)\ddot{\mathbb{G}}(\tau+s_2))d\tau.$$

Partial integration with respect to τ gives

$$\mathbb{K}(s_1,s_2) = \dot{\mathbb{G}}(s_1)\dot{\mathbb{G}}^{-1}(0)\dot{\mathbb{G}}(s_2) + \int_0^\infty \dot{\mathbb{G}}(\tau+s_1)\frac{d}{d\tau}\dot{\mathbb{G}}^{-1}(\tau)\dot{\mathbb{G}}(\tau+s_2)d\tau,$$

which yields a non-negative dissipation since, under our assumptions,

$$\frac{d}{d\tau}\dot{\mathbb{G}}^{-1}(\tau) = -\dot{\mathbb{G}}^{-1}(\tau)\ddot{\mathbb{G}}(\tau)\dot{\mathbb{G}}^{-1}(\tau) \tag{6.15}$$

is a non-positive tensor (i.e. negative semidefinite).

In fact, a more direct demonstration of the fact that Ψ_F is a free energy can be given, though the above, general, approach is used later. From (4.46),

$$\frac{d}{dt}\dot{\breve{\mathbf{I}}}^t(\tau) = -\dot{\mathbb{G}}(\tau)\dot{\mathbf{E}}(t) + \ddot{\breve{\mathbf{I}}}^t(\tau), \tag{6.16}$$

so that

$$
\begin{aligned}
\frac{d}{dt}\Psi_F(\mathbf{I}^t) &= \mathbf{T}(t)\cdot\dot{\mathbf{E}}(t) - \int_0^\infty \dot{\mathbb{G}}^{-1}(\tau)\ddot{\breve{\mathbf{I}}}^t(\tau)\cdot\dot{\breve{\mathbf{I}}}^t(\tau)d\tau \\
&= \mathbf{T}(t)\cdot\dot{\mathbf{E}}(t) + \frac{1}{2}\dot{\mathbb{G}}^{-1}(0)\dot{\breve{\mathbf{I}}}^t(0)\cdot\dot{\breve{\mathbf{I}}}^t(0) \\
&\quad + \frac{1}{2}\int_0^\infty \left[\frac{d}{d\tau}\dot{\mathbb{G}}^{-1}(\tau)\right]\dot{\breve{\mathbf{I}}}^t(\tau)\cdot\dot{\breve{\mathbf{I}}}^t(\tau)d\tau.
\end{aligned}
\tag{6.17}
$$

Thus, we have

$$
D(t) = -\frac{1}{2}\dot{\mathbb{G}}^{-1}(0)\dot{\breve{\mathbf{I}}}^t(0)\cdot\dot{\breve{\mathbf{I}}}^t(0) - \frac{1}{2}\int_0^\infty \left[\frac{d}{d\tau}\dot{\mathbb{G}}^{-1}(\tau)\right]\dot{\breve{\mathbf{I}}}^t(\tau)\cdot\dot{\breve{\mathbf{I}}}^t(\tau)d\tau \geqq 0.
\tag{6.18}
$$

Note that

$$
D(t) \geqq -\frac{1}{2}\int_0^\infty \left[\frac{d}{d\tau}\dot{\mathbb{G}}^{-1}(\tau)\right]\dot{\breve{\mathbf{I}}}^t(\tau)\cdot\dot{\breve{\mathbf{I}}}^t(\tau)d\tau \geqq 0.
\tag{6.19}
$$

Let us further assume that there exists a non-negative $\alpha_1 \in \mathbb{R}^{++}$, such that

$$
\ddot{\mathbb{G}}(\tau) + \alpha_1\dot{\mathbb{G}}(\tau) \geqq 0 \quad \forall\tau \in \mathbb{R}^+.
\tag{6.20}
$$

This yields

$$
\frac{d}{d\tau}\dot{\mathbb{G}}^{-1}(\tau) \leqq \alpha_1\dot{\mathbb{G}}^{-1}(\tau) \leqq 0,
\tag{6.21}
$$

and, from (6.19),

$$
D(t) \geqq \alpha_1\left(\Psi_F(\mathbf{I}^t) - \phi(t)\right).
\tag{6.22}
$$

We can express a family of free energies using a simple generalization of the above procedure. Consider, for a given integer $n \geqq 1$,

$$
\begin{aligned}
\Psi_n(\mathbf{I}^t) &= \phi(t) + \frac{(-1)^n}{2}\int_n \mathbb{G}_n^{-1}(\tau)\breve{\mathbf{I}}_n^t(\tau_n)\cdot\breve{\mathbf{I}}_n^t(\tau_n), \\
\mathbb{G}_n(\tau) &:= \frac{d^n}{d\tau^n}\mathbb{G}(\tau), \qquad \breve{\mathbf{I}}_n^t(\tau) = \frac{d^n}{d\tau^n}\breve{\mathbf{I}}_n^t(\tau), \\
\int_n &:= \int_0^\infty d\tau_1\int_{\tau_1}^\infty d\tau_2\int_{\tau_2}^\infty d\tau_3\cdots\int_{\tau_{n-1}}^\infty d\tau_n.
\end{aligned}
\tag{6.23}
$$

It is assumed that for all τ,

$$
(-1)^{n+1}\mathbb{G}_{n+1}(\tau) \geqq 0.
\tag{6.24}
$$

It follows that

$$
(-1)^m\mathbb{G}_m(\tau) \geqq 0,
\tag{6.25}
$$

where m is any integer in the interval $0 < m \leqq n$. Using a generalization of (6.16):

$$\frac{d}{dt}\breve{\mathbf{I}}^t_n(\tau) = -\mathbb{G}_n(\tau)\dot{\mathbf{E}}(t) + \breve{\mathbf{I}}^t_{n+1}(\tau), \tag{6.26}$$

and we can show that

$$\begin{aligned}\frac{d}{dt}\Psi_n(\mathbf{I}^t) = \mathbf{T}(t)\cdot\dot{\mathbf{E}}(t) + \frac{(-1)^{n-1}}{2}\int_{n-1}\mathbb{G}_n^{-1}(\tau_{n-1})\breve{\mathbf{I}}^t_n(\tau_{n-1})\cdot\breve{\mathbf{I}}^t_n(\tau_{n-1})\\ -\frac{(-1)^n}{2}\int_n\left[\frac{d}{d\tau}\mathbb{G}_n^{-1}(\tau_n)\right]\breve{\mathbf{I}}^t_n(\tau_n)\cdot\breve{\mathbf{I}}^t_n(\tau_n).\end{aligned} \tag{6.27}$$

For $n = 1$, the middle term on the right-hand side of (6.27) is understood to yield the middle term on the right of $(6.17)_2$. We have

$$\begin{aligned}D(t) = \frac{(-1)^n}{2}\int_{n-1}\mathbb{G}_n^{-1}(\tau_{n-1})\breve{\mathbf{I}}^t_n(\tau_{n-1})\cdot\breve{\mathbf{I}}^t_n(\tau_{n-1})\\ +\frac{(-1)^n}{2}\int_n\left[\frac{d}{d\tau_n}\mathbb{G}_n^{-1}(\tau_n)\right]\breve{\mathbf{I}}^t_n(\tau_n)\cdot\breve{\mathbf{I}}^t_n(\tau_n) \geqq 0,\end{aligned} \tag{6.28}$$

which is non-negative since

$$(-1)^n\frac{d}{d\tau}\mathbb{G}_n^{-1}(\tau) = (-1)^{n+1}\mathbb{G}_n^{-1}(\tau)\mathbb{G}_{n+1}(\tau)\mathbb{G}_n^{-1}(\tau) \geqq 0$$

by virtue of (6.24). Because the first term on the right of (6.28) is positive we have

$$D(t) \geqq \frac{(-1)^n}{2}\int_n\left[\frac{d}{d\tau_n}\mathbb{G}_n^{-1}(\tau_n)\right]\breve{\mathbf{I}}^t_n(\tau_n)\cdot\breve{\mathbf{I}}^t_n(\tau_n) \geqq 0. \tag{6.29}$$

Again, if we assume that an $\alpha_n > 0$ exists such that

$$(-1)^{n+1}\left[\mathbb{G}_{n+1}(\tau) + \alpha_n\mathbb{G}_n(\tau)\right] \geqq 0, \quad \forall\tau\in\mathbb{R}^+, \tag{6.30}$$

then

$$(-)^n\frac{d}{d\tau}\mathbb{G}_n^{-1}(\tau) \geqq (-1)^n\alpha_n\mathbb{G}_n(\tau)$$

and (6.22) holds, with α_n replacing α_1.

We deduce from (6.25) that each Ψ_m, $0 < m \leqq n$ is also a free energy if (6.24) holds. Note that if $\mathbb{G}$ is completely monotonic, in other words, if (6.25) holds for all integers $m \geqq 0$ [16, 18], then there is an infinite sequence of free energies given by (6.24).

Finally, we consider

$$\Psi_0(I^t) = \phi(t) + \frac{1}{2}\breve{\mathbb{G}}^{-1}(0)\breve{\mathbf{I}}^t(0)\cdot\breve{\mathbf{I}}^t(0),$$

where the second term is positive by virtue of the fact that $\breve{\mathbb{G}}^{-1}(0) > 0$ (see (4.6)). By similar manipulations to those in (6.13) and (6.14), we obtain

$$\mathbb{G}(s_1, s_2) = \breve{\mathbb{G}}(s_1)\breve{\mathbb{G}}^{-1}(0)\breve{\mathbb{G}}(s_2) + \mathbb{G}_\infty,$$

which clearly obeys (6.9). Also

$$\mathbb{K}(s_1, s_2) = \dot{\mathbb{G}}(s_1)\breve{\mathbb{G}}^{-1}(0)\breve{\mathbb{G}}(s_2) + \breve{\mathbb{G}}(s_1)\breve{\mathbb{G}}^{-1}(0)\dot{\mathbb{G}}(s_2).$$

Let us assume that

$$\dot{\mathbb{G}}(\tau) \leqq 0 \quad \forall \tau \in \mathbb{R}^+, \tag{6.31}$$

which implies that $\breve{\mathbb{G}}(\tau) \geqq 0, \quad \tau \in \mathbb{R}^+$, and further assume that there exists a non-negative $\alpha_0 \in \mathbb{R}^{++}$ such that

$$\dot{\mathbb{G}}(\tau) + \alpha_0 \breve{\mathbb{G}}(\tau) = 0 \quad \forall \tau \in \mathbb{R}^+, \tag{6.32}$$

so that $\breve{\mathbb{G}}$ consists of a single exponential term. We then have (6.22) as an equality with $2\alpha_0$ replacing α_1, which implies a non-negative dissipation. Note that we can write Ψ_0 as

$$\Psi_0(I^t) = \phi(t) + \frac{1}{2}\breve{\mathbb{G}}^{-1}(0)\left(\mathbf{T}(t) - \mathbb{G}_\infty \mathbf{E}(t)\right) \cdot \left(\mathbf{T}(t) - \mathbb{G}_\infty \mathbf{E}(t)\right). \tag{6.33}$$

In [18] (Section 4, Corollary 4.2), it is proved that (6.33) is a functional of the minimal state (which is also a free energy first considered in [4]). They show that it is a free energy if, and only if, $\mathbb{G}$ is of exponential type with $\dot{\mathbb{G}}(0) \leqq \mathbf{0}_4$.

7. The minimum free energy in terms of $\breve{\mathbf{I}}^t$

Consider (5.3), referred to time t rather than the origin. It is convenient to write it over $\mathbb{R}^-$ rather than $\mathbb{R}^+$ so that we obtain

$$\begin{aligned} W(\sigma, P) = \tfrac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{0} \mathbb{G}(|s-u|)\dot{\mathbf{E}}^t(s)\cdot\dot{\mathbf{E}}^t(u)\,du\,ds \\ - \int_{-\infty}^{0} \mathbf{I}^t(-s)\cdot\dot{\mathbf{E}}^t(s)\,ds, \end{aligned} \tag{7.1}$$

where $\dot{\mathbf{E}}^t(s)$ describes the process $\dot{\mathbf{E}}(t+|s|),\ s \in \mathbb{R}^-$ and $\mathbf{I}^t$ is given by (4.43).

We seek the maximum recoverable work respect to the set of processes where $\dot{\mathbf{E}}^t(\cdot) \in \mathcal{H}_G(\mathbb{R}^-)$ given by

$$\dot{\mathbf{E}}^t(s) = \dot{\mathbf{E}}^t_m(s) + \varepsilon \mathbf{e}(s) \quad , \quad s \in \mathbb{R}^-,$$

where ε is a real parameter and $\mathbf{e} \in \mathcal{H}_G(\mathbb{R}^-)$. If $\dot{\mathbf{E}}^t_m$ is the process for which we obtain the maximum recoverable work, we have.

$$\begin{aligned} &\frac{d}{d\varepsilon}\left[-W(\sigma, P)\right]_{\varepsilon=0} = \\ &\quad -\int_{-\infty}^{0}\int_{-\infty}^{0} \mathbb{G}(|s-u|)\dot{\mathbf{E}}^t_m(u)\cdot\mathbf{e}(s)\,du\,ds + \int_{-\infty}^{0}\mathbf{I}^t(-s)\cdot\mathbf{e}(s)\,ds = 0, \end{aligned} \tag{7.2}$$

From (7.1) by virtue of the arbitrariness of $\mathbf{e}(s)$, we obtain

$$\int_{-\infty}^{0} \mathbb{G}(|s-u|)\dot{\mathbf{E}}_m^t(u)\,du = \mathbf{I}^t(-s), \quad s \leqq 0. \tag{7.3}$$

Equation (7.3) is a Wiener-Hopf equation, the solution of which maximizes the recoverable work. The minimum free energy is given by (3.6). From (7.1), (7.3) we obtain

$$\psi_m(\mathbf{E}^t) = \psi_m(t) = \frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{0} \mathbb{G}(|s-u|)\dot{\mathbf{E}}_m^t(u)\cdot\dot{\mathbf{E}}_m^t(s)duds, \tag{7.4}$$

where $\dot{\mathbf{E}}_m^t$ is the solution of (7.3).

