Arch. Rational Mech. Anal. 179 (2006) 389–412
Digital Object Identifier (DOI) 10.1007/s00205-005-0396-z

# *Symmetry Groups and Non-Planar Collisionless Action-Minimizing Solutions of the Three-Body Problem in Three-Dimensional Space*

DAVIDE L. FERRARIO

*Communicated by* P. RABINOWITZ


## Abstract

Periodic and quasi-periodic solutions of the $n$-body problem can be found as minimizers of the Lagrangian action functional restricted to suitable spaces of symmetric paths. The main purpose of this paper is to develop a systematic approach to the equivariant minimization for the three-body problem in three-dimensional space. First we give a finite complete list of symmetry groups fitting to the minimization of the action, with the property that any other symmetry group can be reduced to be isomorphic to one of these representatives. A second step is to prove that the resulting (local and global) symmetric action-minimizers are always collisionless (when they are not already bound to collisions). Furthermore, we prove some results which address the question of whether minimizers are planar or non-planar; as a consequence of our theory we will give general criteria for a symmetry group to yield planar or homographic minimizers (either homographic or not, as in the Chenciner-Montgomery eight solution). On the other hand we will provide a rigorous proof of the existence of some interesting one-parameter families of periodic and quasi-periodic non-planar orbits. These include the choreographic Marchal's $P_{12}$ family with equal masses – together with a less-symmetric choreographic family (which anyway probably coincides with the $P_{12}$ family).


## 1. Introduction

In some recent papers, classical variational methods have been successfully applied in the proof of the existence of periodic or quasi-periodic solutions for the $n$-body problem. Suitable symmetry groups of the Lagrangian action functional have been introduced and exploited in order to apply the aforementioned techniques to the class of all symmetric loops, to name a few, in the articles [2–4, 10–12, 15, 18, 19]. Surveys and further details on this approach can be found for example in [1, 5, 7, 12, 17]. The major problem in the search for equivariant minimizers is that of collisions: a (local or global) minimizer might consist *a priori* of a colliding trajectory. The latest significant breakthrough in this direction has been allowed by Marchal's averaging technique [5, 12]. In the paper [12], the authors develop a general theory for $G$-equivariant minimizers and present a class of groups that yield always, as a consequence of Marchal's averaging technique, collision-free minimizers. In [1], this result was extended to all possible symmetry groups for the planar three-body problem. A naturally related problem is to find and classify all possible symmetry groups and to understand whether the resulting minima are rotating central configurations or if they are new solutions (and, at the same time, to provide rigorous proofs of the existence and of the properties of some solutions whose existence was accepted as a fact after numerical evidence). This has been done for the planar problem by V. BARUTELLO, S. TERRACINI & D. FERRARIO in [1]. The purpose of the paper is to give a complete answer to the classification problem for the three-body problem in space, and at the same time to determine and describe properties of the resulting minimizers. In particular, we focus on non-planar orbits, since planar orbits have been already included in the list of [1]. In order to state the main results, we anticipatively sketch some basic definitions: A symmetry group $G$ of the Lagrangian functional $\mathcal{A}$ (see Definition 2.5 below) is termed *bound to collisions* if all $G$-equivariant loops actually have collisions (see Definition 2.6 below), *fully uncoercive* if for every possible rotation vector $\underline{\omega}$ the action functional $\mathcal{A}^G_{\underline{\omega}}$ in the frame rotating around $\underline{\omega}$ with angular speed $|\underline{\omega}|$ is not coercive in the space of $G$-equivariant loops (that is, its global minimum escapes to infinity – see Definition 2.17); moreover, $G$ is termed *homographic* if all $G$-equivariant loops are constant up to orthogonal motions and rescaling. Note that if there is a rotation axis $\underline{\omega}$ then the group $G$ is implicitly assumed to be a symmetry group of the action functional $\mathcal{A}_{\underline{\omega}}$ in the rotating frame (that is, the functional including the centrifugal force and Coriolis terms); such a group is termed *of type R* (see Definition 2.14 below); finally, the *core* of the group $G$ is the subgroup of all the elements which do not move the time $t \in \mathbb{T}$ (see Definition 2.8 below). In the first theorem we classify symmetry groups, up to a change in rotating frame. For the symbols used, refer to Sections 2 and 3 below.

**Theorem A.** *Symmetry groups not bound to collisions, not fully uncoercive and not homographic are, up to a change of rotating frame, either the three-dimensional extensions of planar groups (if trivial core) listed in Table* 1 *or the vertical isosceles triangle (if non-trivial core) of Definition* 6.2.

The next theorem is the answer to the natural questions about collisions and description of some main features of minimizers.

**Theorem B.** *Let $G$ be a symmetry group not bound to collisions and not fully uncoercive. Then,*

*(i) Local minima of $\mathcal{A}^G_{\underline{\omega}}$ do not have collisions.*
*(ii) In the following cases minimizers are planar trajectories:*
  *(a) $G$ is not of type $R$: $D_6^{+,-}$, $D_6^{-,+}$ and $D_{12}^{-,+}$ (then $G$-equivariant minimizers are Chenciner–Montgomery eights);*

**Table 1.** Space extensions of planar symmetry groups with trivial core

| *Name* | *Extensions* |
|---|---|
| Trivial | $C_1^-$ |
| Line | $L_2^{+,-}$, $L_2^{-,+}$ |
| Isosceles | $H_2^{+,-}$, $H_2^{-,+}$ |
| Hill | $H_4^{+,-}$, $H_4^{-,+}$ |
| 3-choreography | $C_3^+$, $C_3^-$ |
| Lagrange | $L_6^{+,+}$, $L_6^{+,-}$, $L_6^{-,+}$ |
| $D_6$ | $D_6^{+,-}$, $D_6^{-,+}$ |
| $D_{12}$ | $D_{12}^{-,+}$ |

*(b) there is a G-equivariant minimal Lagrange rotating solution: $C_1^-$, $H_2^{+,-}$, $C_3^+$, $L_6^{+,+}$ and $L_6^{+,-}$ (then the Lagrange solution is of course the minimizer);*

*(c) the core is non-trivial and it is not the vertical isosceles, see Definition* 6.2 *(then the minimizers are homographic).*

*(iii) In the following cases minimizers are always non-planar:*

*(a) the groups $L_6^{-,+}$ and $C_3^-$ for all $\omega \in (-1,1)+6\mathbb{Z}$, $\omega \neq 0$ (the minimizers for $L_6^{-,+}$ are the elements of Marchal family $P_{12}$, and the minimizers of $C_3^-$ are a less-symmetric family $P'_{12}$ [1]);*

*(b) the extensions of line and Hill-Euler type groups, for on open subset of mass distributions and angular speeds ω (explicitly given in* (5.8)*): $L_2^{+,-}$, $L_2^{-,+}$, $H_4^{+,-}$ and $H_4^{-,+}$ (for $L_2^{-,+}$ this happens also with equal masses).*

*(c) the vertical isosceles, (see Definition* 6.2*), for suitable choices of masses and ω.*

In this paper, we develop the needed tools and prove these statements, after the necessary explanations about preliminary results and notation. In Section 2, we introduce all the definitions needed and prove some preliminary results. In Section 3, we introduce the concept of a three-dimensional *extension* of a planar symmetry group, so that we can use the classification of [1]. In Section 4, the angular momentum $J$ must be taken into account, and we show how the existence of rotation axes is related to the possibility of $J$ being non-zero. Afterwards, in Section 5, we prove some interesting estimates on second variations, which are remarkably simple (incidentally, they work not only for 3 bodies, but for $n$ arbitrary). It is by an application of these simple estimates that we can prove the fact that the non-planar quasi-periodic orbits listed in Theorem B exist. In Section 6, we come to the classification of three-dimensional space symmetries, which is a proof of Theorem A. The proof of the various items of Theorem B is completed in Section 7. Finally, in Section 8, some concluding remarks are collected.

[1] Highly likely they are not distinct families: this is the recurring phenomenon of "more symmetries than expected" in $n$-body problems.

Before we start with the next section, a few words have to be spent on the existence of the $P_{12}$ family. A different – and very elementary – proof of the existence of the $P_{12}$-family with $D_{12}$-symmetries was presented by A. CHENCINER in [5, 7], which does not require local results on collisions, since collisions are excluded by action level estimates. The advantage of our approach is that it can be plainly extended to the case of any odd number $n \geqq 3$ of bodies in space (see Remark 8.5 below). All other results are, to our knowledge, new: whenever similar methods or results were published elsewhere, it has been remarked in-place.

## 2. Preliminaries

Consider the linear space of configurations with center of mass in 0:

$$\mathcal{X} = \{x = (x_1, x_2, x_3) \in E^3 \mid m_1x_1 + m_2x_2 + m_3x_3 = 0\}.$$

Let $\mathbb{T} = S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ be the unit circle of length $2\pi$. We are dealing with periodic orbits of the Newtonian $n$-body problem, which will be seen as critical points of a suitable functional on the Sobolev space $\Lambda = H^1(\mathbb{T}, \mathcal{X})$ consisting of all $L^2$ loops $\mathbb{T} \to \mathcal{X}$ with $L^2$ derivative. It is an Hilbert space with the scalar product

$$x \cdot y = \int_{\mathbb{T}} (x(t)y(t) + \dot{x}(t)\dot{y}(t))dt.$$

The $\alpha$-homogeneous Newtonian potential can be written as

$$U(x) = \frac{m_1m_2}{|x_1 - x_2|^\alpha} + \frac{m_1m_3}{|x_1 - x_3|^\alpha} + \frac{m_2m_3}{|x_2 - x_3|^\alpha}. \tag{2.1}$$

Let $\underline{\omega} \in E \cong \mathbb{R}^3$ be a vector. The kinetic form in a frame uniformly rotating around $\underline{\omega}$ with angular speed $\omega = |\underline{\omega}|$ is defined by

$$2K(x, \dot{x}) = \sum_{i=1}^{3} m_i |\dot{x}_i + \Omega x_i|^2, \tag{2.2}$$

where $\Omega$ is the matrix

$$\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

obtained by the coefficients $(\omega_1, \omega_2, \omega_3)$ of $\underline{\omega} \in \mathbb{R}^3$. Thus, for every $\underline{\omega}$ the Lagrangian can be written as

$$L_{\underline{\omega}}(x, \dot{x}) = L_{\underline{\omega}} = K_{\underline{\omega}} + U, \tag{2.3}$$

and, finally, the action functional as

$$\mathcal{A}_{\underline{\omega}}(x) = \int_{\mathbb{T}} L_{\underline{\omega}}(x(t), \dot{x}(t))dt. \tag{2.4}$$

**Definition 2.5.** We define the *symmetry group* as every subgroup of $O(\mathbb{T}) \times O(3) \times \Sigma_3$, where $O(\mathbb{T}) = O(2)$ is the orthogonal group of dimension 2 acting on the time circle, $O(3)$ is the orthogonal group of dimension 3 acting on the space $E$ and $\Sigma_3$ is the symmetric group on the three elements $\{1, 2, 3\}$.

