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Symmetries and Global Solvability of the Isothermal Gas Dynamics Equations

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Abstract

We study the Cauchy problem associated with the system of two conservation laws arising in isothermal gas dynamics, in which the pressure and the density are related by the γ-law equation $p(\rho) \sim \rho^\gamma$ with $\gamma = 1$. Our results complete those obtained earlier for $\gamma > 1$. We prove the global existence and compactness of entropy solutions generated by the vanishing viscosity method. The proof relies on compensated compactness arguments and symmetry group analysis. Interestingly, we make use here of the fact that the isothermal gas dynamics system is invariant modulo a linear scaling of the density. This property enables us to reduce our problem to that with a small initial density.

One symmetry group associated with the linear hyperbolic equations describing all entropies of the Euler equations gives rise to a fundamental solution with initial data imposed on the line $\rho = 1$. This is in contrast to the common approach (when $\gamma > 1$) which prescribes initial data on the vacuum line $\rho = 0$. The entropies we construct here are weak entropies, i.e., they vanish when the density vanishes.

Another feature of our proof lies in the reduction theorem, which makes use of the family of weak entropies to show that a Young measure must reduce to a Dirac mass. This step is based on new convergence results for regularized products of measures and functions of bounded variation.

1. Introduction

We consider the Euler equations for compressible fluids

$$\partial_t \rho + \partial_x (\rho u) = 0, \tag{1.1}$$

$$\partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) = 0, \tag{1.2}$$

where $\rho \geqq 0$ denotes the density, u the velocity, and $p(\rho) \geqq 0$ the pressure. We assume that the fluid is governed by the isothermal equation of state

$$p(\rho) = k^2 \rho, \tag{1.3}$$

where $k > 0$ is a constant. Observe that the scaling $u \to k\,u$, $t \to t/k$ allows us to reduce the system (1.1)–(1.3) to the same system with $k = 1$.

The existence of weak solutions (containing jump discontinuities) for the Cauchy problem associated with (1.1)–(1.3) was first established by NISHIDA [24] (in the Lagrangian formulation). The solutions obtained by Nishida have bounded variation and remain bounded away from the vacuum. For background on the BV theory we refer to [6, 16].

By contrast, we are interested here in solutions in a much weaker functional class and in solutions possibly reaching the vacuum $\rho = 0$. Near the vacuum, the system (1.1)–(1.3) is degenerate and, in particular, the velocity u cannot be defined uniquely. Indeed, the present paper is devoted to developing the existence theory in a framework covering solutions satisfying

$$\rho \in L^\infty(\Pi), \quad \rho|u| \leqq C\,(\rho + \rho|\log\rho|), \quad \Pi = \mathbb{R} \times (0,T),$$

with a constant $C > 0$ depending solely on initial data. The time interval $(0,T)$ is arbitrary. Our proof extends DIPERNA'S pioneering work [10] concerned with the pressure law $p(\rho) \sim \rho^\gamma$.

2. Main result

Introducing the momentum variable $m := \rho u$, it is possible to reformulate the Cauchy problem associated with (1.1)–(1.3) as follows:

$$\begin{aligned} \partial_t\rho + \partial_x m &= 0, \\ \partial_t m + \partial_x\Big(\frac{m^2}{\rho} + \rho\Big) &= 0, \end{aligned} \tag{2.1}$$

with initial condition

$$\rho|_{t=0} = \rho_0, \qquad m|_{t=0} = m_0 := \rho_0\,u_0, \tag{2.2}$$

where ρ_0, u_0 are prescribed. Let us first recall the following terminology. A pair of (smooth) functions $\eta = \eta(m,\rho)$, $q = q(m,\rho)$ is called an *entropy pair* if, for any smooth solution (m,ρ) of (2.1), we also have

$$\partial_t\eta(m,\rho) + \partial_x q(m,\rho) = 0.$$

More precisely, we consider entropies $\eta, q \in C^2(\Omega) \cap C^1(\bar\Omega)$ in any domain of the form

$$\Omega := \Big\{0 < \rho < \rho_*, \quad |m| < c_*\rho\,(1 + |\ln\rho|)\Big\}, \quad c_* > 0, \quad \rho_* > 0.$$

It is easily checked that η, q must solve the equations

$$q_m = 2\,\frac{m}{\rho}\eta_m + \eta_\rho, \quad q_\rho = \eta_m - \frac{m^2}{\rho^2}\eta_m, \tag{2.3}$$

which implies that

$$\eta_{\rho\rho} = \frac{p'(\rho)}{\rho^2}\eta_{uu} = \frac{1}{\rho^2}\eta_{uu}. \tag{2.4}$$

A pair (η, q) is said to be a *weak entropy* if $\eta(0,0) = q(0,0) = 0$. It is said to be *convex* if in addition, η is convex with respect to the conservative variables (ρ, m).

Given an initial data $m_0, \rho_0 \in L^\infty(\mathbb{R})$ obeying the inequalities

$$\rho_0(x) \geqq 0, \quad |m_0(x)| \leqq c_0\,\rho_0(x)\,(1 + |\ln \rho_0(x)|), \qquad x \in \mathbb{R} \tag{2.5}$$

for some constant $c_0 > 0$, *an entropy solution* to the Cauchy problem (2.1), (2.2) on the time interval $(0, T)$ is, by definition, a pair of functions $(m, \rho) \in L^\infty(\Pi)$ satisfying the inequalities

$$\rho(x,t) \geqq 0, \quad |m(x,t)| \leqq c\rho(x,t)(1 + |\ln \rho(x,t)|), \qquad (x,t) \in \Pi \tag{2.6}$$

for some positive constant c, together with the inequality

$$\iint_\Pi \Big(\eta(m,\rho)\,\partial_t\varphi + q(m,\rho)\,\partial_x\varphi\Big)\,dxdt + \int_{\mathbb{R}} \eta(m_0,\rho_0)\,\varphi(\cdot,0)\,dx \geqq 0 \tag{2.7}$$

for every convex, weak entropy pair (η, q) and every non-negative function $\varphi \in \mathcal{D}(\mathbb{R} \times [0,T))$ (smooth functions with compact support).

The main results established in the present paper are summarized in Theorems 2.1–2.3 below.

Theorem 2.1 (Cauchy problem in momentum-density variables). *Given an arbitrary time interval* $(0, T)$ *and an initial data* $(m_0, \rho_0) \in L^\infty(\mathbb{R})$ *satisfying the condition* (2.5)*, there exists an entropy solution* (m, ρ) *of the Cauchy problem* (2.1)*,* (2.2) *satisfying the inequalities* (2.6)*, with a constant c depending on* c_0 *only.*

To prove this theorem it will be convenient to introduce the Riemann invariants W and Z by

$$W := \rho e^u, \quad Z := \rho e^{-u},$$

or equivalently

$$\rho = f_1(W,Z) := (WZ)^{1/2}, \qquad \rho u = f_2(W,Z) := (WZ)^{1/2}\ln(W/Z)^{1/2}.$$

We can then reformulate the Cauchy problem (2.1), (2.2) in terms of W, Z, as follows:

$$\begin{gathered}\partial_t f_1(W,Z) + \partial_x f_2(W,Z) = 0,\\ \partial_t f_2(W,Z) + \partial_x(f_3(W,Z) + f_1(W,Z)) = 0,\\ f_3 := (WZ)^{1/2}\big(\ln(W/Z)^{1/2}\big)^2,\end{gathered} \tag{2.8}$$

$$W|_{t=0} = W_0 := \rho_0 e^{u_0}, \qquad Z|_{t=0} = Z_0 := \rho_0 e^{-u_0}. \tag{2.9}$$

A pair of non-negative functions $W, Z \in L^\infty(\Pi)$ is then called an *entropy solution* to the problem (2.8), (2.9) if

$$\iint_\Pi \Big(\widetilde{\eta}(W,Z)\Big)\,\partial_t\varphi + \widetilde{q}(W,Z)\,\partial_x\varphi\Big)\,dxdt + \int_{\mathbb{R}} \widetilde{\eta}(W_0,Z_0)\,\varphi(\cdot,0)\,dx \geqq 0$$

for any non-negative function $\varphi \in \mathcal{D}(\mathbb{R}\times[0,T))$, where

$$\tilde{\eta}(W,Z) := \eta(f_2(W,Z), f_1(W,Z)), \quad \tilde{q}(W,Z) := q(f_2(W,Z), f_1(W,Z)),$$

and (η, q) is any convex, weak entropy pair in the sense introduced above.

Theorem 2.1 above will be obtained as a corollary of the following result.

Theorem 2.2 (Cauchy problem in Riemann invariant variables)**.** *Given non-negative functions $W_0, Z_0 \in L^\infty(\mathbb{R})$, the Cauchy problem* (2.8), (2.9) *has an entropy solution on any time interval (0,T).*

It is checked immediately that, if (W, Z) is an entropy solution given by Theorem 2.2, then the functions $m := f_2(W,Z)$ and $\rho := f_1(W,Z)$ determine an entropy solution of the problem (2.1), (2.2).

One more consequence of Theorem 2.2 concerns the original problem (1.1)–(1.3) in the density-velocity variables. Defining the density and velocity from the Riemann variables by

$$u := \ln(W/Z)^{1/2}, \qquad \rho := (WZ)^{1/2},$$

we deduce also the following result from Theorem 2.2.

Theorem 2.3 (Cauchy problem in velocity-density variables)**.** *Let* $(0,T)$ *be a time interval. Given any measurable functions u_0 and ρ_0 satisfying the conditions*

$$0 \leqq \rho_0 \in L^\infty(\mathbb{R}), \qquad |u_0(x)| \leqq c_0(1 + |\ln\rho_0(x)|), \quad x \in \mathbb{R}$$

for some positive constant c_0, there exist measurable functions $u = u(x,t)$ and $\rho = \rho(x,t)$ such that

$$0 \leqq \rho \in L^\infty(\Pi), \qquad |u(x,t)| \leqq c(1 + |\ln\rho(x,t)|), \quad (x,t) \in \Pi$$

(*where $c > 0$ is a constant depending on c_0*) *and* (u, ρ) *is an entropy solution of the problem* (1.1)–(1.3) *in the sense that the entropy inequality*

$$\iint_\Pi \Big(\eta(\rho, \rho u)\,\partial_t\varphi + q(\rho,\rho u)\,\partial_x\varphi\Big)\,dxdt + \int_{\mathbb{R}} \eta(\rho_0, \rho_0 u_0)\,\varphi(\cdot,0)\,dx \geqq 0$$

holds for any convex, weak entropy pair (η, q) *and any function φ as in Theorem 2.1.*

The novel features of our proof of the above results are :

- the use of symmetry and scaling properties of both the isothermal Euler equations and the entropy-wave equation,
- an analysis of nonconservative products of functions with bounded variation by measures.

We rely on two classical ingredients. The first tool is the compensated compactness method introduced by TARTAR in [32, 33]. (See also MURAT [22].) This method allows us to show that a weakly convergent sequence (of approximate solutions given by the viscosity method) is actually strongly convergent: such a result is achieved by a "reduction lemma" (to point mass measures) for Young measures representing the limiting behavior of the sequence. Tartar's method was applied to systems of conservation laws by DIPERNA [9, 10]. For a completely different approach to the vanishing viscosity method, we refer to BIANCHINI & BRESSAN [2]. Still another (geometric) perspective is introduced in LEFLOCH [17].

The second main tool is the symmetry group analysis of differential equations which goes back to Lie's classical works. The first symmetry property we use concerns the system (1.1)–(1.3) itself: we observe that it is invariant with respect to the scaling $\rho \to \lambda\,\rho$ (λ being an arbitrary parameter). This property allows us to assume that the density is sufficiently small when performing the reduction of the Young measures.

To generate the class of weak entropies, we calculate all the Lie groups associated with the entropy equation (2.4) for the function η. By using one of them we construct the fundamental solution with initial data prescribed on the line $\rho = 1$. This is in contrast with the standard approach which prescribes initial data on the vacuum line $\rho = 0$.

The need of a large family of weak entropies for the Young measure reduction was demonstrated by DiPerna for the isentropic gas dynamics equations with the pressure law $p = \rho^\gamma$, $\gamma > 1$. When $\gamma = \frac{2n+3}{2n+1}$, with n being integer, DiPerna used weak entropies which are progressive waves given by LAX [14]. The method of Tartar and DiPerna was then extended by SERRE [29] to strictly hyperbolic systems of two conservation laws, by CHEN, et al. [3, 8] to fluid equations with $\gamma \in (1, 5/3]$ and by LIONS, PERTHAME, SOUGANIDIS, and TADMOR [18, 19] to the full range $\gamma > 1$. An alternative proof of DiPerna's theorem was provided by MORAWETZ [20]. The theory was extended to real fluid equations by CHEN and LEFLOCH [4, 15, 5]. We also mention the important work by PERTHAME and TZAVARAS on the kinetic formulation for systems of two conservation laws; see [26, 27]. The success of these works relies on a detailed analysis of the fundamental solution of the entropy wave equation (2.4), which is a degenerate, linear wave equation.

When $\gamma = 1$ the analysis developed in [4, 5] for the construction of entropies does not work because (2.4) degenerates at a higher degree and the Cauchy problem at the line $\rho = 0$ becomes highly singular. One novelty of the present paper is to rely on symmetry group argument to identify the entropy kernel.

For the convenience of the reader we summarize now the main steps of the proof of Theorems 2.1–2.3.

Step 1. We rely on the vanishing viscosity method and first construct a sequence of approximate solutions $(u^\varepsilon, \rho^\varepsilon)$, $\varepsilon \downarrow 0$, defined on the strip Π, and such that

$$2\,\varepsilon^r \leqq \rho^\varepsilon \leqq \rho_2 < 1$$

for some $r > 1$. The constant ρ_2 can be chosen to be arbitrarily small by introducing a rescaled, initial density $\lambda\,\rho_0$. We will thus establish first Theorems 2.1

to 2.3 in the case when the initial density is small. Then we will treat the general case by observing that the system (1.1)–(1.3) is invariant via the symmetry $(u, \rho) \to (u, \lambda\rho)$. More precisely, given an entropy solution (u, ρ) of the problem (1.1)–(1.3) with initial data (u_0, ρ_0), the pair $(u', \rho') := (u, \lambda\rho)$ is also an entropy solution with the initial data $(u_0', \rho_0') = (u_0, \lambda\rho_0)$.

Step 2. Next, we prove that there is a sequence $\varepsilon \downarrow 0$ such that

$$W^\varepsilon := \rho^\varepsilon e^{u^\varepsilon} \rightharpoonup W, \quad Z^\varepsilon := \rho^\varepsilon e^{-u^\varepsilon} \rightharpoonup Z \quad \text{weakly} \star \text{ in } L^\infty_{\text{loc}}(\Pi) \tag{2.10}$$

and there exist Young measures $\nu_{x,t}$, associated with the sequence $\varepsilon \downarrow 0$ and defined on the (W, Z)-plane for each point $(x, t) \in \Pi$, such that

$$\lim_{\varepsilon\to 0} F(W^\varepsilon(x,t), Z^\varepsilon(x,t)) = \iint F(\alpha,\beta)d\nu_{x,t} = \langle \nu_{x,t}, F\rangle =: \langle F\rangle \quad \text{in } \mathcal{D}'(\mathbb{R}^2)$$

for any $F(\alpha, \beta) \in C_{\text{loc}}(\mathbb{R}^2)$. The crucial point in the compensated compactness argument is to prove that ν is a point mass measure. In that case the convergence in (2.10) becomes strong in any $L^r_{\text{loc}}(\Pi)$, $1 \leqq r < \infty$.

Step 3. Given two entropy pairs (η_i, q_i), obeying the conditions of Theorem 2.1, we check that Tartar's commutation relations

$$\langle \nu_{x,t}, \eta_1 q_2 - \eta_2 q_1\rangle = \langle \nu_{x,t}, \eta_1\rangle \langle \nu_{x,t}, q_2\rangle - \langle \nu_{x,t}, \eta_2\rangle \langle \nu_{x,t}, q_1\rangle \tag{2.11}$$

hold. Here, we apply the so-called div-curl lemma of MURAT [22] and TARTAR [32, 33]. The objective is to prove that the measure ν is a point mass measure by using a "sufficiently large" class of entropy pairs in (2.11).