We now seek to solve (7.3). We can write it as

$$-\mathbb{G}_\infty\mathbf{E}(t) + \mathbb{G}_\infty\mathbf{E}_m^t(-\infty) + \int_{-\infty}^{0} \breve{\mathbb{G}}(|s-u|)\dot{\mathbf{E}}_m^t(u)\,du = \mathbf{I}^t(-s), \quad s \leqq 0. \tag{7.5}$$

Noting that

$$\dot{\mathbf{E}}_m^t(u) = -\frac{\partial}{\partial u}\mathbf{E}_{mr}^t(u), \qquad \mathbf{E}_{mr}^t(u) = \mathbf{E}_m^t(u) - \mathbf{E}(t),$$

and carrying out a partial integration relation (7.5) yields

$$\int_{-\infty}^{0} \frac{\partial}{\partial s}\mathbb{G}(|s-u|)\mathbf{E}_{mr}^t(u)\,du - \breve{\mathbf{I}}^t(-s) = -\mathbf{J}^t(s), \quad s < 0, \tag{7.6}$$

on utilizing (4.43). The quantity $\mathbf{J}^t$ is defined by (4.50), so that

$$\begin{aligned}\mathbf{J}^t(s) &= \int_{0}^{\infty} \frac{\partial}{\partial s}\mathbb{G}(u+|s|)\mathbf{E}_r^t(u)\,du \\ &= \int_{0}^{\infty} \frac{\partial}{\partial s}\mathbb{G}(|s-u|)\mathbf{E}_r^t(u)\,du, \qquad s < 0.\end{aligned} \tag{7.7}$$

We will use this last form to define $\mathbf{J}^t$ on $\mathbb{R}$, which in fact corresponds to (4.57) or $\mathbf{J}^t(s,-1)$ in the notation in the integrand of $(4.52)_1$.

In order to solve the Wiener-Hopf equation (7.6), we write it in the form [35, 29]

$$\int_{-\infty}^{0} \frac{\partial}{\partial s}\mathbb{G}(|s-u|)\mathbf{E}_{mr}^t(u)\,du + \mathbf{J}^t(s) = \mathbf{R}^t(s), \quad s \in \mathbb{R}, \tag{7.8}$$

where $\mathbf{R}^t$ is, for the moment, unknown apart from the fact that

$$\mathbf{R}^t(s) = 0, \quad s \in \mathbb{R}^-. \tag{7.9}$$

Using (4.14), we take the Fourier transform of (7.8) to obtain

$$\frac{2i}{\omega}\,\mathbb{H}(\omega)\mathbf{E}_{(m)}^t(\omega) + \mathbf{J}_F^t(\omega) = \mathbf{R}_+^t(\omega), \tag{7.10}$$

where $\mathbf{E}^t_{(m)}(\omega)$ is the Fourier transform of the optimal relative continuation, related to the optimal relative process. It is analytic on $\Omega^{(+)}$. Also, $\mathbf{R}^t_+(\omega)$, by virtue of (7.9), is analytic in $\Omega^{(-)}$. The behavior of $\mathbf{J}^t_F(\omega, -1)$ for large ω is given by (4.54), which is important in the present context. Any other value of λ could be used. However, it is undesireable to choose $\mathbf{J}^t_F(\omega) = \mathbf{J}^t_-(\omega)$ (in other words, where the extension is zero). This choice vanishes as ω^{-1} at large ω. Such weak decay to zero causes convergence problems (which can in fact be overcome with care) in integrals introduced later (see (7.19)). Recalling the factorization of $\mathbb{H}(\omega)$ given by (4.26), it is clear that (7.10) may be written in the form

$$\mathbb{H}_-(\omega)\mathbf{E}^t_{(m)}(\omega) + \frac{\omega}{2i}\left[\,\mathbb{H}_+(\omega)\right]^{-1}\mathbf{J}^t_F(\omega) = \frac{\omega}{2i}\left[\,\mathbb{H}_+(\omega)\right]^{-1}\mathbf{R}^t_+(\omega), \tag{7.11}$$

if $\left[\,\mathbb{H}_+(\omega)\right]^{-1}$ exists, which is true in particular for $\omega \in \mathbb{R}\backslash\{0\}$. The factor ω ensures that it is well-defined at the origin. We put

$$\begin{aligned}\mathbf{P}^t(\omega) &:= \frac{\omega}{2i}\left[\,\mathbb{H}_+(\omega)\right]^{-1}\mathbf{J}^t_F(\omega)\\ &= \mathbf{p}^t_-(\omega) - \mathbf{p}^t_+(\omega),\end{aligned} \tag{7.12}$$

where $\mathbf{p}^t_\pm(\omega)$ is analytic on $\Omega^\mp$ respectively. They can be written, with the aid of the Plemelj formulae, in the form

$$\mathbf{p}^t_\pm(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{P}^t(\omega')}{\omega' - \omega^\mp}d\omega', \tag{7.13}$$

the integral being convergent since $\mathbf{J}^t_F(\omega)$ decays as ω^{-3} for large ω, or, more strongly, if $\lambda = 1$. This step is of course identical to (4.30) and (4.31), as can be seen from (4.53) for $\lambda = -1$. By means of a standard argument, [35, 29], we deduce that

$$\mathbf{E}^t_{(m)}(\omega) = -\left[\,\mathbb{H}_-(\omega)\right]^{-1}\mathbf{p}^t_-(\omega). \tag{7.14}$$

It follows from (4.54) and (7.13) that

$$\mathbf{p}^t_-(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\frac{\omega'}{2i}\left[\,\mathbb{H}_+(\omega')\right]^{-1}\mathbf{J}^t_-(\omega')_3}{\omega' - \omega^+}d\omega', \tag{7.15}$$

where (see (A.11))

$$\mathbf{J}^t_-(\omega)_3 = \mathbf{J}^t_-(\omega) + \frac{\mathbf{J}^t(0)}{i\omega} - \frac{\mathbf{J}^t_{(1)}(0)}{\omega^2}, \qquad \mathbf{J}^t_{(1)}(s) := \frac{d}{ds}\mathbf{J}^t(s). \tag{7.16}$$

The quantity $\mathbf{J}^t_-(\omega)_3$ is that part of $\mathbf{J}^t_-(\omega)$ with the terms decaying as ω^{-1}, ω^{-2} removed. It is easy to show that the term $\mathbf{J}^t_+(\omega)_3$, where $\mathbf{J}^t_-(\omega)_3 + \mathbf{J}^t_+(\omega)_3 = \mathbf{J}^t_F(\omega)$, does not contribute to $\mathbf{p}^t_-(\omega)$. If λ is varied from -1, $\mathbf{p}^t_-$ is not affected. However, there is no similar argument for $\mathbf{p}^t_+$, which will change if λ is varied. If $\lambda = -1$, (4.34) holds.

It is convenient to use (7.13) rather than (7.15) in the sequel.

The minimum free energy is given by (4.33). Our objective is to write this quantity in the time domain, as a quadratic form in $\breve{\mathbf{I}}^t$.

Let $\boldsymbol{\Pi}^t$ be the inverse Fourier transform of $\mathbf{P}^t$. Then

$$\mathbf{P}^t(\omega) = \int_{-\infty}^{\infty} \boldsymbol{\Pi}^t(s)e^{-i\omega s}ds \tag{7.17}$$

and

$$\begin{aligned}\mathbf{p}^t_+(\omega) &= -\textstyle\int_0^\infty \boldsymbol{\Pi}^t(s)e^{-i\omega s}ds,\\ \mathbf{p}^t_-(\omega) &= \textstyle\int_{-\infty}^0 \boldsymbol{\Pi}^t(s)e^{-i\omega s}ds.\end{aligned} \tag{7.18}$$

The quantity $\boldsymbol{\Pi}^t$ can be written more explicitly, by means of the Faltung theorem. However, we first need to write $\mathbf{P}^t$ as the product of two functions both in $L^2(\mathbb{R})$. Let us divide and multiply by $(\omega^-)^2$, omitting the superscript for factors in the numerator, where it is irrelevant. Thus, we obtain

$$\mathbf{P}^t(\omega) = \frac{\left[\,\mathbb{H}_+(\omega)\right]^{-1}}{2i\omega^-}\left[\omega^2\mathbf{J}^t_F(\omega)\right], \tag{7.19}$$

where both factors are in $L^2(\mathbb{R})$ and where the first factor has all its singularities in $\Omega^{(+)}$. We define

$$\mathbf{J}^t_{(2)}(s) := \frac{d^2}{ds^2}\mathbf{J}^t(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2\mathbf{J}^t_F(\omega)e^{i\omega s}d\omega. \tag{7.20}$$

Also, let us define $\mathbb{M} \in Lin(Sym)$ by

$$\mathbb{M}(s) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left[\,\mathbb{H}_+(\omega)\right]^{-1}}{2i\omega^-}e^{i\omega s}d\omega, \quad s \in \mathbb{R}. \tag{7.21}$$

The integrand has a quadratic singularity near the origin, owing to the explicit pole term and the factor ω in $\mathbb{H}_+(\omega)$ which is taken, for consistency, to be ω^-. This gives a finite contribution, which is easily calculated. The quantity $\mathbb{M}$ vanishes for $s \in \mathbb{R}^{--}$.

By the Faltung theorem, we have

$$\boldsymbol{\Pi}^t(s) = -\int_{-\infty}^{s}\mathbb{M}(s-u)\mathbf{J}^t_{(2)}(u)du. \tag{7.22}$$

From (7.18) it is clear that only non-positive arguments of $\mathbf{J}^t_{(2)}$ contribute to $\mathbf{p}^t_-$. Also, from (7.18) and Parseval's theorem,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}^t_-(\omega)|^2d\omega = \int_{-\infty}^{0}|\boldsymbol{\Pi}^t(s)|^2ds. \tag{7.23}$$

Extending the upper limit in (7.22) to zero and interchanging integrations, we have

$$\psi_m(t) = \phi(t) + \int_{-\infty}^{0} \int_{-\infty}^{0} \mathbf{J}^t_{(2)}(u) \cdot \mathbb{L}(u, v)\mathbf{J}^t_{(2)}(v)dudv. \tag{7.24}$$

where $\mathbb{L} \in Lin(Sym)$ is given by

$$\begin{aligned} \mathbb{L}(u, v) &:= \int_{-\infty}^{0} \mathbb{M}^*(s-u)\ \mathbb{M}(s-v)ds \\ &= \int_{\max(u,v)}^{0} \mathbb{M}^*(s-u)\ \mathbb{M}(s-v)ds, \end{aligned} \tag{7.25}$$

where $\mathbb{M}^*$ is the hermitean conjugate of $\mathbb{M}$, which also vanishes on $\mathbb{R}^{--}$.

Observe that

$$\mathbb{L}^*(u, v) = \mathbb{L}(v, u), \tag{7.26}$$

which ensures that ψ_m is real. If $\mathbb{M}(s_1)$, $\mathbb{M}^*(s_2)$, $\forall\, s_1,\ s_2 \in \mathbb{R}^+$, commute, then $\mathbb{L}$ is hermitean. Note that

$$\mathbb{L}(u, 0) = \mathbb{L}(0, v) = 0\ \forall\, u, v \in \mathbb{R}^+ \tag{7.27}$$

and also, since $\mathbb{M}$, $\mathbb{M}^* \in L^2(\mathbb{R}^+)$,

$$\mathbb{L}(u, -\infty) = \mathbb{L}(-\infty, v) = 0\ \forall\, u, v \in \mathbb{R}^{--}. \tag{7.28}$$

Therefore, we can write (7.24) as

$$\begin{aligned} \psi_m(t) &= \phi(t) + \int_{-\infty}^{0} \int_{-\infty}^{0} \mathbf{J}^t_{(1)}(u) \cdot \mathbb{L}_{12}(u, v)\mathbf{J}^t_{(1)}(v)dudv \\ \mathbb{L}_{12}(u, v) &= \frac{\partial^2}{\partial u \partial v}\mathbb{L}(u, v). \end{aligned} \tag{7.29}$$

It follows from (7.21) that the quantity $\mathbb{L}_{12}$ will have singular delta distribution terms. This can be avoided by using (7.24). Referring to (4.50), it can be written in terms of $\breve{\mathbf{I}}^t$ as

$$\psi_m(t) = \phi(t) + \int_{0}^{\infty} \int_{0}^{\infty} \breve{\mathbf{I}}^t_{(2)}(u) \cdot \mathbb{K}(u, v)\breve{\mathbf{I}}^t_{(2)}(v)dudv, \tag{7.30}$$

where

$$\breve{\mathbf{I}}^t_{(2)}(s) := \frac{d^2}{ds^2}\breve{\mathbf{I}}^t(s) \tag{7.31}$$

and

$$\begin{aligned} \mathbb{K}(u, v) &= \mathbb{L}(-u, -v) \\ &= \int_{0}^{\min(u,v)} \mathbb{M}^*(u-s)\ \mathbb{M}(v-s)ds. \end{aligned} \tag{7.32}$$

We now give an expression for the rate of dissipation. This quantity is given by [37, 20]

$$D_m(t) = |\mathbf{K}(t)|^2, \tag{7.33}$$

where

$$i\mathbf{K}(t) = \lim_{\omega\to\infty} \omega\mathbf{p}^t_-(\omega). \tag{7.34}$$

A slightly different formula was given in [37, 20] in terms of $\mathbf{q}^t_-(\omega)$, defined by (4.31), which can easily be shown to be equivalent to (7.34). From $(7.18)_2$, we deduce with the aid of (A.11) that

$$\begin{aligned}\mathbf{K}(t) = \mathbf{\Pi}^t(0) &= -\int_{-\infty}^{0} \mathbb{M}(-u)\mathbf{J}^t_{(2)}(u)du \\ &= -\int_0^{\infty} \mathbb{M}(u)\breve{\mathbf{I}}^t_{(2)}(u)du,\end{aligned} \tag{7.35}$$

on using (7.22) and (4.50). Therefore

$$\begin{aligned}D_m(t) &= \left|\int_0^{\infty} \mathbb{M}(u)\breve{\mathbf{I}}^t_{(2)}(u)du\right|^2 \\ &= \int_0^{\infty}\int_0^{\infty} \breve{\mathbf{I}}^t_{(2)}(u)\mathbb{N}(u,v)\breve{\mathbf{I}}^t_{(2)}(v)dudv\end{aligned} \tag{7.36}$$

$$\mathbb{N}(u,v) := \mathbb{M}^*(u)\,\mathbb{M}(v).$$

8. Maximum free energy as a functional of $\breve{\mathbf{I}}^t$

We now consider the maximum free energy, defined in the general theory by (3.8) or (3.10).