Given a subgroup $G \subset O(\mathbb{T}) \times O(3) \times \Sigma_3$, it is possible to define three homomorphisms $\tau\colon G \to O(\mathbb{T})$, $\rho\colon G \to O(3)$ and $\sigma\colon G \to \Sigma_3$ by projections onto the first, second and third factors of the direct product. Given $\tau$, $\rho$ and $\sigma$ we can define in an obvious way a $G$-action on $\mathbb{T}$, $E$ and $\{1, 2, 3\}$, and hence an action on the centered configuration space $\mathcal{X}$, provided that for every $g \in G$, $m_i = m_{gi}$ (that is, the masses are constant in $G$-orbits in the index set $\{1, 2, 3\}$). Thus, there is an induced action of $G$ on the Sobolev space $\Lambda$ of loops, defined by $g(x(t)) = (gx)(g^{-1}t)$. The action is orthogonal on $\Lambda$ so that the Palais theorem says that, for a given group $G$, if the functional $\mathcal{A}_{\underline{\omega}}\colon \Lambda \to \mathbb{R}$ is $G$-invariant and a $G$-equivariant loop $x(t)$ is collisionless and critical for the restriction $\mathcal{A}_{\underline{\omega}}^G = \mathcal{A}_{\underline{\omega}}|\Lambda^G$, then $x(t)$ is critical for $\mathcal{A}_{\underline{\omega}}$.

**Definition 2.6.** A group is termed *bound to collisions* if for every equivariant loop $x(t) \in \Lambda^G$ collisions occur, that is, for each $G$-equivariant $x(t)$ there exist $t_c \in \mathbb{T}$ and $i \neq j \in \mathbf{n} = \{1, 2, 3\}$ such that $x_i(t_c) = x_j(t_c)$.

**Definition 2.7.** A group $G$ is termed *homographic* if every equivariant loop $x(t) \in \Lambda^G$ is constant up to rescaling and orthogonal motions.

**Definition 2.8.** The kernel $\ker\tau$ is termed the *core* of the symmetry group $G$.

**Definition 2.9.** A group $G$ is said to be of *cyclic*, *brake* or *dihedral* type respectively, if $G/\ker\tau$ acts orientation-preserving on the time circle $\mathbb{T}$, if $G/\ker\tau$ has order 2 and acts orientation-reversing on $\mathbb{T}$ or if $G/\ker\tau$ is a dihedral group of order $\geqq 4$.

Consider the following elements in $O(2)$: 1 is the trivial motion, $-1$ is the rotation of angle $\pi$ and $l$ is a reflection along a line. Elements of the symmetric group $\Sigma_3$ will be denoted in the cyclic permutation notion. Let $\ker\det(\tau) \subset G$ denote the subgroup of $G$ of the elements acting orientation-preserving on $\mathbb{T}$. A symmetry group with trivial core will be fully determined once the images $\rho(r)$ and $\sigma(r)$ of a generator $r$ of $\ker\det(\tau)$ rotating a minimal angle are given, together, if it is not of cyclic type, with the images $\rho(h)$ and $\sigma(h)$ of one of the elements not in $\ker\det(\tau)$ (which have order 2). The full list of representatives of the planar classification exposed in [1] can be therefore found in Table 2, with the corresponding generators.

**Definition 2.10.** A planar symmetry group $G$ is said to be *of type R* if the determinant homomorphisms $\det(\rho), \det(\tau)\colon G \to \{+1, -1\}$ coincide, that is, if they coincide as $G$-representations.

**Definition 2.11.** A vector $v \in E \cong \mathbb{R}^3$ is defined as a *rotation axis* with respect to a symmetry group $G$ if the line spanned by $v$ in $E$ is $G$-invariant and the following equality of one-dimensional $G$-representations holds:

$$\det(\tau)\det(\rho) = \det(v),$$

**Table 2.** Planar symmetry groups with trivial core

| *Name* | *Symbol* | $\rho(r)$,$\sigma(r)$ | $\rho(h)$,$\sigma(h)$ |
|---|---|---|---|
| Trivial | $C_1$ | 1,() | |
| Line | $L_2$ | 1,() | 1,() |
| 2-1-choreography | $C_2$ | 1, (1,2) | |
| Isosceles | $H_2$ | 1,() | 1,(1,2) |
| Hill | $H_4$ | 1,(1,2) | 1,(1,2) |
| 3-choreography | $C_3$ | 1,(1,2,3) | |
| Lagrange | $L_6$ | 1,(1,2,3) | 1,(1,2) |
| $C_6$ | $C_6$ | 1,(1,2,3) | |
| $D_6$ | $D_6$ | 1,(1,2,3) | −1,(1,2) |
| $D_{12}$ | $D_{12}$ 1,(1,2,3) | −1,(1,2) | |

where $\det(v)$ denotes the real representation of $G$ induced by restricting $\rho$ to the invariant subspace generated by $v \in E$.

**Lemma 2.12.** *The restriction of a three-dimensional symmetry group G to the orthogonal complement of a rotation axis $v \in E$ is a planar symmetry group of type R. Conversely, if the restriction of G to an invariant plane is of type R, then the orthogonal complement of the invariant plane is a rotation axis for G.*

**Proof.** Let $\tau$, $\rho$ and $\sigma$ be the defining homomorphisms of $G$, where $\rho\colon G \to O(3)$ can be written as $\rho = \rho_2 \times \rho_1$, with $\rho_2\colon G \to O(2)$ induced by restriction to the (invariant) orthogonal complement of $v$ and $\rho_1\colon G \to O(1)$ by restriction to $v$. Since

$$\det(\rho) = \det(\rho_2)\det(\rho_1) = \det(\rho_2)\det(v), \tag{2.13}$$

the planar symmetry group defined by $\tau$, $\rho_2$, $\sigma$ (see Definition 2.10) is of type R if and only if $\det(\rho_2) = \det(\tau)$, and hence if and only if $\det(\rho)\det(v) = \det(\tau)$ as claimed. □

The previous lemma yields the following natural definition.

**Definition 2.14.** A space symmetry group $G$ is said to be *of type R* if it has at least one rotation axis, that is, if it is the extension of a planar group of type R.

**Lemma 2.15.** *Let $\underline{\omega} \in E \cong \mathbb{R}^3$ be a rotation axis for a symmetry group $G$. Then the Lagrangian action functional $\mathcal{A}_{\underline{\omega}}$ (defined in* (2.4)*) in a frame rotating around $\underline{\omega}$ with angular speed $\omega = |\underline{\omega}|$ is $G$-invariant.*

**Proof.** Let $g \in G$ and $x(t) \in \Lambda^G$. Since for every $i \in \{1, 2, 3\}$ ,

$$x_{gi}(\tau(g)t) = \rho(g)x_i(t),$$

the derivative fulfills the equality

$$\det(\tau(g))\dot{x}_{gi}(\tau(g)t) = \rho(g)\dot{x}_i(t).$$

Thus for every $g \in G$ and $t \in \mathbb{T}$,

$$\dot{x}_{gi}(\tau(g)t) + \Omega x_{gi}(\tau(g)(t)) = \det(\tau(g))\rho(g)\dot{x}_i + \Omega\rho(g)x_i(\tau(t)),$$

and hence

$$\left|\dot{x}_{gi}(\tau(g)t) + \Omega x_{gi}(\tau(g)(t))\right|^2 = \left|\dot{x}_i + \det(\tau(g))\rho(g^{-1})\Omega\rho(g)x_i(\tau(t))\right|^2.$$

We can deduce that the action functional $\mathcal{A}_{\underline{\omega}}$ is $G$-invariant if (and only if) for every $g$,

$$\det(\tau(g))\rho(g^{-1})\Omega\rho(g) = \Omega. \tag{2.16}$$

If $\Omega \neq 0$, equation (2.16) holds if and only if $\det(\rho_2) = \det(\tau)$, where as above $\rho_2$ denotes the restriction of $\rho$ to the plane orthogonal to $\underline{\omega}$. But by (2.13) this is equivalent to the identity $\det(\rho)\det(\underline{\omega}) = \det(\tau)$, that is, $\underline{\omega}$ is a rotation axis as in Definition 2.11. □

**Definition 2.17.** A symmetry group $G$ is *fully uncoercive* if for every possible rotation vector $\underline{\omega}$, the action functional $\mathcal{A}^G_{\underline{\omega}}$ is not coercive.

The following proposition is an easy consequence of this definition and (4.1) of [12].

**Lemma 2.18.** *Let $G$ be a symmetry group.*

*(i) If there are no rotation axes and $\mathcal{X}^G \neq 0$, (or, equivalently, $\mathcal{A}^G$ is not coercive), then $G$ is fully uncoercive.*
*(ii) If every rotating axis is uncoercive as a one-dimensional $G$-module and the action on the index set is not transitive, then $G$ is fully uncoercive.*

**Definition 2.19.** If $G$ and $G'$ are two groups conjugated in $O(\mathbb{T}) \times O(3) \times \Sigma_3$, we will write $G \cong G'$. If there exists a change of rotating frame for which a group $G$ can be written as $G'$, which is conjugate to a third group $G''$, we will write $G \sim G''$. It is easy to see that $\cong$ and $\sim$ are equivalence relations, and that $G \cong G' \implies |G| = |G'|$, while the same does not hold for $\sim$ (see [1], Section 3 for further details on changing the coordinates in a rotating frame).

