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Maxwell's Equations with Vector Hysteresis

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Abstract

Electromagnetic processes in magnetic materials are described by Maxwell's equations. In ferrimagnetic insulators, assuming that $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, we have the equation

$$\varepsilon \frac{\partial^2}{\partial t^2}\big(\boldsymbol{H} + 4\pi \boldsymbol{M}\big) + c^2 \mathrm{curl}^2 \boldsymbol{H} = \boldsymbol{f}_{\mathrm{ext}}.$$

In ferromagnetic metals, neglecting displacement currents and assuming Ohm's law, we instead get

$$4\pi\sigma \frac{\partial}{\partial t}\big(\boldsymbol{H} + 4\pi \boldsymbol{M}\big) + c^2 \mathrm{curl}^2 \boldsymbol{H} = \boldsymbol{f}_{\mathrm{ext}}.$$

Alternatively, under quasi-stationary conditions, for either material we can also deal with the magnetostatic equations:

$$\nabla \cdot (\boldsymbol{H} + 4\pi \boldsymbol{M}) = 0, \qquad c\, \mathrm{curl} \boldsymbol{H} = 4\pi \boldsymbol{J}_{\mathrm{ext}}.$$

(Here $\boldsymbol{f}_{\mathrm{ext}}$ and $\boldsymbol{J}_{\mathrm{ext}}$ are prescribed time-dependent fields.) In any of these settings, the dependence of $\boldsymbol{M}$ on $\boldsymbol{H}$ is represented by a constitutive law accounting for hysteresis: $\boldsymbol{M} = \mathcal{F}(\boldsymbol{H})$, $\mathcal{F}$ being a vector extension of the *relay model*. This is characterized by a rectangular hysteresis loop in a prescribed x-dependent direction, and accounts for high anisotropy and nonhomogeneity. The discontinuity in this constitutive relation corresponds to the possible occurrence of *free boundaries*.

Weak formulations are provided for Cauchy problems associated with the above equations; existence of a solution is proved via approximation by time-discretization, derivation of energy-type estimates, and passage to the limit. An analogous representation is given for hysteresis in the dependence of $\boldsymbol{P}$ on $\boldsymbol{E}$ in ferroelectric materials. A model accounting for coupled ferrimagnetic and ferroelectric hysteresis is considered, too.

Introduction

Maxwell's equations

In this paper we deal with models of electromagnetic processes in either ferromagnetic, ferrimagnetic or ferroelectric materials. More specifically, we address the modelling of the electromagnetic evolution of materials that exhibit hysteresis in the constitutive law relating the fields $\boldsymbol{H}$ and $\boldsymbol{M}$, or between $\boldsymbol{E}$ and $\boldsymbol{P}$. The models we propose might be used for the simulation of the electromagnetic behaviour of electric transformers, magnetic tapes, and other devices.

We distinguish three main formulations. For insulating ferrimagnetic substances we get an equation of the form

$$\varepsilon \frac{\partial^2}{\partial t^2}\big(\boldsymbol{H} + 4\pi \boldsymbol{M}\big) + c^2 \nabla \times \nabla \times \boldsymbol{H} = \boldsymbol{f}_{\text{ext}} \quad (\nabla \times := \text{curl}). \tag{1}$$

On the other hand, in ferromagnetic metals Ohm's law must be considered; if the frequency is not too high, the Ohmic current dominates the displacement current, which can then be neglected. This yields

$$4\pi\sigma \frac{\partial}{\partial t}\big(\boldsymbol{H} + 4\pi \boldsymbol{M}\big) + c^2 \nabla \times \nabla \times \boldsymbol{H} = \boldsymbol{f}_{\text{ext}}. \tag{2}$$

If the ferromagnet is surrounded by a nonconducting material, there (1) should be applied. Denoting the characteristic function of Ω by χ_Ω, this leads us to formulate the following equation:

$$(1 - \chi_\Omega)\varepsilon \frac{\partial^2}{\partial t^2}\big(\boldsymbol{H} + 4\pi \boldsymbol{M}\big) + 4\chi_\Omega \pi\sigma \frac{\partial}{\partial t}\big(\boldsymbol{H} + 4\pi \boldsymbol{M}\big) + c^2 \nabla \times \nabla \times \boldsymbol{H} = \boldsymbol{f}_{\text{ext}}. \tag{3}$$

For quasi-stationary processes, the magnetostatic equations can be used for either class of materials:

$$\nabla \cdot (\boldsymbol{H} + 4\pi \boldsymbol{M}) = 0, \qquad c\nabla \times \boldsymbol{H} = 4\pi \boldsymbol{J}_{\text{ext}} \qquad (\nabla \cdot := \text{div}). \tag{4}$$

(Throughout this paper by $\boldsymbol{f}_{\text{ext}}$ and $\boldsymbol{J}_{\text{ext}}$ we denote prescribed time-dependent fields.)

Magnetic and electric hysteresis

In ferromagnetic and ferrimagnetic materials, the dependence of $\boldsymbol{M}$ on $\boldsymbol{H}$ must be represented by a suitable constitutive law accounting for hysteresis, which we synthetically represent in the form

$$\boldsymbol{M} = \mathcal{F}(\boldsymbol{H}) \qquad \text{in } \Omega \times]0, T[; \tag{5}$$

here $\mathcal{F}$ is a scalar *hysteresis operator,* cf. [4, 18, 19, 36], cf. Section 2. This dependence is nonlocal in time but pointwise in space. Under suitable hypotheses on $\mathcal{F}$, by coupling either of the equations (1) and (4) with (5) we get a quasi-linear hyperbolic problem, whereas the system (2), (5) is quasi-linear parabolic. The classification of P.D.E.s (partial differential equations) with hysteresis is illustrated at the end of this introduction.

Some dielectric materials, named *ferroelectrics,* also exhibit hysteresis in the dependence of $\boldsymbol{P}$ on $\boldsymbol{E}$, where $\boldsymbol{P}$ represents the electric polarization. By coupling a hysteresis relation of the form $\boldsymbol{P} = \mathcal{F}(\boldsymbol{E})$ with the Maxwell equations, we get an equation similar to (1), with $\boldsymbol{E}$ and $\boldsymbol{P}$ in place of $\boldsymbol{H}$ and $\boldsymbol{M}$, respectively; quasi-stationary processes can be represented via the electrostatic equations. Ferroelectrics are typically insulators, and accordingly for them the parabolic problem seems to be ruled out.

We also deal with an (insulating) fictitious material which couples ferrimagnetic and ferroelectric properties, by exhibiting hysteresis in both the $\boldsymbol{M}$ vs. $\boldsymbol{H}$ and $\boldsymbol{P}$ vs. $\boldsymbol{E}$ relations. We couple these conditions with the Ampère-Maxwell and Faraday laws,

$$c\nabla\times\boldsymbol{H} = \boldsymbol{J}_{\text{ext}} + \frac{\partial}{\partial t}(\boldsymbol{E} + 4\pi\boldsymbol{P}), \qquad c\nabla\times\boldsymbol{E} = -\frac{\partial}{\partial t}(\boldsymbol{H} + 4\pi\boldsymbol{M}) \qquad \text{in } \mathbf{R}^3_T \,. \tag{6}$$

P.D.E.s with hysteresis

The above setting leads us to consider P.D.E.s with hysteresis. Scalar quasi-linear parabolic equations with hysteresis have been studied for more than twenty years; references can be found in the monographs [4, 19, 36]. On the other hand, existence of a solution for scalar quasi-linear hyperbolic problems has only recently been proved in [38]. Here we deal with the vector setting, and represent hysteresis in a strongly anisotropic, nonhomogeneous material via a vector generalization of the *relay operator,* cf. [8, 21–23, 36], which we allow to depend explicitly on the space variable, x. This means that at each point x of the domain Ω, $\boldsymbol{M}$ is assumed to attain a prescribed x-dependent direction, and the components of $\boldsymbol{H}$ and $\boldsymbol{M}$ along that direction move along a rectangular hysteresis loop; the latter is characterized by a pair of thresholds, which also depend on x.

We provide a weak formulation in the framework of Sobolev spaces for a Cauchy problem for any of the equations (1), (3) and (4), coupled with (5). We approximate any of these problems via a time-discretized scheme, show existence of a solution for the latter, and prove convergence to a solution of the continuous problem. This approximation procedure is quite convenient in the analysis of equations that include a hysteresis operator, since at any time-step we have to solve a stationary problem, in which the hysteresis operator is reduced to the superposition with a nonlinear function. This approximate problem might be fully discretized, and then numerically solved by standard procedures; in this paper however we do not address this issue.

The main difficulty of the existence proof stays in the passage to the limit in the hysteresis relation. In the scalar case this follows from the strong convergence of the approximating magnetic field, cf. [34, 36]. In the vector setting that convergence looks hardly attainable, on account of the structure of the Maxwell equations: we are not able to exclude fast oscillations in space, for no *a priori* estimate seems to be available for the divergence of the magnetic field $\boldsymbol{H}$. Here we overcome this drawback by reformulating the relay operator as a system of two (nonvariational) inequalities, along the lines of [38]. For equation (1) the above existence result and those of [38] are based on the dissipative character of hysteresis, and have no analogue for equations without hysteresis. Thus this P.D.E. turns out to be one of the few known instances in which analysis is made easier by occurrence of hysteresis.

For all of these vector problems, uniqueness of the solution is an open question. For the scalar parabolic problem with hysteresis, uniqueness was first proved by HILPERT [13] for continuous *play-type* operators; this was then extended to discontinuous operators including relays, cf. [36, Chap. VIII]. For the scalar hyperbolic problem, KREJČÍ proved uniqueness under rather strong restrictions [19, Section III.2].

In recent years a different approach to hysteresis has been proposed by Mielke, Theil, Levitas and other researchers in Stuttgart [24–26]; their formulation does not involve hysteresis operators, and is based on coupling the energy balance with a stability condition. As we shall see in Section 4, there are similarities between this model and the formulation of the relay operator of [38], we also use in this paper. The approach based on the energy balance and the stability condition has recently been applied to ferromagnetic hysteresis by EFENDIEV [11] and by ROUBÍČEK & KRUŽÍK [30]. This method indeed looks capable of providing a rather general framework for a number of phenomena.

Preisach models

The large class of scalar *Preisach models* [29] is constructed by assuming that a (possibly infinite) family of relay operators coexist at each point of Ω. The vector extension of relays then induces a corresponding vector extension of those models, see [8, 22, 23, 36]. For $\mathcal{F}$ equal to a Preisach operator, in the scalar setting existence of a solution for the quasi-linear parabolic equation (2) was proved in [34] some time ago, and recently for the quasi-linear hyperbolic equation (1) in [38].

The vector setting looks more challenging, and seems to require a revision of that model. A homogenization procedure looks quite natural: a periodic distribution of the direction and threshold pairs that characterize vector relays may be assumed, and the limit may then be taken as the periodicity step vanishes. Although this produces the vector extension of the Preisach model we mentioned above, it is by no means obvious that the solutions of the corresponding P.D.E. problems converge, even if we allow for extraction of a subsequence. In the scalar setting this convergence had been shown without much effort for the quasi-linear parabolic problem, cf. [36, Section XI.7], whereas in the vector setting it is still under study. A necessary step in this program is to show existence of a solution for the periodic problem, and this is just a particular case of the results proved in this paper.

The relay and Preisach models are essentially phenomenologic, and deal with a length-scale which is intermediate between macroscopic and mesoscopic scales. The model known as *micromagnetics* provides a mesoscopic description of the ferromagnetic behaviour, cf. e.g. [5, 20] and the physical monographs quoted below.

The vector-relay operator represents the limit of this model as the anisotropy coefficient tends to infinity. Although the system obtained by coupling the Maxwell equations with the Landau and Lifshitz dynamic equation of micromagnetics has a solution, cf. [1, 3, 15, 16, 35], its asymptotic behaviour as the anisotropy coefficient diverges has not yet been studied. The interested reader may find information about the physics of magnetism in a number of monographs, e.g., [2, 6, 7, 10, 12, 14, 27]. In recent years research on models of hysteresis phenomena has been progressing, see, e.g., the monographs [4, 18, 19, 36] and [2, 10, 22] for mathematically and physically oriented approaches, respectively.

Plan of the paper

In Section 1 we derive the basic equations from Maxwell's and Ohm's laws. In Section 2 we review the scalar relay, introduce the corresponding vector operator, and reformulate it in terms of inequalities. In Section 3 we derive some compactness results to be used afterwards. In Section 4 we briefly study the stationary problem, and formulate a problem for the quasi-stationary equation (4) in the framework of Sobolev spaces; we then prove existence of a solution. In Section 5 we formulate a Cauchy problem for the parabolic-hyperbolic equation (3), and prove existence of a solution. Similarly, in Section 6 we formulate a Cauchy problem for the hyperbolic equation (1), and prove an existence result. In Section 7 we then deal with a Cauchy problem for the system (6) coupled with hysteresis relations between $\boldsymbol{M}$ and $\boldsymbol{H}$ and between $\boldsymbol{P}$ and $\boldsymbol{E}$, and prove existence of a solution in this case too. Finally, in Section 8 we draw some conclusions and outline some open questions. Some of these results have been announced in [39, 40].

Classification of P.D.E.s with hysteresis

Above we referred to either parabolic or hyperbolic P.D.E.s with hysteresis. It seems in order to explain how the standard classification of nonlinear P.D.E.s can be extended to equations that contain hysteresis operators.

Any scalar hysteresis operator, $\mathcal{F}$, is reduced to a superposition operator on any time interval in which the input function is monotone (either increasing or decreasing). Let us denote by $S_{\mathcal{F}}$ this class of superposition operators; in typical examples, they are associated with (possibly multivalued) nondecreasing functions. We then say that a scalar equation that includes $\mathcal{F}$ is parabolic (hyperbolic, resp.) whenever it would be so if the operator $\mathcal{F}$ were replaced by any element of $S_{\mathcal{F}}$. As these equations are nonlinear, by the same criterion we also extend the usual denomination of semi-linearity, quasi-linearity and full nonlinearity.

If a vector input function evolves monotonically along any fixed (possibly x-dependent) direction, the vector-relay and the vector Preisach operator are reduced to a (maximal monotone and multivalued) superposition operator. The above definitions can then be applied to this case, too.

1. Maxwell's equations

Magnetic hysteresis

In *ferromagnetic* materials (e.g., iron, cobalt and nickel) strongly coupled atomic dipole moments tend to be aligned parallel. As a result a *spontaneous magnetization* exists in such materials, below a critical temperature. *Ferrimagnetic* materials (e.g., ferrites, garnets, spinels) also exhibit spontaneous magnetization on a microscopic length-scale, but their atomic dipoles are not aligned parallel; we can then distinguish two or more sublattices, each one characterized by a parallel alignment of dipoles. The net magnetic moment is the sum of those of the sublattices; if this sum vanishes, the material is labelled as *antiferromagnetic*. Ferromagnetic materials are metals, hence good conductors of electricity, whereas ferrimagnetic materials are poor conductors.

Let us outline a classic experimental procedure, cf., e.g., [17]. By applying an electric current through a conducting solenoid wound around a ring-shaped sample of either ferromagnetic or ferrimagnetic material, we can enforce a co-axial magnetic field, $\boldsymbol{H}$, and control its axial intensity. This magnetic field determines a magnetic induction field, $\boldsymbol{B}$, in the sample. By winding a secondary coil around the ring and connecting it to a fluxometer, we can then measure the axial intensity of $\boldsymbol{B}$. We can assume that the fields $\boldsymbol{H}$ and $\boldsymbol{B}$ are uniform within the sample, and regard them just as functions of time. As $\boldsymbol{B} = \boldsymbol{H} + 4\pi \boldsymbol{M}$ (in Gauss units), we can equivalently deal with the pair $(\boldsymbol{H}, \boldsymbol{M})$ instead of $(\boldsymbol{H}, \boldsymbol{B})$.

Tests of this sort show that the dependence of $\boldsymbol{M}$ on $\boldsymbol{H}$ exhibits *hysteresis:* at any instant $\boldsymbol{M}$ depends on the previous evolution of $\boldsymbol{H}$, and this relation is rate-independent. The relation between the axial components M and H of $\boldsymbol{M}$ and $\boldsymbol{H}$ can then synthetically be represented in the form $M = \mathcal{G}(H)$, where $\mathcal{G}$ is a scalar *hysteresis operator,* cf. [4, 18, 19, 36]. We extrapolate this relation to space-distributed systems by assuming that at each point and at any instant M only depends on the previous evolution of H at that point. In the next section we shall introduce a specific vector model of magnetic hysteresis.

Maxwell equations

We deal with processes in a magnetic material which occupies a Euclidean domain Ω in a time interval $]0, T[$, and set $\Omega_T := \Omega\times]0, T[$, $\mathbf{R}^3_T := \mathbf{R}^3\times]0, T[$. We denote the magnetic field by $\boldsymbol{H}$, the magnetization by $\boldsymbol{M}$, and the magnetic induction by $\boldsymbol{B}$; in Gauss units, $\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}$. We also denote the electric field by $\boldsymbol{E}$, the electric displacement by $\boldsymbol{D}$, the electric current density by $\boldsymbol{J}$, the electric charge density by $\hat{\rho}$, and the speed of light in vacuum by c. The Maxwell equations read

$$c\nabla\times\boldsymbol{H} = 4\pi\boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t} \qquad \text{in } \mathbf{R}^3_T, \tag{1.1}$$

$$c\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \qquad \text{in } \mathbf{R}^3_T, \tag{1.2}$$

$$\nabla\cdot\boldsymbol{B} = 0, \qquad \nabla\cdot\boldsymbol{D} = 4\pi\hat{\rho} \qquad \text{in } \mathbf{R}^3_T. \tag{1.3}$$

These equations must be coupled with appropriate constitutive relations, with initial conditions for $\boldsymbol{D}$ and $\boldsymbol{B}$, and with suitable restrictions on the behaviour of $\boldsymbol{H}$ and $\boldsymbol{E}$ at infinity. We assume that the material is surrounded by vacuum, that the electric permittivity, ε, is constant (and scalar, just for the sake of simplicity), and that $\boldsymbol{J}$ equals a prescribed time-dependent field, $\boldsymbol{J}_{\text{ext}}$, outside Ω, that may represent an electric current circulating in an exterior conductor. We consider three classes of equations.