Definition 3.1 has been adopted for the free energies, and the set of such free energies has been labelled by $\mathcal{F}$. A slightly different definition (which takes a free energy to be a lower potential for the work done on processes with respect to a suitable norm) has been given in [18] (Section 2, Definition 2.2): the set of functions obeying this definition was called $\mathcal{F}_2(\mathbb{G})$. There, $\mathbb{G}$ appeared in order to emphasize the circumstance that any definition of free energy is relative to a given relaxation function. In [16], further progress was made. In particular, besides a normalizing constant (which in our case is zero), it has been pointed out that the quantity $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$, defined in (4.21), is the maximal element in a suitable set of functions. Such a set is denoted by $\mathcal{F}_1(\mathbb{G})$ and this is not, in general, coincident with $\mathcal{F}_2(\mathbb{G})$: indeed, elements in $\mathcal{F}_1(\mathbb{G})$ need not be functions of state. The fact that $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$ is the maximal element of $\mathcal{F}_1(\mathbb{G})$, is no longer, in general, true for free energies defined according to Definition 3.1, although this is certainly the case when the minimal state coincides with the history-deformation pair.

Recent progress in the characterization of the maximal element of $\mathcal{F}_2(\mathbb{G})$ has been achieved in [15]. There, the author characterizes the maximum free energy by introducing the concept of relaxed work: this is the minimum work required to approach a given history by continuations of another history. Provided that a suitable dissipation postulate holds, the relaxed work from the natural state, and the opposite of the relaxed work to the natural state, are the maximum and the minimum free energies, respectively. They both turn out to be functions of state.

Here, we take the initial state to be the zero state. Our objective is to derive an (explicit) expression for this quantity, both in terms of the history and in terms of $\breve{\mathbf{I}}^t$. There are two distinct cases related to those above: (i) where the maximum free energy is equal to the work function; this occurs when the set of minimal states (Definition 2.4) is a singleton; and (ii) when it is less than the work function, which is true for materials for which the space of minimal states contains more than one member.

It was shown in [29, 26] that the latter situation prevails for discrete spectrum materials (the relaxation function is given by a sum of decaying exponentials), by deriving and solving a Weiner-Hopf equation, subject to a constraint enforced by a Lagrange multiplier function. A simpler, and more general argument, is now developed.

For continuous spectrum materials (the relaxation function is given by an integral of a density function, with some continuity characteristics, multiplying a decaying exponential), it is shown in [22] that the first situation applies.

A material can be characterized by the singularity structure of $\dot{\mathbb{G}}_F$ on $\Omega^{(+)}$, as can be seen, at least in the case where $\dot{\mathbb{G}}_F$ is analytic at infinity, by evaluating

$$\dot{\mathbb{G}}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \dot{\mathbb{G}}_F(\omega) e^{i\omega s} d\omega \tag{8.1}$$

by contour integration on Ω^+, drawing the contours past the singularities to infinity.

The possible types of singularities are: isolated singularities, discontinuities associated with branch cuts and essential singularities. The first type, if the poles are simple and on the imaginary axis of $\Omega^{(+)}$, corresponds to sums of decaying exponentials in $\dot{\mathbb{G}}$. The argument here allows for general isolated singularities, so that each element of $\dot{\mathbb{G}}_F$ is a meromorphic function of ω, with a finite or even an infinite number of singularities of various orders.[2] This corresponds for example to decaying exponentials multiplied by polynomials or convergent power series. This is observed in the scalar case by multiple differentiations of $\dot{\mathbb{G}}_F$ with simple poles with respect to the positions of the poles. This corresponds to the differentiation of $\dot{\mathbb{G}}(s)$ with respect to the coefficients of s in the exponentials. Also, the singularities may be, in principle, anywhere on $\Omega^{(+)}$ but subject to the condition that $\dot{\mathbb{G}}$ be real or that $\bar{\dot{\mathbb{G}}}_F(\omega) = \dot{\mathbb{G}}_F(-\bar{\omega})$. This constraint in fact means that the singularities are symmetric under reflection in the positive imaginary axis.

The second type yields integrals over exponentials which, for cuts along the imaginary axis in $\Omega^{(+)}$, gives the standard continuous spectrum form. Again, the branch cuts may now be anywhere on $\Omega^{(+)}$, subject to the constraint that $\dot{\mathbb{G}}$ is real.

It should be pointed out that singularities off the imaginary axis may yield oscillatory behavior in $\dot{\mathbb{G}}$, combined with relaxation behavior, because of trigonometric

[2] If the number is infinite, they must form a divergent sequence. Finite points of accumulation of singularities produce an essential singularity [53], which we exclude.

functions mutiplying the exponentials. This is not excluded by thermodynamic principles [42, 19].

Essential singularities at infinity of a certain kind are associated with finite memory, i.e., where $\dot{\mathbb{G}}(s)$, or some term in that function, vanishes for $s > d > 0$, the quantity d being the duration of the memory [30]. For simplicity, such singularities are excluded here.

Essential singularities at finite points on $\Omega^{(+)}$ are the remaining possibility. It is difficult to imagine a choice of relaxation behavior that would generate such behavior in $\dot{\mathbb{G}}_F$. Such singularities are excluded from consideration in this context.

One difference between isolated and branch-cut singularities is that the former always have infinite behavior associated with them, while the latter are characterized by generally finite discontinuities, though, in fact, infinities may occur at branch points and indeed on the cut. However, there is the following clear-cut distinction which is important in the present context.

Remark 8.1. If $\mathbb{F}$, generally a tensorial quantity, has isolated singularities at a set of points, then $\mathbb{F}^{-1}$ or more precisely its determinant, will have zeros at these points. If $\mathbb{F}$ has a branch cut between two branch points, then $\mathbb{F}^{-1}$ will also have a branch cut between these two branch points. The converse of these two statements also holds.

In [29, 26], many different factorizations of $\mathbb{H}$ were obtained (in [29], only the scalar case was considered) by interchanging some or all of the zeros of det $\mathbb{H}_+$ and det $\mathbb{H}_-$. These were obtained under the assumption that the eigenspaces of $\dot{\mathbb{G}}$ are time-independent, which implies that $\mathbb{H}_+$ and $\mathbb{H}_-$ commute. In practical terms, it would seem that we must make such an assumption in order to obtain explicit representations of the factors. We will do so for purposes of the present discussion. Two factorizations will be considered, that given by (4.26) and

$$\mathbb{H}(\omega) = \mathbb{H}_+^{(e)}(\omega)\, \mathbb{H}_-^{(e)}(\omega), \qquad \mathbb{H}_+^{(e)}(\omega)\, \mathbb{H}_-^{*(e)}(\omega), \tag{8.2}$$

where det $\mathbb{H}_+^{(e)}$ has all the zeros of det $\mathbb{H}_-$ and det $\mathbb{H}_-^{(e)}$ has all the zeros of det $\mathbb{H}_+$. In other words, all the zeros have been interchanged. On individual eigenspaces, the manipulations are in effect those of the scalar case.

Let us write (4.40) and (4.41) in the following way: let $(\mathbf{E}(t), \mathbf{E}^t)$ be a given current strain, strain history couple. Then $(\mathbf{E}_1(t), \mathbf{E}_1^t)$ is in the same minimal state as $(\mathbf{E}(t), \mathbf{E}^t)$ if

$$\mathbf{E}_d(t) = 0, \qquad \int_0^\infty \dot{\mathbb{G}}(|s-u|)\mathbf{E}_d^t(u)du = 0, \quad s \in \mathbb{R}^-, \tag{8.3}$$

where $\mathbf{E}_d(t) = \mathbf{E}_1(t) - \mathbf{E}(t)$, $\mathbf{E}_d^t(u) = \mathbf{E}_{r1}^t(u) - \mathbf{E}_r^t(u)$. We write $(8.3)_2$ in the form

$$\begin{aligned} \int_0^\infty \dot{\mathbb{G}}(|s-u|)\mathbf{E}_d^t(u)du &= \mathbf{R}(s), \quad s \in \mathbb{R}^{++} \\ &= 0, \quad s \in \mathbb{R}^- \end{aligned} \tag{8.4}$$

and take Fourier transforms, using (4.14), to obtain

$$\frac{2i}{\omega}\ \mathbb{H}(\omega)\mathbf{E}^t_{d+}(\omega) = \mathbf{R}_+(\omega). \tag{8.5}$$

Equations (4.30) and (4.31) can now be utilized to write

$$\frac{2i}{\omega}\Big[\ \mathbb{H}^{(1)}_+(\omega)(\mathbf{p}^t_{d-}(\omega) - \mathbf{p}^t_{d+}(\omega))\Big] = \mathbf{R}_+(\omega), \tag{8.6}$$

where the factorization $\mathbb{H}(\omega) = \ \mathbb{H}^{(1)}_+(\omega)\ \mathbb{H}^{(1)}_-(\omega)$ can mean either (4.26) or (8.2), and (*cf.* (4.31) and (7.13))

$$\mathbf{p}^t_{d\pm}(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbb{H}^{(1)}_-(\omega')\mathbf{E}^t_{d+}(\omega')d\omega'}{\omega' - \omega^{\mp}}. \tag{8.7}$$

It was shown in [20] that [3]

$$\mathbf{p}^t_{d-}(\omega) = 0, \quad \omega \in \mathbb{R} \tag{8.8}$$

for the factorization (4.26) if $(\mathbf{E}(t), \mathbf{E}^t)$ and $(\mathbf{E}_1(t), \mathbf{E}^t_1)$ are in the same minimal state. The argument in [20] was extended to any other factorization based on an interchange of zeros in [26] for the case of simple poles and zeros. It applies without modification to the case where $\mathbb{H}$ is a general memomorphic function with the correct symmetries and behavior at zero and infinity. Thus, we have from (8.6)

$$\mathbf{R}_+(\omega) = -\frac{2i}{\omega}\ \mathbb{H}^{(1)}_+(\omega)\mathbf{p}^t_{d+}(\omega), \tag{8.9}$$

and, by virtue of (8.5),

$$\mathbf{E}^t_{d+}(\omega) = -\Big[\ \mathbb{H}^{(1)}_-(\omega)\Big]^{-1}\mathbf{p}^t_{d+}(\omega). \tag{8.10}$$

Now $\mathbf{E}^t_{d+}$, if it is non-zero, must be analytic in Ω^- and convergent at infinity in this half-plane . Therefore, $\mathbb{H}^{(1)}_-$ can have no branch cut singularities. It follows from (4.27) and $(8.2)_2$ that $\mathbb{H}^{(1)}_+$ and $\mathbb{H}$ have the same property. Also, the zeros of det $\mathbb{H}^{(1)}_-(\omega)$ must be in $\Omega^{(+)}$ which means that the factorization is given by (8.2). We have thus

Proposition 8.2. *For a material where $\dot{\mathbb{G}}_F$ has time-independent eigenspaces, the set of minimal states has more than one member only if $\dot{\mathbb{G}}_F$ possesses no branch cut singularities.*

In other words, $\dot{\mathbb{G}}_F$ can have only isolated singularities.

If $\dot{\mathbb{G}}_F$ has branch cut singularities, then the set of minimum states is a singleton, $\mathbf{E}^t_d$ is zero and the work function is the maximum free energy. In this case, the state is defined by $(\mathbf{E}(t), \mathbf{E}^t)$ and the work function is a function of state.

[3] Actually $\mathbf{q}^t_{d-}(\omega) = 0$, but (8.8) follows immediately from $(8.3)_1$.