**Proposition 2.20.** *Let $G$ be a symmetry group such that a Lagrange rotating solution $x(t) = \{x_j(t)\} = \{e^{ikt}\zeta_3^j\}$ is $G$-equivariant and $|k+\omega|$ is minimal (as $k$ varies in $\mathbb{Z}$) and non-zero. Then $x(t)$ is the absolute minimum of the action functional.*

**Proof.** This is, for the three-dimensional plane, proposition (4.1) of [1] (see also [8]). Actually, for the three body problem the proof is straightforward: assume that $\sum_i m_i = 1$; since the center of mass is in zero, $\sum_i m_i x_i = 0$, and hence $\sum_i m_i \dot{x}_i = 0$. The kinetic energy can be written in terms of the differences

$$\frac{1}{2}\sum_i m_i|\dot{x}_i + \Omega x_i|^2 = \frac{1}{2}\sum_{i<j} m_i m_j |\dot{x}_i - \dot{x}_j + \Omega(x_i - x_j)|^2. \tag{2.21}$$

Thence the action functional is written as the sum of three terms of the type

$$\frac{1}{2} m_i m_j |\dot{x}_i - \dot{x}_j + \Omega x_i - \Omega x_j|^2 + m_i m_j |x_i - x_j|^{-\alpha},$$

which is a Kepler (one-center) problem for the variable $y = x_i - x_j$ in the rotating frame. Since a rotating solution $x(t)$ with $|k + \omega|$ minimal exists by assumption, it yields three (identical, up to a time-shift) rotating solutions in $y$, with $|k + \omega|$ minimal. It is easy to conclude the proof and to show that every trajectory has an action which is at least three times the action of a minimal one-center $y$. □

## 3. Three-dimensional extensions of planar symmetry groups

In this section we will take planar groups, listed in Table 2, and define some extensions acting on the three-dimensional space. Of all the resulting groups, we will take into account only the extensions with trivial core, not bound to collisions and not fully uncoercive. The outcome is the list of Table 1. We proceed as follows. Consider one of the planar groups in Table 2. It can be extended to a group acting on the three-dimensional space simply by adding a one-dimensional real representation. Now, since the groups have trivial core as in Table 2 we can assume that the symmetry group is generated by two elements $r$ and $h$ with the following properties: $\tau(r)$ is a time-shift in $\mathbb{T}$ of minimal angle and $\tau(h)$ is a time-reflection (which exists only if the group is not of cyclic type). Up to conjugacy or change of orientation in $\mathbb{T}$, the choice of $r$ and $h$ yields uniquely back the symmetry group $G$.

Consider first the case of cyclic type, and let $r$ denote the cyclic generator above (in the notation of Table 2, groups of cyclic type are $C_1$, $C_2$, $C_3$ and $C_6$). Now, $\rho_2(r) \in O(2)$ can be extended in two ways to a matrix in $O(3)$: adding either a trivial one-dimensional representation or a non-trivial one. Thus, for each cyclic group $C_i$ listed above, there exist two corresponding groups, denoted by $C_i^+$ and $C_i^-$, which are generated by the element $(\tau(r), \rho_2(r), \varepsilon, \sigma(r))$ in $O(\mathbb{T}) \times O(2) \times O(1) \times \Sigma_3$, for $\varepsilon = \rho_1(r) \in \{\pm 1\}$. The other cases can be dealt with in an analogous way; the choices are $2^2$: a sign for $\rho_1(r)$ and a sign for $\rho_1(h)$. So, if $G$ is a symmetry group not of cyclic type, its three-dimensional extensions groups will be denoted by the symbol $G^{\varepsilon_1, \varepsilon_2}$, where $\varepsilon_1$ is the sign of $\rho_1(r)$ and $\varepsilon_2$ the sign of $\rho_1(h)$. By 2.12 the third axis will be a rotation axis if and only if the planar symmetry group is of type R. Furthermore, it is easy to see that if the action of the group on the index set is not transitive, extensions of type $+, +, +$ are fully uncoercive. The list of remaining symmetry groups is therefore: $C_1^-$, $C_1^{+,-}$, $C_1^{-,+}$, $C_1^{-,-}$, $L_2^{+,-}$, $L_2^{-,+}$, $L_2^{-,-}$ $C_2^-$ $H_2^{+,-}$, $H_2^{-,+}$, $H_2^{-,-}$ $H_4^{+,-}$, $H_4^{-,+}$, $H_4^{-,-}$ $C_3^+$, $C_3^-$ $L_6^{+,+}$, $L_6^{+,-}$, $L_6^{-,+}$, $L_6^{-,-}$, $C_6^+$, $C_6^-$ $D_6^{+,+}$, $D_6^{+,-}$, $D_6^{-,+}$, $D_6^{-,-}$, $D_{12}^{+,+}$ , $D_{12}^{+,-}$, $D_{12}^{-,+}$ and $D_{12}^{-,-}$. Now, some of them are the the same after a change in coordinates: $C_1^{-,-} \cong C_1^{-,+}$, $L_2^{-,+} \cong L_2^{-,-}$, $H_2^{-,+} \cong H_2^{-,-}$, $H_4^{-,+} \cong H_4^{-,-}$, $C_6^+ \cong C_3^-$, $C_6^- \sim C_3$, $D_6^{+,+} \cong L_6^{+,-}$, $D_6^{-,+} \cong D_6^{-,-} \cong D_{12}^{+,-}$, $L_6^{-,+} \cong L_6^{-,-} \cong D_{12}^{+,+}$ and $D_{12}^{-,-} \sim L_6^{+,-}$. Furthermore, $C_1^{+,-}$, $C_1^{-,+}$ and $C_2^-$ are clearly fully uncoercive. Hence the following lemma holds.

**Lemma 3.1.** *Of all the three-dimensional extensions of planar symmetry groups, those with trivial core, not bound to collisions and not fully uncoercive are listed in Table* 1.

**Remark 3.2.** The order of the space group now does not necessarily coincide with the order of the planar group: for example, the order of $C_3^-$ is 6 and not 3.

The following lemma will be used as a key-step for the classification below.

**Lemma 3.3.** *Let G a symmetry group for the three-body problem with trivial core. Then, up to a change of rotating frame, $\rho$ is the sum of one-dimensional real representations.*

**Proof.** Since $\ker\tau = 1$, $G$ is isomorphic to a subgroup of a finite dihedral group, and hence its orthogonal irreducible representations have dimension at most 2. So, $\rho$ can be written as $\rho_2 \times \rho_1$, where $\rho_2\colon G \to O(2)$ and $\rho_1\colon G \to O(1)$. Now, by (5.1) of [1] up to a change in rotating frame we can assume that $\rho_2(g)^2 = 1$ for every $g \in G$, so that $\rho_2$ is reducible as a sum of two one-dimensional $G$-representations. □

## 4. Groups without rotation axes

As we have seen in Lemma 3.1, the list of candidates for space symmetry groups is given in Table 1. Now we consider the 10 groups yielded by extending the three planar groups not of type R.

First, we prove a three-dimensional analogue of Proposition 3.9 of [1]. Given a path $x(t) \in \Lambda$, its angular momentum $J$ is the function of $t \in \mathbb{T}$ given by

$$J(t) = \sum_{i \in \mathbf{n}} m_i x_i \times \dot{x}_i \,, \tag{4.1}$$

where $\times$ is the vector product in $E \cong \mathbb{R}^3$. If $x$ is a (generalized) solution, then the angular momentum is constant.

**Lemma 4.2.** *For every equivariant $x(t) \in \Lambda^G$ and every $t \in \mathbb{T}$,*

$$J(gt) = \det(\rho(g)) \det(\tau(g)) \rho(g) J(t).$$

**Proof.** The lemma follows from the chain of equalities below:

$$\begin{aligned} J(\tau(g)t) &= \sum_{i=1}^{3} m_i x_i(\tau(g)t) \times \dot{x}_i(\tau(g)t) \\ &= \sum_{i=1}^{3} m_i \left[ (\rho(g) x_{g^{-1}i}(t)) \times (\det(\tau(g)) \rho(g) \dot{x}_{g^{-1}i}(t)) \right] \\ &= \det(\tau(g)) \sum_{i=1}^{3} m_{g^{-1}i} \left[ (\rho(g) x_{g^{-1}i}(t)) \times (\rho(g) \dot{x}_{g^{-1}i}(t)) \right] \end{aligned}$$

$$= \det(\tau(g)) \sum_{i=1}^{3} m_{g^{-1}i} \det(\rho(g))\rho(g) \left[x_{g^{-1}i}(t) \times \dot{x}_{g^{-1}i}(t)\right]$$
$$= \det(\tau(g)) \det(\rho(g))\rho(g) J(t). \quad \square$$

**Lemma 4.3.** *Let $x \in \Lambda^G$ be a $G$-equivariant periodic orbit with angular momentum $J$. Then $J$ belongs to the subspace $E^* \subset E$ fixed by the $G$-representation* $\det(\rho)\det(\tau)\rho$.

**Proof.** By Lemma 4.2, for every $g \in G$ $J = \det(\tau(g))\det(\rho(g))\rho(g)J$, and hence $J \in E^*$. $\square$

**Lemma 4.4.** *Let $G$ be an extension of a planar symmetry group not of type R and let $V \subset E$ denote the invariant plane. Then $E^* \subset V$.*

**Proof.** Let $V^*$ denote the orthogonal complement of the invariant plane. Since $\det(\rho) = \det(\rho_2)\det(\rho_1)$ with $\det(\rho_2)\det(\tau) \neq 1$, the projection of $E^*$ on $V^*$ is fixed by the action of $G$ under the non-trivial homomorphism $\det(\tau)\det(\rho)\det(\rho_1) = \det(\tau)\det(\rho_2)$. Hence $E^* \subset V$. $\square$

By Lemma 4.4, we need to consider the vectors in the plane $V \subset E$ fixed by

$$\det(\rho_2(r))\varepsilon_1\rho_2(r) \quad \text{and} \quad -\det(\rho_2(h))\varepsilon_2\rho_2(h)$$

(the latter only if the action type is not cyclic), where $\varepsilon_1$ and $\varepsilon_2$ are as above the elements $\rho_1(r)$ and $\rho_1(h)$.