(i) *Maxwell's Equations without Displacement Current.* Dealing with a ferromagnetic metal, let us denote by $\boldsymbol{E}_{\text{app}}$ a prescribed applied electromotive force, due, e.g., to a battery, by σ the electric conductivity, and set $\boldsymbol{J}_{\text{ext}} := \boldsymbol{0}$ in Ω. *Ohm's law* then reads

$$\boldsymbol{J} = \sigma\big(\boldsymbol{E} + \boldsymbol{E}_{\text{app}}\big) + \boldsymbol{J}_{\text{ext}} \qquad \text{in } \mathbf{R}^3_T. \tag{1.4}$$

We assume that $\sigma = 0$ outside Ω, and that the prescribed field $\boldsymbol{E}_{\text{app}}$ and the unknown field $\boldsymbol{E}$ do not vary too rapidly in Ω. As in metals σ is very large, in Ω the Ohmic current $\boldsymbol{J}$ then dominates the displacement current $\frac{\partial \boldsymbol{D}}{\partial t}$, which can then be neglected (the so-called *eddy-current approximation*). As $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, (1.1) then yields

$$\begin{aligned} c\nabla\times\boldsymbol{H} &= 4\pi\sigma\big(\boldsymbol{E} + \boldsymbol{E}_{\text{app}}\big) && \text{in } \Omega_T, \\ c\nabla\times\boldsymbol{H} &= 4\pi\boldsymbol{J}_{\text{ext}} + \varepsilon\frac{\partial \boldsymbol{E}}{\partial t} && \text{in } \mathbf{R}^3_T \setminus \Omega_T. \end{aligned} \tag{1.5}$$

This system will be coupled with the Faraday law (1.2) and with initial conditions for $\boldsymbol{E}$ in $\mathbf{R}^3 \setminus \Omega$ and for $\boldsymbol{B}$ in $\mathbf{R}^3$. As the fields $\boldsymbol{B}$ and $\boldsymbol{H}$ will be related by a constitutive law with hysteresis in Ω and by the relation $\boldsymbol{B} = \boldsymbol{H}$ outside Ω, this problem will be parabolic with hysteresis in Ω, and linear hyperbolic outside.

Dealing with electromagnetic processes, in general it is not natural to formulate a boundary-value problem on a bounded domain. In fact the evolution of the exterior fields may affect the interior process; it is then difficult to represent this interaction at a distance by prescribing appropriate boundary conditions. For this reason we have chosen an approach in which the Maxwell equations are set in the whole space, but are coupled with different constitutive relations inside and outside Ω.

(ii) *Maxwell's Equations with Displacement Current.* In a ferrimagnetic insulator $\sigma = 0$, hence $\boldsymbol{J} = \boldsymbol{0}$ in Ω. Equations (1.1) and (1.2) then yield

$$c\nabla\times\boldsymbol{H} = 4\pi\boldsymbol{J}_{\text{ext}} + \varepsilon\frac{\partial \boldsymbol{E}}{\partial t}\,, \qquad c\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{in } \mathbf{R}^3_T, \tag{1.6}$$

with $\boldsymbol{J}_{\text{ext}} = \boldsymbol{0}$ in Ω_T. This system will be coupled with initial conditions for $\boldsymbol{E}$ and $\boldsymbol{B}$ in $\mathbf{R}^3$. By appending the above-mentioned $\boldsymbol{B}$ vs. $\boldsymbol{H}$ relation, we get a problem that is nonlinear hyperbolic in Ω_T and linear hyperbolic outside.

(iii) *Magnetostatic equations.* For either a ferromagnetic or ferrimagnetic material, under quasi-stationary conditions we can deal with the magnetostatic equations

$$\nabla\cdot\boldsymbol{B} = 0, \qquad c\nabla\times\boldsymbol{H} = 4\pi\boldsymbol{J} \qquad \text{in } \mathbf{R}^3_T; \tag{1.7}$$

here $\boldsymbol{J}$ is a prescribed field.

For any of the above problems, in the next section we shall introduce a constitutive law relating the fields $\boldsymbol{H}$ and $\boldsymbol{M}$. In order to make the formulae more readable, henceforth we omit the physical coefficients c, ε, σ, as well as the constant 4π in (1.1), (1.3) and in the identity $\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}$. The conductivity, σ, is accordingly replaced by χ_Ω, the characteristic function of Ω: $\chi_\Omega = 1$ in Ω, $\chi_\Omega = 0$ outside Ω. Setting $\boldsymbol{g}_{\text{ext}} = \chi_\Omega \boldsymbol{E}_{\text{app}} + \boldsymbol{J}_{\text{ext}}$, equations (1.5) read

$$\nabla\times\boldsymbol{H} = \chi_\Omega\boldsymbol{E} + (1-\chi_\Omega)\frac{\partial\boldsymbol{E}}{\partial t} + \boldsymbol{g}_{\text{ext}} \qquad \text{in } \mathbf{R}^3_T. \tag{1.8}$$

This procedure should not be regarded as a renormalization, but rather as an abuse of notation. However, it has no effect on our analysis, which could easily be extended if the actual physical coefficients were displayed in the equations.

In each of the above problems, we shall relate $\boldsymbol{H}$ and $\boldsymbol{M}$ by a constitutive law with hysteresis of the form $\boldsymbol{M} = \mathcal{F}(\boldsymbol{H})$. Under natural assumptions on $\mathcal{F}$, the systems (1.2), (1.5) and (1.6) are respectively parabolic-hyperbolic and hyperbolic; cf. equations (3) and (1) of the Introduction.

Ferroelectrics

If the region Ω is occupied by a dielectric material, a linear relation can be assumed between the fields $\boldsymbol{H}$ and $\boldsymbol{B}$; that is, $\boldsymbol{B} = \mu\boldsymbol{H}$, where μ is a positive constant or, more generally, a positive-definite matrix depending on $x \in \Omega$. The Faraday law (1.2) then reads

$$c\nabla\times\boldsymbol{E} = -\mu\frac{\partial\boldsymbol{H}}{\partial t} \qquad \text{in } \mathbf{R}^3_T. \tag{1.9}$$

As dielectrics are electric insulators, $\boldsymbol{J} = \mathbf{0}$ in Ω_T; on the other hand, we can assume that $\boldsymbol{J}$ is prescribed outside Ω. For slow processes, however, we can deal with the electrostatic equations:

$$\nabla\cdot\boldsymbol{D} = 4\pi\hat{\rho}, \qquad c\nabla\times\boldsymbol{E} = \mathbf{0} \qquad \text{in } \mathbf{R}^3_T. \tag{1.10}$$

For some dielectrics (e.g., Rochelle salt and barium titanate) the relation between $\boldsymbol{E}$ and the electric polarization vector, $\boldsymbol{P} := (\boldsymbol{D}-\boldsymbol{E})/4\pi$, exhibits hysteresis. The analogy with the behaviour of ferromagnetic materials is obvious, and indeed these materials are named *ferroelectrics*. We can then relate $\boldsymbol{E}$ and $\boldsymbol{P}$ by a constitutive law with hysteresis of the form $\boldsymbol{P} = \mathcal{G}(\boldsymbol{E})$, $\mathcal{G}$ being a *hysteresis operator*.

The results of this paper concerning the quasi-stationary and parabolic-hyperbolic problems could easily be extended to that setting. In Section 7 we also deal with a fictitious material which exhibits both ferroelectric and ferrimagnetic hysteresis. In this case the Maxwell system (1.1)–(1.3) is coupled with the constitutive relations

$$\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}, \quad \boldsymbol{M} = \mathcal{F}(\boldsymbol{H}); \quad \boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}, \quad \boldsymbol{P} = \mathcal{G}(\boldsymbol{E}) \qquad \text{in } \Omega_T. \tag{1.11}$$

2. Hysteresis

In this section we review the definition of relay operator, and specify the functional framework. We refer to [36, Chap. VI] for a more detailed presentation.

Scalar relay

Let us set $\mathcal{P} := \{\rho := (\rho_1, \rho_2) \in \mathbf{R}^2 : \rho_1 < \rho_2\}$, and fix $\rho \in \mathcal{P}$. For any $u \in C^0([0, T])$ and any $\xi \in \{-1, 1\}$, let us set $X_u(t) := \{\tau \in]0, t] : u(\tau) = \rho_1$ or $\rho_2\}$ and define the function $w = h_\rho(u, \xi) : [0, T] \to \{-1, 1\}$ as follows:

$$w(0) := \begin{cases} -1 & \text{if } u(0) \leqq \rho_1, \\ \xi & \text{if } \rho_1 < u(0) < \rho_2, \\ 1 & \text{if } u(0) \geqq \rho_2, \end{cases} \tag{2.1}$$

$$w(t) := \begin{cases} w(0) & \text{if } X_u(t) = \emptyset, \\ -1 & \text{if } X_u(t) \neq \emptyset \text{ and } u(\max X_u(t)) = \rho_1, \\ 1 & \text{if } X_u(t) \neq \emptyset \text{ and } u(\max X_u(t)) = \rho_2, \end{cases} \qquad \forall t \in]0, T], \tag{2.2}$$

cf. Fig. 1. We call h_ρ a (delayed) *relay operator.* Any function $u \in C^0([0, T])$ is uniformly continuous, hence it can only oscillate at most a finite number of times between the thresholds ρ_1 and ρ_2. Therefore w can jump just a finite number of times between -1 and 1, if at all. Hence the total variation of w in $[0, T]$ is finite, i.e., $w \in BV(0, T)$. All scalar- or vector-valued functions having finite total variation in time will be assumed to be continuous from the right at any $t > 0$.

Closure

On account of its discontinuity, the relay operator $h_\rho : C^0([0, T]) \to L^1(0, T)$ is not closed. Following [36, Chap. VI], we then introduce the (multivalued) *completed relay operator,* k_ρ. For any $u \in C^0([0, T])$ and any $\xi \in [-1, 1]$, we set $w \in k_\rho(u, \xi)$ if and only if w is measurable in $]0, T[$,

$$w(0) := \begin{cases} -1 & \text{if } u(0) < \rho_1, \\ \xi & \text{if } \rho_1 \leqq u(0) \leqq \rho_2, \\ 1 & \text{if } u(0) > \rho_2, \end{cases} \tag{2.3}$$

and, for any $t \in]0, T]$,

$$w(t) \in \begin{cases} \{-1\} & \text{if } u(t) < \rho_1, \\ [-1, 1] & \text{if } \rho_1 \leqq u(t) \leqq \rho_2, \\ \{1\} & \text{if } u(t) > \rho_2, \end{cases} \tag{2.4}$$

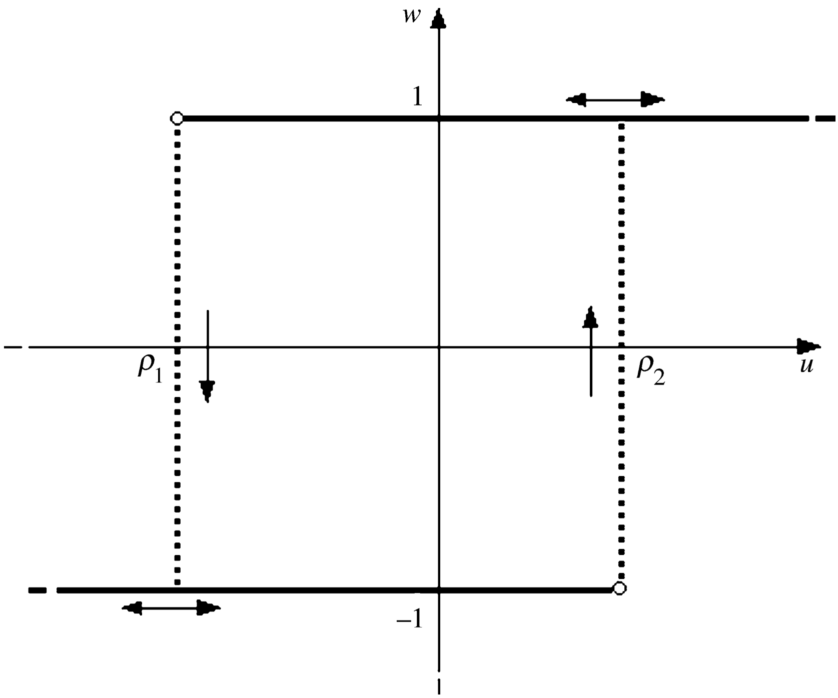

Fig. 1. Relay operator.

$$\begin{array}{lll} \text{if } u(t) \neq \rho_1, \rho_2, & \text{then } w \text{ is constant in a neighbourhood of } t, \\ \text{if } u(t) = \rho_1, & \text{then } w \text{ is nonincreasing in a neighbourhood of } t, \\ \text{if } u(t) = \rho_2, & \text{then } w \text{ is nondecreasing in a neighbourhood of } t. \end{array} \tag{2.5}$$

Notice that $w \in BV(0,T)$ for any $u \in C^0([0,T])$. The graph of k_ρ in the (u,w)-plane invades the whole rectangle $[\rho_1, \rho_2] \times [-1, 1]$, cf. Fig. 2. This operator is the closure of h_ρ with respect to the strong topology of $C^0([0,T])$ and the sequential weak star topology of $BV(0,T)$, cf. [36, Theorem 1.2 of Chap. VI], and turns out to be appropriate for the analysis of differential equations.

Reformulation of the scalar relay

In view of the coupling with P.D.E.s, it is convenient to reformulate the conditions (2.4) and (2.5). It is easy to see that (2.4) is equivalent to

$$\left.\begin{array}{ll} |w| & \leqq 1 \\ (w-1)(u-\rho_2) & \geqq 0 \\ (w+1)(u-\rho_1) & \geqq 0 \end{array}\right\} \qquad \text{a.e. in }]0,T[; \tag{2.6}$$

moreover, as $\mathrm{d}w = (\mathrm{d}w)^+ - (\mathrm{d}w)^-$ and $|\mathrm{d}w| = (\mathrm{d}w)^+ + (\mathrm{d}w)^-$, (2.5) is equivalent to

$$\int_0^t u \,\mathrm{d}w = \int_0^t \rho_2 \,(\mathrm{d}w)^+ - \int_0^t \rho_1 \,(\mathrm{d}w)^-$$

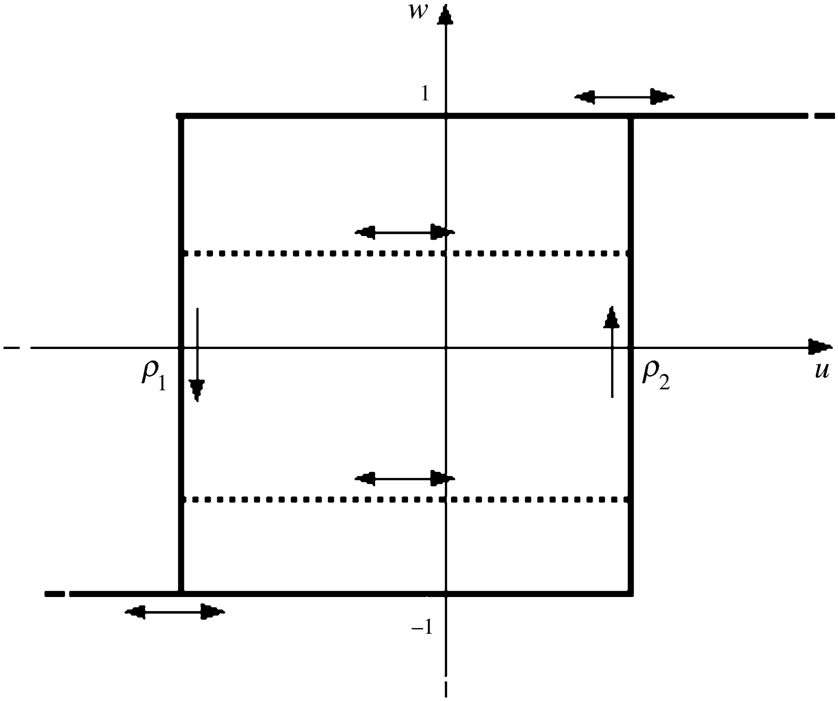

Fig. 2. Completed relay operator. Here the pair (u, w) can attain any value of the rectangle $[\rho_1, \rho_2] \times [-1, 1]$.

$$
\begin{aligned}
&= \frac{\rho_2 + \rho_1}{2} \int_0^t \mathrm{d}w + \frac{\rho_2 - \rho_1}{2} \int_0^t |\mathrm{d}w| \\
&= \frac{\rho_2 + \rho_1}{2} [w(t) - w(0)] + \frac{\rho_2 - \rho_1}{2} \int_0^t |\mathrm{d}w| \\
&=: \Psi_\rho(w; [0, t]) \qquad \forall t \in]0, T]
\end{aligned}
\tag{2.7}
$$

(these are Stieltjes integrals); cf. [38], where (2.6) and (2.7) were respectively labelled *confinement* and *dissipation* conditions. Notice that $\Psi_\rho(w; [0, t])$ depends on $w|_{[0,t]}$, hence also $w(0)$. Condition (2.4) entails the inequality $u\,\mathrm{d}w \leqq \rho_2\,(\mathrm{d}w)^+ - \rho_1\,(\mathrm{d}w)^-$, whence $\int_0^t u\,\mathrm{d}w \leqq \Psi_\rho(w; [0, t])$, independently from the dynamics; the opposite inequality is then equivalent to (2.7). Therefore the system (2.4) and (2.5) is equivalent to (2.6) coupled with the inequality

$$\int_0^t u\,\mathrm{d}w \geqq \Psi_\rho(w; [0, t]) \qquad \forall t \in]0, T]. \tag{2.8}$$

Vector relay

Let us set $\rho := (\rho_1, \rho_2)$ and $\boldsymbol{\theta} := (\theta_1, \theta_2, \theta_3) \in S^2 := \{\boldsymbol{\theta} \in \mathbf{R}^3 : |\boldsymbol{\theta}| = 1\}$. For any $(\rho, \boldsymbol{\theta}) \in \mathcal{P} \times S^2$, following [8] and [36, Section IV.5], we introduce the *vector-relay operator:*

$$\boldsymbol{h}_{(\rho,\theta)} : C^0([0, T])^3 \times \{\pm 1\} \to L^\infty(0, T) : (\boldsymbol{u}, \xi) \mapsto h_\rho(\boldsymbol{u}\cdot\boldsymbol{\theta}, \xi)\boldsymbol{\theta}. \tag{2.9}$$

Thus the component of the input $\boldsymbol{u}(x,\cdot)$ in the direction $\boldsymbol{\theta}(x)$ is assumed as input for the scalar relay $h_\rho(x)$; the output of the latter is then applied to the same direction $\boldsymbol{\theta}(x)$.

The operator $\boldsymbol{h}_{(\rho,\theta)}$ is Borel measurable with respect to $(\rho,\boldsymbol{\theta})$, and inherits several properties from h_ρ, see [36, Chap. IV]. Similarly to the scalar operator h_ρ, the vector-relay operator is not closed in any natural function space. Its (multivalued) closure $\boldsymbol{k}_{(\rho,\theta)}$ is simply obtained replacing h_ρ by k_ρ in (2.9).