We now seek the choice of state $(\mathbf{E}_1(t), \mathbf{E}_1^t)$ such that the work done to achieve this state is least among members of the minimal state which has $(\mathbf{E}(t), \mathbf{E}^t)$ as a member. We have from (4.23) and by analogy with (4.34)

$$\widetilde{W}\left(\mathbf{E}_1(t), \mathbf{E}_1^t\right) = W_1(t) = \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{E}}_{r1+}^t(\omega) \cdot \mathbb{H}(\omega)\mathbf{E}_{r1+}^t(\omega)$$
$$= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[|\mathbf{p}_{1-}^t(\omega)|^2 + |\mathbf{p}_{1+}^t(\omega)|^2\right] d\omega, \tag{8.11}$$

where

$$\mathbf{p}_{1\pm}^t(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbb{H}_-^{(e)}(\omega')\mathbf{E}_{r1+}^t(\omega')d\omega'}{\omega' - \omega^{\mp}}. \tag{8.12}$$

The minimum of W_1, gives the maximum free energy (see (3.10)) [9, 27, 16, 29, 26] Now, recalling (8.8), we see $\mathbf{p}_{1-}^t$ is fixed by virtue of the fact that it is equal to $\mathbf{p}_{e-}^t$, defined by (8.12) but with $\mathbf{E}_{r1+}^t$ replaced by $\mathbf{E}_{r+}^t$. However, $\mathbf{p}_{1+}^t$ can be varied and the choice which minimizes W_1 is clearly

$$\mathbf{p}_{m+}^t(\omega) = 0, \quad \omega \in \mathbb{R}, \tag{8.13}$$

where $\mathbf{p}_{m+}^t$ is the optimal choice, corresponding to an optimal history $\mathbf{E}_m^t$. Noting that

$$\mathbf{p}_{d+}^t(\omega) = \mathbf{p}_{m+}^t(\omega) - \mathbf{p}_{e+}^t(\omega) = -\mathbf{p}_{e+}^t(\omega), \tag{8.14}$$

we see (8.10) gives

$$\mathbf{E}_{m+}^t(\omega) = \mathbf{E}_{r+}^t(\omega) + \left[\mathbb{H}_-^{(e)}(\omega)\right]^{-1} \mathbf{p}_{e+}^t(\omega)$$
$$= \left[\mathbb{H}_-^{(e)}(\omega)\right]^{-1} \left\{\mathbf{p}_{e-}^t(\omega) - \mathbf{p}_{e+}^t(\omega) + \mathbf{p}_{e+}^t(\omega)\right\}$$
$$= \left[\mathbb{H}_-^{(e)}(\omega)\right]^{-1} \mathbf{p}_{e-}^t(\omega). \tag{8.15}$$

Proposition 8.3. *For materials where $\dot{\mathbb{G}}_F$ has time-independent eigenspaces and only isolated singularities, the maximum free energy is given by*

$$\psi_M(t) = \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}_{e-}^t(\omega)|^2 d\omega \tag{8.16}$$

and the Fourier transformed optimal history associated with this quantity has the form

$$\mathbf{E}_{m+}^t(\omega) = \left[\mathbb{H}_-^{(e)}(\omega)\right]^{-1} \mathbf{p}_{e-}^t(\omega). \tag{8.17}$$

where

$$\mathbf{p}_{e-}^t(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbb{H}_-^{(e)}(\omega')\mathbf{E}_+^t(\omega')d\omega'}{\omega' - \omega^{+}}. \tag{8.18}$$

If $\dot{\mathbb{G}}_F$ has branch cut singularities, then the maximum free energy is equal to the work function.

It remains to express ψ_M in terms of $\breve{\mathbf{I}}^t$. We can proceed very similarly to Section 7. Instead of (7.12), we introduce

$$\begin{aligned}\mathbf{P}_e^t(\omega) &:= \frac{\omega}{2i}\left[\mathbb{H}_+^{(e)}(\omega)\right]^{-1}\mathbf{J}_F^t(\omega)\\ &= \mathbb{H}_-^{(e)}(\omega)\mathbf{E}_{r+}^t(\omega)\end{aligned} \tag{8.19}$$

by virtue of (4.53), taking $\mathbf{J}_F^t(\omega) = \mathbf{J}_F^t(\omega, -1)$. The argument then goes through unchanged, to yield

$$\psi_M(t) = \phi(t) + \int_0^\infty \int_0^\infty \breve{\mathbf{I}}_{(2)}^t(u) \cdot \mathbb{K}_e(u, v)\breve{\mathbf{I}}_{(2)}^t(v)dudv \tag{8.20}$$

instead of (7.30), with

$$\begin{aligned}\mathbb{K}_e(u, v) &= \int_0^{min(u,v)} \mathbb{M}_e^*(u - s)\, \mathbb{M}_e(v - s)ds,\\ \mathbb{M}_e(s) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left[\mathbb{H}_+^{(e)}(\omega)\right]^{-1}}{2i\omega^-} e^{i\omega s} d\omega,\end{aligned} \tag{8.21}$$

replacing (7.32) and (7.21).

Also, the rate of dissipation corresponding to ψ_M is given by

$$\begin{aligned}D_M(t) &= \left|\int_0^\infty \mathbb{M}_e(u)\breve{\mathbf{I}}_{(2)}^t(u)du\right|^2\\ &= \int_0^\infty \int_0^\infty \breve{\mathbf{I}}_{(2)}^t(u)\mathbb{N}_e(u, v)\breve{\mathbf{I}}_{(2)}^t(v)dudv\\ \mathbb{N}_e(u, v) &:= \mathbb{M}_e^*(u)\, \mathbb{M}_e(v)\end{aligned} \tag{8.22}$$

instead of (7.36).

Remark 8.4. Similar representations can be given for the family of free energies introduced in [29, 26] obtained by exchanging some but not all zeros in the factorization and indeed generalizations of these to the case of non-simple poles.

Finally, for materials with branch-cut singularities, the maximum free energy, which is the work function, is given by $(4.23)_1$ or

$$\begin{aligned}W(t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{E}}_{r+}^t(\omega) \cdot \mathbb{H}(\omega)\mathbf{E}_{r+}^t(\omega)d\omega\\ &= \phi(t) + \frac{1}{8\pi}\int_{-\infty}^{\infty} \omega^2\bar{\mathbf{J}}_F^t(\omega) \cdot [\,\mathbb{H}(\omega)]^{-1}\,\mathbf{J}_F^t(\omega)d\omega\end{aligned} \tag{8.23}$$

on comparing the two right-hand sides of (8.19). Let us define

$$\mathbb{M}_h(s) = \mathbb{M}_h(-s) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{[\,\mathbb{H}(\omega)]^{-1}}{\omega^2} e^{i\omega s} d\omega. \tag{8.24}$$

Then

$$W(t) = \phi(t) + \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \breve{\mathbf{I}}^t_{(2)}(u) \cdot \mathbb{M}_h(u-v)\breve{\mathbf{I}}^t_{(2)}(v)dudv. \qquad (8.25)$$

This depends on $\breve{\mathbf{I}}^t$ defined on $\mathbb{R}$ which is not a function of the minimal state, a property which holds for $\breve{\mathbf{I}}^t$ restricted to $\mathbb{R}^+$ (see (4.43) and (4.44)). However, we have $\breve{\mathbf{I}}^t(s), \quad s \in \mathbb{R}$, given by (4.57) which is a function of the state $(\mathbf{E}(t), \mathbf{E}^t)$. This is the minimal state for the type of material under consideration.

9. Applications to initial and boundary value problems in linear viscoelasticity

The usefulness of studying linear viscoelastic materials via the new state function $\mathbf{I}^t$, instead of the history $\mathbf{E}^t$, becomes apparent when considering the partial differential equations which describe the behavior of bodies composed of these materials. In the classic approach, the problem is set up in the space-time domain $Q_T = \mathcal{B} \times (0, T)$ by means of the equation:

$$\ddot{\mathbf{u}}(\mathbf{x}, t) = \nabla \cdot \mathbf{T}(\mathbf{x}, t), \qquad (9.1)$$

where $\mathbf{T}(\mathbf{x}, t)$ is given by (4.2). Together with equation (9.1), initial and boundary conditions are given by

$$\begin{aligned} \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}), \ \ \dot{\mathbf{u}}(\mathbf{x}, 0) = \dot{\mathbf{u}}_0(\mathbf{x}), \\ \mathbf{u}^{t=0}(\mathbf{x}, s) &= \mathbf{u}^0(\mathbf{x}, s), \ \ s \in \mathbb{R}^+, \ \ \mathbf{x} \in \mathcal{B}, \\ \mathbf{u}|_{\partial\mathcal{B}}(\mathbf{x}, s) &= 0, \end{aligned} \qquad (9.2)$$

so that besides specifying $\mathbf{u}_0(\mathbf{x})$ and $\dot{\mathbf{u}}_0(\mathbf{x})$ at the outset, it is also necessary to assign the history $\mathbf{u}^0(\mathbf{x}, \cdot)$, which is a function defined for all $\mathbf{x} \in \mathcal{B}$ in the time interval $\mathbb{R}^+$. Thus, such an approach carries with it the conceptual difficulty of having to assign all the past history of the displacement on the infinite interval $(-\infty, 0]$.

If we set up the problem using the new definition of state, then (9.1) and (9.2) in the domain Q_T become, with the aid of (4.2) and (5.2),

$$\begin{aligned} \ddot{\mathbf{u}}(\mathbf{x}, t) &= \nabla \cdot (\mathbb{G}_0(\mathbf{x})\nabla\mathbf{u}(\mathbf{x}, t) + \int_0^{\infty} \dot{\mathbb{G}}(\mathbf{x}, s)\nabla\mathbf{u}^t(\mathbf{x}, s)ds) \\ &= \nabla \cdot (\mathbb{G}_0(\mathbf{x})\nabla\mathbf{u}(\mathbf{x}, t) + \int_0^{t} \dot{\mathbb{G}}(\mathbf{x}, s)\nabla\mathbf{u}^t(\mathbf{x}, s)ds) - \nabla \cdot \tilde{\mathbf{I}}^0(\mathbf{x}, t), \end{aligned} \qquad (9.3)$$

where, (see (4.45))

$$\tilde{\mathbf{I}}^0(\mathbf{x}, \tau) = -\int_0^{\infty} \dot{\mathbb{G}}(\mathbf{x}, s+\tau)\nabla\mathbf{u}^{t=0}(\mathbf{x}, s)ds \qquad (9.4)$$

and the initial and boundary conditions are now given by

$$\begin{aligned}\mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x},0) = \dot{\mathbf{u}}_0(\mathbf{x}),\\ \tilde{\mathbf{I}}^{t=0}(\mathbf{x},\tau) &= \tilde{\mathbf{I}}^0(\mathbf{x},\tau), \quad \tau \in (0,T),\\ \mathbf{u}|_{\partial\mathcal{B}}(\mathbf{x},\tau) &= 0.\end{aligned} \tag{9.5}$$

It now appears that in order to study problem (9.3), (9.5) in the domain Q_T, besides assigning initial conditions $\mathbf{u}_0(\mathbf{x})$, $\dot{\mathbf{u}}_0(\mathbf{x})$ it suffices to give the function $\tilde{\mathbf{I}}^0(\mathbf{x},\cdot)$, in the interval $(0,T)$ only, instead of the whole past history $\mathbf{u}^0(\mathbf{x},s)$, $s \in \mathbb{R}^+$.

As we can see from (9.4), the function $\tilde{\mathbf{I}}^0$ is an observable quantity. It carries a precise physical meaning: it represents the negative of the stress associated with the null process $\mathbf{E}(\cdot) = 0$ on $(0,\tau)$. It is therefore convenient that the initial condition $\tilde{\mathbf{I}}^0$ is required only over the finite time interval $(0,T)$.

This new definition of state also has an advantage in the study of stability of solutions where we consider (9.3), (9.5) in the domain $Q = \mathcal{B} \times \mathbb{R}^+$. In fact, it allows us to set up the problem in a very large space of initial conditions.

Usually, this differential problem is written in a weak sense, by a variational form derived from a stationary principle. Now we show that the system (9.3), (9.5) can be represented in a weak form by means of a virtual power principle. This new framework allows us to pose the problem in natural spaces, related to the domain of definition of the free energy given by the BRUEUR and ONAT equation in [3], namely (7.4). Moreover, it is possible to obtain existence, uniqueness and stability theorems in more regular spaces.

The virtual power principle related to the system (9.3), (9.5) in the domain $Q_\infty := \Omega \times (0,+\infty)$ assumes the following form:

$$\begin{aligned}\int_0^\infty \int_{\mathcal{B}} (\mathbf{u}_{tt}(\mathbf{x},t) \cdot \boldsymbol{\varphi}_t(\mathbf{x},t) + \mathbb{G}_0(\mathbf{x})\nabla\mathbf{u}(\mathbf{x},t) \cdot \nabla\boldsymbol{\varphi}_t(\mathbf{x},t)\\ + \int_0^t \dot{\mathbb{G}}(\mathbf{x},s)\nabla\mathbf{u}^t(\mathbf{x},s)ds \cdot \nabla\boldsymbol{\varphi}_t(\mathbf{x},t))d\mathbf{x}dt\\ = \int_0^\infty \int_{\mathcal{B}} \tilde{\mathbf{I}}^0(\mathbf{x},t) \cdot \nabla\boldsymbol{\varphi}_t(\mathbf{x},t)d\mathbf{x}dt,\end{aligned} \tag{9.6}$$

where a subscript time variable indicates differentiation with respect to that variable and $\phi_t : Q_\infty \mapsto \mathbb{R}^3$ is any "test" function. Before providing a precise definition of a solution, we have to rewrite (9.6) in the equivalent form

$$\begin{aligned}\int_0^\infty \int_{\mathcal{B}} (\mathbf{u}_t(\mathbf{x},t) \cdot \boldsymbol{\varphi}_{tt}(\mathbf{x},t) - \int_0^t \mathbb{G}(\mathbf{x},t-\tau)\nabla\mathbf{u}_\tau(\mathbf{x},\tau)d\tau \cdot \nabla\boldsymbol{\varphi}_t(\mathbf{x},t))d\mathbf{x}dt\\ = -\int_{\mathcal{B}} \mathbf{u}_t(\mathbf{x},0) \cdot \boldsymbol{\varphi}_t(\mathbf{x},0)d\mathbf{x} - \int_0^\infty \int_{\mathcal{B}} \mathbf{I}^0(\mathbf{x},t) \cdot \nabla\boldsymbol{\varphi}_t(\mathbf{x},t)d\mathbf{x}dt\end{aligned} \tag{9.7}$$

obtained from (9.6) after integration by parts with respect to both time variables, where $\mathbf{I}^0(\mathbf{x},t) = -\mathbb{G}(\mathbf{x},t)\nabla\mathbf{u}(\mathbf{x},0) + \tilde{\mathbf{I}}^0(\mathbf{x},t)$ (see (4.45)), assuming that