Step 4. To produce a large family of entropy pairs, we have to construct a fundamental solution $\chi(R, u-s)$ (where $R := \ln\rho$) of the entropy equation (2.4). To this end, we rely on symmetry group arguments for the equation (2.4). We find that it has an invariant solution

$$\eta(u,\rho) = \sqrt{\rho}\, f(u^2 - \ln^2\rho), \qquad \text{where} \qquad \xi f''(\xi) + f'(\xi) + \frac{1}{16} f(\xi) = 0.$$

Then we define

$$\chi = e^{R/2} f(|u-s|^2 - R^2)\,\mathbf{1}_{|u-s|<|R|}.$$

The function $f(\xi)$ can be represented by a Bessel function of zero index.

Step 5. Then we search for the entropy pairs in the form

$$\eta = \int \chi(R, u-s)\,\psi(s)\,ds, \qquad q = \int \sigma(R,u,s)\,ds, \tag{2.12}$$

where $\psi \in L^1(\mathbb{R})$ is arbitrary and we describe properties of the kernels χ, σ. In particular, we find that $\sigma = u\,\chi(R, u-s) + h(R, u-s)$, where the function h is given by an explicit formula. We also will show that

$$P\chi := \partial_s\chi = e^{R/2}\left(\delta_{s=u-|R|} - \delta_{s=u+|R|}\right) + G^\chi(R, u-s)\,\mathbf{1}_{|u-s|<|R|},$$

$$Ph := \partial_s h = e^{R/2}\left(\delta_{s=u-|R|} + \delta_{s=u+|R|}\right) + G^h(R, u-s)\,\mathbf{1}_{|u-s|<|R|},$$

where $G^\chi(R, v)$ and $G^h(R, v)$ are bounded, continuous functions.

Step 6. Finally, we plug in the entropy pairs (2.12) in Tartar's commutation relations, but in the form derived by CHEN and LEFLOCH [4]. We arrive after cancellation of ψ at the following equality in $\mathcal{D}'(\mathbb{R})^3$:

$$\begin{aligned}&\langle \chi_1 P_2 h_2 - h_1 P_2 \chi_2 \rangle \langle P_3 \chi_3 \rangle + \langle h_1 P_3 \chi_3 - \chi_1 P_3 h_3 \rangle \langle P_2 \chi_2 \rangle \\ &\quad = -\langle P_3 h_3 P_2 \chi_2 - P_3 \chi_3 P_2 h_2 \rangle \langle \chi_1 \rangle,\end{aligned}$$

where the notation $g_i := g(R, u, s_i)$ and $P_i g_i := \partial_{s_i} g(R, u, s_i)$ is used. Then we test this equality with the function

$$\frac{1}{\delta^2} \psi(s_1) \varphi_2 (\frac{s_1 - s_2}{\delta}) \varphi_3 (\frac{s_1 - s_3}{\delta}),$$

where $\psi \in \mathcal{D}(\mathbb{R})$ and φ_j are molifiers such that

$$Y := \int_{-\infty}^{\infty} \int_{-\infty}^{s_2} \big(\varphi_2(s_2)\, \varphi_3(s_3) - \varphi_3(s_2)\, \varphi_2(s_3)\big)\, ds_2 ds_3 \neq 0.$$

This identity involves products of measures by functions of bounded variation. Such products were earlier discussed by DAL MASO, LEFLOCH, & MURAT [7].

By letting δ go to zero we obtain the equalities

$$Y \iint_{W,Z} D(\rho)\rho \iint_{\{W'<W\}\cap\{Z'<1/W\}} \sqrt{\rho'} d\nu(W', Z') d\nu(W, Z) = 0, \quad (2.13)$$

$$Y \iint_{W,Z} D(\rho)\rho \iint_{\{W'<1/Z\}\cap\{Z'<Z\}} \sqrt{\rho'} d\nu(W', Z') d\nu(W, Z) = 0, \quad (2.14)$$

where

$$\rho = (WZ)^{1/2}, \quad D(\rho) = \sqrt{\rho} \left(-\frac{1}{2} + \frac{15}{8} \ln \frac{1}{\rho} \right), \quad \rho' = (W'Z')^{1/2},$$

and the measure $d\nu(W', Z')$ is a copy of $d\nu$ on the (W', Z')-plane. At this point we choose the constant ρ_2 (see Step 1) small enough to ensure the inequality $D(\rho) \geqq \sqrt{\rho}/2$. Hence, it follows from (2.13), (2.14) that $d\nu_{x,t} = \alpha \delta_P + \mu_{x,t}$ and $\alpha(1 - \alpha) = 0$, where $P(x, t)$ is a point on the (W, Z)-plane and the support of the measure supp $\mu_{x,t}$ lies in the set$\{\rho = 0\}$. This representation formula for the measure $\nu_{x,t}$ enables us to justify the passage to the limit as $\varepsilon \downarrow 0$. We summarize Step 6 in the following key result.

Theorem 2.4. *Let* (m_n, ρ_n) *be a bounded in* $L^\infty(\Pi)$ *sequence of entropy solutions of the problem* (2.1) *and such that*

$$0 \leqq \rho_n, \quad |m_n| \leqq c\rho_n (1 + |\ln \rho_n|)$$

uniformly in n. Then, passing to a subsequence if necessary, (m_n, ρ_n) *converges almost everywhere in* Π *to an entropy solution* (m, ρ) *of* (2.1).

Remark 2.5. A recent paper by HUANG & WANG [12] claims to establish the existence and compactness of weak solutions to (1.1)–(1.3), with initial data allowing for vacuum states. Their proof however is not complete as it stands, for the following reason. They make use of the entropies

$$\eta_* = \rho^{-\gamma}(e^{\beta u} + e^{-\beta u}), \quad \gamma = \frac{4\alpha^2}{4\alpha+1}, \quad \beta = \frac{\alpha(4\alpha+2)}{4\alpha+1}, \quad 0 < \alpha < \frac{1}{4}.$$

and claim that $\eta_* \in L(d\nu_{t,x})$. The exact meaning of this inclusion is the following limit relationship (for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$):

$$\lim_{\varepsilon\to 0} \iint_\Pi \eta_*(m^\varepsilon, \rho^\varepsilon)\varphi(x,t)\,dxdt = \iint_\Pi \varphi(x,t) \int_{m,\rho} \eta_*(m,\rho)\,d\nu_{x,t}\,dxdt \tag{2.15}$$

for the Young measures $\nu_{x,t}$ associated with the weakly convergent sequence $(m^\varepsilon, \rho^\varepsilon)$, where $(m^\varepsilon, \rho^\varepsilon)$ is either a sequence of solutions to a viscous approximation problem or a sequence of bounded entropy solutions. But the uniform *a priori* estimates

$$0 \leqq \rho^\varepsilon \leqq c, \quad |m^\varepsilon| \leqq c$$

do not imply both that the function $\eta_*(m^\varepsilon, \rho^\varepsilon)$ is bounded uniformly with respect to ε in $L^1_{\text{loc}}(\Pi)$ and that the limit in the left-hand side of (2.15) exists.

The following example shows that the above strong entropies cannot be used in the proof of the compactness result. Consider the bounded sequence of constant entropy solutions

$$\rho^\varepsilon = \varepsilon, \quad m^\varepsilon = \varepsilon \ln \varepsilon, \quad \varepsilon \to 0.$$

Clearly the limit in the left-hand side of (2.15) is equal to ∞. On the other hand, the method in [12] can probably be applied when the vacuum is avoided, that is, under the extra assumption $\rho^\varepsilon \geqq c = \text{const} > 0$.

3. Vanishing viscosity method

Given parameters $\varepsilon, \varepsilon_1 > 0$ we consider the Cauchy problem

$$\rho_t + (\rho u)_x = \varepsilon \rho_{xx} + 2\varepsilon_1 u_x, \tag{3.1}$$

$$(\rho u)_t + (\rho u^2)_x + \rho_x = \varepsilon(\rho u)_{xx} + \varepsilon_1 (u^2)_x + 2\varepsilon_1 (\ln \rho)_x, \tag{3.2}$$

with initial condition

$$\rho|_{t=0} = \rho_0^\varepsilon + 2\,\varepsilon_1, \qquad u|_{t=0} = u_0^\varepsilon. \tag{3.3}$$

In this section we establish the existence of smooth solutions to this problem. Later in this section we will assume that $\varepsilon_1 = \varepsilon^r$ for some $r > 1$. The positivity of the density will be obtained by the following argument.

Lemma 3.1 (Positivity for convection-diffusion equations)**.** *If $v = v(x,t)$ is a smooth bounded solution of the Cauchy problem*

$$v_t + (u\,v)_x = \varepsilon v_{xx}, \qquad v|_{t=0} = v_0(x), \tag{3.4}$$

where $u = u(x,t) \in L^\infty(\Pi)$ and $u_0 \in L^\infty(\mathbb{R})$, then $v \geqq 0$ provided $v_0 \geqq 0$.

Proof. Given $R > 0$, let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be a non-increasing function of class C^2 such that $\psi(x) = 1$ for $x \in [0, R]$, $\psi(x) = e^{-x}$ for $x \geqq 2R$, and $\psi(x)$ is a non-negative polynomial for $R \leqq x \leqq 2R$. Define $\Psi(x) = \psi(|x|)$ for $x \in \mathbb{R}$. Clearly,

$$|\Psi^{'}(x)| \leqq \frac{c_1}{R}\Psi(x), \quad |\Psi^{''}(x)| \leqq \frac{c_1}{R^2}\Psi(x) \tag{3.5}$$

for some constant $c_1 > 0$. The map

$$U_\mu(v) = \begin{cases} \sqrt{v^2+\mu^2} - \mu, & v \leqq 0, \\ 0, & v > 0, \end{cases}$$

is a regularization of the mapping $v \mapsto v_- := \max\{-v, 0\}$.

Using (3.4) and (3.5) we can compute the t-derivative of the integral $\int \Psi U_\mu(v)\,dx$:

$$\begin{aligned} &\frac{d}{dt}\int \Psi U_\mu dx + \varepsilon \int \Psi v_x^2 \frac{\partial^2 U_\mu}{\partial v^2} dx \\ &= \int \frac{\partial^2 U_\mu}{\partial v^2} v v_x (\varepsilon \Psi_x + u\Psi) dx + \int v \frac{\partial U_\mu}{\partial v}(\varepsilon \Psi_{xx} + u\Psi_x) dx \\ &\leqq \int \frac{\partial^2 U_\mu}{\partial v^2} v|v_x|\Psi(\varepsilon c_1/R + |u|) dx \\ &\quad + \int v \left|\frac{\partial U_\mu}{\partial v}\right| \Psi(\varepsilon c_1/R^2 + |u| c_1/R) dx. \end{aligned} \tag{3.6}$$

Observe that

$$\varepsilon v_x^2 - v|v_x|(\varepsilon c_1/R + |u|) = \varepsilon\left(|v_x| - v\left(\frac{c_1}{2R} + \frac{|u|}{2\varepsilon}\right)\right)^2 - v^2\left(\frac{c_1}{2R} + \frac{|u|}{2\varepsilon}\right)^2,$$

$$v^2 \frac{\partial^2 U_\mu}{\partial v^2} \leqq \frac{\mu^2 v^2}{(v^2+\mu^2)^{3/2}},$$

and

$$v\frac{\partial U_\mu}{\partial v} \to v_- \quad \text{as} \quad \mu \to 0.$$

We integrate (3.6) with respect to t and let μ tend to zero, by taking into account that $U_\mu(v_0) = 0$:

$$\int \Psi v_-\, dx \leqq \int_0^t \int \Psi v_- \left(\frac{\varepsilon\, c_1}{R^2} + \frac{|u|\, c_1}{R}\right) dx d\tau.$$

By Gronwall's lemma, $\int \Psi v_-\, dx = 0$. We thus conclude that $v \geqq 0$. □

As a consequence of Lemma 3.1, we deduce that any bounded solution (u, ρ) of the problem (3.1)–(3.3) has the following property:

$$\rho \geqq 2\,\varepsilon_1 \quad \text{uniformly in } \varepsilon. \tag{3.7}$$

Namely, this is clear since the function $v = \rho - 2\,\varepsilon_1$ solves the problem

$$v_t + (u\,v)_x = \varepsilon\, v_{xx}, \qquad v|_{t=0} \geqq 0.$$

From now on, we assume that the initial data ρ_0^ε and u_0^ε belong to the Hölder space $H^{2+\beta}(\mathbb{R})$ for some $\beta \in (0,1)$ and satisfy

$$0 \leqq \rho_0^\varepsilon \leqq M, \qquad \|u_0^\varepsilon\|_\infty \leqq u_1,$$

and

$$u_0^\varepsilon \to u_0, \quad \rho_0^\varepsilon \to \rho_0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}),$$

where $u_1 := \|u_0\|_\infty$ and $M := \|\rho_0\|_\infty$.

Lemma 3.2. *Let* (u, ρ) *be a smooth bounded solution of the Cauchy problem* (3.1)–(3.3). *Then there exist positive constants* c_1, ρ_1, W_1, *and* Z_1 *such that*

$$\begin{gathered}
2\varepsilon_1 \leqq \rho \leqq \rho_1, \quad |m| := \rho|u| \leqq c_1\rho(1+|\ln\rho|) \leqq m_1,\\
\rho_1 := (2\varepsilon_1 + M)e^{u_1}, \quad m_1 := c_1 \sup_{0\leqq\rho\leqq\rho_1} \rho(1+|\ln\rho|),\\
0 \leqq W := \rho e^u \leqq W_1, \quad 0 \leqq Z := \rho e^{-u} \leqq Z_1,
\end{gathered} \tag{3.8}$$

uniformly in ε.

Proof. Passing to the Riemann invariant variables

$$w := u + \ln\rho, \quad z := u - \ln\rho,$$

we can rewrite the system (3.1), (3.2) as

$$\begin{aligned}
w_t + w_x\left(u + 1 - \frac{2\varepsilon_1}{\rho} + \frac{\varepsilon z_x}{2} - \frac{3\varepsilon w_x}{4}\right) &= \varepsilon w_{xx} - \frac{\varepsilon z_x^2}{4},\\
z_t + z_x\left(u - 1 + \frac{2\varepsilon_1}{\rho} - \frac{\varepsilon w_x}{2} + \frac{3\varepsilon z_x}{4}\right) &= \varepsilon z_{xx} + \frac{\varepsilon w_x^2}{4}.
\end{aligned}$$

By the maximum principle,

$$w \leqq \max w_0(x), \quad z \geqq \min z_0(x).$$

Now, the estimates (3.8) are a simple consequence of these inequalities. □

By the estimates (3.8) there exist sequences W^{ε_n}, Z^{ε_n}, ρ^{ε_n}, and $m^{\varepsilon_n} := \rho^{\varepsilon_n} u^{\varepsilon_n}$ and a family of non-negative probability measures $\nu_{x,t}$, called Young measures, defined on the (W, Z)-plane, such that

$$W^{\varepsilon_n} \rightharpoonup W, \quad Z^{\varepsilon_n} \rightharpoonup Z, \quad \rho^{\varepsilon_n} \rightharpoonup \rho, \quad \rho^{\varepsilon_n} u^{\varepsilon_n} \rightharpoonup m \quad \text{weakly} \star \text{ in } L^\infty_{loc}(\Pi), \tag{3.9}$$

and

$$\iint_\Pi \Big(F(W^{\varepsilon_n}(x,t), Z^{\varepsilon_n}(x,t)) - \langle F \rangle \Big) \varphi(x,t)\, dt dx \to 0,$$

where we have set $\langle F \rangle := \int_{W,Z} F(W,Z) d\nu_{x,t}$ for any test function $\varphi \in \mathcal{D}(\mathbb{R}^2)$ and any continuous function $F(W,Z) \in C_{loc}(\mathbb{R}^2)$. Moreover,

$$\operatorname{supp} \nu_{x,t} \subset \{(W,Z) : 0 \leqq W \leqq W_1, \quad 0 \leqq Z \leqq Z_1\}.$$

For a proof that a Young measure $\mu_{x,t}$ can be associated with each bounded sequence $v_n(x,t)$, we refer to Tartar [32] and Ball [1]; see also [30].