For $C_6^+$, for example, $E^*$ is the subspace fixed by $-\rho_2(r)$, which is a reflection along a line. Hence $C_6^+$, even if there is an extension of a planar symmetry group not of type R, might have minimizers with non-zero angular momentum. In fact, it is not difficult to see that up to a change in coordinates $C_6^+ = C_3^-$. On the other hand, for $C_6^-$ it happens that $E^*$ is the subspace fixed by $\rho_2(r)$, which is again a line. Again, as above, $C_6^-$ can be written as $C_3^-$ with a suitable choice of $\omega$ for $C_3^-$ (which is of type R).

Now we can consider the extensions of $D_6$ and $D_{12}$. For $D_6$, we have that $\det(\rho_2(r))\varepsilon_1\rho_2(r)$ and $-\det(\rho_2(h))\varepsilon_2\rho_2(h)$ are respectively equal to $\varepsilon_1$ and $\varepsilon_2$ (seen as $2\times 2$ matrices), and hence equivariant minimizers of $D_6^{+,-}$, $D_6^{-,+}$ and $D_6^{-,-}$ have zero angular momentum. On the other hand it is easy to see that after a change of coordinates $D_6^{+,+} = L_6^{+,-}$. For $D_{12}$, $\det(\rho_2(r))\varepsilon_1\rho_2(r)$ and $-\det(\rho_2(h))\varepsilon_2\rho_2(h)$ are respectively equal to $-\varepsilon_1\rho_2(r)$ (which is a reflection along a line) and $\varepsilon_2$ (seen as a matrix). Thus, if $\varepsilon_2 = -1$, orbits have zero angular momentum. Otherwise, for $\varepsilon_2 = 1$, this is not necessary. Furthermore, it is true that $D_{12}^{+,+} = L_6^{-,-}$ and $D_{12}^{-,+} = L_6^{+,-}$ for a suitable $\underline{\omega}$.

The following definition is the natural extension of the corresponding property for planar groups.

**Definition 4.5.** A symmetry group $G$ is said to be *of type R* if there is a rotation axis for $G$ (and the restriction of the action of $G$ on the invariant plane orthogonal to the axis is a planar symmetry group of type R). A symmetry group $G$ is said to be *not of type R* if there are no rotation axes for $G$ in $E$.

The following lemma follows immediately from the previous arguments.

**Lemma 4.6.** *If $G$ does not have rotation axes, i.e., it is not of type $R$, then all $G$-equivariant trajectories have zero angular momentum and hence they are planar.*

**Lemma 4.7.** *Let $G$ be any space symmetry group not of type $R$. Then every $G$-equivariant non-collinear orbit is contained in a (unique) $G$-invariant plane.*

**Proof.** By Lemma 4.6, the angular momentum is zero and hence the orbit is planar. It is only left to show that the plane containing the orbit is $G$-invariant. But since for every $g \in G$ and every $t \in \mathbb{T}$

$$\begin{aligned} &\pm g\left[(x_1(t) - x_2(t)) \times (x_1(t) - x_3(t))\right] \\ &\quad = (gx_1(t) - gx_2(t)) \times (gx_1(t) - gx_3(t)) \\ &\quad = (x_{g1}(gt) - x_{g2}(gt)) \times (x_{g1}(gt) - x_{g3}(gt), \end{aligned}$$

it follows that the plane containing the configuration $x_1(t)$, $x_2(t)$ and $x_3(t)$ is $G$-invariant. □

## 5. The vertical variation

Let $(z, w)$ be a system of coordinates for the Euclidean space $E \cong \mathbb{R}^3 \cong \mathbb{C} \oplus \mathbb{R}$, with $z \in \mathbb{C}$ and $w \in \mathbb{R}$. For a planar central configuration $\bar{x}$, consider the planar rotating periodic path $x(t) = e^{ikt}\bar{x}$, with $k \in \mathbb{Z}$. In space the orbit can be written for $i = 1, 2, 3$ as $(x_i(t), w_i(t)) \in \mathbb{R}^2 \times \mathbb{R}$ with $w_i = 0$. Now consider three periodic $H^1$-functions $\varphi_i : \mathbb{T} \to \mathbb{R}$. There is a corresponding path in $\Lambda$, which will be denoted by $(x(t), \varepsilon\varphi(t))$, obtained by adding the vertical variation $\varepsilon\varphi$ to the rotation configuration $\bar{x}$.

**Lemma 5.1.** *Let $\mathcal{A}(\varepsilon)$ denote the action of the path $(x(t), \varepsilon\varphi(t))$ in $[0, 2\pi]$. Then the second derivative of $\mathcal{A}(\varepsilon)$ evaluated with $\varepsilon = 0$ is*

$$\left.\frac{d^2\mathcal{A}}{d\varepsilon^2}\right|_{\varepsilon=0} = \int_0^{2\pi} \left[ \sum_{i \in \mathbf{n}} m_i \dot{\varphi}_i^2 - \alpha \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^{\alpha+2}} \left(\varphi_i - \varphi_j\right)^2 \right] dt.$$

**Proof.** The second derivative of the kinetic part is

$$\frac{d^2}{d\varepsilon^2} \sum_{i \in \mathbf{n}} \frac{1}{2} m_i (|\dot{x}_i + \Omega x_i|^2 + \varepsilon^2 \dot{\varphi}_i^2) = \sum_{i \in \mathbf{n}} m_i \dot{\varphi}_i^2.$$

Now, it is easy to see that

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \left[(a + \varepsilon^2 b)^c\right] = 2a^{c-1}bc;$$

moreover, the terms in the potential part contain expressions of the type

$$m_i m_i \left[ (x_i - x_j)^2 + \varepsilon^2 (\varphi_i - \varphi_j)^2 \right]^{-\alpha/2},$$

with $a = (x_i - x_j)^2$, $b = (\varphi_i - \varphi_j)^2$ and $c = -\alpha/2$. Hence ,

$$\begin{aligned}
&\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \sum_{i<j} m_i m_j |(x_i, \varphi_i) - (x_j, \varphi_j)|^{-\alpha} \\
&= \sum_{i<j} m_i m_j 2 \left[ (x_i - x_j)^2 \right]^{-\alpha/2-1} (\varphi_i - \varphi_j)^2 \left( -\frac{\alpha}{2} \right) \\
&= -\alpha \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^{\alpha+2}} (\varphi_i - \varphi_j)^2.
\end{aligned}$$

Thus, the claim is proved. □

**Lemma 5.2.** *Consider the path $x(t) = e^{ikt}\bar{x}$ as above. For a unit vector $\mathbf{e} \in \mathbb{C} \subset E$ define $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ by the scalar product $\varphi_i(t) = x_i(t/k) \cdot \mathbf{e}$ for $i = 1, 2, 3$. Then, the second variation of Lemma 5.1 is*

$$\frac{d^2 \mathcal{A}}{d\varepsilon^2}\bigg|_{\varepsilon=0} = \pi \left( I(\bar{x}) - \alpha U(\bar{x}) \right),$$

*where $I(\bar{x}) = \sum_i m_i \bar{x}^2$ is the momentum of inertia of $\bar{x}$ and $U(\bar{x})$ is the value of the potential function.*

**Proof.** Define $\beta = 1/k$. Then, $\varphi_i(t) = x_i(\beta t) \cdot \mathbf{e} = (e^{it}\bar{x}_i) \cdot \mathbf{e}$ and therefore

$$\int_0^{2\pi} m_i \dot{\varphi}_i^2 \, dt = \int_0^{2\pi} m_i \bar{x}^2 \sin^2(t + \delta_i) \, dt = \pi m_i \bar{x}^2$$

for some suitable $\delta_i$, which implies that

$$\int_0^{2\pi} \sum_{i=1}^{3} m_i \dot{\varphi}_i^2 = \pi \sum_{i=1}^{3} m_i \bar{x}^2 = \pi I(\bar{x}).$$

As for the second part of the expression in Lemma 5.1, since the norms $|x_i - x_j|$ are constant we obtain

$$\begin{aligned}
&\int_0^{2\pi} \frac{m_i m_j}{|x_i(t) - x_j(t)|^{\alpha+2}} (\varphi_i - \varphi_j)^2 \, dt \\
&= \int_0^{2\pi} \frac{m_i m_j}{|x_i(t) - x_j(t)|^{\alpha}} \left( \frac{\varphi_i - \varphi_j}{|x_i(t) - x_j(t)|} \right)^2 dt \\
&= \frac{m_i m_j}{|\bar{x}_i - \bar{x}_j|^{\alpha}} \int_0^{2\pi} \left( \frac{x_i(\beta t) - x_j(\beta t)}{|x_i(\beta t) - x_j(\beta t)|} \cdot \mathbf{e} \right)^2 dt \\
&= \frac{m_i m_j}{|\bar{x}_i - \bar{x}_j|^{\alpha}} \int_0^{2\pi} \cos^2(t + \delta_{ij}) \, dt \\
&= \pi \frac{m_i m_j}{|\bar{x}_i - \bar{x}_j|^{\alpha}}.
\end{aligned}$$

with a suitable choice for the shift constant $\delta_{ij}$. Thus, by summing up we obtain

$$\int_0^{2\pi} \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^{\alpha+2}} \left(\varphi_i - \varphi_j\right)^2 dt = \pi \sum_{i<j} \frac{m_i m_j}{|\bar{x}_i - \bar{x}_j|^{\alpha}} = \pi U(\bar{x}).$$

Hence conclusion follows. □

Until now we did not assume anything else on $x(t)$ other than it rotating $k$ times during the interval $[0, 2\pi]$. Now we assume that it is a minimizer in a suitable *linear* class of paths (such as $\Lambda^G$ for a suitable $G$ acting on the plane or the space). Then the following equation holds:

