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Uniqueness and Existence Results on the Hele-Shaw and the Stefan Problems

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Abstract

In this paper we introduce a notion of viscosity solutions for the one-phase Hele-Shaw and Stefan problems when there is no surface tension. We prove the uniqueness and existence of the solutions for both problems and the uniform convergence of solutions of porous medium equation to those of the Hele-Shaw problem.

0. Introduction

Let Ω_0 be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega_0$ having two disjoint parts Γ_0 and Γ_1, both of them being closed hypersurfaces in $\mathbb{R}^n$. For convenience, we let $\Gamma_1 = \{x \in \mathbb{R}^n : |x| = 1\}$ (See Fig. 1).

Let $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$, $f \in C(\Gamma_1)$ and consider a nonnegative function $u_0 \in C(\bar{\Omega})$ such that

$$\begin{aligned} u_0 = f > 0 \quad &\text{on } \Gamma_1 \\ \{u_0 > 0\} = \Omega_0. \quad & \end{aligned} \tag{0.1}$$

In this paper we study the one-phase Hele-Shaw and Stefan problems with initial data given as in (0.1). The classical Hele-Shaw problem, in $n = 2$, models an incompressible viscous fluid which occupies part of the space between two parallel, narrowly placed plates. In this case u_0 denotes the initial pressure of the fluid and f denotes the rate of injection from Γ_1 into Ω. (For convenience we assume that f is time-independent.) As more fluid is injected through a fixed boundary, the region occupied by the fluid will grow as time increases. Let us assume that the equilibrium temperature is zero. Assuming no surface tension, then the pressure of the fluid $u(x, t)$ solves the following free boundary problem:

$$\begin{aligned} -\Delta u = 0 \qquad & \text{in } \{u > 0\}, \\ V = u_t/|Du| = -Du \cdot \hat{n} = |Du| \quad & \text{on } \partial\{u = 0\}, \end{aligned} \tag{0.2}$$

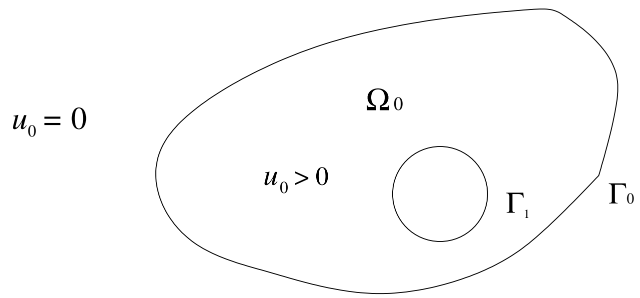

Fig. 1.

where V is the normal velocity and $\hat{n}$ is the outward unit normal vector of the free boundary $\partial\{u = 0\}$.

The classical Stefan problem accounts for phase transitions between solid and fluid states, such as the melting of ice in contact with water, or the freezing of water in contact with ice.

Here we assume that the temperature varies in only one phase, which gives us the following one-phase Stefan problem for the temperature u:

$$\begin{aligned} u_t - \Delta u &= 0 \qquad \text{in } \{u > 0\}, \\ V = u_t/|Du| = -(Du)\cdot\hat{n} &= |Du| \quad \text{on } \partial\{u = 0\}, \end{aligned} \tag{0.3}$$

where V and $\hat{n}$ are defined the same as in (0.2).

In both models initially smooth free boundaries may develop singularities in finite time, and therefore classical solutions are not expected to exist globally in time. This fact motivates the study of the solutions in a generalized sense, i.e., the weak solutions. For the Hele-Shaw problem (0.2), the short-time existence of classical solutions when the initial interface Γ_0 is $C^{2+\alpha}$ was proved by ESCHER & SIMONETT [ES]. When $n = 2$, ELLIOT & JANOVSKY [EJ] showed the existence and uniqueness of weak solutions in $H^1(Q)$. An extensive amount of work has been done on the Stefan problem (0.3). When the initial data and the interface are $C^{2+\alpha}$, HANZAWA [H] showed the short-time existence of classical solutions. KAMENOMOSTSKAJA [K] introduced the notion of weak solutions of this problem in H^1, and proved its global existence and uniqueness. Her work was generalized by OLEINIK [O] and FRIEDMAN [F]. The formulation of the problem as a parabolic variational inequality was initiated by DUVAUT [D]. This method was developed by, for example, FRIEDMAN & KINDERLEHRER [FK] who used variational inequality to prove the existence and uniqueness of weak solution in $L^\infty(0, T; H^{2,1})$.

In this paper we apply a notion of viscosity solutions to describe the global-time behavior of the free boundary problems (0.2) and (0.3) past singularities. The notion of viscosity solutions, introduced by [CL], has been used very successfully to study nonlinear elliptic and parabolic equations. The analytical heart of the theory lies in a comparison principle derived from maximal-principle-type arguments, which in turn leads to uniqueness and existence results.

The Hele-Shaw problem can be also derived as a limiting case of the porous medium equation (in short PME). Let the domain $Q = \Omega \times (0, \infty)$, where Ω is given as above. We consider a sequence of viscosity solutions $\{u_m\}_m \geqq 0 \in C(\bar{Q})$, where u_m satisfies

$$(\mathrm{PME})_m \qquad \begin{array}{rl} u_t - |Du|^2 - (m-1)u\Delta u = 0 & \text{in } Q, \\ u = f & \text{on } \Gamma_1, \\ u(x,0) = u_0(x) & \text{in } \bar{Q} \cap \{t = 0\}. \end{array}$$

Here $u_m = \frac{m}{m-1} v^{m-1}$, where $v \geqq 0$ satisfies the usual form of porous medium equation:

$$v_t = \Delta(v)^m \quad \text{in } Q.$$

CAFFARELLI & VAZQUEZ [CV] proved that there is a unique viscosity solution for the porous medium equation for $m > 1$ when the initial data is given. Here viscosity solutions are defined by comparison with the classical solutions of PME, in particular with the Barenblatt solutions (see Section 1). As $m \to \infty$, by a formal computation we can easily see that the limiting equation leads to the Hele-Shaw problem (0.2). However, the Barenblatt solutions become strictly positive as $m \to \infty$ and thus they cannot be used as test functions for the free boundary problem (0.2).

In Section 1 we define the viscosity solution of (0.2). Variational inequalities are used to describe the free boundary condition and to make the viscosity solutions stable through limit operations in various settings (see, for example, the proof of Theorem 1.5). The difference between our definition and that for PME in [CV] comes from (i) the presence of an additional equation on the free boundary (ii) lack of well-known classical solutions. We also show that the limit of viscosity sub(super)solutions of $(\mathrm{PME})_m$ as $m \to \infty$ is the viscosity sub(super)solution of (0.2) in our definition.

In Section 2 we show a comparison result between two viscosity solutions of (0.2) using the sup- and inf-convolutions. Here we require the solutions to be initially *strictly separated.* The choice of test functions plays an important role in the proof. For general cases, a more careful analysis is required to prove the comparison principle, which we explain in Section 3.

In Section 3 a uniqueness result for the viscosity solution of (0.2), and the uniqueness of the global time free boundary is proved when the initial free boundary $\partial\{u_0 > 0\}$ expands immediately. This condition holds, for example, if u_0 satisfies

$$\begin{array}{rl} -\Delta u_0 = 0 & \text{in } \Omega_0, \\ |Du_0| > 0 & \text{on } \Gamma_0. \end{array} \tag{0.4}$$

Also we show that in this case the sequence $\{u_m\}$ with initial data u_0 uniformly converge to u as $m \to \infty$, which is the unique viscosity solution of (0.2).

In Section 4 we turn to the Stefan problem (0.3) and state the corresponding uniqueness and existence theorems. Here the main difficulty lies in dealing with the scaling properties of (0.3) to produce the uniqueness result. We point out the essential differences between (0.2) and (0.3) in proving each result.

1. The Hele-Shaw problem

Here we define a notion of viscosity sub- and supersolutions for (0.2) by using variational inequalities. Recall that $Q = \{|x| > 1\} \times (0, \infty)$ and $\Gamma_1 = \{|x| = 1\}$. $u \in C(\bar{Q})$ is a viscosity solution of (0.2) if it is both viscosity sub- and supersolution.

Definition 1.1. (1) A *nonnegative uppersemicontinuous function* u defined in $\bar{Q}$ is a viscosity subsolution of (0.2) with initial data u_0 and fixed boundary data f if

(i) $u = u_0$ at $t = 0$, $u \leqq f$ for $x \in \Gamma_1$;
(ii) $\overline{\{u > 0\}} \cap \{t = 0\} = \overline{\{u(x,0) > 0\}}$;
(iii) for each $T \geqq 0$ the set $\overline{\{u > 0\}} \cap \{t \leqq T\}$ is bounded; and
(iv) for every $\phi \in C^{2,1}(Q)$ that has a local maximum of $u - \phi$ in $\overline{\{u > 0\}} \cap \{t \leqq t_0\} \cap Q$ at (x_0, t_0),

(a) $-\Delta\phi(x_0, t_0) \leqq 0$ if $u(x_0, t_0) > 0$.
(b) $\min(-\Delta\phi, \phi_t - |D\phi|^2)(x_0, t_0) \leqq 0$ if $(x_0, t_0) \in \partial\{u > 0\}$, $u(x_0, t_0) = 0$.

(2) *A nonnegative lowersemicontinuous function* v defined in $\bar{Q}$ is a viscosity supersolution of (0.2) with initial data u_0 and fixed boundary data f if

(i) $v = v_0$ at $t = 0$, $v \geqq f$ for $x \in \Gamma_1$ and
(ii) if for every $\phi \in C^{2,1}(Q)$ that has a local minimum of $v - \phi$ in $\overline{\{v > 0\}} \cap \{t \leqq t_0\} \cap Q$ at (x_0, t_0),

(a) $-\Delta\phi(x_0, t_0) \geqq 0$ if $(x_0, t_0) \in \{v > 0\}$,
(b) If $(x_0, t_0) \in \partial\{v > 0\}$ and if

$$|D\phi|(x_0, t_0) \neq 0 \text{ and } \{\phi > 0\} \cap \{v > 0\} \cap B(x_0, t_0) \neq \emptyset \tag{1.1}$$

for any ball $B(x_0, t_0)$, then

$$\max(-\Delta\phi, \phi_t - |D\phi|^2)(x_0, t_0) \geqq 0.$$

Remark. The conditions (ii) and (iii) in (1) control the behavior of the free boundary $\partial\{u > 0\}$ respectively at $t = 0$ and at infinity.

