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The Navier-Stokes Equations in $\mathbb{R}^n$ with Linearly Growing Initial Data

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Abstract

Consider the equations of Navier-Stokes on $\mathbb{R}^n$ with initial data U_0 of the form $U_0(x) = u_0(x) - Mx$, where M is an $n \times n$ matrix with constant real entries and $u_0 \in L^p_\sigma(\mathbb{R}^n)$. It is shown that under these assumptions the equations of Navier-Stokes admit a unique local solution in $L^p_\sigma(\mathbb{R}^n)$. Moreover, if $\|e^{tM}\| \leqq 1$ for all $t \geqq 0$, then this mild solution is even analytic in x. This is surprising since the underlying semigroup of Ornstein-Uhlenbeck type is not analytic, in contrast to the Stokes semigroup.

1. Introduction

In this paper we consider the flow of an incompressible, viscous fluid in $\mathbb{R}^n$ for initial data which grow linearly at infinity. The equations governing the flow are the equations of Navier-Stokes, i.e.,

$$\begin{aligned} \partial_t U - \Delta U + U \cdot \nabla U + \nabla P = 0 &\quad \text{in } \mathbb{R}^n \times (0,T), \\ \nabla \cdot U = 0 &\quad \text{in } \mathbb{R}^n \times (0,T), \\ U(0) = U_0 &\quad \text{with } \nabla \cdot U_0 = 0 \quad \text{in} \quad \mathbb{R}^n. \end{aligned} \tag{1.1}$$

Here, $U = (U^1, \ldots, U^n)$ and P represent the unknown velocity and the unknown pressure of the fluid and U_0 is the given initial velocity. There is a vast literature on existence of solutions of (1.1) in $\mathbb{R}^n$, see, e.g., [1, 5–7, 9, 13, 14, 21, 22, 24, 28–30]. All these results assume that the initial data decay as $|x| \to \infty$. On the other hand, OKAMOTO [27] showed that for certain concrete flow problems there exist many exact solutions u which have the property that u grows linearly as $|x| \to \infty$.

It is the aim of this paper to construct mild solutions to the equations of Navier-Stokes in $L^p(\mathbb{R}^n)$ for the case where the initial data may grow as Mx, where M

is a constant $n \times n$ matrix. We hence assume throughout this paper that the initial velocity is of the form

$$U_0(x) = u_0(x) - Mx, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $u_0 \in L^p(\mathbb{R}^n)^n$ is a function satisfying $\nabla \cdot u_0 = 0$ and $M = (m_{ij})_{1 \leqq i,j \leqq n}$ denotes an $n \times n$ matrix with constant real coefficients.

In the case $M = 0$, it is well known that there exists a local smooth solution to (1.1) provided the initial data U_0 belongs to $L^p_\sigma(\mathbb{R}^n)$ and $p \geqq n$ (see, e.g., the above list of articles). As usual $L^p_\sigma(\mathbb{R}^n)$ denotes the closure of the set $\{u \in C_c^\infty(\mathbb{R}^n), \operatorname{div} u = 0\}$ with respect to the $\|\cdot\|_p$ norm.

If $M \neq 0$, the situation is more complicated. Notice first that if $\operatorname{tr} M = 0$, then by the substitution $u := U + Mx$ the pair (U, P) satisfies (1.1) if and only if (u, p) satisfies

$$\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u - Mx \cdot \nabla u - Mu + \nabla p = 0 \quad &\text{in } \mathbb{R}^n \times (0, T), \\ \nabla \cdot u = 0 \quad &\text{in } \mathbb{R}^n \times (0, T), \\ u(0) = u_0 \quad \text{with } \nabla \cdot u_0 = 0 \quad &\text{in } \mathbb{R}^n, \end{aligned} \tag{1.3}$$

where p has to be defined in a suitable way; see the beginning of Section 4.

In the particular case where M describes pure rotation, i.e., $M = R$ and R denotes the rotation matrix, the problem (1.3) was investigated by Hishida and by Babin, Mahalov and Nicolaenko. Indeed, HISHIDA constructed in [17–19] a local mild solution to (1.3) in L^2 provided the initial data u_0 belongs to a certain fractional power space. BABIN, MAHALOV & NICOLAENKO [3, 4] proved the existence of a local mild solution to (1.3) provided u_0 is in $L^p_\sigma(\mathbb{R}^n)$ or u_0 is a periodic function satisfying certain properties. We recall that these results were proved for the particular case of $Mx = \omega \times x$, where $\omega = (1, 0, 0)$ and $x \in \mathbb{R}^n$.

An interesting example of M is $M = R + J$ where

$$R = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} -b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 2b \end{pmatrix}$$

for $a, b \in \mathbb{R}$. Note that R corresponds to pure rotation and describes the Coriolis force, see [17, 4]. On the other hand, following MAJDA [23], $M = J$ for $b > 0$ corresponds to the drain along the x_1- and x_2-axes and to the jet along the x_3-axis of the fluid. He showed that $U = Mx$ is an exact solution of (1.1) provided that the pressure is chosen appropriately. GIGA & KAMBE [11] investigated the axisymmetric irrotational flow and studied the stability of the vortex, when the velocity field of the fluid U is expressed as $U = Mx + V$, where V is a two-dimensional velocity field.

In [31], SAWADA proved the existence of a local mild solution of (1.3), still for $M = R$, provided u_0 belongs to the Besov space $\dot{B}^0_{\infty,1}$. It seems to be an open problem to prove the existence of a local mild solution in this space for arbitrary M.

2. Main results

In this section we state the main results of this paper. To this end, recall that M denotes an $n \times n$ matrix with real coefficients. Let $u_0 \in L^p_\sigma(\mathbb{R}^n)$ for some p satisfying $1 < p < \infty$. We consider the equation

$$\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u - Mx \cdot \nabla u - Mu + \nabla p = 0 \quad &\text{in } \mathbb{R}^n \times (0, T), \\ \nabla \cdot u = 0 \quad &\text{in } \mathbb{R}^n \times (0, T), \\ u(0) = u_0 \quad \text{with } \nabla \cdot u_0 = 0 \quad &\text{in } \mathbb{R}^n \end{aligned} \tag{2.1}$$

and define the operator A in $L^p_\sigma(\mathbb{R}^n)^n$ as

$$Au := -\Delta u - Mx \cdot \nabla u + Mu \tag{2.2}$$

with domain $D(A) := \{u \in W^{2,p}(\mathbb{R}^n)^n \cap L^p_\sigma(\mathbb{R}^n)^n;\ Mx \cdot \nabla u \in L^p(\mathbb{R}^n)^n\}$.