Choosing the constitutive relation $\boldsymbol{M} \in \boldsymbol{k}_{(\rho,\theta)}(\boldsymbol{H}\boldsymbol{\theta},\xi)$ we find that $\boldsymbol{M}$ is parallel to $\boldsymbol{\theta}$. This is representative of what we name *strong anisotropy.*

Reformulation of the vector relay

The characterization (2.6), (2.7) of the scalar relay can easily be extended to vectors. For any $(\boldsymbol{u},\xi) \in C^0([0,T])^3 \times [-1,1]$ and any $(\rho,\boldsymbol{\theta}) \in \mathcal{P} \times S^2$, by (2.3), (2.6) and (2.8) we have $\boldsymbol{w} \in \boldsymbol{k}_{(\rho,\theta)}(\boldsymbol{u},\xi)$ if and only if $\boldsymbol{w}(t) := w(t)\boldsymbol{\theta}$ for any t, and

$$w(0) = \begin{cases} -1 & \text{if } \boldsymbol{u}(0)\cdot\boldsymbol{\theta} < \rho_1, \\ \xi_{(\rho,\theta)} & \text{if } \rho_1 \leqq \boldsymbol{u}(0)\cdot\boldsymbol{\theta} \leqq \rho_2, \\ 1 & \text{if } \boldsymbol{u}(0)\cdot\boldsymbol{\theta} > \rho_2, \end{cases} \tag{2.10}$$

$$\left.\begin{array}{ll} |w(t)| & \leqq 1 \\ (w(t)-1)\big(\boldsymbol{u}(t)\cdot\boldsymbol{\theta} - \rho_2\big) & \geqq 0 \\ (w(t)+1)\big(\boldsymbol{u}(t)\cdot\boldsymbol{\theta} - \rho_1\big) & \geqq 0 \end{array}\right\} \qquad \forall t \in [0,T], \tag{2.11}$$

$$\int_0^t \boldsymbol{u}\cdot\boldsymbol{\theta}\,\mathrm{d}w \geqq \Psi_\rho(w;[0,t]) \qquad \forall t \in\,]0,T]. \tag{2.12}$$

This formulation of the vector-relay operator can also be extended to space-distributed systems, just assuming that $u(x,\cdot) \in C^0([0,T])$, $w(x,\cdot) \in BV(0,T)$, and (2.10), (2.11) and (2.12) hold a.e. in Ω. Let us set $\Omega_t := \Omega\times]0,t[$ for any $t > 0$. Recalling (2.7), Equation (2.12) may also be extended by requiring that

$$\begin{aligned} {}_{C^0(\overline{\Omega_t})}\left\langle \boldsymbol{u}, \frac{\partial \boldsymbol{w}}{\partial \tau}\right\rangle_{C^0(\overline{\Omega_t})'} \geqq{}& \frac{\rho_2+\rho_1}{2}\int_\Omega [w(x,t) - w^0(x)]\,\mathrm{d}x \\ &+ \frac{\rho_2-\rho_1}{2}\left\|\frac{\partial w}{\partial \tau}\right\|_{C^0(\overline{\Omega_t})'} \quad \forall t \in\,]0,T]. \end{aligned} \tag{2.13}$$

Notice that the second member coincides with $\int_{\bar{\Omega}} \Psi_\rho(w;[0,t])\,\mathrm{d}x$, cf. (2.7). Moreover,

$${}_{C^0(\overline{\Omega_t})}\left\langle \boldsymbol{u}, \frac{\partial \boldsymbol{w}}{\partial \tau}\right\rangle_{C^0(\overline{\Omega_t})'} = {}_{C^0(\overline{\Omega_t})}\left\langle \boldsymbol{u}\cdot\boldsymbol{\theta}, \frac{\partial w}{\partial \tau}\right\rangle_{C^0(\overline{\Omega_t})'}.$$

3. Some compactness results

In this section we collect some compactness results we shall use afterwards. Although these results might be set in the framework of MURAT and TARTAR's *compensated compactness* (cf. e.g. [28, 32]), here we derive them via a seemingly simpler approach based on a well-known compactness theorem due to Aubin and J.L. Lions; see, e.g., SIMON [31].

We remind the reader that $L^2_{\rm rot}(\mathbf{R}^3)^3 := \{v \in L^2(\mathbf{R}^3)^3 : \nabla\times v \in L^2(\mathbf{R}^3)^3\}$ ($\nabla\times :=$ curl) equipped with the graph norm is a Hilbert space. Let us denote by $\mathcal{B}$ an open ball of $\mathbf{R}^3$, by $\boldsymbol{\nu}$ an outward-oriented unit vector field orthogonal to $\partial\mathcal{B}$, and set $\mathcal{B}_T := \mathcal{B}\times]0,T[$. It is known, cf., e.g., [9, Chap. IX], [33, Chap. 1], that the space $L^2(\mathcal{B})^3$ is the direct sum of the orthogonal subspaces

$$\nabla\times H^1(\mathcal{B})^3 := \{\nabla\times\boldsymbol{\psi} : \boldsymbol{\psi}\in H^1(\mathcal{B})^3\},$$
$$\mathrm{Ker}(\nabla\times) := \{\boldsymbol{v}\in L^2(\mathcal{B})^3 : \nabla\times\boldsymbol{v} = \mathbf{0} \text{ in } \mathcal{D}'(\mathcal{B})^3,\ \boldsymbol{\nu}\times\boldsymbol{v} = \mathbf{0} \text{ on } \partial\mathcal{B}\}$$

(here $\boldsymbol{\nu}\times\boldsymbol{v}\in H^{-1/2}(\partial\mathcal{B})^3$), which are the image and the kernel of the curl operator. Thus

$$\boldsymbol{u} = \boldsymbol{u}^{\rm rot} + \boldsymbol{u}^{\rm irr}, \quad \boldsymbol{u}^{\rm rot}\in\nabla\times H^1(\mathcal{B})^3, \quad \boldsymbol{u}^{\rm irr}\in\mathrm{Ker}(\nabla\times) \qquad \forall\boldsymbol{u}\in L^2(\mathcal{B})^3; \tag{3.1}$$

$\boldsymbol{u}^{\rm rot}$ and $\boldsymbol{u}^{\rm irr}$ are the rotational and irrotational components of $\boldsymbol{u}$, respectively.

Proposition 3.1. *Let the sequences* $\{\boldsymbol{u}_m\}$ *and* $\{\boldsymbol{w}_m\}$ *be such that, for some* $\alpha, \beta > 0$,

$$\boldsymbol{u}_m \to \boldsymbol{u} \qquad \text{weakly in } L^2\big(0,T; L^2_{\rm rot}(\mathbf{R}^3)^3\big), \tag{3.2}$$
$$\boldsymbol{w}_m \to \boldsymbol{w} \qquad \text{weakly in } L^2(\mathbf{R}^3_T)^3 \cap H^\alpha\big(0,T; H^{-\beta}(\mathbf{R}^3)^3\big), \tag{3.3}$$
$$\nabla\cdot\boldsymbol{w}_m = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3), \text{ a.e. in }]0,T[. \tag{3.4}$$

Then

$$\iint_{\mathcal{B}_T} \boldsymbol{w}_m\cdot\boldsymbol{u}_m\,\mathrm{d}x\mathrm{d}t \to \iint_{\mathcal{B}_T} \boldsymbol{w}\cdot\boldsymbol{u}\,\mathrm{d}x d\mathrm{d}t \qquad \forall \text{ ball } \mathcal{B}\subset\mathbf{R}^3. \tag{3.5}$$

Proof. Let us fix any ball $\mathcal{B}\subset\mathbf{R}^3$. By (3.2),

$$\boldsymbol{u}_m^{\rm rot} \to \boldsymbol{u}^{\rm rot} \qquad \text{weakly in } L^2\big(0,T; H^1(\mathcal{B})^3\big).$$

By (3.3), the classic Aubin-Lions compactness theorem, see e.g. SIMON [31], yields

$$\boldsymbol{w}_m \to \boldsymbol{w} \qquad \text{strongly in } L^2\big(0,T; H^1(\mathcal{B})^3\big)'.$$

Therefore

$$\iint_{\mathcal{B}_T} \boldsymbol{w}_m\cdot\boldsymbol{u}_m^{\rm rot}\,\mathrm{d}x\,\mathrm{d}t \to \iint_{\mathcal{B}_T} \boldsymbol{w}\cdot\boldsymbol{u}^{\rm rot}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.6}$$

As the $\boldsymbol{w}_m$'s are divergence-free, $\boldsymbol{w}_m, \boldsymbol{w} \in \nabla \times H^1(\mathcal{B})^3$ a.e. in $]0, T[$ for any m. Hence

$$\iint_{\mathcal{B}_T} \boldsymbol{w}_m \cdot \boldsymbol{u}_m \,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathcal{B}_T} \boldsymbol{w}_m \cdot \boldsymbol{u}_m^{\mathrm{rot}} \,\mathrm{d}x\mathrm{d}t,$$
$$\iint_{\mathcal{B}_T} \boldsymbol{w} \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathcal{B}_T} \boldsymbol{w} \cdot \boldsymbol{u}^{\mathrm{rot}} \,\mathrm{d}x\,\mathrm{d}t.$$

(3.6) then entails (3.5). □

Lemma 3.2. *Let $\mathcal{B}$ be any open ball of $\mathbf{R}^3$, and the (scalar) sequences $\{u_m\}$ and $\{w_m\}$ be such that*

$$u_m \to u \qquad \text{weakly in } L^2(\mathcal{B}_T) \cap H^{-1}\big(0, T; H^1(\mathcal{B})\big), \tag{3.7}$$

$$w_m \to w \qquad \text{weakly star in } L^\infty(\mathcal{B}_T), \tag{3.8}$$

$$w_m \in L^1(\mathcal{B}; BV(0,T)), \qquad \|w_m\|_{L^1(\mathcal{B}; BV(0,T))} \leqq \text{Constant}. \tag{3.9}$$

Then

$$\iint_{\mathcal{B}_T} w_m u_m \,\mathrm{d}x\,\mathrm{d}t \to \iint_{\mathcal{B}_T} wu \,\mathrm{d}x\,\mathrm{d}t. \tag{3.10}$$

We refer to [38, Section 5] for the argument, which is based on Banach-space interpolation and on the compactness of Sobolev embeddings.

Proposition 3.3. *Let $\{\boldsymbol{u}_m\}$ and $\{\boldsymbol{z}_m\}$ be sequences such that*

$$\boldsymbol{u}_m \to \boldsymbol{u} \qquad \text{weakly in } L^2(\mathbf{R}^3_T)^3 \cap H^{-1}\big(0, T; L^2_{\mathrm{rot}}(\mathbf{R}^3)^3\big), \tag{3.11}$$

$$\boldsymbol{z}_m \to \boldsymbol{z} \qquad \text{weakly star in } L^\infty(\mathbf{R}^3_T)^3, \tag{3.12}$$

$$\boldsymbol{z}_m \in L^1\big(\mathbf{R}^3; BV(0,T)^3\big), \qquad \|\boldsymbol{z}_m\|_{L^1(\mathbf{R}^3; BV(0,T)^3)} \leqq \text{Constant}, \tag{3.13}$$

$$\nabla \cdot (\boldsymbol{u}_m + \boldsymbol{z}_m) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3), \text{ a.e. in }]0, T[, \forall m. \tag{3.14}$$

Then

$$\limsup_{m\to\infty} \iint_{\mathcal{B}_T} \boldsymbol{z}_m \cdot \boldsymbol{u}_m \,\mathrm{d}x\,\mathrm{d}t \leqq \iint_{\mathcal{B}_T} \boldsymbol{z} \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t \qquad \forall \text{ ball } \mathcal{B} \subset \mathbf{R}^3. \tag{3.15}$$

Proof. Let us fix any ball $\mathcal{B} \subset \mathbf{R}^3$. As $\boldsymbol{u}_m + \boldsymbol{z}_m$ and $\boldsymbol{u} + \boldsymbol{z}$ are divergence-free, $\boldsymbol{u}_m + \boldsymbol{z}_m, \boldsymbol{u} + \boldsymbol{z} \in \nabla \times H^1(\mathcal{B})^3$ for any m. Hence

$$\iint_{\mathcal{B}_T} \big(\boldsymbol{u}_m + \boldsymbol{z}_m\big) \cdot \boldsymbol{u}_m \,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathcal{B}_T} \big(\boldsymbol{u}_m + \boldsymbol{z}_m\big) \cdot \boldsymbol{u}_m^{\mathrm{rot}} \,\mathrm{d}x\,\mathrm{d}t \qquad \forall m,$$
$$\iint_{\mathcal{B}_T} \big(\boldsymbol{u} + \boldsymbol{z}\big) \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathcal{B}_T} \big(\boldsymbol{u} + \boldsymbol{z}\big) \cdot \boldsymbol{u}^{\mathrm{rot}} \,\mathrm{d}x\,\mathrm{d}t.$$

Therefore

$$\begin{aligned}\iint_{\mathcal{B}_T} \boldsymbol{z}_m \cdot \boldsymbol{u}_m \,\mathrm{d}x\,\mathrm{d}t &= \iint_{\mathcal{B}_T} \big[\big(\boldsymbol{u}_m + \boldsymbol{z}_m\big) \cdot \boldsymbol{u}_m - |\boldsymbol{u}_m|^2\big] \,\mathrm{d}x\,\mathrm{d}t \\ &= \iint_{\mathcal{B}_T} \big[\big(\boldsymbol{u}_m + \boldsymbol{z}_m\big) \cdot \boldsymbol{u}_m^{\mathrm{rot}} - |\boldsymbol{u}_m|^2\big] \,\mathrm{d}x\,\mathrm{d}t \\ &= \iint_{\mathcal{B}_T} \big[\big(|\boldsymbol{u}_m^{\mathrm{rot}}|^2 + \boldsymbol{z}_m \cdot \boldsymbol{u}_m^{\mathrm{rot}}\big) - |\boldsymbol{u}_m|^2\big] \,\mathrm{d}x\,\mathrm{d}t \\ &= \iint_{\mathcal{B}_T} \boldsymbol{z}_m \cdot \boldsymbol{u}_m^{\mathrm{rot}} \,\mathrm{d}x\,\mathrm{d}t - \iint_{\mathcal{B}_T} |\boldsymbol{u}_m^{\mathrm{irr}}|^2 \,\mathrm{d}x\,\mathrm{d}t \quad \forall m; \quad (3.16)\end{aligned}$$

similarly, $\iint_{\mathcal{B}_T} \boldsymbol{z} \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathcal{B}_T} \boldsymbol{z} \cdot \boldsymbol{u}^{\mathrm{rot}} \,\mathrm{d}x\,\mathrm{d}t - \iint_{\mathcal{B}_T} |\boldsymbol{u}^{\mathrm{irr}}|^2 \,\mathrm{d}x\,\mathrm{d}t$. By (3.11),

$$\boldsymbol{u}_m^{\mathrm{rot}} \to \boldsymbol{u}^{\mathrm{rot}} \qquad \text{weakly in } L^2(\mathcal{B}_T)^3 \cap H^{-1}\big(0, T; H^1(\mathcal{B})^3\big).$$

Applying Lemma 3.2 to the Cartesian components of $\boldsymbol{z}_m$ and $\boldsymbol{u}_m^{\mathrm{rot}}$, we then have

$$\begin{aligned}\iint_{\mathcal{B}_T} \boldsymbol{z}_m \cdot \boldsymbol{u}_m^{\mathrm{rot}} \,\mathrm{d}x\,\mathrm{d}t &= \sum_{i=1}^{3} \iint_{\mathcal{B}_T} (z_m)_i \cdot (u_m^{\mathrm{rot}})_i \,\mathrm{d}x\,\mathrm{d}t \to \sum_{i=1}^{3} \iint_{\mathcal{B}_T} z_i \cdot (u^{\mathrm{rot}})_i \,\mathrm{d}x\,\mathrm{d}t \\ &= \iint_{\mathcal{B}_T} \boldsymbol{z} \cdot \boldsymbol{u}^{\mathrm{rot}} \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Therefore, using also the lower semicontinuity of the norm,

$$\begin{aligned}&\limsup_{m\to\infty} \iint_{\mathcal{B}_T} \boldsymbol{z}_m \cdot \boldsymbol{u}_m \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \lim_{m\to\infty} \iint_{\mathcal{B}_T} \boldsymbol{z}_m \cdot \boldsymbol{u}_m^{\mathrm{rot}} \,\mathrm{d}x\,\mathrm{d}t - \liminf_{m\to\infty} \iint_{\mathcal{B}_T} |\boldsymbol{u}_m^{\mathrm{irr}}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqq \iint_{\mathcal{B}_T} \boldsymbol{z} \cdot \boldsymbol{u}^{\mathrm{rot}} \,\mathrm{d}x\,\mathrm{d}t - \iint_{\mathcal{B}_T} |\boldsymbol{u}^{\mathrm{irr}}|^2 \,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathcal{B}_T} \boldsymbol{z} \cdot \boldsymbol{u} \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

□

Remark. Proposition 3.3 holds also if (3.14) is replaced by the condition that there exists a sequence $\{\boldsymbol{\psi}_m\}$ in $L^2(\mathbf{R}^3_T)^3$ such that

$$\begin{aligned}&\nabla \cdot (\boldsymbol{u}_m + \boldsymbol{z}_m + \boldsymbol{\psi}_m) = 0 \quad \text{ in } \mathcal{D}'(\mathbf{R}^3), \text{ a.e. in }]0, T[, \forall m, \\ &\boldsymbol{\psi}_m \to \boldsymbol{\psi} \quad \text{ weakly in } L^2(\mathbf{R}^3_T)^3.\end{aligned} \qquad (3.17)$$

This can easily be checked by replacing $\boldsymbol{z}_m$ with $\boldsymbol{z}_m + \boldsymbol{\psi}_m$ in the above argument.

4. Stationary and quasi-stationary problems

In this section we provide a weak formulation of the quasi-stationary magnetostatic equations coupled with a space-dependent vector-relay operator. This can represent processes in either a ferromagnetic or ferrimagnetic nonhomogeneous material, in the presence of a slowly varying electric current field. We anticipate the formulation of a corresponding stationary problem without hysteresis, and prove the existence of a solution for it.

Henceforth we assume that our magnetic material occupies a bounded domain Ω of Lipschitz class, and is characterized by an anisotropy axis, $\boldsymbol{\theta}(x) \in S^2$, and a threshold pair, $\rho(x) \in \mathcal{P}$, at a.a. $x \in \Omega$. Of course we require the mapping $(\rho, \boldsymbol{\theta}) : \Omega \to \mathcal{P} \times S^2$ to be measurable.

Stationary Problem

We prescribe the electric current density $\boldsymbol{J} := \nabla \times \boldsymbol{K}_{\text{ext}}$, with $\boldsymbol{K}_{\text{ext}} \in L^2(\mathbf{R}^3)^3$, and constrain $\boldsymbol{M}(x)$ to be parallel to $\boldsymbol{\theta}(x)$ at a.a. $x \in \Omega$.

