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Binary Decompositions for Planar N-Body Problems and Symmetric Periodic Solutions

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Abstract

A binary decomposition for a system of N masses is a way of treating the system as $\binom{N}{2}$ binaries with the total action exactly the same as that of the original system. By considering binary decompositions, we are able to provide effective lower-bound estimates for the action of collision paths in several spaces of symmetric loops. As applications, we use our estimates to prove the existence of some new classes of symmetric periodic solutions for the N-body problem.

1. Introduction

The planar Newtonian N-body problem concerns the motion of N ($\geqq 2$) mass points $m_1, m_2, \dots, m_N$ moving in $\mathbb{C}$ in accordance with Newton's law of gravitation:

$$m_k \ddot{x}_k = \frac{\partial}{\partial x_k} U(x), \quad k = 1, \dots, N, \tag{1}$$

where $x_k \in \mathbb{C}$ is the position of m_k and

$$U(x) = U(x_1, \dots, x_N) = \sum_{1 \leqq i < j \leqq N} \frac{G m_i m_j}{|x_i - x_j|}$$

is the potential energy, and $G > 0$ is the gravitation constant. The kinetic energy is given by

$$K(\dot{x}) = \sum_{i=1}^{N} \frac{1}{2} m_i |\dot{x}_i|^2.$$

Equations (1) are the Euler-Lagrange equations for the action functional $\mathcal{A}$ defined by

$$\mathcal{A}(x) = \int_0^T K(\dot{x}) + U(x)\, dt\,, \tag{2}$$

where T is a positive constant. In this article we will primarily be dealing with periodic orbits. Therefore, the ground space over which $\mathcal{A}$ is defined is selected as $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$.

The major purpose of this article is to provide a systematic way of estimating the lower bound for the action functional on collision paths in $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$. This was carried out by treating each mass as a compound of particles, each of which interact with exactly one particle in the system. The new system consists of $\binom{N}{2}$ binaries with total action exactly the same as the action of the original system. When a path in $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$ possesses certain symmetry, we may classify the binaries and then estimate their action accordingly. As applications, we use our lower-bound estimates to prove the existence of some new classes of symmetric periodic solutions with zero angular momentum. Our study is motivated by recent discoveries made by CHENCINER, MONTGOMERY, SIMÓ *et al.* [1, 6–8].

This paper is organized as follows. The main results can be found in Sections 6 and 7. In Section 5 we define binary decompositions for a system of N masses, based on which we derive in Section 6 lower-bound estimates for the action functional on some spaces of symmetric loops. Sections 7, 8 contain some applications of our lower-bound estimates. In particular, we prove the existence of infinitely many nontrivial double choreographic solutions (defined in Section 5) for (1). Sections 2, 3, 4 are preparations for the proof of the results in Sections 6, 7, 8.

2. An extension of Gordon's theorem

In the simplest case, the Kepler problem ($N = 2$), the action functional $\mathcal{A}$ defined on $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^2)$ takes the form

$$\begin{aligned}\mathcal{A}(x) &= \int_0^T \frac{1}{2}(m_1|\dot{x}_1|^2 + m_2|\dot{x}_2|^2) + \frac{Gm_1m_2}{|x_1 - x_2|}\, dt \\ &= \mathcal{A}^0(\mathbf{r}) + \mathcal{A}^1(\bar{x}),\end{aligned}$$

where $\mathbf{r} = x_2 - x_1$, $\bar{x}$ is the mass center, and

$$\mathcal{A}^0(\mathbf{r}) = \int_0^T \frac{m_1m_2}{2(m_1+m_2)}|\dot{\mathbf{r}}|^2 + \frac{Gm_1m_2}{|\mathbf{r}|}\, dt, \tag{3}$$

$$\mathcal{A}^1(\bar{x}) = \int_0^T \frac{m_1+m_2}{2}|\dot{\bar{x}}|^2\, dt. \tag{4}$$

Granting that linear momentum is an integral of motion, it is customary to drop the integral $\mathcal{A}^1$ and consider critical points of $\mathcal{A}^0$ with loops $\mathbf{r}$ in $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$.

Let Λ_T be the space of loops in $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}\setminus\{0\})$ with nonzero winding number around 0. In [11] GORDON proved

Gordon's Theorem. *The functional $\mathcal{A}^0$ attains its infimum over Λ_T at elliptical Keplerian orbits with prime period T, and attains its infimum over $\partial\Lambda_T$, the boundary of Λ_T, at collision-ejection Keplerian orbits with prime period T. The values of $\mathcal{A}^0$ over these orbits are all equal to*

$$3\left(\frac{G^2\pi^2}{2(m_1+m_2)}\right)^{\frac{1}{3}} m_1m_2T^{\frac{1}{3}}. \tag{5}$$

Periodic collision-ejection Keplerian orbits can be considered to be degenerate elliptical orbits with eccentricity 1. It is easy to see that without the topological constraint the action functional, in spite of being weakly lower-semicontinuous and bounded from below, has no minimum. In what follows we present an extension of Gordon's theorem in a larger space. Our generalization has no use in the Kepler problem, but we will see later that it is quite useful when $N \geqq 3$ (see Theorems 1 and 2).

Restriction to the case with $\mathcal{A}^1 = 0$ involves no loss of generality because we are aiming for critical points of $\mathcal{A}$ and it follows from Kepler's equation that linear momentum is a first integral. If critical points of $\mathcal{A}$ are not the only things we are concerned about, then it is less natural to make such an assumption. This concern will become more apparent in Sections 5 and 6. For this reason we consider loops in $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^2)$ without assuming *a priori* that $\mathcal{A}^1$ is zero.

Let $\mathcal{O}(x_2 - x_1) \subset \mathbb{C}$ denote the orbit of $\mathbf{r} = x_2 - x_1$ for $x = (x_1, x_2) \in H^1$. Consider

$$\Sigma_T = \left\{ x \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^2) : \begin{array}{l} \mathcal{O}(x_2 - x_1) \cap L \neq \emptyset \\ \text{for any straight line } L \text{ containing } 0 \end{array} \right\}. \tag{6}$$

Clearly Σ_T is a closed subset in H^1 that contains Λ_T in its interior. It contains lots of loops with zero winding number around 0.

Proposition 1. *The action functional $\mathcal{A}$, and hence $\mathcal{A}^0$, attains its infimum over Σ_T. A minimizer x of $\mathcal{A}$ or $\mathcal{A}^0$ is either an elliptical Keplerian orbit with prime period T, a collision-ejection Keplerian orbit with prime period T, or is traveling back and forth along one half of an elliptical Keplerian orbit with prime period T. The infimum value of $\mathcal{A}$ over Σ_T is given by* (5).

Proof. We write Σ for Σ_T and Λ for Λ_T here for simplicity. Since $\bar{\Lambda}$ is a subset of Σ, by Gordon's theorem all we need to show is

$$\inf_{x\in\Sigma\setminus\bar{\Lambda}} \mathcal{A}(x) \geqq 3\left(\frac{G^2\pi^2}{2(m_1+m_2)}\right)^{\frac{1}{3}} m_1m_2T^{\frac{1}{3}},$$

and minimizers in $\Sigma \setminus \bar{\Lambda}$ are orbits that travel back and forth along one half of an elliptical Keplerian orbit with prime period T.

Note that $\mathcal{A}^0$, Σ, and Λ are invariant under translations in $\mathbb{C}$; that is, for any $\xi \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$, $\mathcal{A}^0$ has the same value at $x_\xi = x - (\xi, \xi)$ as at x, and x_ξ belongs to Σ (resp. Λ) if x does. Therefore minimizers of $\mathcal{A}$ on $\Sigma \setminus \Lambda$, if they exist, have to be zeros of $\mathcal{A}^1$.

Let $x \in \Sigma \setminus \bar{\Lambda}$ be a zero of $\mathcal{A}^1$. We may assume that the mass center is at the origin. By definition of Σ, x satisfies

$$\max_{t\in[0,T]} \operatorname{Arg}(\mathbf{r}(t)) - \min_{t\in[0,T]} \operatorname{Arg}(\mathbf{r}(t)) \geqq \pi.$$

Here $\operatorname{Arg}(w) \in [0, 2\pi)$ denotes the argument of $w \in \mathbb{C} \setminus \{0\}$. By rotating about the origin properly and translating the time variable (neither of which affect the action), we may assume that

$$\begin{aligned}&\operatorname{Arg}(\mathbf{r}(0)) = \min_{t\in[0,T]} \operatorname{Arg}(\mathbf{r}(t)) = 0, \text{ and}\\ &\operatorname{Arg}(\mathbf{r}(\tau)) = \pi\end{aligned}$$

for some $\tau \in (0, T)$. The assumption that the mass center is at the origin implies that x_1, x_2 are both on the real axis at $t = 0, \tau, T$.

The winding number of $\mathbf{r}$ about 0 is zero since $x \notin \Lambda$. Now we define a new loop $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in \Sigma$ as follows:

$$\tilde{x}(t) = \begin{cases} (x_1(t), x_2(t)) \text{ for } t \in [0, \tau],\\ (\bar{x}_1(t), \bar{x}_2(t)) \text{ for } t \in (\tau, T].\end{cases}$$

This path reflects a portion of x by complex conjugation. By the construction, $\tilde{x}$ belongs to Σ, the winding number of $\tilde{\mathbf{r}} = \tilde{x}_2 - \tilde{x}_1$ about the origin is nonzero (so $\tilde{x} \in \Lambda$), and $\mathcal{A}(x) = \mathcal{A}(\tilde{x})$. This proves the inequality we claimed. In order that the infimum is achieved at $x \in \Sigma \setminus \bar{\Lambda}$, the corresponding $\tilde{x}$ defined above has to be an elliptical Keplerian orbit with prime period T, in which case x has to travel back and forth along half of the orbit of $\tilde{x}$. This completes the proof. □

3. Three topological lemmas

From either a theoretic point of view or from numerical simulations, many periodic orbits for the N-body problem have been discovered to, or are expected to, possess certain symmetries, especially in cases where equal masses are present. In many cases some masses share a single orbit. This motivates us to investigate the relationships between particles sharing the same orbit. For this purpose we prove three topological lemmas in this section. These lemmas together with Proposition 1 show how these masses behave compared to Keplerian orbits, and will be used in Section 6 to prove Theorems 1 and 2.