We prove in the following section that $-A$ generates a C_0-semigroup $(e^{-tA})_{t \geqq 0}$ on $L^p_\sigma(\mathbb{R}^n)^n$, which is not analytic.

Applying the Helmholtz projection $\mathbb{P}$ to (2.1) we may rewrite (2.1) as

$$\begin{aligned} u'(t) + Au + \mathbb{P}u \cdot \nabla u - 2\mathbb{P}Mu &= 0, \\ u(0) &= u_0. \end{aligned} \tag{2.3}$$

In the given situation of $\mathbb{R}^n$, the Helmholtz projection $\mathbb{P}$ can be expressed explicitly by $\mathbb{P} := (\delta_{ij} + R_i R_j)_{1 \leqq i,j \leqq n}$, where δ_{ij} denotes the Kronecker's delta, and R_i is the i-th Riesz transform on $\mathbb{R}^n$ defined by $R_i := \partial_i(-\Delta)^{-1/2}$, $i = 1, \ldots, n$. Note that A and $\mathbb{P}$ commute, since $\nabla \cdot Au = 0$ if $\nabla \cdot u = 0$. For $T > 0$ we call a function $u \in C([0, T); L^p_\sigma(\mathbb{R}^n))$ a *mild solution* of (2.3) if u satisfies the integral equation

$$\begin{aligned} u(t) = e^{-tA}u_0 &- \int_0^t e^{-(t-s)A}\mathbb{P}u(s) \cdot \nabla u(s)ds \\ &+2\int_0^t e^{-(t-s)A}\mathbb{P}Mu(s)ds, \qquad t > 0, \end{aligned} \tag{2.4}$$

and $u(0) = u_0$.

We now state the local existence and uniqueness result for mild solutions of (2.3) in L^p spaces. Note that the underlying semigroup $(e^{-tA})_{t \geqq 0}$ is *not* analytic; hence it is *a priori* not obvious that the classical iteration procedure due to Kato is applicable in the given situation.

Theorem 2.1 (Local Existence and Uniqueness)**.** *Let $n \geqq 2$, $p \in [n, \infty)$ and $q \in [p, \infty]$. Let M be an $n \times n$ matrix with real coefficients and assume that $u_0 \in L^p_\sigma(\mathbb{R}^n)$. Then there exist $T_0 > 0$ and a unique mild solution u of* (2.3) *such that*

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}u \in C([0, T_0); L^q_\sigma(\mathbb{R}^n)), \tag{2.5}$$

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}}\nabla u \in C([0, T_0); L^q(\mathbb{R}^n)). \tag{2.6}$$

Remark 2.2. (a) The functions defined in (2.5) and (2.6) are continuous in t, moreover, they vanish at $t = 0$ provided $q \neq p$ in (2.5).
(b) In the case $p = \infty$, it can also be shown that there exists a unique $u \in C_w([0, T_0); L^\infty_\sigma(\mathbb{R}^n))$ satisfying (2.4) provided $u_0 \in L^\infty_\sigma(\mathbb{R}^n)$.

We now give a corollary of the above Theorem 2.1 concerning higher-order regularity of the mild solution. Observe first that it seems not to be known whether the above mild solution to (2.3) is a C^∞-function. However, we may deduce certain regularity properties of u as long as we couple the size of the time interval with the order of differentiation.

Corollary 2.3. *Assume that $n, m \in \mathbb{N}$ with $n, m \geqq 2$. Let $u_0 \in L^n_\sigma(\mathbb{R}^n)$ and $q \in [n, \infty]$. Denote by u the mild solution of* (2.3). *Then there exist constants $T_1(m) \in (0, T_0)$ satisfying $T_1(m) \geqq C_1 m^{-m}$ for some $C_1 > 0$ and $C_2 > 0$ such that*

$$\|\nabla^m u(t)\|_q \leqq C_2 t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{m}{2}}, \qquad t \in (0, T_1(m)]. \tag{2.7}$$

The estimates given in Proposition 3.3 below show that in general the linear term of (2.3) grows exponentially. Hence it seems to be difficult to obtain results on global existence of mild solutions to (2.3).

Considering questions analogous to those above for exterior domains Ω instead of $\mathbb{R}^n$, leads us to interesting applications such as spin-coating of fluids. This will be the content of a forthcoming publication; in [16] we prove that $-A$ generates a C_0-semigroup on $L^p(\Omega)$ for $1 < p < \infty$.

We now turn to regularity questions of the mild solution obtained above. The estimates for higher-order derivatives given in the following result imply that the mild solution to (2.3) is analytic in the space variable x, see Corollary 2.6 below.

Theorem 2.4. *Let $n \geqq 2$, $u_0 \in L^n_\sigma(\mathbb{R}^n)$, $r \in (n, \infty)$ and $q \in [n, \infty]$. Assume that $\|e^{tM}\| \leqq 1$ for all $t \geqq 0$. Let u be the local mild solution of* (2.1) *for some $T > 0$. Assume further that there exist constants $M_1, M_2 \geqq 0$ such that*

$$\sup_{0<t<T} \|u(t)\|_n \leqq M_1 < \infty \qquad \text{and} \sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})}\|u(t)\|_r \leqq M_2 < \infty.$$

Then there exist constants K_1 and K_2 such that

$$\|\nabla^m u(t)\|_q \leqq K_1(K_2 m)^m t^{-\frac{m}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}, \qquad t \in (0, T),\ m \in \mathbb{N}_0. \tag{2.8}$$

Remark 2.5. Suppose that M_1 and M_2 do not depend on T. Then K_1 and K_2 do not depend on T either and the estimate (2.8) yields estimates for higher-order derivatives of mild solutions in the class $C([0, T); L^n_\sigma(\mathbb{R}^n)) \cap C((0, T); L^r_\sigma(\mathbb{R}^n))$.

It follows from the above Theorem 2.4 that the mild solution u of (2.3) is analytic in x. More precisely, we have the following estimate on the radius of analyticity of u.

Corollary 2.6. *Let u be a mild solution of* (2.3) *satisfying the assumptions of Theorem 2.4. Then u is analytic in x. Moreover, there exists a constant $C > 0$ such that the radius $\rho(t)$ of analyticity is given by*

$$\rho(t) \geqq \limsup_{m\to\infty} \left(\frac{\|\nabla^m u(t)\|_\infty}{m!} \right)^{-1/m} \geqq C\sqrt{t}.$$

The proof of the above results relies heavily on certain smoothing properties of the underlying semigroup $(e^{-tA})_{t\geqq 0}$. It should be noted that – due to the unbounded coefficient in the drift term – the underlying semigroup $(e^{-tA})_{t\geqq 0}$ is not analytic. Hence estimates for $\|\nabla e^{-tA}\|$ do not follow automatically as in the situation of the classical Stokes semigroup from the theory of semigroups. We hence have to give explicit proofs for decay estimates for $(e^{-tA})_{t\geqq 0}$; this will be done in Section 3.