Lemma 3.3 (Entropy dissipation estimate)**.** *The smooth solution* (u, ρ) *of the Cauchy problem* (3.1)–(3.3) *satisfies the estimate*

$$\left\| \frac{\varepsilon \rho_x^2}{\rho} + \varepsilon\, \rho u_x^2 \right\|_{L^1_{\mathrm{loc}}(\Pi)} \leqq c \tag{3.10}$$

uniformly in ε.

Proof. The identity

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(\frac{\rho u^2}{2} + (1 + \rho \ln \rho - \rho) \Big) + \frac{\varepsilon \rho_x^2}{\rho} + \varepsilon \rho u_x^2 \\ &\quad = -\frac{\partial}{\partial x} \left\{ \frac{\rho u^3}{2} + u\rho \ln \rho - \varepsilon \rho_x \ln \rho - 2\varepsilon_1 u \ln \rho - \varepsilon \left(\frac{\rho u^2}{2} \right)_x - \frac{\varepsilon_1 u^3}{3} \right\} \\ &\quad =: -J_x \end{aligned} \tag{3.11}$$

follows immediately from (3.1) and (3.2). Multiplying this identity by the function $\Psi(x)$ introduced in the proof of Lemma 3.1 and integrating with respect to x we deduce, in view of the estimates (3.7) and (3.8),

$$\int J \Psi_x \, dx \leqq \frac{1}{2} \int \Psi \left(\frac{\varepsilon \rho_x^2}{\rho} + \varepsilon \rho u_x^2 \right) dx + c \int \Psi \left(1 + \frac{\rho u^2}{2} \right) dx.$$

Hence, we have

$$\int_0^T \int \Psi (\frac{\varepsilon \rho_x^2}{\rho} + \varepsilon\, \rho u_x^2)\, dx dt \leqq c,$$

which yields the desired estimate. □

We rewrite the equations (3.1), (3.2) as a quasi-linear parabolic system:

$$u_t + a_1(u, \rho, u_x \rho_x) = \varepsilon\, u_{xx}, \quad \rho_t + a_2(u, \rho, u_x \rho_x) = \varepsilon\, \rho_{xx}, \tag{3.12}$$

where we have set

$$a_1 := u u_x - \frac{\rho_x}{\rho} - \frac{2\varepsilon \rho_x u_x}{\rho} - \frac{2\varepsilon_1 \rho_x}{\rho^2}, \qquad a_2 := (\rho u)_x - 2\varepsilon_1\, u_x.$$

In view of (3.7) and (3.8), we obtain the global *a priori* estimates

$$2\,\varepsilon_1 \leqq \rho \leqq \rho_1, \qquad |u| \leqq c(u_1, \rho_1, \varepsilon).$$

With these estimates at hand, it is a standard matter to derive estimates in Hölder's norms, depending on ε, by standard techniques of the theory of quasi-linear parabolic equations [13]. We will only sketch the derivation. Let $\zeta(x,t)$ be a smooth function such that $\zeta \neq 0$ only if $x \in \omega$, where ω is an interval $[x_0 - \sigma, x_0 + \sigma]$. Define

$$u^{(n)} := \max\{u - n, 0\}.$$

Multiplying the second equation in (3.12) by $\zeta^2 \rho^{(n)}$ and integrating with respect to x, we obtain

$$\frac{d}{dt}\int \zeta^2 |\rho^{(n)}|^2 dx + \varepsilon \int \zeta^2 |\rho_x^{(n)}|^2 dx \leqq \gamma \int (\zeta_x^2 + \zeta|\zeta_t|)|\rho^{(n)}|^2 dx + \gamma \int \zeta \mathbf{1}_{\rho \geqq n} dx.$$

Similarly, for the velocity variable we get

$$\begin{aligned} &\frac{d}{dt}\int \zeta^2 |u^{(n)}|^2\, dx + \varepsilon \int \zeta^2 |u_x^{(n)}|^2\, dx \\ &\quad \leqq \gamma \int (\zeta_x^2 + \zeta|\zeta_t|)|u^{(n)}|^2 + \zeta \mathbf{1}_{\rho \geqq n} + \varepsilon \zeta^2 |\rho_x^{(n)}|^2\, dx. \end{aligned}$$

These inequalities imply that u and ρ belong to a class $\mathcal{B}_2(Q, M, \gamma, r, \delta, n)$ [13] (Chapter II, Section 7, formula (7.5)), for some parameters Q, M, γ, r, δ, and n. Then it follows that the estimate

$$\|u, \rho\|_{H^{\alpha,\alpha/2}(\omega \times [0,T])} \leqq c$$

holds for some $\alpha \in (0, 1)$.

In the same manner, we can estimate the Hölder norm of the derivatives u_x, u_{xx}, u_t, ρ_x, ρ_{xx}, and ρ_t, in the same way as done in [11] for a general class of parabolic systems.

We now arrive at the main existence result, concerning the viscous approximation (3.1)–(3.3).

Lemma 3.4 (Existence of smooth solution of the regularized system)**.** *Let* u_0^ε, $\rho_0^\varepsilon \in L^\infty \cap H^\beta_{\text{loc}}$, $0 < \beta < 1$. *Then the Cauchy problem* (3.1)–(3.3) *has a unique solution such that*

$$u, \rho \in L^\infty(\Pi) \cap H^{2+\beta, 1+\beta/2}_{\text{loc}}(\Pi).$$

Now, we set $\varepsilon_1 = \varepsilon^r$, $r > 1$, and study compactness of the viscous solutions $(u^\varepsilon, \rho^\varepsilon)$ when $\varepsilon \to 0$.

Lemma 3.5. *Given an entropy/entropy-flux pair $(\eta(m, \rho), q(m, \rho))$, $m = \rho u$, the sequence*

$$\theta^\varepsilon := \frac{\partial \eta^\varepsilon}{\partial t} + \frac{\partial q^\varepsilon}{\partial x}$$

is compact in $W^{-1,2}_{\mathrm{loc}}(\Pi)$, where $\eta^\varepsilon = \eta(m^\varepsilon, \rho^\varepsilon)$, $q^\varepsilon = q(m^\varepsilon, \rho^\varepsilon)$.

Proof. We use the following lemma due to Murat [23].

Let $Q \subset \mathbb{R}^2$ be a bounded domain, $Q \in C^{1,1}$. Let A be a compact set in $W^{-1,2}(Q)$, B be a bounded set in the space of bounded Radon measures $M(Q)$, and C be a bounded set in $W^{-1,p}(Q)$ for some $p \in (2, \infty]$. Further, let $D \subset \mathcal{D}'(Q)$ be such that

$$D \subset (A + B) \cap C.$$

Then there exists E, a compact set in $W^{-1,2}(Q)$ such that $D \subset E$.

By definition, the functions $\eta(m, \rho)$ and $q(m, \rho)$ solve the system

$$q_m = \frac{2m}{\rho}\eta_m + \eta_\rho, \quad q_\rho = \eta_m - \frac{m^2}{\rho^2}\eta_m.$$

Hence, calculations show that

$$\begin{aligned}
\theta^\varepsilon &= 2\varepsilon_1 m_x \left(\frac{\eta^\varepsilon_\rho}{\rho} + \frac{m\eta^\varepsilon_m}{\rho^2}\right) + 2\varepsilon_1 \left(-\frac{m\eta^\varepsilon_\rho}{\rho^2} - \frac{m^2\eta^\varepsilon_m}{\rho^3} + \frac{\eta^\varepsilon_m}{\rho}\right) \\
&\quad + \varepsilon\eta^\varepsilon_\rho \rho_{xx} + \varepsilon\eta^\varepsilon_m m_{xx} \\
&= \varepsilon_1 u_x (q^\varepsilon_m + \eta^\varepsilon_\rho) - 2\varepsilon_1 \frac{\rho_x \eta^\varepsilon_m}{\rho} + \varepsilon\eta^\varepsilon_{xx} \\
&\quad - \varepsilon[\eta^\varepsilon_{\rho\rho}\rho_x^2 + \eta^\varepsilon_{mm} m_x^2 + 2\eta^\varepsilon_{\rho m}\rho_x m_x].
\end{aligned} \tag{3.13}$$

We check the conditions of Murat's lemma. By Lemma 3.2, the sequence θ^ε is bounded in $W^{-1,\infty}_{\mathrm{loc}}(\Pi)$. Hence, it is enough to show that $\varepsilon\eta^\varepsilon_x \to 0$ in $L^2_{\mathrm{loc}}(\Pi)$ and the residual sequence $\theta^\varepsilon - \varepsilon\eta^\varepsilon_{xx}$ is bounded in $L^1_{\mathrm{loc}}(\Pi)$.

We have

$$\varepsilon\eta^\varepsilon_x = \varepsilon\rho u_x \eta^\varepsilon_m + \varepsilon\rho_x \frac{q^\varepsilon_m + \eta^\varepsilon_\rho}{2}.$$

Thus, by estimates (3.8) and (3.10), $\varepsilon\eta^\varepsilon_x \to 0$ in L^2_{loc}.

Consider the sequence $\theta^\varepsilon - \varepsilon\eta^\varepsilon_{xx}$. We have

$$\theta^\varepsilon - \varepsilon\eta^\varepsilon_{xx} = -\varepsilon[\eta^\varepsilon_{\rho\rho}\rho_x^2 + \eta^\varepsilon_{mm}m_x^2 + 2\eta^\varepsilon_{\rho m}\rho_x m_x] + \varepsilon_1 u_x (q^\varepsilon_m + \eta^\varepsilon_\rho) - 2\varepsilon_1 \frac{\rho_x \eta^\varepsilon_m}{\rho}.$$

Each term on the right-hand side is bounded in L^1_{loc} provided $\varepsilon_1 = \varepsilon$. Indeed, by (3.7),

$$2\varepsilon_1|u_x| = \frac{2\varepsilon_1\rho^{1/2}|u_x|}{\rho^{1/2}} \leqq \sqrt{2\varepsilon}\rho^{1/2}|u_x|, \quad \frac{2\varepsilon_1|\rho_x|}{\rho} \leqq \frac{\sqrt{2\varepsilon}|\rho_x|}{\rho^{1/2}}.$$

Moreover, if $\varepsilon_1 = 0(\varepsilon)$,

$$\varepsilon_1 u_x(q_m^\varepsilon + \eta_\rho^\varepsilon) - 2\varepsilon_1 \frac{\rho_x \eta_m^\varepsilon}{\rho} \to 0 \quad \text{in} \quad L^2_{\text{loc}}(\Pi). \tag{3.14}$$

The other terms are treated similarly. This completes the proof. □

Given two entropy pairs $(\eta_i(m,\rho), q_i(m,\rho))$, $(i = 1, 2)$, from Lemma 3.5, we define

$$\tilde\eta_i(W,Z) = \eta_i(f_2(W,Z), f_1(W,Z)), \quad \tilde q_i(W,Z) = q_i(f_2(W,Z), f_1(W,Z)).$$

Clearly, the functions

$$\partial_t \tilde\eta_i^\varepsilon + \partial_x \tilde q_i^\varepsilon$$

are compact in $W^{-1,2}_{\text{loc}}(\Pi)$. Hence, by the div-curl lemma [32], Tartar's commutation relation

$$\langle \tilde\eta_1 \tilde q_2 - \tilde\eta_2 \tilde q_1 \rangle = \langle \tilde\eta_1 \rangle \langle \tilde q_2 \rangle - \langle \tilde\eta_2 \rangle \langle \tilde q_1 \rangle \tag{3.15}$$

is valid.

For their convenience, we remind readers that the div-curl lemma states the following.

Let $Q \subset \mathbb{R}^2$ be a bounded domain, $Q \in C^{1,1}$. Let

$$w_1^k \rightharpoonup w, \quad w_2^k \rightharpoonup w_2, \quad v_1^k \rightharpoonup v_1, \quad v_2^k \rightharpoonup v_2,$$

weakly in $L^2(Q)$, as $k \to \infty$. With $\operatorname{curl}(w_1, w_2)$ *denoting* $\partial w_2/\partial x_1 - \partial w_1/\partial x_2$, *suppose that the sequences* $\operatorname{div}(v_1^k, v_2^k)$ *and* $\operatorname{curl}(w_1^k, w_2^k)$ *lie in a compact subset E of $W^{-1,2}(Q)$. Then, for a subsequence,*

$$v_1^k w_1^k + v_2^k w_2^k \to v_1 w_1 + v_2 w_2 \quad \text{in } \mathcal{D}'(Q) \quad \text{as} \quad k \to \infty.$$

The further claim is due to the fact that system (1.1)–(1.3) is invariant with respect to the scaling $\rho \to \lambda\rho$.

Lemma 3.6. *If (m, ρ) is an entropy solution with initial data (m_0, ρ_0), then $(cm, c\rho)$ is also the entropy solution with the initial data $(cm_0, c\rho_0)$, where c is an arbitrary positive constant.*

Proof. The claim follows easily from the fact that the pair $(\eta(cm, c\rho), q(cm, c\rho))$ is an entropy/entropy-flux pair as soon as the pair $(\eta(m,\rho), q(m,\rho))$ is an entropy/entropy-flux pair. □

Given $\lambda > 0$, let us consider the auxiliary problem:

$$\rho_t + (\rho u)_x = \varepsilon \rho_{xx} + 2\varepsilon_2 u_x, \tag{3.16}$$

$$(\rho u)_t + (\rho u^2)_x + \rho_x = \varepsilon(\rho u)_{xx} + \varepsilon_2 (u^2)_x + 2\varepsilon_2 (\ln \rho)_x, \tag{3.17}$$

$$\rho|_{t=0} = \lambda \rho_0^\varepsilon(x) + 2\varepsilon_2, \quad u|_{t=0} = u_0^\varepsilon(x), \tag{3.18}$$

where $\varepsilon_2 = \lambda\varepsilon_1 = \lambda\varepsilon^r$.

The main feature of the auxiliary problem is the following. If the functions $(u_\varepsilon, \rho_\varepsilon)$ solve the problem (3.1)–(3.3) then the functions $(u_\varepsilon, \rho_\varepsilon^{'})$ solve the problem (3.16)–(3.18) with $\rho_\varepsilon^{'} = \lambda\rho_\varepsilon$.

The solution $(u_\varepsilon, \rho_\varepsilon)$ of problem (3.16)–(3.18) obeys the estimates

$$\begin{aligned} 2\varepsilon_2 \leqq \rho_\varepsilon &\leqq (2\varepsilon_2 + \lambda\|\rho_0\|_\infty)e^{\|u_0\|_\infty} =: \rho_2, \\ |u_\varepsilon \rho_\varepsilon| &\leqq c\rho_\varepsilon(1 + |\ln \rho_\varepsilon|) \end{aligned} \tag{3.19}$$

uniformly in ε. Lemmas 3.3–3.5 are also valid for $(u_\varepsilon, \rho_\varepsilon)$. The corresponding Young measure $\nu_{x,t}$ has a finite support:

$$\operatorname{supp} \nu_{x,t} \subset \{(W, Z) : 0 \leqq W \leqq W_2, \quad 0 \leqq Z \leqq Z_2\} := K. \tag{3.20}$$

We impose the following smallness conditions for λ:

$$\rho_2 < 1, \quad \ln \frac{1}{\rho_2} \geqq \frac{8}{15}. \tag{3.21}$$

Assume that the solution $(u_\varepsilon, \rho_\varepsilon)$ of the auxiliary problem converges to an entropy solution (m, ρ) of the problem (2.1):

$$(u_\varepsilon \rho_\varepsilon, \rho_\varepsilon) \to (m, \rho) \quad \text{almost everywhere in} \quad \Pi.$$

The initial data for (m, ρ) are

$$\rho|_{t=0} = \lambda\rho_0, \quad m|_{t=0} = \lambda m_0.$$

By Lemma 3.6, the functions $(m^{'}, \rho^{'}) = (m/\lambda, \rho/\lambda)$ are an entropy solution of the same problem with the initial data

$$\rho^{'}|_{t=0} = \rho_0, \quad m^{'}|_{t=0} = m_0.$$

Thus, it is enough to study convergence of the solutions to the auxiliary problem.