Lemma 1. *Let $C \subset \mathbb{C}$ be a piecewise smooth closed curve, possibly with self intersections. Let $\gamma_1, \gamma_2 : \mathbb{R}/T\mathbb{Z} \to C$ be two absolutely continuous parametrizations of C with the same orientation. Then the orbit $\mathcal{O}(\gamma_2 - \gamma_1) \subset \mathbb{C}$ of $\gamma_2 - \gamma_1$ intersects with every straight line passing through the origin.*

Proof. The case $0 \in \mathcal{O}(\gamma_2 - \gamma_1)$ is obvious. Consider the case $0 \notin \mathcal{O}(\gamma_2 - \gamma_1)$. Without loss of generality, we may assume

$$\operatorname{Arg}(\gamma_2(0) - \gamma_1(0)) = \min_{t\in[0,T]} \operatorname{Arg}(\gamma_2(t) - \gamma_1(t)) = 0.$$

All we need to show is

$$\max_{t\in[0,T]} \operatorname{Arg}(\gamma_2(t) - \gamma_1(t)) \geqq \pi. \tag{7}$$

Consider a moving frame along $\gamma_1(t)$ with its coordinate axes parallel to the real and imaginary axis. See Fig. 1. Equation (7) holds if we can show that for some t the point $\gamma_2(t)$ is inside the third or fourth quadrant or on the negative real axis of the moving frame. Define

$$b = \max\{\beta : \alpha + \beta i \in C, \alpha, \beta \in \mathbb{R}\}, \quad a = \max\{\alpha : \alpha + bi \in C, \alpha \in \mathbb{R}\}.$$

Let $P = (a, b)$; then $P \in C$ and there exists some $t_P \in [0, T]$ such that $\gamma_1(t_P) = P$. Because $\gamma_2(t_P)$ cannot be equal to $\gamma_1(t_P)$, at this moment $\gamma_2(t_P)$ has to be inside the third or fourth quadrant or on the negative real axis of the moving frame. This proves (7). □

From the proof Lemma 1 it looks plausible to strengthen the conclusion by showing

$$\max_{t\in[0,T]} \operatorname{Arg}(\gamma_2(t) - \gamma_1(t)) > \pi.$$

But this is not the case, as the following example demonstrates.

Example 1. Consider the lemiscate C with the standard parametric equation:

$$(\alpha(t), \beta(t)) = \left(\frac{\cos t}{1 + \sin^2 t}, \frac{\cos t \sin t}{1 + \sin^2 t}\right).$$

Let $\theta_0 = \tan^{-1}(\frac{1}{\sqrt{2}}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and let $\varphi(t)$ be a smooth nonnegative function with support $[\theta_0, \pi - \theta_0]$ and with $|\varphi'(t)| < 1$ for any t. Define two parametrizations $\gamma_1, \gamma_2 : [-\theta_0, 2\pi - \theta_0) \to C$ by

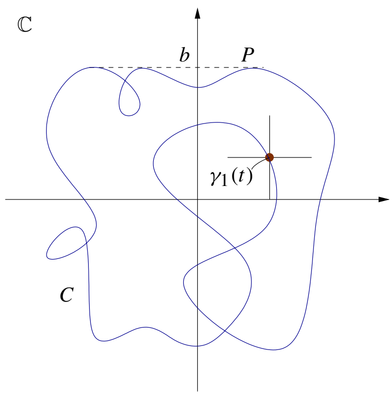

Fig. 1.

$$\gamma_1(t) = \begin{cases} \alpha(t) + \beta(t)i & \text{for } t \in [-\theta_0, \theta_0] \cup [\pi - \theta_0, \pi + \theta_0], \\ \alpha(t + \varphi(t)) + \beta(t + \varphi(t))i & \text{for } t \in (\theta_0, \pi - \theta_0) \cup (\pi + \theta_0, 2\pi - \theta_0), \end{cases}$$
$$\gamma_2(t) = \alpha(t + \pi) + \beta(t + \pi)i.$$

It is easy to verify that $\gamma_1(t) \neq \gamma_2(t)$ for any t and

$$\min_{t \in [0,T]} \operatorname{Arg}(\gamma_2(t) - \gamma_1(t)) = 0, \qquad \max_{t \in [0,T]} \operatorname{Arg}(\gamma_2(t) - \gamma_1(t)) = \pi.$$

If the parametrizations γ_1 and γ_2 are in phase, that is, there is some τ such that $\gamma_1(t + \tau) = \gamma_2(t)$ for any t, then Lemma 1 can be strengthened:

Lemma 2. *Under the assumptions in Lemma 1. Suppose γ_1, γ_2 are in phase and $\gamma_1(t) \neq \gamma_2(t)$ for any t, then*

$$\max_{t \in [0,T]} \operatorname{Arg}(\gamma_2(t) - \gamma_1(t)) - \min_{t \in [0,T]} \operatorname{Arg}(\gamma_2(t) - \gamma_1(t)) > \pi. \tag{8}$$

Proof. Choose $\tau \in (0, T)$ such that $\gamma_1(t + \tau) = \gamma_2(t)$ for all t. Without loss of generality, we may assume that

$$\operatorname{Arg}(\gamma_2(0) - \gamma_1(0)) = \min_{t \in [0,T]} \operatorname{Arg}(\gamma_2(t) - \gamma_1(t)) = 0.$$

Suppose the inequality in (8) is false, then from Lemma 1 equation (8) would be an equality, and by assumption for any $t \in [0, T]$ the imaginary part $\operatorname{Im}(\gamma_2(t))$ of $\gamma_2(t)$ is greater than or equal to the imaginary part $\operatorname{Im}(\gamma_1(t))$ of $\gamma_1(t)$. But

$$\begin{aligned} \int_0^T \operatorname{Im}(\gamma_2(t))\, dt &\geqq \int_0^T \operatorname{Im}(\gamma_1(t))\, dt \\ &= \int_0^T \operatorname{Im}(\gamma_1(t + \tau))\, dt \\ &= \int_0^T \operatorname{Im}(\gamma_2(t))\, dt. \end{aligned}$$

This implies that $\operatorname{Im}(\gamma_1(t)) = \operatorname{Im}(\gamma_2(t))$ for any t, which means that the image of $\operatorname{Arg}(\gamma_2(t) - \gamma_1(t))$ is $\{0, \pi\}$. This is impossible since $\gamma_1(t) \neq \gamma_2(t)$ for any t. □

Lemma 3. *Suppose $\gamma_1, \gamma_2, \gamma_3 : \mathbb{R}/T\mathbb{Z} \to \mathbb{C}$ are three continuous closed curves such that $\mathcal{O}(\gamma_2 - \gamma_3)$ intersect with every straight line passing through the origin. Given $\mu \in [0, 1]$, let $\gamma_{1j}^{\mu} = \mu\gamma_1 + (1 - \mu)\gamma_j$, $j = 2, 3$. Then $\mathcal{O}(\gamma_{12}^{\mu} - \gamma_{13}^{\mu})$ also intersect with every straight line passing through the origin.*

Proof. This is obvious because $\gamma_{12}^{\mu} - \gamma_{13}^{\mu} = (1 - \mu)(\gamma_2 - \gamma_3)$. □

4. Coercivity of the action functional

It is not hard to see that the action functional $\mathcal{A}$ is weakly lower-semicontinuous on $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$ but not coercive. To find periodic solutions with certain symmetries, the appropriate variational problem for (1) is to constrain $\mathcal{A}$ on some subspace Y of $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$. If $\mathcal{A}$ is coercive on Y, then it follows from a standard argument in calculus of variations that $\mathcal{A}$ attains its infimum on $\bar{Y}$. In the following we provide a simple criterion for coercivity that will be used in the applications of Theorems 1 and 2.

We say a subspace Y of $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$ is *noncentral* if there exists some $\nu \in (0, 2]$ such that, for any $x \in Y$, there corresponds a $t_x \in (0, T]$ satisfying

$$x(0) \cdot x(t_x) \leqq (1 - \nu)|x(0)| \cdot |x(t_x)|. \tag{9}$$

It is understood that the left-hand side is the standard scalar product in $(\mathbb{R}^2)^N \cong \mathbb{C}^N$. Clearly (9) is always valid if $\nu \leqq 0$ and fails if $\nu > 2$. The constant $\nu \in (0, 2]$ is independent of $x \in Y$, and therefore each path in Y has to move away from its initial position by a certain angle (relative to the origin). The following proposition states that the minimizing problem of the action functional on a noncentral subspace of $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$ is solvable.

Proposition 2. *Suppose the subspace Y of $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$ is noncentral and weakly closed. Then the action functional $\mathcal{A}$ restricted to Y is coercive and attains its minimum.*

Proof. As remarked above, it suffices to show coercivity.

Consider the function $\delta : Y \to \mathbb{R}$ defined by

$$\delta(x) := \max_{s_1, s_2 \in [0,T]} |x(s_1) - x(s_2)|.$$

This function measures the "size" of the curve $x(t)$.

We first consider the case $\nu \in (0, 2)$. The case $\nu = 2$ is actually easier. Let $t_x \in (0, T]$ be chosen so that (9) is satisfied. If $x(0) \neq 0$ and $x(t_x) \neq 0$ and θ is the angle between $x(0)$ and $x(t_x)$, $0 < \theta \leqq \pi$, then clearly

$$|x(0) - x(t_x)| \geqq \sin(\theta)|x(0)|$$

and the equality holds only when $x(0) - x(t_x)$ is perpendicular to $x(t_x)$. From (9) it can easily be seen that

$$\cos(\theta) \leqq 1 - \nu, \quad \sin(\theta) \geqq C_\nu := \sqrt{\nu(2 - \nu)}.$$

In the other situation, where $x(0) = 0$ or $x(t_x) = 0$, we have $|x(0) - x(t_x)| \geqq |x(0)| \geqq C_\nu|x(0)|$. In both situations, we obtain

$$|x(0) - x(t_x)| \geqq C_\nu|x(0)|.$$

Note that $C_\nu > 0$ because $\nu \in (0, 2)$. For any $t \in [0, T]$,

$$|x(t)| \leqq |x(0)| + \delta(x) \leqq \frac{1}{C_\nu}|x(0) - x(t_x)| + \delta(x) \leqq \left(\frac{1}{C_\nu} + 1\right)\delta(x),$$

and hence

$$\int_0^T |x|^2 dt \leqq \left(\frac{1}{C_\nu}+1\right)^2 \delta(x)^2 T.$$

On the other hand, by the Cauchy-Schwarz inequality

$$\delta(x)^2 \leqq \left(\int_0^T |\dot{x}|\,dt\right)^2 \leqq T\int_0^T |\dot{x}|^2\,dt.$$

Therefore the H^1 norm of x is controlled by its action:

$$\begin{aligned}\|x\|_{H^1}^2 &= \int_0^T |x|^2 + |\dot{x}|^2\,dt \\ &\leqq \left(\left(\frac{1}{C_\nu}+1\right)^2 T^2+1\right)\int_0^T |\dot{x}|^2\,dt \\ &< \left(\left(\frac{1}{C_\nu}+1\right)^2 T^2+1\right)\left(\frac{2}{m}\right)\mathcal{A}(x).\end{aligned}$$

Here $m = \min_i\{m_i\}$. This implies that $\mathcal{A}$ restricted to Y is coercive.