3. Estimates for the semigroup e^{-tA}

Let M be an $n \times n$ matrix with constant real entries. We define the realization of the operator

$$\mathcal{L}u(x) := -\Delta u(x) - < Mx, \nabla u(x) >, \qquad x \in \mathbb{R}^n, \tag{3.1}$$

in $L^p(\mathbb{R}^n)$ as follows. Set

$$\begin{aligned} Lu &:= \mathcal{L}u, \\ D(L) &:= \{u \in W^{2,p}(\mathbb{R}^n); < Mx, \nabla u > \in L^p(\mathbb{R}^n)\}. \end{aligned}$$

Then the following result on the Ornstein-Uhlenbeck semigroup was proved by METAFUNE, PALLARA & PRIOLA [25] and by METAFUNE, PRÜSS, RHANDI & SCHNAUBELT [26].

Proposition 3.1. *Let $1 < p < \infty$. Then the operator $-L$ generates a C_0-semigroup $(e^{-tL})_{t\geqq 0}$ on $L^p(\mathbb{R}^n)$ satisfying $\|e^{-tL}\| \leqq e^{-\frac{t}{p} trM}$ for all $t \geqq 0$. Moreover, the semigroup $(e^{-tL})_{t\geqq 0}$ is given by*

$$\begin{aligned} e^{-tL} f(x) := \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} &\int_{\mathbb{R}^n} f(e^{tM}x - y) \\ &\times e^{-\frac{1}{4}(Q_t^{-1}y,y)} dy, \quad x \in \mathbb{R}^n, t > 0, \end{aligned}$$

where Q_t for $t > 0$ is given by $Q_t := \int_0^t e^{sM} e^{sM^T} ds$.

Remark 3.2. (a) It is a well-known fact that the semigroup $(e^{-tL})_{t\geqq 0}$ is not analytic.

(b) By Young's inequality, the family $(e^{-tL})_{t\geqq 0}$ is also a semigroup on $L^\infty(\mathbb{R}^n)$ which, however, is not strongly continuous.

By the above proposition, the semigroup $(e^{-tL})_{t\geqq 0}$ acts on the space $L^p(\mathbb{R}^n)$. For the iteration scheme described in Section 4, it is however essential that the semigroup maps an L^p-function u with $\nabla \cdot u = 0$ into the space of L^p-functions which are divergence free. To this end, we introduce the operator $\mathcal{A}$ by

$$\mathcal{A}u := D_{\mathcal{L}}u + Mu,$$

where $u = (u_1, \dots, u_n) \in L^p(\mathbb{R}^n)^n$ and $D_{\mathcal{L}}$ is the $n \times n$ diagonal matrix operator with entries $\mathcal{L}$. Observe that

$$\nabla \cdot \{-Mx \cdot \nabla u + Mu\} = 0 \qquad \text{provided} \quad \nabla \cdot u = 0.$$

Hence we define the realization of A of $\mathcal{A}$ in $L^p_\sigma(\mathbb{R}^n)^n$ as

$$\begin{aligned} Au &:= \mathcal{A}u, \\ D(A) &:= D(L)^n \cap L^p_\sigma(\mathbb{R}^n)^n \end{aligned} \tag{3.2}$$

and by standard perturbation theory it follows that $-A$ generates a C_0-semigroup on $L^p_\sigma(\mathbb{R}^n)^n$. Indeed, we have the following lemma.

Lemma 3.3. *The operator $-A$ generates a C_0-semigroup on $L^p_\sigma(\mathbb{R}^n)^n$ which is given by*

$$\begin{aligned} (e^{-tA}u)(x) := \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} e^{-tM} \int_{\mathbb{R}^n} u(e^{tM}x - y) \\ \times e^{-\frac{1}{4}(Q_t^{-1}y,y)}dy, \qquad x \in \mathbb{R}^n. \end{aligned} \tag{3.3}$$

Note that the semigroup $(e^{-tA})_{t\geqq 0}$ is not analytic. This is due to the fact that $(e^{-tL})_{t\geqq 0}$ is not analytic. In order to simplify our notation, we do not distinguish in the following between the spaces $L^p_\sigma(\mathbb{R}^n)$ and $L^p_\sigma(\mathbb{R}^n)^n$. Notice also that as before $(e^{-tA})_{t\geqq 0}$ extends to a semigroup on $L^\infty(\mathbb{R}^n)$ which is not strongly continuous.

We now turn to $(L^p$–$L^q)$-smoothing properties for the semigroup $(e^{-tL})_{t\geqq 0}$ as well as gradient estimates for e^{-tA}. Note that due to the non-analyticity of $(e^{-tA})_{t\geqq 0}$, gradient estimates for e^{-tA} do not follow from the general theory of semigroups. Notice also that in the special case where $M = \mathrm{Id}$, $(L^p$–$L^q)$-smoothing estimates as well as gradient estimates for e^{-tA} were obtain by GALLAY & WAYNE [8].

We introduce the following condition on the matrix M: assume that there exists a constant $C > 0$ such that

$$\|e^{-tM}\| \leqq C \text{ for all } t > 0. \tag{3.4}$$

We note that the above condition is satisfied if and only if $\mathrm{Re}\lambda \geqq 0$ for all eigenvalues λ of M and the algebraic and geometric multiplicities coincide for those eigenvalues λ which satisfy $\mathrm{Re}\lambda = 0$.

Since $Q_t = \int_0^t e^{sM}e^{sM^T}ds$ is symmetric and positive definite for all $t > 0$, condition (3.4) implies that that there exists a constant $C > 0$ such that

$$\det Q_t \geqq Ct^n, \qquad t > 0. \tag{3.5}$$

We start with the following result.

Proposition 3.4. *Let* $1 < p < \infty$ *and* $p \leqq q \leqq \infty$.