With the condition (3.21) at hand, the function

$$D(R) := \left(-\frac{1}{2} + \frac{15|R|}{8}\right) e^{R/2}, \quad R := \ln \rho,$$

from Section 5 admits the estimate $D(R) \geqq \frac{1}{2}e^{R/2}$. Hence, $D(R)$ vanishes only at the vacuum points $\rho = 0$.

To conclude the section, we remark that the parameter ε_1 serves as a regularizer for the hyperbolic system (1.1)–(1.3) with $\varepsilon = 0$ due to the estimate (3.7) (cf. [21]).

4. A large class of mathematical entropies

4.1. Symmetry group analysis

We already pointed out that a pair (η, q) is a mathematical entropy if and only if η satisfies

$$\eta_{\rho\rho} = \frac{1}{\rho^2}\eta_{uu}. \tag{4.1}$$

In order to derive an explicit formula for the weak entropies of the Euler system we rely on symmetry group analysis, following [31]. Using the Riemann invariants

$$w := u + \ln\rho, \quad z := u - \ln\rho,$$

we write the equation (4.1) for the entropies in the form

$$F(\eta_w, \eta_z, \eta_{wz}) := \eta_{wz} - A\,(\eta_z - \eta_w) = 0, \quad A := \frac{1}{4}. \tag{4.2}$$

In the more general case where A is a function of w and z, complete group analysis arguments were developed in OVSYANNIKOV's monograph [28]. In our case, A is a constant and this analysis is simpler. We only present the results of the formal derivation and we refer to [28] for further details on the theory.

We look for a one-parameter group determined by the infinitesimal operator

$$X = \xi(w, z, \eta)\frac{\partial}{\partial w} + \tau(w, z, \eta)\frac{\partial}{\partial z} + \varphi(w, z, \eta)\frac{\partial}{\partial \eta}.$$

Calculation of the first and the second prolongations of this operator yields

$$X^1 = X + \zeta^{\eta_w}\frac{\partial}{\partial \eta_w} + \zeta^{\eta_z}\frac{\partial}{\partial \eta_z}, \quad X^2 = X^1 + \zeta^{\eta_{ww}}\frac{\partial}{\partial \eta_{ww}} + \zeta^{\eta_{wz}}\frac{\partial}{\partial \eta_{wz}} + \zeta^{\eta_{zz}}\frac{\partial}{\partial \eta_{zz}},$$

where

$$\zeta^{\eta_w} = D_w\varphi - \eta_w D_w\xi - \eta_z D_w\tau, \quad D_w = \frac{\partial}{\partial w} + \eta_w\frac{\partial}{\partial \eta},$$
$$\zeta^{\eta_z} = D_z\varphi - \eta_w D_z\xi - \eta_z D_z\tau, \quad D_z = \frac{\partial}{\partial z} + \eta_z\frac{\partial}{\partial \eta},$$

and

$$\begin{aligned}\zeta^{\eta_{wz}} &= D_z\varphi_w + \eta_w D_z\varphi_\eta + \varphi_\eta\eta_{wz} - \eta_{ww}D_z\xi - \eta_{wz}(D_w\xi + D_z\tau)\\ &\quad -\eta_w(D_z\xi_w + \eta_w D_z\xi_\eta + \xi_\eta\eta_{wz}) - \eta_{zz}D_w\tau\\ &\quad -\eta_z(D_z\tau_w + \eta_w D_z\tau_\eta + \eta_{wz}\tau_\eta).\end{aligned}$$

Note that we need not calculate the coefficients $\zeta^{\eta_{ww}}$ and $\zeta^{\eta_{zz}}$. Application of the operator X^2 to F and analysis of this application on the manifold $F = 0$ enable us to conclude that (4.2) admits four one-dimensional groups G_i and one infinite-dimensional group G_5, associated with the infinitesimal operators

$$\frac{\partial}{\partial w}, \quad \frac{\partial}{\partial z}, \quad \eta\frac{\partial}{\partial \eta}, \quad E := w\frac{\partial}{\partial w} - z\frac{\partial}{\partial z} + A(w+z)\eta\frac{\partial}{\partial \eta}, \quad \beta(w,z)\frac{\partial}{\partial \eta},$$

where β is a solution to (4.2). The fact that (4.2) admits the group G_i means the following: if $\eta(w,z)$ solves (4.2), then for any $c, \xi \in R$ the following functions are solutions of (4.2) as well:

$$\begin{aligned}&\eta(w+c,z), \quad \eta(w,z+c), \quad c\eta(w,z),\\ &\eta(e^{-\xi}w, e^{\xi}z)\exp\left(Aw(1-e^{-\xi}) - Az(1-e^{\xi})\right), \quad \eta(w,z)+\beta(w,z).\end{aligned}$$

Note that, once this assertion is obtained, its validity can be checked directly without referring to group analysis.

Let us find an invariant solution to (4.2), by applying the one-dimensional group G_4 associated with the infinitesimal operator E. First, we look for invariants $I(w,z,\eta)$ of the group G_4 as solutions of the equation $E(I) = 0$. By the method of characteristics, we derive the system of ordinary differential equations

$$\frac{dw}{w} = -\frac{dz}{z} = \frac{d\eta}{A(w+z)}$$

and obtain easily the following two invariants:

$$I_1 = wz, \quad I_2 = \eta e^{-A(w-z)}.$$

Then, we look for an invariant solution of (4.2) in the form (see again [28]) $I_2 = f(I_1)$, or equivalently

$$\eta = e^{A(w-z)} f(wz). \tag{4.3}$$

Plugging (4.3) in (4.2), we arrive at the following condition for the function $f(x)$:

$$xf''(x) + f'(x) + A^2 f(x) = 0. \tag{4.4}$$

In conclusion, equation (4.2) admits the solution

$$\eta = \rho^{1/2} f(u^2 - \ln^2 \rho),$$

where the function f satisfies equation (4.4).

4.2. Mathematical entropies

We search for entropies $\eta = \eta(\rho, u)$ having the form

$$\eta(\rho, u) = \rho^{1/2} f(u^2 - \ln^2 \rho).$$

It is straightforward to see that η solves the entropy equation (4.1) if and only if the function $f = f(m)$ is a solution of the ordinary differential equation

$$mf'' + f' + A^2 f = 0, \quad A^2 = \frac{1}{16}. \tag{4.5}$$

With the notation

$$R := \ln \rho$$

the entropy then takes the form

$$\eta = \eta(R, u) = e^{R/2} f(u^2 - R^2),$$

while the entropy equation (4.1) reads

$$\mathbf{L}(\eta) := \eta_{RR} - \eta_{uu} - \eta_R = 0. \tag{4.6}$$

One solution to the second-order equation (4.5) is given by the expansion series

$$f(m) := \sum_{n=0}^{\infty} (\frac{A^n}{n!})^2 (-1)^n m^n,$$

with

$$f(0) = 1, \quad f'(0) = -A^2, \qquad f(-y^2) = \sum_{n=0}^{\infty} \left(\frac{A^n y^n}{n!}\right)^2.$$

Observe that $f(m)$ can be represented by the Bessel function of zero order. Given any function $g : \mathbb{R} \to \mathbb{R}$, we introduce the notation

$$\overline{g}(m) = \begin{cases} g(m), & m \leqq 0, \\ 0, & m > 0. \end{cases}$$

In particular, this defines the function $\overline{f}$. We denote by δ the Dirac measure concentrated at the point $m = 0$ and, more generally, by $\delta_{m=a}$ the Dirac measure concentrated at the point a. We denote by $\mathcal{D}(\mathbb{R})$ the space of smooth functions with compact support and by $\mathcal{D}'(\mathbb{R})$ the space of distributions.

Lemma 4.1. *The function $\overline{f}$ solves the ordinary differential equation* (4.5) *in* $\mathcal{D}'(\mathbb{R})$.

Proof. Given a test function $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle m\,\overline{f}'', \varphi\rangle := \int_{\mathbb{R}} (m\varphi)''\,\overline{f}\,dm = \int_{-\infty}^{0} (m\varphi)''\,f\,dm = \langle f(0)\,\delta + \overline{m\,f''}, \varphi\rangle$$

and

$$\langle \overline{f}', \varphi\rangle = \langle -f(0)\,\delta + \overline{f'}, \varphi\rangle.$$

Hence, we find

$$\langle m\,\overline{f}'' + \overline{f}' + A^2\,\overline{f}, \varphi\rangle = \langle \overline{m\,f'' + f' + A^2\,f}, \varphi\rangle = 0. \qquad \square$$

Motivated by Lemma 4.1 we introduce the function

$$\begin{aligned}\chi(R,u) &:= e^{R/2}\,\overline{f}(u^2 - R^2) = e^{R/2}\,\mathbf{1}_{R^2-u^2\geqq 0}\,f(R^2-u^2)\\ &= e^{R/2}\sum_{n=0}^{\infty}\Big(\frac{A^n}{n!}\Big)^2\,(R^2-u^2)_+^n,\end{aligned} \tag{4.7}$$

where

$$\lambda_+ := \begin{cases}\lambda, & \lambda \geqq 0,\\ 0, & \lambda < 0,\end{cases}$$

and $\mathbf{1}_{g\geqq 0}$ denotes the characteristic function of the set $\{g \geqq 0\}$.

Theorem 4.2 (Existence of the entropy kernel**).** *The function* χ *defined by* (4.7) *is a fundamental solution of the equation* (4.6) *in* $\mathcal{D}'(\mathbb{R}^2)$. *More precisely,* $\mathbf{L}(\chi) = 4\,\delta_{(R,u)=(0,0)}$.

Proof. From the definition (4.7) of χ and since the multiplicative factor $e^{R/2}$ is a smooth function, it is straightforward to obtain

$$\mathbf{L}(\chi) = e^{R/2}\left(\overline{f}_{RR} - \overline{f}_{uu} - \frac{\overline{f}}{4}\right)$$

in the sense of distributions. Note that, throughout the calculation, $f = f(u^2 - R^2)$ and that the variables u and R describe $\mathbb{R}$. We compute each term in the right-hand side of the above identity successively. We have first

$$\begin{aligned}\langle \overline{f}_{RR}, \varphi\rangle = \langle \overline{f}, \varphi_{RR}\rangle &= \iint_{u^2-R^2\leqq 0} f\,\varphi_{RR}\,du\,dR\\ &= \iint_{|R|>|u|} f\,\varphi_{RR}\,dR\,du\\ &= \iint_{|R|>|u|} (\varphi_R\,f)_R + 2R\,f'\,\varphi_R\,dR\,du.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\langle \overline{f}_{RR}, \varphi\rangle &= f(0) \int_{\mathbb{R}} \big(\varphi_R(-|u|, u) - \varphi_R(|u|, u)\big)\, du \\ &\quad + \iint_{|R|>|u|} \big((2R\,\varphi f')_R - \varphi\,(2f' - 4R^2\, f'')\big)\, dR du \\ &= f(0) \int_{\mathbb{R}} \big(\varphi_R(-|u|, u) - \varphi_R(|u|, u)\big)\, du \\ &\quad -2f'(0) \int_{\mathbb{R}} \big(\varphi(-|u|, u)\,|u| + \varphi(|u|, u)\,|u|\big)\, du \\ &\quad + \iint_{|R|>|u|} \varphi\,(4R^2\, f'' - 2f')\, dR du.\end{aligned}$$

Thus, we have established that

$$\overline{f}_{RR} = \overline{4R^2\, f'' - 2\, f'} + J_1, \tag{4.8}$$

where J_1 is the distribution defined by

$$\begin{aligned}\langle J_1, \varphi\rangle &= f(0)\Big(\int_{-\infty}^{0} \big(\varphi_R(u, u) - \varphi_R(-u, u)\big)\, du \\ &\quad + \int_0^{+\infty} \big(\varphi_R(-u, u) - \varphi_R(u, u)\big)\, du\Big) \\ &\quad + 2\, f'(0)\Big(\int_{-\infty}^{0} \big(u\,\varphi(u, u) + u\,\varphi(-u, u)\big)\, du \\ &\quad - \int_0^{+\infty} \big(u\,\varphi(-u, u) + u\,\varphi(u, u)\big)\, du\Big).\end{aligned}$$

The derivative in u is determined in a similar fashion. We get

$$\begin{aligned}\langle \overline{f}_{uu}, \varphi\rangle &= \iint_{|u|<|R|} f\,\varphi_{uu}\, du dR \\ &= \int_{-\infty}^{+\infty} \int_{-|R|}^{+|R|} \big((\varphi_u f)_u - 2u\,\varphi_u\, f'\big)\, du dR \\ &= f(0) \int_{-\infty}^{+\infty} \big(\varphi_u(R, |R|) - \varphi_u(R, -|R|)\big)\, dR \\ &\quad + \iint_{|u|<|R|} \big(\varphi\,(2f' + 4u^2\, f'') - 2\,(u\, f'\,\varphi)_u\big)\, du dR\end{aligned}$$

and thus

$$\overline{f}_{uu} = \overline{2f' + 4u^2\, f''} + J_2, \tag{4.9}$$

where the distribution J_2 is given by

$$\begin{aligned}\langle J_2, \varphi\rangle = f(0)\Big(\int_{-\infty}^{0} &\varphi_u(R,-R) - \varphi_u(R,R)dR \\ &+ \int_0^{+\infty} \varphi_u(R,R) - \varphi_u(R,-R)dR\Big) \\ &+2f'(0)\Big(\int_{-\infty}^{0}\big(R\,\varphi(R,-R) + R\,\varphi(R,R)\big)\,dR \\ &- \int_0^{+\infty}\big(R\,\varphi(R,R) + R\,\varphi(R,-R)\big)\,dR\Big).\end{aligned}$$

Now, since the function f satisfies the differential equation (4.5) it follows from (4.8), (4.9) that

$$\overline{f}_{RR} - \overline{f}_{uu} - \overline{f}/4 = J_1 - J_2.$$

To conclude, we observe that

$$\begin{aligned}\int_{-\infty}^{0} u\,\varphi(-u,u)\,du &= -\int_0^{\infty} R\,\varphi(R,-R)\,dR,\\ \int_0^{\infty} u\,\varphi(-u,u)\,du &= -\int_{-\infty}^{0} R\,\varphi(R,-R)dR,\\ \frac{d}{dR}\varphi(R,R) &= \varphi_u(R,R) + \varphi_R(R,R),\\ \frac{d}{dR}\varphi(R,-R) &= -\varphi_u(R,-R) + \varphi_R(R,-R),\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{0} \varphi_u(R,R)\,dR &= \varphi(0) - \int_{-\infty}^{0} \varphi_R(R,R)\,dR,\\ \int_0^{\infty} \varphi_u(R,R)\,dR &= -\varphi(0) - \int_0^{\infty} \varphi_R(R,R)\,dR,\\ \int_{-\infty}^{0} \varphi_u(R,-R)\,dR &= -\varphi(0) + \int_{-\infty}^{0} \varphi_R(R,-R)\,dR,\\ \int_0^{\infty} \varphi_u(R,-R)\,dR &= \varphi(0) + \int_0^{\infty} \varphi_R(R,-R)\,dR.\end{aligned}$$

We find that

$$\langle J_1 - J_2, \varphi\rangle = 4f(0)\,\varphi(0).$$

Since $f(0) = 1$ and $e^{R/2} = 1$ when $R = 0$, this completes the proof of Lemma 4.2. □

Lemma 4.3. *The kernel χ vanishes on the vacuum*

$$\lim_{R\to-\infty} \chi(R,u) = 0 \textit{ for every } u,$$

and, at the origin $R = 0$, satisfies

$$\lim_{R\to 0} \chi(R,\cdot) = 0, \qquad \lim_{R\to 0\pm} \chi_R(R,\cdot) \to \pm 2\,\delta_{u=0} \tag{4.10}$$

in the distributional sense in u. Moreover, for any fixed R, χ has a compact support, precisely

$$\chi(R,u) = 0, \qquad |u| > R.$$

It is smooth everywhere except along the boundary of its support where it has a jump of strength $\pm e^{R/2}$.