The other case, $\nu = 2$, is similar. Let t_x be as before, then it follows easily from (9) that

$$|x(0)| \leqq |x(0) - x(t_x)| \leqq \delta(x).$$

Thus

$$|x(t)| \leqq |x(0)| + \delta(x) \leqq 2\delta(x)$$

for any $t \in [0, T]$. The fact that $\mathcal{A}$ is coercive on Y follows by the same argument as above. □

5. Binary decompositions for a system of N bodies

Consider a system of N ($\geqq 2$) mass points $m_1, m_2, \dots, m_N$ moving in $\mathbb{C}$. A *binary decomposition* of this system of N bodies is a selection of two nonnegative $N \times N$ matrices $\mathbf{M} = [m_{ij}]$ and $\Lambda = [\lambda_{ij}]$ satisfying

$$\begin{array}{rll} m_{ii} = \lambda_{ii} = 0 & & \text{for any } i, \\ m_{ij} > 0 & & \text{for any } i \neq j, \\ 0 \leqq \lambda_{ij} = \lambda_{ji} \leqq 1 & & \text{for any } i \neq j, \\ \sum_{j=1}^N m_{ij} = m_i & & \text{for any } i. \end{array}$$

The matrices $\mathbf{G^0} = [G^0_{ij}]$, $\mathbf{G^1} = [G^1_{ij}]$ defined by

$$G^0_{ij} = \frac{G\lambda_{ij} m_i m_j}{m_{ij} m_{ji}}, \tag{10}$$

$$G^1_{ij} = \frac{G(1-\lambda_{ij}) m_i m_j}{m_{ij} m_{ji}} \tag{11}$$

are called the *matrices of attraction constants* for the binary decomposition. The special case

$$\lambda_{ij} = \lambda, \quad m_{ij} = \frac{m_i}{N-1} \quad \text{for any } i \neq j \tag{12}$$

is called the *standard binary decomposition* of the system. The constant $\lambda \in [0, 1]$ is called the *weight* of the standard binary decomposition.

Regard $\{m_{ij}\}_{i \neq j}$ as a system of distinct $N(N-1)$ elementary particles moving in space; m_{ij} and m_{ji} constitute a pair of particle-antiparticle with attraction constant $G_{ij} = G^0_{ij} + G^1_{ij}$, and they do not interact with any other particle. Fixing any i, we bind the subsystem $\{m_{ij} : j \neq i\}$ so that the particles all have the same position x_i. The potential and kinetic energy for the binary $\{m_{ij}, m_{ji}\}$ are given by

$$U_{ij}(x) = \frac{G_{ij} m_{ij} m_{ji}}{|x_i - x_j|},$$
$$K_{ij}(\dot{x}) = \frac{1}{2}(m_{ij}|\dot{x}_i|^2 + m_{ji}|\dot{x}_j|^2).$$

Due to the binding force for the subsystems, the binary $\{m_{ij}, m_{ji}\}$ do not move as an isolated Newtonian mechanical system, and therefore their path is not a critical point (except for $N = 2$) of their action

$$\mathcal{A}_{ij}(x) = \int_0^T K_{ij}(\dot{x}) + U_{ij}(x)\, dt. \tag{13}$$

Clearly $U_{ij} = U_{ji}$, $K_{ij} = K_{ji}$, $\mathcal{A}_{ij} = \mathcal{A}_{ji}$, and from (13) the total action of the decomposed system is exactly the same as the action of the original system:

$$\mathcal{A}(x) = \sum_{\substack{(i,j) \\ i<j}} \mathcal{A}_{ij}(x). \tag{14}$$

Similar to (3),(4), we may decompose the action $\mathcal{A}_{ij}$ of each binary into $\mathcal{A}^0_{ij}$, $\mathcal{A}^1_{ij}$ as follows:

$$K^0_{ij}(\dot{x}) = \frac{m_{ij} m_{ji}}{2(m_{ij} + m_{ji})} |\dot{x}_i - \dot{x}_j|^2, \qquad K^1_{ij}(\dot{x}) = \frac{m_{ij} + m_{ji}}{2} |\dot{\bar{x}}_{ij}|^2,$$
$$U^0_{ij}(x) = \frac{G^0_{ij} m_{ij} m_{ji}}{|x_i - x_j|}, \qquad U^1_{ij}(x) = \frac{G^1_{ij} m_{ij} m_{ji}}{|x_i - x_j|},$$

where $\bar{x}_{ij} = \frac{1}{m_{ij}+m_{ji}}(m_{ij}x_i + m_{ji}x_j)$ is the mass center of the binary $\{m_{ij}, m_{ji}\}$. When the decomposition is standard, $\bar{x}_{ij}$ is the same as the mass center of $\{m_i, m_j\}$. Let

$$\mathcal{A}^0_{ij}(x) = \int_0^T K^0_{ij}(\dot{x}) + U^0_{ij}(x)\, dt,$$
$$\mathcal{A}^1_{ij}(x) = \int_0^T K^1_{ij}(\dot{x}) + U^1_{ij}(x)\, dt,$$
$$\mathcal{A}^0(x) = \sum_{\substack{(i,j) \\ i<j}} \mathcal{A}^0_{ij}, \quad \mathcal{A}^1(x) = \sum_{\substack{(i,j) \\ i<j}} \mathcal{A}^1_{ij}.$$

Then

$$\mathcal{A}(x) = \mathcal{A}^0(x) + \mathcal{A}^1(x).$$

According to Gordon's theorem, if $x_j - x_i : \mathbb{R}/T\mathbb{Z} \to \mathbb{C}$ belongs to $\bar{\Lambda}_T$ then (5) provides a lower-bound estimate for $\mathcal{A}_{ij}$ and $\mathcal{A}^0_{ij}$. This condition, unfortunately, is invalid in many cases. This prompts us to select another path space:

$$X_T = \{x = (x_1, \dots, x_N) \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N) : (x_i, x_j) \in \Sigma_T \text{ for any } i \neq j\}, \tag{15}$$

where the set Σ_T is defined in (6). The examples below show that many interesting cases fall into this category.

Example 2. A periodic solution is called *self-similar* or *homographic* if the configuration remains similar at any instant. If the whole system moves as a rigid body the solution is called a *relative equilibrium*. By stopping a relative equilibrium and releasing all masses with zero initial velocity, the configuration will shrink homothetically to a total collapse. These types of solutions are called *homothetic solutions* and their configurations are called *central configurations*. It is quite obvious that the set of T-periodic self-similar solutions, including collision-ejection homothetic solutions, are in X_T.

Example 3. A periodic solution is called a *simple choreographic solution* if all masses chase along a single closed curve. If the orbit consists of two or more closed curves, each of which is the trajectory of at least two masses, then we call the solution a *multiple choreographic solution*. Placing N equal masses on the vertexes of a regular N-gon can result in a relative equilibria which is clearly a simple choreographic solution; many other relative equilibria with nested polygons are multiple choreographic solutions. These types of relative equilibria will be referred to as *trivial* choreographic solutions. We will be most interested in nontrivial choreographic solutions.

The first nontrivial simple choreographic solution ever found is the figure-8 orbit with three equal masses [7]. A nontrivial double choreographic solution for the four-body problem is given in [1]. In [15, 16], TERRACINI & VENTURELLI prove the existence of a nonplanar simple choreographic solution for the four-body problem. Many simple choreographic solutions were numerically discovered [6, 14] without any analytical proof for the existence.

All T-periodic simple choreographic solutions are in X_T. This is a direct consequence of Lemma 1. Among all the numerical discoveries of simple choreographic solutions, the masses are all equal and in phase, in which cases Lemma 2 can be applied.

Example 4. We say a loop $x \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$ has *d-fold rotation symmetry*, $d \geqq 2$, if $x(t) = e^{\frac{2\pi i}{d}} x(t + \frac{T}{d})$ for any t. The case $d = 2$ was first studied by COTI ZELATI [9] and then many others, mostly focusing on Newtonian-type potentials with stronger forces.

Clearly orbits with some d-fold rotation symmetry belong to X_T. Relative equilibria are the only solutions that have d-fold rotation symmetry for every $d \geqq 2$.

We say a loop $x \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$ has *mirror symmetry* if there is a straight line L such that $x(t)$ and $x(t+\frac{T}{2})$ are symmetric with respect to L for any t. No self-similar loops have mirror symmetry except the collinear ones.

Numerical findings indicate that many choreographic solutions belong to one or the other category. A loop with mirror symmetry is not necessarily in X_T, but usually it is easy to determine whether it is in X_T if there exists additional symmetry.

6. Lower-bound estimates for the action functional

In this section we provide lower-bound estimates for the action functional over loops in X_T with rotation or mirror symmetry.

For any $x = (x_1, \dots, x_N) \in X_T, i \neq j$, we use the symbol "$i \bowtie j$" to indicate that $x_i(t) = x_j(t)$ for some t. The set I_x of *collision indexes* of x is defined by

$$I_x = \{(i, j) : i < j \text{ and } i \bowtie j\}. \tag{16}$$

6.1. Closed loops with rotation symmetry

Proposition 3. *Consider a system of N mass points $m_1, \dots, m_N$ with positions $x_1, \dots, x_N \in \mathbb{C}$. Suppose $[m_{ij}]$, $[\lambda_{ij}]$ is a binary decomposition of the system. Let $x = (x_1, \dots, x_N)$ and I_x be collision indexes of x. Suppose x has d-fold rotation symmetry. Then*

$$\mathcal{A}^0(x) \geqq 3\left(\frac{G^2\pi^2}{2}\right)^{\frac{1}{3}} \left[\left(d^{\frac{2}{3}} \sum_{\substack{(i,j)\in I_x \\ i<j}} + \sum_{\substack{(i,j)\notin I_x \\ i<j}}\right)\left(\frac{\lambda_{ij}^2 m_i^2 m_j^2 m_{ij} m_{ji}}{m_{ij}+m_{ji}}\right)^{\frac{1}{3}}\right] T^{\frac{1}{3}}. \tag{17}$$

If all masses are equal, $m_1 = \cdots = m_N = 1$, and $\lambda_{ij} = \lambda$ for any $i \neq j$, then

$$\mathcal{A}^0(x) \geqq 3\left(\frac{G^2\lambda^2\pi^2}{4(N-1)}\right)^{\frac{1}{3}} \left((d^{\frac{2}{3}} - 1)|I_x| + \binom{N}{2}\right) T^{\frac{1}{3}}. \tag{18}$$

Proof. First note that d-fold rotation symmetry implies that x belongs to X_T, and therefore by Proposition 1 the action of each binary in the decomposition is comparable to Keplerian ones.

It follows directly from the definition of X_T, G_{ij}, and Proposition 1 that, for any $i \neq j$,

$$\mathcal{A}^0_{ij}(x) \geqq 3\left(\frac{G^2\pi^2}{2}\right)^{\frac{1}{3}} \left(\frac{\lambda_{ij}^2 m_i^2 m_j^2 m_{ij} m_{ji}}{m_{ij}+m_{ji}}\right)^{\frac{1}{3}} T^{\frac{1}{3}}.$$

If $i \bowtie j$, then by the symmetry assumption there exists some τ such that $x_i\left(\tau + \frac{kT}{d}\right) = x_j\left(\tau + \frac{kT}{d}\right)$ for $k = 0, 1, \dots, d-1$. The path $x_j - x_i$ is a closed loop that begins and ends at the origin on the interval $\left[\tau + \frac{kT}{d}, \tau + \frac{(k+1)T}{d}\right]$ for any k, and its orbits on these intervals are identical except that they differ by an angle. Thus, by rotating x_i, x_j once after every $\frac{kT}{d}$, (x_i, x_j) can be considered to be a loop in $\Sigma_{\frac{T}{d}}$. By Proposition 1,

$$\begin{aligned}\mathcal{A}^0_{ij}(x) &= d\int_0^{\frac{T}{d}} K^0_{ij}(\dot{x}) + U^0_{ij}(x)\,dt \\ &\geqq 3d\left(\frac{G^2\pi^2}{2}\right)^{\frac{1}{3}}\left(\frac{\lambda^2_{ij}m_i^2m_j^2m_{ij}m_{ji}}{m_{ij}+m_{ji}}\right)^{\frac{1}{3}}\left(\frac{T}{d}\right)^{\frac{1}{3}}.\end{aligned}$$

Equation (17) is obtained by summing up $\mathcal{A}^0_{ij}$ over all $i < j$.