(a) *Let* $T > 0$. *Then there exists a constant* $C > 0$ *such that*

$$\|e^{-tA} f\|_q \leqq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_p, \ \ 0 < t < T, \tag{3.6}$$

$$\|\nabla e^{-tA} f\|_p \leqq Ct^{-\frac{1}{2}}\|f\|_p, \ \ 0 < t < T. \tag{3.7}$$

(b) *Assume in addition that* (3.4) *holds. Then there exists a constant* $C > 0$ *such that*

$$\|e^{-tA} f\|_q \leqq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_p, \ \ t > 0, \tag{3.8}$$

$$\|\nabla e^{-tA} f\|_p \leqq Ct^{-\frac{1}{2}}\|f\|_p, \ \ t > 0. \tag{3.9}$$

(c) *Moreover, for* $1 < p < q \leqq \infty$ *and* $f \in L^p(\mathbb{R}^n)^n$,

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|e^{-tA} f\|_q \to 0 \qquad \text{as } t \to 0, \tag{3.10}$$

$$t^{\frac{1}{2}}\|\nabla e^{-tA} f\|_p \to 0 \qquad \text{as } t \to 0. \tag{3.11}$$

Proof. We start by showing (3.8). Notice first that by (3.5) there exists a constant $C > 0$ such that $\det Q_t \geqq Ct^n$ for $t > 0$. By Young's inequality,

$$\|e^{-tA} f\|_q \leqq \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}}\|e^{-tM}\|\Big(\int_{\mathbb{R}^n} e^{-\frac{r}{4}(Q_t^{-1}y,y)}dy\Big)^{1/r}\|f(e^{tM}\cdot)\|_p,$$

where $r \in (1,\infty)$ with $1/q = 1/r + 1/p - 1$. Note further that $\|f(e^{tM}\cdot)\|_p = e^{-t\frac{trM}{p}}\|f\|_p$. By the change of variables $y = Q_t^{1/2}z$ we obtain

$$\begin{aligned}\Big(\int_{\mathbb{R}^n} e^{-\frac{r}{4}(Q_t^{-1}y,y)}dy\Big)^{1/r} &= \Big(\int_{\mathbb{R}^n} e^{-\frac{r|z|^2}{4}}(\det Q_t)^{1/2}dz\Big)^{1/r}\\ &\leqq Ce^{\omega t}t^{\frac{n}{2}(1-\frac{1}{p}+\frac{1}{q})}, \quad t > 0\end{aligned}$$

for some constant $C > 0$. We thus proved (3.8).

In order to prove the gradient estimates (3.9), we verify that

$$\begin{aligned}&\nabla e^{-tA} f(x)\\ &= \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}}e^{-tM}\int_{\mathbb{R}^n} f(e^{tM}x - y)\nabla_y e^{-\frac{1}{4}(Q_t^{-1}y,y)}dy\\ &= \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}}e^{-tM}\int_{\mathbb{R}^n} f(e^{tM}x - y)(-\frac{1}{2})Q_t^{-1}ye^{-\frac{1}{4}(Q_t^{-1}y,y)}dy.\end{aligned}$$

Similarly as above we obtain

$$\begin{aligned}\|\nabla e^{-tA} f\|_p &\leqq \frac{1}{(4\pi)^{n/2}}\|e^{-tM}\|\frac{1}{2}\int_{\mathbb{R}^n}\|Q_t^{-1/2}\||y|e^{-\frac{1}{4}(Q_t^{-1}y,y)}dy\|f(e^{tM}\cdot)\|_p\\ &\leqq Ct^{-1/2}\|f\|_p\end{aligned}$$

for some constant $C > 0$. Here we used the fact that $\|Q_t^{-1}\| \leqq \frac{C}{t}$ for $t > 0$ and some $C > 0$.

Assertions (3.6) and (3.7) are proved in a similar way.

In order to prove (3.10) we note first that, without loss of generality, we may assume $q < \infty$. Let $g \in C_c^\infty(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ and $t < 1$. The triangle inequality and (3.8) imply that there exists a constant $C > 0$ such that

$$\begin{aligned} t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|e^{-tA}f\|_q &\leqq t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|e^{-tA}f - e^{-tA}g\|_q + t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|e^{-tA}g\|_q \\ &\leqq C\|f-g\|_p + Ct^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|g\|_q \to 0, \end{aligned}$$

by sending first $t \to 0$ and then approximating f by g. The prove of (3.11) follows the above lines and is therefore omitted. □

The following estimates for higher-order derivatives of semigroup, i.e., for $\nabla^m e^{-tA}f$, are very useful in Section 4 where we consider smoothing properties of mild solutions of the Navier-Stokes equations. The main difficulty is again as in the proof of Proposition 3.4 that the semigroup e^{-tA} and the derivatives in ∇ do not, in general, commute. Nevertheless, we obtain estimates similar to those which are known for the classical Stokes operator.

Lemma 3.5. *Let $1 < p < \infty$ and $p \leqq q \leqq \infty$. Then there exist constants $C_1, C_2, C_3 > 0$, $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{R}$ such that*

$$\|\nabla^m e^{-tA}f\|_q \leqq C_1 e^{(\omega_1+\omega_2 m)t} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\nabla^m f\|_p, \qquad t > 0, m \in \mathbb{N}, \tag{3.12}$$

for $f \in W^{m,p}(\mathbb{R}^n)$ and

$$\|\nabla^m e^{-tA}f\|_q \leqq C_2(C_3 m)^{m/2} e^{(\omega_3+\omega_4 m)t} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}}\|f\|_p, \qquad t > 0, m \in \mathbb{N} \tag{3.13}$$

for $f \in L^p(\mathbb{R}^n)$.

Proof. Consider first the case $p = q$. Since $\|e^{tM}\| = \|e^{tM^T}\| \leqq Ce^{\omega t}$ for all $t \geqq 0$ and some constants $C > 0$, $\omega_2 \in \mathbb{R}$, it follows that

$$\|\nabla^m e^{-tA}f\|_p \leqq \|e^{tM}\|^m \|e^{-tA}\nabla^m f\|_p \leqq Ce^{\omega_2 mt}e^{\omega_1 t}\|\nabla^m f\|_p \tag{3.14}$$

for some $\omega_1 \in \mathbb{R}$. The case $p < q$ follows by combining (3.14) for $p = q$ with (3.8). This shows assertion (3.12).

In order to prove (3.13), we write

$$\begin{aligned} \|\nabla^m e^{-tA}f\|_q &= \|\nabla e^{-\frac{t}{2m}A}e^{(m-1)tM}\nabla^{m-1}e^{-(1-\frac{1}{2m})tA}f\|_q \\ &\leqq C\Big(\frac{t}{2m}\Big)^{-1/2} e^{\frac{\omega t}{2m}} Ce^{\omega(m-1)t}\|\nabla^{m-1}e^{-(1-\frac{1}{2m})tA}f\|_q. \end{aligned}$$

Iterating this procedure, we see that there exist constant $C > 0$, $\omega_3, \omega_4 \in \mathbb{R}$ such that

$$\|\nabla^m e^{-tA}f\|_q \leqq C^m m^{m/2} e^{\omega_4 mt}e^{\omega_3 t}t^{-m/2}\|e^{-\frac{t}{2}A}f\|_q.$$

Applying (3.8) we finally obtain (3.13). □

Remark 3.6. The assertion (3.12) holds true with $\omega_2 = 0$ provided $\|e^{tM}\| \leqq 1$ for all $t > 0$.