**Lemma 5.3.** *If $x(t) = e^{ikt}\bar{x}$ is a minimizer of $\mathcal{A}_\omega$, then (Kepler's law)*

$$(k+\omega)^2 I(\bar{x}) = \alpha U(\bar{x}).$$

**Proof.** It is easy to see that the action is

$$\frac{1}{2\pi}\mathcal{A}_\omega = \frac{1}{2}(k+\omega)^2 I(\bar{x}) + U(\bar{x}).$$

By deriving the expression in $R = \sqrt{I}$ we find that the minimum, as $R > 0$ varies, is achieved when $(k+\omega)^2 R^2 = \alpha U(\bar{x})$. Otherwise, we can also use homogeneity and directly Newton's equations. □

**Proposition 5.4.** *Assume that for a symmetry group $G$ every rotating $G$-equivariant central configuration $x(t) = e^{ikt}\bar{x}$ is such that $(k+\omega)^2 > 1$. Then rotating central configurations cannot be minimizers of $\mathcal{A}^G$.*

**Proof.** By Lemma 5.2 the second variation is $\left.\frac{d^2\mathcal{A}}{d\varepsilon^2}\right|_{\varepsilon=0} = \pi(I(\bar{x}) - \alpha U(\bar{x}))$. But by Lemma 5.3

$$\alpha U(\bar{x}) = (k+\omega)^2 I(\bar{x}),$$

so that

$$\left.\frac{d^2\mathcal{A}}{d\varepsilon^2}\right|_{\varepsilon=0} = \pi I(\bar{x})(1 - (k+\omega)^2) < 0,$$

which shows that $x(t)$ cannot be a minimizer. □

Now we consider a different vertical variation, which can be readily used for a vertical isosceles triangle. Consider now the variation $\varphi$ (of Lemma 5.1) defined as follows: $\varphi = v \sin t$, where $v \in \mathbb{R}^n$ is a one-dimensional configuration with $\sum_{i\in\mathbf{n}} m_i v_i = 0$. Without loss of generality we assume that $\sum_{i\in\mathbf{n}} m_i = 1$.

**Lemma 5.5.** *Let $G$ be a symmetry group and $x(t) = e^{ikt}\bar{x}$ be a planar $G$-equivariant rotating central configuration. If $(x, \varphi)$ is $G$-equivariant and*

$$\sum_{i<j} m_i m_j (v_i - v_j)^2 \left(1 - \alpha|\bar{x}_i - \bar{x}_j|^{\alpha+2}\right) < 0, \tag{5.6}$$

*then $x(t)$ is not a minimizer.*

**Proof.** Since $\sum_{i\in\mathbf{n}} m_i\varphi_i = 0$ and $\sum_{i\in\mathbf{n}} m_i = 1$ by assumption, we can write as in (2.21) the kinetic energy in terms of differences, and therefore the equation of Lemma 5.1 can be read as

$$\left.\frac{d^2\mathcal{A}}{d\varepsilon^2}\right|_{\varepsilon=0} = \pi\left[\sum_{i\in\mathbf{n}} m_i m_j (v_i - v_j)^2 - \alpha\sum_{i<j}\frac{m_i m_j}{|x_i - x_j|^{\alpha+2}}\left(v_i - v_j\right)^2\right],$$

since $\int_0^{2\pi}\sin^2 t = \int_0^{2\pi}\cos^2 t = \pi$. This implies the claim. □

**Proposition 5.7.** *In the hypotheses of Lemma* 5.5, *consider the following case:* $n = 3$, $m_1 = m_2$, $x_1 = -x_2$ *(and hence* $x_3 = 0$*) and* $v_1 = v_2$. *Then* $\left.\frac{d^2\mathcal{A}}{d\varepsilon^2}\right|_{\varepsilon=0} < 0$ *if and only if*

$$(k+\omega)^2 > \frac{m_1}{2^{\alpha+1}} + 1 - 2m_1. \tag{5.8}$$

**Proof.** If $x(t) = e^{ikt}\bar{x}$ is a minimum, then $\bar{x} = (R, -R, 0)$, with

$$R^{\alpha+2} = \frac{\alpha}{2\beta}, \tag{5.9}$$

where $\beta = \dfrac{(k+\omega)^2}{m_1 2^{-\alpha} + 2m_3}$. Since $v_1 = v_2$, the left term of (5.6) is equal to

$$2m_1 m_3 (v_1 - v_3)^2\left(1 - \alpha|\bar{x}_1 - \bar{x}_3|^{\alpha+2}\right),$$

and therefore $\left.\frac{d^2\mathcal{A}}{d\varepsilon^2}\right|_{\varepsilon=0} < 0$ if and only if $R^{\alpha+2} < \alpha$. But by (5.9) this is true if and only if

$$(k+\omega)^2 > \frac{m_1}{2^{\alpha+1}} + 1 - 2m_1,$$

as claimed. □

**Remark 5.10.** The right-hand side of (5.8) is a linear function of $m_1$, which is defined for $m_1 \in (0, 1/2)$ and goes from a limit value of 1 (for $m_1 = 0$) to a limit value of $2^{-(2+\alpha)}$ (for $m_1 = 1/2$). Hence it is always possible to find mass distributions $(m_1, m_2, m_3)$ for which minimizers are not rotating Euler solutions, provided that for the minimal $k$ we have $(k+\omega)^2 > 2^{-(2+\alpha)}$. For example, if $\alpha = 1$ then we find that there is a non-trivial interval of values of $\omega$ for which minimizers are non-trivial for all $m \in \left(\frac{3}{7}, \frac{1}{2}\right)$ (in the case that $k$ can be any integer) while the same happens for all $m \in \left(0, \frac{1}{2}\right)$ if the symmetry group implies a constraint on $k$ such that $k = 0 \mod 2$. For equal masses we have $m = \frac{1}{3} \notin \left(\frac{3}{7}, \frac{1}{2}\right)$ and so we must assume that $k = 0 \mod 2$, inequality (5.8) becomes

$$(k+\omega)^2 > \frac{5}{12},$$

and thus non-planar orbits exist for $\omega \in \left(\sqrt{\frac{5}{12}}, 2 - \sqrt{\frac{5}{12}}\right)$. We will apply this simple argument below in Lemma 7.7 to prove the existence of non-planar (quasi)-periodic orbits for when two masses are approximately equal.

## 6. Space symmetries

In this section we will describe space symmetries and prove Theorem A.

Consider the case of groups with trivial core. Let $r \subset G$ denote the $\mathbb{T}$-cyclic generator, and, if it exists, let $h \in G$ denote one of the time reflections. Consider $r_\Sigma = \sigma(r)$, $h_\Sigma = \sigma(h)$, $r_V = \rho(r)$ and $h_V = \rho(h)$. By Lemma 3.3, the $G$-representation $\rho$ is the sum of one-dimensional components, hence $r_V$ and $h_V$ can be written as matrices with $\pm 1$ diagonal elements. Thus a choice of the generators $r$ and $h$ yields a $3 \times 2$ matrix $\begin{bmatrix} {r_V}^1 & {h_V}^1 \\ {r_V}^2 & {h_V}^2 \\ {r_V}^3 & {h_V}^3 \end{bmatrix}$ where the entries ${r_V}^i$ and ${h_V}^i$ are the diagonal entries of the matrices $r_V$ and $h_V$ respectively. Conversely, if such a matrix is given, the elements $r$ and $h$ can be obtained by the permutations $r_\Sigma$ and $h_\Sigma$ in $\Sigma_3$ (analogously for cyclic and brake action types). The number of such matrices is the number of unordered 3-tuples of elements chosen in the set $\{[++], [+-], [-+], [--]\}$, which is $\binom{4+3-1}{3} = 20$. Under the identification $0 = [++]$, $1 = [+-]$, $2 = [-+]$ and $3 = [--]$, it is possible to represent such matrices by 3 digit numbers as in Table 3.

If the group $G$ is of cyclic type, then there are 12 possible cases for $r_V$ and $r_\Sigma$ (4 for $r_V$ times 3 for $r_\Sigma$): $r_\Sigma \in \{(), (1,2), (1,2,3)\}$ and $r_V \in \{[+++], [++-], [+--], [---]\}$. Is not difficult to see that the resulting groups are

**Table 3.** The list of all dihedral symmetries

$$\overset{000:}{\begin{bmatrix}+&+\\+&+\\+&+\end{bmatrix}} \equiv \overset{001:}{\begin{bmatrix}+&+\\+&+\\+&-\end{bmatrix}} \equiv \overset{002:}{\begin{bmatrix}+&+\\+&+\\-&+\end{bmatrix}} \equiv \overset{003:}{\begin{bmatrix}+&+\\+&+\\-&-\end{bmatrix}} \equiv \overset{011:}{\begin{bmatrix}+&+\\+&-\\+&-\end{bmatrix}}$$

$$\overset{012:}{\begin{bmatrix}+&+\\+&-\\-&+\end{bmatrix}} \equiv \overset{013:}{\begin{bmatrix}+&+\\+&-\\-&-\end{bmatrix}} \equiv \overset{022:}{\begin{bmatrix}+&+\\-&+\\-&+\end{bmatrix}} \equiv \overset{023:}{\begin{bmatrix}+&+\\-&+\\-&-\end{bmatrix}} \equiv \overset{033:}{\begin{bmatrix}+&+\\-&-\\-&-\end{bmatrix}}$$

$$\overset{111:}{\begin{bmatrix}+&-\\+&-\\+&-\end{bmatrix}} \equiv \overset{112:}{\begin{bmatrix}+&-\\+&-\\-&+\end{bmatrix}} \equiv \overset{113:}{\begin{bmatrix}+&-\\+&-\\-&-\end{bmatrix}} \equiv \overset{122:}{\begin{bmatrix}+&-\\-&+\\-&+\end{bmatrix}} \equiv \overset{123:}{\begin{bmatrix}+&-\\-&+\\-&-\end{bmatrix}}$$

$$\overset{133:}{\begin{bmatrix}+&-\\-&-\\-&-\end{bmatrix}} \equiv \overset{222:}{\begin{bmatrix}-&+\\-&+\\-&+\end{bmatrix}} \equiv \overset{223:}{\begin{bmatrix}-&+\\-&+\\-&-\end{bmatrix}} \equiv \overset{233:}{\begin{bmatrix}-&+\\-&-\\-&-\end{bmatrix}} \equiv \overset{333:}{\begin{bmatrix}-&-\\-&-\\-&-\end{bmatrix}}$$

three-dimensional extensions as listed in Table 4 (where, as in Definition 2.19, from now on the symbol '$\sim$' means that the symmetry group is equivalent to the group in question after a change in rotating coordinates).