$\boldsymbol{\varphi}_t(\mathbf{x}, \infty) = 0$. Now we can consider for the system (9.7) the function $\mathbf{v}:=\mathbf{u}_t$ as unknown and $\mathbf{w}:=\boldsymbol{\varphi}_t$ as a test function. We then have

$$\begin{aligned}\int_0^\infty \int_{\mathcal{B}} (\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{w}_t(\mathbf{x}, t) - \int_0^t \mathbb{G}(\mathbf{x}, t-\tau)\nabla \mathbf{v}(\mathbf{x}, \tau) d\tau \cdot \nabla \mathbf{w}(\mathbf{x}, t)) d\mathbf{x} dt \\ = - \int_{\mathcal{B}} \mathbf{v}(\mathbf{x}, 0) \cdot \mathbf{w}(\mathbf{x}, 0) d\mathbf{x} - \int_0^\infty \int_{\mathcal{B}} \mathbf{I}^0(\mathbf{x}, t) \cdot \nabla \mathbf{w} d\mathbf{x} dt. \qquad (9.8)\end{aligned}$$

This problem corresponds to the differential system

$$\mathbf{v}_t(\mathbf{x}, t) = \nabla \cdot \int_0^t \mathbb{G}(\mathbf{x}, t-\tau)\nabla \mathbf{v}(\mathbf{x}, \tau) d\tau - \nabla \cdot \mathbf{I}^0(\mathbf{x}, t), \qquad (9.9)$$

with initial and boundary conditions

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{v}(\mathbf{x}, t)|_{\partial\mathcal{B}} = 0. \qquad (9.10)$$

The following frequency domain quantities and results will be used later. From (4.17), (4.18), using the notation of (A.6) and $\breve{\mathbb{G}}_+(\mathbf{x}, \omega) = \mathbb{G}_+(\mathbf{x}, \omega) - \frac{1}{i\omega^-}\mathbb{G}_\infty(\mathbf{x})$, we find that

$$\mathbb{G}_+(\mathbf{x}, \omega) = \frac{1}{i\omega^-}\left[\mathbb{G}(\mathbf{x}, 0) + \dot{\mathbb{G}}_F(\mathbf{x}, \omega)\right] = \mathbb{G}_c(\mathbf{x}, \omega) - i\,\mathbb{G}_s(\mathbf{x}, \omega), \quad (9.11)$$

where, by virtue of (4.4),

$$\mathbb{G}_c(\mathbf{x}, \omega) = -\frac{1}{\omega}\dot{\mathbb{G}}_s(\mathbf{x}, \omega) > 0. \qquad (9.12)$$

From (4.8), we have

$$\mathbb{G}_c(\mathbf{x}, \omega) \sim -\frac{\dot{\mathbb{G}}_0(\mathbf{x})}{\omega^2} \qquad (9.13)$$

at large ω if $\dot{\mathbb{G}}(\mathbf{x}, 0) < \mathbf{0}_4$. Also, from (4.19),

$$\mathbb{G}_s(\mathbf{x}, \omega) = \frac{1}{\omega^-}\left[\mathbb{G}_0(\mathbf{x}) + \dot{\mathbb{G}}_c(\mathbf{x}, \omega)\right], \qquad (9.14)$$

and at large ω

$$\mathbb{G}_s(\omega) \sim \frac{\mathbb{G}_0(\mathbf{x})}{\omega}. \qquad (9.15)$$

Thus

$$\mathbb{G}_s(\mathbf{x}, \omega) = -\omega\mathbb{G}_0(\mathbf{x})\left[\dot{\mathbb{G}}_0(\mathbf{x})\right]^{-1}\mathbb{G}_c(\mathbf{x}, \omega) + o\left(\frac{1}{\omega}\right). \qquad (9.16)$$

Now, we need to fix the domain of the functions $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{w}(\mathbf{x}, t)$. As we will see, the free energy (7.4) characterizes the set of solutions.

We introduce the spaces (the first related of course to (5.6), used in the argument leading to (7.4), and its dual to (5.9))

$$\begin{aligned}\mathcal{H}_G(\mathbb{R}^+; H_0^1(\mathcal{B})) := \Big\{ & \mathbf{v} \in L^2_{loc}(\mathbb{R}^+; H_0^1(\mathcal{B})); \\ & \int_0^\infty \int_0^\infty \int_{\mathcal{B}} \mathbb{G}(\mathbf{x}, |\tau - \tau'|) \nabla \mathbf{v}(\mathbf{x}, \tau') \cdot \nabla \mathbf{v}(\mathbf{x}, \tau) d\mathbf{x} d\tau' d\tau \\ & = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{\mathcal{B}} \mathbb{G}_c(\mathbf{x}, \omega) \nabla \mathbf{v}(\mathbf{x}, \omega) \cdot \nabla \mathbf{v}(\mathbf{x}, \omega) d\mathbf{x} d\omega < \infty \Big\}, \\ & \mathcal{F}(Q) := H^{\frac{1}{2}}(\mathbb{R}^+; L^2(\mathcal{B})) \cap \mathcal{H}_G(\mathbb{R}^+; H_0^1(\mathcal{B})).\end{aligned} \tag{9.17}$$

Moreover, by means of the state function $\mathbf{I}^0$, we introduce the linear functional on $\mathcal{H}_G(\mathbb{R}^+; H_0^1(\mathcal{B}))$ defined by

$$\mathbf{l}(\mathbf{I}^0, \nabla \mathbf{v}) := \int_0^\infty \int_{\mathcal{B}} \mathbf{I}^0(\mathbf{x}, \tau) \cdot \nabla \mathbf{v}(\mathbf{x}, \tau) d\mathbf{x} d\tau.$$

From the Riesz theorem, there exists an element $\nabla \mathbf{v}^{\mathbf{I}^0} \in \mathcal{H}'_G(\mathbb{R}^+; H_0^1(\mathcal{B}))$ such that for all $\nabla \mathbf{v}(\mathbf{x}, \tau) \in \mathcal{H}_G(\mathbb{R}^+; H_0^1(\mathcal{B}))$

$$\mathbf{l}(\mathbf{I}^0, \nabla \mathbf{v}) = \int_0^\infty \int_0^\infty \int_{\mathcal{B}} \mathbb{G}(\mathbf{x}, |\tau - \tau'|) \nabla \mathbf{v}^{\mathbf{I}^0}(\mathbf{x}, \tau') \cdot \nabla \mathbf{v}(\mathbf{x}, \tau) d\mathbf{x} d\tau' d\tau, \tag{9.18}$$

from which we define the function

$$\mathbf{F}^0(\mathbf{x}, \tau) = \int_0^\infty \mathbb{G}(\mathbf{x}, |\tau - \tau'|) \nabla \mathbf{v}^{\mathbf{I}^0}(\mathbf{x}, \tau') d\tau', \ \ \tau \in \mathbb{R}. \tag{9.19}$$

Then, for any element $\mathbf{I}^0$, we have only one function $\mathbf{F}^0$ defined by (9.19) such that $\mathbf{I}^0(\mathbf{x}, \tau) = \mathbf{F}^0(\mathbf{x}, \tau),\ \tau \in \mathbb{R}^+$ and moreover

$$\begin{aligned}\mathbf{l}(\mathbf{I}^0, \nabla \mathbf{v}) &= \int_0^\infty \int_{\mathcal{B}} \mathbf{I}^0(\mathbf{x}, \tau) \cdot \nabla \mathbf{v}(\mathbf{x}, \tau) d\mathbf{x} d\tau \\ &= \int_0^\infty \int_{\mathcal{B}} \mathbf{F}^0(\mathbf{x}, \tau) \cdot \nabla \mathbf{v}(\mathbf{x}, \tau) d\mathbf{x} d\tau\end{aligned}$$

for all $\nabla \mathbf{v}(\mathbf{x}, \tau) \in \mathcal{H}_G(\mathbb{R}^+; H_0^1(\mathcal{B}))$.

Let us consider the new function $\mathbb{G}^*(\mathbf{x}, s) : \mathcal{B} \times \mathbb{R}^+ \to \text{Sym}$ defined as

$$\int_{-\infty}^{\infty} \mathbb{G}(\mathbf{x}, |t - s|) \mathbb{G}^*(\mathbf{x}, |s|) ds = \mathbf{I} \delta(t),$$

where δ is the Dirac measure, or

$$\mathbb{G}_c(\mathbf{x}, \omega) \mathbb{G}_c^*(\mathbf{x}, \omega) = \mathbf{I},$$

where **I** is the unit tensor. Now, we can introduce the spaces

$$\mathcal{S}_G(\mathbb{R}; L^2(\mathcal{B})) := \Big\{\mathbf{F}^0 \in L^2_{loc}(\mathbb{R}; L^2(\mathcal{B}));$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbb{G}^*(\mathbf{x}, |\tau-\tau'|)\mathbf{F}^0(\mathbf{x},\tau')\cdot\mathbf{F}^0(\mathbf{x},\tau)d\mathbf{x}d\tau' d\tau$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbb{G}_c^{-1}(\mathbf{x},\omega)\,\mathbf{F}^0_F(\mathbf{x},\omega)\cdot\overline{\mathbf{F}^0_F(\mathbf{x},\omega)}d\mathbf{x}d\omega < \infty\Big\}$$

and

$$\tilde{\mathcal{S}}_G(\mathbb{R}; L^2(\mathcal{B})) = \Big\{\mathbf{F}^0 \in \mathcal{S}_G(\mathbb{R}; L^2(\mathcal{B}));$$
$$\int_{-\infty}^{\infty}\int_{\mathcal{B}}(1+|\omega|)\mathbb{G}_c^{-1}(\mathbf{x},\omega)\mathbf{F}^0_F(\mathbf{x},\omega)\cdot\overline{\mathbf{F}^0_F(\mathbf{x},\omega)}d\mathbf{x}d\omega < \infty\Big\}.$$

The spaces $\mathcal{H}_G$ and $\mathcal{S}_G$ are Hilbert spaces with respect to the inner products

$$(\mathbf{v}_1,\mathbf{v}_2)_{\mathcal{H}_G} = \int_0^{\infty}\int_0^{\infty}\int_{\mathcal{B}} \mathbb{G}(\mathbf{x}, |\tau-\tau'|)$$
$$\times\left[\nabla\mathbf{v}_1(\mathbf{x},\tau')\cdot\nabla\mathbf{v}_2(\mathbf{x},\tau)+\nabla\mathbf{v}_1(\mathbf{x},\tau)\cdot\nabla\boldsymbol{v}_2(\mathbf{x},\tau')\right]d\mathbf{x}d\tau' d\tau,$$

$$(\mathbf{F}^0_1,\mathbf{F}^0_2)_{\mathcal{S}_G} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbb{G}^*(\mathbf{x}, |\tau-\tau'|)\check{\mathbf{F}}^0_1(\mathbf{x},\tau')\cdot\check{\mathbf{F}}^0_2(\mathbf{x},\tau)d\mathbf{x}d\tau' d\tau.$$

Finally, the space $\mathcal{F}(Q)$, defined by $(9.17)_2$, will be an Hilbert space with respect to the inner product

$$(\mathbf{v}_1,\mathbf{v}_2)_{\mathcal{F}}$$
$$= \frac{1}{2}\int_0^{\infty}\int_{\mathcal{B}}(\mathbf{v}_1(\mathbf{x},t)\cdot\boldsymbol{v}_{t2}(\mathbf{x},t)+\mathbf{v}_2(\mathbf{x},t)\cdot\mathbf{v}_{t1}(\mathbf{x},t))\,d\mathbf{x}dt + (\mathbf{v}_1,\mathbf{v}_2)_{\mathcal{H}_G}$$
$$= -\frac{1}{2}\int_{\mathcal{B}}\mathbf{v}_1(\mathbf{x},0)\cdot\mathbf{v}_2(\mathbf{x},0)d\mathbf{x} + (\mathbf{v}_1,\mathbf{v}_2)_{\mathcal{H}_G}$$

Definition 9.1. A function $\mathbf{v} \in \mathcal{F}(Q)$ *is called a virtual power solution of the problem (9.9)–(9.10) with data* $\mathbf{v}_0 \in L^2(\mathcal{B})$, $\mathbf{F}^0 \in \mathcal{S}_G(\mathbb{R}^+; L^2(\mathcal{B}))$, *if it satisfies (9.8) for any* $\mathbf{w} \in \mathcal{F}(Q)$.

In the following we suppose that [4]

$$\mathbf{v}_0(\mathbf{x}) = 0. \tag{9.20}$$

[4] These initial conditions are not restrictive. If **v** is a solution of (9.9) with initial data $\mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x})$, then the function $\mathbf{z} = \mathbf{v}+\mathbf{w}$, with $\mathbf{w} \in C^{\infty}(\mathbb{R}^+, H^1_0(\mathcal{B}))$ and such that

$$\mathbf{w}(\mathbf{x},0) = -\mathbf{v}_0(\mathbf{x}),$$

where $\mathbf{w}(\mathbf{x},\cdot) = 0$ for every $t > t_0$, satisfies a problem formally identical to (9.9)–(9.10) with initial data

$$\mathbf{z}(\mathbf{x},0) = 0.$$

Now we are in a position to give the following

Theorem 9.2. *Assume that the kernel* $\breve{\mathbb{G}}(\mathbf{x},\cdot) \in L^1(\mathbb{R}^+, L^\infty(\mathcal{B}))$ *satisfies the thermodynamic condition* $\breve{\mathbb{G}}_c(\mathbf{x},\omega) > 0$, *for any* $\omega \in \mathbb{R}$ *Then there exists a unique virtual power solution* $\mathbf{v} \in \mathcal{F}(Q)$ *of the problem (9.9), (9.10) with* $\mathbf{v}_0(\mathbf{x}) = 0$ *and* $\mathbf{F}^0 \in \tilde{\mathcal{S}}_G(\mathbb{R}^+; L^2(\mathcal{B}))$.