The following estimate on $\|\nabla e^{-tA}\mathbb{P}\|$ will be of central importance in the proof of Corollary 2.6.

Lemma 3.7. *Let $1 \leqq p \leqq \infty$ and let A and $\mathbb{P}$ as above. Then there exist constants $C_p > 0$, $w \in R$ such that*

$$\|\nabla e^{-tA}\mathbb{P}\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \leqq \frac{C_p}{t^{1/2}} e^{wt}, \qquad t > 0.$$

We remark that this estimate was shown in [10] for the case $A = -\Delta$.

Proof. Denote by $\mathcal{M}_p(\mathbb{R}^n)$ the space of all Fourier multipliers for $L^p(\mathbb{R}^n)$; see, e.g., the monographs [32] or [2]. Given $a \in \mathcal{M}_p(\mathbb{R}^n)$, define a_t by $a_t(\xi) := a(e^{tM}\xi)$ where $t > 0$ and $\xi \in \mathbb{R}^n$. Then $a_t \in \mathcal{M}_p(\mathbb{R}^n)$ for all $t > 0$ and $\|a_t\|_{\mathcal{M}_p(\mathbb{R}^n)} = \|a\|$ for all $t > 0$. This follows from the fact that

$$\mathcal{F}^{-1}(a_t\mathcal{F}) = J_t^{-1}\mathcal{F}^{-1}a\mathcal{F}J_t,$$

where J_t is the isometry $(J_t f)(x) = f(e^{tM}x)e^{\frac{t}{p}trM}$ on $L^p(\mathbb{R}^n)$ and $\mathcal{F}$ denotes the Fourier transform. Thus we have

$$\|\nabla e^{-tA}\mathbb{P}\|_{\mathcal{L}(L^p(\mathbb{R}^n))} = e^{wt}\|Q_t^{-1/2}\|\|a\|_{\mathcal{M}_p(\mathbb{R}^n)},$$

where

$$a(\xi) := \xi_i \frac{\xi_j}{|\xi|}\frac{\xi_k}{|\xi|}e^{-\frac{-|\xi|^2}{4}}, \quad \xi \in \mathbb{R}^n$$

for some $i, j, k \in \{1, \dots, n\}$. Now $a \in \mathcal{M}_p(\mathbb{R}^n)$ for all p satisfying $1 \leqq p \leqq \infty$ by Proposition 8.2.3 and Lemma 8.2.2 of [2]. □

After these preparations, we are now in the position to show that (2.3) admits a local mild solution and to investigate its properties.

4. Mild solutions of the Navier-Stokes equations

For a given matrix M, we denote by M^{sym} and M^{ssym} the symmetric and skew-symmetric part of M, respectively, i.e.

$$M^{\text{sym}} := \tfrac{1}{2}(M + M^T) \quad \text{and} \quad M^{\text{ssym}} := \tfrac{1}{2}(M - M^T).$$

Here M^T denotes the transposed matrix of M.

Setting $u := U + Mx$, we see that (U, P) is a solution of (1.1) if and only if u satisfies (1.3), where p is defined by $p(x, t) := P(x, t) - (\Pi x, x)$, and $\Pi := \frac{1}{2}((M^{\text{sym}})^2 + (M^{\text{ssym}})^2)$. We thus consider in the following, (1.3) and its abstract formulation in (2.3).

Proof of Theorem 2.1. Let $n \geqq 2$ and $u_0 \in L^n_\sigma(\mathbb{R}^n)$. For $j \geqq 1$ and $t > 0$, we define functions u_{j+1} by

$$u_{j+1}(t) := e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathbb{P}u_j(s)\cdot\nabla u_j(s)ds + 2\int_0^t e^{-(t-s)A}\mathbb{P}u_j(s)ds \tag{4.1}$$

provided the above integrals exist and where $u_1(t) := e^{-tA}u_0$. Since $(e^{-tA})_{t\geq 0}$ acts on $L^p_\sigma(\mathbb{R}^n)$ it follows from the definition of the Helmholtz projection that the functions u_j are divergence-free for all $t > 0$ and all j.

For $T \in (0,1]$ and $\delta \in (0,1)$, we define

$$A_0 := \sup_{0<t\leqq T} t^{\frac{1-\delta}{2}}\|e^{-tA}u_0\|_{n/\delta} \quad \text{and} \quad A_0' := \sup_{0<t\leqq T} t^{\frac{1}{2}}\|\nabla e^{-tA}u_0\|_n,$$

as well as $A_j := A_j(T)$ and $A_j' := A_j'(T)$, where

$$\left.\begin{aligned} A_j(T) &:= \sup_{0<t\leqq T} t^{\frac{1-\delta}{2}}\|u_j(t)\|_{n/\delta} \\ A_j'(T) &:= \sup_{0<t\leqq T} t^{1/2}\|\nabla u_j(t)\|_n \end{aligned}\right\} j \geqq 1.$$

We thus obtain the following from (4.1), the $(L^p$–$L^q)$-smoothing of the semigroup, and the boundedness of $\mathbb{P}$ from $L^p(\mathbb{R}^n)$ into $L^p_\sigma(\mathbb{R}^n)$:

$$\begin{aligned} \|u_{j+1}(t)\|_{n/\delta} &\leqq \|e^{t\Delta}u_0\|_{n/\delta} + \int_0^t \|e^{-(t-s)A}\mathbb{P}u_j(s)\cdot\nabla u_j(s)\|_{n/\delta}ds \\ &\quad +2\int_0^t \|e^{-(t-s)A}\mathbb{P}Mu_j(s)\|_{n/\delta}ds \\ &\leqq t^{-\frac{1-\delta}{2}}A_0 + C\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{\delta}{n})}\|u_j(s)\cdot\nabla u_j(s)\|_r ds \\ &\quad +C\int_0^t \|u_j(s)\|_{n/\delta}ds, \end{aligned}$$

where $r = \frac{n}{1+\delta}$. In order to estimate the second term on the right-hand side of the last inequality, we now apply Hölder's inequality to conclude that

$$\|u_j(s)\cdot\nabla u_j(s)\|_r \leqq \|u_j(s)\|_{n/\delta}\|\nabla u_j(s)\|_n \leqq A_jA_j's^{-\frac{1-\delta}{2}-\frac{1}{2}}.$$

This implies

$$\|u_{j+1}(t)\|_{n/\delta} \leqq t^{-\frac{1-\delta}{2}}A_0 + CA_jA_j'\int_0^t (t-s)^{-\frac{1}{2}}s^{-1+\frac{\delta}{2}}ds + CA_j\int_0^t s^{-\frac{1-\delta}{2}}ds.$$

Multiplying with $t^{\frac{1-\delta}{2}}$ and taking $\sup_{0<t\leqq T}$ on both sides, we obtain

$$A_{j+1} \leqq A_0 + C_1A_jA_j' + C_2TA_j \tag{4.2}$$

with some positive constants C_1, C_2 independent of j and T.