Given a symmetry group $G$, consider the elements $r_V$, $r_\Sigma$, $h_V$ and $h_\Sigma$ defined above. The matrix $\begin{bmatrix} {r_V}^1 & {h_V}^1 \\ {r_V}^2 & {h_V}^2 \\ {r_V}^3 & {h_V}^3 \end{bmatrix}$ associated with $r_V$ and $h_V$ is one of the 20 matrices of Table 3. Furthermore, it is easy to prove that up to permutations the pair $[r_\Sigma, h_\Sigma]$ can be chosen from the set

$$\{[(1,2,3),(1,2)],[(),(1,2)],[(1,2),()],[(1,2),(1,2)],[(),()]\}.$$

Consider first the case $[r_\Sigma, h_\Sigma] = [(1,2,3),(1,2)]$. Since the matrices

$$\begin{bmatrix} {r_V}^1 & {h_V}^1 \\ {r_V}^2 & {h_V}^2 \\ {r_V}^3 & {h_V}^3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} {r_V}^1 & {h_V}^1 {r_V}^1 \\ {r_V}^2 & {h_V}^2 {r_V}^2 \\ {r_V}^3 & {h_V}^3 {r_V}^3 \end{bmatrix} \tag{6.1}$$

yield the same symmetry group as the matrix up to a change of coordinates, it is possible to assume that $[r_V, h_V]$ belongs to one of the 13 items: 000, 001, 002 ($\cong$ 003), 011, 012 ($\cong$ 013), 022 ($\cong$ 033), 023, 111, 112 ($\cong$ 113), 122 ($\cong$ 133), 123, 222 ($\cong$ 333), 223 ($\cong$ 233).

Now, since we are excluding the case of groups which are bound to collisions, we have to rule out the cases in which $h_V$ or the product $r_V h_V$ is trivial, which means we have to exclude the four cases 000, 002 $\cong$ 003, 022 $\cong$ 033, 222 $\cong$ 333; thus we are left with a list of 9 matrices.

We can now take the rotation axes into account. Let $e_1$, $e_2$ and $e_3$ denote the canonical elements of the base of the vector space $E$. Table 5 shows the list of elements of $\{e_1, e_2, e_3\}$ which are rotation axes for the corresponding group. Using the rotating frame change of coordinates, it is therefore possible to show that 023 $\sim$ 001, 123 $\sim$ 011 and 223 $\sim$ 012, and hence we are left with a choice among the 6 items: 001 $= L_6^{+,-}$, 011 $= D_6^{+,+} \cong L_6^{+,-}$, 111 $= D_6^{+,-}$, 012 $= D_{12}^{+,+} \cong L_6^{-,+}$, 112 $= D_6^{-,+}$ and 122 $= D_{12}^{-,+}$. Consider now the case $[r_\Sigma, h_\Sigma] = [(),(1,2)]$. We shall proceed in a similar way, analyzing case by case until we are left with a small

**Table 4.** Symmetry groups of cyclic type

| | () | (1, 2) | (1, 2, 3) |
|---|---|---|---|
| $[+++]$ | $C_1^+$ (fully uncoercive) | $C_2^+$ (fully uncoercive) | $C_3^+$ |
| $[++-]$ | $C_1^-$ | $C_2^-$ (fully uncoercive) | $C_3^-$ |
| $[+--]$ | $L_2^-$ (fully uncoercive) | $H_2^-$ (fully uncoercive) | $C_6^- \sim C_3^+$ |
| $[---]$ | $\sim C_1^-$ | $\sim C_2^-$ | $\sim C_3^-$ |

**Table 5.** Rotation axes

| 001 | 011 | 111 | 012 | 112 | 023 | 122 | 123 | 223 |
|---|---|---|---|---|---|---|---|---|
| $e_1, e_2$ | $e_2, e_3$ | | $e_3$ | | $e_1$ | | $e_1$ | $e_2, e_3$ |

number of significative choices. As above, 000, 002, 022 and 222 yield a group which is bound to collisions, and up to a change of coordinates we can choose among the same following 9 items: 001, 011, 012 ($\cong$ 013), 023, 111, 112 ($\cong$ 113), 122 ($\cong$ 133), 123, 223 ($\cong$ 233).

The rotation axes are the same as those of Table 5 above, and again in a suitable rotating frame 023 $\sim$ 001, 123 $\sim$ 011 and 223 $\sim$ 012, so that we can choose just among the 6 items: 001, 011, 111, 012, 112 and 122. First an easy computation shows that 111, 112 and 122 yield symmetry groups without rotation axes which are at the same time not coercive (thus fully uncoercive by Lemma 2.18). Furthermore, the choice of 001, 011 or 012 yields groups $G$ conjugate to the three-dimensional extensions $H_2^{+,+}$, $H_2^{+,-}$ and $H_2^{-,+}$ of the planar Isosceles symmetry group $H_2$. As a third possibility, now consider the cases $[r_\Sigma, h_\Sigma] = [(1,2),(1,2)]$ and $[r_\Sigma, h_\Sigma] = [(1,2),()]$. We can consider just the case $[r_\Sigma, h_\Sigma] = [(1,2),(1,2)]$, up to a change of coordinates (but we will not be able to use the argument of (6.1) to reduce the number of matrices). As above, we start with the list of all possibilities, refer to Table 3. Since $h_\Sigma = (1,2)$, if $h_V$ is trivial then the resulting group is bound to collisions, and therefore we cancel the four matrices 000, 002, 022 and 222. From the 16 matrices left the following 8 do not have a rotation axis: 003, 033, 111, 112, 113, 122, 133 and 333. Since the action on $\{1,2,3\}$ is not transitive, by Lemma 2.18 all those with a row equal to $[++]$ are fully uncoercive (that is, 003 and 033). Furthermore, for the matrices 111, 112 and 122, the resulting symmetry group is bound to collisions (since $r_V h_V$ is the antipodal map, while $r_\Sigma h_\Sigma$ is the trivial permutation). The three remaining items 113, 133 and 333 are fully uncoercive simply because they contain a row equal to $[--]$ (which yields a one-dimensional non-coercive symmetry group).

So, we are left with 8 choices, all with rotation axes: 001, 011, 012, 013, 023, 123, 223, 233. After a change in rotating coordinates we can see that 023 $\sim$ 001, 123 $\sim$ 011, 223 $\sim$ 012 and 233 $\sim$ 013; furthermore, it is easy to see that 001 and 013 yield fully uncoercive symmetry group. As a consequence, the remaining matrices are 011 and 012. In the notation of Table 1, they are respectively the symmetry groups $H_4^{+,-}$ and $H_4^{-,+}$. At last, we can consider the case $[r_\Sigma, h_\Sigma] = [(),()]$ where the resulting group acts trivially on the index set. The matrices 111, 113, 133 and 333 yield groups which are bound to collisions. By the same argument as (6.1), we do not consider the duplicates 003, 013, 033, 112, 122, 222, 233. Of the remaining 9 matrices, three do not have rotation axes (000, 002 and 022) and yield groups which are not coercive, while by a change in rotating frames the other six can be reduced to the following three: 001($\sim$ 023), 011($\sim$ 011) and 012($\sim$ 012). The group induced by 001 is fully uncoercive. So, we can as before reduce the list of 20 matrices to the two cases 011 and 012 which yield respectively the groups $L_2^{+,-}$ and $L_2^{-,+}$ of Table 1.

**Definition 6.2.** We say that a symmetry group is of *vertical isosceles* type if its core is generated by an element $k$ such that $\rho(k)$ is the rotation of angle $\pi$ around an axis and $\sigma(k)$ is conjugate to the permutation $(1,2)$.

To conclude the proof it is left to prove the easy fact that if the core is non-trivial, then the group is homographic, provided that it is not of vertical isosceles type.

**Lemma 6.3.** *Let $G$ be a symmetry group which is not bound to collisions and not fully uncoercive. Then either $G$-equivariant trajectories are always homographic, or the group is of vertical isosceles type, see Definition* 6.2.

**Proof.** As in the proof of the (similar) Proposition 5.4 for planar groups of [1], $\ker\tau$ is isomorphic to a subgroup of $\Sigma_3$ and hence its three-dimensional representation in $E$ is reducible: that is, there is a $\ker\tau$-invariant line $\mathbb{R}$ in $E$. Now, since $\ker\tau$ is normal in $G$, either $g\mathbb{R} \subset \mathbb{R}$ for every $g \in G$, or $E$ is the sum of three copies of $\mathbb{R}$, which are permuted by the elements in $G$ (hence $\ker\tau$ has order 2 and acts as the antipodal map $-1$ on $E$). After an analysis of a few cases, it is clear that the possible actions of the core are the following: $\ker\tau = \langle(a_3, (1, 2))\rangle$, $\ker\tau = \langle(r_2, (1, 2))\rangle$, $\ker\tau = \langle(r_3, (1, 2, 3))\rangle$, $\ker\tau = \langle(r_3, (1, 2, 3), (h_2, (1, 2))\rangle$ (which, incidentally, are extensions of planar analogues), where $a_3$ is the antipodal map $a_3 = -1$, $r_2$ the rotation of $\pi$ around a fixed axis, $r_3$ the rotation of $2\pi/3$ around a fixed axis $\mathbb{R}$, and $h_2$ the reflection with respect to a plane containing (or with respect to a line orthogonal to $\mathbb{R}$). In the first case $\ker\tau$-invariant configurations are antipodal binaries with a third mass at the origin (this is equivalent to a spatial Kepler problem), which is a homographic group; the second case of the isosceles triangle (which is not homographic) is the well known; the third and the fourth cases yield homographic symmetry groups. Thus the proof is completed. □

## 7. Proof of Theorem B

In this section we will complete the proof of Theorem B, proving one-by-one all its parts.