Equation (18) follows easily from (17) by considering the standard decomposition. □

Now we estimate $\mathcal{A}^1(x)$:

$$\begin{aligned}\sum_{\substack{(i,j)\\ i<j}} K^1_{ij}(\dot{x}) &= \sum_{\substack{(i,j)\\ i<j}} \frac{m_{ij}+m_{ji}}{2}|\dot{\bar{x}}_{ij}|^2 \\ &= \sum_{\substack{(i,j)\\ i\neq j}} \frac{1}{2}m_{ij}|\dot{\bar{x}}_{ij}|^2 \\ &= \frac{1}{2(N-2)}\sum_{\substack{(i,j,k)\\ i,j,k\text{ distinct}}} m_{ij}|\dot{\bar{x}}_{ij}|^2 \\ &= \frac{1}{2(N-2)}\sum_{\substack{(i,j,k)\\ i,j,k\text{ distinct}}} \frac{1}{2}(m_{ik}|\dot{\bar{x}}_{ik}|^2 + m_{jk}|\dot{\bar{x}}_{jk}|^2) \\ &= \frac{1}{N-2}\sum_{k=1}^{N}\sum_{\substack{(i,j)\\ i<j\\ i,j\neq k}} \frac{1}{2}(m_{ik}|\dot{\bar{x}}_{ik}|^2 + m_{jk}|\dot{\bar{x}}_{jk}|^2),\\ \sum_{\substack{(i,j)\\ i<j}} U^1_{ij}(x) &= \sum_{\substack{(i,j)\\ i<j}} \frac{G^1_{ij}m_{ij}m_{ji}}{|x_i - x_j|} \\ &= \frac{1}{N-2}\sum_{k=1}^{N}\sum_{\substack{(i,j)\\ i<j\\ i,j\neq k}} \frac{G^1_{ij}m_{ij}m_{ji}}{|x_i - x_j|}.\end{aligned}$$

This process essentially treats all mass centers $\bar{x}_{ij}$ as real masses and then considers their binary decomposition. The problem here is that $|x_i - x_j|$ may not be expressed as a fixed multiple of $|\bar{x}_{ik} - \bar{x}_{jk}|$. In the case of equal masses this can be most easily resolved, and the corresponding lower-bound estimate for $\mathcal{A}$ is given in the following theorem.

Theorem 1. *Consider a systems of N equal masses $m_1 = \cdots = m_N = 1$ with positions $x_1, \ldots, x_N \in \mathbb{C}$. Let I_x be collision indexes of x. Suppose x has d-fold rotation symmetry. Then*

$$\mathcal{A}(x) \geqq 3\left(\frac{5G^2\pi^2}{16(N-1)}\right)^{\frac{1}{3}}\left((d^{\frac{2}{3}}-1)|I_x| + \binom{N}{2}\right)T^{\frac{1}{3}}. \tag{19}$$

Proof. Consider a standard decomposition with $\lambda_{ij} = \lambda$, $m_{ij} = \frac{1}{N-1}$ for any $i \neq j$. We first prove the following lower-bound estimate for $\mathcal{A}^1$:

$$\mathcal{A}^1(x) \geqq 3\left(\frac{G^2(1-\lambda)^2\pi^2}{16(N-1)}\right)^{\frac{1}{3}}\left(\left(d^{\frac{2}{3}}-1\right)|I_x| + \binom{N}{2}\right)T^{\frac{1}{3}}. \tag{20}$$

Clearly $|\bar{x}_{jk} - \bar{x}_{ik}| = \frac{1}{2}|x_j - x_i|$ for any $k \neq i, j$. If $i \bowtie j$, then by Lemma 3 $\bar{x}_{jk} - \bar{x}_{ik}$ can actually be considered a loop in $\Sigma_{\frac{T}{d}}$ that begins and ends at the origin. Note that the symmetry assumption automatically implies $x \in X_T$. By the same argument as in the proof of Proposition 3,

$$\begin{aligned}
&\mathcal{A}^1(x)\\
&= \sum_{\substack{(i,j)\\ i<j}} \int_0^T K^1_{ij}(\dot{x}) + U^1_{ij}(x)\,dt\\
&= \frac{1}{N-2}\sum_{k=1}^{N}\sum_{\substack{(i,j)\\ i<j\\ i,j\neq k}} \int_0^T \frac{1}{2}(m_{ik}|\dot{\bar{x}}_{ik}|^2 + m_{jk}|\dot{\bar{x}}_{jk}|^2) + \frac{G^1_{ij}m_{ik}m_{jk}}{2|\bar{x}_{ik} - \bar{x}_{jk}|}\,dt\\
&= \frac{1}{(N-2)(N-1)}\sum_{\substack{(i,j)\\ i<j}}\sum_{\substack{k=1\\ k\neq i,j}}^{N} \int_0^T \frac{1}{2}(|\dot{\bar{x}}_{ik}|^2 + |\dot{\bar{x}}_{jk}|^2) + \frac{\frac{G}{2}(1-\lambda)(N-1)}{|\bar{x}_{ik} - \bar{x}_{jk}|}\,dt\\
&= \frac{1}{(N-2)(N-1)}\left(d\sum_{\substack{(i,j)\in I_x\\ i<j}}\sum_{\substack{k=1\\ k\neq i,j}}^{N}\int_0^{\frac{T}{d}}\cdots + \sum_{\substack{(i,j)\notin I_x\\ i<j}}\sum_{\substack{k=1\\ k\neq i,j}}^{N}\int_0^{T}\cdots\right)\\
&\geqq \frac{3}{N-1}\left(\frac{G^2(1-\lambda)^2(N-1)^2\pi^2}{2^4}\right)^{\frac{1}{3}}\left(d^{\frac{2}{3}}|I_x| + \binom{N}{2} - |I_x|\right)T^{\frac{1}{3}}.
\end{aligned}$$

The last inequality is obtained by applying Proposition 1 to both integrals. This proves (20). Equation (19) follows easily by adding (18) to (20) and then maximizing over $\lambda \in [0, 1]$. □

6.2. Closed loops with mirror symmetry

The arguments in Proposition 3 and Theorem 1 with $d = 2$ work for closed loops in X_T with mirror symmetry. We list the corresponding lower-bound estimates here and omit the proof.

Proposition 4. *Consider a systems of N mass points $m_1, \dots, m_N$ with positions $x_1, \dots, x_N \in \mathbb{C}$. Suppose $x = (x_1, \dots, x_N) \in X_T$ and $[m_{ij}]$, $[\lambda_{ij}]$ is a binary decomposition of the system. Let I_x be collision indexes of x. Suppose x has mirror symmetry. Then*

$$\mathcal{A}^0(x) \geqq 3\left(\frac{G^2\pi^2}{2}\right)^{\frac{1}{3}}\left[\left(2^{\frac{2}{3}}\sum_{\substack{(i,j)\in I_x\\ i<j}} + \sum_{\substack{(i,j)\notin I_x\\ i<j}}\right)\left(\frac{\lambda_{ij}^2 m_i^2 m_j^2 m_{ij} m_{ji}}{m_{ij}+m_{ji}}\right)^{\frac{1}{3}}\right]T^{\frac{1}{3}}. \tag{21}$$

If all masses are equal, $m_1 = \cdots = m_N = 1$, and $\lambda_{ij} = \lambda$ for any $i \neq j$, then

$$\mathcal{A}^0(x) \geqq 3\left(\frac{G^2\lambda^2\pi^2}{4(N-1)}\right)^{\frac{1}{3}}\left((2^{\frac{2}{3}}-1)|I_x| + \binom{N}{2}\right)T^{\frac{1}{3}}, \tag{22}$$

$$\mathcal{A}^1(x) \geqq 3\left(\frac{G^2(1-\lambda)^2\pi^2}{16(N-1)}\right)^{\frac{1}{3}}\left((2^{\frac{2}{3}}-1)|I_x| + \binom{N}{2}\right)T^{\frac{1}{3}}. \tag{23}$$

Theorem 2. *Consider a systems of N equal masses $m_1 = \cdots = m_N = 1$ with positions $x_1, \dots, x_N \in \mathbb{C}$. Let I_x be collision indexes of x. Suppose $x \in X_T$ has mirror symmetry. Then*

$$\mathcal{A}(x) \geqq 3\left(\frac{5G^2\pi^2}{16(N-1)}\right)^{\frac{1}{3}}\left((2^{\frac{2}{3}}-1)|I_x| + \binom{N}{2}\right)T^{\frac{1}{3}}. \tag{24}$$

Example 5. The discovery of the figure-8 orbit [7] for the three-body problem with equal masses has attracted much attention in recent years. Apart from Chenciner and Montgomery's original proof, there have been several other existence proofs [2, 4, 17]. The estimate in ZHANG & ZHOU [17] is the sharpest. Further results and open questions related to the figure-8 orbit can be found in [5].

The figure-8 orbit is a simple choreographic solution that satisfies the following symmetry:

$$\begin{aligned}(x_1, x_2, x_3)(t) &= -(\bar{x}_3, \bar{x}_1, \bar{x}_2)\left(t + \tfrac{T}{6}\right)\\ &= -(x_2, x_1, x_3)(-t).\end{aligned}$$

The first equation shows that it has mirror symmetry. It belongs to the space X_T since it is simple choreographic (Lemma 1). The space Y of loops satisfying the symmetry described above is noncentral. This can be seen by choosing $t_x = \frac{T}{3}$ and $\nu = \frac{1}{2}$ in (9). By Proposition 2, the action functional $\mathcal{A}$ attains its infimum on Y, and by Palais' principle (see Section 7) the minimizers are solutions to the three-body problem.

Set $G = T = 1$. By Theorem 2, the action of a collision path is at least

$$\frac{9}{2}(5\pi^2)^{\frac{1}{3}} \approx 16.5058.$$

This estimate is not as sharp as the one in [17], but is higher than the action of the test paths selected in [2, 7]. According to [7], the numerical value of the action of a figure-8 orbit with period $T = 1$ is approximately 13.2078. In fact, we may choose a simpler test path by parametrizing the lemiscate.

7. Multiple choreographic solutions with equal masses

In this section we show some applications of Theorem 1. Consider a system of N equal masses $m_1 = \cdots = m_N = 1$. Let G be a group of linear transformations on $H = H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$. The space H^G of G-invariant loops is defined by:

$$H^G = \{x \in H : g \cdot x = x \text{ for any } g \in G\}.$$

Critical points of $\mathcal{A}$ restricted to H^G, called *G-critical points*, are not necessarily critical points of $\mathcal{A}$ on H. In this regard we quote a result of PALAIS [13]:

Palais' Principle of Symmetric Criticality. *Suppose the group G is orthogonal and $\mathcal{A}$ is G-invariant, then G-critical points of $\mathcal{A}$ are critical points of $\mathcal{A}$ on H.*

For all the applications presented herein, the assumptions in Palais' principle are met.