Proof. Detailed behavior of χ as $R \to -\infty$ can be derived from the asymptotic formula [25]

$$\sum_0^\infty \left(\frac{x^n}{n!}\right)^2 = \frac{e^{2x}}{2\sqrt{\pi x}} \left(1 + O\left(\frac{1}{x}\right)\right) \quad \text{when } x \uparrow \infty.$$

It follows that

$$\chi(R,u) = \mathbf{1}_{R^2-u^2 \geqq 0} \frac{e^{(-|R|+\sqrt{R^2-u^2})/2}}{\sqrt{\pi}(R^2-u^2)^{1/4}} \left(1 + O\left(\frac{1}{\sqrt{R^2-u^2}}\right)\right) \text{ when } R \downarrow -\infty.$$

Next, given $\varphi = \varphi(u)$, $\psi = \psi(R) \in \mathcal{D}(\mathbb{R})$ we have

$$\begin{aligned}\langle \chi_R, \varphi\,\psi\rangle &= -\int_{\mathbb{R}} \varphi(u) \int_{|R|>|u|} e^{R/2}\, f(m)\, \psi_R\, dR du \\ &= -J + \int_{|R|>|u|} \varphi\,\psi\, e^{R/2} \left(\frac{f}{2} - 2R\, f'\right) du dR,\end{aligned}$$

where

$$\begin{aligned} J &= \int_{\mathbb{R}} \varphi(u) \Big\{ \int_{-\infty}^{-|u|} + \int_{|u|}^{\infty} \Big\} (e^{R/2}\, f\psi)_R\, dR du \\ &= \int_{\mathbb{R}} \varphi(u) \left(e^{-|u|/2}\, \psi(-|u|) - e^{|u|/2}\, \psi(|u|)\right) du.\end{aligned}$$

We calculate

$$\begin{aligned}\int_{\mathbb{R}} \varphi(u)\, e^{-|u|/2}\, \psi(-|u|)\, du &= \int_{\mathbb{R}} e^{R/2} \psi(R) \left(\varphi(R) + \varphi(-R)\right) \mathbf{1}_{R<0}\, dR, \\ \int_{\mathbb{R}} \varphi(u)\, e^{|u|/2}\, \psi(|u|)\, du &= \int_{\mathbb{R}} e^{R/2} \psi(R) \left(\varphi(R) + \varphi(-R)\right) \mathbf{1}_{R>0}\, dR.\end{aligned}$$

It follows that, for each R, χ_R is a distribution in the variable u, given by the formula

$$\begin{aligned}\langle \chi_R(R,\cdot), \varphi(u)\rangle = &\int_{|u|<|R|} \varphi(u)\, e^{R/2} \left(\frac{f(u^2-R^2)}{2} - 2R\, f'(u^2-R^2)\right) du \\ &+ e^{R/2} \big(\varphi(R) + \varphi(-R)\big) \big(\mathbf{1}_{R>0} - \mathbf{1}_{R<0}\big).\end{aligned}$$

This completes the proof of (4.10) and thus the proof of Lemma 4.3. □

Since (4.6) is invariant under the transformations $u \mapsto u - s$ for every constant s, we deduce immediately from Lemma 4.2 that, for every $s \in \mathbb{R}$, the function

$$\chi(R, u - s) = e^{R/2}\, \overline{f}(|u - s|^2 - R^2)$$

satisfies the partial differential equation

$$\mathbf{L}(\chi)(R, u - s) = 4\, \delta_{(R,u)=(0,s)} \tag{4.11}$$

in $\mathcal{D}'(\mathbb{R}^2)$. We arrive at:

Theorem 4.4 (The class of weak entropies to the isothermal Euler equations)**.** *Restrict attention to the region* $R < 0$ (*respectively,* $R > 0$)*. The formula*

$$\eta(R, u) = \int_{\mathbb{R}} \chi(R, u - s)\, \psi(s)\, ds,$$

where ψ *is an arbitrary function in* $L^1(\mathbb{R})$*, describes the class of all weak entropies to the Euler equations for isothermal fluids* (1.1)–(1.3)*. In particular, for all* $u \in \mathbb{R}$*,*

$$\lim_{R\to 0} \eta(R, u) = 0, \qquad \lim_{R\to 0\pm} \eta_R(R, u) = \pm 2\, \psi(u), \qquad \lim_{R\to -\infty} \eta(R, u) = 0. \tag{4.12}$$

Proof. It follows from (4.11) that, for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$,

$$\int_{\mathbb{R}} \mathbf{L}(\eta)\varphi\, dR du = 4 \int_{\mathbb{R}} \psi(s)\, \varphi(s, 0)\, ds,$$

which implies that

$$\mathbf{L}(\eta) = 0, \quad R \neq 0.$$

Since, for any fixed s, R, the fundamental solution $\chi(R, u - s)$ has a compact support in the variable u, we also have

$$\int_{\mathbb{R}} \chi(R, u - s)\, \psi(s)\, ds \to 0, \qquad R \to 0. \qquad \square$$

4.3. Mathematical entropy-flux functions

We look for the entropy-flux kernel σ which should generate the class of entropy-flux functions q via the general formula

$$q(R, u) = \int_{\mathbb{R}} \sigma(R, u, s)\, \psi(s)\, ds.$$

In the variables (R, u), the system of equations characterizing the entropies

$$q_\rho = u\, \eta_\rho + \rho^{-1}\, \eta_u, \qquad q_u = \rho\, \eta_\rho + u\, \eta_u$$

reads, by setting $Q := q - u\, \eta$,

$$Q_R = \eta_u, \qquad Q_u = \eta_R - \eta. \tag{4.13}$$

It is clear that the entropy flux can be deduced from the entropy by integration in R and u. We focus attention on the region $0 \leqq \rho \leqq 1$, that is, $R \leqq 0$. We will use the notation

$$a \vee b := \max(a, b).$$

Theorem 4.5 (Entropy-flux kernel)**.** *The entropy-flux kernel has the form*

$$\sigma(R,u,s) = u\,\chi(R,u-s) + h(R,u-s),$$

where the function h admits the following representation formulas:

$$h = -\mathrm{sgn}(u-s) + \frac{\partial}{\partial u}\int_0^R \chi(r,u-s)\,dr,$$

or equivalently

$$h = \frac{\partial}{\partial s}H(|u-s|,R), \quad H = |u-s| + \int_{-(|R|\vee|u-s|)}^{-|u-s|} e^{r/2}\,f(|u-s|^2-r^2)\,dr, \tag{4.14}$$

or still

$$\begin{aligned} h = {} & \mathrm{sgn}(u-s)\left(e^{-|u-s|/2}\,\mathbf{1}_{|u-s|<|R|}-1\right) \\ & -2\int_{-(|R|\vee|u-s|)}^{-|u-s|}(u-s)\,e^{r/2}\,f'(|u-s|^2-r^2)\,dr. \end{aligned} \tag{4.15}$$

Proof. In view of (4.13) we can calculate any value $Q_* = Q(R_*,u_*)$ via an integral, as follows. We have

$$Q_* = \int_{l_*}\eta_u\,dR + (\eta_R-\eta)\,du, \quad l_* = l^- \cup l_0,$$

where l^- and l_0 are the curves in the (R,u)–plane given by

$$\begin{aligned} l^- &: R = 0, \qquad u = \lambda, \quad -\infty < \lambda < u_*, \\ l_0 &: R = \lambda R_*, \quad u = u_*, \quad 0 < \lambda < 1. \end{aligned}$$

It follows from (4.12) that

$$Q_* = -2\int_{-\infty}^{u_*}\psi(u)\,du + \int_0^{R_*}\eta_u(R,u_*)\,dR.$$

Replacing l^- by

$$l^+ : u = \lambda, \quad u_* < \lambda < \infty, \quad R = 0,$$

We obtain similarly

$$Q_* = -2\int_{\infty}^{u_*}\psi(u)du + \int_0^{R_*}\eta_u(u_*,R)dR.$$

Observe, that

$$\int_{-\infty}^{u_*}\psi(u)\,du + \int_{\infty}^{u_*}\psi(u)\,du = \int_{\mathbb{R}}\psi(u)\mathrm{sgn}(u_*-u)\,du.$$

Hence,

$$Q(R,u) = -\int_{\mathbb{R}} \psi(s)\mathrm{sgn}(u-s)\,ds + \int_{\mathbb{R}} \psi(s)\,\frac{\partial}{\partial u}\int_0^R \chi(r,u-s)\,drds,$$

Next, we have

$$\int_0^R \chi(r,u-s)\,dr = -\int_{-|R|}^0 e^{r/2} f(|u-s|^2-r^2)\mathbf{1}_{r<-|u-s|}\mathbf{1}_{r>-|R|}dr = -H_1,$$

where H_1 is the last integral in (4.14) and, therefore, the first formula is established.

To calculate

$$\frac{\partial}{\partial u}H_1 = \frac{\partial}{\partial u}\int_{-(|R|\vee|u-s|)}^{-|u-s|} e^{r/2}\, f(|u-s|^2-r^2)\,dr,$$

we observe that

$$\frac{\partial}{\partial u}(|R|\vee|u-s|) = \mathbf{1}_{|u-s|>|R|}\,\mathrm{sgn}(u-s).$$

Hence, we have

$$\begin{aligned}\frac{\partial}{\partial u}H_1 &= 2\int_{-(|R|\vee|u-s|)}^{-|u-s|}(u-s)e^{r/2}f'(|u-s|^2-r^2)dr - f(0)e^{-|u-s|/2}\mathrm{sgn}(u-s)\\ &\quad+\mathbf{1}_{|u-s|\geqq|R|}e^{-(|R|\vee|u-s|)/2}\, f(|u-s|^2-(|R|\vee|u-s|)^2)\,\mathrm{sgn}(u-s).\end{aligned}$$

The last term coincides with

$$\mathbf{1}_{|u-s|\geqq|R|}\, e^{-|u-s|/2}\,\mathrm{sgn}(u-s).$$

Thus, the representation formula (4.15) is proved and the proof of Theorem 4.5 is completed. □

4.4. Singularities of entropy and entropy-flux kernels

From the above results we see that the functions χ and h are continuous everywhere except along the boundary of their support, that is, the lines $u = s \pm |R|$. The most singular parts (measures and BV part) of the first-order derivatives of the functions χ and h with respect to the variable s are now computed.

Theorem 4.6 (Singularities of the entropy kernels)**.** *The derivatives χ_s and h_s in $\mathcal{D}'(\mathbb{R})$ are as follows:*

$$\chi_s = e^{R/2}\left(\delta_{s=u-|R|} - \delta_{s=u+|R|}\right) + G^\chi(R,u-s)\,\mathbf{1}_{|u-s|<|R|}, \qquad (4.16)$$

$$h_s = e^{R/2}\left(\delta_{s=u-|R|} + \delta_{s=u+|R|}\right) + G^h(R,u-s)\,\mathbf{1}_{|u-s|<|R|}, \qquad (4.17)$$

where, for all $|v| \leqq |R|$,

$$G^\chi(R,v) := 2e^{R/2}\, v\, f'(v^2-R^2)$$

and

$$G^h(R,v) := e^{-|v|/2}\,(1/2 - 2|v|) \\ +2\int_{-(|R|\vee|v|)}^{-|v|} \Big(e^{r/2}\, f'(v^2-r^2) + 2e^{r/2}\, v^2\, f''(v^2-r^2)\Big)\, dr.$$

It will be convenient to extend the functions G^χ and G^h by continuity outside the region $|v| < |R|$ by setting

$$G^\chi(R,v) = \begin{cases} 2|R|\, f'(0)\, e^{R/2}, & v \geqq |R|, \\ -2|R|\, f'(0)\, e^{R/2}, & v \leqq -|R|, \end{cases}$$

and

$$G^h(R,v) = e^{R/2}\,(2R + 1/2), \qquad |v| \geqq |R|.$$

Proof. Given a test function $\varphi = \varphi(s)$, we can write

$$\begin{aligned} \int_{\mathbb{R}} \chi\, \varphi'(s)\, ds &= e^{R/2} \int_{u-|R|}^{u+|R|} \varphi'(s)\, f(|u-s|^2 - R^2)\, ds \\ &= e^{R/2}\big(\varphi(u+|R|) - \varphi(u-|R|)\big) \\ &\quad + e^{R/2} \int_{u-|R|}^{u+|R|} 2\varphi f'(|u-s|^2 - R^2)(u-s)ds, \end{aligned}$$

which yields the first formula (4.16).

Next, it follows from (4.14) that

$$\begin{aligned} h_s = {} & \operatorname{sgn}(u-s)\, e^{-|u-s|/2} \Big(\delta_{s=u-|R|} - \delta_{s=u+|R|} + \frac{1}{2}\operatorname{sgn}(u-s)\mathbf{1}_{|u-s|<|R|}\Big) \\ & -2\int_{-(|R|\vee|u-s|)}^{-|u-s|} \frac{\partial}{\partial s}\Big((u-s)\, e^{r/2}\, f'(|u-s|^2 - r^2)\Big)\, dr \\ & +2\, f'(0)\, e^{-|u-s|/2}\,(u-s)\operatorname{sgn}(s-u) \\ & -2e^{-(|R|\vee|u-s|)/2}\mathbf{1}_{|u-s|\geqq|R|} f'\big(|u-s|^2 - (|R|\vee|u-s|)^2\big)(u-s)\operatorname{sgn}(u-s). \end{aligned}$$

The last term above coincides with

$$-2\, e^{-|u-s|/2}\, \mathbf{1}_{|u-s|\geqq|R|}\, f'(0)|u-s|$$

and, therefore, the second formula (4.17) is also established. □

5. Reduction of the support of the Young measure

5.1. Tartar's commutation relations

We now turn to investigating Tartar's commutation relation for Young measures, following the approach in CHEN & LEFLOCH [4, 5]. In the previous section we constructed the class of weak entropies η and entropy fluxes q in terms of the variables ρ and u. We can also express η and q as functions of the Riemann invariants W and Z, via the following change of variables:

$$\bar{\eta}(W, Z) := \eta(u, \rho), \quad \bar{q}(W, Z) = q(u, \rho),$$
$$W := \rho e^{u}, \quad Z := \rho e^{-u}.$$

To simplify notation, it is convenient to adopt the following convention. In the rest of this section we will write $\langle F \rangle = \int F(u, \rho)\, d\nu$ instead of $\int \bar{F}(W, Z)\, d\nu$, by assuming that ρ, u are the functions of the variables W, Z given by

$$\rho = (WZ)^{1/2}, \quad u = \frac{1}{2} \ln \frac{W}{Z}.$$

We will prove:

Theorem 5.1 (Reduction of the support of the Young measure). *Let $\nu = \nu(W, Z)$ be a probability measure with support included in the region*

$$\{(W, Z) : 0 \leqq W \leqq W_2, \quad 0 \leqq Z \leqq Z_2\}$$

and such that

$$\langle \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \eta_1 \rangle \langle q_2 \rangle - \langle \eta_2 \rangle \langle q_1 \rangle \tag{5.1}$$

(where $\langle F \rangle := \langle \nu, F \rangle$) for any two weak entropy pairs (η_1, q_1) and (η_2, q_2) of the Euler equations (1.1), (1.2). *Then, the support of ν in the (W, Z)-plane is either a single point or a subset of the vacuum line $\{\rho = 0\} = \{WZ = 0\}$.*

The proof of Theorem 5.1 will be based on *cancellation properties* associated with the entropy and entropy-flux pairs of systems of conservation laws. The key idea (going back to DIPERNA [10]) is that, nearby the diagonal $\{s_2 = s_3\}$, the function

$$E(\rho, v; s_2, s_3) := \chi(\rho, v - s_2)\, \sigma(\rho, v, s_3) - \chi(\rho, v - s_2)\, \sigma(\rho, v, s_3)$$

is much more regular than the kernels χ and σ themselves.