Similarly, applying ∇ to (4.1) and estimating it in the L^n norm, by (3.9) we obtain

$$A'_{j+1} \leqq A'_0 + C_3 A_j A'_j + C_4 T A_j \tag{4.3}$$

with some positive constants C_3 and C_4. The estimates (3.10) and (3.11) imply that for any $\lambda > 0$, there exists $\tilde{T}_0 > 0$ such that $A_0, A'_0 \leqq \lambda$ for all $T \leqq \tilde{T}_0$. More precisely, we may choose $\tilde{T}_0 \leqq \min(1, \frac{1}{3C_2}, \frac{1}{3C_4})$ provided $\lambda \leqq \min(\frac{1}{9C_1}, \frac{1}{9C_3})$. Therefore, we obtain bounds for $A_j(T)$ and $A'_j(T)$ for any $T \leqq \tilde{T}_0$ uniformly in j provided that $\tilde{T}_0$ is small enough.

Using the uniform bounds of A_j and A'_j we just obtained, it follows that $t^{\frac{1}{2}-\frac{n}{2q}}\|u_j(t)\|_q$ as well as $t^{1-\frac{n}{2q}}\|\nabla u_j(t)\|_q$ are bounded for $q \in [n, \infty)$, $t \leqq \tilde{T}_0$ and all $j \in \mathbb{N}$. The continuity of the above functions follows from similar calculations and (3.10).

We finally derive estimates for the differences $u_{j+1} - u_j$. Indeed, we now put

$$\left.\begin{aligned} L_j(T) &:= \sup_{0<t\leqq T} t^{\frac{1-\delta}{2}}\|u_{j+1}(t) - u_j(t)\|_{n/\delta} \\ L'_j(T) &:= \sup_{0<t\leqq T} t^{1/2}\|\nabla u_{j+1}(t) - \nabla u_j(t)\|_n \end{aligned}\right\} j \geqq 1.$$

Similarly as before, we have for all $j \geqq 1$,

$$\begin{aligned} L_j(T) &\leqq C_5\lambda(L_{j-1} + L'_{j-1}) + C_6 T L_{j-1}, \\ L'_j(T) &\leqq C_7\lambda(L_{j-1} + L'_{j-1}) + C_8 T L_{j-1} \end{aligned}$$

with some positive constants C_5, C_6, C_7 and C_8. We now choose $T_0 \leqq \tilde{T}_0$ small enough so that $T_0 \leqq \min(\frac{1}{3C_6}, \frac{1}{3C_8})$ provided $3(C_5 + C_7)\lambda \leqq 1$. Hence we have $(L_j + L'_j)/(L_{j-1} + L'_{j-1}) \leqq 1/2$ for all j and $T \leqq T_0$. This implies that L_j and L'_j tend to zero as $j \to \infty$. It thus follows that the above sequences are Cauchy sequences and we conclude that there are unique limit functions

$$t^{\frac{1}{2}-\frac{n}{2q}}u(t) \in C([0, T_0]; L^q), \qquad t^{1-\frac{n}{2q}}v(t) \in C([0, T_0]; L^q),$$

of the sequences $(t^{\frac{1}{2}-\frac{n}{2q}}u_j(t))_{j\geqq 1}$ and $(t^{1-\frac{n}{2q}}\nabla u_j(t))_{j\geqq 1}$. Finally, note that $v(t) = t^{1/2}\nabla u(t)$ and that u is a mild solution of (2.3) on $[0, T_0]$.

Uniqueness of mild solutions follows as in [12] from Gronwall's inequality. This completes the proof of Theorem 2.1. □

Proof of Corollary 2.3. The proof of Corollary 2.3 essentially follows the same lines as above. In fact, in order to prove the assertion we replace the estimates for $\nabla e^{-tA} f$ given in Proposition 3.3 by the estimates for $\nabla^m e^{-tA} f$ given in Lemma 3.4. □

We now turn to the proof of Theorem 2.4. In the situation of the classical Stokes operator, i.e., M = 0, it was recently proved by GIGA & SAWADA [15] that mild solutions to (2.3) are analytic in x. The following proof is a modification of that proof to this situation. Note, however, that in contrast to the heat semigroup the semigroup $(e^{-tA})_{t\geqq 0}$ is not analytic. On the other hand, the smoothing properties of $(e^{-tA})_{t\geqq 0}$ given in Proposition 3.3 and Lemma 3.4 allow us to follow the strategy of [15].

Proof of Theorem 2.4. We start by proving the assertion under the additional assumption that the mild solution of (2.3) is already smooth. More precisely, we prove first the following result. □

Proposition 4.1. *Suppose that the assumptions of Theorem 2.4 are satisfied. Assume furthermore that*

$$\partial_x^\alpha u \in C((0,T); L^q(\mathbb{R}^n)) \tag{4.4}$$

for all $q \in [n,\infty]$ *and all* $\alpha \in \mathbb{N}_0^n$. *Then, given* $\delta \in (\frac{1}{2}, 1]$, *there exist constants* $K_1 > 0$, $K_2 > 0$ (*depending only on n, r, M,* M_1, M_2, *T and* δ) *such that*

$$\|\nabla^m u(t)\|_q \leqq K_1(K_2 m)^{m-\delta} t^{-\frac{m}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}, \qquad t \in (0,T], m \in \mathbb{N}_0 \tag{4.5}$$

for all $q \in [n,\infty]$.

Proof. We use an induction argument with respect to m. We may argue that $\nabla^m u$ is continuous up to $t = 0$ and has a value in $L^q(\mathbb{R}^n)$ by considering $u((e^{-tA})_{t\geqq 0})$ for $(e^{-tA})_{t\geqq 0} > 0$ as initial data and sending $(e^{-tA})_{t\geqq 0} \to 0$. To this end, let $k_0 \geqq 2$ (depending only on n and M). Then (4.5) follows for all $m \leqq k_0$, provided K_1 is chosen large enough.

Assume hence that $k \geqq k_0$. We suppose by assumption that (4.5) holds for all $q \in [n,\infty]$ and all $m \leqq k-1$. We claim that (4.5) holds for $m = k$.