The condition (1.1) is to ensure that near (x_0, t_0) the function $\phi_+ = \max(\phi, 0)$ (x_0, t_0) is nontrivial in $\{v > 0\}$. For example, ϕ satisfies the condition if there is a vector $\nu \in \mathbb{R}^n$ such that

$$\begin{aligned} (x_0 + h\nu, t_0) \in \{v > 0\} &\quad \text{for } 0 < h \ll 1, \\ \partial\phi/\partial\nu > 0 &\quad \text{at } (x_0, t_0). \end{aligned}$$

For a real-valued function $f(x, t)$ in domain D, we define

$$f^*(x,t) = \limsup_{(y,s)\in D \to (x,t)} f(y,s) \quad \text{and} \quad f_*(x,t) = \liminf_{(y,s)\in D \to (x,t)} f(y,s).$$

Next we define a viscosity solution of (0.2):

Definition 1.2. A lowersemicontinuous function u is a *viscosity solution* of (0.2) if u^* is a viscosity subsolution and if $u = u_*$ is a viscosity supersolution of (0.2).

Remark. We mention that by definition of viscosity sub- and supersolutions it follows that $u^*(x, 0) = u_*(x, 0) = u_0(x)$. Thus $u(x, t)$ given above converges uniformly to $u_0(x, 0)$ as $t \to 0^+$.

Note that we did not define u to be continuous. This is because we expect u to be discontinuous when the free boundary of u develops a singularity. An easy example is when $n = 1$, where Ω_0 is given as two disjoint interval with fixed boundary inside each interval (see Fig. 2).

For later use, we introduce the sup- and inf-convolutions of u and v. Given a viscosity subsolution u and a constant $r > 0$, we define

$$\bar{u}_r(x, t) = \sup_{B_r(x,t)} u(y, \tau),$$

where $B_r(x, t) = \{(y, \tau) : |y - x|^2 + (t - \tau)^2 \leqq r^2\}$. Similarly, given a viscosity supersolution v and $\delta \ll r$ we define

$$\underline{v}_r(x, t) = \inf_{B_{r-\delta t}(x,t)} v(y, \tau).$$

This kind of construction has been used, for example, in [ACS] and in [CV]. We point out that for the analysis of free boundary speed it is necessary to involve the entire space-time balls including future times $\tau > t$ in the above definitions. Let D be a bounded domain in Q. We say that D has the *interior*(*exterior*)*-ball property* at $P = (x, T) \in \partial D$ if there is a closed $(n+1)$-dimensional (space-time) ball $B \subset \bar{D}$ ($\overline{D^c}$) such that $B \cap \overline{D^c}(\bar{D}) = P$. For a ball B with radius r, we denote kB as a ball with the same center as B and radius kr. Similary we define *interior*(*exterior*) *ellipsoid property*.

Lemma 1.3. *In the domain* $\{x : |x| > 1 + r\} \times (r, r/\delta)$, $\bar{u}_r$ *is a viscosity subsolution and* $\underline{v}_r$ *is a viscosity supersolution of* (0.2) *with corresponding initial and fixed boundary data. Moreover, on the free boundary the positivity set of* $\bar{u}_r$ ($\underline{v}_r$) *has the interior-ball and exterior-ellipsoid properties. At points of the free boundaries of* u, v *where these balls are centered we have the complementary results.*

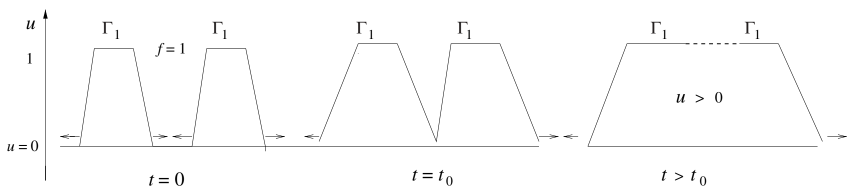

Fig. 2.

Proof.

Step 1. The exterior(interior)-ball properties are clear from the definition (see Figs. 3 and 4).

Step 2. To show that $\bar{u}_r$ is a viscosity subsolution, first observe that conditions (1)(ii) and (1)(iii) in Definition 1.1 follow from the corresponding properties of u as a viscosity subsolution. Thus we only have to show that (1)(iv) in Definition 1.1 holds for $\bar{u}_r$.

Step 3. We consider a smooth test function $\phi \in C^{2,1}(Q)$. Suppose that $\bar{u}_r - \phi$ has a local maximum at (x_0, t_0) in $\overline{\{\bar{u}_r > 0\}}$. By definition and upper semicontinuity of u,

$$\bar{u}_r(x_0, t_0) = \sup_{B_r(x_0,t_0)} u(x, t) = u(x_1, t_1) \quad \text{for some } (x_1, t_1) \in B_r(x_0, t_0).$$

But then $u(x, t) - \phi(x - x_1 + x_0, t - t_1 + t_0)$ has a local maximum at (x_1, t_1) in $\overline{\{u > 0\}}$. This leads to our conclusion.

Step 4. Suppose that $\underline{v}_r - \phi$ has a local minimum zero at (x_0, t_0) in $\overline{\{\underline{v}_r > 0\}}$. From the definition and the lower semicontinuity of v,

$$\underline{v}_r(x_0, t_0) = \inf_{B_{r-\delta t_0}(x_0,t_0)} v(x, t) = v(x_2, t_2) \quad \text{for some } (x_2, t_2) \in B_{r-\delta t_0}(x_0, t_0).$$

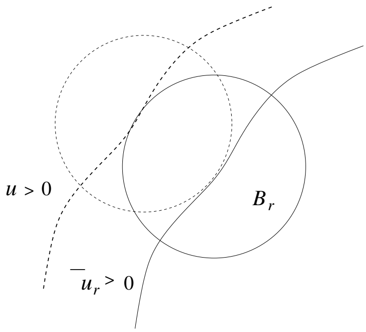

Fig. 3.

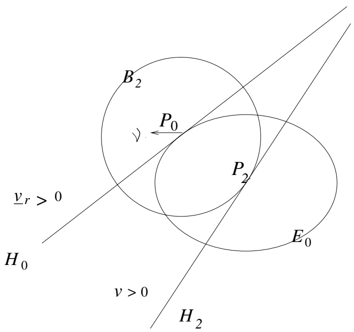

Fig. 4.

If $(x_0, t_0) \in \{\underline{v}_r > 0\}$, then $v(x,t) - \phi(x - x_2 + x_0, t - t_2 + t_0)$ has a local minimum at (x_2, t_2), and thus we get

$$-\Delta\phi(x_0, t_0) \geqq 0.$$

If $(x_0, t_0) \in \partial\{\underline{v}_r > 0\}$ and ϕ satisfies (1.1) with respect to $\underline{v}_r$ at (x_0, t_0), then $(x_2, t_2) \in \partial\{v > 0\}$ and the function $\phi(x - x_2, t - t_2 + t_0)$ satisfies (1.1) with respect to v at (x_2, t_2). Thus $v(x,t) - \varphi(x - x_2 + x_0, t - t_2 + t_0)$ has its local minimum at (x_2, t_2), and this leads to the desired inequality

$$\max(-\Delta\phi, \quad \phi_t - |D\phi|^2)(x_0, t_0) \geqq 0. \qquad \square$$

By definition, for $P_0 = (x_0, t_0) \in \partial\{\underline{v}_r > 0\}$ there is a point $P_2 = (x_2, t_2) \in \partial\{v > 0\}$ such that at P_0 the set $\{\underline{v}_r > 0\}$ has an exterior space-time ellipsoid E_0

$$(x - x_2)^2 + (t - t_2)^2 \leqq (r - \delta t)^2$$

and at P_2 the set $\{v > 0\}$ has an interior space-time ball B_2 of radius $r - \delta t_0$ centered at P_0 (see Fig. 4.).

Let us denote H_0 as the tangent hyperplane of E_0 at P_0 and (ν, m) as the inward normal vector to H_0 with respect to E_0 at P_0 with $|\nu| = 1$. We denote by m the *advancing speed* of the free boundary $\underline{v}_r$ at P_0. Observe that the tangent hyperplane H_2 of B_2 at P_2 has outward normal vector $(\nu', m - \delta)$, $|\nu'| = 1$ with respect to B_2. (In other words, the free boundary of v_r propagates faster than that of v by δ). In the following lemma we show that in fact the advancing speed m is strictly positive.

Lemma 1.4. *The free boundary of $\underline{v}_r$ has positive advancing speed, that is, $m \geqq \delta$.*

Proof. If $m < \delta$, then at $P_2 = (x_2, t_2)$ v has an interior ball B_2 with negative advancing speed. Note that by the lowersemicontinuity $v(P_2) = 0$. Moreover, since $v > 0$ in B_2, v has a positive lower bound in $\frac{1}{4}\bar{B}_2$.

Now for $\tau \ll 1$, we consider $h(x,t)$ on $B_2 \cap [t_2 - \tau, t_2]$ such that

$$\begin{aligned} -\Delta h > 0 &\quad \text{outside } \tfrac{1}{4}B_2, \\ 0 < h > v &\quad \text{inside } \tfrac{1}{4}\bar{B}_2, \\ \{h > 0\} = B_2, &\quad |Dh| \neq 0 \text{ on } \partial B_2. \end{aligned}$$

(Refer to Appendix A for the construction of h.)

Note that $h < 0$ outside B_2, and thus $v - h > 0$ on $\partial\{v > 0\} \cap \{t \leqq t_0\}$ except at P_2. Also by the maximum principle of harmonic functions, $v - h > 0$ inside B_2. Therefore $v - h$ has its local minimum zero at P_2 in $\overline{\{v > 0\}} \cap \{t \leqq t_0\}$, but this contradicts the fact that at P_2

$$-\Delta h < 0, \quad h_t - |Dh|^2 \leqq -|Dh|^2 < 0. \qquad \square$$

Next we prove that the limit of viscosity solutions of $(\text{PME})_m$ as $m \to \infty$, if it exists, is a viscosity solution of (0.2). We recall the definition of viscosity solutions for $(\text{PME})_m$ in [CV].

A nonnegative function $u \in C(\bar{Q})$ is a *classical moving free boundary solution* of $(\text{PME})_m$ if

(i) $u \in C^{2,1}(\overline{\{u > 0\}})$ solves the equation in the classical sense in $\{u > 0\}$,
(ii) the free boundary of u, $\Gamma = \partial\{u = 0\} \cap Q$, is a $C^{2,1}$ hypersurface in space-time, and
(iii) on Γ we have $u_t = |Du|^2$ and $|Du| \neq 0$.

Definition 1.5. (1) A nonnegative continuous function u defined in $\bar{Q}$ is a *viscosity subsolution* of (0.2) if $u \leqq f$ on Γ_1 and for every $\phi \in C^{2,1}(Q)$ that has a local maximum zero of $u - \phi$ at (x_0, t_0),

$$(\phi_t - m\phi\Delta\phi - |D\phi|^2)(x_0, t_0) \leqq 0.$$

(2) A nonnegative continuous function v defined in $\bar{Q}$ is a *viscosity supersolution* of (0.2) if $v \geqq f$ on Γ_1 and

(a) for every $\phi \in C^{2,1}(Q)$ that has a local minimum zero of $v - \phi$ at $(x_0, t_0) \in \{v > 0\}$,

$$(\phi_t - m\phi\Delta\phi - |D\phi|^2)(x_0, t_0) \geqq 0.$$

(b) Any classical moving free-boundary solution that lies below v at a time $t = t_1 \geqq 0$ cannot cross v at a later time $t_2 > t_1$.