The principal scheme can be explained as follows. Given functions $\psi_i \in \mathcal{D}(\mathbb{R})$, $(i = 1, 2, 3)$, we define the entropy pairs

$$\eta_i(u, R) = \int \chi(u - s_i, R)\psi_i(s_i) ds_i, \quad q_i(u, R) = \int \sigma(u, R, s_i)\psi_i(s_i) ds_i,$$

and deduce from Tartar's relations (5.1) the following remarkable identity (see CHEN & LEFLOCH [4], as well as the earlier work [19])

$$\langle \eta_1 q_2 - q_1 \eta_2 \rangle \langle \eta_3 \rangle + \langle q_1 \eta_3 - \eta_1 q_3 \rangle \langle \eta_2 \rangle + \langle q_3 \eta_2 - \eta_3 q_2 \rangle \langle \eta_1 \rangle = 0.$$

Next, replacing $\psi_i(s_i)$ with $-\psi_i'(s_i) = -P_i\psi_i(s_i)$ and defining $F_i = F(u, R, s_i)$, we arrive, after cancellation of the arbitrary functions $\psi_i(s_i)$, at the equality

$$\begin{aligned}&\langle \chi_1 P_2 h_2 - h_1 P_2 \chi_2\rangle\langle P_3\chi_3\rangle + \langle h_1 P_3\chi_3 - \chi_1 P_3 h_3\rangle\langle P_2\chi_2\rangle\\ &\quad= -\langle P_3 h_3 P_2\chi_2 - P_3\chi_3 P_2 h_2\rangle\langle\chi_1\rangle,\end{aligned} \tag{5.2}$$

which is valid in $\mathcal{D}'(\mathbb{R})^3$. In view of the expression of the distributional derivative of σ and h (Theorem 4.6), each term in (5.2) can be calculated explicitly. Writing

$$w = u - |R|, \quad z = u + |R|,$$

we find

$$\begin{aligned}\chi_1 P_2 h_2 - h_1 P_2\chi_2 &= e^{R/2}(h_1 - \chi_1)\delta_{s_2=w} - e^{R/2}(h_1 + \chi_1)\delta_{s_2=z}\\ &\quad + (h_1 G_2^\chi - \chi_1 G_2^h)\mathbf{1}_{|u-s_2|<|R|}\end{aligned} \tag{5.3}$$

and, similarly,

$$\begin{aligned}\chi_1 P_3 h_3 - h_1 P_3\chi_3 &= e^{R/2}(h_1 - \chi_1)\delta_{s_3=w} - e^{R/2}(h_1 + \chi_1)\delta_{s_3=z}\\ &\quad + (h_1 G_3^\chi - \chi_1 G_3^h)\mathbf{1}_{|u-s_3|<|R|}.\end{aligned} \tag{5.4}$$

Moreover, we have

$$\begin{aligned}P_3\chi_3 P_2 h_2 - P_3 h_3 P_2\chi_2 &= 2e^R\Big(\delta_{s_2=z}\,\delta_{s_3=w} - \delta_{s_2=w}\,\delta_{s_3=z}\Big)\\ &+ e^{R/2}\Big(\delta_{s_2=w}(G_3^\chi - G_3^h) + \delta_{s_2=z}(G_3^\chi + G_3^h)\Big)\mathbf{1}_{|u-s_3|<|R|}\\ &+ e^{R/2}\Big(\delta_{s_3=w}(G_2^h - G_2^\chi) - \delta_{s_3=z}(G_2^h + G_2^\chi)\Big)\mathbf{1}_{|u-s_2|<|R|}\\ &+ (G_3^\chi G_2^h - G_2^\chi G_3^h)\mathbf{1}_{|u-s_2|<|R|}\mathbf{1}_{|u-s_3|<|R|}.\end{aligned}$$

In view of the formulas (5.3) and (5.4) the right-hand side of (5.2) contains products of functions with bounded variation (involving σ and h) and Dirac masses plus smoother terms. Such products were earlier discussed by DAL MASO, LEFLOCH & MURAT [7]. On the other hand, the right-hand side of (5.2) is *more singular* and involves products of measures, product of BV functions by measures, and smoother contributions; see (5.2). Our calculation below will show that the left-hand side of (5.2) tends to zero in the sense of distributions if $s_2 \to s_1$ and $s_3 \to s_1$, while the right-hand side tends to a (possibly) non-trivial limit.

We test the equality (5.2) with the function

$$\psi(s)\,\varphi_2^\varepsilon(s - s_2)\,\varphi_3^\varepsilon(s - s_3) := \psi(s)\,\frac{1}{\varepsilon^2}\,\varphi_2\left(\frac{s - s_2}{\varepsilon}\right)\varphi_3\left(\frac{s - s_3}{\varepsilon}\right) \tag{5.5}$$

of the variables $s = s_1, s_2, s_3$, where $\psi \in \mathcal{D}(\mathbb{R})$ and $\varphi_j : \mathbb{R} \to \mathbb{R}$ is a molifier such that

$$\varphi_j(s_j) \geqq 0, \qquad \int_{\mathbb{R}} \varphi_j(s_j)\,ds_j = 1, \qquad \text{supp } \varphi_j(s_j) \subset (-1, 1).$$

5.2. Nonconservative products

To provide testing of equality (5.2) by the function (5.5), we will need the following technical observations.

Lemma 5.2. *Let $\psi, F : \mathbb{R} \to \mathbb{R}$ and $f : [a', b'] \to \mathbb{R}$ be continuous functions. Then, for every interval $[a, b] \subset \mathbb{R}$, the integral*

$$I^{\varepsilon}(a, b, a', b') := \int_{a'}^{b'} \psi(s_1)\, f(s_1)\, \varphi_2^{\varepsilon}(s_1 - a) \int_a^b F(s_3)\, \varphi_3^{\varepsilon}(s_1 - s_3)\, ds_3 ds_1$$

has the limit, when $\varepsilon \to 0$,

$$\psi(a)\, F(a) \left(A_{2,3}^{-}\, f(a)\, \mathbf{1}_{a'<a<b'} + B_{2,3}^{-}\, f(a'+)\, \mathbf{1}_{a=a'} + C_{2,3}^{-}\, f(b'-)\, \mathbf{1}_{a=b'} \right),$$

where $A_{2,3}^{-} := B_{2,3}^{-} + C_{2,3}^{-}$ and the coefficients B^{-} and C^{-} depend only on the mollifying functions:

$$B_{2,3}^{-} := \int_0^{\infty} \int_{-\infty}^{y_1} \varphi_2(y_1)\varphi_3(y_3) dy_3 dy_1,$$
$$C_{2,3}^{-} := \int_{-\infty}^{0} \int_{-\infty}^{y_1} \varphi_2(y_1)\varphi_3(y_3) dy_3 dy_1.$$

Formally the integral I^{ε} has the form

$$I(a, b, a', b') := \int_{s_1=a'}^{b'} \psi(s_1)\, f(s_1)\, \delta_{s_1=a} \int_{s_3=a}^{b} F(s_3)\, \delta_{s_1=s_3}.$$

Lemma 5.2 shows that this term cannot be defined in a classical manner and that, by regularization of the Dirac masses, different limits may be obtained, depending on the choice of the mollifying functions.

Similarly we have

Lemma 5.3. *Let $\psi, F : \mathbb{R} \to \mathbb{R}$ and $f : [a', b'] \to \mathbb{R}$ be continuous functions. Then, for every interval $[a, b] \subset \mathbb{R}$, the integral*

$$J^{\varepsilon}(a, b, a', b') := \int_{a'}^{b'} \psi(s_1)\, f(s_1)\, \varphi_2^{\varepsilon}(s_1 - b) \int_a^b F(s_3)\, \varphi_3^{\varepsilon}(s_1 - s_3)\, ds_3 ds_1$$

has the following limit when $\varepsilon \to 0$:

$$\psi(b)\, F(b) \left(A_{2,3}^{+}\, f(b)\, \mathbf{1}_{a'<b<b'} + B_{2,3}^{+}\, f(a'+)\, \mathbf{1}_{a=a'} + C_{2,3}^{+}\, f(b'-)\, \mathbf{1}_{a=b'} \right),$$

where $A_{2,3}^{+} := B_{2,3}^{+} + C_{2,3}^{+}$ and

$$B_{2,3}^{+} := \int_0^{\infty} \int_{y_1}^{\infty} \varphi_2(y_1)\varphi_3(y_3) dy_3 dy_1,\; C_{2,3}^{+} := \int_{-\infty}^{0} \int_{y_1}^{\infty} \varphi_2(y_1)\varphi_3(y_3)\, dy_3 dy_1.$$

Along the same lines we have also:

Lemma 5.4. *Let $\psi, F : \mathbb{R} \to \mathbb{R}$ be continuous functions and let the function $f : \mathbb{R} \to \mathbb{R}$ be continuous everywhere except possibly at two points a and b with $a < b$. Then, for every real α, the integral*

$$K^\varepsilon(a,b,\alpha) := \int_{\mathbb{R}} \psi(s_1)\, f(s_1)\, \varphi_3^\varepsilon(s_1-\alpha) \int_a^b F(s_2)\, \varphi_2^\varepsilon(s_1-s_2)\, ds_2 ds_1$$

has the following limit when $\varepsilon \to 0$:

$$\psi(\alpha)\, F(\alpha) \Big(f(\alpha)\, \mathbf{1}_{a<\alpha<b} + \big(C_{2,3}^- \, f(a-) + B_{2,3}^- f(a+)\big)\, \mathbf{1}_{\alpha=a} \\ + \big(C_{2,3}^+ \, f(b-) + B_{2,3}^+ \, f(b+)\big)\, \mathbf{1}_{\alpha=b} \Big).$$

We only give the proof of this last statement. Lemma 5.2 and 5.3 can be checked similarly.

Proof. Making first the change of variables $s_2 = s_1 - \varepsilon y_2$ and then $s_1 = \varepsilon y_1 + \alpha$, we can write

$$K^\varepsilon = -\int_{\mathbb{R}} \psi(\varepsilon y_1 + \alpha) f(\varepsilon y_1 + \alpha)\varphi_3(y_1) \\ \int_{y_1+(\alpha-a)/\varepsilon}^{y_1+(\alpha-b)/\varepsilon} F(\varepsilon(y_1-y_2)+\alpha)\varphi_2(y_2) dy_2 dy_1.$$

Clearly, we have $K^\varepsilon \to 0$ when $\alpha < a$ or $\alpha > b$.

Now, if $\alpha = a$, we can write

$$K^\varepsilon = \sum_1^3 K_i^\varepsilon = -\left(\int_{-\infty}^a + \int_a^b + \int_b^\infty\right) \psi(s_1) f(s_1) \varphi_3^\varepsilon(s_1-a) \\ \int_{\frac{s_1-a}{\varepsilon}}^{\frac{s_1-b}{\varepsilon}} F(s_1-\varepsilon y_2)\varphi_2(y_2)\, dy_2 ds_1.$$

Consider the first term K_1^ε (obtained by taking $s_1 = \varepsilon y_1 + a$)

$$K_1^\varepsilon = -\int_{-\infty}^0 \psi(\varepsilon y_1 + a) f(\varepsilon y_1 + a)\varphi_3(y_1) \\ \int_{y_1}^{y_1 - \frac{b-a}{\varepsilon}} F(\varepsilon(y_1-y_2)+a)\varphi_2(y_2)\, dy_2 dy_1,$$

which satisfies

$$K_1^\varepsilon \to \psi(a)\, F(a) f(a-) \int_{-\infty}^0 \varphi_3(y_1) \int_{-\infty}^{y_1} \varphi_2(y_2)\, dy_2 dy_1.$$

Similarly, we see that

$$K_2^\varepsilon \to \psi(a)F(a)f(a+) \int_0^\infty \varphi_3(y_1) \int_{-\infty}^{y_1} \varphi_2(y_2) dy_2 dy_1, \quad K_3^\varepsilon \to 0.$$

The other values of α can be studied by the same arguments and this completes the proof of Lemma 5.4. $\square$

5.3. Proof of Theorem 5.1

Step 1. First, we consider the right-hand side of (5.2). Let us denote

$$d\nu := d\nu(W,Z), \quad d\nu' := d\nu(W',Z'), \quad w' = u' - |R'|, \quad z' = u' + |R'|.$$

Applying the distribution $\langle P_3h_3P_2\chi_2 - P_3\chi_3P_2h_2\rangle\langle\chi_1\rangle$ to the test function (5.5), we write the integral

$$\int_{\mathbb{R}^3} \langle P_3h_3(s_3)P_2\chi_2(s_2) - P_3\chi_3(s_3)P_2h_2(s_2)\rangle\langle\chi_1(s)\rangle\psi(s)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)ds ds_2 ds_3 \quad (5.6)$$

as the sum $\sum_1^4 I_i^\varepsilon$, in which, in view of Theorem 4.6, we can distinguish between products of Dirac measures

$$I_1^\varepsilon := \int \psi\langle\chi_1\rangle\Big\langle 2e^R\varphi_2^\varepsilon(s-z))\varphi_3^\varepsilon(s-w)) - 2e^R\varphi_2^\varepsilon(s-w))\varphi_3^\varepsilon(s-z))\Big\rangle ds,$$

products of Dirac measure by functions with bounded variation

$$\begin{aligned} I_2^\varepsilon &:= \int \psi\langle\chi_1\rangle\Big\langle e^{R/2}\varphi_2^\varepsilon(s-w))\int(G_3^\chi - G_3^h)\varphi_3^\varepsilon(s-s_3)\mathbf{1}_{|u-s_3|<|R|}ds_3\Big\rangle ds \\ &\quad - \int \psi\langle\chi_1\rangle\Big\langle e^{R/2}\varphi_3^\varepsilon(s-w))\int(G_2^\chi - G_2^h)\varphi_2^\varepsilon(s-s_2)\mathbf{1}_{|u-s_2|<|R|}ds_2\Big\rangle ds \\ &=: I_{2,1}^\varepsilon - I_{2,2}^\varepsilon, \\ I_3^\varepsilon &:= \int \psi\langle\chi_1\rangle\Big\langle e^{R/2}\varphi_2^\varepsilon(s-z))\int(G_3^\chi + G_3^h)\varphi_3^\varepsilon(s-s_3)\mathbf{1}_{|u-s_3|<|R|}\,ds_3\Big\rangle ds \\ &\quad - \int \psi\langle\chi_1\rangle\Big\langle e^{R/2}\varphi_3^\varepsilon(s-z))\int(G_2^\chi + G_2^h)\varphi_2^\varepsilon(s-s_2)\mathbf{1}_{|u-s_2|<|R|}\,ds_2\Big\rangle ds, \end{aligned}$$

and a smoother remainder

$$I_4^\varepsilon = \int_{\mathbb{R}^3} \psi\langle\chi_1\rangle\Big\langle(G_3^\chi G_2^h - G_2^\chi G_3^h)\mathbf{1}_{|u-s_3|<|R|}\mathbf{1}_{|u-s_2|<|R|}\Big\rangle\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)ds ds_2 ds_3.$$