For simplicity, we first prove the assertion under the additional assumptions that $T \leqq 1$, $n \geqq 3$ and $q < \infty$. The claim then follows by minor modifications of the proof given below.

We start by noticing that, given $q \in [n,\infty)$ and $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
\|\nabla^k u(t)\|_q \;\leqq\; & \|\nabla^k e^{-tA} u_0\|_q + \left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t\right) \|\nabla^k e^{-(t-s)A}\mathbb{P}u\cdot\nabla u(s)\|_q ds \\
& + 2\left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t\right) \|\nabla^k e^{-(t-s)A}\mathbb{P}Mu(s)\|_q ds \\
=:\; & B_1 + B_2 + B_3 + B_4 + B_5.
\end{aligned}$$

We shall estimate each of the above terms $B_1 - B_5$ separately.

The estimates for B_1 are derived from (3.13) as follows:

$$\begin{aligned}
B_1 &\leqq C_2(C_3 k)^{k/2} e^{\omega_3 k t}\|u_0\|_n t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}} \\
&\leqq C_4(C_5 k)^{k-\delta} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}, \qquad t \in (0,T),
\end{aligned}$$

for constants $C_4 := C_2\|u_0\|_n \leqq C_2 M_1$ and $C_5 := C_3 e^{\omega_3}$. This follows since $k/2 \leqq k-\delta$ for $k \geqq 2$ and $\delta \leqq 1$.

Similarly, we estimate B_4 as

$$\begin{aligned}
B_4 &\leqq 2C_2(C_3 k)^{k/2} e^{\omega_3 k t}\int_0^{(1-\varepsilon)t} \|\nabla^k e^{-(t-s)A}\|_{\mathcal{L}(L^n,L^q)}\|\mathbb{P}M\|_{\mathcal{L}(L^n)}\|u(s)\|_n ds \\
&\leqq C_6(C_7 k/\varepsilon)^{k/2} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}
\end{aligned}$$

for $t \in (0, T)$ and some positive constants $C_6 := C_6(n, q, M, M_1)$ and $C_7 := C_7(n, M)$. Here we used the boundedness of the Helmholtz projection and the Riesz transform in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Estimate (3.13) implies also the following estimate for B_5:

$$B_5 \leqq 2C_1 e^{\omega_1 t} \int_{(1-\varepsilon)t}^{t} \|e^{-(t-s)A}\mathbb{P}M\|_{\mathcal{L}(L^q, L^p)} \|\nabla^k u(s)\|_q ds$$
$$\leqq C_8 \int_{(1-\varepsilon)t}^{t} \|\nabla^k u(s)\|_q ds$$

for some constant $C_8 := C_8(n, q, M) > 0$. Note that C_8 does not depend on k, since we assumed that $\|e^{tM}\| \leqq C$; see Remark 3.6(a).

In order to estimate B_2 note that

$$B_2 \leqq \int_0^{(1-\varepsilon)t} \|\nabla^k e^{-(t-s)A}\|_{\mathcal{L}(L^{n/2}, L^q)} \|\mathbb{P}\|_{\mathcal{L}(L^{n/2})} \|u \cdot \nabla u(s)\|_{n/2} ds$$
$$\leqq C_9 (C_{10} k/\varepsilon)^{k/2} t^{-\frac{n}{2}(\frac{1}{n} - \frac{1}{q}) - \frac{k}{2}}$$

for some constants $C_9 := C_6(n, p, q, M, M_1, M_2)$ and $C_{10} := C_{10}(n, M)$. Here we have used Hölder's inequality to obtain $\|u \cdot \nabla u\|_{n/2} \leqq \|u\|_n \|\nabla u\|_n$ and we also made use of the estimate $\|\nabla u(s)\|_n \leqq Cs^{-1/2}$ for some constant $C := C(n, p, M, M_1, M_2)$. See also the proof of Step 2 of Proposition 3.1 in [15].

We now estimate B_3. Similarly as in the estimates for B_5, we obtain by (3.12),

$$B_3 \leqq C_1 e^{\omega_1 t} \int_{(1-\varepsilon)t}^{t} \|\nabla e^{-(t-s)A}\mathbb{P}\|_{\mathcal{L}(L^q)} \|\nabla^k (u \otimes u)(s)\|_q ds$$
$$\leqq C_{11} \int_{(1-\varepsilon)t}^{t} (t-s)^{-1/2} \|\nabla^k (u \otimes u)(s)\|_q ds$$

with some $C_{11} := C_{11}(n, M)$. Since $\|e^{tM}\| \leqq 1$, the constant C_{11} can be chosen independently not only of k but also of q. Note next that by Lemma 3.7 for all $q \in [1, \infty]$ there exist constant $C > 0$ and $w \in \mathbb{R}$ such that $\|\nabla e^{-tA}\mathbb{P}\|_{\mathcal{L}(L^q)} \leqq Ct^{-1/2}e^{wt}$.

We now calculate $\nabla^k (u \otimes u)$ by Leibniz's rule. We divide the sum into two parts:

$$B_3 \leqq 2C_{11} \int_{(1-\varepsilon)t}^{t} (t-s)^{-1/2} \|\nabla^k u(s)\|_q \|u(s)\|_\infty ds$$
$$+ C_{11} \int_{(1-\varepsilon)t}^{t} (t-s)^{-1/2} \max_{|\beta|=k} \sum_{0<\gamma<\beta} \binom{\beta}{\gamma} \|\partial_x^\gamma u(s)\|_q \|\partial_x^{\beta-\gamma} u(s)\|_\infty ds$$
$$=: B_{3a} + B_{3b}.$$

Here, $\gamma < \beta$ means $\gamma_i \leqq \beta_i$ for all i and $|\gamma| < |\beta|$ for multi-indices β and γ

Consider B_{3a}. Then there exists $C > 0$ (depending only on n, p, M, M_1, M_2 such that $\|u(s)\|_\infty \leqq Cs^{-1/2}$; see Step 1 of the proof of Proposition 3.1 in [15]. Thus

$$B_{3a} \leqq C_{12} \int_{(1-\varepsilon)t}^{t} (t-s)^{-1/2} s^{-1/2} \|\nabla^k u(s)\|_q ds$$

with some constant $C_{12} := C_{12}(n, p, q, M, M_1, M_2)$.