The comparison principle and the uniqueness result for viscosity solutions of $(\text{PME})_m$ are proved in [CV]. Next we consider

$$u_1(x, t) = \limsup_{(y,s)\to(x,t)} u_m(y, s),$$
$$u_2(x, t) = \liminf_{(y,s)\to(x,t)} u_m(y, s),$$

where $(y, s) \in Q$.

Theorem 1.6. *The functions u_1, u_2 are respectively a viscosity sub- and supersolution of* (0.2).

To prove Theorem 1.6 the following lemma plays an important role.

Lemma 1.7. *Let v be a viscosity supersolution* (*subsolution*) *of* $(\text{PME})_m$. *Suppose that ϕ is a smooth function and $v - \phi$ has a local minimum (maximum) in $\overline{\{v > 0\}}$ at $(x_0, t_0) \in \partial\{v > 0\}$. If ϕ satisfies* (1.1) *at (x_0, t_0), then*

$$(\phi_t - |D\phi|^2)(x_0, t_0) \geqq 0 (\leqq 0).$$

Proof. We only prove the supersolution part. The subsolution part can be shown with a parallel argument, by comparison with a supersolution of the form $A(|x| - ct - 1)_+$, where $c > A > 0$.

Step 1. For $r, \delta > 0$, we prove the lemma for

$$W(x, t) = \inf_{B_{r-\delta t}(x,t)} v(y, \tau).$$

Then the lemma follows by first taking $\delta \to 0$ and then $r \to 0$. Suppose that $W - \phi$ has a local minimum in $\overline{\{W > 0\}}$ at $P_0 = (x_0, t_0) \in \partial\{W > 0\}$ with ϕ satisfying (1.1). Suppose that $(\phi_t - |D\phi|^2)(x_0, t_0) < 0$.

Step 2. By adding $\varepsilon(t-t_0)-\varepsilon(x-x_0)^2$ to ϕ if necessary, we assume that $W-\phi$ has a strict local minimum zero at P_0. By condition (1.1), ϕ_+ is nontrivial in $\{W > 0\}$ and it has a smooth free boundary near P_0.

Let H be the tangent hyperplane of the free boundary of ϕ_+ at P_0. Let (ν, α) be the inward normal vector of H with respect to $\{\phi_+ > 0\}$ with $|\nu| = 1$. Then α is the advancing speed of the free boundary of ϕ. Note that $\alpha \geqq \delta > 0$ by Lemma 1.4.

Step 3. Observe that from a previous assumption we have

$$\alpha = -\phi_t/\phi_\nu(x_0, t_0) < \phi_\nu(x_0, t_0),$$

and therefore near (x_0, t_0) we have the nontangential estimate

$$W(x, t_0) \geqq \phi_+(x, t_0) > \alpha[(x - x_0) \cdot \nu]_+ + O(|x - x_0|^2). \tag{1.2}$$

Step 4. Next we introduce the Barenblatt solutions given by the formula

$$S(x, t; \tau, C) = (t + \tau)^{-\lambda nm} \left(C - \kappa \frac{x^2}{(t + \tau)^{2\lambda}} \right)_+,$$

where $\lambda = (mn + 2)^{-1}$, $\kappa = \lambda/2$ and C and τ are arbitrary. They are classical solutions of $(\mathrm{PME})_m$. Observe that C controls the size of the support of S, and τ controls the advancing speed of the free boundary of S.

At $t = t_0$, by the regularity of the free boundary at P_0 the set $\{x : \phi(x, t_0) > 0\}$ has a space interior ball B with radius $0 < r_1 < d_1$. Note that $H \cap \{t = t_0\}$ is tangent to B at P_0. We may replace the origin so that $(0, t_0)$ is the center of B. Choose C, τ such that supp $S(x, t_0) = B$ and the free boundary of $S(x, t)$ at P_0 has the advancing speed α. Let $\tilde{H}$ be the hyperplane with a normal vector $(\nu, \alpha(1 - \delta))$ such that $\tilde{H} \cap \{t = t_0\} = H \cap \{t = t_0\}$.

Since the support of S is tangent to $\tilde{H} \cap \{t = t_0\}$ at P_0 and advancing faster than $\tilde{H}$ near P_0, we have $0 < \theta < \pi/2$ such that for $k = \sin\theta$ the support of $\tilde{S}(x, t) = kS(k^{-1}x, k^{-1}t)$ crosses $\tilde{H}$ at $\tilde{P} = (\tilde{x}, \tau), \tau > t_0$. Note that $\tau \to t_0$ as $\theta \to \pi/2$. Finally observe that $\mathrm{supp}\tilde{S}(x, t_0) = (\sin\theta)B \subset K$, where K is a nontangential space cone with vertex P_0, axis e_1 and aperture θ. Note that θ does not depend on the size of r_1.

Step 5. Since $\tilde{S}(\tilde{P}) = 0$ and $\tilde{S}$ satisfies $(\mathrm{PME})_m$, we have

$$\tilde{S}_t/|D\tilde{S}|(\tilde{P}) = |D\tilde{S}|(\tilde{P}) = \alpha.$$

Due to (1.2) we can put S below W at $t = t_0$ when r is small enough. By definition of W it follows that a translate of $\tilde{S}$ crosses the free boundary of v near the point P_2, which leads to a contradiction. □

Proof of Theorem 1.6.
Step 1. First we show the supersolution part. Suppose that there is a smooth function ϕ such that $u_2 - \phi$ has a local minimum zero at (x_0, t_0) in $\overline{\{u_2 > 0\}}$. Adding $\varepsilon(t - t_0) - \varepsilon(x - x_0)^2$ to ϕ if necessary, we assume that $u_2 - \phi$ has a strict local minimum zero at $(x_0, t_0) \cap B_r(x_0, t_0)$ for small $r > 0$. Then for large enough m, along a subsequence $u_m - \phi$ has its minimum at (x_m, t_m) in $\overline{\{u_m > 0\}} \cap B_r(x_0, t_0)$ with $(x_m, t_m) \to (x_0, t_0)$.

Step 2. If $(x_0, t_0) \in \{u_2 > 0\}$, then $(x_m, t_m) \in \{u_m > 0\}$ for large m. By definition

$$\left[\frac{1}{m-1}(\phi_t - |D\phi|^2) - \phi\Delta\phi\right](x_m, t_m) \geqq 0$$

and since $\phi(x_0, t_0) > 0$, in the limit we get

$$-\Delta\phi(x_0, t_0) \geqq 0$$

as desired.

Step 3. Suppose that $(x_0, t_0) \in \partial\{u_2 > 0\}$ and (1.1) holds for ϕ. If we have $\max(-\Delta\phi, \phi_t - |D\phi|^2)(x_0, t_0) < 0$, then for large enough m

$$[\phi_t - (m-1)\phi\Delta\phi - |D\phi|^2](x_m, t_m) < 0.$$

Thus $(x_m, t_m) \in \partial\{u_m > 0\}$. Moreover ϕ satisfies (1.1) at (x_m, t_m) for large m since ϕ is smooth in its support. This and the above inequality contradict Lemma 1.6.

Step 4. To show the subsolution part, we first observe that for each $T > 0$ and for large $m = m(T) > 0$, the family of sets $(\{u_m > 0\} \cap \{t \leqq T\})_m$ are uniformly bounded. This can be easily shown by comparing u_m with a barrier function φ and using the finite-propagation property of $\{u_m > 0\}$. For example, we let $\varphi(x, t) = h(r(t)|x|)$, where $h(r) = M^2 - r^2$ and M large enough that $u_0 \prec \varphi$ at $t = 0$. Now if we choose $r(t)$ to satisfy $r(0) = 1$ and $r = e^{-8Mt}$, then on $\partial\{\varphi(x, t) > 0\} = \{|x| = M/r(t)\}$,

$$\varphi_t = -r'(t)M\varphi_r > 2(\varphi_r)^2.$$

Thus φ is a supersoluton of $(\mathrm{PME})_m$ for large $m = m(T)$, and $\overline{\{u_m > 0\}} \subset \{\varphi > 0\}$ for $m > m(T)$ and $t \leqq T$. Similarly, we can also show that $\overline{\{u_1 > 0\}} \cap \{t = 0\} = \overline{\{u_0 > 0\}}$. Suppose not, and there is a point x_0 such that $u_0 = 0$ in $D_{2r}(x_0)$ and $u_1(x_0, t) > 0$ for $t > 0$. This leads to a contradiction by comparing u_m with $\varphi(x, t) = M(r(t)|x - x_0|^2 - r^2)$, where $M = M(r)$ is large enough that $u_0 \prec \varphi$ and $r(t) = 1/(1 - Mt)$.

The rest of the proof can be shown as in the supersolution part. □

2. Comparison principle for *separated* initial data

Definition 2.1. A pair of functions u_0, v_0 are (*strictly*) *separated* if

(i) the support of u_0, $supp(u_0)$, is a compact subset of $\mathbb{R}^n$ and

$$\text{supp}(u_0(x)) \subset \text{Int}(\text{supp}(v_0(x))).$$

(ii) inside the support of u_0 the functions are strictly ordered:

$$u_0(x) < v_0(x).$$

We represent such a strict ordering, or separation, by the symbol $u_0 \prec v_0$.

Theorem 2.2. *Let* u, v *be respectively viscosity sub- and supersolutions of* (0.2) *in* Q. *If their initial data are strictly separated* ($u_0 \prec v_0$,) *then the solutions remain separated for all time:*

$$u(x,t) \prec v(x,t) \quad \textit{for } t \geqq 0.$$

To prove the theorem, we use sup- and inf-convolutions as in Section 1. For technical reasons concerning semicontinuous functions, we apply inf- and sup-convolution twice, first in the space balls $D_r(x,t) = \{(y,t) : |x-y| \leqq r^2\}$ and then in the space-time balls $B_r(x,t) = \{(y,s) : |x-y|^2 + |t-s|^2 \leqq r^2\}$. More precisely, in the domain $Q_r = \{x : |x| > 1+2r\} \times [r, r/\delta)$ let us define functions Z and W as given below:

$$\begin{aligned} Z(x,t) &= \sup\nolimits_{B_r(x,t)} U(y,s) \quad &&\text{where } U(x,t) = \sup\nolimits_{D_r(x,t)} u(y,t),\\ W(x,t) &= \inf\nolimits_{B_{r-\delta t}(x,t)} V(y,s) \quad &&\text{where } V(x,t) = \inf\nolimits_{D_r(x,t)} v(y,t). \end{aligned}$$

Note that Z, U and W, V are respectively viscosity sub- and supersolutions of (0.2).