Problem 4.1. Find $\boldsymbol{H} \in L^2(\mathbf{R}^3)^3$ and $\boldsymbol{M} \in L^\infty(\Omega)^3$ such that, setting $\boldsymbol{M} := \mathbf{0}$ outside Ω and $\boldsymbol{B} := \boldsymbol{H} + \boldsymbol{M}$ a.e. in $\mathbf{R}^3$,

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{in } \mathcal{D}'\big(\mathbf{R}^3\big), \tag{4.1}$$

$$\nabla \times \boldsymbol{H} = \nabla \times \boldsymbol{K}_{\text{ext}} \qquad \text{in } \mathcal{D}'(\mathbf{R}^3)^3, \tag{4.2}$$

$$\boldsymbol{M} \times \boldsymbol{\theta} = \mathbf{0} \qquad \text{a.e. in } \Omega, \tag{4.3}$$

$$|\boldsymbol{M}| \leqq 1, \qquad \left.\begin{array}{lll} \boldsymbol{M}\cdot\boldsymbol{\theta} = -1 & \text{if} & \boldsymbol{H}\cdot\boldsymbol{\theta} < \rho_1 \\ \boldsymbol{M}\cdot\boldsymbol{\theta} = 1 & \text{if} & \boldsymbol{H}\cdot\boldsymbol{\theta} > \rho_2 \end{array}\right\} \qquad \text{a.e. in } \Omega. \tag{4.4}$$

Because of the discontinuity of the $\boldsymbol{M}$ vs. $\boldsymbol{H}$ relation (4.4), this is the weak formulation of a *free boundary problem*; see, e.g., [37, Section IV.8].

Theorem 4.1. *For any $\boldsymbol{K}_{\text{ext}} \in L^2(\mathbf{R}^3)^3$ Problem 4.1 has a solution.*

Proof. For any $v \in \mathbf{R}$, any $z \in [-1, 1]$, and a.a. $x \in \Omega$, let us set (cf. Fig. 3)

$$G_\rho(v; x, z) := \begin{cases} \{-1\} & \text{if } v < \rho_1(x), \\ [-1, z] & \text{if } v = \rho_1(x), \\ \{z\} & \text{if } \rho_1(x) < v < \rho_2(x), \\ [z, 1] & \text{if } v = \rho_2(x), \\ \{1\} & \text{if } v > \rho_2(x). \end{cases} \tag{4.5}$$

Let us fix any measurable function $z : \Omega \to [-1, 1]$, and replace (4.4) by the stronger condition $\boldsymbol{M}\cdot\boldsymbol{\theta} \in G_\rho\big(\boldsymbol{H}\cdot\boldsymbol{\theta}; z\big)$ a.e. in Ω. (Henceforth we shall not display the dependence on x.) Hence

$$\boldsymbol{B} := \boldsymbol{H} + \boldsymbol{M} \in \boldsymbol{H} + G_\rho(\boldsymbol{H}\cdot\boldsymbol{\theta}; z)\boldsymbol{\theta} =: \boldsymbol{F}_z(\boldsymbol{H}) \qquad \text{a.e. in } \Omega;$$

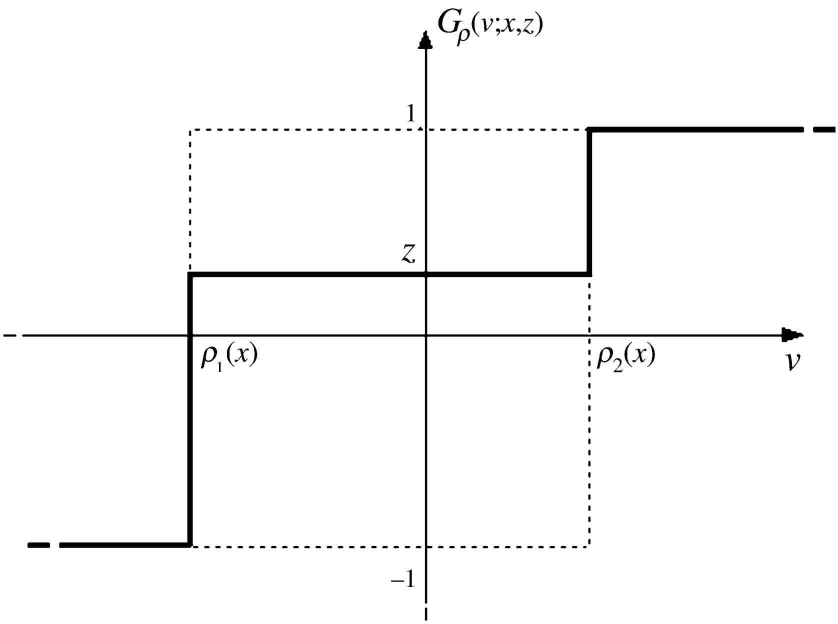

Fig. 3. Graph of the multivalued function $G_\rho(\cdot; x, z)$, for any fixed $x \in \Omega$ and any $z \in [-1, 1]$.

let us also set $\boldsymbol{F}_z(\boldsymbol{H}) := \boldsymbol{H}$ in $\mathbf{R}^3 \setminus \Omega$. As the multivalued vector mapping $G_\rho(\cdot; z)$ is cyclically monotone, the same holds for $\boldsymbol{F}_z \big(= \boldsymbol{F}_z(\cdot, x)\big)$ a.e. in $\mathbf{R}^3$, and for its inverse, $\boldsymbol{F}_z^{-1}$. The latter is then the subdifferential of a proper, lower semicontinuous, convex function, $\Phi_z : \mathbf{R}^3 \to \mathbf{R}$, which depends measurably on x via $z(x)$; thus $\boldsymbol{F}_z^{-1} = \partial\Phi_z$. The functional

$$\Phi : L^2(\mathbf{R}^3)^3 \to \mathbf{R} : \boldsymbol{v} \mapsto \int_{\mathbf{R}^3} \Phi_z\big(\boldsymbol{v}\big)\, \mathrm{d}x - \int_{\mathbf{R}^3} \boldsymbol{K}_{\mathrm{ext}} \cdot \boldsymbol{v}\, \mathrm{d}x \tag{4.6}$$

is then convex and lower semicontinuous. As G_ρ is uniformly bounded, Φ is coercive, i.e., $\Phi(v) \to +\infty$ as $\|v\|_{L^2(\mathbf{R}^3)^3} \to +\infty$. The restriction of Φ to $V := \{\boldsymbol{v} \in L^2(\mathbf{R}^3)^3 : \nabla\cdot\boldsymbol{v} = 0 \text{ in } \mathcal{D}'(\mathbf{R}^3)\}$ has then a minimum point, $\boldsymbol{B}$. Setting $I_V := 0$ in V and $I_V := +\infty$ outside V, the minimum condition $\partial(\Phi + I_V)\big(\boldsymbol{B}\big) \ni 0$ in V' also reads $\boldsymbol{F}_z^{-1}(\boldsymbol{B}) - \boldsymbol{K}_{\mathrm{ext}} + \partial I_V(\boldsymbol{B}) \ni 0$. Therefore

$$\boldsymbol{B} \in V, \qquad \exists \boldsymbol{H} \in \boldsymbol{F}_z^{-1}\big(\boldsymbol{B}\big) : \forall \boldsymbol{v} \in V, \int_{\mathbf{R}^3} \big(\boldsymbol{H} - \boldsymbol{K}_{\mathrm{ext}}\big)\cdot\boldsymbol{v}\, \mathrm{d}x = 0.$$

As the space of curl-free fields is the orthogonal complement in $L^2(\mathbf{R}^3)^3$ of the space of divergence-free fields, the magnetostatic equations (4.1) and (4.2) follow. Finally, by the above construction, $\boldsymbol{M} := \boldsymbol{B} - \boldsymbol{H}$ fulfils (4.3) and (4.4). □

Remark. The above argument holds for any choice of the function z, hence in general the stationary Problem 4.1 has several solutions. This *multistability* is at the basis of occurrence of hysteresis in evolution.

Quasi-stationary problem

We now provide a weak formulation of a quasi-stationary problem for the magnetostatic equations (1.7), coupled with a space-dependent vector-relay operator. This is obtained by coupling the conditions of the stationary Problem 4.1 with the dissipation condition (2.12).

We make the following assumptions on the initial data and on the exterior field $\boldsymbol{J}_{\text{ext}} = \nabla \times \boldsymbol{K}_{\text{ext}}$:

$$\begin{aligned} &\boldsymbol{K}_{\text{ext}} \in H^1\big(0, T; L^2(\mathbf{R}^3)^3\big), \quad \boldsymbol{H}^0 \in L^2(\mathbf{R}^3)^3, \\ &\boldsymbol{M}^0 \in L^\infty(\Omega)^3, \quad |\boldsymbol{M}^0| \leqq 1 \quad \text{a.e. in } \Omega, \end{aligned} \tag{4.7}$$

and, setting $\boldsymbol{M}^0 := \mathbf{0}$ outside Ω and $\boldsymbol{B}^0 := \boldsymbol{H}^0 + \boldsymbol{M}^0$ a.e. in $\mathbf{R}^3$,

$$\nabla \cdot \boldsymbol{B}^0 = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3). \tag{4.8}$$

Problem 4.2. Find $\boldsymbol{H} \in L^2\big(0, T; L^2_{\text{rot}}(\mathbf{R}^3)^3\big)$ and $\boldsymbol{M} \in L^\infty(\Omega_T)^3$ such that $\frac{\partial \boldsymbol{M}}{\partial t} \in \big(C^0(\overline{\Omega_T})^3\big)'$. Moreover, setting $\boldsymbol{M} := \mathbf{0}$ outside Ω and $\boldsymbol{B} := \boldsymbol{H} + \boldsymbol{M}$ a.e. in $\mathbf{R}^3_T$, we require that (4.1)–(4.4) hold a.e. in $]0, T[$, and

$$\begin{aligned} &\frac{1}{2}\int_{\mathbf{R}^3} \big(|\boldsymbol{H}(x,t)|^2 - |\boldsymbol{H}^0(x)|^2\big)\, \mathrm{d}x + \int_{\bar{\Omega}} \Psi_{\rho(x)}(\boldsymbol{M}(x,\cdot)\cdot\boldsymbol{\theta}(x); [0,t])\, \mathrm{d}x \\ &\quad \leqq \int_{\mathbf{R}^3} \big(\boldsymbol{B}(x,t)\cdot\boldsymbol{K}_{\text{ext}}(x,t) - \boldsymbol{B}^0(x)\cdot\boldsymbol{K}_{\text{ext}}(x,0)\big)\, \mathrm{d}x \\ &\qquad - \int_0^t\!\!\int_{\mathbf{R}^3} \boldsymbol{B}\cdot\frac{\partial \boldsymbol{K}_{\text{ext}}}{\partial \tau}\, \mathrm{d}x\, \mathrm{d}\tau \qquad \text{for a.a. } t \in\,]0, T[, \end{aligned} \tag{4.9}$$

$$\boldsymbol{M}(x,0) = \boldsymbol{M}^0(x) \qquad \text{for a.a. } x \in \Omega. \tag{4.10}$$

Interpretation. By (4.1) and (4.2), $\boldsymbol{B}$ and $\boldsymbol{H} - \boldsymbol{K}_{\text{ext}}$ are orthogonal in $L^2(\mathbf{R}^3)^3$ a.e. in $]0, T[$. The same then holds for $\frac{\partial \boldsymbol{B}}{\partial t}$ and $\boldsymbol{H} - \boldsymbol{K}_{\text{ext}}$:

$$\int_0^t\!\!\int_{\mathbf{R}^3} \frac{\partial \boldsymbol{B}}{\partial \tau}\cdot(\boldsymbol{H} - \boldsymbol{K}_{\text{ext}})\, \mathrm{d}x\, \mathrm{d}\tau = 0 \qquad \forall t \in\,]0, T], \tag{4.11}$$

provided that this integral has a meaning. However, we do not know whether $\frac{\partial \boldsymbol{B}}{\partial t} \in L^2(\mathbf{R}^3)^3$; this interpretation should then be regarded just as *formal* (i.e., nonrigorous). Along the same lines, (4.11) allows us to write (4.9) in the equivalent form

$$\int_0^t\!\!\int_\Omega \frac{\partial \boldsymbol{M}}{\partial \tau}\cdot\boldsymbol{H}\, \mathrm{d}x\, \mathrm{d}\tau \geqq \int_{\bar{\Omega}} \Psi_{\rho(x)}(\boldsymbol{M}(x,\cdot)\cdot\boldsymbol{\theta}(x); [0,t])\, \mathrm{d}x \qquad \forall t \in\,]0, T], \tag{4.12}$$

which can be compared with (2.12). The latter condition coupled with (4.3), (4.4) a.e. in $]0, T[$ and with (4.10) accounts for the hysteresis relation

$$\boldsymbol{M} \in \boldsymbol{k}_{(\rho,\theta)}\big(\boldsymbol{H}, \boldsymbol{M}^0\big) \qquad \text{a.e. in } \Omega_T. \tag{4.13}$$

In conclusion, Problem 4.2 is a weak formulation of the magnetostatic equations (1.7), coupled with the hysteresis relation (4.13).

Equation (4.9) represents the energy balance: the first integral equals the variation of magnetic energy, the second one is the dissipated energy, and the second member is the energy provided by the exterior field $\boldsymbol{K}_{\mathrm{ext}}$. It is also easy to see that (4.4) is equivalent to

$$\Big(\boldsymbol{H}\cdot\boldsymbol{\theta} - \frac{\rho_1+\rho_2}{2}\Big)\,\big(\boldsymbol{M}\cdot\boldsymbol{\theta} - v\big) + \frac{\rho_2-\rho_1}{2}|\boldsymbol{M}\cdot\boldsymbol{\theta} - v| \geqq 0 \quad \forall v \in [-1,1],\ \text{a.e. in } \Omega_T. \tag{4.14}$$

The magnetostatic equations (4.1) and (4.2) might also be incorporated into this variational principle, by suitably reformulating it. The resulting variational inequality might then be compared with the *stability condition* that, coupled with an energy balance analogous to (4.9), Mielke and other researchers recently proposed as a general framework for a number of hysteresis phenomena, cf. [24–26].

Existence result for the quasi-stationary problem

Theorem 4.2. *Assume that* (4.7) *and* (4.8) *hold, and that*

$$\boldsymbol{K}_{\mathrm{ext}} \in L^2\big(0,T; L^2_{\mathrm{rot}}(\mathbf{R}^3)^3\big), \qquad \frac{\partial \boldsymbol{K}_{\mathrm{ext}}}{\partial t} \in L^\infty(\Omega_T)^3. \tag{4.15}$$

Then there exists a solution of Problem 4.2 such that

$$\boldsymbol{H} \in H^1\big(0,T; L^2(\mathbf{R}^3)^3\big) \cap L^2\big(0,T; L^2_{\mathrm{rot}}(\mathbf{R}^3)^3\big). \tag{4.16}$$

Proof. (i) *Approximation.* Let us fix any $m \in \mathbf{N}$, set

$$k := \frac{T}{m}, \quad \boldsymbol{H}^0_m := \boldsymbol{H}^0, \quad \boldsymbol{M}^0_m := \boldsymbol{M}^0,$$
$$\boldsymbol{K}^n_{\mathrm{ext\,m}} := k^{-1}\int_{(n-1)k}^{nk} \boldsymbol{K}_{\mathrm{ext}}(\xi)\,\mathrm{d}\xi \quad \text{for } n = 1,\ldots,m,$$

and define the multivalued function G_ρ as in (4.5). We now introduce a time-discretization scheme.

Problem 4.2$_{\mathbf{m}}$. Find $\boldsymbol{H}^n_m \in L^2_{\mathrm{rot}}(\mathbf{R}^3)^3$ and $\boldsymbol{M}^n_m \in L^\infty(\Omega)^3$ for $n = 1, ..., m$, such that, setting $\boldsymbol{M}^n_m := \mathbf{0}$ outside Ω and $\boldsymbol{B}^n_m := \boldsymbol{H}^n_m + \boldsymbol{M}^n_m$ a.e. in $\mathbf{R}^3$,

$$\nabla\cdot\boldsymbol{B}^n_m = 0 \qquad \text{in } \mathcal{D}'\big(\mathbf{R}^3\big), \text{ for } n = 1, ..., m, \tag{4.17}$$
$$\nabla\times\boldsymbol{H}^n_m = \nabla\times\boldsymbol{K}^n_{\mathrm{ext\,m}} \qquad \text{in } \mathcal{D}'(\mathbf{R}^3)^3, \text{ for } n = 1, ..., m, \tag{4.18}$$
$$\boldsymbol{M}^n_m \in G_\rho\big(\boldsymbol{H}^n_m\cdot\boldsymbol{\theta}; \boldsymbol{M}^{n-1}_m\cdot\boldsymbol{\theta}\big)\boldsymbol{\theta} \qquad \text{a.e. in } \Omega, \text{ for } n = 1, ..., m. \tag{4.19}$$

For any n, by the argument of Theorem 4.1 there exists a solution of this problem. It is not difficult to see that $\boldsymbol{H}$ is uniquely determined; but this does not mean that $\boldsymbol{M}$ is also unique, since the mapping G_ρ is multivalued.