We next estimate B_{3b}. By assumption of induction, we obtain

$$\begin{aligned} B_{3b} &\leqq C_{11} \int_{(1-\varepsilon)t}^{t} (t-s)^{-\frac{1}{2}} \max_{|\beta|=k} \sum_{0<\gamma<\beta} \binom{\beta}{\gamma} K_1 (K_2|\gamma|)^{|\gamma|-\delta} s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{|\gamma|}{2}} \\ &\quad \times K_1 (K_2|\beta-\gamma|)^{|\beta-\gamma|-\delta} s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q}-\frac{|\beta-\gamma|}{2}} ds \\ &\leqq C_{11} K_1^2 K_2^{k-2\delta} \sum_{0<\gamma<\beta} \binom{\beta}{\gamma} |\gamma|^{|\gamma|-\delta} |\beta-\gamma|^{|\beta-\gamma|-\delta} \\ &\quad \times \int_{(1-\varepsilon)t}^{t} (t-s)^{-\frac{1}{2}} s^{-1-\frac{n}{2q}-\frac{k}{2}} ds. \end{aligned}$$

For the multiplication of multi-sequences we apply Kahane's lemma [20, Lemma 2.1] and obtain

$$B_{3b} \leqq C_{13} K_1^2 K_2^{k-2\delta} k^{k-\delta} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}} I(\varepsilon),$$

where $I(\varepsilon) := \int_{1-\varepsilon}^{1} (1-\tau)^{-\frac{1}{2}} \tau^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}-\frac{1}{2}} d\tau$ and C_{13} depends only on C_{11} and δ. Note that C_{13} in δ is proportional to $\sum_{j=1}^{\infty} j^{-1/2-\delta/2}$.

Combining the estimates for $B_1, \ldots, B_5$ and defining b_ε by

$$b_\varepsilon := C_4(C_5 k)^{k-\delta} + C_6(C_7 k/\varepsilon)^{k/2} + C_9(C_{10} k/\varepsilon)^{k/2} + C_{13} K_1^2 K_2^{k-2\delta} k^{k-\delta} I(\varepsilon),$$

we obtain

$$\|\nabla^k u(t)\|_q \leqq b_\varepsilon t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}} + (C_8 + C_{12}) \int_{(1-\varepsilon)t}^{t} (t-s)^{-1/2} s^{-1/2} \|\nabla^k u(s)\|_q ds.$$

Applying a Gronwall-type inequality (see [15, Lemma 2.4]), we see that there exists $\varepsilon_k \in (0, 1)$ such that

$$\|\nabla^k u(t)\|_q \leqq 2 b_{\varepsilon_k} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}, \qquad t \in (0, T). \tag{4.6}$$

If $\varepsilon_k := 1/k$ then $I(1/k) \leqq \frac{1}{2(C_8+C_{12})}$ for sufficiently large k, say $k \geqq k_0 := k_0(n, p, M, M_1, M_2)$.

Finally, we show $2b_{1/k} \leqq K_1(K_2 k)^{k-\delta}$ for any k with suitable constants K_1 and K_2. When K_1 is large enough, (4.5) holds for $k \leqq k_0$, i.e., there exists a constant $K_0 > 0$ (depending only on n, p, M, M_1 and M_2) such that $\|\nabla^k u(t)\|_q \leqq K_0$ for $k \leqq k_0$. Since $I(1/k) \leqq 2$ for all $k \geqq 2$, we have

$$2b_{1/k} \leqq 2\{C_4 C_5^{k-\delta} + C_6 C_7^{k-\delta} + C_9 C_{10}^{k-\delta} + 2C_{13} K_1^2 K_2^{k-2\delta}\} k^{k-\delta}.$$

Choosing the constants K_1 and K_2 as

$$K_1 := \max\big(K_0, 4(C_4+C_6+C_9)\big) \quad \text{and} \quad K_2 := \max\big(C_5, C_7, C_{10}, (8C_{13}K_1)^{\delta}\big),$$

we obtain (4.5) for all k. The proof is complete. □

Proposition 4.2. *Assume that the assumptions of Theorem 2.4 are satisfied. Then the mild solution u of* (2.3) *satisfies* (4.4). *There even exist constants $\tilde{K}_1, \tilde{K}_2 > 0$ such that*

$$\|\nabla^m u(t)\|_q \leqq \tilde{K}_1(\tilde{K}_2 m)^m t^{-\frac{m}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}, \qquad t \in (0,T), m \in \mathbb{N}_0, q \in [n,\infty]. \tag{4.7}$$

Proof. Let $u_0 \in L^n_\sigma(\mathbb{R}^n)$. As in the proof of Theorem 2.1, we derive an *a priori* estimate in order to obtain uniform bounds for $\|u_j(t)\|_q$ and $\|\nabla u_j(t)\|_q$. More precisely, there exists $T > 0$ such that

$$\sup_j t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}\|u_j(t)\|_q \leqq 3\|u_0\|_n \quad \sup_j t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})+\frac{1}{2}}\|\nabla u_j(t)\|_q \leqq 3\|u_0\|_n, \quad t \in (0,T)$$

for $q \in [n,\infty]$. Moreover, there exists a unique u such that

$$\lim_{j\to\infty} u_j(t) = u(t) \qquad \text{in} \quad L^n(\mathbb{R}^n)$$

for all $t \in (0,T)$.

We fix $k \in \mathbb{N}_0$, $q \in [n,\infty]$ and set $\psi_j(t) := \|\nabla^k u_j(t)\|_q$. Similarly as the proof of Proposition 4.1 we conclude that, for $j \geqq 0$, $t \in (0,T)$,

$$\psi_{j+1}(t) \leqq \tilde{b}_\varepsilon t^{-\frac{k}{2}+\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} + \tilde{C}\int_{(1-\varepsilon)t}^t (t-s)^{-1/2}s^{-1/2}\psi_j(s)ds,$$

where $\tilde{C}$ is a constant depending only on n, u_0 and T and $\tilde{b}_\varepsilon$ is chosen similarly to b_ε. Applying a Gronwall-type inequality (see [15, Lemma 2.4]), we find that there exists ε_k such that $\psi_j(t) \leqq 2\tilde{b}_{\varepsilon_k} t^{-\frac{k}{2}+\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}$ for all j. We thus have

$$\|\nabla^k u_j(t)\|_q \leqq \tilde{K}_1(\tilde{K}_2 k)^{k-\delta} t^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}$$

for all $j \geqq 0$ and $t \in (0,T)$, provided we are able to choose constants $\tilde{K}_1$ and $\tilde{K}_2$ uniformly in j, similarly as in the end of the proof of Proposition 4.1.

Since the limit is unique, we obtain $\partial_x^\beta u(t) =: v(t)$ and $v(t)$ also satisfies

$$\|v(t)\|_q \leqq \tilde{K}_1(\tilde{K}_2 k)^{k-\delta} t^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}$$

for all $t \in (0,T)$. We thus obtain (4.7) for arbitrary $k \in \mathbb{N}_0$ and $q \in [n,\infty]$. □

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