Proof. Let us consider the Fourier transform of the system (9.9), (9.10)

$$-i\omega\mathbf{v}_+(\mathbf{x},\omega) + \nabla\cdot\mathbb{G}_+(\mathbf{x},\omega)\nabla\mathbf{v}_+(\mathbf{x},\omega) = \nabla\cdot\mathbf{I}^0_+(\mathbf{x},\omega), \tag{9.21}$$

with

$$\mathbf{v}(\mathbf{x},0) = 0, \qquad \mathbf{v}_+(\mathbf{x},\omega)|_{\partial\mathcal{B}} = 0. \tag{9.22}$$

If for any fixed $\omega \in \mathbb{R}$ the bilinear form

$$a(\mathbf{v}_+(\omega),\mathbf{v}_+(\omega)) := \int_{\mathcal{B}} \Big[i\omega\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{\mathbf{v}_+(\mathbf{x},\omega)} + \mathbb{G}_c(\mathbf{x},\omega)\nabla\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{\nabla\mathbf{v}_+(\mathbf{x},\omega)}\Big]\,d\mathbf{x} \tag{9.23}$$

is bounded and coercive in $H^1_0(\mathcal{B})$, then the problem (9.21), (9.22) has one and only one solution $\mathbf{v}_+(\cdot,\omega) \in H^1_0(\mathcal{B})$. It is easy to verify that the bilinear form $a(\cdot,\cdot)$ is bounded. In order to demonstrate coercivity, we have to prove that the inequality for any fixed $\omega \in \mathbb{R}$

$$|a(\mathbf{v}_+(\omega),\mathbf{v}_+(\omega))| \geqq k(\omega)\,\|\mathbf{v}_+(\omega)\|^2_{H^1_0}$$

holds for all $\mathbf{v}_+ \in H^1_0(\mathcal{B})$ where $k(\omega)$ is a positive scalar. From the definition (9.23) we have

$$\begin{aligned}|a(\mathbf{v}_+(\omega),\mathbf{v}_+(\omega))| &\geqq \int_{\mathcal{B}} \mathbb{G}_c(\mathbf{x},\omega)\nabla\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{\nabla\mathbf{v}_+(\mathbf{x},\omega)}d\mathbf{x}\\ &\geqq k(\omega)\,\|\mathbf{v}_+(\omega)\|^2_{H^1_0},\end{aligned}$$

where $k(\omega)$ is the least eigenvalue of $\mathbb{G}_c(\mathbf{x},\omega)$, which is a symmetric, positive-definite tensor.

This proves that for any fixed $\omega \neq 0$, the problem (9.21), (9.22) yields a solution $\mathbf{v}_+ \in H^1_0(\mathcal{B})$, if the supply $\nabla\cdot\mathbf{I}^0_+(\mathbf{x},\omega)$ is an element of $H^{-1}(\mathcal{B})$. Now, in order to study the behavior of $\mathbf{v}_+$ when $\omega \to \infty$, we apply Parseval's theorem to (9.8), obtaining

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{\mathcal{B}} \Big\{\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{i\omega\mathbf{w}_+(\mathbf{x},\omega)} - \mathbb{G}_+(\mathbf{x},\omega)\nabla\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{\nabla\mathbf{w}_+(\mathbf{x},\omega)}\Big\}\,d\mathbf{x}d\omega\\ &\quad = -\int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbf{F}^0_+(\mathbf{x},\omega)\cdot\overline{\nabla\mathbf{w}_+(\mathbf{x},\omega)}d\mathbf{x}d\omega.\end{aligned} \tag{9.24}$$

When $\mathbf{w}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t)$ we have

$$\int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbb{G}_{+}(\mathbf{x}, \omega)\nabla\mathbf{v}_{+}(\mathbf{x}, \omega) \cdot \overline{\nabla\mathbf{v}_{+}(\mathbf{x}, \omega)}d\mathbf{x}d\omega$$
$$= \int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbf{F}^{0}_{+}(\mathbf{x}, \omega) \cdot \overline{\nabla\mathbf{v}_{+}(\mathbf{x}, \omega)}d\mathbf{x}d\omega, \tag{9.25}$$

from which we get

$$\int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbb{G}_{c}(\mathbf{x}, \omega)\nabla\mathbf{v}_{+}(\mathbf{x}, \omega) \cdot \overline{\nabla\mathbf{v}_{+}(\mathbf{x}, \omega)}d\mathbf{x}d\omega = \pi\,\|\nabla\mathbf{v}_{+}\|^{2}_{\mathcal{H}_G}$$
$$\leqq |\int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbf{F}^{0}_{+}(\mathbf{x}, \omega) \cdot \overline{\nabla\mathbf{v}_{+}(\mathbf{x}, \omega)}d\mathbf{x}d\omega|$$
$$= |\int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbb{G}_{c}^{-\frac{1}{2}}(\mathbf{x}, \omega)\mathbf{F}^{0}_{+}(\mathbf{x}, \omega) \cdot \mathbb{G}_{c}^{\frac{1}{2}}(\mathbf{x}, \omega)\nabla\mathbf{v}_{+}(\mathbf{x}, \omega)d\mathbf{x}d\omega|$$
$$\leqq \pi\,\|\mathbf{F}^{0}_{+}\|_{\mathcal{S}_G}\|\nabla\mathbf{v}_{+}\|_{\mathcal{H}_G} \tag{9.26}$$

by Schwartz inequality. It follows that

$$\int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbb{G}_{c}(\mathbf{x}, \omega)\nabla\mathbf{v}_{+}(\mathbf{x}, \omega) \cdot \overline{\nabla\mathbf{v}_{+}(\mathbf{x}, \omega)}d\mathbf{x}d\omega$$
$$\leqq \int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbb{G}_{c}^{-1}(\mathbf{x}, \omega)\mathbf{F}^{0}_{+}(\mathbf{x}, \omega) \cdot \overline{\mathbf{F}^{0}_{+}(\mathbf{x}, \omega)}d\mathbf{x}d\omega. \tag{9.27}$$

Therefore, if $\mathbf{F}^0 \in \mathcal{S}_G(\mathbb{R}^+; L^2(\mathcal{B}))$ we have that $\mathbf{v} \in \mathcal{H}_G(\mathbb{R}^+; H^1_0(\mathcal{B}))$. Moreover, from Poincare Lemma, there is a constant $p(\mathcal{B})$ such that

$$p(\mathcal{B})\int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbb{G}_{c}(\mathbf{x}, \omega)\mathbf{v}_{+}(\mathbf{x}, \omega) \cdot \overline{\mathbf{v}_{+}(\mathbf{x}, \omega)}d\mathbf{x}d\omega$$
$$\leqq \int_{-\infty}^{\infty}\int_{\mathcal{B}} \mathbb{G}_{c}^{-1}(\mathbf{x}, \omega)\mathbf{F}^{0}_{+}(\mathbf{x}, \omega) \cdot \overline{\mathbf{F}^{0}_{+}(\mathbf{x}, \omega)}d\mathbf{x}d\omega. \tag{9.28}$$

Next, we consider in (9.24) $\mathbf{w} = i\ (sign\,\omega)\,\mathbf{v}_{+}$

$$\int_{-\infty}^{\infty}\int_{\mathcal{B}} \Big\{\mathbf{v}_{+}(\mathbf{x}, \omega) \cdot \overline{|\omega|\,\mathbf{v}_{+}(\mathbf{x}, \omega)}$$
$$+(sign\,\omega)\mathbb{G}_{s}(\mathbf{x}, \omega)\nabla\mathbf{v}_{+}(\mathbf{x}, \omega) \cdot \overline{\nabla\mathbf{v}_{+}(\mathbf{x}, \omega)}\Big\}\,d\mathbf{x}d\omega$$
$$= Im\left\{\int_{-\infty}^{\infty}\int_{\mathcal{B}} (sign\omega)\mathbf{F}^{0}_{+}(\mathbf{x}, \omega) \cdot \overline{\nabla\mathbf{v}_{+}(\mathbf{x}, \omega)}d\mathbf{x}d\omega\right\}. \tag{9.29}$$

Recalling (9.16) with $\dot{\mathbb{G}}_0(\mathbf{x}) \neq \mathbf{0}_4$, we see there is a constant C such that

$$\left|\mathbb{G}_{s}(\mathbf{x}, \omega)\nabla\mathbf{v}_{+}(bx, \omega) \cdot \overline{\nabla\mathbf{v}_{+}(\mathbf{x}, \omega)}\right|$$
$$\leqq C(1 + |\omega|)\mathbb{G}_{c}(\mathbf{x}, \omega)\nabla\mathbf{v}_{+}(\mathbf{x}, \omega) \cdot \overline{\nabla\mathbf{v}_{+}(\mathbf{x}, \omega)}. \tag{9.30}$$

Then, from (9.29) and (9.26), there is a constant C_2 such that

$$\int_{-\infty}^{\infty}\int_{\mathcal{B}}|\omega|\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{\mathbf{v}_+(\mathbf{x},\omega)}d\mathbf{x}d\omega$$
$$\leqq C_2\int_{-\infty}^{\infty}\int_{\mathcal{B}}\Big[(1+|\omega|)\mathbb{G}_c(\mathbf{x},\omega)\nabla\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{\nabla\mathbf{v}_+(\mathbf{x},\omega)}$$
$$+\mathbb{G}_c^{-1}(\mathbf{x},\omega)\mathbf{F}_+^0(\mathbf{x},\omega)\cdot\overline{\mathbf{F}_+^0(\mathbf{x},\omega)}\Big]d\mathbf{x}d\omega. \tag{9.31}$$

Moreover, if we put in (9.24) $\mathbf{w}_+(\mathbf{x},\omega)=|\omega|\,\mathbf{v}_+(\mathbf{x},\omega)$ we obtain

$$\int_{-\infty}^{\infty}\int_{\mathcal{B}}|\omega|\,\mathbb{G}_c(\mathbf{x},\omega)\nabla\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{\nabla\mathbf{v}_+(\mathbf{x},\omega)}d\mathbf{x}d\omega$$
$$=\int_{-\infty}^{\infty}\int_{\mathcal{B}}|\omega|\,\mathbf{F}_+^0(\mathbf{x},\omega)\cdot\overline{\nabla\mathbf{v}_+(\mathbf{x},\omega)}d\mathbf{x}d\omega, \tag{9.32}$$

from which, by a similar argument to that used in (9.26), we find that

$$\int_{-\infty}^{\infty}\int_{\mathcal{B}}|\omega|\,\mathbb{G}_c(\mathbf{x},\omega)\nabla\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{\nabla\mathbf{v}_+(\mathbf{x},\omega)}d\mathbf{x}d\omega$$
$$\leqq\int_{-\infty}^{\infty}\int_{\mathcal{B}}\mathbb{G}_c^{-1}(\mathbf{x},\omega)\,|\omega|\,\mathbf{F}_+^0(\mathbf{x},\omega)\cdot\overline{\mathbf{F}_+^0(\mathbf{x},\omega)}d\mathbf{x}d\omega. \tag{9.33}$$

Then, by means of the equations (9.28)–(9.33), we obtain

$$\int_{-\infty}^{\infty}\int_{\mathcal{B}}|\omega|\;\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{\mathbf{v}_+(\mathbf{x},\omega)}$$
$$+|\omega|\,\mathbb{G}_c(\mathbf{x},\omega)\nabla\mathbf{v}_+(\mathbf{x},\omega)\cdot\overline{\nabla\mathbf{v}_+(\mathbf{x},\omega)}d\mathbf{x}d\omega$$
$$\leqq C\int_{-\infty}^{\infty}\int_{\mathcal{B}}(1+|\omega|)\mathbb{G}_c^{-1}(\mathbf{x},\omega)\,\mathbf{F}_+^0(\mathbf{x},\omega)\cdot\overline{\mathbf{F}_+^0(\mathbf{x},\omega)}d\mathbf{x}d\omega, \tag{9.34}$$

where C is a suitable constant.

Hence, if the supply $\mathbf{F}^0\in\tilde{\mathcal{S}}_G(\mathbb{R}^+;L^2(\mathcal{B}))$, then we see that the function $\mathbf{v}$ belongs to $\mathcal{F}(Q)$ and it is a virtual power solution of the problem (9.9), (9.10) in the sense of Definition 9.1. □

Finally, we are interested in studying the properties of the function $\mathbf{u}(\mathbf{x},t)$, which is related to $\mathbf{v}(\mathbf{x},t)$ by means of the system

$$\dot{\mathbf{u}}(\mathbf{x},t)=\mathbf{v}(\mathbf{x},t),\;\;\mathbf{u}(\mathbf{x},t)|_{\partial\mathcal{B}}=0\,,\;\text{where }\mathbf{v}\in\mathcal{H}_{\mathrm{G}}(\mathbb{R}^+;\mathrm{H}_0^1(\mathcal{B})) \tag{9.35}$$

and subject to the initial condition

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})\,,\;\;\mathbf{x}\in\mathcal{B}.$$

Lemma 9.1. *If* $\dot{\mathbb{G}}(\mathbf{x},0)\neq 0,\;\dot{\mathbb{G}}(\mathbf{x},0)<\infty,\;\ddot{\mathbb{G}}(\mathbf{x},\cdot)\geqq 0$*, then the solution* $\mathbf{u}$ *of the problem (9.35) must be an element of* $L^2(\mathbb{R}^+;H_0^1(\mathcal{B}))$.