(ii) *A Priori Estimates.* Let us first define time-interpolate functions. For any family $\{v^n_m\}_{n=1,\ldots,m}$ of functions $\mathbf{R}^3 \to \mathbf{R}$, let us denote by v_m the piecewise linear time-interpolate of $v_m^0 := v^0$, ..., v_m^m a.e. in $\mathbf{R}^3$, and define the piecewise constant function $\bar{v}_m(\cdot, t) := v_m^n$ a.e. in $\mathbf{R}^3$, if $(n-1)k < t \leqq nk$, for $n = 1, \dots, m$. We use this notation for vector functions, too. By (4.17) and (4.18), $\boldsymbol{B}_m^n - \boldsymbol{B}_m^{n-1}$ and $\boldsymbol{H}_m^n - \boldsymbol{K}_{\mathrm{ext}\,m}^n$ are orthogonal in $L^2(\mathbf{R}^3)^3$, that is,

$$\int_{\mathbf{R}^3} \left(\boldsymbol{B}_m^n - \boldsymbol{B}_m^{n-1}\right)\cdot\left(\boldsymbol{H}_m^n - \boldsymbol{K}_{\mathrm{ext}\,m}^n\right) \mathrm{d}x = 0 \qquad \text{for } n = 1, \dots, m. \tag{4.20}$$

Setting $M_m^n := \boldsymbol{M}_m^n \cdot \boldsymbol{\theta}$, we have $\boldsymbol{M}_m^n = M_m^n \boldsymbol{\theta}$ a.e. in Ω, and by (4.19)

$$\begin{aligned}
&\sum_{n=1}^{\ell} \left(\boldsymbol{M}_m^n - \boldsymbol{M}_m^{n-1}\right)\cdot \boldsymbol{H}_m^n \\
&\quad = \sum_{n=1}^{\ell} \left(M_m^n - M_m^{n-1}\right) \boldsymbol{H}_m^n \cdot \boldsymbol{\theta} \\
&\quad \geqq \sum_{n=1}^{\ell} \left[\left(M_m^n - M_m^{n-1}\right)^+ \rho_2 - \left(M_m^n - M_m^{n-1}\right)^- \rho_1\right] \\
&\quad = \Psi_{\rho(x)}(M_m; [0, \ell k])
\end{aligned} \tag{4.21}$$

a.e. in Ω, for $\ell = 1, \dots, m$. Summing (4.20) with respect to n, we then get

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbf{R}^3} \left(|\boldsymbol{H}_m^\ell|^2 - |\boldsymbol{H}^0|^2\right) \mathrm{d}x + \int_\Omega \Psi_{\rho(x)}(M_m; [0, \ell k])\, \mathrm{d}x \\
&\quad \leqq \int_{\mathbf{R}^3} \boldsymbol{B}_m^\ell \cdot \boldsymbol{K}_{\mathrm{ext}\,m}\, \mathrm{d}x - \int_{\mathbf{R}^3} \boldsymbol{B}^0 \cdot \boldsymbol{K}_{\mathrm{ext}}^0\, \mathrm{d}x \\
&\qquad - \sum_{n=1}^{\ell} \int_{\mathbf{R}^3} \boldsymbol{B}_m^{n-1} \cdot \left(\boldsymbol{K}_{\mathrm{ext}\,m}^n - \boldsymbol{K}_{\mathrm{ext}\,m}^{n-1}\right) \mathrm{d}x
\end{aligned} \tag{4.22}$$

for $\ell = 1, \dots, m$. By (4.15), a standard calculation then yields

$$\|\boldsymbol{H}_m\|_{L^\infty(0,T;L^2(\mathbf{R}^3)^3)}\,, \|\Psi_{\rho(x)}(M_m(x, \cdot); [0, t])\|_{L^\infty(0,T;L^1(\Omega))} \leqq C_1. \tag{4.23}$$

(By C_1, C_2, ... we denote suitable positive constants independent of m.) By (4.18) and (4.23) we also have

$$\|\boldsymbol{H}_m\|_{L^2(0,T;L^2_{\mathrm{rot}}(\mathbf{R}^3)^3)}\,, \left\|\frac{\partial \boldsymbol{M}_m}{\partial t}\right\|_{L^1(\Omega_T)^3} \leqq C_2. \tag{4.24}$$

(iii) *Further A Priori Estimates.* Similarly to (4.20),

$$\int_{\mathbf{R}^3} \left(\boldsymbol{B}_m^n - \boldsymbol{B}_m^{n-1}\right)\cdot\left(\boldsymbol{H}_m^n - \boldsymbol{H}_m^{n-1} - \boldsymbol{K}_{\mathrm{ext}\,m}^n + \boldsymbol{K}_{\mathrm{ext}\,m}^{n-1}\right) \mathrm{d}x = 0 \qquad \text{for } n = 1, \dots, m. \tag{4.25}$$

The monotonicity of $G_\rho(\cdot; M_m^{n-1})$ entails

$$\left(\boldsymbol{M}_m^n - \boldsymbol{M}_m^{n-1}\right)\cdot\left(\boldsymbol{H}_m^n - \boldsymbol{H}_m^{n-1}\right) \geqq 0 \qquad \text{a.e. in } \Omega. \tag{4.26}$$

Summing (4.25) with respect to n, we then have

$$\begin{aligned}
&\sum_{n=1}^{\ell} \int_{\mathbf{R}^3} \left|\boldsymbol{H}_m^n - \boldsymbol{H}_m^{n-1}\right|^2 \mathrm{d}x \\
&\quad \leqq \sum_{n=1}^{\ell} \int_{\mathbf{R}^3} \left(\boldsymbol{B}_m^n - \boldsymbol{B}_m^{n-1}\right)\cdot\left(\boldsymbol{H}_m^n - \boldsymbol{H}_m^{n-1}\right) \mathrm{d}x \\
&\quad = \sum_{n=1}^{\ell} \int_{\mathbf{R}^3} \left(\boldsymbol{B}_m^n - \boldsymbol{B}_m^{n-1}\right)\cdot\left(\boldsymbol{K}_{\mathrm{ext}\,m}^n - \boldsymbol{K}_{\mathrm{ext}\,m}^{n-1}\right) \mathrm{d}x \\
&\quad \leqq \left(k\sum_{n=1}^{\ell} \|\boldsymbol{H}_m^n - \boldsymbol{H}_m^{n-1}\|^2_{L^2(\mathbf{R}^3)^3}\right)^{1/2} \left\|\frac{\partial \boldsymbol{K}_{\mathrm{ext}\,m}}{\partial t}\right\|_{L^2(\mathbf{R}^3_T)^3} \\
&\qquad + \left(k\sum_{n=1}^{\ell} \|\boldsymbol{M}_m^n - \boldsymbol{M}_m^{n-1}\|^2_{L^1(\mathbf{R}^3)^3}\right)^{1/2} \left\|\frac{\partial \boldsymbol{K}_{\mathrm{ext}\,m}}{\partial t}\right\|_{L^\infty(\Omega_T)^3}. \qquad (4.27)
\end{aligned}$$

By (4.15) and (4.23), we then get

$$\|\boldsymbol{H}_m\|_{H^1(0,T;L^2(\mathbf{R}^3)^3)} \leqq C_3. \qquad (4.28)$$

For any $\alpha \in]0, 1/2[$ and any $s > 3/2$, $BV(0,T) \subset H^\alpha(0,T)$ and $L^1(\mathbf{R}^3) \subset H^{-s}(\mathbf{R}^3)$; hence

$$L^1\left(\mathbf{R}^3; BV(0,T)^3\right) \subset H^{-s}\left(\mathbf{R}^3; H^\alpha(0,T)^3\right) = H^\alpha\left(0,T; H^{-s}(\mathbf{R}^3)^3\right). \qquad (4.29)$$

Equations (4.24) and (4.28) then yield

$$\|\boldsymbol{B}_m\|_{H^\alpha(0,T;H^{-s}(\mathbf{R}^3)^3)}, \|\boldsymbol{M}_m\|_{H^\alpha(0,T;H^{-s}(\Omega)^3)} \leqq C_4 \qquad \forall\alpha \in]0, 1/2[, \forall s > 3/2. \qquad (4.30)$$

(iv) *Limit Procedure.* By the above estimates there exist $\boldsymbol{H}$ and $\boldsymbol{M}$ such that, as $m \to \infty$ along a suitable sequence, setting $\boldsymbol{B} := \boldsymbol{H} + \boldsymbol{M}$, for any $\alpha \in]0, 1/2[$ and any $s > 3/2$,

$$\boldsymbol{H}_m \to \boldsymbol{H} \quad \text{weakly in } H^1\left(0,T; L^2(\mathbf{R}^3)^3\right) \cap L^2\left(0,T; L^2_{\mathrm{rot}}(\mathbf{R}^3)^3\right), \qquad (4.31)$$

$$\boldsymbol{M}_m \to \boldsymbol{M} \quad \text{weakly star in } L^\infty(\Omega_T)^3 \cap H^\alpha\left(0,T; H^{-s}(\Omega)^3\right), \qquad (4.32)$$

$$\bar{\boldsymbol{B}}_m, \boldsymbol{B}_m \to \boldsymbol{B} \quad \text{weakly in } L^2(\mathbf{R}^3_T)^3 \cap H^\alpha\left(0,T; H^{-s}(\mathbf{R}^3)^3\right). \qquad (4.33)$$

Let $\mathcal{B} \subset \Omega$ be any ball of $\mathbf{R}^3$ and set $\mathcal{B}_T := \mathcal{B}\times]0,T[$. Let us fix any nonnegative function $\psi \in C^\infty([0,T])$. As the $\boldsymbol{B}_m$'s are divergence-free, we can apply Proposition 3.1 to the sequences $\{\bar{\boldsymbol{H}}_m\psi\}$ and $\{\bar{\boldsymbol{B}}_m\}$, getting

$$\iint_{\mathcal{B}_T} \bar{\boldsymbol{B}}_m\cdot\bar{\boldsymbol{H}}_m\psi(t)\,\mathrm{d}x\,\mathrm{d}t \to \iint_{\mathcal{B}_T} \boldsymbol{B}\cdot\boldsymbol{H}\psi(t)\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.34)$$

This statement will be used afterwards in this proof.

Let us set $M_m^n := \boldsymbol{M}_m^n\cdot\boldsymbol{\theta}$ and $M := \boldsymbol{M}\cdot\boldsymbol{\theta}$ a.e. in Ω_T for any m, n. By passing to the limit in (4.27) and (4.28), we get the magnetostatic equations (4.1) and (4.2) a.e. in $]0,T[$. By (4.19), $\boldsymbol{M}_m\times\boldsymbol{\theta} = 0$ a.e. in Ω_T; (4.3) then follows. We are left with the proof of (4.4) a.e. in $]0, T[$ and of (4.9). The fact that $(4.4)_1$ holds a.e. in $]0, T[$ is a direct consequence of (4.19), which also entails $(\bar{M}_m + 1)\big(\bar{\boldsymbol{H}}_m\cdot\boldsymbol{\theta} - \rho_1\big) \geqq 0$ a.e. in Ω_T. For any nonnegative function $\psi \in C^\infty([0, T])$, we then have

$$\iint_{\mathcal{B}_T} (\bar{M}_m + 1)\big(\bar{\boldsymbol{H}}_m\cdot\boldsymbol{\theta} - \rho_1\big)\psi(t)\,\mathrm{d}x\,\mathrm{d}t \geqq 0. \tag{4.35}$$

By (4.34) and by the lower semicontinuity of the norm,

$$\begin{aligned}
&\limsup_{m\to\infty} \iint_{\mathcal{B}_T} \bar{M}_m\ \bar{\boldsymbol{H}}_m\cdot\boldsymbol{\theta}\psi(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \limsup_{m\to\infty} \iint_{\mathcal{B}_T} \bar{\boldsymbol{M}}_m\cdot\bar{\boldsymbol{H}}_m\psi(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \lim_{m\to\infty} \iint_{\mathcal{B}_T} \bar{\boldsymbol{B}}_m\cdot\bar{\boldsymbol{H}}_m\psi(t)\,\mathrm{d}x\,\mathrm{d}t - \liminf_{m\to\infty} \iint_{\mathcal{B}_T} |\bar{\boldsymbol{H}}_m|^2\psi(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqq \iint_{\mathcal{B}_T} \boldsymbol{B}\cdot\boldsymbol{H}\psi(t)\,\mathrm{d}x\,\mathrm{d}t - \iint_{\mathcal{B}_T} |\boldsymbol{H}|^2\psi(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \iint_{\mathcal{B}_T} \boldsymbol{M}\cdot\boldsymbol{H}\ \psi(t)\,\mathrm{d}x\,\mathrm{d}t = \iint_{\mathcal{B}_T} M\ \boldsymbol{H}\cdot\boldsymbol{\theta}\ \psi(t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.36}$$

Passing to the superior limit in (4.35), we then get

$$\iint_{\mathcal{B}_T} \big(M + 1\big)\big(\boldsymbol{H}\cdot\boldsymbol{\theta} - \rho_1\big)\psi(t)\,\mathrm{d}x\,\mathrm{d}t \geqq 0.$$

As this holds for any ball $\mathcal{B}$ and any nonnegative smooth function ψ, this inequality is equivalent to $(M + 1)\big(\boldsymbol{H}\cdot\boldsymbol{\theta} - \rho_1\big) \geqq 0$ a.e. in Ω_T, i.e., $(4.4)_2$ a.e. in $]0, T[$. Equation $(4.4)_3$ can be derived similarly. Finally, (4.22) also reads

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^3} \big(|\bar{\boldsymbol{H}}_m(x, t)|^2 - |\boldsymbol{H}^0(x)|^2\big)\,\mathrm{d}x + \int_\Omega \Psi_{\rho(x)}(M_m; [0, \ell k])\,\mathrm{d}x \\
&\quad\leqq \int_{\mathbf{R}^3} \bar{\boldsymbol{B}}_m(x, t)\cdot\bar{\boldsymbol{K}}_{\mathrm{ext}\,m}(x, t)\,\mathrm{d}x - \int_{\mathbf{R}^3} \boldsymbol{B}^0(x)\cdot\boldsymbol{K}_{\mathrm{ext}\,m}(x, 0)\,\mathrm{d}x \\
&\qquad - \int_0^t\!\!\int_{\mathbf{R}^3} \bar{\boldsymbol{B}}_m\cdot\frac{\partial\boldsymbol{K}_{\mathrm{ext}\,m}}{\partial\tau}\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned} \tag{4.37}$$

for any $t \in\ ]0, T]$. Passing to the inferior limit as $m \to \infty$, we get (4.9) by lower semicontinuity. More precisely, at first we multiply (4.37) by any smooth positive function of time, ψ, integrate it in time, and then pass to the inferior limit as $m \to \infty$. As this holds for any ψ, (4.9) follows. □

Remark. Notice that we have proved existence of a solution $(\boldsymbol{H}, \boldsymbol{M})$ of Problem 4.2 such that, cf. (4.27),

$$\begin{aligned}
\int_0^{\tilde{t}} \Big|\frac{\partial\boldsymbol{H}}{\partial t}\Big|^2\,\mathrm{d}t \leqq \int_0^{\tilde{t}} \bigg(&\Big\|\frac{\partial\boldsymbol{H}}{\partial t}\Big\|_{L^2(\mathbf{R}^3)^3}\Big\|\frac{\partial\boldsymbol{K}_{\mathrm{ext}}}{\partial t}\Big\|_{L^2(\mathbf{R}^3)^3}\bigg)\,\mathrm{d}t \\
&+ \Big\|\frac{\partial\boldsymbol{M}}{\partial t}\Big\|_{(C^0(\overline{\Omega_T})^3)'}\Big\|\frac{\partial\boldsymbol{K}_{\mathrm{ext}}}{\partial t}\Big\|_{L^\infty(\overline{\Omega_T})^3},
\end{aligned} \tag{4.38}$$

for any $\tilde{t} \in\ ]0, T]$.

5. Parabolic-hyperbolic problem

In this section we provide a weak formulation of a Cauchy problem for the eddy-current problem for a strongly anisotropic, nonhomogeneous ferromagnetic material, which occupies a bounded domain Ω of Lipschitz class. We assume that a measurable function $(\rho, \boldsymbol{\theta}) : \Omega \to \mathcal{P} \times S^2$ is prescribed and that, setting $\boldsymbol{g}_{\text{ext}} := \chi_\Omega \boldsymbol{E}_{\text{ext}} + \boldsymbol{J}_{\text{ext}}$,

$$\boldsymbol{g}_{\text{ext}} \in L^2\big(0,T; L^2(\mathbf{R}^3)^3\big), \quad \nabla\cdot\boldsymbol{g}_{\text{ext}} = 0 \text{ in } \mathcal{D}'(\mathbf{R}^3)^3, \text{ a.e. in }]0,T[, \tag{5.1}$$

$$\boldsymbol{E}^0 \in L^2(\mathbf{R}^3 \setminus \Omega)^3,\ \boldsymbol{H}^0 \in L^2(\mathbf{R}^3)^3,\ \boldsymbol{M}^0 \in L^\infty(\Omega)^3,\ |\boldsymbol{M}^0| \leqq 1 \text{ a.e. in } \Omega. \tag{5.2}$$

Moreover we set $\boldsymbol{M}^0 := \mathbf{0}$ outside Ω, $\boldsymbol{B}^0 := \boldsymbol{H}^0 + \boldsymbol{M}^0$ a.e. in $\mathbf{R}^3$, and assume that

$$\nabla\cdot\boldsymbol{B}^0 = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3)^3. \tag{5.3}$$

Problem 5.1. Find $\boldsymbol{E}, \boldsymbol{H} \in L^2(\mathbf{R}^3_T)^3$ and $\boldsymbol{M} \in L^\infty(\Omega_T)^3$ such that $\frac{\partial \boldsymbol{M}}{\partial t} \in \big(C^0(\overline{\Omega_T})^3\big)'$. Moreover, setting $\boldsymbol{M} := \mathbf{0}$ outside Ω_T and $\boldsymbol{B} := \boldsymbol{H} + \boldsymbol{M}$ a.e. in $\mathbf{R}^3_T$, we require that

$$\iint_{\mathbf{R}^3_T} \Big[\boldsymbol{H}\cdot\nabla\times\boldsymbol{v} - \big(\chi_\Omega\boldsymbol{E} + \boldsymbol{g}_{\text{ext}}\big)\cdot\boldsymbol{v} + (1-\chi_\Omega)\big(\boldsymbol{E} - \boldsymbol{E}^0\big)\cdot\frac{\partial \boldsymbol{v}}{\partial t}\Big]\,\mathrm{d}x\,\mathrm{d}t = 0$$
$$\forall \boldsymbol{v} \in H^1(\mathbf{R}^3_T)^3,\ \boldsymbol{v}(\cdot,T) = \mathbf{0} \ \text{ in } \mathbf{R}^3, \tag{5.4}$$

$$\iint_{\mathbf{R}^3_T} \Big[\boldsymbol{E}\cdot\nabla\times\boldsymbol{v} + \big(\boldsymbol{B}^0 - \boldsymbol{B}\big)\cdot\frac{\partial \boldsymbol{v}}{\partial t}\Big]\,\mathrm{d}x\,\mathrm{d}t = 0$$
$$\forall \boldsymbol{v} \in H^1(\mathbf{R}^3_T)^3,\ \boldsymbol{v}(\cdot,T) = \mathbf{0} \ \text{ in } \mathbf{R}^3, \tag{5.5}$$

$$\boldsymbol{M}\times\boldsymbol{\theta} = \mathbf{0} \qquad \text{a.e. in } \Omega_T, \tag{5.6}$$

$$\left.\begin{array}{lll} |\boldsymbol{M}| & \leqq 1 & \\ \boldsymbol{M}\cdot\boldsymbol{\theta} & = -1 & \text{if } \boldsymbol{H}\cdot\boldsymbol{\theta} < \rho_1 \\ \boldsymbol{M}\cdot\boldsymbol{\theta} & = 1 & \text{if } \boldsymbol{H}\cdot\boldsymbol{\theta} > \rho_2 \end{array}\right\} \qquad \text{a.e. in } \Omega_T, \tag{5.7}$$

$$\frac{1}{2}\int_{\mathbf{R}^3} \big(|\boldsymbol{H}(x,t)|^2 - |\boldsymbol{H}^0(x)|^2\big)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbf{R}^3\setminus\Omega} \big(|\boldsymbol{E}(x,t)|^2 - |\boldsymbol{E}^0(x)|^2\big)\,\mathrm{d}x$$
$$+ \int_{\bar{\Omega}} \Psi_{\rho(x)}(\boldsymbol{M}(x,\cdot)\cdot\boldsymbol{\theta}(x); [0,t])\,\mathrm{d}x + \int_0^t\!\!\int_\Omega |\boldsymbol{E}|^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ \int_0^t\!\!\int_{\mathbf{R}^3} \boldsymbol{g}_{\text{ext}}\cdot\boldsymbol{E}\,\mathrm{d}x\,\mathrm{d}\tau \leqq 0 \qquad \text{for a.a. } t \in]0,T[, \tag{5.8}$$

$$\boldsymbol{M}(x,0) = \boldsymbol{M}^0(x) \qquad \text{for a.a. } x \in \Omega. \tag{5.9}$$

Interpretation. Equations (5.4) and (5.5) respectively entail the equations

$$\nabla\times\boldsymbol{H} = \chi_\Omega\boldsymbol{E} + (1-\chi_\Omega)\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{g}_{\text{ext}} \quad \text{in } L^2\big(0,T; \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)'\big), \tag{5.10}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \text{in } L^2\big(0,T; \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)'\big), \tag{5.11}$$

which are a weak form of (1.8) and (1.2), respectively. A comparison of the terms of these equations yields

$$\boldsymbol{H}\big|_{\Omega_T} \in L^2\big(0,T; L^2_{\mathrm{rot}}(\Omega)^3\big), \quad (1-\chi_\Omega)\frac{\partial \boldsymbol{E}}{\partial t}, \frac{\partial \boldsymbol{B}}{\partial t} \in L^2\big(0,T; \big(L^2_{\mathrm{rot}}(\mathbf{R}^3)^3\big)'\big). \tag{5.12}$$

If $\boldsymbol{g}_{\mathrm{ext}} := \chi_\Omega \boldsymbol{E}_{\mathrm{app}} + \boldsymbol{J}_{\mathrm{ext}}$, (5.10) is equivalent to (1.8). By integrating by parts in (5.4) and (5.5), we get the initial conditions

$$(1-\chi_\Omega)\boldsymbol{E}\big|_{t=0} = \boldsymbol{E}^0, \quad \boldsymbol{B}\big|_{t=0} = \boldsymbol{B}^0 \qquad \text{in } \big(L^2_{\mathrm{rot}}(\mathbf{R}^3)^3\big)'. \tag{5.13}$$

Conversely, (5.10), (5.11) and (5.13) yield (5.4) and (5.5). The system (5.4) and (5.5) is thus a weak formulation of a Cauchy problem for the eddy-current system (1.2) and (1.8).