Suppose that u crosses v from below at some point. Then we have $0 < T < \infty$ such that

$$T = \sup\{t : u(x,\tau) \prec v(x,\tau), 0 < \tau < t\}.$$

Since $\overline{\{u > 0\}} \cap \{t = 0\} = \overline{\{u_0 > 0\}}$, $v > u$ on ∂Q and $u - v$ is upper semicontinuous, we can take r, δ small enough that $r < T < r/\delta$ and $W > Z$ on $\partial Q_r \times [r, T]$. Now consider the contact time

$$0 < t_0 = \sup\{t : Z(x,t) \prec W(x,t)\} \leqq T.$$

Before proceeding to further arguments, we need the following two observations at $t = t_0$.

Lemma 2.3. *For any* $T > 0$,

$$\partial\{Z > 0; t \geqq T\} \cap \{t = T\} \subset \overline{\{Z > 0; t < T\}} \cap \{t = T\}.$$

Proof. Suppose that the lemma does not hold. Then there is a point $(x_1, T) \in \partial\{Z > 0\}$ and $h > 0$ such that (x_1, t) belongs to the interior of $\{Z = 0\}$ for $T - h < t < T$. If $Z(x_1, T) > 0$, then we can choose $(x_1', T) \in \partial\{Z(\cdot, T) > 0\}$ which belongs to the interior of $\{Z = 0\}$ for $T - h < t < T$. (Note that this is possible since $\{Z(\cdot, T) > 0\}$ is bounded.) By definition of Z there is $(x_2, t_2) \in \partial\{u > 0\}$ such that

$(*)$ there is a cylinder $C = D_r(x_2) \times [t_2 - h, t_2]$ such that $u = 0$ in C for $t < t_2$.

Moreover, at (x_2, t_2) the set $\overline{\{u > 0\}}$ has a exterior cylinder $C' = D_r(x_3) \times [t_2 - h, t_2]$. Now at $t = t_2$ we can construct a strict superharmonic function $\varphi(x) = \varphi(|x - x_3|)$ in $2D_r(x_3) - D_r(x_3)$ with the boundary data

$$\varphi = \sup_{2D_r} u \quad \text{on } \partial(2D_r) \quad \text{and } \varphi = 0 \quad \text{on } \partial D_r.$$

Due to $(*)$, we can extend $\varphi(x, t)$ for $t_2 - \tau \leqq t \leqq t_2$ with

$$\varphi(x, t) = \varphi_+(k(t - t_2)(|x - x_3| - 2r) + |x - x_3|),$$

where $k = (2\tau)^{-1}$ and $\varphi \geqq 0 = u$ for $t < t_2$. If we choose τ sufficiently small, then

$$\varphi_t > kr = \sup_{2D_r} |D\varphi|^2(x).$$

This choice of $\varphi(x, t)$ leads to a contradiction since $u - \varphi_+$ has a local maximum in the set

$$(2D_r - D_r) \cap \overline{\{u > 0\}} \times (t_2 - \tau, t_2] \quad \text{at } t = t_2.$$

□

By Lemma 2.3 and Lemma 1.4, $\overline{\{Z(\cdot, t_0) > 0\}} \subseteq \overline{\{W(\cdot, t_0) > 0\}}$. Consider $P_0 = (x_0, t_0)$, where the nonnegative maximum of $Z - W$ is attained in $\overline{\{Z > 0\}} \cap \{t \leqq t_0\}$. If $Z(P_0) \geqq W(P_0) > 0$, at $t = t_0$ the function $Z - W$ has a maximum in $\{Z > 0\} \cap \{W > 0\}$. Then using the definition of Z, W we get a contradiction by the maximum principle of harmonic functions. Hence $W(P_0) = 0$ and $P_0 \in \partial\{Z > 0\} \cap \partial\{W > 0\}$.

Lemma 2.4. The function $Z(P_0) = 0$.

Proof.
Step 1. Suppose that the lemma does not hold. Then $Z(x_0, t_0) > 0$. By definition there is $(x_1, t_1) \in \partial\{u > 0\}$ with $u(x_1, t_1) = Z(x_0, t_0) > 0$. Moreover, by definition of Z and by Lemma 1.4 there is a cylinder $C = D_r(x_2) \times [t_1 - h, t_1]$ such that $C \cap \overline{\{u > 0\}} = (x_1, t_1)$.

Step 2. Let $\varphi(x, t) = \varphi(x)$ be a smooth and strictly superharmonic function $2C - C$ such that $\varphi = 0$ on $\partial_x C$, $\varphi > 0$ outside C and $\varphi > u$ on the parabolic boundary of $2C$. Since $u - \varphi$ is positive at (x_1, t_1), $u - \varphi$ has a positive maximum at (x_3, t_3) in the set $2C - C$ and this contradicts the definition of u. □

Now we come back to the analysis of W and Z at the contact time $t = T$. By Lemma 2.3 and 2.4, P_0 is the contact point of the free boundaries of W and Z and $Z \leqq W$ at $t = t_0$. (From now on we do not need the convolution in the space ball and therefore for simplicity we denote U by u and V by v.) By Lemma 1.3, at P_0 the set $\{Z > 0\}$ has an interior space-time ball of radius r, centered at $P_1 \in \partial\{u > 0\}$. Also at P_1, the set $\{u > 0\}$ has an exterior space-time ball B_1 of radius r centered at P_0 (see Fig. 5.)

By choosing appropriate origin and coordinates, we may assume that $P_0 = (0, t_0)$ and the space projection of $\overline{P_0 P_1} = d_1 e_1$, where $e_1 = (1, 0, \ldots, 0)$. Let us write $x = (x_1, x')$, $x' \in \mathbb{R}^{n-1}$. Finally, if H is the tangent hyperplane to the interior ball of Z at P_0, let us write (e_1, m) as the internal normal vector to H with respect to $\{Z > 0\}$ at P_0 with $m = \tan\alpha$ for some $0 < \alpha < \pi/2$. Then m is the advancing speed of $\{Z > 0\}$ and $P_1 = (x_1, t_1) = (r\cos\alpha, 0, t_0 + r\sin\alpha)$, $0 \in \mathbb{R}^{n-1}$.

Moreover by Lemma 1.3 at P_0 the set$\{W > 0\}$ has a exterior space-time ball B centered at $P_2 \in \partial\{v > 0\}$, and at P_2 the set $\{v > 0\}$ has an interior space-time ball B_2 centered at P_0. Note that the space projection of $\overline{P_2 P_0} = d_2 e_1$, $d_2 > 0$.

Lemma 2.5. *The tangent hyperplane H is neither vertical nor horizontal.*

Proof.
Step 1. By Lemma 1.4, m is bigger than δ and therefore H is not vertical.

Step 2. Suppose H is horizontal. Then $\{u > 0\}$ has an exterior ball B_1 at P_1 with horizontal tangency. Recall that by Lemma 2.3 $x_1 \in \partial\{u(\cdot, t_1) > 0\}$ and by Lemma 2.4 $u(x_1, t_1) = 0$.

After parabolic scaling $(x, t) \to (\lambda(x - x_1), \lambda^2(t - t_1) + 1)$, for any $\delta > 0$ we can build up a new subsolution ω in the unit cylinder $C_1 = B_1 \times [0, 1]$ with $P_1 = (0, 1) \in \partial\{\omega(\cdot, 1) > 0\}$, $0 \in \mathbb{R}^n$, which takes on the value 0 on the bottom, and less than δ on the lateral boundary. Consider

$$\phi(x, t) = g(|x| + \tfrac{1}{4}t),$$

where $g(r) : \mathbb{R}^+ \to \mathbb{R}$ is such that

$$g(1) > \delta, \quad g < 0 \text{ if } 0 < r < \tfrac{1}{2}, \quad \text{and } -g'' - \tfrac{n-1}{r}g' > 0 \text{ for } r > 0.$$

(For example, let $g = 2\delta(2 - r^{-2n})$.)

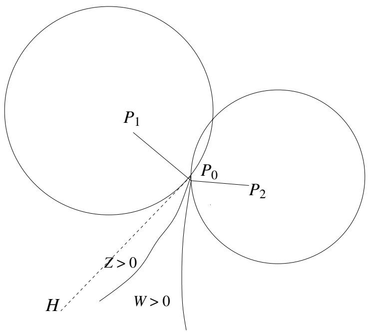

Fig. 5.

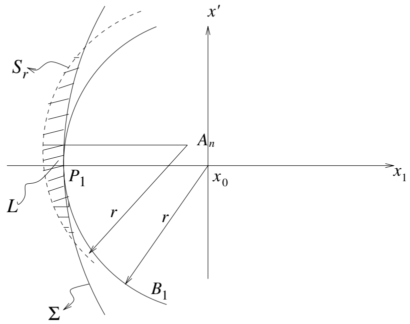

Fig. 6.

Consider $\phi_+ = \max(\phi, 0)$. Since $P_1 \in \partial\{\omega(\cdot, 1) > 0\}$ and $\{\phi(\cdot, 1) > 0\} \subset \{|x| > 1/4\}$, we have $\sup(\omega - \phi_+) > \omega(P_1) = 0$ in C_1. Observe that $\omega \leqq \phi_+$ on the lateral boundary and bottom of C_1. Therefore $\omega - \phi$ has a local maximum zero in $\overline{\{\omega > 0\}} \cap \{t \leqq T\}$ at $(\bar{x}, T)$.

Since $-\Delta\phi = -g'' - \frac{n-1}{r} g > 0$, $(\bar{x}, T) \in \partial\{\omega > 0\} \cap \partial\{\phi > 0\}$. By taking small δ, we can make g' small enough so that

$$\phi_t - |D\phi|^2 = \tfrac{1}{4} g' - (g')^2 > 0 \quad \text{on } \partial\{\phi > 0\},$$

which contradicts the definition of ω. □

The following lemma is essential for proving Theorem 2.2. The main idea of the proof is drawn from [CV], but we state the full proof to present the role of test functions. A nontangential cone at P_0 is a space cone with vertex P_0, axis e_1 and aperture $0 < \theta < \pi/2$. Recall that we set $P_0 = (x_0, t_0) = (0, t_0)$.