By change of variable we see that the integral

$$I_1^\varepsilon = \frac{2}{\varepsilon}\int_{W,Z}\int_{W',Z'}\int e^R\,\psi(\varepsilon y+z)\,\chi(R', u' - (\varepsilon y + z)) \Big(\varphi_2(y)\,\varphi_3(y+2|R|/\varepsilon) - \varphi_2(y+2|R|/\varepsilon)\,\varphi_3(y)\Big)\,dy d\nu d\nu'.$$

tends to zero : $I_1^\varepsilon \to 0$. The same is true for the smoothest term I_4^ε, in view of the identity

$$\begin{aligned} I_4^\varepsilon = \int\int\int\Big(&\int_w^z G^\chi(R,u,s_3)\,\varphi_3^\varepsilon(s-s_3)\,ds_3\int_w^z G^h(R,u,s_2)\,\varphi_2^\varepsilon(s-s_2)\,ds_2 \\ &- \int_w^z G^\chi(R,u,s_2)\,\varphi_2^\varepsilon(s-s_2)\,ds_2\int_w^z G^h(R,u,s_3)\,\varphi_3^\varepsilon(s-s_3)\,ds_3\Big) \\ &\psi\chi(R', u'-s)\,ds d\nu d\nu', \end{aligned}$$

which clearly tends to

$$\int\int\int \psi\, \chi(R', u'-s)\, \mathbf{1}_{|u-s|<|R|} \Big(G^\chi(R,u,s)\, G^h(R,u,s) \\ - G^\chi(R,u,s)\, G^h(R,u,s) \Big)\, ds d\nu d\nu' = 0.$$

We define

$$Q^\pm := G^\chi \pm G^h, \quad F_i = F(R,u,s_i), \quad F_i' = F(R',u',s_i).$$

Let us now consider the term $I_2^\varepsilon = I_{2,1}^\varepsilon - I_{2,2}^\varepsilon$ in (5.6). We have

$$I_{2,1}^\varepsilon = \int\int\int \psi e^{R/2} \chi_1' \,\varphi_2^\varepsilon(s-w) \int_w^z Q_3^- \,\varphi_3^\varepsilon(s-s_3)\, ds_3 ds d\nu d\nu'.$$

Therefore, in view of Lemma 5.2, we find that $I_{2,1}^\varepsilon$ tends to

$$\iint e^{R/2} \psi(w) Q^-(w) \\ \Big(\chi'(w) \mathbf{1}_{w'<w<z'} A_{2,3}^- + \chi'(w'+) \mathbf{1}_{w=w'} B_{2,3}^- + \chi'(z'-) \mathbf{1}_{w=z'} C_{2,3}^- \Big) d\nu d\nu',$$

and $I_{2,2}^\varepsilon$ tends to

$$\iint e^{R/2} \psi(w)\, Q^-(w) \\ \Big(\chi'(w) \mathbf{1}_{w'<w<z'} A_{3,2}^- + \chi'(w'+) \mathbf{1}_{w=w'} B_{3,2}^- + \chi'(z'-) \mathbf{1}_{w=z'} C_{3,2}^- \Big) d\nu d\nu',$$

as $\varepsilon \to 0$. We conclude that the limit of I_2^ε is equal to

$$\iint e^{R/2}\, \psi(w)\, Q^-(w) \\ \Big(\chi'(w)\, \mathbf{1}_{w'<w<z'}\, A^- + \chi'(w'+)\, \mathbf{1}_{w=w'}\, B^- + \chi'(z'-)\, \mathbf{1}_{w=z'}\, C^- \Big)\, d\nu d\nu',$$

where

$$A^- = A_{2,3}^- - A_{3,2}^-, \quad B^- = B_{2,3}^- - B_{3,2}^-, \quad C^- = C_{2,3}^- - C_{3,2}^-.$$

By Lemma 5.3 we can determine similarly that $\lim_{\varepsilon\to 0} I_3^\varepsilon$ is equal to

$$\iint e^{R/2}\, \psi(z)\, Q^+(z) \\ \Big(\chi'(z)\, \mathbf{1}_{w'<z<z'} A^+ + \chi'(w'+)\, \mathbf{1}_{z=w'}\, B^+ + \chi'(z'-)\, \mathbf{1}_{z=z'}\, C^+ \Big) d\nu d\nu',$$

where

$$A^+ := A_{2,3}^+ - A_{3,2}^+, \quad B^+ := B_{2,3}^+ - B_{3,2}^+, \quad C^+ := C_{2,3}^+ - C_{3,2}^+.$$

In conclusion, we have identified the limit of the term (5.10). It is equal to

$$\iint e^{R/2}\psi(w)Q^-(w)$$
$$\Big(\chi'(w)\mathbf{1}_{w'<w<z'}A^- + \chi'(w'+)\mathbf{1}_{w=w'}B^- + \chi'(z'-)\mathbf{1}_{w=z'}C^-\Big)d\nu d\nu'$$
$$+\iint e^{R/2}\psi(z)Q^+(z)$$
$$\Big(\chi'(z)\mathbf{1}_{w'<z<z'}A^+ + \chi'(w'+)\mathbf{1}_{z=w'}B^+ + \chi'(z'-)\mathbf{1}_{z=z'}C^+\Big)d\nu d\nu'.$$

Step 2. We now proceed by studying the two terms in the left-hand side of (5.2). We apply the distribution $\langle \chi_1(s_1)P_2h_2(s_2) - h_1(s_1)P_2\chi_2(s_2)\rangle\,\langle P_3\chi_3(s_3)\rangle$ to the test function (5.5). We write the integral

$$\int_{\mathbb{R}^3}\langle \chi_1(s_1)P_2h_2(s_2) - h_1(s_1)P_2\chi_2(s_2)\rangle$$
$$\langle P_3\chi_3(s_3)\rangle\,\psi(s_1)\,\varphi_2^\varepsilon(s_1-s_2)\,\varphi_3^\varepsilon(s_1-s_3)\,ds_1ds_2ds_3$$

as the sum $\sum_1^3 J_i^\varepsilon$, where

$$J_1^\varepsilon := \int_{\mathbb{R}^3}\langle e^{R/2}\,(h_1-\chi_1)\,\delta_{s_2=w}\rangle\,\langle P_3\chi_3\rangle\,\psi(s)\,\varphi_2^\varepsilon(s-s_2)\,\varphi_3^\varepsilon(s-s_3)\,dsds_2ds_3$$
$$J_2^\varepsilon := -\int_{\mathbb{R}^3}\langle e^{R/2}(h_1+\chi_1)\delta_{s_2=z}\rangle\,\langle P_3\chi_3\rangle\,\psi(s)\,\varphi_2^\varepsilon(s-s_2)\,\varphi_3^\varepsilon(s-s_3)\,dsds_2ds_3,$$

and

$$J_3^\varepsilon := \int_{\mathbb{R}^3}\langle (h_1G_2^\chi - \chi_1G_2^h)\mathbf{1}_{|u-s_2|<|R|}\rangle\langle P_3\chi_3\rangle\psi(s)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)dsds_2ds_3.$$

The application of the distribution $\langle \chi_1P_3h_3^\varepsilon - h_1P_3\chi_3^\varepsilon\rangle\langle P_2\chi_2^\varepsilon\rangle$ to the test function (5.5) can be represented similarly. The integral

$$\int_{\mathbb{R}^3}\langle \chi_1(s)P_3h_3(s_3) - h_1(s)P_3\chi_3(s_3)\rangle$$
$$\langle P_2\chi_2(s_2)\rangle\,\psi(s)\,\varphi_2^\varepsilon(s-s_2)\,\varphi_3^\varepsilon(s-s_3)\,dsds_2ds_3$$

is equal to $\sum_1^3 K_i^\varepsilon$, where

$$K_1^\varepsilon := \int_{\mathbb{R}^3}\langle e^{R/2}\,(h_1-\chi_1)\,\delta_{s_3=w}\,\rangle\langle P_2\chi_2\rangle\,\psi(s)\,\varphi_2^\varepsilon(s-s_2)\,\varphi_3^\varepsilon(s-s_3)\,dsds_2ds_3,$$
$$K_2^\varepsilon := -\int_{\mathbb{R}^3}\langle e^{R/2}\,(h_1+\chi_1)\,\delta_{s_3=z}\rangle\,\langle P_2\chi_2\rangle\,\psi(s)\varphi_2^\varepsilon(s-s_2)\,\varphi_3^\varepsilon(s-s_3)\,dsds_2ds_3,$$

and

$$K_3^\varepsilon := \int_{\mathbb{R}^3}\langle (h_1G_3^\chi - \chi_1G_3^h)\mathbf{1}_{|u-s_3|<|R|}\rangle\langle P_2\chi_2\rangle\psi(s)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)dsds_2ds_3.$$

Some further decomposition of these integral terms will be necessary:

$$J_i^\varepsilon = \sum_1^3 J_{i,j}^\varepsilon, \quad K_i^\varepsilon = \sum_1^3 K_{i,j}^\varepsilon,$$

where

$$\begin{aligned}
J_{1,1}^\varepsilon &:= \int_{\mathbb{R}^3} \langle e^{R/2}(h_1 - \chi_1)\delta_{s_2=w}\rangle \langle e^{R/2}\delta_{s_3=w}\rangle \\
&\qquad \psi(s_1)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)\, ds ds_2 ds_3,\\
J_{1,2}^\varepsilon &:= -\int_{\mathbb{R}^3} \langle e^{R/2}(h_1 - \chi_1)\delta_{s_2=w}\rangle \langle e^{R/2}\delta_{s_3=z}\rangle \\
&\qquad \psi(s_1)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)\, ds ds_2 ds_3,\\
J_{1,3}^\varepsilon &:= \int_{\mathbb{R}^3} \langle e^{R/2}(h_1 - \chi_1)\delta_{s_2=w}\rangle \langle G_3^\chi \mathbf{1}_{|u-s_3|<|R|}\rangle \\
&\qquad \psi(s)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)\, ds ds_2 ds_3,\\
J_{2,1}^\varepsilon &:= -\int_{\mathbb{R}^3} \langle e^{R/2}(h_1 + \chi_1)\delta_{s_2=z}\rangle \langle e^{R/2}\delta_{s_3=w}\rangle \\
&\qquad \psi(s)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)\, ds ds_2 ds_3,\\
J_{2,2}^\varepsilon &:= \int_{\mathbb{R}^3} \langle e^{R/2}(h_1 + \chi_1)\delta_{s_2=z}\rangle \langle e^{R/2}\delta_{s_3=z}\rangle \\
&\qquad \psi(s)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)\, ds ds_2 ds_3,\\
J_{2,3}^\varepsilon &:= -\int_{\mathbb{R}^3} \langle e^{R/2}(h_1 + \chi_1)\delta_{s_2=z}\rangle \langle G_3^\chi \mathbf{1}_{|u-s_3|<|R|}\rangle \\
&\qquad \psi(s)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)\, ds ds_2 ds_3,\\
J_{3,1}^\varepsilon &:= \int_{\mathbb{R}^3} \langle (h_1 G_2^\chi - \chi_1 G_2^h)\mathbf{1}_{|u-s_2|<|R|}\rangle \langle e^{R/2}\delta_{s_3=w}\rangle \\
&\qquad \psi(s)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)\, ds ds_2 ds_3,\\
J_{3,2}^\varepsilon &:= -\int_{\mathbb{R}^3} \langle (h_1 G_2^\chi - \chi_1 G_2^h)\mathbf{1}_{|u-s_2|<|R|}\rangle \langle e^{R/2}\delta_{s_3=z}\rangle \\
&\qquad \psi(s)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)\, ds ds_2 ds_3,\\
J_{3,3}^\varepsilon &:= \int_{\mathbb{R}^3} \langle (h_1 G_2^\chi - \chi_1 G_2^h)\mathbf{1}_{|u-s_2|<|R|}\rangle \langle G_3^\chi \mathbf{1}_{|u-s_3|<|R|}\rangle \\
&\qquad \psi(s)\varphi_2^\varepsilon(s-s_2)\varphi_3^\varepsilon(s-s_3)\, ds ds_2 ds_3.
\end{aligned}$$

The terms $K_{1,1}^\varepsilon$, $K_{1,2}^\varepsilon$, etc. are defined in a completely analogous fashion.

We can put $J_{1,1}^\varepsilon$ in the form

$$\begin{aligned}
J_{1,1}^\varepsilon &= \frac{1}{\varepsilon^2}\iiint \psi\, e^{(R+R')/2}\,(h_1-\chi_1)\,\varphi_2\left(\frac{s-w)}{\varepsilon}\right)\varphi_3\left(\frac{s-w'}{\varepsilon}\right))\, d\nu d\nu' ds\\
&= \frac{1}{\varepsilon}\iiint \psi(s) e^{(R+R')/2}\,(h-\chi)(s)\\
&\qquad \varphi_2(y_1)|_{s=\varepsilon y_1 + w}\varphi_3\left(y_1 + \frac{w-w'}{\varepsilon}\right) d\nu d\nu' dy_1.
\end{aligned}$$

A similar representation formula is valid for $K^{\varepsilon}_{1,1}$. In consequence we find

$$J^{\varepsilon}_{1,1} - K^{\varepsilon}_{1,1} = \frac{1}{\varepsilon} \iiint \psi(s)\, e^{(R+R')/2}(h-\chi)(s)|_{s=\varepsilon y_1 + w} \left(\varphi_2(y_1)\varphi_3\left(y_1 + \frac{w-w'}{\varepsilon}\right) - \varphi_3(y_1)\varphi_2\left(y_1 + \frac{w-w'}{\varepsilon}\right)\right) d\nu d\nu' dy_1.$$

Clearly, we have

$$J^{\varepsilon}_{1,1} - K^{\varepsilon}_{1,1} \to 0.$$

The terms $J^{\varepsilon}_{k,l}$ and $K^{\varepsilon}_{k,l}$ contain the product of measures or the product of BV-functions and can be treated in the same manner. In turn, we obtain

$$J^{\varepsilon}_{1,2} - K^{\varepsilon}_{1,2} \to 0,\; J^{\varepsilon}_{2,1} - K^{\varepsilon}_{2,1} \to 0,\; J^{\varepsilon}_{2,2} - K^{\varepsilon}_{2,2} \to 0,\; J^{\varepsilon}_{3,3} - K^{\varepsilon}_{3,3} \to 0.$$

Let us consider the terms $J^{\varepsilon}_{k,l}$ and $K^{\varepsilon}_{k,l}$, containing the product of a measure and a BV-function. By Lemma 5.4, the term

$$J^{\varepsilon}_{1,3} = \iiint \psi(s) e^{R/2}\, (h_1 - \chi_1)\, \varphi^{\varepsilon}_2(s-w) \int_{w'}^{z'} G^{\chi'}_3\, \varphi^{\varepsilon}_3(s-s_3)\, ds_3 ds d\nu d\nu'$$

converges toward

$$\iint e^{R/2}\, \psi(w)\, (h-\chi)(w)\, G^{\chi'}(w) \left(\mathbf{1}_{w'<w<z'} + \mathbf{1}_{w=w'}\,(C^{-}_{2,3} + B^{-}_{2,3}) + \mathbf{1}_{w=z'}\,(C^{+}_{2,3} + B^{+}_{2,3})\right) d\nu d\nu',$$

hence,

$$\lim_{\varepsilon\to 0}(J^{\varepsilon}_{1,3} - K^{\varepsilon}_{1,3}) = \iint e^{R/2}\, \psi(w)\, (h-\chi)(w)\, G^{\chi'}(w) \left(\mathbf{1}_{w=w'}\,(C^{-} + B^{-}) + \mathbf{1}_{w=z'}(C^{+} + B^{+})\right) d\nu d\nu'.$$

By the same argument we find that the term

$$J^{\varepsilon}_{3,1} = \iiint \psi(s)\, e^{R'/2} \varphi^{\varepsilon}_3(s-w') \int_{w}^{z} (h_1 G^{\chi}_2 - \chi_1 G^{h}_2)\, \varphi^{\varepsilon}_2(s-s_2)\, ds_2 ds d\nu d\nu'$$

tends toward

$$\iint e^{R'/2}\, \psi(w')\, G^{\chi}(w')\, L\, d\nu d\nu' - \iint e^{R'/2}\, \psi(w')\, G^{h}(w')\, S\, d\nu d\nu',$$

where

$$L := \mathbf{1}_{w<w'<z}\, h(w') + \mathbf{1}_{w'=w}\left(h(w-)\, C^{-}_{3,2} + h(w+)\, B^{-}_{3,2}\right) + \mathbf{1}_{w'=z}\left(h(z-)\, C^{+}_{3,2} + h(z+)\, B^{+}_{3,2}\right)$$

and

$$S := \mathbf{1}_{w<w'<z}\,\chi(w') + \mathbf{1}_{w'=w}\Big(\chi(w-)\,C^-_{3,2} + \chi(w+)\,B^-_{3,2}\Big)$$
$$+\mathbf{1}_{w'=z}\Big(\chi(z-)C^+_{3,2} + \chi(z+)B^+_{3,2}\Big).$$

Hence,

$$J^\varepsilon_{3,1} - K^\varepsilon_{3,1} \to -\iint e^{R'/2}\psi(w')\,(\mathbf{1}_{w'=w}M_w + \mathbf{1}_{w'=z}\,M_z)\,d\nu d\nu',$$

where

$$M_w := G^\chi(w)\Big(h(w-)C^- + h(w+)B^-\Big) - G^h(w)\Big(\chi(w-)C^- + \chi(w+)B^-\Big),$$
$$M_z = G^\chi(z)\Big(h(z-)C^+ + h(z+)B^+\Big) - G^h(z)\Big(\chi(z-)C^+ + \chi(z+)B^+\Big).$$

It can be seen that the integrals, containing the functions $\mathbf{1}_{w'=w}$ and $\mathbf{1}_{w'=z}$ cancel each other.