**Lemma 7.1.** *Let $G$ be a symmetry group of the three-body problem which is not bound to collisions. Then all $G$-equivariant minimizers are collisionless.*

**Proof.** Consider first the case of a group with trivial core. Let $H$ be one of its maximal $\mathbb{T}$-isotropy groups; $H$ is generated by the non-trivial element $h \in H$ with $\sigma(h) \in \{(), (1, 2)\}$. The orthogonal motion $\rho(h)$ is of order at most two, and hence there are at least three invariant orthogonal lines (equivalently, the representation of $H$ is the sum of one-dimensional $H$-representations). An immediate consequence is that if it is not bound to collisions (that is, if $(\rho(h), \sigma(h)) \neq (1, (1, 2))$), the subgroup $H$ has the rotating circle property (10.1) of [12]. Thus, by (10.10) of the same paper, $G$-equivariant minimizers are collisionless. Now assume that the kernel of $\tau$ is not trivial. As in the proof of Lemma 6.3 it is possible to assume that $\ker\tau$ acts as one of the four cases listed, where the first case yields a one-center (Kepler) problem in space, the third and fourth cases yield a one-center planar problem, and the second case is the isosceles triangle, see Definition 6.2. We can readily see that Theorem 10.1 of [12] can be applied in all of the cases, and hence the thesis is proved. □

**Lemma 7.2.** *For every $\omega$, minimizers for $C_1^-$, $H_2^{+,-}$, $C_3^+$, $L_6^{+,+}$, $L_6^{+,-}$ are Lagrange homographic solutions.*

**Proof.** It is easy to see that the hypotheses of Proposition 2.20 are fulfilled, hence the lemma is proved. □

**Lemma 7.3.** *Minimizers for $D_6^{+,-}$, $D_6^{-,+}$ and $D_{12}^{-,+}$ are zero-angular momentum planar solutions: the Chenciner-Montgomery figure-eight solution.*

**Proof.** Since these groups do not have rotation axes, by Lemma 4.6, their minimizers are planar orbits with zero angular momentum, contained in a $G$-invariant plane by Lemma 4.7. All planes are $G$-invariant for $D_6^{+,-}$, and the restricted group is $D_6$. Hence by (4.15) of [1] the minimizer for $D_6^{+,-}$ is the $D_{12}$-symmetric Chenciner-Montgomery figure eight [10]. Next, if $G = D_{12}^{-,+}$, a $G$-orbit (which is collisionless by 7.1, has to be contained in the $G$-invariant plane where $G$ acts as $D_{12}$, since otherwise it would be bound to collisions. Hence the minimum for $D_{12}^{-,+}$ is again the Chenciner-Montgomery figure eight. At last, consider $D_6^{-,+}$, which is a group of order 12. In any one of the infinitely many invariant planes with $D_{12}$-action as restriction there is a CM-eight minimum, while in the other invariant plane there is a redundant $D_6$-action (hence if the minimum were to be contained in this plane, it would have implied that the action of the $D_6$-eight is less than the action of the $D_{12}$-eight, which is not true by [1]). Thus as above the minimum is the $D_{12}$-symmetric CM-eight. □

**Lemma 7.4.** *For $\omega \in (-1, 1) + 6\mathbb{Z}$, minimizers for $L_6^{-,+}$ and its subgroup $C_3^- \subset L_6^{-,+}$ are the non-planar (if $\omega \neq 0$) families of quasi-periodic solutions called $P_{12}$ and $P'_{12}$, respectively which might be likely to coincide.*

**Proof.** By Lemma 7.1, minima for $C_3^-$ and $L_6^{-,+}$ exist and are collisionless. If they are planar, then in both cases they have to be a Lagrange rotating solution (rotating at angular speed $k$, with $k = \pm 2 \mod 6$, minimizing the number $(k + \omega)^2$). The proof can be concluded by applying Proposition 5.4. Action levels for the resulting

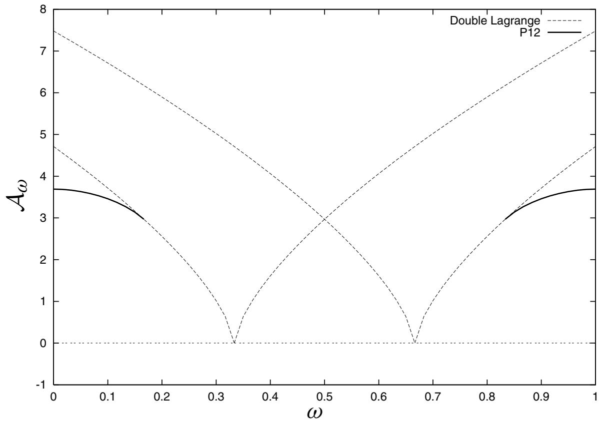
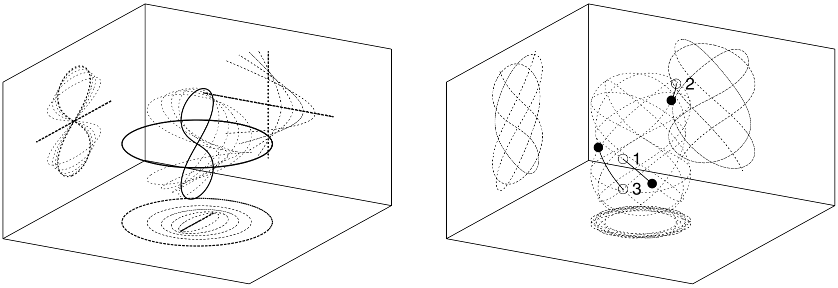



**Fig. 1.** Action-levels for the $L_6^{-,+}$-symmetric $P_{12}$ minimizers.




(a) Some representatives of the $P_{12}$ family in the rotating frame

(b) One of representatives of the $P_{12}$ family on inertial frame

**Fig. 2.** The $P_{12}$ family.

minima are depicted in Fig. 1: in the intervals $(1/6, 1/2)$ and $(1/2, 5/6)$ the minimum is a rotating Lagrange solution, while otherwise it is a non-planar orbit (in the graph the period is rescaled to $12\pi$, so that values of $\omega$ need to be multiplied by a factor of 6). It is remarkable how the estimate of Proposition 5.4 seems to be sharp, since when its hypothesis is not fulfilled we can find (numerically) that minimizers are in fact planar rotating Lagrange triangles. □

**Remark 7.5.** In [14][2] MARCHAL actually introduced, under a different notation, the family corresponding to minimizers (as $\omega$ varies) under $L_6^{-,+}$-symmetries, naming it $P_{12}$, – see also [13]. The fact that there is also a $C_3^-$-equivariant family (as well as the fact that there is a group action of cyclic type yielding a figure-eight orbit) apparently was not known. As for the planar eights, it seems that (numerically) these two families coincide, and that for $\omega = 0$ we find the CM-eight and the cyclic eight (which as well should likely coincide). Some of these were questions raised in the last section of [14] (see also Section 4.(iv) of [5]), questions which probably still need to be answered. For example, as said at the end of section 4 of [5], we should prove that for $\omega = 0$ the minimum is planar and that minimizers are a continuous family (from Fig. 1 and 2(a) it seems that action levels depend continuously on $\omega$, as well as the corresponding trajectories). In Fig. 2(a) the minimizers corresponding to the values $\omega = j/5$ for $j = 0 \ldots 5$ are shown, together with their projections. The curves with $j = 0$ and $j = 5$ have a bigger width. On its side, there is one of the orbits (actually, corresponding to the non-integer value $j = 2.5$) in the inertial frame.

**Remark 7.6.** In [9], CHENCINER, FÉJOZ & MONTGOMERY found, under an assumption (at the moment numerically evaluated) of non-degeneracy for the CM-eight,

[2] The proof of the existence, claimed in [14] and later in [15], was completed in [7, 5] by action level estimates on colliding paths, since the main result of [15] cannot be applied to the $P_{12}$ family, which has a symmetry group of dihedral type. The action levels graph of [7] is a qualitative picture of Fig. 1.

three families of periodic orbits in rotating frames, derived from three different symmetry-breaks of the planar eight. The family there termed $\Gamma_1$ is $P_{12}$.

**Lemma 7.7.** *For every $\omega \notin \mathbb{Z}$ there is a mass distribution such that minimizers of $L_2^{+,-}$, $L_2^{-,+}$, $H_4^{+,-}$ and $H_4^{-,+}$ are not planar (where for the first groups two masses do not need to be equal, while in the remaining groups two masses need to be equal). Conversely, for every mass distribution with two equal masses, there is an $\omega$ such that minimizers which are symmetric under the groups $L_2^{-,+}$ are not planar.*

**Proof.** This follows directly from Remark 5.10. □

## 8. Remarks

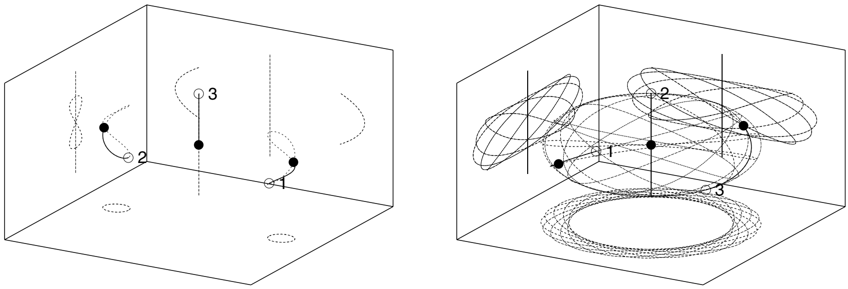
We did not prove general and complete results about planarity or non-planarity of minimizers for the following symmetry groups: $L_2^{+,-}$, $L_2^{-,+}$, $H_2^{-,+}$, $H_4^{+,-}$ and $H_4^{-,+}$. In fact, we have proved that for some choices of masses and angular speed, minima are non-planar, but we could not prove that minima are planar for other choices (as apparently they are: all the Hill-type orbits and Euler solutions exposed in, for example, [1]). We describe now, among other remarks, some properties of their minimizers with a little bit more details.