Lemma 2.6. *In any nontangential cone K,*

$$\liminf_{x \to 0, x \in K} \frac{Z(x, t_0)}{m(x_1)_+} \geqq 1. \tag{2.1}$$

Proof.
Step 1. It will be convenient to displace the t axis so that $t_1 = 0$, $P_1 = (x_1, t_1) = (r\cos\alpha e_1, 0)$ and $P_0 = (x_0, t_0) = (0, -r\sin\alpha)$. Suppose that (2.1) is not true. Then there is a sequence of points $A_n = (x_{1n}, x'_n)$ converging to $0 \in \mathbb{R}^n$ and lying in a nontangential cone K such that

$$Z(Q_n) \leqq m(1-\varepsilon)(x_{1n})^+ \quad \text{for some } \varepsilon > 0 \tag{2.2}$$

with $Q_n = (A_n, t_0)$. By definition of Z we have

$$u \leqq m(1-\varepsilon)(x_{1n})^+ \quad \text{in} \quad B_r(Q_n).$$

Besides, $u = 0$ in B_1. Moreover, since $v = 0$ at P_2, at time t_0 the function $W = 0$ in the space ball B' of radius $0 < d < r - \delta t_0$ centered at Q' and tangent to H at $(0, t_0)$. Since $Z \leqq W$ at $t = t_0$ so does Z. By definition of Z we conclude

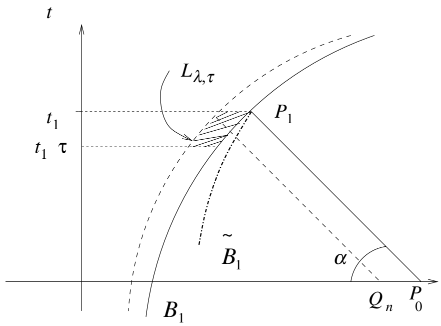

Fig. 7.

that $u = 0$ in the set Σ : union of the space-time balls of radius r centered at the points of B'.

Step 2. As a consequence of both estimates above and taking $x_{1n} = \lambda > 0$ small in (2.2) we conclude that there is a set $L = L_\lambda$ in space-time as portrayed in Fig. 7 where $u \leqq \mu_\lambda = m(1 - \varepsilon)\lambda$.

Moreover, the boundary of L_λ contains a concave part closer to the origin, which is a piece of the boundary part of Σ containing the point P_1, and there we have $u = 0$. The farther boundary part of L_λ is formed by a piece of the sphere $S_r(Q_\lambda)$, boundary of $B_r(Q_\lambda)$ where $u \leqq \mu_\lambda$. From a straightforward computation it turns out that L_λ is of depth λ and width $O(\sqrt{\lambda})$ in space. Observe that Σ contains B_1 with $\partial\Sigma \cap \bar{B}_1 = \{P_1\}$.

Step 3. Consider a smooth function ϕ such that

$$\begin{array}{ll} \phi(x,t) \quad \begin{cases} > 0 \\ = 0 \end{cases} & \begin{array}{l} \text{outside } B_1, \\ \text{on } \partial B_1, \end{array} \\ -\Delta\phi > 0 & \text{outside } B_1, \\ \phi(x,t) = \phi(r,t) & \text{where } r = |x|, \\ \phi_r = m(1 - \varepsilon/3) & \text{at } P_1. \end{array} \tag{2.3}$$

Since B_1 has the outward normal vector (e_1, m) at P_1, we have

$$\phi_t/|D\phi|(P_1) = m > |D\phi|(P_1) = \phi_r(P_1) = m(1 - \varepsilon/3).$$

(See Appendix A for the construction of ϕ.)

We compare ϕ with u in $L_{\lambda,\tau} = L \cap \{-\tau \leqq t \leqq 0\}$. First we compare them on the boundary of $L_{\lambda,\tau}$. Since Σ contains B_1, $\phi \geqq 0$ on $\partial\Sigma$, and thus ϕ is above u on Σ.

Next we compare them on the other part of the boundary, $S_r(Q_\lambda) \cap \{-\tau \leqq t \leqq 0\}$.

Claim 1. *For $\lambda, \tau > 0$ (let $\varepsilon\tau = \lambda$) small enough compared to ε,*

$$\phi \geqq m(1 - \varepsilon)\lambda \quad on \quad S_r \cap \{-\tau \leqq t \leqq 0\}. \tag{2.4}$$

Proof of the claim. By definition, S_r consists of points (x, t) such that

$$|x_1 - \lambda|^2 + |x' - k\lambda|^2 + (t + r\sin\alpha)^2 = r^2$$

with k : bounded independently of λ. Then for $d_1 = r\cos\alpha$

$$|x|^2 = x_1^2 + (x')^2 = d_1^2 + 2x_1\lambda + 2kx'\lambda - 2tr\sin\alpha - \lambda^2(1+k^2)t^2.$$

Note that $r\sin\alpha = d_1 m$. In the first approximation we have $x_1 = d_1 + \lambda$ and $x' = O(\sqrt{\lambda})$ from the previous argument. Therefore up to terms of higher order in λ and $-\tau \leqq t \leqq 0$ we have

$$|x| = d_1 + \lambda - mt + o(\lambda).$$

Moreover,

$$\begin{aligned}
\phi((d_1+\lambda-mt)e_1, t) &= \phi((d_1+\lambda)e_1, 0) + t(-m\phi_r + \phi_t)((d_1+\lambda)e_1, 0) + O(t^2)\\
&= \phi((d_1+\lambda)e_1, 0) + t(-m\phi_r + \phi_t)(d_1 e_1, 0) + O(t\lambda) + O(t^2)\\
&= \lambda\phi_r(d_1 e_1, 0) + O(\lambda^2, \tau\lambda, \tau^2)\\
&= m(1-\varepsilon/3)\lambda + o(\lambda)\\
&> m(1-\varepsilon)\lambda \qquad \text{for small enough } \lambda.
\end{aligned}$$

□

Hence at each time $t \in [-\tau, 0]$, $\phi \geqq u$ on $\partial L_{\lambda,\tau}$ for small λ and therefore

$$\phi \geqq u \quad \text{in} \quad L_{\lambda,\tau}.$$

But then $u - \phi$ has its maximum zero at P_1 in $L_{\lambda,\tau}$. Since $L_{\lambda,\tau}$ is a neighborhood of P_1 in $\overline{\{u > 0\}} \cap \{0 \leqq t \leqq t_1\}$, by definition of u we need

$$\min(-\Delta\phi, \phi_t - |D\phi|^2)(P_1) \leqq 0,$$

which is not true. □

Proof of Theorem 2.2.
Step 1. At $P_2 = (x_2, t_2)$, for any $0 < r_2 < d_2$ the set $\{x : v(x,t) > 0\}$ has an interior space-time ball B_2 with radius r_2 (see Fig. 8). Note that by definition B_2 has the outward normal vector $(e_1, m - \delta)$ at P_2, i.e., B_2 has the *advancing speed* $(m - \delta)$ at P_2. Since $Z \leqq W$ at $t = t_0$, by Lemma 2.4 and the definition of W, for any $\varepsilon_0 > 0$ we can choose r_2 such that $0 < r_2 < d_2$ and

$$m(1-\varepsilon_0)d(x, \partial B_2 \cap \{t = s\}) < v(x, s) \quad \text{for } (x, s) \in B_2. \tag{2.5}$$

Step 2. Take $0 < \varepsilon_0 < \delta/2$ and consider φ such that

$$\begin{aligned}
&\phi(x,t) \quad \begin{cases} > 0 & \text{outside } B_2,\\ = 0 & \text{on } \partial B_2,\end{cases}\\
&-\Delta\varphi < 0 \qquad\qquad \text{outside } \tfrac{1}{4}B_2,\\
&\varphi(x,t) = \varphi(r,t),\\
&\varphi_r = m(1-\varepsilon_0) \qquad \text{on } \partial[B_2 \cap \{t = t_2\}].
\end{aligned} \tag{2.6}$$

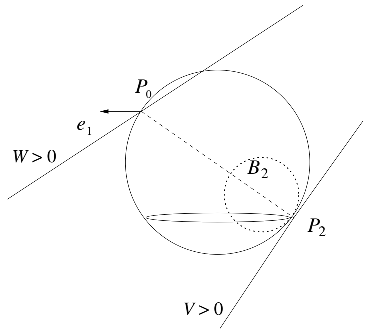

Fig. 8.

(See Appendix A for the construction of φ.) Here $r = |x - C|$, C being the space projection of the center of B_2. Since B_2 has the advancing speed $m - \delta$ at $t = t_2$, we can choose τ so small that

$$-\varphi_t/\varphi_r < m(1 - \delta/2) < -\varphi_r \quad \text{on} \quad \partial B_2 \cap \{t_2 - \tau \leqq t \leqq t_2\}. \tag{2.7}$$

Step 3. By (2.5) and the last condition of (2.6), for small τ there is $1/4 < r < 1$ such that

$$\varphi(x, t) < v(x, t) \quad \text{on} \quad \partial(rB_2) \cap \{t_2 - \tau \leqq t \leqq t_2\}. \tag{2.8}$$

Since $\varphi = 0 \leqq v$ on B_2, by the maximal principle of harmonic functions $v - \varphi \geqq 0$ on $B_2 - rB_2$ for $t_2 - \tau \leqq t \leqq t_2$. Thus the function $v(x, t) - \varphi(x, t)$ has a local minimum zero at P_2 in the closure of Σ_2, where

$$\Sigma_2 = (B_2 - rB_2) \cap \{t_2 - \tau \leqq t \leqq t_2\}.$$

However, by (2.6) and (2.7) we have

$$\max(-\Delta\varphi, \varphi_t - |D\varphi|^2)(P_2) < 0,$$

which leads to a contradiction. □

3. Uniqueness and existence results

Due to Theorem 2.2 and the scaling properties of (0.2), we now complete the comparison principle as below. For a real-valued function $f(x, t)$ we define

$$\bar{f}(x, t) = \lim_{\varepsilon \to 0^+} \sup_{t \leqq s < t + \varepsilon} f(x, s); \quad \underline{f}(x, t) = \lim_{\varepsilon \to 0^+} \inf_{t - \varepsilon < s \leqq t} f(x, s).$$

Theorem 3.1. *Let u, v be respectively viscosity sub- and supersolutions of* (0.2) *in Q with initial data $u_0(x)$, $v_0(x)$. Suppose that $\{v_0 > 0\}$ is bounded, $v_0 \in C^1(\overline{\{v_0 > 0\}})$ and*

$$|Dv_0| \neq 0 \quad on \quad \partial\{v_0 = 0\}. \tag{3.1}$$

If $u_0 \leqq v_0$, then

$$u(x, t) \leqq \bar{v}(x, t); \quad \underline{u}(x, t) \leqq v(x, t) \quad for \quad t \geqq 0.$$

Proof.
Step 1. For $\varepsilon > 0$, define

$$v_\varepsilon(x,t) = (1+\varepsilon)v(x,(1+\varepsilon)t+\varepsilon).$$

Then v_ε is also a viscosity supersolution of (0.2) with the initial data

$$v_\varepsilon(x,0) = (1+\varepsilon)v(x,\varepsilon).$$

Step 2. From (3.1) it follows that the initial free boundary of v expands with strictly positive speed $|Dv_0|$. Moreover v is harmonic in $\{v > 0\}$, and thus $v_0 \leqq v(\cdot,\varepsilon)$ for any $\varepsilon > 0$. Hence for any $\varepsilon > 0$ we have $u_0 \prec v_\varepsilon(\cdot,0)$, and from Theorem 2.2

$$u(x,t) \leqq v_\varepsilon(x,t) \quad \text{for} \quad t \geqq 0.$$

Now we conclude by sending $\varepsilon \to 0$. □

Remark. If v_0 is a superharmonic function in $\{v_0 > 0\}$, by standard barrier arguments (for example, see Chapter 2 of [GT]) it turns out that v_0 satisfies the condition (3.1) if $\{v_0 > 0\}$ satisfies the *interior-ball property* on its free boundary (in other words, if $\{v > 0\}$ has interior-ball property at every point on its free boundary).