Given a finite orthogonal group G acting on H, it is usually easy to determine coercivity of the action $\mathcal{A}$ restricted to H^G (Proposition 2). For instance, if H^G is contained in the space of loops with some d-fold rotation symmetry, then we can easily verify that H^G is noncentral.

Suppose $\mathcal{A}$ is coercive on H^G and $x \in H^G$ is a minimizer. Let $[\tau_1, \tau_2]$ be a *fundamental domain* of the action, that is, a smallest closed time interval over which the projection $H^G \to H^1([\tau_1, \tau_2], \mathbb{C}^N)$ is injective. Then x is clearly also a minimizer for the fixed-ends problem:

$$\inf\{\mathcal{A}(y) : y \in H^1([\tau_1, \tau_2], \mathbb{C}^N),\ y(\tau_1) = \xi_1,\ y(\tau_2) = \xi_2\}, \tag{25}$$

where $\xi_1 = x(\tau_1)$ and $\xi_2 = x(\tau_2)$. A fundamental result by MARCHAL [12] (see also [4]) states

Marchal's Theorem. *Given any $\xi_1, \xi_2 \in \mathbb{C}^N$. Minimizers of the fixed-ends problem* (25) *are collision-free on the interval* (τ_1, τ_2).

Based on the estimates in the previous section and the theorems of PALAIS and MARCHAL, in what follows we show the existence of some highly symmetric periodic solutions for the N-body problem.

7.1. Double choreographic solutions with $2n$ equal masses

Let n be any odd number. In this subsection we will show the existence of double choreographic solutions with $N = 2n$ equal masses. Throughout this section the gravitation constant is assumed to be 1.

Let $n = 3$. Consider the group K_3 generated by σ_3 and τ_3:

$$\sigma_3 \cdot x(t) = e^{\frac{\pi i}{3}}(x_6, x_1, x_2, x_3, x_4, x_5)(-t)$$
$$\tau_3 \cdot x(t) = (x_3, x_6, x_5, x_2, x_1, x_4)\left(t + \frac{T}{3}\right).$$

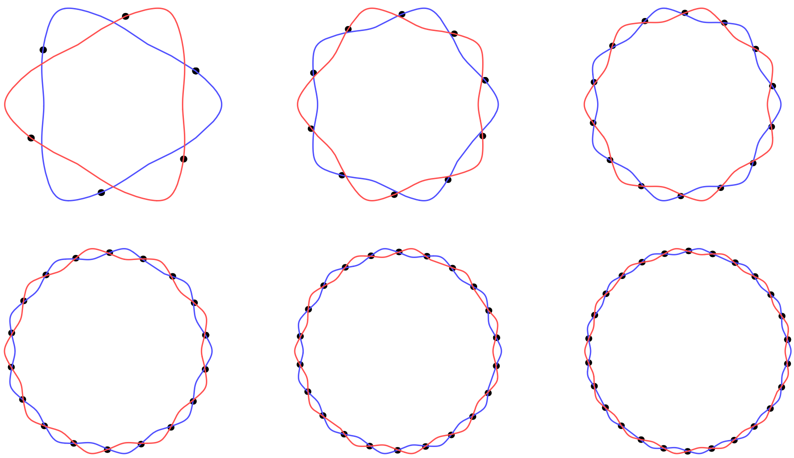

Fig. 2. Double choreographic loops with $2n$ equal masses.

More generally, for any odd integer n, we define the group K_n generated by σ_n and τ_n:

$$\begin{aligned}\sigma_n \cdot x(t) &= e^{\frac{\pi i}{n}}(x_{2n}, x_1, x_2, \dots, x_{2n-1})(-t)\\ \tau_n \cdot x(t) &= (x_3, x_{2n}, x_5, x_2, \dots, x_{2k-1}, x_{2(k-2)},\\ &\qquad \dots, x_{2n-1}, x_{2(n-2)}, x_1, x_{2(n-1)})\left(t + \frac{T}{n}\right).\end{aligned}$$

Figure 2 shows some paths that are numerically the minimizers of $\mathcal{A}$ on H^{K_n}.

Theorem 3. *The action functional $\mathcal{A}$ attains its infimum on H^{K_n} for any odd number $n \geqq 3$. All minimizers are collision-free double choreographic solutions for the $2n$-body problem with zero angular momentum.*

Proof. The assumption that all masses are equal ensures that $\mathcal{A}$ is K_n-invariant, and all requirements in Palais' principle are met. The symmetry assumptions immediately implies that $\mathcal{A}$ is coercive on H^{K_n} (Proposition 2), and therefore minimizers of $\mathcal{A}$ on H^{K_n} exist and are solutions of the $2n$-body problem. Invariance under the action of σ_n implies that the solutions have zero angular momentum. Also, $\{m_1, m_3, \dots, m_{2n-1}\}$ share the same orbit, and $\{m_2, m_4, \dots, m_{2n}\}$ share another identical orbit which differs from the orbit of $\{m_1, m_3, \dots, m_{2n-1}\}$ by an angle of $\frac{\pi}{n}$. If we can show that minimizers are collision-free, then they have to be double choreographic solutions.

A fundamental domain of the group action is $\left[0, \frac{T}{2n}\right]$. At $t = 0$, the configuration is a regular $2n$-gon. At $t = \frac{T}{2n}$, we have

$$
\begin{aligned}
& e^{\frac{\pi i}{n}}(x_{2n}, x_1, x_2, \dots, x_{2n-1})\left(\frac{T}{2n}\right) \\
& \quad = x\left(-\frac{T}{2n}\right) \\
& \quad = (x_3, x_{2n}, x_5, x_2, \dots, x_{2k-1}, x_{2(k-2)}, \\
& \qquad \dots, x_{2n-1}, x_{2(n-2)}, x_1, x_{2(n-1)})\left(\frac{T}{2n}\right),
\end{aligned}
$$

which implies that the configuration is a regular $2n$-gon as well. According to Marchal's theorem, all we need to show is that minimizers cannot begin or end at a total collapse; that is, a collision of all $2n$ bodies.

Suppose $x \in H^{K_n}$ begins or ends with a total collapse. Then the size of the set I_x of collision indexes is $\binom{2n}{2} = n(2n-1)$. Note that each pair (x_i, x_j), $i \neq j$, can be considered as a loop in $\Sigma_{\frac{T}{n}}$, and thus x is contained in X_T. This case is no different from the case with n-fold rotation symmetry. By Theorem 1,

$$
\mathcal{A}(x) \geqq 3\left(\frac{5\pi^2}{16(2n-1)}\right)^{\frac{1}{3}} n^{\frac{2}{3}}\binom{2n}{2} T^{\frac{1}{3}} \;=:\; M_n .
$$

The only thing that remains is to select an appropriate test path in H^{K_n} for each n that has smaller action than the lower-bound estimate M_n. Below is a short list of the approximate values of the lower-bound estimate M_n for the case $T = 1$ and the action $\mathcal{A}_{\text{test}}$ of the test paths. All data are accurate to the third decimal place.

$N = 2n$	M_n	$\mathcal{A}_{\text{test}}$	α	β	γ
6	79.681	61.380	0.39	0.64	0.83
10	276.238	170.504	0.50	0.55	0.44
14	618.439	327.029	0.57	0.55	0.32
18	1124.280	527.263	0.65	0.55	0.30

In each case we select a test path $x = x_{\text{test}}$ by setting $x_1(t) = r_1(t)e^{i\theta_1(t)}$ defined on $\left[-\frac{1}{4n}, \frac{1}{4n}\right]$, extending it periodically to $\left[-\frac{1}{4n}, \frac{4n-1}{4n}\right]$, and then defining each x_k by symmetry. More precisely, let

$$
\begin{aligned}
r_1(t) &= \alpha\left(1 + \frac{\beta}{n}\sin(2n\pi t)\right), \\
\theta_1(t) &= \frac{1}{2n}(\pi(1-4nt) - \gamma\cos(2n\pi t)), \\
r_2(t) &= r_1(-t), \\
\theta_2(t) &= \theta_1(-t) + \frac{\pi}{n},
\end{aligned}
$$

$$\begin{aligned} x_1(t) &= r_1(t)e^{i\theta_1(t)}, \\ x_2(t) &= r_2(t)e^{i\theta_2(t)}, \\ x_{2k+1}(t) &= x_1(t)e^{2k\frac{\pi}{n}i}, \\ x_{2k+2}(t) &= x_2(t)e^{2k\frac{\pi}{n}i}, \quad k = 1, \dots, n-1. \end{aligned}$$

The values of α, β, γ are given in the table above. The resulting test path belongs to H^{K_n} and the configuration remains a union of two regular n-gons. The estimates of their action $\mathcal{A}_{\text{test}}$ in the above table is straightforward, and they are all below the value of M_n. This proves the theorem for $n = 3, 5, 7, 9$. To prove the theorem for general n, we need to give a general formula for our test paths and provide a direct upper-bound estimate for their action. The formula we provide below actually works for all $n \geqq 5$.

By symmetry the polar form for the kinetic energy $K(\dot{x})$ is given by

$$K(\dot{x}) = \frac{n}{2}(\dot{r}_1^2 + \dot{r}_2^2 + r_1^2\dot{\theta}_1^2 + r_2^2\dot{\theta}_2^2).$$

The way test paths are selected makes it easy to find the exact contribution of the kinetic energy to the total action:

$$\begin{aligned} \int_0^1 K(\dot{x})\,dt &= \frac{n}{2}\int_0^1 (\dot{r}_1^2 + \dot{r}_2^2 + r_1^2\dot{\theta}_1^2 + r_2^2\,\dot{\theta}_2^2)\,dt \\ &= \frac{\pi^2\alpha^2}{8n}\Big(4(8 + 4\beta^2 + \gamma^2)n^2 - 32\beta\gamma n + 16\beta^2 + 3\beta^2\gamma^2\Big) =: K_n \end{aligned}$$

The estimates for the contribution of the potential energy $U(x)$ to the total action is more delicate. By symmetry of the test path and Lemma 4, Lemma 5,

$$\begin{aligned} &\int_0^1 U(x)\,dt \\ &\quad = n\int_0^1 2\sum_{k=1}^{\frac{n-1}{2}}\left(\frac{1}{|x_1 - x_{2k+1}|} + \frac{1}{|x_2 - x_{2(k+1)}|}\right) + \sum_{k=1}^{n}\frac{1}{|x_1 - x_{2k}|}\,dt \\ &\quad = n\int_0^1 \sum_{k=1}^{\frac{n-1}{2}}\left(\frac{1}{r_1\sin(\frac{k\pi}{n})} + \frac{1}{r_2\sin(\frac{k\pi}{n})}\right) + \left(\sum_{k=1}^{\frac{n-1}{2}} + \sum_{k=\frac{n+1}{2}}^{n-1}\frac{1}{|x_1 - x_{2k}|}\right) \\ &\qquad\qquad + \frac{1}{|x_1 - x_{2n}|}\,dt \\ &\quad \leqq n\left(\frac{2}{\alpha(1 - \frac{\beta}{n})}\sum_{k=1}^{\frac{n-1}{2}}\csc(\frac{k\pi}{n}) + \frac{n}{4\alpha}\sum_{k=1}^{\frac{n-1}{2}}\frac{1}{k - \frac{1}{2}} + \frac{n}{4\alpha}\sum_{k=\frac{n+1}{2}}^{n-1}\frac{1}{k - \frac{n}{2}} + \frac{n}{2\alpha\beta}\right) \end{aligned}$$

$$\begin{aligned}
&\leqq \frac{n}{\alpha}\left(\frac{n\ln n}{1-\frac{\beta}{n}} + \frac{n}{2}\sum_{k=1}^{\frac{n-1}{2}}\frac{1}{k-\frac{1}{2}} + \frac{n}{2\beta}\right) \\
&\leqq \frac{n^2}{\alpha}\left(\frac{\ln n}{1-\frac{\beta}{n}} + \frac{1}{2}\left(2+\int_2^{\frac{n+1}{2}}\frac{1}{s-1}\,ds\right) + \frac{1}{2\beta}\right) \\
&= \frac{n^2}{\alpha}\left(\frac{\ln n}{1-\frac{\beta}{n}} + \frac{1}{2}\big(2-\ln 2+\ln(n-1)\big) + \frac{1}{2\beta}\right) =: U_n.
\end{aligned}$$

The action $\mathcal{A}_{\text{test}}$ of the test path is bounded from above by $U_n + K_n$. This is valid regardless of the value of $\alpha > 0$, $\beta \in (0, \frac{2}{\pi}]$, and $\gamma > 0$. Observe that $M_n = O(n^{\frac{7}{3}})$ and $U_n + K_n = O(n^2 \ln n)$ as n approaches infinity. This already shows the theorem is valid for every large n.