Let us assume that $\frac{\partial \boldsymbol{E}}{\partial t}, \frac{\partial \boldsymbol{B}}{\partial t} \in L^2\big(0,T; L^2(\mathbf{R}^3)^3\big)$. Multiplying (5.10) by $\boldsymbol{E}$, (5.11) by $-\boldsymbol{H}$, summing these equalities and integrating in time, we get the energy-integral formula

$$\begin{aligned}\int_0^t\!\!\int_{\mathbf{R}^3} \frac{\partial \boldsymbol{B}}{\partial \tau}\cdot \boldsymbol{H}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_{\mathbf{R}^3\setminus\Omega} \big(|\boldsymbol{E}(x,t)|^2 - |\boldsymbol{E}^0|^2\big)\,\mathrm{d}x \\ + \int_0^t\!\!\int_\Omega |\boldsymbol{E}|^2\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_{\mathbf{R}^3} \boldsymbol{g}_{\mathrm{ext}}\cdot\boldsymbol{E}\,\mathrm{d}x\,\mathrm{d}\tau = 0 \qquad \forall t \in\,]0,T].\end{aligned} \tag{5.14}$$

This is just *formal,* for $\boldsymbol{E}$ and $\boldsymbol{B}$ may not have the required regularity. By (5.14), (5.8) formally reads

$$\int_0^t\!\!\int_{\mathbf{R}^3} \frac{\partial \boldsymbol{M}}{\partial \tau}\cdot \boldsymbol{H}\,\mathrm{d}x\,\mathrm{d}\tau \geqq \int_{\bar\Omega} \Psi_{\rho(x)}(\boldsymbol{M}(x,\cdot)\cdot\boldsymbol{\theta}(x); [0,t])\,\mathrm{d}x \qquad \forall t \in\,]0,T].$$

As we saw, this inequality can be regarded as a weak formulation of (2.12) for $\boldsymbol{M}$ and $\boldsymbol{H}$. The latter inequality, (5.6), (5.7) and the initial condition (5.9) formally account for the hysteresis relation

$$\boldsymbol{M} \in \boldsymbol{k}_{(\rho(x),\theta(x))}\big(\boldsymbol{H}, \boldsymbol{M}^0\big) \qquad \text{a.e. in } \Omega_T. \tag{5.15}$$

In conclusion, Problem 5.1 is a weak formulation of a Cauchy problem associated with the system (5.10), (5.11) and (5.15). This problem is parabolic with hysteresis in Ω, linear hyperbolic outside. The discontinuity of the constitutive relation accounts for the possible onset of moving fronts, which separate regions characterized by different values of the magnetization field $\boldsymbol{M}$; see e.g. [37, Sect. IV.8].

Existence result for the parabolic-hyperbolic problem

Theorem 5.1. *Assume that* (5.1)–(5.3) *hold. Then there exists a solution of Problem 5.1 such that*

$$\begin{aligned} &\boldsymbol{E} \in L^2(\mathbf{R}^3_T)^3 \cap L^\infty\big(0,T;L^2(\mathbf{R}^3\setminus\Omega)^3\big),\\ &\boldsymbol{H} \in L^\infty\big(0,T;L^2(\mathbf{R}^3)^3\big) \cap L^2\big(0,T;L^2_{\text{rot}}(\Omega)^3\big). \end{aligned} \tag{5.16}$$

Proof. (i) *Approximation.* Let us fix any $m \in \mathbf{N}$, and let us set

$$\begin{aligned} &k := T/m, \quad \boldsymbol{H}^0_m := \boldsymbol{H}^0, \quad \boldsymbol{E}^0_m := \boldsymbol{E}^0,\\ &\boldsymbol{M}^0_m := \boldsymbol{M}^0, \quad \boldsymbol{g}^n_{\text{ext},m} := k^{-1}\int_{(n-1)k}^{nk} \boldsymbol{g}_{\text{ext}}(\xi)\,\mathrm{d}\xi \end{aligned} \qquad \text{for } n = 1,\dots,m,$$

and define G_ρ as in (4.5). We then introduce a time-discretization scheme of implicit type for our problem.

Problem 5.1$_\mathbf{m}$. Find $\boldsymbol{E}^n_m, \boldsymbol{H}^n_m \in L^2(\mathbf{R}^3)^3$ and $\boldsymbol{M}^n_m \in L^\infty(\Omega)^3$ for $n = 1, ..., m$, such that, setting $\boldsymbol{M}^n_m := \mathbf{0}$ outside Ω and $\boldsymbol{B}^n_m := \boldsymbol{H}^n_m + \boldsymbol{M}^n_m$ a.e. in $\mathbf{R}^3$, we have

$$\nabla\times\boldsymbol{H}^n_m = \chi_\Omega\boldsymbol{E}^n_m + (1-\chi_\Omega)\frac{\boldsymbol{E}^n_m - \boldsymbol{E}^{n-1}_m}{k} + \boldsymbol{g}^n_{\text{ext},m} \quad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ for } n = 1, ..., m, \tag{5.17}$$

$$\nabla\times\boldsymbol{E}^n_m = \frac{\boldsymbol{B}^{n-1}_m - \boldsymbol{B}^n_m}{k} \quad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ for } n = 1, ..., m, \tag{5.18}$$

$$\boldsymbol{M}^n_m \in G_\rho\big(\boldsymbol{H}^n_m\cdot\boldsymbol{\theta};\boldsymbol{M}^{n-1}_m\cdot\boldsymbol{\theta}\big)\boldsymbol{\theta} \qquad \text{a.e. in } \Omega, \text{ for } n = 1, ..., m. \tag{5.19}$$

By eliminating the field $\boldsymbol{E}^n_m$ between (5.17) and (5.18), we get an equation of the form

$$\begin{aligned} &\frac{\boldsymbol{B}^n_m - \boldsymbol{B}^{n-1}_m}{k} + \nabla\times\left(\frac{k}{k\chi_\Omega + 1 - \chi_\Omega}\nabla\times\boldsymbol{H}^n_m\right)\\ &\quad = \nabla\times\boldsymbol{\Phi}^n_m \quad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ for } n = 1, ..., m, \end{aligned} \tag{5.20}$$

where $\boldsymbol{\Phi}^n_m$ only depends on $\boldsymbol{E}^{n-1}_m$ and $\boldsymbol{g}^n_{\text{ext},m}$; at the n-th step it is thus a known function. By arguing as we did for Theorem 4.1, we can then see that Problem 5.1$_m$ has a solution, which can be constructed step by step.

We claim that this solution is unique, and prove it step by step. Let us assume that at the step n this problem has two solutions. We label these (1) and (2). Taking the difference between the corresponding equations (5.20), multiplying it by $\boldsymbol{H}^{n(1)}_m - \boldsymbol{H}^{n(2)}_m$ and summing with respect to n, by the monotonicity of G_ρ we easily get $\boldsymbol{H}^{n(1)}_m = \boldsymbol{H}^{n(2)}_m$ a.e. in Ω for any n. By (5.20) it follows that $\boldsymbol{B}^{n(1)}_m = \boldsymbol{B}^{n(2)}_m$ a.e. in Ω, whence $\boldsymbol{M}^{n(1)}_m = \boldsymbol{M}^{n(2)}_m$, $\boldsymbol{E}^{n(1)}_m = \boldsymbol{E}^{n(2)}_m$ a.e. in Ω.

(ii) *A Priori Estimates.* Using the notation of Section 4 for time-interpolate functions, (5.17) also reads

$$\nabla\times\bar{\boldsymbol{H}}_m = \chi_\Omega\bar{\boldsymbol{E}}_m + (1-\chi_\Omega)\frac{\partial\boldsymbol{E}_m}{\partial t} + \bar{\boldsymbol{g}}_{\text{ext},m} \qquad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ a.e. in }]0,T[, \tag{5.21}$$

$$\nabla\times\bar{\boldsymbol{E}}_m = -\frac{\partial \boldsymbol{B}_m}{\partial t} \qquad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ a.e. in }]0,T[. \tag{5.22}$$

Comparisons within (5.17) and in (5.18) show that $\boldsymbol{E}^n_m, \boldsymbol{H}^n_m \in L^2_{\text{rot}}(\mathbf{R}^3)^3$ for any m, n. Let us now multiply (5.17) by $k\boldsymbol{E}^n_m$, (5.18) by $k\boldsymbol{H}^n_m$, and sum for $n = 1, \dots, \ell$, for any $\ell \in \{1, \dots, m\}$. By (4.21), we get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbf{R}^3}\big(|\boldsymbol{H}^\ell_m|^2 - |\boldsymbol{H}^0|^2\big)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbf{R}^3\setminus\Omega}\big(|\boldsymbol{E}^\ell_m|^2 - |\boldsymbol{E}^0|^2\big)\,\mathrm{d}x \\ &\quad + \int_\Omega \Psi_{\rho(x)}(M_m;[0,\ell k])\,\mathrm{d}x + k\sum_{n=1}^{\ell}\int_\Omega |\boldsymbol{E}^n_m|^2\,\mathrm{d}x \\ &\leqq -k\sum_{n=1}^{\ell}\langle \boldsymbol{g}^n_{\text{ext},m}, \boldsymbol{E}^n_m\rangle \\ &\leqq \Big(k\sum_{n=1}^{\ell}\|\boldsymbol{g}^n_{\text{ext},m}\|^2_{L^2(\mathbf{R}^3)^3}\Big)^{1/2}\Big(k\sum_{n=1}^{\ell}\|\boldsymbol{E}^n_m\|^2_{L^2(\mathbf{R}^3)^3}\Big)^{1/2} \end{aligned} \tag{5.23}$$

for $\ell = 1, \dots, m$. A standard calculation then yields

$$\begin{aligned} &\|\boldsymbol{E}_m\|_{L^2(0,T;L^2(\Omega)^3)\cap L^\infty(0,T;L^2(\mathbf{R}^3\setminus\Omega)^3)}\,, \\ &\|\boldsymbol{H}_m\|_{L^\infty(0,T;L^2(\mathbf{R}^3)^3)}\,, \|\Psi_{\rho(x)}(M_m;[0,t])\|_{L^\infty(0,T;L^1(\Omega))} \leqq C_5. \end{aligned} \tag{5.24}$$

By (5.18) we then have

$$\|\boldsymbol{B}_m\|_{H^1(0,T;(L^2_{\text{rot}}(\Omega)^3)')}\,, \left\|\frac{\partial \boldsymbol{M}_m}{\partial t}\right\|_{L^1(\Omega_T)^3} \leqq C_6. \tag{5.25}$$

On account of (4.29), the two latter formulae yield

$$\|\boldsymbol{H}_m\|_{H^\alpha(0,T;H^{-s}(\Omega)^3)}\,, \|\boldsymbol{M}_m\|_{H^\alpha(0,T;H^{-s}(\Omega)^3)} \leqq C_7 \qquad \forall\alpha\in]0,1/2[, \forall s>3/2. \tag{5.26}$$

(iii) *Limit Procedure.* By the above estimates there exist $\boldsymbol{H}$ and $\boldsymbol{M}$ such that, as $m\to\infty$ along a suitable sequence, for any $\alpha\in]0,1/2[$ and any $s>3/2$,

$$\boldsymbol{E}_m \to \boldsymbol{E} \quad \text{weakly star in } L^2\big(0,T;L^2(\Omega)^3\big)\cap L^\infty\big(0,T;L^2(\mathbf{R}^3\setminus\Omega)^3\big), \tag{5.27}$$

$$\boldsymbol{H}_m \to \boldsymbol{H} \quad \text{weakly in } L^2\big(0,T;L^2_{\text{rot}}(\Omega)^3\big)\cap H^\alpha(0,T;H^{-s}(\mathbf{R}^3)^3), \tag{5.28}$$

$$\boldsymbol{M}_m \to \boldsymbol{M} \quad \text{weakly star in } L^\infty(\Omega_T)^3\cap H^\alpha\big(0,T;H^{-s}(\Omega)^3\big), \tag{5.29}$$

$$\bar{\boldsymbol{B}}_m, \boldsymbol{B}_m \to \boldsymbol{B} \quad \text{weakly in } L^2(\mathbf{R}^3_T)^3\cap H^\alpha\big(0,T;H^{-s}(\mathbf{R}^3)^3\big). \tag{5.30}$$

Passing to the limit in (5.21) and (5.22), we get (5.10) and (5.11). By (5.3) and (5.22), $\boldsymbol{B}_m$ is divergence-free. For any ball $\mathcal{B}\subset\mathbf{R}^3$ and any nonnegative function $\psi\in C^\infty([0,T])$, setting $\mathcal{B}_T := \mathcal{B}\times]0,T[$, by Proposition 3.1 we then have

$$\iint_{\mathcal{B}_T}\bar{\boldsymbol{B}}_m\cdot\bar{\boldsymbol{H}}_m\,\psi(t)\,\mathrm{d}x\,\mathrm{d}t \to \iint_{\mathcal{B}_T}\boldsymbol{B}\cdot\boldsymbol{H}\,\psi(t)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.31}$$

This allows one to derive (5.7) as we did in the proof of Theorem 4.1, cf. (4.35) and (4.36).

Equation (5.23) also reads

$$\frac{1}{2}\int_{\mathbf{R}^3}\big(|\bar{\boldsymbol{H}}_m(x,t)|^2-|\boldsymbol{H}^0|^2\big)\,\mathrm{d}x+\frac{1}{2}\int_{\mathbf{R}^3\setminus\Omega}\big(|\bar{\boldsymbol{E}}_m(x,t)|^2-|\boldsymbol{E}^0|^2\big)\,\mathrm{d}x$$
$$+\int_\Omega \Psi_{\rho(x)}(M_m;[0,t])\,\mathrm{d}x+\int_0^t\!\!\int_{\mathbf{R}^3}\big(\chi_\Omega|\bar{\boldsymbol{E}}_m|^2+\bar{\boldsymbol{g}}_{\mathrm{ext},m}\cdot\bar{\boldsymbol{E}}_m\big)\,\mathrm{d}x\,\mathrm{d}\tau \leqq 0$$
$$\text{for a.a. } t\in\,]0,T[. \tag{5.32}$$

Passing to the inferior limit as $m\to\infty$, by lower semicontinuity we get the time primitive of (5.8), hence the inequality (5.8) itself. More precisely, this is accomplished as follows: at first we multiply (5.32) by any positive smooth function of time, ψ, integrate in time, and then pass to the inferior limit as $m\to\infty$; by the arbitrariness of ψ, this yields (5.8). □

In the interior of Ω, the system (5.10), (5.11), (5.15) is parabolic (with hysteresis). We can then establish a regularity result.