**Remark 8.1.** If the masses are equal, for all $\omega$ minimizers under the symmetry group $L_2^{+,-}$ are planar (and they are the Euler and Hill retrograde orbits described in [1] – note that it is easy to prove by Proposition 5.7 that for all these symmetry groups there are choices of masses and angular speed $\omega$ for which minimizers are not planar). Also, it turns out that there are other local minimizers (planar and non-planar).

**Remark 8.2.** The symmetry group $L_2^{-,+}$ imposes that possible rotating configurations (which are Euler collinear) have to rotate an even number of loops, i.e. the cyclic part of the symmetry group imposes that a rotating central configuration has to have $k = 0 \mod 2$, and hence by Remark 5.10 there is a continuum of choices for $\omega$, for every choice of non-zero masses, such that the minimizer is non-planar. An example is shown in Fig. 3.

**Remark 8.3.** If the symmetry group is $H_2^{-,+}$, then again it implies that a rotating central configuration has to rotate $k = 0 \mod 2$ times. The possible rotating configuration is a Lagrange triangle and it is not possible to apply Proposition 5.4 to show that it is not a minimum since it is always possible to find $k$ with $|k+\omega| \leqq 1$. On the other hand the constraint $k = 0 \mod 2$ prevents us from applying Proposition 2.20 to show that the minimum is in fact a Lagrange solution. Numerical experiments show that this is the case.

**Remark 8.4.** Now consider the symmetry groups $H_4^{+,-}$ and $H_4^{-,+}$. As in the planar case, we find non-homographic minimizers for some interval of values of $\omega$ (and approximately equal masses – see Fig. 4 of [1]). Homographic solutions have




(a) Rotating frame (b) Inertial frame

**Fig. 3.** A non-planar $L_2^{-,+}$-symmetric orbit with equal masses, $\omega = 0.9$.

to be Euler-Moulton rotating collinear configurations, and hence we can literally repeat the arguments applied above to $L_2^{+,-}$ and $L_2^{-,+}$, and obtain the fact that for two equal masses it is always possible to find intervals of angular speed $\omega$ for which minimizers are not planar. This time rotating central configurations do not necessarily have $k = 0 \mod 2$ since $H_4$ has already a cyclic part of order 2. Hence we cannot prove with the vertical variation above that there are non-planar orbits for equal masses (for equal masses, after numerical experiments it seems that local minima under $H_4^{+,-}$ are planar, while local minima under the action of $H_4^{-,+}$ can be non-planar for some $\omega$).

**Remark 8.5.** In the proofs of Proposition 5.4 and Proposition 5.5 there is clearly no need to assume that there are three bodies. In fact, the same vertical variation yields interesting non-planar orbits for every number of bodies. For example, it is easy to show by a straightforward extension of Lemma 5.2 that families corresponding to the $P_{12}$ and $P'_{12}$ ones exist for any odd number $n$ of bodies. In fact, consider the cyclic group $C$ of order $2n$, acting by a cyclic permutation of the $n$ bodies on the index set and by a reflection along a plane $p$ in the space $E$. Then, $C$-symmetric loops are choreographies in $E$ consisting of $n$ bodies, and if we choose as rotation axis $\underline{\omega}$ the line orthogonal to the plane $p$, we obtain a family of coercive $n$-body problems such that for $\omega \in (-1, 1) \mod 2n$ minima are not equilibrium solutions. Since they are collisionless due to [12], they are periodic orbits (non-planar for $\omega \neq 0$). It is likely that they behave like the $P_{12}$, namely that they connect a eight solution with a (twice rotating in the rotating frame) homographic solution, but this is probably hard to prove (already for $n = 3$ nobody has published a proof that for $\omega = 0$ the $P_{12}$ is a planar eight and that the family is a continuous one).

**Remark 8.6.** With regards to Proposition 5.4, a similar proposition was used by CHENCINER in [6] to show that minimizers for the $n \geqq 4$ anti-symmetric loops are non-planar, following MOECKEL'S theorem on central configurations [16]. While the computation is very similar, here we use a different type of vertical variation, which yields solutions, in particular, also when the rotating central configuration

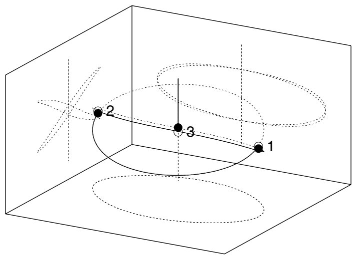



**Fig. 4.** The simplest non-planar solution with three equal masses.

minimizes the reduced potential $\tilde{U}$, due to the greater number of loops that symmetry constraints impose on rotating central configurations. Furthermore, in Section 3 of [6] there is an interesting short remark proving the existence of a non-planar periodic solution for 3 bodies in the vertical isosceles problem under the antisymmetry constraint. Since this constraint coincides with the group $C_1^-$ with angular speed $\omega = 1$ (a group which implies $k = 0 \mod 2$ for any $C_1^-$-equivariant equilibrium solution), we can use Proposition 5.7, to obtain that for all $\omega \in \left(\sqrt{\frac{5}{12}}, 2 - \sqrt{\frac{5}{12}}\right)$ (as in Remark 5.10 above) and all equal masses, a $C_1^-$-symmetric vertical isosceles minimizer is not planar. Fig. 4 shows the solution in the inertial frame – probably this is the simplest non-planar periodic solution of the three-body problem.

**Remark 8.7.** In this paper we have sometimes rescaled the period to a number different than $2\pi$. It is worth mentioning that, because of the homogeneity of the potential, a minimizer in a frame rotating with angular velocity $\omega$ and period $k2\pi$ corresponds to a minimizer with period $2\pi$ in a frame rotating with angular velocity $k\omega$. The reason for rescaling the period is that in our numerical experiments we have decided to rescale the period in order to have a fundamental domain of length $\pi$. The data for the minimizers used for the figures were obtained by a custom optimization program running on a Linux cluster. The symmetries are computed by a package written in GAP and python.

**Remark 8.8.** In Theorem B nothing is stated about $H_2^{-,+}$. Numerically we can find that $H_2^{-,+}$-symmetric minima are rotating Lagrange triangles, but Proposition 2.20 cannot be applied to this case.

*Acknowledgements.* The author was partially supported by MIUR, National Project "Variational Methods and Nonlinear Differential Equations".

## References

1. Barutello, V., Ferrario, D., Terracini, S.: *Symmetry groups of the planar 3-body problem and action–minimizing trajectories.* Preprint: www.arxiv.org, PaperId: math.DS/0404514.
2. Bessi, U., Coti Zelati, V.: Symmetries and noncollision closed orbits for planar $N$-body-type problems. *Nonlinear Anal.* **16**, 587–598 (1991)

3. CHEN, K.-C.: Action-minimizing orbits in the parallelogram four-body problem with equal masses. *Arch. Ration. Mech. Anal.* **158**, 293–318 (2001)
4. CHEN, K.-C.: Variational methods on periodic and quasi-periodic solutions for the $N$-body problem. *Ergodic Theory Dynam. Systems* **23**, 1691–1715 (2003)
5. CHENCINER, A.: Action minimizing solutions of the Newtonian $n$-body problem: from homology to symmetry. In: *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)* Higher Ed. Press, Beijing, 2002, pp. 279–294
6. CHENCINER, A.: Simple non-planar periodic solutions of the $n$-body problem. In: *Proceedings of the NDDS Conference.* Kyoto, 2002
7. CHENCINER, A.: Some facts and more questions about the "Eight". In: *Proceedings of the conférence "Nonlinear functional analysis", Taiyuan*. World Scientific, 2002
8. CHENCINER, A., DESOLNEUX, N.: Minima de l'intégrale d'action et équilibres relatifs de *n* corps. *C. R. Acad. Sci. Paris Sér. I Math.* **326**, 1209–1212 (1998)
9. CHENCINER, A., FÉJOZ, J., MONTGOMERY, R.: Rotating eights I: the three $\Gamma_i$ families. Preprint, 2004
10. CHENCINER, A., MONTGOMERY, R.: A remarkable periodic solution of the three-body problem in the case of equal masses. *Ann. of Math. (2)* **152**, 881–901 (2000)
11. CHENCINER, A., VENTURELLI, A.: Minima de l'intégrale d'action du problème newtonien de 4 corps de masses égales dans $\mathbf{r}^3$: orbites "hip-hop". *Celestial Mech. Dynam. Astronom.* **77**, 139–152 (2001)
12. FERRARIO, D. L., TERRACINI, S.: On the existence of collisionless equivariant minimizers for the classical n-body problem. *Invent. Math.* **155**, 305–362 (2004)
13. MARCHAL, C.: *The three-body problem*, vol. 4 of *Studies in Astronautics*. Elsevier Science Publishers B.V., Amsterdam, 1990. With a foreword by Victor Szebehely, With French, Russian, German, Spanish, Japanese, Chinese and Arabic summaries.
14. MARCHAL, C.: The family $P_{12}$ of the three-body problem—the simplest family of periodic orbits, with twelve symmetries per period. *Celestial Mech. Dynam. Astronom.* **78**, 279–298 (2001) New developments in the dynamics of planetary systems Badhofgastein, 2000
15. MARCHAL, C.: How the method of minimization of action avoids singularities. *Celestial Mech. Dynam. Astronom.* **83**, 325–353 (2002)
16. MOECKEL, R.: On central configurations. *Math. Z.* **205**), 499–517 (1990)
17. MONTGOMERY, R.: A new solution to the three-body problem. *Notices Amer. Math. Soc.* **48**, 471–481 (2001)
18. MONTGOMERY, R.: Action spectrum and collisions in the planar three-body problem. In: *Celestial mechanics (Evanston, IL, 1999)*, *Contemp. Math.* **292**. Amer. Math. Soc., Providence, RI, 2002, pp. 173–184
19. SERRA, E., TERRACINI, S.: Collisionless periodic solutions to some three-body problems. *Arch. Ration. Mech. Anal.* **120**, 305–325 (1992)



Dipartimento di Matematica e Applicazioni
University of Milano–Bicocca
Via Cozzi 53
20125 Milano (Italy)
e-mail: davide.ferrario@unimib.it



(*Accepted June 17, 2005*)
*Published online September 16, 2005* – © *Springer-Verlag* (*2005*)