By choosing $\alpha = \beta = \gamma = 0.6$ and considering single variable functions M_n, U_n, K_n in n, we can easily verify that

$$M_n \geqq U_n + K_n \text{ for any } n \geqq 5$$

and, by Newton's method, the largest real solution for $M_n = U_n + K_n$ is at $n \approx 4.0879$. This proves the theorem for $n \geqq 5$. $\square$

Lemma 4. *Let $n \geqq 3$ be an odd number. Then*

$$\sum_{j=1}^{\frac{n-1}{2}} \csc\left(\frac{j\pi}{n}\right) \leqq \frac{n}{2}\ln n.$$

Proof. This follows easily from the following two elementary inequalities:

$$\begin{aligned}
\csc(s) &\leqq \frac{\pi}{2s} \quad \text{for any } s \in \left(0, \frac{\pi}{2}\right]; \\
\sum_{j=1}^{k}\frac{1}{j} &\leqq \ln(2k+1) \quad \text{for any integer } k \geqq 1. \quad \square
\end{aligned}$$

Lemma 5. *Let x be the test path defined in the proof of Theorem 3. Let $n \geqq 3$ be an odd number and $0 < \beta \leqq \frac{2}{\pi}$. Then, for any $t \in [0, \frac{1}{4n}]$,*

$$|x_1(t) - x_{2k}(t)| \geqq \begin{cases} \dfrac{4\alpha(k-\frac{1}{2})}{n} & \textit{for } k = 1, 2, \ldots, \frac{n-1}{2}; \\ \dfrac{4\alpha(k-\frac{n}{2})}{n} & \textit{for } k = \frac{n+1}{2}, \frac{n+3}{2} \cdots, n-1; \\ \dfrac{2\alpha\beta}{n} & \textit{for } k = n. \end{cases}$$

Proof. Define

$$h_k(t) = 1 - \cos\left(\frac{2k-1}{n}\pi + 4\pi t\right) + \frac{\beta^2}{n^2}\sin^2(2n\pi t)\left(1 + \cos\left(\frac{2k-1}{n}\pi + 4\pi t\right)\right).$$

Then

$$\begin{aligned}|x_1 - x_{2k}|^2 &= r_1^2 + r_2^2 - 2r_1r_2\cos\left(\theta_1 - \theta_2 - \frac{2(k-1)\pi}{n}\right)\\ &= \alpha^2\left(1 + \frac{\beta}{n}\sin(2n\pi t)\right)^2 + \alpha^2\left(1 - \frac{\beta}{n}\sin(2n\pi t)\right)^2\\ &\quad - 2\alpha^2\left(1 - \frac{\beta^2}{n^2}\sin^2(2n\pi t)\right)\cos\left(4\pi t + \frac{2k-1}{n}\pi\right)\\ &= 2\alpha^2 h_k(t).\end{aligned}$$

When $k = 1, 2, \dots, \frac{n-1}{2}$, it is quite obvious that

$$\begin{aligned}&\frac{1}{4\pi}\dot{h}_k(t)\\ &\quad= \sin\left(4\pi t + \frac{2k-1}{n}\pi\right)\left[1 - \frac{\beta^2}{n^2}\sin^2(2n\pi t)\right]\\ &\qquad+\frac{\beta^2}{2n}\sin(4n\pi t)\left(1 + \cos\left(4\pi t + \frac{2k-1}{n}\pi\right)\right)\end{aligned}$$

is nonnegative on $[0, \frac{1}{4n}]$. Therefore, the minimum value of $|x_1 - x_{2k}|^2$ on $[0, \frac{1}{4n}]$ is

$$\begin{aligned}|x_1(0) - x_{2k}(0)|^2 &= 2\alpha^2\left[1 - \cos\left(\frac{2k-1}{n}\pi\right)\right]\\ &\geqq 2\alpha^2\left(\frac{2}{\pi^2}\right)\left(\frac{2k-1}{n}\pi\right)^2\\ &= 16\alpha^2\left(\frac{k-\frac{1}{2}}{n}\right)^2.\end{aligned}$$

The second line uses the fact that $1 - \cos(s) \geqq \frac{2s^2}{\pi^2}$ for $s \in [0, \pi]$. This proves the first inequality in the lemma.

Consider $k = \frac{n+1}{2}, \frac{n+3}{2} \cdots, n$. We will show that the minimum value of $|x_1 - x_{2k}|$ on $[0, \frac{1}{4n}]$ is at $t = \frac{1}{4n}$. Note that $1 - \cos(4\pi t) \leqq \sin(4\pi t)$ for any $t \in [0, \frac{1}{4n}]$. When $k = \frac{n+1}{2}$,

$$\begin{aligned}&\frac{1}{4\pi}\dot{h}_{\frac{n+1}{2}}(t)\\ &\quad= -\sin(4\pi t)\left[1 - \frac{\beta^2}{n^2}\sin^2(2n\pi t)\right] + \frac{\beta^2}{2n}\sin(4n\pi t)\left(1 - \cos(4\pi t)\right)\end{aligned}$$

$$
\begin{aligned}
&\leqq -\sin(4\pi t)\left[1-\frac{1}{n^2}\right]+\frac{1}{2n}\sin(4\pi t)\\
&= -\sin(4\pi t)\left[1-\frac{2+n}{2n^2}\right]\\
&\leqq 0.
\end{aligned}
$$

When $k = \frac{n+3}{2}, \frac{n+5}{2} \cdots, n-1$, the bound $2 \leqq 2k-n-1 \leqq n-3$ and the assumptions $0 \leqq t \leqq \frac{1}{4n}$, $0 < \beta \leqq \frac{2}{\pi}$ yield

$$
\begin{aligned}
&\frac{1}{4\pi}\dot{h}_{\frac{n+1}{2}}(t)\\
&\quad\leqq -\sin\left(4\pi t+\frac{2k-n-1}{n}\pi\right)\left[1-\frac{\beta^2}{n^2}\right]\\
&\qquad+\frac{\beta^2}{2n}\left(1-\cos\left(4\pi t+\frac{2k-n-1}{n}\pi\right)\right)\\
&\quad\leqq -\sin\left(\frac{2\pi}{n}\right)\left[1-\frac{1}{n^2}\right]+\frac{2}{n\pi^2}\left(1-\cos\left(\frac{\pi}{n}+\frac{2k-n-1}{n}\pi\right)\right)\\
&\quad\leqq -\frac{2}{\pi}\left(\frac{2\pi}{n}\right)\left[1-\frac{1}{n^2}\right]+\frac{2}{n\pi^2}\left(\frac{1}{2}\right)\left(\frac{2k-n}{n}\pi\right)^2\\
&\quad\leqq -\frac{4(n^2-1)}{n^3}+\frac{(n-2)^2}{n^3}\\
&\quad< 0.
\end{aligned}
$$

In the above we used the inequality $1-\cos(s) \leqq \frac{s^2}{2}$ for any $s \in [0, \pi]$. We have proved that, for $k = \frac{n+1}{2}, \frac{n+3}{2} \cdots, n-1$, the minimum value of $|x_1 - x_{2k}|^2$ on $[0, \frac{1}{4n}]$ is

$$
\begin{aligned}
\left|x_1\left(\frac{1}{4n}\right)-x_{2k}\left(\frac{1}{4n}\right)\right|^2 &= 2\alpha^2 h_k\left(\frac{1}{4n}\right)\\
&= 2\alpha^2\left[1-\cos\left(\frac{2k\pi}{n}\right)+\frac{\beta^2}{n^2}\left(1+\cos\left(\frac{2k\pi}{n}\right)\right)\right]\\
&\geqq 2\alpha^2\left[1+\cos\left(\frac{2n-2k}{n}\pi\right)\right]\\
&= 4\alpha^2\cos^2\left(\frac{n-k}{n}\pi\right)\\
&\geqq 4\alpha^2\left[1-\frac{2}{\pi}\left(\frac{n-k}{n}\pi\right)\right]^2\\
&= 16\alpha^2\left(\frac{k-\frac{n}{2}}{n}\right)^2.
\end{aligned}
$$

This proves the second inequality in the lemma.

When $k = n$, for any $t \in [0, \frac{1}{4n}]$,

$$\begin{aligned}\frac{1}{4\pi}\dot{h}_n(t) &= -\sin\left(\frac{\pi}{n} - 4\pi t\right)\left[1 - \frac{\beta^2}{n^2}\sin^2(2n\pi t)\right] \\ &\quad + \frac{\beta^2}{2n}\sin(4n\pi t)\left(1 + \cos\left(4\pi t - \frac{\pi}{n}\right)\right) \\ &\leqq -2\left(\frac{1}{n} - 4t\right)\left[1 - \frac{\beta^2}{n^2}\sin^2(2n\pi t)\right] + \frac{\beta^2}{n}\sin(4n\pi t) \\ &=: g_n(t).\end{aligned}$$

The function $g_n(t)$ in the last row is negative when $t = 0$, zero when $t = \frac{1}{4n}$. With the upper bound for β, it is an easy exercise to verify that g_n is increasing on $[0, \frac{1}{4n}]$ for any $n \geqq 3$, implying that g_n (and hence $\dot{h}_n$) is nonpositive on $[0, \frac{1}{4n}]$. This shows that the minimum value of $|x_1 - x_{2n}|^2$ on $[0, \frac{1}{4n}]$ is

$$\left|x_1\left(\frac{1}{4n}\right) - x_{2n}\left(\frac{1}{4n}\right)\right|^2 = 2\alpha^2 h_n\left(\frac{1}{4n}\right) = \frac{4\alpha^2\beta^2}{n^2}.$$

This completes the proof. □

7.2. Multiple choreographic solutions with 4n *equal masses*

Let $N = 4n$, $n = 1, 2, \ldots$. Consider the group G_n generated by σ_n and τ_n:

$$\begin{aligned}\sigma_n \cdot x(t) &= e^{\frac{\pi i}{2n}}(x_{4n}, x_1, x_2, \ldots, x_{4n-1})(-t) \\ \tau_n \cdot x(t) &= (x_{2n+1}, x_{2n+2}, \ldots, x_{4n}, x_1, x_2, \ldots, x_{2n})\left(t + \frac{T}{2}\right).\end{aligned}$$

The group G_n is isomorphic to $\mathbb{Z}_{4n} \times \mathbb{Z}_2$. The assumption that all masses are equal implies that $\mathcal{A}$ is G_n-invariant. Clearly the group G_n is orthogonal. Therefore the requirements in Palais' principle are met. Note that the configuration of any x in H^{G_n} begins with a regular $4n$-gon and the ordering of masses are reversed after $\frac{T}{2}$. In particular, a loop in H^{G_n} cannot be self-similar. Masses with odd indexes remain in the configuration of a regular $2n$-gon and masses with even indexes remain in the configuration of another. These two regular $2n$-gons rotate in opposite directions. Figure 3 shows several loops in H^{G_n} whose action values are approximately the infimum of $\mathcal{A}$ on H^{G_n}.