Proof. Consider the functional on the space $\mathcal{H}_G(\mathbb{R}^+; H_0^1(\mathcal{B}))$

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \int_0^\infty \int_{\mathcal{B}} \breve{\mathbb{G}}(\mathbf{x}, \left|\tau - \tau'\right|)\nabla\mathbf{v}(\mathbf{x}, \tau') \cdot \nabla\mathbf{v}(\mathbf{x}, \tau) d\mathbf{x} d\tau' d\tau \\
&\quad = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{\mathcal{B}} \breve{\mathbb{G}}_c(\mathbf{x}, \omega)\nabla\mathbf{v}_+(\mathbf{x}, \omega) \cdot \overline{\nabla\mathbf{v}_+(\mathbf{x}, \omega)} d\omega d\mathbf{x} \\
&\quad = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{\mathcal{B}} \breve{\mathbb{G}}_c(\mathbf{x}, \omega)\nabla\left(i\omega\mathbf{u}_+(\mathbf{x}, \omega)\right. \\
&\qquad \left. -\mathbf{u}(\mathbf{x}, 0)\right) \cdot \nabla\overline{(i\omega\mathbf{u}_+(\mathbf{x}, \omega) - \mathbf{u}(\mathbf{x}, 0))} d\omega d\mathbf{x} \\
&\quad \leqq \frac{1}{\pi}\int_{-\infty}^\infty \int_{\mathcal{B}} (\omega^2 \breve{\mathbb{G}}_c(\mathbf{x}, \omega)\nabla\mathbf{u}_+(\mathbf{x}, \omega) \cdot \nabla\overline{\mathbf{u}_+(\mathbf{x}, \omega)} \\
&\qquad + \breve{\mathbb{G}}_c(\mathbf{x}, \omega)\nabla\mathbf{u}(\mathbf{x}, 0) \cdot \nabla\mathbf{u}(\mathbf{x}, 0)) d\omega d\mathbf{x} \\
&\qquad < \infty.
\end{aligned} \tag{9.36}$$

Because

$$\int_{-\infty}^\infty \int_{\mathcal{B}} \breve{\mathbb{G}}_c(\mathbf{x}, \omega)\nabla\mathbf{u}(\mathbf{x}, 0) \cdot \nabla\mathbf{u}(\mathbf{x}, 0) d\mathbf{x} d\omega < \infty, \tag{9.37}$$

then from (9.36) we have

$$\int_{-\infty}^\infty \int_{\mathcal{B}} (\omega^2 \breve{\mathbb{G}}_c(\mathbf{x}, \omega)\nabla\mathbf{u}_+(\mathbf{x}, \omega) \cdot \nabla\overline{\mathbf{u}_+(\mathbf{x}, \omega)} d\mathbf{x} d\omega < \infty. \tag{9.38}$$

Moreover, because (see $(4.19)_1$, (4.24) and (4.25))

$$\lim_{\omega\to\infty} \omega^2 \breve{\mathbb{G}}_c(\mathbf{x}, \omega) = -\dot{\mathbb{G}}(\mathbf{x}, 0)$$

and $\dot{\mathbb{G}}(\mathbf{x}, 0) \neq 0$, $\dot{\mathbb{G}}(\mathbf{x}, 0) < \infty$, then by means of (9.37), (9.38) we obtain $\mathbf{u} \in L^2(\mathbb{R}^+; H_0^1(\mathcal{B}))$. □

10. Application of semi-group theory to linear viscoelastic systems

The differential system $(9.3)_1$ can be rewritten in the form

$$\begin{aligned}
\dot{\mathbf{u}}(\mathbf{x}, t) &= \mathbf{v}(\mathbf{x}, t) \\
\dot{\mathbf{v}}(\mathbf{x}, t) &= \nabla \cdot (\mathbb{G}_\infty \nabla\mathbf{u}(\mathbf{x}, t) + \int_0^\infty \dot{\mathbb{G}}(\mathbf{x}, s)\nabla\mathbf{u}_r^t(\mathbf{x}, s) ds), \\
\dot{\mathbf{u}}_r^t(\mathbf{x}, s) &= -\frac{d}{ds}\mathbf{u}_r^t(\mathbf{x}, s) - \dot{\mathbf{u}}(\mathbf{x}, t),
\end{aligned} \tag{10.1}$$

where $\mathbf{u}_r^t(\mathbf{x}, s) = \mathbf{u}^t(\mathbf{x}, s) - \mathbf{u}(\mathbf{x}, t)$. This system is supplemented by Dirichlet boundary conditions

$$\mathbf{u}|_{\partial\mathcal{B}} = 0. \tag{10.2}$$

The problem (10.1)–(10.2) can be considered using semi-group theory, where the state is given by the triple $\chi := (\mathbf{u}, \mathbf{v}, \mathbf{u}_r^t) \in \mathcal{G} = H_0^1(\mathcal{B}) \times L^2(\mathcal{B}) \times \mathcal{K}$. In [11, 32, 25] the space $\mathcal{K}$ is the domain of definition of Graffi free energy ψ_G given by (6.1), and the exponential decay of solutions is proved for initial conditions $\chi \in \mathcal{G}$, under the following restrictions on the kernel $\mathbb{G}$:

$$\dot{\mathbb{G}}(\mathbf{x}, s) < 0, \quad \ddot{\mathbb{G}}(\mathbf{x}, s) \geqq 0, \quad \text{for all } (\mathbf{x}, \mathrm{s}) \in \mathcal{B} \times \mathrm{R}^+ \tag{10.3}$$

(see Remark 6.1), and there exists $\alpha \in \mathbb{R}^{++}$ such that

$$\int_{\mathcal{B}} \left[\ddot{\mathbb{G}}(\mathbf{x}, s) + \alpha \dot{\mathbb{G}}(\mathbf{x}, s)\right] \nabla \mathbf{u}(\mathbf{x}, s) \cdot \nabla \mathbf{u}(\mathbf{x}, s) d\mathbf{x} \geqq 0. \tag{10.4}$$

When we use the function $\mathbf{I}^t(\cdot, \cdot)$, the system (10.1) can be reduced to the following problem:

$$\begin{aligned}
\dot{\mathbf{u}}(\mathbf{x}, t) \quad &= \mathbf{v}(\mathbf{x}, t) \\
\dot{\mathbf{v}}(\mathbf{x}, t) \quad &= \nabla \cdot [\mathbb{G}_\infty(\mathbf{x}) \nabla \mathbf{u}(\mathbf{x}, t)] - \nabla \cdot \breve{\mathbf{I}}^t(\mathbf{x}, 0), \\
\tfrac{d}{dt} \breve{\mathbf{I}}^t(\mathbf{x}, \tau) &= \frac{d}{d\tau} \breve{\mathbf{I}}^t(\mathbf{x}, \tau) - \breve{\mathbb{G}}(\tau) \nabla \mathbf{v}(\mathbf{x}, t),
\end{aligned} \tag{10.5}$$

(see (4.46)), with the boundary condition (10.2), while the initial conditions are given by

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \;\; \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \;\; \breve{\mathbf{I}}^{t=0}(\mathbf{x}, \tau) = \breve{\mathbf{I}}^0(\mathbf{x}, \tau), \;\; \tau \in \mathbb{R}^+. \tag{10.6}$$

For this problem, the state is given by the triple $\chi := (\mathbf{u}, \mathbf{v}, \breve{\mathbf{I}}^t)$ which is an element of the Hilbert space $\mathcal{F} := H_0^1(\mathcal{B}) \times L^2(\mathcal{B}) \times \mathcal{H}_F^*(\mathbb{R}^+; L^2(\mathcal{B}))$ with inner product

$$\begin{aligned}
&\langle \mathbf{u}_1(t), \mathbf{v}_1(t), \breve{\mathbf{I}}_1^t; \mathbf{u}_2(t), \mathbf{v}_2(t), \breve{\mathbf{I}}_2^t \rangle \\
&= \int_{\mathcal{B}} (\mathbb{G}_\infty(\mathbf{x}) \nabla \mathbf{u}_1(\mathbf{x}, t) \cdot \nabla \mathbf{u}_2(\mathbf{x}, t) + \mathbf{v}_1(\mathbf{x}, t) \cdot \mathbf{v}_2(\mathbf{x}, t)) d\mathbf{x} \\
&\quad - \int_0^\infty \int_{\mathcal{B}} \dot{\mathbb{G}}^{-1}(\mathbf{x}, \tau) \breve{\mathbf{I}}_{1\tau}^t(\mathbf{x}, \tau) \cdot \breve{\mathbf{I}}_{2\tau}^t(\mathbf{x}, \tau) d\mathbf{x} d\tau
\end{aligned} \tag{10.7}$$

Theorem 10.1. *Under the hypotheses (10.3), (10.4), for any initial condition $\chi_0 \in \mathcal{F}$, there exists a solution $\chi = (\mathbf{u}, \mathbf{v}, \breve{\mathbf{I}}^t)$ such that*

$$\|\mathbf{v}(t)\|_{L^2} + 2\Psi_F(\mathbf{I}^t) \leqq M e^{-\mu t} (\|\mathbf{v}(0)\|_{L^2} + 2\Psi_F(\mathbf{I}^0)),$$

where M and μ are suitable constants and (cf. (6.12)*)*

$$\Psi_F(\mathbf{I}^t) = \frac{1}{2} \mathbb{G}_\infty(\mathbf{x}) \nabla \mathbf{u}(\mathbf{x}, t) \cdot \nabla \mathbf{u}(\mathbf{x}, t) - \frac{1}{2} \int_0^\infty \dot{\mathbb{G}}^{-1}(\mathbf{x}, \tau) \breve{\mathbf{I}}_\tau^t(\mathbf{x}, \tau) \cdot \breve{\mathbf{I}}_\tau^t(\mathbf{x}, \tau) d\tau$$

Proof. Consider the functional ζ defined by

$$\zeta(\mathbf{v}(t), \mathbf{I}^t) := \frac{1}{2}\mathbf{v}^2(\mathbf{x}, t) + \Psi_F(\mathbf{I}^t), \tag{10.8}$$

which satisfies the equality (see (6.15) and (6.17))

$$\begin{aligned}
&\dot{\zeta}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t) \cdot \dot{\mathbf{v}}(\mathbf{x}, t) - \mathbf{T}(\mathbf{x}, t) \cdot \nabla\mathbf{v}(\mathbf{x}, t) \\
&= \frac{1}{2}\dot{\mathbb{G}}^{-1}(\mathbf{x}, 0)\breve{\mathbf{I}}^t_\tau(\mathbf{x}, 0) \cdot \breve{\mathbf{I}}^t_\tau(\mathbf{x}, 0) \\
&\quad -\frac{1}{2}\int_0^\infty \dot{\mathbb{G}}^{-1}(\mathbf{x}, \tau)\ddot{\mathbb{G}}(\mathbf{x}, \tau)\dot{\mathbb{G}}^{-1}(\mathbf{x}, \tau)\breve{\mathbf{I}}^t_\tau(\mathbf{x}, \tau) \cdot \breve{\mathbf{I}}^t_\tau(\mathbf{x}, \tau)d\tau.
\end{aligned} \tag{10.9}$$

By means of hypotheses (10.3), (10.4) and relations (9.1), (10.2) we obtain

$$\int_{\mathcal{B}} \dot{\zeta}(\mathbf{x}, t)d\mathbf{x} \leqq \frac{\alpha_1}{2}\int_{\mathcal{B}}\int_0^\infty \dot{\mathbb{G}}^{-1}(\mathbf{x}, \tau)\breve{\mathbf{I}}^t_\tau(\mathbf{x}, \tau) \cdot \breve{\mathbf{I}}^t_\tau(\mathbf{x}, \tau)d\tau d\mathbf{x} \leqq 0, \tag{10.10}$$

which is in effect (6.22). Thus, if we introduce the total energy

$$\mathcal{E}(t) = \int_{\mathcal{B}} \zeta(\mathbf{x}, t)d\mathbf{x}, \tag{10.11}$$

then

$$0 \leqq \mathcal{E}(t) \leqq \mathcal{E}(0).$$

Moreover, integrating (10.10) on $(0, \infty)$, we have

$$-\frac{\alpha_1}{2}\int_0^\infty\int_{\mathcal{B}}\int_0^\infty \dot{\mathbb{G}}^{-1}(\mathbf{x}, \tau)\breve{\mathbf{I}}^t_\tau(\mathbf{x}, \tau) \cdot \breve{\mathbf{I}}^t_\tau(\mathbf{x}, \tau)d\mathbf{x}d\tau dt \leqq \mathcal{E}(0). \tag{10.12}$$

Hence, by Theorem 9.2 and Lemma 9.1 we have that the solution $\mathbf{u}$ is an element of $H^{\frac{3}{2}}(\mathbb{R}^+; L^2(\mathcal{B})) \cap L^2(\mathbb{R}^+; H^1_0(\mathcal{B}))$ and

$$\begin{aligned}
\int_0^\infty \mathcal{E}(t)dt &= \int_0^\infty\int_{\mathcal{B}}\left\{\frac{1}{2}\mathbf{v}^2(\mathbf{x}, t) + \Psi_F(\mathbf{I}^t(\mathbf{x}))\right\}d\mathbf{x}dt \\
&= \frac{1}{2}\int_0^\infty\int_{\mathcal{B}}\Big\{(\mathbf{v}^2(\mathbf{x}, t) + \mathbb{G}_\infty(\mathbf{x})\nabla\mathbf{u}(\mathbf{x}, t) \cdot \nabla\mathbf{u}(\mathbf{x}, t)) \\
&\quad -\int_0^\infty \dot{\mathbb{G}}^{-1}(\mathbf{x}, \tau)\breve{\mathbf{I}}^t_\tau(\mathbf{x}, \tau) \cdot \breve{\mathbf{I}}^t_\tau(\mathbf{x}, \tau)d\mathbf{x}d\tau\Big\}d\mathbf{x}dt < \infty.
\end{aligned} \tag{10.13}$$

Now, following [32], we write the system (10.5) in the form

$$\dot{\chi}(t) = A\chi(t), \tag{10.14}$$

where A denotes the operator represented by the right-hand side of (10.5), which is defined on the domain

$$\mathcal{D}(A) = \Big\{ \chi \in \mathcal{F};\ \mathbf{v} \in H_0^1(\mathcal{B}), \text{ and } \breve{\mathbf{I}}^t \in L^2(\mathcal{B}) \text{ such that} \\ -\int_0^\infty \int_{\mathcal{B}} \dot{\mathbb{G}}^{-1}(\mathbf{x},\tau)\breve{\mathbf{I}}^t(\mathbf{x},\tau)\cdot\breve{\mathbf{I}}^t(\mathbf{x},\tau)d\mathbf{x}d\tau < \infty \Big\}.$$