Propositin 5.2. *Assume that* (5.1)–(5.3) *are fulfilled, and that*

$$(\nabla\times\boldsymbol{H}^0)\big|_\Omega\in L^2_{\mathrm{loc}}(\Omega)^3. \tag{5.33}$$

Then the solution of Problem 5.1 obtained in Theorem 5.1 satisfies, in addition to (5.1),

$$\boldsymbol{H}\in H^1\big(0,T;L^2_{\mathrm{loc}}(\Omega)^3\big),\qquad \nabla\times\boldsymbol{H}\in L^\infty\big(0,T;L^2_{\mathrm{loc}}(\Omega)^3\big). \tag{5.34}$$

Proof. Let us fix any ball $\mathcal{B}$ such that $\bar{\mathcal{B}}\subset\Omega$, and any function $\zeta\in\mathcal{D}(\Omega)$ such that $0\leqq\zeta\leqq 1$ in Ω and $\zeta\equiv 1$ in $\mathcal{B}$. Multiplying (5.21) by $\zeta^2\nabla\times\partial\boldsymbol{H}_m/\partial t$ and (5.22) by $\zeta^2\partial\boldsymbol{H}_m/\partial t$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^3}\zeta^2|\nabla\times\bar{\boldsymbol{H}}_m|^2\,\mathrm{d}x=\int_\Omega\zeta^2(\bar{\boldsymbol{E}}_m+\bar{\boldsymbol{g}}_{\mathrm{ext},m})\cdot\nabla\times\frac{\partial\boldsymbol{H}_m}{\partial t}\,\mathrm{d}x\quad\text{a.e. in }]0,T[,$$
$$\int_{\mathbf{R}^3}\bar{\boldsymbol{E}}_m\cdot\nabla\times\Big(\zeta^2\frac{\partial\boldsymbol{H}_m}{\partial t}\Big)\,\mathrm{d}x=-\int_{\mathbf{R}^3}\zeta^2\frac{\partial\boldsymbol{B}_m}{\partial t}\cdot\frac{\partial\boldsymbol{H}_m}{\partial t}\,\mathrm{d}x\quad\text{a.e. in }]0,T[; \tag{5.35}$$

by (4.26) the latter equality entails

$$\int_{\mathbf{R}^3}\Big(\zeta^2\bar{\boldsymbol{E}}_m\cdot\nabla\times\frac{\partial\boldsymbol{H}_m}{\partial t}+\bar{\boldsymbol{E}}_m\times\frac{\partial\boldsymbol{H}_m}{\partial t}\cdot 2\zeta\nabla\zeta\Big)\,\mathrm{d}x$$
$$+\int_{\mathbf{R}^3}\zeta^2\Big|\frac{\partial\boldsymbol{H}_m}{\partial t}\Big|^2\,\mathrm{d}x\leqq 0\quad\text{a.e. in }]0,T[. \tag{5.36}$$

Summing (5.35) and (5.36) and integrating in time, we get

$$\int_0^t\!\!\int_\Omega\zeta^2\Big|\frac{\partial\boldsymbol{H}_m}{\partial\tau}\Big|^2\,\mathrm{d}x\,\mathrm{d}\tau+\frac{1}{2}\int_\Omega\zeta^2|\nabla\times\bar{\boldsymbol{H}}_m(x,t)|^2\,\mathrm{d}x$$
$$\leqq\frac{1}{2}\int_\Omega\zeta^2|\nabla\times\boldsymbol{H}^0(x)|^2\,\mathrm{d}x$$
$$+\int_0^t\Big(\|\bar{\boldsymbol{g}}_{\mathrm{ext},m}\|_{L^2(\mathbf{R}^3)^3}+\|\bar{\boldsymbol{E}}_m\|_{L^2(\mathbf{R}^3)^3}(|\zeta|$$
$$+2|\nabla\zeta|)\Big)\Big\|\zeta\frac{\partial\boldsymbol{H}_m}{\partial\tau}\Big\|_{L^2(\Omega)^3}\,\mathrm{d}\tau\quad\forall t\in\,]0,T]. \tag{5.37}$$

By (5.16) and (5.24), the latter inequality entails a uniform estimate for its left-hand side, and this yields (5.34). □

6. Quasi-linear hyperbolic problem

In this section we provide a weak formulation of a Cauchy problem for the system of Maxwell equations for a strongly anisotropic, nonhomogeneous, insulating ferrimagnetic material which occupies a bounded domain Ω of Lipschitz class. We assume that a measurable mapping $(\rho, \boldsymbol{\theta}) : \Omega \to \mathcal{P} \times S^2$ is prescribed, jointly with the data

$$\boldsymbol{J}_{\text{ext}} \in L^2\big(0, T; L^2(\mathbf{R}^3)^3\big), \quad \nabla\cdot \boldsymbol{J}_{\text{ext}} = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^3), \text{ a.e. in }]0, T[, \tag{6.1}$$

$$\boldsymbol{H}^0, \boldsymbol{E}^0 \in L^2(\mathbf{R}^3)^3, \quad \boldsymbol{M}^0 \in L^\infty(\Omega)^3, \quad |\boldsymbol{M}^0| \leqq 1 \quad \text{a.e. in } \Omega. \tag{6.2}$$

We set $\boldsymbol{M}^0 := \mathbf{0}$ outside Ω and $\boldsymbol{B}^0 := \boldsymbol{H}^0 + \boldsymbol{M}^0$ a.e. in $\mathbf{R}^3$, and also assume that

$$\nabla\cdot \boldsymbol{B}^0 = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3). \tag{6.3}$$

Problem 6.1. Find $\boldsymbol{H}, \boldsymbol{E} \in L^2\big(0, T; L^2(\mathbf{R}^3)^3\big)$ and $\boldsymbol{M} \in L^\infty(\mathbf{R}^3_T)^3$ such that $\frac{\partial \boldsymbol{M}}{\partial t} \in \big(C^0(\overline{\Omega_T})^3\big)'$, and, setting $\boldsymbol{M} := \mathbf{0}$ outside Ω_T and $\boldsymbol{B} := \boldsymbol{H} + \boldsymbol{M}$ a.e. in $\mathbf{R}^3_T$, we have

$$\iint_{\mathbf{R}^3_T} \Big(\boldsymbol{H}\cdot\nabla\times\boldsymbol{v} - \boldsymbol{J}_{\text{ext}}\cdot\boldsymbol{v} + \big(\boldsymbol{E} - \boldsymbol{E}^0\big)\cdot\frac{\partial \boldsymbol{v}}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t = 0$$
$$\forall \boldsymbol{v} \in H^1(\mathbf{R}^3_T)^3,\ \boldsymbol{v}(\cdot, T) = \mathbf{0} \ \text{ in } \mathbf{R}^3, \tag{6.4}$$

$$\iint_{\mathbf{R}^3_T} \Big(\boldsymbol{E}\cdot\nabla\times\boldsymbol{v} + \big(\boldsymbol{B}^0 - \boldsymbol{B}\big)\cdot\frac{\partial \boldsymbol{v}}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t = 0$$
$$\forall \boldsymbol{v} \in H^1(\mathbf{R}^3_T)^3,\ \boldsymbol{v}(\cdot, T) = \mathbf{0} \ \text{ in } \mathbf{R}^3, \tag{6.5}$$

$$\boldsymbol{M}\times\boldsymbol{\theta} = \mathbf{0} \qquad \text{a.e. in } \Omega_T, \tag{6.6}$$

$$\left.\begin{array}{ll} |\boldsymbol{M}| \leqq 1 & \\ \boldsymbol{M}\cdot\boldsymbol{\theta} = -1 & \text{if } \boldsymbol{H}\cdot\boldsymbol{\theta} < \rho_1 \\ \boldsymbol{M}\cdot\boldsymbol{\theta} = 1 & \text{if } \boldsymbol{H}\cdot\boldsymbol{\theta} > \rho_2 \end{array}\right\} \qquad \text{a.e. in } \Omega_T, \tag{6.7}$$

$$\begin{aligned} &\frac{1}{2}\int_{\mathbf{R}^3} \big(|\boldsymbol{H}(x,t)|^2 + |\boldsymbol{E}(x,t)|^2 - |\boldsymbol{H}^0(x)|^2 - |\boldsymbol{E}^0(x)|^2\big)\,\mathrm{d}x \\ &\quad + \int_{\bar\Omega} \Psi_{\rho(x)}(\boldsymbol{M}(x,\cdot)\cdot\boldsymbol{\theta}(x); [0,t])\,\mathrm{d}x \\ &\qquad + \int_0^t\!\!\int_{\mathbf{R}^3} \boldsymbol{J}_{\text{ext}}\cdot\boldsymbol{E}\,\mathrm{d}x\,\mathrm{d}\tau \leqq 0 \qquad \text{for a.a. } t \in]0, T[, \end{aligned} \tag{6.8}$$

$$\boldsymbol{M}(x, 0) = \boldsymbol{M}^0(x) \qquad \text{for a.a. } x \in \Omega. \tag{6.9}$$

Interpretation. Equations (6.4) and (6.5) entail the equations

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}_{\text{ext}} + \frac{\partial \boldsymbol{E}}{\partial t} \qquad \text{in } L^2\big(0, T; \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)'\big), \tag{6.10}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{in } L^2\big(0, T; \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)'\big). \tag{6.11}$$

A comparison within these equations yields

$$\frac{\partial \boldsymbol{E}}{\partial t}, \frac{\partial \boldsymbol{B}}{\partial t} \in L^2\big(0, T; \big(L^2_{\mathrm{rot}}(\mathbf{R}^3)^3\big)'\big). \tag{6.12}$$

Integrating (6.4) and (6.5) by parts in time, we then get the initial conditions

$$\boldsymbol{E}\big|_{t=0} = \boldsymbol{E}^0, \quad \boldsymbol{B}\big|_{t=0} = \boldsymbol{B}^0 \qquad \text{in } \big(L^2_{\mathrm{rot}}(\mathbf{R}^3)^3\big)'. \tag{6.13}$$

Conversely, (6.10)–(6.13) yield (6.4) and (6.5).

The remainder of this interpretation is reminiscent of that of Problem 4.2. Let us assume that $\frac{\partial \boldsymbol{E}}{\partial t}, \frac{\partial \boldsymbol{B}}{\partial t} \in L^2\big(0, T; L^2(\mathbf{R}^3)^3\big)$. Multiplying (6.10) by $\boldsymbol{E}$, (6.11) by $\boldsymbol{H}$, summing and integrating in time, we get

$$\frac{1}{2}\int_{\mathbf{R}^3} \big(|\boldsymbol{E}(x,t)|^2 - |\boldsymbol{E}^0(x)|^2\big)\,\mathrm{d}x + \int_0^t\!\!\int_{\mathbf{R}^3} \Big[\frac{\partial \boldsymbol{B}}{\partial \tau}\cdot \boldsymbol{H} + \boldsymbol{J}_{\mathrm{ext}}\cdot \boldsymbol{E}\Big]\,\mathrm{d}x\,\mathrm{d}\tau \leqq 0 \quad \text{for a.a. } t \in\,]0, T[. \tag{6.14}$$

By (6.14), (6.8) is equivalent to

$$\int_0^t\!\!\int_{\mathbf{R}^3} \frac{\partial \boldsymbol{M}}{\partial \tau}\cdot \boldsymbol{H}\,\mathrm{d}x\,\mathrm{d}\tau \geqq \int_{\bar{\Omega}} \Psi_{\rho(x)}(\boldsymbol{M}(x,\cdot)\cdot\boldsymbol{\theta}(x); [0,t])\,\mathrm{d}x \qquad \text{for a.a. } t \in\,]0, T[.$$

As we pointed out in Section 5, this derivation is just *formal*. The latter inequality can be compared with (2.12). Therefore (6.6)–(6.9) formally account for the hysteresis relation

$$\boldsymbol{M} \in \boldsymbol{k}_{(\rho(x),\theta(x))}\big(\boldsymbol{H}, \boldsymbol{M}^0\big) \qquad \text{a.e. in } \Omega_T. \tag{6.15}$$

In conclusion, Problem 6.1 is a weak formulation of a Cauchy problem associated with the system (6.10), (6.11) and (6.15).

Existence result for the quasi-linear hyperbolic problem

Theorem 6.1. *Let us assume that* (6.1)–(6.3) *hold, and that*

$$\exists \delta > 0 : \rho_2(x) - \rho_1(x) \geqq \delta \quad \textit{for a.a. } x \in \Omega. \tag{6.16}$$

Then there exists a solution of Problem 6.1 such that

$$\boldsymbol{E}, \boldsymbol{H} \in L^\infty\big(0, T; L^2(\mathbf{R}^3)^3\big). \tag{6.17}$$

As we shall see in the proof, the regularity $\frac{\partial \boldsymbol{M}}{\partial t} \in \big(C^0(\overline{\Omega_T})^3\big)'$ follows from the nondegeneracy condition (6.16), which then cannot be dispensed with.

Proof. (i) *Approximation.* Let us fix any $m \in \mathbf{N}$, let us set

$$\begin{aligned} &k := T/m, \quad \boldsymbol{H}^0_m := \boldsymbol{H}^0, \quad \boldsymbol{E}^0_m := \boldsymbol{E}^0, \quad \boldsymbol{M}^0_m := \boldsymbol{M}^0, \\ &(\boldsymbol{J}_{\mathrm{ext}})^n_m := k^{-1}\!\int_{(n-1)k}^{nk} \boldsymbol{J}_{\mathrm{ext}}(\xi)\,\mathrm{d}\xi \end{aligned} \qquad \text{for } n = 1, \ldots, m,$$

and define G_ρ as in (4.5).

Problem 6.1$_{\mathbf{m}}$. Find $\boldsymbol{E}_m^n, \boldsymbol{H}_m^n \in L^2(\mathbf{R}^3)^3$ and $\boldsymbol{M}_m^n \in L^\infty(\Omega)^3$ for $n = 1, \dots, m$, such that, setting $\boldsymbol{M}_m^n := \mathbf{0}$ outside Ω and $\boldsymbol{B}_m^n := \boldsymbol{H}_m^n + \boldsymbol{M}_m^n$ a.e. in $\mathbf{R}^3$, we have

$$\boldsymbol{M}_m^n \in G_\rho\big(\boldsymbol{H}_m^n \cdot \boldsymbol{\theta}; \boldsymbol{M}_m^{n-1} \cdot \boldsymbol{\theta}\big)\boldsymbol{\theta} \qquad \text{for a.a. } x \in \Omega, \text{ for } n = 1, ..., m, \tag{6.18}$$

$$\nabla \times \boldsymbol{H}_m^n = (\boldsymbol{J}_{\text{ext}})_m^n + \frac{\boldsymbol{E}_m^n - \boldsymbol{E}_m^{n-1}}{k} \qquad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ for } n = 1, ..., m, \tag{6.19}$$

$$\nabla \times \boldsymbol{E}_m^n = \frac{\boldsymbol{B}_m^n - \boldsymbol{B}_m^{n-1}}{k} \qquad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ for } n = 1, ..., m. \tag{6.20}$$

Interpretation. Setting $\boldsymbol{f}_m^n := \nabla \times \boldsymbol{E}^0 - k \sum_{j=1}^n \nabla \times (\boldsymbol{J}_{\text{ext}})_m^j$, the system (6.19), (6.20) is equivalent to

$$\boldsymbol{H}_m^n + \boldsymbol{M}_m^n - \boldsymbol{B}_m^{n-1} + k^2 \nabla \times \nabla \times \sum_{j=1}^n \boldsymbol{H}_m^j = k \boldsymbol{f}_m^n \qquad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \tag{6.21}$$

for $n = 1, ..., m$. For any n and m, setting $\boldsymbol{Z}_m^n := k \sum_{j=1}^n \boldsymbol{H}_m^j$ and denoting by $\hat{G}_{(\rho,x,m,n)}$ a primitive of $G_\rho\big(\cdot; \boldsymbol{M}_m^{n-1}(x) \cdot \boldsymbol{\theta}(x)\big)$ for a.a. $x \in \Omega$, the functional

$$\begin{aligned}\Gamma_m^n : L^2_{\text{rot}}(\mathbf{R}^3)^3 \to \mathbf{R} : \boldsymbol{v} \mapsto \int_\Omega \hat{G}_{(\rho,x,m,n)}\big(\boldsymbol{v} \cdot \boldsymbol{\theta}(x)\big)\, \mathrm{d}x \\ + \int_{\mathbf{R}^3} \left(\frac{1}{2}|\boldsymbol{v}|^2 + \frac{k^2}{2}|\nabla \times \boldsymbol{v}|^2 + k \nabla \times \boldsymbol{Z}_m^{n-1} \cdot \nabla \times \boldsymbol{v} - \boldsymbol{B}_m^{n-1} \cdot \boldsymbol{v} - k \boldsymbol{f}_m^n \cdot \boldsymbol{v}\right) \mathrm{d}x\end{aligned} \tag{6.22}$$

is (strictly) convex, lower semicontinuous and coercive on $L^2_{\text{rot}}(\mathbf{R}^3)^3$. Hence it has a (unique) minimizer $\boldsymbol{H}_m^n$, and $\partial \Gamma_m^n(\boldsymbol{H}_m^n) \ni 0$ in $\big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)'$. This inclusion is equivalent to the system (6.18), (6.21) (notice that $\boldsymbol{Z}_m^n = k\boldsymbol{H}_m^n + \boldsymbol{Z}_m^{n-1}$). Problem 6.1$_m$ has thus a (unique) solution.

(ii) *A Priori Estimates.* Using the notation of Section 4 for time-interpolate functions, (6.19) and (6.20) also read

$$\nabla \times \bar{\boldsymbol{H}}_m = (\bar{\boldsymbol{J}}_{\text{ext}})_m + \frac{\partial \boldsymbol{E}_m}{\partial t} \qquad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ a.e. in }]0, T[, \tag{6.23}$$

$$\nabla \times \bar{\boldsymbol{E}}_m = -\frac{\partial \boldsymbol{B}_m}{\partial t} \qquad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ a.e. in }]0, T[. \tag{6.24}$$

Let us multiply equation (6.19) by $k\boldsymbol{E}_m^n$, (6.20) by $k\boldsymbol{H}_m^n$, and sum for $n = 1, \dots, \ell$, for any $\ell \in \{1, \dots, m\}$. By (4.21), we get

$$\begin{aligned}&\frac{1}{2}\int_{\mathbf{R}^3} \big(|\boldsymbol{H}_m^\ell|^2 + |\boldsymbol{E}_m^\ell|^2 - |\boldsymbol{H}^0|^2 - |\boldsymbol{E}^0|^2\big)\, \mathrm{d}x \\ &\quad + \int_\Omega \Psi_{\rho(x)}(\boldsymbol{M}_m(x, \cdot) \cdot \boldsymbol{\theta}(x); [0, \ell k])\, \mathrm{d}x \\ &\leqq k \sum_{n=1}^{\ell} \langle (\boldsymbol{J}_{\text{ext}})_m^n, \boldsymbol{H}_m^n \rangle\end{aligned}$$

$$\leqq \sqrt{\ell k}\, \|(\bar{\boldsymbol{J}}_{\text{ext}})_m\|_{L^2(0,T;L^2(\mathbf{R}^3)^3)} \max_{n=1,\dots,\ell} \left(\int_{\mathbf{R}^3} |\boldsymbol{H}_m^n|^2 \, \mathrm{d}x \right)^{1/2} \tag{6.25}$$

for any $\ell \in \{1, \dots, m\}$. A standard calculation then yields

$$\begin{aligned} &\|\boldsymbol{E}_m\|_{L^\infty(0,T;L^2(\mathbf{R}^3)^3)}\,, \|\boldsymbol{H}_m\|_{L^\infty(0,T;L^2(\mathbf{R}^3)^3)}\,, \\ &\|\Psi(\boldsymbol{M}_m \cdot \boldsymbol{\theta}; [0,t])\|_{L^\infty(0,T;L^1(\Omega))} \leqq C_8. \end{aligned} \tag{6.26}$$

By comparison in (6.19) and (6.20) and by (6.16), we then have

$$\|\boldsymbol{H}_m\|_{H^{-1}(0,T;L^2_{\text{rot}}(\mathbf{R}^3)^3)}\,, \|\boldsymbol{E}_m\|_{H^{-1}(0,T;L^2_{\text{rot}}(\mathbf{R}^3)^3)}\,, \left\| \frac{\partial \boldsymbol{M}_m}{\partial t} \right\|_{L^\infty(0,T;L^1(\Omega))} \leqq C_9. \tag{6.27}$$

(iii) *Limit Procedure.* By the above estimates, there exist $\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{M}$ such that, as $m \to \infty$ along a suitable sequence,

$$\begin{aligned} &\boldsymbol{E}_m \to \boldsymbol{E}, \quad \boldsymbol{H}_m \to \boldsymbol{H} \\ &\text{weakly star in } L^\infty\big(0,T; L^2(\mathbf{R}^3)^3\big) \cap H^{-1}(0,T; L^2_{\text{rot}}(\mathbf{R}^3)^3), \end{aligned} \tag{6.28}$$

$$\boldsymbol{M}_m \to \boldsymbol{M} \qquad \text{weakly star in } L^\infty(\Omega_T)^3, \tag{6.29}$$

$$\frac{\partial \boldsymbol{M}_m}{\partial t} \to \frac{\partial \boldsymbol{M}}{\partial t} \qquad \text{weakly star in } \big(C^0(\overline{\Omega_T})^3\big)'. \tag{6.30}$$

Passing to the limit in (6.20) and (6.21) we get (6.4) and (6.5). Equations (6.6) and $(6.7)_1$ follow from (6.18).