Theorem 4. *The action functional $\mathcal{A}$ attains its infimum on H^{G_n} for $n = 1, 2, \ldots, 10$. All minimizers are collision-free multiple choreographic solutions for the* 4n*-body problem with zero angular momentum.*

Proof. By Proposition 2 the action functional $\mathcal{A}$ is coercive (choose $t_x = \frac{T}{2}$ and $\nu = 2$ in (9)), and therefore minimizers on H^{G_n} exist. Minimizers of $\mathcal{A}$ on H^{G_n} are multiple choreographic solutions because all requirements in Palais' principle are satisfied and x_i and x_{2n+i} share the same orbit. Invariance under the action of σ_n implies minimizers have zero angular momentum, which shows they cannot be

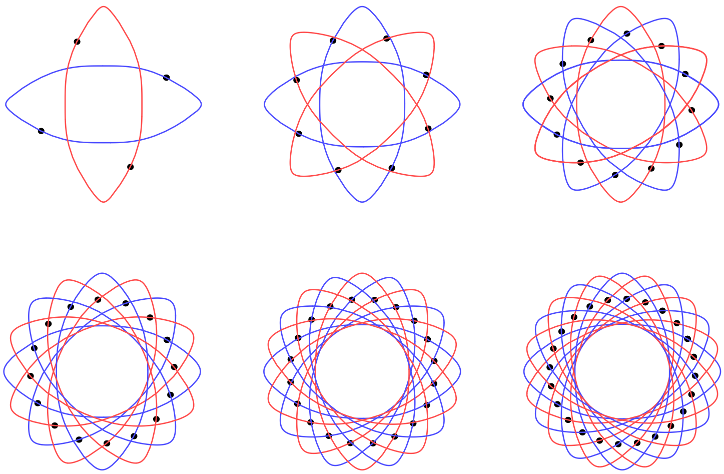

Fig. 3. Multiple choreographic loops with $4n$ equal masses.

relative equilibria. A fundamental domain of the group action is $[0, \frac{T}{4}]$. At $t = 0$, the configuration is a regular $4n$-gon. At $t = \frac{T}{4}$, we have

$$\begin{aligned} & e^{\frac{\pi i}{2n}}(x_{4n}, x_1, x_2, \ldots, x_{4n-1})\left(\frac{T}{4}\right) \\ & \quad = x\left(-\frac{T}{4}\right) \\ & \quad = (x_{2n+1}, x_{2n+2}, \ldots, x_{4n}, x_1, x_2, \ldots, x_{2n})\left(\frac{T}{4}\right), \end{aligned}$$

which implies the configuration is a regular $4n$-gon as well. According to Marchal's theorem, all we need to show is that minimizers cannot begin or end at a total collapse.

Suppose $x \in H^{G_n}$ begins or ends with a total collapse. Then the size of the set I_x of collision indexes is $\binom{4n}{2} = 2n(4n-1)$. Observe that the space H^{G_n} has a two-fold rotation symmetry, and hence is contained in X_T. By Theorem 1,

$$\mathcal{A}(x) \geqq 3\left(\frac{5\pi^2}{16(4n-1)}\right)^{\frac{1}{3}} 2^{\frac{2}{3}} \binom{4n}{2} T^{\frac{1}{3}} =: L_n.$$

The only thing that remains is to select appropriate test paths in H^{G_n} that have smaller action than this lower-bound estimate. Below is a list of the approximate values of L_n for the case $T = 1$ and the action $\mathcal{A}_{\text{test}}$ of the test paths. All data are accurate to the third decimal place.

$N = 4n$	L_n	$\mathcal{A}_{\text{test}}$	α	β	γ
4	28.838	22.158	0.29	0.436	1.53
8	101.465	84.188	0.40	0.394	1.41
12	205.717	178.528	0.48	0.378	1.38
16	337.294	301.986	0.54	0.365	1.32
20	493.583	452.550	0.59	0.354	1.30
24	672.756	628.670	0.64	0.350	1.28
28	873.431	829.246	0.68	0.345	1.27
32	1094.508	1053.316	0.72	0.340	1.25
36	1335.085	1300.113	0.75	0.336	1.24
40	1594.401	1568.928	0.79	0.332	1.23

In each case we select a test path $x = x_{\text{test}}$ by setting $x_1(t) = r_1(t)e^{i\theta_1(t)}$ defined on $[-\frac{1}{8}, \frac{1}{8}]$, extend it periodically to $[-\frac{1}{8}, \frac{7}{8}]$, and then define each x_k by symmetry. More precisely, on $[-\frac{1}{8}, \frac{1}{8}]$, let

$$\begin{aligned}
r_1(t) &= \alpha(1 + \beta \sin(4\pi t)),\\
\theta_1(t) &= \frac{1}{4}(\pi(1 - 8t) - \gamma \cos(4\pi t)),\\
r_2(t) &= r_1(-t),\\
\theta_2(t) &= \theta_1(-t) + \frac{\pi}{2n},\\
x_1(t) &= r_1(t)e^{i\theta_1(t)},\\
x_2(t) &= r_2(t)e^{i\theta_2(t)},\\
x_{2k-1}(t) &= x_1(t)e^{2(k-1)\frac{\pi}{2n}i},\\
x_{2k}(t) &= x_2(t)e^{2(k-1)\frac{\pi}{2n}i}, \quad k = 1, \dots, 2n.
\end{aligned}$$

The values of α, β, γ are given in the table above. The resulting test path remains a union of two regular $2n$-gons. This makes the evaluation of their action much simpler.

By symmetry the kinetic energy $K(\dot{x}) = \frac{1}{2}|\dot{x}|^2$ satisfies

$$K(\dot{x}) = \frac{1}{2}\sum_{j=1}^{2n}(|\dot{x}_{2j-1}|^2 + |\dot{x}_{2j}|^2) = n(|\dot{x}_1|^2 + |\dot{x}_2|^2).$$

Thus, in polar form,

$$K(\dot{x}) = n(\dot{r}_1^2 + \dot{r}_2^2 + r_1^2\dot{\theta}_1^2 + r_2^2\dot{\theta}_2^2). \tag{26}$$

The polar form of the potential energy is

$$\begin{aligned}
U(x) = n\Bigg\{&\left(\frac{1}{2} + \sum_{j=1}^{n-1}\csc\left(\frac{j\pi}{2n}\right)\right)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\\
&+ 2\sum_{j=0}^{2n-1}\frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2 + \frac{j\pi}{n})}}\Bigg\}
\end{aligned}$$

This combined with (26) gives the polar form of the Lagrangian. With the numerical values of α, β, γ provided in the table, we can easily verify the numerical value of their action $\mathcal{A}_{\text{test}}$ listed in the table. The values are all below the lower bound L_n we obtained. □

Remark 1. Numerical estimates indicate that L_n is relatively close to the infimum of $\mathcal{A}$ on H^{G_n}. This means that, in general, a crude estimate of $\mathcal{A}_{\text{test}}$ (as done in the proof of Theorem 4) will not be sufficient to extend the theorem to general n.

Remark 2. Double choreographic solutions in Theorem 3 and the case $N = 4$ in Theorem 4 have zero angular momentum because the path spaces are invariant under some action σ that reverses the time variable, and in the meanwhile σ^2 stabilizes exactly two groups of indexes. These two groups of masses move in opposite directions along identical curves so that the total angular momentum vanishes. This type of symmetry is impossible when N is odd, for otherwise σ^2 as a permutation of indexes would be of the form $(j_1, j_2, \ldots, j_k)(j_{k+1}, \ldots, j_N)$ for some permutation j of $\{1, \ldots, N\}$ and even number k. Then, as permutation of indexes, $1 = \sigma^{2kn} = (j_{k+1}, \ldots, j_N)^{kn}$, which is impossible since kn is even and $N - k$ is odd. From this point of view it would be interesting if we could construct double choreographic solutions with zero angular momentum for an odd number of masses.

8. An example in Ferrario & Terracini [10]

Even though the case $n = 3$ in Theorem 3 is the only example in Section 7 that was specifically included in the recent article by Ferrario & Terracini [10], the major theorem (Theorem 10.8) in [10] can actually be applied to every example in there. Below we discuss some examples in which their theorem cannot fully apply. Theorem 10.8 in [10] will be referred to as the *Ferrario - Terracini theorem.*

First we remark that the symmetry group for the figure-8 orbit in Example 5 is isomorphic to the dihedral group D_6 (denoted by D_{12} in [10]) of order 12. Two maximal subgroups are D_3 and $\mathbb{Z}_6$. The Ferrario-Terracini theorem can be applied to the figure-8 orbit with *less* symmetry – either with a D_3 or $\mathbb{Z}_6$ symmetry – but not for the figure-8 with D_6 symmetry. See Example (11.2) in [10]. (Thanks to Chenciner for pointing this out!) Our argument in Example 5 applies to all these cases since the mirror symmetry and the property of being simple choreographic are the only properties our arguments rely on. Existence of any additional symmetry does not effect the validity of the proof.

Another example in which the Ferrario-Terracini theorem cannot apply is example (11.4) in [10]. Consider four equal masses $m_1 = m_2 = m_3 = m_4 = 1$ moving in $\mathbb{C}$. E_n Let E be the group generated by σ, τ, and δ:

$$\begin{aligned}
\sigma \cdot x(t) &= (\bar{x}_2, \bar{x}_1, \bar{x}_4, \bar{x}_3)(-t),\\
\tau \cdot x(t) &= e^{\frac{\pi i}{3}}(x_3, x_4, x_1, x_2)\left(t + \frac{T}{6}\right),\\
\delta \cdot x(t) &= -(x_2, x_1, x_4, x_3)(t).
\end{aligned}$$

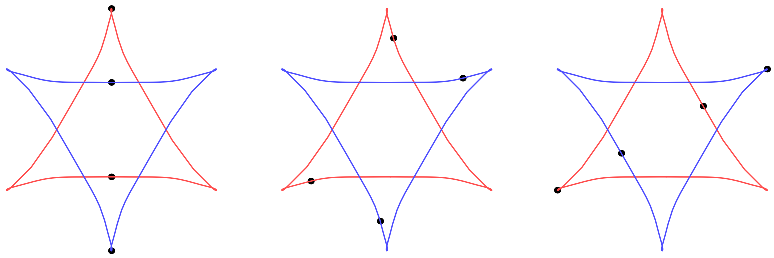

Fig. 4. A double choreographic loop with 3-fold rotation symmetry.