Corollary 3.2. *Let u, v be viscosity solutions of* (0.2) *with initial data v_0 given as in Theorem 3.1. Then $u = v$ and $u^* = \bar{u}$.*

Due to Theorem 3.1 we can prove the following result. Recall that in our initial setting the boundary of Ω_0 has two components Γ_1 and Γ_0, where $u_0 = f > 0$ on Γ_1 and $u_0 = 0$ on Γ_0.

Theorem 3.3. *Let $u_m(x,t)$ be the viscosity solution of* (PME)$_m$ *with $u_m(x,0) \equiv u_0(x)$, where $u_0(x)$ has a compact support with*

$$\begin{aligned} -\Delta u_0 &= 0 \quad \text{in } \Omega_0, \\ |Du_0| &> 0 \quad \text{on } \Gamma_0. \end{aligned} \tag{0.4}$$

Consider u_1, u_2 defined as in Theorem 1.6. Then u_2 is the unique viscosity solution of (0.2) *with the initial data u_0. Moreover, $\overline{u_2} = u_1$.*

Proof.
Step 1. Let u_1, u_2 be defined as in Theorem 1.5, where $u_m(x,0) \equiv u_0(x)$. According to Theorem 1.5, u_1 (u_2) is a viscosity subsolution (supersolution) of (0.2). We claim that $u_1 = u_2 = f$ on ∂Q and $u_1 = u_2 = u_0$ at $t = 0$. The following lemma is easily deduced from the definition of u_1, u_2 and the stability property of the viscosity solutions (see [CIL]).

Lemma 3.4. *The functions u_1, u_2 satisfies the following inequalities in the viscosity sense:*

$$\begin{aligned} \min(u_1 - u_0, -\Delta u_1)(x,0) &\leqq 0, \\ \min(u_1 - f, -\Delta u_1) &\leqq 0 \quad \text{on } x \in \Gamma_1, \\ \max(u_2 - u_0, -\Delta u_2)(x,0) &\geqq 0 \quad \text{if } u_2(x,0) > 0, \\ \max(u_2 - u_0, -\Delta u_2, (u_2)_t - |Du_2|^2)(x,0) &\geqq 0 \quad \text{if } u_2(x,0) = 0, \\ \max(u_2 - f, -\Delta u_2) &\geqq 0 \quad \text{on } x \in \Gamma_1. \end{aligned}$$

Remark. In the lemma the partial differential equations on u_1, u_2 are used in the viscosity sense. For instance, the first inequality means that, if $u_1 > u_0$ at $x = x_0$, then for smooth ϕ that has a local maximum of $u_1(x,0) - \phi(x)$ at $x = x_0$ in $\overline{\{u_1 > 0\}}$ we have $-\Delta\phi(x_0) \leqq 0$.

Step 2. We apply Lemma 3.3 to show that $u_1 = u_2 = f$ on $x \in \Gamma_1$ and $u_1 = u_2 = u_0$ at $t = 0$.

Suppose that for some $x_0 \in \Gamma_1$ we have $u_1(x_0, t_0) > f(x_0, t_0)$. Consider a smooth function ϕ: $C^2(\{1 < |x| < 2\})$ solving the following Dirichlet problem:

$$\begin{aligned} \phi &= f^\varepsilon \geqq f && \text{on} \quad \Gamma_1 = \{|x| = 1\}, \\ \phi &= M = \max\nolimits_{|x|=1} f && \text{on} \quad \{|x| = 2\}, \text{ and} \\ -\Delta\phi &= 1 && \text{in} \quad \{1 < |x| < 2\}. \end{aligned}$$

where f^ε is C^3 in the neighborhood of Γ_1, and $f^\varepsilon \to f$ locally uniformly as $\varepsilon \to 0$. In fact ϕ is C^2 up to the boundary Γ_1 (see [CC]).

Since $u_1 \leqq M$, by the maximal principle of harmonic functions $u_1(x,t) - \phi(x)$ has its positive maximum in $\{1 \leqq |x| < 2\} \times [0, t_0]$ for small ε. But this contradicts Lemma 3.3. We showed that $u_1 \leqq f$ on $x \in \Gamma_1$. Since $u_m = f$ on $x \in \Gamma_1$ for all m, we get $u_1 \geqq f$, and thus we conclude that $u_1 = f$ on ∂Q. Similarly we can also prove that $u_2 = f$ on ∂Q.

Step 3. Next suppose that $u_1(x,0) - u_0(x)$ has a positive maximum at $x = x_0$. Since the initial free boundary of $(u_m)_m$ moves with the normal velocity $|Du_0|$, it is easy to see that $\overline{\{u_1(x,0) > 0\}} = \overline{\{u_0 > 0\}}$. If $x_0 \in \text{int}\{u_0 = 0\}$, then it contradicts the fact that u_1 is subharmonic in a neighborhood of x_0. Thus $x_0 \in \overline{\{u_0 > 0\}}$. Let $\mu = Du_0(x_0) \in \mathbb{R}^N$. Then there is a sequence $(x_\varepsilon, y_\varepsilon) \to (x_0, x_0)$ where the function

$$u_1(x,0) - u_0(y) - \varepsilon^{-4}|x - y + \varepsilon\mu|^2$$

has a positive maximum in $\overline{\{u_0 > 0\}} \times \overline{\{u_0 > 0\}}$.

By continuity of u_0, for small ε, $u_1(x_\varepsilon, 0) > u_0(x_\varepsilon) \geqq 0$, and thus

$$-\Delta u_1(x_\varepsilon, 0) \leqq 0.$$

Moreover, since $|x_\varepsilon - y_\varepsilon + \varepsilon\mu| = o(\varepsilon^2)$, we have $u_0(y_\varepsilon) = u_0(x_\varepsilon) + \varepsilon\mu^2 + o(\varepsilon) > 0$, and thus

$$-\Delta u_0(y_\varepsilon) = 0 \quad \text{for small} \quad \varepsilon > 0.$$

Now a standard viscosity-solutions argument (refer to Lemma 3.2 in [CIL]) leads to a contradiction as ε goes to zero. We proved that $u_1 \leqq u_0$, and therefore $u_1 = u_0$.

Step 4. Finally suppose that $u_2(x,0) - u_0(x)$ has a negative minimum at $x = x_0$. Note that $u_0(x_0) > 0$ in this case, since $u_2 \geqq 0$. If $-\Delta u_2(x_0) \geqq 0$, then we proceed as above to make a contradiction. If not, then $u_2(x_0, 0) = 0$ and there is a smooth function $\phi(x,t)$ such that $u_2 - \phi$ has a minimum at $(x_0, 0)$ and $-\Delta\phi < 0$, $\phi_t - |D\phi|^2 \geqq 0$ at $(x_0, 0)$. This leads to a contradiction since then for any $\lambda > 0$ $u_2 + \lambda t - \phi$ has a minimum at $(x_0, 0)$ and we have

$$(\phi_t - |D\phi|^2)(x_0, 0) \geqq \lambda.$$

Step 5. Thus $u_2 = u_0$. Now we can apply Theorem 3.1 to show that $u_1 \leqq \overline{u_2}$ and $\underline{u_1} \leqq u_2$, which means that

$$u_1 = \overline{u_2} \quad \text{and} \quad u_2 = \underline{u_1}. \qquad \square$$

Theorem 3.5. *Let Γ_0 be a closed C^1 hypersurface in Ω. Then for given continuous boundary data $f(x) > 0$ on Γ_1, there exists a unique viscosity solution of* (0.2) *with its initial free boundary Γ_0 and its initial data u_0 satisfying* (0.4). *In fact for any viscosity solution u of* (0.2), *its initial data u_0 is a viscosity solution of*

$$-\Delta u_0 = 0 \qquad \textit{in } \{u_0 > 0\}.$$

Proof.
Step 1. We prove the second assertion first. Suppose we have a viscosity solution of (0.2) with initial data u_0 and take a positive time sequence $t_n \to 0$ as $n \to \infty$. Then by definition of u we have

$$u_0 \leqq \liminf_{n\to\infty} u(x, t_n) \leqq \limsup_{n\to\infty} \bar{u}(x, t_n) \leqq u_0,$$

and thus the sequence $u_n(x) = u(x, t_n)$ and $\bar{u}_n(x) = \bar{u}(x, t_n)$ uniformly converges to $u_0(x)$. Since $-\Delta u_n(x) \geqq 0$ and $-\Delta \bar{u}_n(x) \leqq 0$ in $\{u_n > 0\}$ for each n, we can conclude by the stability property of viscosity solutions.

Step 2. Recall that Ω_0 is a subset of Ω such that $\partial\Omega_0 = \Gamma_0 \cup \Gamma_1$. Since Γ_0, Γ_1 is C^1, we can solve the Dirichlet problem

$$\begin{aligned} -\Delta u_0 &= 0 && \text{in } \Omega_0, \\ u_0 &= f && \text{in } \Gamma_1, \\ u_0 &= 0 && \text{on } \Gamma_0. \end{aligned}$$

Due to the regularity of Γ_0, the set $\{u_0 > 0\}$ has the interior-ball condition on its free boundary and thus condition (0.4) holds. Now we apply Theorem 3.2 to obtain the unique viscosity solution of (0.2). $\square$

For a closed C^1 hypersurface Γ_0 in $\{x : |x| > 1\}$ and a positive continuous function f on Γ_1, let $\Omega_t(\Gamma_0, f) = \text{int}\{x : u(x,t) > 0\}$, where $u(x,t) = u(\Gamma_0, f; x, t)$ is the unique viscosity solution of (0.2) in Theorem 3.4 . From previous arguments we have the following results for Ω_t:

Corollary 3.6.
(a) *If $\Omega_0(\Gamma_0, f) \subset \Omega_0(\tilde{\Gamma}_0, f)$, then*

$$\Omega_t(\Gamma_0, f) \subset \Omega_t(\tilde{\Gamma}_0, f) \quad \textit{for } t \geqq 0.$$

(b) *If $f(x) > g(x) > 0$, then*

$$\Omega_t(\Gamma_0, g) \subsetneq \Omega_t(\Gamma_0, f) \quad \textit{for } t > 0.$$

Proof.
Step 1. For (a), observe that $u_0(\Gamma_0, f) \leqq u_0(\tilde{\Gamma}_0, f)$ by the maximum principle of harmonic functions. Now (a) follows from Theorem 2.1.