In a similar way, we can treat the other terms and arrive at the final equality

$$\iint e^{R/2}\,\psi(w)\,Q^-(w)\,\chi'(w)\,A^-\,\mathbf{1}_{w'<w<z'}$$
$$+e^{R/2}\,\psi(z)\,Q^-(z)\,\chi'(z)\,A^+\,\mathbf{1}_{w'<z<z'}\,d\nu d\nu' = 0,$$

resulting from (5.2).

Step 3. Observing that $A^+ = -A^-$ and

$$Q^-(w) = -Q^+(z) = e^{R/2}\Big(-\frac{f(0)}{2} + 2|R| + 2|R|f'(0)\Big) =: D(R),$$

we can write

$$\iint Y\,e^{R/2}\,D(R)\Big(\psi(w)\,\chi'(w)\,\mathbf{1}_{w'<w<z'} + \psi(z)\,\chi'(z)\,\mathbf{1}_{w'<z<z'}\Big)\,d\nu d\nu' = 0, \tag{5.7}$$

where

$$Y = \int_{-\infty}^{+\infty}\int_{-\infty}^{s_2} \varphi_2(s_2)\varphi_3(s_3) - \varphi_2(s_3)\varphi_3(s_3)ds_2ds_3.$$

As observed in [5], the functions φ_2 and φ_3 can easily be chosen in such a way that $Y \neq 0$.

In accordance with our notation, the equality (5.7) means precisely

$$\iint D\left(\frac{1}{2}\ln(WZ)\right)\,(W\,Z)^{1/4}\,(W'\,Z')^{1/4}$$
$$\left(f\left(\left(-\ln\frac{W'}{W}\,\ln(Z'W)\right)\,\psi(\ln W)\,\mathbf{1}_{W'<W<1/Z'}\right.\right.$$
$$\left.+f\Big(-\ln(W'Z)\,\ln\frac{Z'}{Z}\Big)\,\psi(-\ln Z)\,\mathbf{1}_{W'<1/Z<1/Z'}\right)\,d\nu d\nu' = 0.$$

The conditions (3.21) guarantee that $|D(R)| \geqq e^{R/2}/2$. Since $f(-x^2) \geqq 1$ and the function ψ is arbitrary, we conclude from the last equality that

$$\iint_{W,Z} (WZ)^{1/2} \iint_{\{W'<W\}\cap\{Z'<1/W\}} (W'Z')^{1/4}\, d\nu(W',Z') d\nu(W,Z) = 0$$

and

$$\iint_{W,Z} (WZ)^{1/2} \iint_{\{W'<1/Z\}\cap\{Z'<Z\}} (W'Z')^{1/4}\, d\nu(W',Z')\, d\nu(W,Z) = 0.$$

We arrive at the following important claim: whenever

$$W^*Z^* \neq 0, \quad (W^*, Z^*) \in \operatorname{supp} \nu,$$

we have

$$\iint_{W,Z} (WZ)^{1/2} d\nu(W,Z) \neq 0$$

and therefore

$$\begin{aligned} &\iint_{\{W'<W^*\}\cap\{Z'<1/W^*\}} (W'Z')^{1/4}\, d\nu(W',Z') = 0, \\ &\iint_{\{W'<1/Z^*\}\cap\{Z'<Z^*\}} (W'Z')^{1/4}\, d\nu(W',Z') = 0. \end{aligned} \tag{5.8}$$

We will now conclude from (5.8) that the Young measure is a Dirac mass or a measure concentrated at the vacuum. At this stage, it is useful to draw a picture on the W, Z-plane, with the W-axis being horizontal. We draw two hyperbolas, $WZ = 1$ and $WZ = \rho_2^2$, keeping in mind that $\operatorname{supp}\nu$ lies below the hyperbola $WZ = \rho_2^2$, where the constant $\rho_2 < 1$ is defined in Section 3. The hyperbola $WZ = 1$ helps us to picture the set

$$\begin{aligned} M^* := &\Big(\{0 < W' < W^*\} \cap \{0 < Z' < 1/W^*\}\Big) \\ &\cup\Big(\{0 < W' < 1/Z^*\} \cap \{0 < Z' < Z^*\}\Big), \end{aligned}$$

a union of two rectangles. The relations (5.8) imply that M^* does not intersect the support of ν:

$$W^*Z^* \neq 0 \quad \text{and} \quad (W^*, Z^*) \in \operatorname{supp} \nu \quad \Longrightarrow M^* \cap \operatorname{supp} \nu = \emptyset. \tag{5.9}$$

By construction, the hyperbola $WZ = 1$ does not intersect $\operatorname{supp}\nu$. The inclusion $\operatorname{supp}\nu \subset \{\rho = 0\}$ holds if no hyperbola $WZ = \delta$, $0 < \delta < 1$, intersects $\operatorname{supp}\nu$. If $\operatorname{supp}\nu$ contains a point (W, Z) such that $\rho(W, Z) \neq 0$, there is a number $0 < \delta < 1$ such that the hyperbola $WZ = \delta$ intersects $\operatorname{supp}\nu$. Let $0 < \delta_0 < 1$ be the largest number such that the hyperbola $WZ = \delta_0$ intersects $\operatorname{supp}\nu$. By (5.9), the intersection

$$\operatorname{supp}\nu \cap \{WZ = \delta_0\}$$

may contain only one point (W^*, Z^*) and

$$\operatorname{supp} \nu \cap \{0 < WZ < \delta_0\} = \emptyset.$$

Thus

$$\nu = \alpha\delta_* + \mu, \tag{5.10}$$

with $\operatorname{supp} \mu \subset \{\rho = 0\}$. Throughout the paper we use only weak entropies. Hence, putting (5.10) into Tartar's commutation relation (5.1), we find that any two entropy pairs satisfy the equality

$$\alpha(q_2\eta_1 - q_1\eta_2) = \alpha^2(q_2\eta_1 - q_1\eta_2) \tag{5.11}$$

at the point (W^*, Z^*). Let us choose the following entropy pair (as in [31]):

$$\eta_i = \rho^{B_i} e^{A_i u}, \quad q_i = -\frac{A_i}{B_i - 1}\rho^{B_i - 1} e^{A_i u}, \quad A_i = \sqrt{B_i(B_i - 1)}, \quad B_1 \neq B_2.$$

Now, the equality(5.11) is rewritten as

$$\alpha(1-\alpha)\rho_*^{B_1+B_2-1} e^{(A_1+A_2)u_*}\left(\sqrt{\frac{B_1}{B_1 - 1}} - \sqrt{\frac{B_2}{B_2 - 1}}\right) = 0.$$

Hence, $\alpha = 0$ or $\alpha = 1$. This completes the proof of Theorem 5.1.

6. Convergence and compactness of solutions

Due to the decomposition (5.10) of the Young measures, the convergence formulas (3.9) imply that

$$W^\varepsilon \rightharpoonup W, \quad Z^\varepsilon \rightharpoonup Z, \quad F(W^\varepsilon, Z^\varepsilon) \rightharpoonup F(W, Z) \quad \text{weakly} \star \text{ in } L^\infty(\Pi),$$

for any function $F(\alpha, \beta)$, $F \in C(K)$, such that $F = 0$ at the vacuum set $\alpha\beta = 0$. (See formula (3.20) for the definition of the compact set K.) Hence, for almost all $(x, t) \in \Pi$,

$$\begin{aligned}
\rho^\varepsilon &:= (W^\varepsilon Z^\varepsilon)^{1/2} \to \rho = (WZ)^{1/2} =: f_1(W, Z),\\
m^\varepsilon &:= (W^\varepsilon\, Z^\varepsilon)^{1/2} \ln\left(\frac{W^\varepsilon}{Z^\varepsilon}\right)^{1/2} \to m\\
&= (WZ)^{1/2} \ln\left(\frac{W}{Z}\right)^{1/2} =: f_2(W, Z),\\
\frac{(m^\varepsilon)^2}{\rho^\varepsilon} &:= (W^\varepsilon\, Z^\varepsilon)^{1/2}\left(\ln\left(\frac{W^\varepsilon}{Z^\varepsilon}\right)^{1/2}\right)^2 \to \frac{m^2}{\rho}\\
&= f_3(W, Z) := (WZ)^{1/2}\left(\ln\left(\frac{W}{Z}\right)^{1/2}\right)^2.
\end{aligned}$$

Moreover,

$$F(m^\varepsilon,\rho^\varepsilon)\to F(m,\rho)\quad\text{for almost all}\quad (x,t)\in\Pi \tag{6.1}$$

for any function $F(m,\rho)$ such that

$$\tilde F(\alpha,\beta):=F(f_2(\alpha,\beta),f_1(\alpha,\beta))\in C(K),\quad \tilde F|_{\alpha\beta=0}=0. \tag{6.2}$$

Indeed, the convergence (6.1) can be derived from the following fact:

$$\begin{aligned} &v^\varepsilon\to v\quad\text{and}\quad (v^\varepsilon)^2\to v^2\quad\text{weakly in } L^2_{\mathrm{loc}}(\Pi)\\ \Longrightarrow\ &v^\varepsilon\to v\quad\text{strongly in } L^2_{\mathrm{loc}}(\Pi). \end{aligned}$$

Let us show that (m,ρ) is an entropy solution of problem (2.1). To this end we let ε and ε_1 go to zero in (3.13). (More exactly we should do it in the similar equality relevant to the auxiliary approximation.) If functions $\eta(m,\rho)$, $q(m,\rho)$ obey the restrictions (6.2), we obtain

$$\begin{aligned} &\int(\eta(m^\varepsilon,\rho^\varepsilon)-\eta(m_0^\varepsilon,\rho_0^\varepsilon))\varphi_t+q(m^\varepsilon,\rho^\varepsilon)\varphi_x\,dxdt\\ &\quad\to\int(\eta(m,\rho)-\eta(m_0,\rho_0))\varphi_t+q(m,\rho)\varphi_x\,dxdt,\\ &\varepsilon\int\eta(m^\varepsilon,\rho^\varepsilon)\varphi_{xx}\,dxdt\to 0 \end{aligned}$$

for any $\varphi\in\mathcal{D}(\mathbb{R}^2)$.

From now on we assume that $\varepsilon_1=\varepsilon^r$, $r>1$. If a function $\eta(m,\rho)$ meets the conditions of Theorem 2.1, the derivatives $\eta_m(m,\rho)$ and $m_\rho(m,\rho)$ are continuous on any closed set

$$\{0\leqq\rho\leqq\rho_1,\quad |m|\leqq c_1\rho(1+|\ln\rho|)\},\quad \rho_1>0,\quad c_1>0.$$

Hence, by estimate (3.19),

$$\begin{aligned} |\varepsilon_1u_x(q_m+\eta_\rho)|&=\left|2\varepsilon_1u_x\left(\frac{m}{\rho}\eta_m+\eta_\rho\right)\right|\leqq c\varepsilon_1(|uu_x|+|u_x|)\\ &\leqq\varepsilon^{1/2}\rho^{1/2}|u_x|(\varepsilon^{\frac{r-1}{2}}+|u|\rho^\gamma\varepsilon^\delta), \end{aligned}$$

where $2\gamma<\frac{r-1}{r}$, $2\delta=r(1-2\gamma)-1$. Besides,

$$\varepsilon_1\rho^{-1}|\eta_m\rho_x|\leqq c\varepsilon^{1/2}\rho^{-1/2}|\rho_x|\varepsilon^{\frac{r-1}{r}}.$$

Now, it follows from Lemma 3.3 and estimates (3.19) that

$$\varepsilon_1u_x(q_m+\eta_\rho)-2\varepsilon_1\eta_m\rho^{-1}\rho_x\to 0\quad\text{in}\quad L^2_{\mathrm{loc}}(\Pi).$$

Taking into account the convexity of the function $\eta(m,\rho)$, we send ε to zero in (3.13) to deduce that the pair (m,ρ) is an entropy solution of (2.1), (2.2). The proof of Theorems 2.1 to 2.3 is completed.

We conclude by giving a proof of Theorem 2.4. Let (m_n,ρ_n) be a sequence of bounded in $L^\infty(\Pi)$ entropy solutions of the problem (2.1) obeying the restriction of Theorem 2.4. We introduce the sequences

$$W_n:=\rho_ne^{m_n/\rho_n},\quad Z_n=\rho_n\,e^{-m_n/\rho_n}.$$

Clearly, we have

$$W_n \rightharpoonup W, \quad Z_n \rightharpoonup Z \quad \text{weakly} \star \text{ in } L^\infty_{\mathrm{loc}}(\Pi),$$

and there exist Young measures $\nu_{x,t}$ such that, for all $F(\alpha, \beta) \in C_{\mathrm{loc}}(\mathbb{R}^2)$,

$$F(W_n(x,t), Z_n(x,t)) \rightharpoonup \langle \nu_{x,t}, F \rangle.$$

Given two entropy pairs $(\eta_i(m,\rho), q_i(m,\rho))$ from Theorem 2.1, the sequences of measures

$$\theta_i^n := \partial_t \eta_i(m_n, \rho_n) + \partial_x q_i(m_n, \rho_n) = \partial_t \tilde{\eta}(W_n, Z_n) + \partial_x \tilde{q}(W_n, Z_n),$$

satisfy the conditions of Murat's lemma and are compact in $W^{-1,2}_{\mathrm{loc}}(\Pi)$. (We recall the notation $\tilde{q}(W,Z) := q(f_2(W,Z), f_1(W,Z))$.) By the div-curl lemma, the Tartar commutation relation (3.15) is valid for the Young measures $\nu_{x,t}$. Then we argue like in the proof of Theorem 2.1 to arrive at the decomposition (5.10) for $\nu_{x,t}$. Hence,

$$F(W_n(x,t), Z_n(x,t)) \to F(W(x,t), Z(x,t)) \quad \text{almost everywhere in } \Pi$$

for any $F(\alpha, \beta) \in C_{\mathrm{loc}}(\mathbb{R}^2)$ such that $F(\alpha,\beta) = 0$ if $\alpha\beta = 0$. Writing

$$\rho = f_1(W,Z), \quad m = f_2(W,Z), \quad \rho_n = f_1(W_n, Z_n), \quad m_n = f_2(W_n, Z_n),$$

we pass to the limit, as $n \to \infty$, in the inequality

$$\iint \eta(m_n, \rho_n)\, \partial_t \varphi + q(m_n, \rho_n)\, \partial_x \varphi \quad dxdt + \int \eta(m_0, \rho_0)\, \varphi(x, 0)\, dx \geqq 0$$

and check that (m, ρ) is an entropy solution of the problem (2.1), (2.2). The proof of Theorem 2.4 is completed.

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