In [10] the transformation $\tau = \tau_n$ is given by

$$\tau_n \cdot x(t) = e^{\frac{\pi i}{n}} (x_3, x_4, x_1, x_2) \left(t + \frac{T}{2n} \right)$$

for an odd integer $n \geqq 3$. We only treat the case $n = 3$ here.

The group E is isomorphic to $D_6 \times \mathbb{Z}_2$, where D_6 is generated by σ, τ, and $\mathbb{Z}_2$ is generated by δ. Let $H = H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^4)$. Clearly paths in H^E have 3-fold rotation symmetry. Figure 4 shows a loop in H^E that is numerically a minimizer for $\mathcal{A}$ on H^E.

Theorem 5. *The action functional $\mathcal{A}$ attains its infimum on H^E. All minimizers are collision-free and nontrivial double choreographic solutions for the 4-body problem with 3-fold rotation symmetry.*

Proof. Without loss of generality we assume $G = T = 1$. By Proposition 2 the action functional $\mathcal{A}$ is coercive (choose $t_x = \frac{1}{2}$ and $\nu = 2$ in (9)), and therefore minimizers on H^E exist. Clearly $\mathcal{A}$ is E-invariant and E is orthogonal. Therefore the requirements in Palais' principle are satisfied. Unless there are collisions, minimizers of $\mathcal{A}$ on H^E are double choreographic solutions since $\{m_1, m_4\}$ share one orbit and $\{m_2, m_3\}$ share another.

A fundamental domain of the group action is $[0, \frac{1}{12}]$. At $t = 0$, the configuration is collinear and masses are aligned on the imaginary axis. Moreover,

$$(\bar{x}_2, \bar{x}_1, \bar{x}_4, \bar{x}_3)\left(\frac{1}{12}\right) = x\left(-\frac{1}{12}\right) = e^{\frac{\pi i}{3}}(x_3, x_4, x_1, x_2)\left(\frac{1}{12}\right). \tag{27}$$

This restricts the configuration for $x \in H^E$ at $t = \frac{1}{12}$ or any $\frac{k}{6} + \frac{1}{12}$, $k \in \mathbb{Z}$. Relative equilibria can be easily excluded by the boundary constraints on any fundamental domain. According to Marchal's theorem, all we need to show is that minimizers cannot begin or end with a collision on the interval $[0, \frac{1}{12}]$. $\square$

Let $x \in H^E$ be a path that begins or ends with a collision on $[0, \frac{1}{12}]$. There are four possibilities:

Case 1: x begins or ends with a total collapse.

The size of the set of collision indexes I_x is 6. Using the fact that paths in H^E have 3-fold rotation symmetry, by Theorem 1,

$$\mathcal{A}(x) \geqq 3\left(\frac{5\pi^2}{48}\right)^{\frac{1}{3}} 3^{\frac{2}{3}}\binom{4}{2} \approx 37.7888.$$

Note that this case includes the following cases:

$$\begin{aligned} x_1(0) = x_2(0) \ \text{ and } \ x_3(0) = x_4(0);\\ x_1\left(\frac{1}{12}\right) &= x_2\left(\frac{1}{12}\right);\\ x_3\left(\frac{1}{12}\right) &= x_4\left(\frac{1}{12}\right). \end{aligned}$$

The case $x_1(\frac{1}{12}) = x_2(\frac{1}{12})$ implies $x_3(\frac{1}{12}) = x_4(\frac{1}{12})$ and vice versa because of (27). They all results in a total collapse since $x_1 = -x_2$, $x_3 = -x_4$.

Case 2: $x_1(0) = x_3(0)$ (or $x_2(0) = x_4(0)$).

Clearly the case $x_1(0) = x_3(0)$ implies $x_2(0) = x_4(0)$ and vice versa. The size of the set of collision indexes I_x is at least 2. By symmetry, $x_1(\frac{k}{6}) = x_3(\frac{k}{6})$ and $x_2(\frac{k}{6}) = x_4(\frac{k}{6})$ for any $k \in \mathbb{Z}$. The path x actually has 6-fold symmetry instead of just 3. By Theorem 1,

$$\mathcal{A}(x) \geqq 3\left(\frac{5\pi^2}{6}\right)^{\frac{1}{3}}\left(6^{\frac{2}{3}} + 2\right) \approx 32.1066.$$

Case 3: $x_1(\frac{1}{12}) = x_3(\frac{1}{12})$ (or $x_2(\frac{1}{12}) = x_4(\frac{1}{12})$).

Clearly the case $x_1(\frac{1}{12}) = x_3(\frac{1}{12})$ implies $x_2(\frac{1}{12}) = x_4(\frac{1}{12})$ and vice versa. The estimates for this case is identical to Case 2.

Case 4: $x_3(0) = x_4(0)$ (or $x_1(0) = x_2(0)$).

The cases $x_3(0) = x_4(0)$ and $x_1(0) = x_2(0)$ are similar. Assume $x_3(0) = x_4(0)$. Then $x_3(\frac{k}{3}) = x_4(\frac{k}{3})$ and $x_1(\frac{k}{3} + \frac{1}{6}) = x_2(\frac{k}{3} + \frac{1}{6})$ for any $k \in \mathbb{Z}$. Instead of applying Theorem 1, we will follow the lines of the proof for Proposition 3 and obtain a better estimate by using symmetry.

Consider the standard decomposition with $\lambda_{ij} = 1$ for any $i \neq j$. As in the proof of Proposition 3,

$$\mathcal{A}^0_{34}(x) = 3\int_0^{\frac{1}{3}} \frac{1}{12}|\dot{x}_3 - \dot{x}_4|^2 + \frac{1}{|x_3 - x_4|}\,dt \geqq 3\left(\frac{\pi^2}{12}\right)^{\frac{1}{3}} 3^{\frac{2}{3}}.$$

Similarly,

$$\mathcal{A}^0_{12}(x) \geqq 3\left(\frac{\pi^2}{12}\right)^{\frac{1}{3}} 3^{\frac{2}{3}}.$$

By the invariance under the action of τ,

$$(x_1 - x_3)\left(\frac{k}{6}\right) = (-1)^k e^{-\frac{k\pi i}{3}}(x_1 - x_3)(0) \quad \text{for any } k \in \mathbb{Z}.$$

When $k = 3$, $(x_1 - x_3)(\frac{1}{2}) = (x_1 - x_3)(0)$. This shows that (x_1, x_3) can be considered a closed loop in $\Sigma_{\frac{1}{2}}$ and

$$\mathcal{A}^0_{13}(x) = 2\int_0^{\frac{1}{2}} \frac{1}{12}|\dot{x}_1 - \dot{x}_3|^2 + \frac{1}{|x_1 - x_3|}\,dt \geqq 3\left(\frac{\pi^2}{12}\right)^{\frac{1}{3}} 2^{\frac{2}{3}}.$$

The same estimate holds for $\mathcal{A}^0_{24}(x)$ by symmetry. Similarly, $(x_1, x_4), (x_2, x_3) \in \Sigma_1$ and

$$\mathcal{A}^0_{14}(x),\ \mathcal{A}^0_{23}(x) \geqq 3\left(\frac{\pi^2}{12}\right)^{\frac{1}{3}}.$$

Combining all these,

$$\mathcal{A}(x) \geqq \sum_{\substack{(i,j)\\ i<j}} \mathcal{A}^0_{ij} \geqq 3\left(\frac{\pi^2}{12}\right)^{\frac{1}{3}}\left(2\cdot 3^{\frac{2}{3}} + 2\cdot 2^{\frac{2}{3}} + 2\right) \approx 26.2386.$$

To finish the proof it suffices to select a test path $x = x_{\text{test}}$ with action lower than the bound we obtain in Case 4. Let

$$\begin{aligned}
r(t) &= 0.257(1 + 0.413\cos(6\pi t) - 0.042\cos(12\pi t)),\\
\theta(t) &= \frac{-1}{3}(6\pi t - 1.43\sin(6\pi t) + 0.3\sin(12\pi t) - 0.07\sin(18\pi t)) + \frac{\pi}{2},\\
x_1(t) &= r(t)e^{i\theta(t)},\\
x_2(t) &= -r(t)e^{i\theta(t)},\\
x_3(t) &= r\left(t - \frac{1}{6}\right)e^{i(\theta(t-\frac{1}{6})-\frac{\pi}{3})},\\
x_4(t) &= -r\left(t - \frac{1}{6}\right)e^{i(\theta(t-\frac{1}{6})-\frac{\pi}{3})}.
\end{aligned}$$

It can easily be verified that $x = (x_1, x_2, x_3, x_4)$ belongs to H^E. The action of this test path is approximately 26.0476 (accurate to the fourth decimal place), which is smaller than the lower-bound estimate we obtained.

Remark 3. For the case $n = 5$, following the proof of Theorem 5 with a modification for Case 4, we obtain a lower bound $\approx$36.2246 for the action of collision paths. By choosing the test path

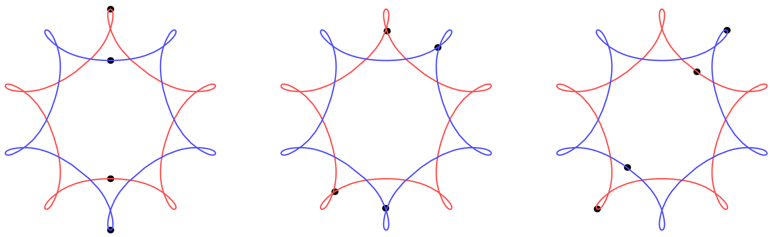

Fig. 5. A double choreographic loop with 5-fold rotation symmetry.

$$
\begin{aligned}
r(t) &= 0.248(1 + 0.282\cos(10\pi t)),\\
\theta(t) &= \frac{-1}{5}(10\pi t - 1.449\sin(10\pi t)) + \frac{\pi}{2},\\
x_1(t) &= r(t)e^{i\theta(t)},\\
x_2(t) &= -r(t)e^{i\theta(t)},\\
x_3(t) &= r\left(t - \frac{1}{10}\right)e^{i(\theta(t-\frac{1}{10})-\frac{\pi}{5})},\\
x_4(t) &= -r\left(t - \frac{1}{10}\right)e^{i(\theta(t-\frac{1}{10})-\frac{\pi}{5})},
\end{aligned}
$$

it can be shown that the value of its action is approximately 33.8819. This proves the theorem for $n = 5$. See [3] for general n and further examples. Figure 5 shows a path that is numerically a minimizer of the action functional.

Note added. Around the same time this article was completed, I received an article by FERRARIO & TERRACINI [10] which includes existence proofs for a number of choreographic solutions. Their results can be applied to examples in Section 7, but their proof is totally different from the one here. Except for Section 8, which discusses some examples that distinguish these two approaches, the present work is independent of [10].

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