Step 2. For (b), there is an $\varepsilon > 0$ such that $f > (1+\varepsilon)g$ on Γ_1. For this ε define

$$u_\varepsilon(\Gamma_0, g; x, t) = (1+\varepsilon)u(\Gamma_0, g; x, (1+\varepsilon)t). \tag{3.2}$$

Since $|Du(x,0)| > 0$ on Γ_0 (see the remark below Theorem 3.1), the initial free boundary expands immediately and thus we have

$$\Omega(\Gamma_0, g; x, t) \subsetneq \Omega(\Gamma_0, g; x, (1+\varepsilon)t) \quad \text{for every } t \geqq 0.$$

On the other hand, note that $u_\varepsilon(\Gamma_0, g; x, t) = u(\Gamma_0, (1+\varepsilon)g; x, t)$. From (3.2) and (a),

$$\Omega_t(\Gamma_0, g; x, (1+\varepsilon)t) = \Omega_t(\Gamma_0, (1+\varepsilon)g; x, t) \subset \Omega_t(\Gamma_0, f; x, t)$$

and hence we conclude the proof. □

4. The Stefan problem

In this section we study the Stefan problem (0.3) stated as below:

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } \{u > 0\}, \\ u_t - |Du|^2 &= 0 && \text{on } \partial\{u > 0\}, \\ u(x,0) &= u_0(x). \end{aligned} \tag{4.1}$$

Definition 4.1. (1) A nonnegative uppersemicontinuous function u in $\mathbb{R}^n \times [0,\infty)$ is a *viscosity subsolution* of (4.1) if (i) $u(x,0) = u_0$, (ii) $\overline{\{u > 0\}} \cap \{t = 0\} = \overline{\{u_0 > 0\}}$ and (iii) for $\phi \in C^{2,1}(Q)$ that has a local maximum of

$u - \phi$ in $\overline{\{u > 0\}} \cap \{t \leqq t_0\} \cap Q$ at (x_0, t_0),

(a) $\phi_t - \Delta\phi(x_0, t_0) \leqq 0$ if $u(x_0, t_0) > 0$,
(b) $\min(\phi_t - \Delta\phi, \phi_t - |D\phi|^2)(x_0, t_0) \leqq 0$ if $(x_0, t_0) \in \partial\{u > 0\}$ and $u(x_0, t_0) = 0$.

(2) A nonnegative lowersemicontinuous function v defined in $\bar{Q}$ is a *viscosity supersolution* of (4.1) if $v(x,0) = u_0$ and for $\phi \in C^{2,1}(Q)$ that has a local minimum of
$v - \phi$ in $\overline{\{v > 0\}} \cap \{t \leqq t_0\} \cap Q$ at (x_0, t_0),

(a) $\phi_t - \Delta\phi(x_0, t_0) \geqq 0$ if $(x_0, t_0) \in \{v > 0\}$,
(b) If $(x_0, t_0) \in \partial\{v > 0\}$ and if (1.1) holds, then

$$\max(\phi_t - \Delta\phi, \phi_t - |D\phi|^2)(x_0, t_0) \geqq 0.$$

(3) The function u is a viscosity solution of (4.1) if u^* is a viscosity supersolution and u_* is a viscosity supersolution of (4.1).

Theorem 4.2. *Let u, v be respectively viscosity sub- and supersolutions of* (4.1) *in* $Q \times (0, \infty)$ *with strictly separated initial data* $u_0 \prec v_0$ *in* Ω. *Then the solutions remain ordered for all time:*

$$u(x,t) \prec v(x,t) \quad \textit{for } t \geqq 0.$$

As before, in the domain $Q_r = \{x : |x| > 1 + 2r\} \times [r, r/\delta)$ let us define functions Z and W as

$$Z(x,t) = \sup_{B_r(x,t)} U(y,s) \quad \text{where } U(x,t) = \sup_{D_r(x,t)} u(y,t),$$
$$W(x,t) = \inf_{B_{r-\delta t}(x,t)} V(y,s) \quad \text{where } V(x,t) = \inf_{D_r(x,t)} v(y,t).$$

Note that Z, U and W, V are respectively viscosity sub- and supersolutions of (P). Since $u_0 \prec v_0$ and $u - v$ is uppersemicontinuous, $Z \prec W$ at $t = r$ if r is small.

Suppose that u crosses v from below at some point. Then we have $0 < T < \infty$ such that

$$T = \sup\{t : u(x,t) \prec v(x,t)\}.$$

Since $\overline{\{u > 0\}} \cap \{t = 0\} = \overline{\{u_0 > 0\}}$, $v > u$ on ∂Q and $u - v$ is upper semi-continuous, we can take r, δ small enough that $r < T < r/\delta$ and $W > Z$ on $\partial Q_r \cap [r, T]$. Now consider the contact time

$$0 < t_0 = \sup\{t : Z(x,t) \prec W(x,t)\} \leqq T.$$

Lemma 4.3. *For any* $T > 0$,

$$\partial\{Z > 0; t \geqq T\} \cap \{t = T\} \subseteq \overline{\{Z > 0; t < T\}} \cap \{t = T\}.$$

Proof. Suppose the lemma does not hold. Then there is a point $(x_1, T) \in \partial\{Z > 0\}$ such that (x_1, t) belongs to the interior of $\{Z = 0\}$ for $T - h \leqq t < T$. If $Z(x_1, T) = 0$, then we proceed as in Lemma 2.3 to conclude. If $Z(x_1, T) = \delta > 0$, then

(∗) there is a cylinder $C = D_r(x_1) \times [T - h, T]$ such that $Z = 0$ in C for $T - h \geqq t < T$.

Now for $T - h < t < T$ we can solve the heat equation for $\varphi(x,t)$ in $D_r(x_1)$ with the boundary data bigger than $Z(x_1, T)$ on ∂D_r and with $\varphi(x_1, T - h) < \delta/2$. Now if h is small enough then $\varphi(x_1, T) + \delta h < \delta$ and we have a contradiction. □

Due to Lemma 4.3, $\overline{\{Z(\cdot, t_0) > 0\}} \subseteq \overline{\{W(\cdot, t_0) > 0\}}$. Moreover from Lemma 4.3 and by a barrier argument we can easily show that the set $\{Z > 0\} \cap \{t \leqq t_0\}$ is bounded, and thus at $t = t_0$ the intersection $\partial\{Z(\cdot, t_0) > 0\} \cap \partial\{W(\cdot, t_0) > 0\}$ is nonempty. By the maximal principle of the heat equation and by a parallel result of Lemma 2.4, we can show that $Z \leqq W$ at $t = t_0$.

Now to apply barrier arguments as in Theorem 2.2, we only have to choose appropriate test functions, which are essentially (local) smooth sub- and supersolutions of (4.1). For the construction of such functions, refer to Appendix B.

Next we proceed to the comparison result for general initial data. For the Stefan problem, the scaling property of the Hele-Shaw problem (0.2) does not hold, and therefore more careful analysis is required to prove the following theorem. Let $\bar{v}$ be defined the same as in Theorem 3.1.

Theorem 4.4. *Let u, v be respectively viscosity sub- and supersolutions of* (4.1) *in Q with initial data $u_0(x)$, $v_0(x)$. Suppose that v_0 satisfies*

$$v_0 \in C^2(\{v_0 > 0\}) \text{ and } -\Delta v_0 > 0 \text{ on } \partial\{v_0 > 0\}. \tag{4.2}$$

If $u_0 \leqq v_0$, then

$$u \leqq \bar{v} \quad \text{for } t \geqq 0.$$

Remark. Observe that from Theorem 4.2 and from the standard viscosity theory (for example, see the proof of Theorem 4.6) the functions

$$U(x,t) = \sup\{\alpha(x,t) | \alpha : \text{ a viscosity subsolution of (4.1) with } \alpha(x,0) \leqq v_0(x)\}$$

and

$$V(x,t) = \inf\{\beta(x,t) | \beta : \text{ a viscosity supersolution of (4.1) with } v_0(x) \leqq \beta(x,0)$$

are respectively the maximal and minimal viscosity solutions with initial data $v_0(x)$ with $U = U^*$ and $V = V_*$. Therefore it is enough to prove the theorem when $u = U$ and $v = V$. In the following proposition we first prove that the free boundary $\partial\{v(\cdot,t) > 0\}$ strictly expands. Then we use the main idea of the proof of the proposition to prove Theorem 4.2.

The condition (4.2) is to guarantee the smooth behavior of v at $t = 0$. The formula (4.2) can be replaced by the short-time existence of a classical solution of (4.1) with initial data v_0. Note that the condition (4.2) implies (3.1) if the initial free boundary satisfies the interior-ball condition.

In the following proposition we first prove that the free boundary of a viscosity solution strictly expands. Then we use the main idea of the proof of the proposition to prove Theorem 4.2.

Proposition 4.5. *Let v be a viscosity solution of* (4.1) *with initial data v_0, where* (3.1) *and* (4.2) *hold for v_0. Then the free boundary of v strictly expands, i.e.,*

$$\{v^*(\cdot,t) > 0\} \subset\subset \{v_*(\cdot,t+s) > 0\} \quad \text{if } t \geqq 0,\ s > 0.$$

Proof.
Step 1. Suppose this is not the case. Then there is $T < \infty$ such that $T = \inf_s\{s \in M\}$, where

$$M = \{s : \partial\{v^*(\cdot,s) > 0\} \cap \partial\{v_*(\cdot,s+\varepsilon) > 0\} \neq 0 \text{ for } 0 < \varepsilon < \varepsilon_0 = \varepsilon(s)\}.$$

Note that by (3.1), $T > 0$.

For simplicity we first assume that $T \in M$ and $\varepsilon(T) = \varepsilon_0 > 0$. Observe that for $\delta > 0$ and for $\alpha = \alpha(\delta)$ satisfying

$$v(x,0) \prec (1+\alpha)v_*(x,\varepsilon) \quad \text{for } 0 < \varepsilon \leqq \delta,$$

we have $v^*(x,t) \prec (1+\alpha)v_*(x,t+\varepsilon)$ for $t < T$, $0 < \varepsilon \leqq \delta$ by the maximum principle of heat equation.