Under the hypotheses (10.3) and (10.4), the operator $A : \mathcal{D}(A) \to \mathcal{F}$ *is a maximal dissipative operator on* $\mathcal{F}$, i.e. for A we may prove the following result. □

Lemma 10.1. *a* $\langle A\chi, \chi\rangle \leqq 0,$ *for any* $\chi \in \mathcal{D}(A)$*;*
b the range of $A - I$ *is* $\mathcal{F}$*, where* I *is the identity operator.*

Proof. We have, from (10.7), (10.8), (10.11) and (10.14),

$$\mathcal{E}(t) = \frac{1}{2}\langle \chi(t), \chi(t)\rangle, \qquad \frac{d}{dt}\mathcal{E}(t) = \langle A\chi(t), \chi(t)\rangle. \tag{10.15}$$

Integrating (10.9) over $\mathcal{B}$, we obtain

$$\langle A\chi(t), \chi(t)\rangle = \frac{1}{2}\int_{\mathcal{B}} \dot{\mathbb{G}}^{-1}(\mathbf{x},0)\breve{\mathbf{I}}_\tau^t(\mathbf{x},0)\cdot\breve{\mathbf{I}}_\tau^t(\mathbf{x},0)d\mathbf{x} \\ -\frac{1}{2}\int_0^\infty \int_{\mathcal{B}} \dot{\mathbb{G}}^{-1}(\mathbf{x},\tau)\ddot{\mathbb{G}}(\mathbf{x},\tau)\dot{\mathbb{G}}^{-1}(\mathbf{x},\tau) \\ \times \breve{\mathbf{I}}_\tau^t(\mathbf{x},\tau)\cdot\breve{\mathbf{I}}_\tau^t(\mathbf{x},\tau)d\tau \leqq 0. \tag{10.16}$$

Moreover, under the hypotheses (10.4), we have from (10.10) that

$$\langle A\chi(t), \chi(t)\rangle \leqq \frac{\alpha_1}{2}\int_0^\infty \int_{\mathcal{B}} \dot{\mathbb{G}}^{-1}(\mathbf{x},\tau)\mathbf{I}_\tau^t(\mathbf{x},\tau)\cdot\mathbf{I}_\tau^t(\mathbf{x},\tau)d\mathbf{x}d\tau \leqq 0,$$

for any solution χ.

The proof of point b is analogous to the case considered by DAFERMOS in [11]. Hence, by means of the Lumer-Phillips Theorem (see PAZY [49]), the operator A generates a strongly continuous semigroup of linear contraction operators $S(t)$ on $\mathcal{F}$ (see also [32]), so that the solutions of the system (10.5), (10.6) have the form

$$\chi(t) = S(t)\chi_0.$$

Moreover, from (10.13) we obtain that the total energy

$$\mathcal{E}(t) = \frac{1}{2}\langle S(t)\chi_0, S(t)\chi_0\rangle$$

satisfies the restriction

$$\int_0^\infty \langle S(t)\chi_0, S(t)\chi_0\rangle\, dt < \infty \tag{10.17}$$

for any $\chi_0 \in \mathcal{F}$. Then $\mathcal{D}(A)$ is dense in $\mathcal{F}$. □

Now we recall the following Lemma proved by DATKO [10].

Lemma 10.2. *Given a strongly continuous semigroup of linear operators $S(t)$ on a Hilbert space $\mathcal{F}$, then there exist two constants C, γ such that*

$$\langle S(t)\chi_0, S(t)\chi_0\rangle \leqq C\exp(-\gamma t)\,\langle \chi_0, \chi_0\rangle, \text{ for any } \chi_0 \in \mathcal{F} \tag{10.18}$$

if, and only if, the integral $\int_0^\infty \langle S(t)\chi_0, S(t)\chi_0\rangle\, dt$ is convergent for any $\chi_0 \in \mathcal{F}$.

Because, for any initial condition χ_0 such that $\frac{1}{2}\langle \chi_0, \chi_0\rangle = \mathcal{E}(0) < \infty$ we have from (10.17) that $\int_0^\infty \langle S(t)\chi_0, S(t)\chi_0\rangle\, dt < \infty$, and the inequality (10.18) follows.

Appendix: Definitions and notation

Let *Sym* be the space of symmetric second-order tensors acting on $\mathbb{R}^3$ viz. $Sym := \{\mathbf{M} \in Lin(\mathbb{R}^3) : \mathbf{M} = \mathbf{M}^\top\}$, where the superscript "$\top$" denotes the transpose. Operating on *Sym* is the space of the fourth-order tensors $Lin(Sym)$.

It is well known that *Sym* is isomorphic to $\mathbb{R}^6$. In particular, for every $\mathbf{L}, \mathbf{M} \in Sym$, if $\mathbf{C}_i$, $i = 1, \cdots, 6$ is an orthonormal basis of *Sym* with respect to the usual inner product in $Lin(\mathbb{R}^3)$, namely $tr(\mathbf{L}\mathbf{M}^\top)$, it is clear that the representation

$$\mathbf{L} = \sum_{i=1}^{6} L_i \mathbf{C}_i\,, \qquad \mathbf{M} = \sum_{i=1}^{6} M_i \mathbf{C}_i, \tag{A.1}$$

is such that $tr(\mathbf{L}\mathbf{M}^\top) = \sum_{i=1}^{6} L_i M_i$. Therefore, we treat each tensor of *Sym* as a vector in $\mathbb{R}^6$ and denote by $\mathbf{L}\cdot\mathbf{M}$ the inner product between elements of *Sym*, viz.

$$\mathbf{L}\cdot\mathbf{M} = tr(\mathbf{L}\mathbf{M}^\top) = tr(\mathbf{L}\mathbf{M}) = \sum_{i=1}^{6} L_i M_i \tag{A.2}$$

and $|\mathbf{M}|^2 = \mathbf{M}\cdot\mathbf{M}$. Consequently [45], any fourth-order tensor $\mathbb{K} \in Lin(Sym)$ will be identified with an element of $Lin(\mathbb{R}^6)$ by the representation

$$\mathbb{K} = \sum_{i,i=1}^{6} K_{ij}\mathbf{C}_i \otimes \mathbf{C}_j, \tag{A.3}$$

and $\mathbb{K}^\top$ means the transpose of $\mathbb{K}$ as an element of $Lin(\mathbb{R}^6)$. The norm $|\mathbb{K}|$ of $\mathbb{K} \in Lin(Sym)$ may be given by

$$|\mathbb{K}|^2 = tr\left(\mathbb{K}\mathbb{K}^\top\right) = \left(\sum_{i,j=1}^{6} K_{ij}K_{ji}\right).$$

We also deal with complex valued tensors. Let Ω be the complex plane and $Sym(\Omega)$ and $Lin(Sym(\Omega))$ the tensors represented by the forms (A.1) and (A.3) with $L_i, M_i, K_{ij} \in \Omega$. Then, for $\mathbf{L}, \mathbf{M} \in Sym(\Omega)$, we have from (A.2),

$$\mathbf{L} \cdot \overline{\mathbf{M}} = tr\left(\mathbf{L}\mathbf{M}^*\right) = tr\left(\mathbf{L}\overline{\mathbf{M}}\right) = \sum_{i=1}^{6} L_i \overline{M}_i, \tag{A.4}$$

where the overhead bar indicates a complex conjugate and $\mathbf{M}^* = \overline{\mathbf{M}}^\top$ is the hermitian conjugate.

The symbols $\mathbb{R}^+$ and $\mathbb{R}^{++}$ denote the non-negative reals and the strictly positive reals, respectively, while $\mathbb{R}^-$ and $\mathbb{R}^{--}$ denote the non-positive and strictly negative reals.

For any function $f : \mathbb{R} \to \mathcal{V}$, where $\mathcal{V}$ is a finite-dimensional vector space, in particular in the present context Sym or $Lin(Sym)$, let f_F, denote its *Fourier transform* viz. $f_F(\omega) = \int_{-\infty}^{\infty} f(s)e^{-i\omega s} ds$. Also, we define

$$f_+(\omega) = \int_0^\infty f(s)e^{-i\omega s} ds, \qquad f_-(\omega) = \int_{-\infty}^0 f(s)e^{-i\omega s} ds \tag{A.5}$$

$$f_s(\omega) = \int_0^\infty f(s) \sin \omega s \, ds, \qquad f_c(\omega) = \int_0^\infty f(s) \cos \omega s \, ds \tag{A.6}$$

The relations defining f_F and (A.6) are to be understood as applying to each component of the tensor quantities involved. Only very weak assumptions need be imposed on f for it to be Fourier-transformable. Indeed, f may even be a tempered distribution. The Fourier transforms of functions of bounded variation are considered, for example, in [20]. In the present context, it is generally assumed that all components of tensors in the time domain belong to $L^2(\mathbb{R})$ (or $L^2(\mathbb{R}^\pm)$ in the case of $f_\pm$ so that in the frequency domain, they belong to $L^2(\mathbb{R})$ (or $L^2(\mathbb{R}^\pm)$) [51, 50]. Further restrictions on the allowed function spaces are introduced where required.

For $f : \mathbb{R}^+ \to \mathcal{V}$ we can always extend the domain of f to $\mathbb{R}$, by considering its *causal* extension viz.

$$f(s) = \begin{cases} f(s) \text{ for } s \geqq 0, \\ 0 \quad \text{for } s < 0, \end{cases} \tag{A.7}$$

in which case

$$f_F(\omega) = f_+(\omega) = f_c(\omega) - i f_s(\omega). \tag{A.8}$$

We shall need to consider the Fourier transform of functions that do not go to zero at large times and thus do not belong to L^2 for the appropriate domain. In particular, let $f(s)$ in (A.7) be given by a constant a for all s. The standard procedure is adopted of introducing an exponential decay factor, calculating the Fourier transform and then letting the time decay constant tend to infinity. Thus, we obtain

$$f_+(\omega) = \frac{a}{i\omega^-},$$
$$\omega^- = \lim_{\alpha\to 0}(\omega - i\alpha). \tag{A.9}$$

The corresponding result for a constant function defined on $\mathbb{R}^-$ is obtained by taking the complex conjugates of this relationship. Also, if f is a function defined on $\mathbb{R}^-$ and if $\lim_{s\to-\infty} f(s) = b$, the components of the function $g : \mathbb{R}^- \to \mathcal{V}$ defined by $g(s) := f(s) - b$ belong to $L^2(\mathbb{R}^+)$, then

$$f_-(\omega) = g_-(\omega) - \frac{b}{i\omega^+}. \tag{A.10}$$

Again, taking complex conjugates gives the result for functions defined on $\mathbb{R}^+$. This procedure amounts to defining the Fourier transform of such functions as the limit of the transforms of a sequence of functions in L^2. The limit is to be taken after integrations over ω are carried out if the ω^{-1} results in a singularity in the integrand. Generally, in the present application, the ω^{-1} produces no such singularity and the limiting process is redundant.

If $f_\pm(\omega)$ is analytic at infinity, and f is differentiable N times at the origin, then we have the asymptotic behavior

$$f_\pm(\omega)\underset{\omega\to\infty}{\longrightarrow} \pm\sum_{n=0}^{N} \frac{f^{(n)}(0)}{(i\omega)^{n+1}} + O\left(\frac{1}{\omega^{N+1}}\right), \tag{A.11}$$

where $f^{(n)}$ is the n^{th} derivative of f. Thus

$$f_c(\omega) \underset{\omega\to\infty}{\longrightarrow} \sum_{n\ \text{odd}}^{N} \frac{f^{(n)}(0)}{(i\omega)^{n+1}} + O\left(\frac{1}{\omega^{N+1}}\right),$$
$$f_s(\omega) \underset{\omega\to\infty}{\longrightarrow} i\sum_{n\ \text{even}}^{N} \frac{f^{(n)}(0)}{(i\omega)^{n+1}} + O\left(\frac{1}{\omega^{N+1}}\right). \tag{A.12}$$

The complex ω plane, denoted by Ω, will play an important role in our discussions. We define the following sets:

$$\Omega^+ = \left\{\zeta \in \Omega : Im\zeta \geqq 0\right\}, \quad \Omega^{(+)} = \{\zeta \in \Omega : Im\zeta > 0\}. \tag{A.13}$$

Analogous meanings are assigned to Ω^- and $\Omega^{(-)}$.

The quantities $f_\pm$ defined by (A.6) are analytic in $\Omega^{(\mp)}$, respectively. This analyticity is extended by assumption to $\Omega^\mp$. The function f_+ may be defined by (A.6) and analytic on a portion of Ω^+ if, for example, f decays exponentially at large times. Over the remaining portion of Ω^+, on which the integral definition is meaningless, f_+ is defined by analytic continuation.

Acknowledgments. L. DESERI gratefully acknowledges (i) the support of the Center for Nonlinear Analysis at Carnegie-Mellon University, Pittsburgh PA, USA, (ii) the support and hospitality of the Department of Theoretical and Applied Mechanics at Cornell University, Ithaca NY, USA, and (iii) the hospitality of the Department of Engineering of the University of Ferrara, Italy.

L. DESERI and M. FABRIZIO acknowledge the Italian grant COFIN 2002 *Modelli Matematici per la Scienza dei Materiali*, coordinated by Prof. P. PODIO-GUIDUGLI.

The research efforts of M. J. GOLDEN were supported by International Collaboration grants from Enterprise Ireland.

The authors wish to thank G. AMENDOLA for his suggestions.

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(*Received May 27, 2004 / Accepted July 11, 2005*)
Published online January 27, 2006 – © *Springer-Verlag* (*2006*)