For any domain $\tilde{\Omega} \subset \Omega$ and any nonnegative function $\psi \in C^\infty([0,T])$, applying Proposition 3.3 to the sequences $\{\boldsymbol{H}_m \psi\}$ and $\{\bar{\boldsymbol{M}}_m\}$ we get

$$\begin{aligned} &\limsup_{m\to\infty} \iint_{\tilde{\Omega}_T} \bar{M}_m \bar{\boldsymbol{H}}_m \cdot \boldsymbol{\theta}\, \psi(t) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = \limsup_{m\to\infty} \iint_{\tilde{\Omega}_T} \bar{\boldsymbol{M}}_m \cdot \bar{\boldsymbol{H}}_m \, \psi(t) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \leqq \iint_{\tilde{\Omega}_T} \boldsymbol{M} \cdot \boldsymbol{H} \, \psi(t) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad = \iint_{\tilde{\Omega}_T} M \boldsymbol{H} \cdot \boldsymbol{\theta} \, \psi(t) \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{6.31}$$

Equations $(6.7)_2$ and $(6.7)_3$ can then be proved via the procedure we used for Theorem 4.1, cf. (4.35) and (4.36).

Equation (6.25) yields

$$\begin{aligned} &\frac{1}{2} \int_{\mathbf{R}^3} \big(|\bar{\boldsymbol{H}}_m(x,t)|^2 + |\bar{\boldsymbol{E}}_m(x,t)|^2 - |\boldsymbol{H}^0|^2 - |\boldsymbol{E}^0|^2 \big) \, \mathrm{d}x \\ &\quad + \int_\Omega \Psi_{\rho(x)}(\boldsymbol{M}_m(x,\cdot) \cdot \boldsymbol{\theta}(x); [0,t]) \, \mathrm{d}x \\ &\quad + \int_0^t \langle (\bar{\boldsymbol{J}}_{\text{ext}})_m, \bar{\boldsymbol{H}}_m \rangle \, \mathrm{d}\tau \leqq 0 \qquad \text{for a.a. } t \in \,]0,T[. \end{aligned} \tag{6.32}$$

Let us now multiply this inequality by any positive smooth function of time, ψ, integrate in time, and pass to the inferior limit as $m \to \infty$. By the arbitrariness of ψ, (6.8) then follows. □

7. Problem with double hysteresis

In this section we provide a weak formulation of a Cauchy problem for the system of Maxwell equations for an (insulating) material which exhibits hysteresis in both the $\boldsymbol{M}$ vs. $\boldsymbol{H}$ and $\boldsymbol{P}$ vs. $\boldsymbol{E}$ constitutive relations. We assume that the material is strongly anisotropic and nonhomogeneous, and represent both relations via x-dependent vector relays.

We prescribe two measurable mappings $(\rho', \boldsymbol{\theta}'), (\rho'', \boldsymbol{\theta}'') : \Omega \to \mathcal{P} \times S^2$ (with $\rho' := (\rho'_1, \rho'_2)$ and $\rho'' := (\rho''_1, \rho''_2)$), and the data

$$\boldsymbol{J}_{\text{ext}} \in L^2\big(0, T; L^2(\mathbf{R}^3)^3\big), \quad \nabla\cdot \boldsymbol{J}_{\text{ext}} = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^3), \text{ a.e. in }]0, T[, \tag{7.1}$$

$$\boldsymbol{H}^0, \boldsymbol{E}^0 \in L^2(\mathbf{R}^3)^3, \quad \boldsymbol{M}^0, \boldsymbol{P}^0 \in L^\infty(\Omega)^3, \quad |\boldsymbol{M}^0| \leqq 1, |\boldsymbol{P}^0| \leqq 1 \quad \text{a.e. in } \Omega. \tag{7.2}$$

We set

$$\boldsymbol{M}^0 := \mathbf{0}, \;\; \boldsymbol{P}^0 := \mathbf{0} \quad \text{in } \mathbf{R}^3 \setminus \Omega, \tag{7.3}$$

$$\boldsymbol{B}^0 := \boldsymbol{H}^0 + \boldsymbol{M}^0, \;\; \boldsymbol{D}^0 := \boldsymbol{E}^0 + \boldsymbol{P}^0 \quad \text{in } \Omega, \tag{7.4}$$

and assume that

$$\nabla\cdot \boldsymbol{B}^0 = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^3), \qquad \nabla\cdot \boldsymbol{D}^0 \in L^2(\mathbf{R}^3)^3. \tag{7.5}$$

Problem 7.1. Find $\boldsymbol{H}, \boldsymbol{E} \in L^2\big(0, T; L^2(\mathbf{R}^3)^3\big)$ and $\boldsymbol{M}, \boldsymbol{P} \in L^\infty(\mathbf{R}^3_T)^3$ such that $\frac{\partial \boldsymbol{M}}{\partial t}, \frac{\partial \boldsymbol{P}}{\partial t} \in \big(C^0(\overline{\Omega_T})^3\big)'$, and, setting

$$\boldsymbol{M} := \mathbf{0}, \boldsymbol{P} := \mathbf{0} \;\; \text{in } \mathbf{R}^3_T \setminus \Omega_T, \qquad \boldsymbol{B} := \boldsymbol{H} + \boldsymbol{M}, \;\; \boldsymbol{D} := \boldsymbol{E} + \boldsymbol{P} \;\; \text{a.e. in } \mathbf{R}^3_T,$$

we have

$$\iint_{\mathbf{R}^3_T} \Big(\boldsymbol{H}\cdot\nabla\times\boldsymbol{v} - \boldsymbol{J}_{\text{ext}}\cdot\boldsymbol{v} + \big(\boldsymbol{D} - \boldsymbol{D}^0\big)\cdot\frac{\partial \boldsymbol{v}}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t = 0$$
$$\forall \boldsymbol{v} \in H^1(\mathbf{R}^3_T)^3, \boldsymbol{v}(\cdot, T) = \mathbf{0} \;\; \text{in } \mathbf{R}^3, \tag{7.6}$$

$$\iint_{\mathbf{R}^3_T} \Big(\boldsymbol{E}\cdot\nabla\times\boldsymbol{v} + \big(\boldsymbol{B}^0 - \boldsymbol{B}\big)\cdot\frac{\partial \boldsymbol{v}}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t = 0$$
$$\forall \boldsymbol{v} \in H^1(\mathbf{R}^3_T)^3, \boldsymbol{v}(\cdot, T) = \mathbf{0} \;\; \text{in } \mathbf{R}^3, \tag{7.7}$$

$$\boldsymbol{M}\times\boldsymbol{\theta}' = \mathbf{0} \qquad \boldsymbol{P}\times\boldsymbol{\theta}'' = \mathbf{0} \qquad \text{a.e. in } \Omega_T, \tag{7.8}$$

$$\left.\begin{array}{lll} |\boldsymbol{M}| & \leqq 1 & \\ \boldsymbol{M}\cdot\boldsymbol{\theta}' & = -1 & \text{if } \boldsymbol{H}\cdot\boldsymbol{\theta}' < \rho'_1 \\ \boldsymbol{M}\cdot\boldsymbol{\theta}' & = 1 & \text{if } \boldsymbol{H}\cdot\boldsymbol{\theta}' > \rho'_2 \end{array}\right\} \quad \text{a.e. in } \Omega_T, \tag{7.9}$$

$$\left.\begin{array}{lll} |\boldsymbol{P}| & \leqq 1 & \\ \boldsymbol{P}\cdot\boldsymbol{\theta}'' & = -1 & \text{if } \boldsymbol{E}\cdot\boldsymbol{\theta}'' < \rho''_1 \\ \boldsymbol{P}\cdot\boldsymbol{\theta}'' & = 1 & \text{if } \boldsymbol{E}\cdot\boldsymbol{\theta}'' > \rho''_2 \end{array}\right\} \quad \text{a.e. in } \Omega_T, \tag{7.10}$$

$$\frac{1}{2}\int_{\mathbf{R}^3} \big(|\boldsymbol{H}(x,t)|^2 + |\boldsymbol{E}(x,t)|^2 - |\boldsymbol{H}^0(x)|^2 - |\boldsymbol{E}^0(x)|^2\big)\,\mathrm{d}x$$

$$
\begin{aligned}
&+\int_{\bar{\Omega}} \Psi_{\rho'(x)}(\boldsymbol{M}(x,\cdot)\cdot\boldsymbol{\theta}'(x);[0,t])\,\mathrm{d}x \\
&+\int_{\bar{\Omega}} \Psi_{\rho''(x)}(\boldsymbol{P}(x,\cdot)\cdot\boldsymbol{\theta}''(x);[0,t])\,\mathrm{d}x \\
&+\int_0^t\!\!\int_{\mathbf{R}^3} \boldsymbol{J}_{\text{ext}}\cdot\boldsymbol{E}\,\mathrm{d}x\,\mathrm{d}\tau \leqq 0 \qquad \text{for a.a. } t \in\,]0,T[, \qquad (7.11)
\end{aligned}
$$

$$
\boldsymbol{M}(x,0)=\boldsymbol{M}^0(x), \quad \boldsymbol{P}(x,0)=\boldsymbol{P}^0(x) \qquad \text{for a. } x\in\Omega. \qquad (7.12)
$$

Interpretation. In Section 6 we saw that (7.5) and (7.6) are equivalent to the Ampère-Maxwell and Faraday laws (6.10) and (6.11), coupled with the initial conditions (6.13).

For a moment let us assume that $\frac{\partial \boldsymbol{E}}{\partial t}, \frac{\partial \boldsymbol{B}}{\partial t} \in L^2\big(0,T;L^2(\mathbf{R}^3)^3\big)$. Multiplying (6.10) by $\boldsymbol{E}$, (6.11) by $\boldsymbol{H}$, summing and integrating in time, we get

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^3} \big(|\boldsymbol{H}(x,t)|^2+|\boldsymbol{E}(x,t)|^2-|\boldsymbol{H}^0(x)|^2-|\boldsymbol{E}^0(x)|^2\big)\,\mathrm{d}x \\
&\quad+\int_0^t\!\!\int_{\mathbf{R}^3}\Big(\frac{\partial \boldsymbol{M}}{\partial\tau}\cdot\boldsymbol{H}+\frac{\partial \boldsymbol{P}}{\partial\tau}\cdot\boldsymbol{E}+\boldsymbol{J}_{\text{ext}}\cdot\boldsymbol{E}\Big)\,\mathrm{d}x\,\mathrm{d}\tau = 0 \quad \text{for } t\in\,]0,T[.
\end{aligned} \qquad (7.13)
$$

Equation (7.10) is then formally equivalent to

$$
\begin{aligned}
&\int_0^t\!\!\int_{\mathbf{R}^3}\Big(\frac{\partial \boldsymbol{M}}{\partial\tau}\cdot\boldsymbol{H}+\frac{\partial \boldsymbol{P}}{\partial\tau}\cdot\boldsymbol{E}\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad\geqq \int_{\bar{\Omega}} \Psi_{\rho'(x)}(\boldsymbol{M}(x,\cdot)\cdot\boldsymbol{\theta}'(x);[0,t])\,\mathrm{d}x \\
&\qquad+\int_{\bar{\Omega}} \Psi_{\rho''(x)}(\boldsymbol{P}(x,\cdot)\cdot\boldsymbol{\theta}''(x);[0,t])\,\mathrm{d}x \qquad \text{for } t\in\,]0,T[. \qquad (7.14)
\end{aligned}
$$

In Section 2 we saw that the *confinement condition* (2.6) yields $\int_0^t u\,dw \leqq \Psi_\rho(w;[0,t])$. Thus in our case (7.8) and (7.9) formally entail

$$
\left.
\begin{aligned}
&\int_0^t\!\!\int_{\Omega}\frac{\partial \boldsymbol{M}}{\partial\tau}\cdot\boldsymbol{H}\,\mathrm{d}x\,\mathrm{d}\tau \leqq \int_{\bar{\Omega}} \Psi_{\rho'(x)}(\boldsymbol{M}(x,\cdot)\cdot\boldsymbol{\theta}'(x);[0,t])\,\mathrm{d}x, \\
&\int_0^t\!\!\int_{\Omega}\frac{\partial \boldsymbol{P}}{\partial\tau}\cdot\boldsymbol{E}\,\mathrm{d}x\,\mathrm{d}\tau \leqq \int_{\bar{\Omega}} \Psi_{\rho''(x)}(\boldsymbol{P}(x,\cdot)\cdot\boldsymbol{\theta}''(x);[0,t])\,\mathrm{d}x
\end{aligned}
\right\} \quad \text{for } t\in\,]0,T[.
$$

From (7.13) we then infer the opposite inequalities. Therefore (7.7)–(7.11) formally account for the hysteresis relations

$$
\boldsymbol{M}\in \boldsymbol{k}_{(\rho'(x),\boldsymbol{\theta}'(x))}\big(\boldsymbol{H},\boldsymbol{M}^0\big), \quad \boldsymbol{P}\in \boldsymbol{k}_{(\rho''(x),\boldsymbol{\theta}''(x))}\big(\boldsymbol{E},\boldsymbol{P}^0\big) \qquad \text{a.e. in } \Omega_T. \qquad (7.15)
$$

In conclusion, Problem 7.1 is a weak formulation for a Cauchy problem associated with the system (6.10), (6.11) and (7.14).

Theorem 7.1. *Let us assume that* (7.1)–(7.4) *hold, and that*

$$\exists\delta > 0 : \rho_2'(x) - \rho_1'(x) \geqq \delta, \quad \rho_2''(x) - \rho_1''(x) \geqq \delta \quad \textit{for a.a. } x \in \Omega. \tag{7.16}$$

Then there exists a solution of Problem 7.1 such that

$$\boldsymbol{E}, \boldsymbol{H} \in L^\infty\big(0, T; L^2(\mathbf{R}^3)^3\big). \tag{7.17}$$

Outline of the Proof. This argument is similar to that of Theorem 6.1. We approximate Problem 7.1 via an implicit time-discretization scheme analogous to Problem 6.1_m. This essentially consists in a Cauchy problem for the system

$$\begin{aligned} &\boldsymbol{M}_m^n \in G_{\rho'}\big(\boldsymbol{H}_m^n\cdot\boldsymbol{\theta}'; \boldsymbol{M}_m^{n-1}\cdot\boldsymbol{\theta}'\big)\boldsymbol{\theta}',\\ &\boldsymbol{P}_m^n \in G_{\rho''}\big(\boldsymbol{E}_m^n\cdot\boldsymbol{\theta}''; \boldsymbol{P}_m^{n-1}\cdot\boldsymbol{\theta}''\big)\boldsymbol{\theta}'' \end{aligned} \qquad \text{for a.a. } x \in \Omega, \text{ for } n = 1, \dots, m, \tag{7.18}$$

$$\begin{aligned} \nabla\times\boldsymbol{H}_m^n = (\boldsymbol{J}_{\text{ext}})_m^n &+ \frac{\boldsymbol{E}_m^n - \boldsymbol{E}_m^{n-1}}{k}\\ &+ \frac{\boldsymbol{P}_m^n - \boldsymbol{P}_m^{n-1}}{k} \quad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ for } n = 1, \dots, m, \end{aligned} \tag{7.19}$$

$$\nabla\times\boldsymbol{E}_m^n = \frac{\boldsymbol{B}_m^n - \boldsymbol{B}_m^{n-1}}{k} \qquad \text{in } \big(L^2_{\text{rot}}(\mathbf{R}^3)^3\big)', \text{ for } n = 1, \dots, m. \tag{7.20}$$

This problem is equivalent to the minimization of a lower-semicontinuous convex functional, and has a solution $(\boldsymbol{H}_m^n, \boldsymbol{E}_m^n, \boldsymbol{M}_m^n, \boldsymbol{P}_m^n)$. The energy estimate can be derived by the standard procedure we already used in Section 6. This yields weak convergence as $m \to \infty$ along a suitable subsequence, and this allows us to pass to the limit in (7.18) and (7.19).

The $\boldsymbol{M}$ vs. $\boldsymbol{H}$ hysteresis relation can be obtained as in Section 6. The same procedure allows us to derive the $\boldsymbol{P}$ vs. $\boldsymbol{E}$ relation; here the Remark following Proposition 3.3 is also used, by taking, cf. (3.17),

$$\boldsymbol{\psi}_m(x,t) = \int_0^t (\boldsymbol{J}_{\text{ext}})_m(x,\tau)\,\mathrm{d}\tau - \boldsymbol{E}_m^0(x) - \boldsymbol{P}_m^0(x) \qquad \text{for a.a. } (x,t) \in \mathbf{R}^3_T, \forall m.$$

□

8. Conclusions and open questions

We represented electromagnetic processes in either ferromagnetic or ferrimagnetic strongly anisotropic, nonhomogeneous materials, by coupling the system of Maxwell and Ohm laws with an $\boldsymbol{M}$ vs. $\boldsymbol{H}$ constitutive relation with hysteresis. More specifically, we considered a vector extension of the scalar relay, and represented it by two conditions, which we expressed via a system of two (nonvariational) inequalities: one of them accounts for the rectangular shape of the hysteresis loop, the other one for the dissipative dynamics along that loop. This formulation of the hysteresis relation turned out to be especially convenient for the analysis of related P.D.E.s.

Displacement currents can be neglected in processes in ferromagnetic metals, whereas they must be included when dealing with ferrimagnetic insulators. These two settings respectively correspond to quasi-linear parabolic and hyperbolic equations with hysteresis. However, for quasi-stationary processes we considered the magnetostatic equations, too. We also dealt with a quasi-linear hyperbolic equation with $\boldsymbol{M}$ vs. $\boldsymbol{H}$ and $\boldsymbol{P}$ vs. $\boldsymbol{E}$ hysteresis relations, for a material which exhibits both ferrimagnetic and ferroelectric properties. For each of these problems, we provided a weak formulation in Sobolev spaces, and proved existence of a solution via approximation by time-discretization, derivation of energy-type *a priori* estimates, and passage to the limit.

Concerning the hyperbolic problems, the existence result for these vector problems and those of [38] for the scalar problem rest on the dissipative character of hysteresis: as is well known, the area of the region bounded by the hysteresis loop represents the amount of electromagnetic energy that is transformed into heat at any cycle. This provides a bound for the number of cyles that can be closed in the process, and entails an *a priori* estimates that has no analogue in the case without hysteresis. It is this that makes the analysis of these equations so different from that of the same equations without hysteresis. For more general hyperbolic problems with hysteresis, it might be of some interest to combine more typical hyperbolic techniques with those of the present work.

For the above vector problems, uniqueness of the solution, its large-time behaviour, and existence of a time-periodic solution are open questions. A more complete description of magnetic hysteresis should also include *magnetostriction,* namely, the interaction between mechanic and electromagnetic phenomena, and heat exchange due to dissipation of electromagnetic energy. Another question I am currently studying is the extension of the above results to the case in which at each space point x the hysteresis relation is represented by a *Preisach operator,* namely, by a family of coexisting relays.

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