Step 2. Now consider the function

$$w(x,t) = (1+\alpha)v_*(x,(1+\alpha)t-\beta)$$

where $\alpha = \alpha(\varepsilon_0)$ and β is such that $(1+\alpha)T - \beta = T + \varepsilon_0$ (see Fig. 9.) Take τ so small that at $t = T-\tau$ we have $t < (1+\alpha)t-\beta$, and thus $v^*(x, T-\tau) \prec w(x, T-\tau)$. Observe that w satisfies

$$\begin{aligned} w_t - (1+\alpha)\Delta w &\geqq 0 \quad &&\text{in } \{w>0\}, \\ \max(w_t - (1+\alpha)\Delta w, w_t - |Dw|^2) &\geqq 0 \quad &&\text{on } \partial\{w>0\}. \end{aligned} \tag{4.3}$$

Take $r > 0$, $0 < \delta \ll r$ small enough that

$$W(x.t) = \inf_{B_{r-\delta t}(x,t)} w(y,s)$$

and

$$Z(x,t) = \sup_{B_r(x,t)} v^*(y,s)$$

satisfies $Z \prec W_1$ at $t = T-\tau$. By definition, Z crosses W from below at $T-\tau < t \leqq T$. If Z crosses W with $Z = W > 0$ at $t = t_0$, then for

$$W_2 = \inf_{B_{r-\delta t}} (1+\alpha)v_*(x, t+\gamma)$$

with γ such that $t_0 + \gamma = (1+\alpha)t_0 - \beta$ (see Fig. 6), $Z - W_2$ has a local maximum zero at $t = t_0$ in $\{Z_1 > 0\} \cap \{t \leqq t_0\}$. Observe that for $T \leqq t \leqq t_0$, the free boundary of Z does not cross W_2 since $t+\gamma > (1+\alpha)t - \beta$ for $t \leqq t_0$. But this contradicts the fact that $Z_1 - W_2$ is a subsolution of the heat equation in $\{Z_1 > 0\} \subseteq \{W_2 > 0\}$ for $T - \tau \leqq t \leqq t_0$.

Step 3. Thus the contact point is on the boundary: in other words, W and Z intersect at $P_0 = (x_0, t_0)$ on the free boundary with $Z \leqq W$ for $T-\tau \leqq t = t_0 \leqq T$. Thus we expect the free boundary of Z to advance faster than that of W at the contact point P_0. But roughly speaking, the free boundary of W advances with speed larger than $|DW| + \delta$, whereas that of Z advances with speed smaller than $|DZ|$. This

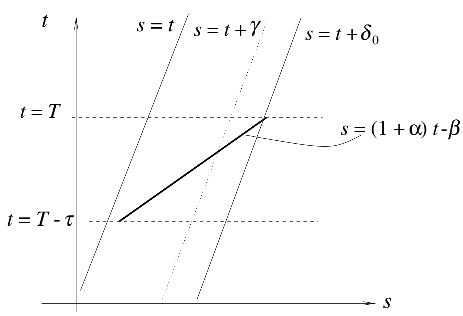

Fig. 9.

implies that $|DW| < |DZ|$, which contradicts the fact that $Z \leqq W$ for $t \leqq t_0$. Following the steps of the proof for Theorem 2.2, a contradiction occurs if there is a corresponding test function for (4.3) to compare with W. Such test functions can be built with a slight modification from the construction in Appendix B. Now we proceed as in the proof of Theorem 2.2 to conclude.

Step 4. To prove the general case, i.e., when $T \notin M$, we argue at time $t = T_\delta$ instead of $t = T$ and replace $t = T - \tau$ by $t = T$, where T_δ is given as

$$T_\delta = \inf_s \{s : \partial\{v^*(\cdot, s) > 0\} \cap \partial\{v_*(\cdot, s+\delta) > 0\} \neq 0\}.$$

(For details refer to the proof of Theorem 4.4 below.) To apply the above arguments we only need to show that if δ is small enough then there is $\alpha > 0$ such that

$$v_0 \prec (1+\alpha)v_*(x, \varepsilon) \quad \text{for } 0 < \varepsilon \leqq \delta, \text{ and } \alpha|T - T_\delta| < \delta. \tag{4.4}$$

This is always possible if $\Delta v_0 \geqq 0$, since then the interior regularity of v at $t = 0$ and straightforward computation implies that we can choose $\alpha = o(\delta)$. Thus we assume that Δv_0 is negative at some point and therefore at that point v is decreasing. Let $\bar{x} = \bar{x}(\delta)$ be the point where $v_*(x, \delta) - v(x, 0)$ has its negative minimum. Note that by (4.2) v strictly increases near the free boundary for $0 < t < \tau$ for small τ and thus $v_0(\bar{x}) > c_0$ for small δ, where c_0 is independent of δ. Now if we take δ so small that $|T - T_\delta| < 1/4M$, where $M = -\kappa/c_0 > 0$ and κ is the strictly negative minimum of Δv_0, then $\alpha = 2M\delta$ satisfies (4.4). □

Now we turn to the proof of Theorem 4.4.

Proof of Theorem 4.4.
Step 1. As mentioned above, it is enough to consider the case when $u = U$ and $v = V$. (In particular $v(x, 0) = u(x, 0) = v_0$ and $v \leqq u$.) For given δ, take $\alpha > 0$ so that

$$v_0(x) \prec (1+\alpha)v_*(x, \varepsilon) \quad \text{for } 0 < \varepsilon \leqq \delta.$$

Step 2. Let

$$v_\delta(x, t) = (1+\alpha)v(x, t+\delta).$$

Suppose that u crosses $\bar{v}$ from below at $t = T$. Then for small $\delta > 0$, u crosses v_δ at $P_\delta = (x_\delta, T_\delta)$, where $T < T_\delta$.

As shown above, we can choose δ small enough so that $\alpha(T_\delta - T) < \delta$. As in Lemma 4.3 we consider

$$w(x, t) = v_\delta(x, (1+\alpha)t - \alpha T_\delta).$$

Step 3. Then $u(x, T) \prec w(x, T)$ and w crosses u from above at $\bar{P} = (\bar{x}, \bar{T})$ with $T < \bar{T} \leqq T_\delta$. If $u = w > 0$ at $\bar{P}$, then

$$f(x, t) \equiv u - (1+\alpha)v(x, t+\gamma) = 0 \quad \text{at } \bar{P},$$

where $\gamma = \gamma_\delta$ is chosen to satisfy $f(x, t) = (u - w)(x, t)$ at $t = \bar{T}$. Note that f is a subsolution of the heat equation in the domain $\{u > 0\} \times (T, \bar{T}]$. From the choice of α and by the fact that $\gamma \leqq \delta$, $u \prec (1+\alpha)v(x, t+\gamma)$ at $t = T$. Also for $T \leqq t < \bar{T}$, we have $(1+\alpha)(t-T) + T + \delta < t + \gamma$ and thus

$$\{u(x,t) > 0\} \subsetneq \{w(x,t) > 0\} \subset \{v(x, t+\gamma) > 0\}.$$

This and the maximal principle of the heat equation lead to a contradiction.

Step 4. Hence $u = w = 0$ at $\bar{P}$, i.e., the contact point is on the free boundary. Now we proceed as in Proposition 4.5 to get a contradiction, observing that u and w have the same motion law on the free boundary. □

Corollary 4.6. *If u,v are lowersemicontinuous viscosity solutions of* (4.1) *with initial data* v_0*, then*

$$u = v \quad \text{and } \bar{u} = u^*.$$

Theorem 4.7. *Let* Γ_0 *be a closed* $C^{1,1}$ *hypersurface in* $\mathbb{R}^n$ *and let* Ω_0 *be the region bounded by* Γ_0*. Then there exists a unique lowersemicontinuous viscosity solution* v *of* (4.1) *for given initial data* $u_0 \geqq 0$ *with* $\{u_0 > 0\} = \Omega_0$ *and with the condition* (4.2).

Proof.
Step 1. We apply Perron's method to show the existence part. First consider Ψ: a solution of heat equation in $Q_0 = \Omega_0 \times (0, \infty)$ with initial data u_0 and zero on the lateral boundary. Then $\Psi(x,t)$ is a supersolution of (4.1) since $\Psi_t - |D\psi|^2 = -|D\psi|^2 < 0$ on $\partial\{\Psi = 0\}$. Let

$$U = \sup\{z : z \text{ is a subsolution of (4.1)}, \quad z_0 = u_0 \text{ and } \Psi \leqq z\}.$$

From barrier arguments at each point on Γ_0 we can easily check that $U^*(x,0) = u_0(x)$ and $\{U^* > 0\} = \{u_0 > 0\}$. Since U is a supremum of viscosity subsolutions, it follows that U^* is a viscosity subsolution with initial data u_0. Hence by definition of U we have $U = U^*$.

Step 2. Next we claim that

$$U_*(x,t) = \liminf_{(y,s)\to(x,t)} U(y,s) \text{ is a supersolution of (4.1).}$$

Since $\Psi \leqq U$, we get $U_* = u_0$. We only have to show that U_* is a supersolution on the free boundary of U_*. (For the arguments in the interior of positive set see for example [CIL].) Suppose this is not the case. Then there is a smooth function

$$\phi(x,t) = s(t-t_0) + < p, x - x_0 > +1/2$$
$$< X(x-x_0), x - x_0 > +o(|x-x_0|^2 + |t-t_0|),$$

where $(s, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^n$ such that $U_* - \phi$ has its local maximum zero in $\overline{\{U_* > 0\}} \cap \{t \leqq t_0\} \cap Q$ at $(x_0, t_0) \in \partial\{U_* = 0\}$, ϕ satisfies (1.1) and

$$s - |p|^2 > 0 \quad \text{and } -\text{trace}X > 0. \tag{4.5}$$

Here S^n is the set of $n \times n$ symmetric matrices. Then, by (4.5) and (1.1), the function

$$\tilde{U}_{\delta,\gamma}(x,t) = (\phi(x,t) + \delta - \gamma(|x - x_0|^2 + |t - t_0|^2))_+$$

is a supersolution of (4.1) in $B_r = \{(x,t) : |x - x_0|^2 + |t - t_0|^2 < r^2\}$ for all small $r, \delta, \gamma > 0$. Note that by (1.1), we can choose $\delta = \delta_0 > 0$ such that $\tilde{U} = 0$ outside $\{U > 0\} \cup B_{r/2}$. Finally, observe that by definition there is a sequence $(x_n, t_n) \to (x_0, t_0)$ such that $U(x_n, t_n) \to U_*(x_0, t_0)$ and thus

$$-\delta = \lim_{n\to\infty} (\tilde{U}(x_n, t_n) - U(x_n, t_n)) \geqq \tilde{U}(x_0, t_0) - U(x_0, t_0).$$

Hence we have a bigger subsolution U_2 if we let

$$U_2(x) = \begin{cases} \max\{U, \bar{U}\} & \text{if } |x - x_0|^2 + |t - t_0|^2 < r^2, \\ U(x) & \text{otherwise.} \end{cases}$$

with $U_2 = U$ on ∂Q for small r. This contradicts the definition of U.

Step 3. By Theorem 4.4 we obtain $U* \leqq \bar{U}_*$, and therefore $U^* = \bar{U}_* = {U_*}^*$ and $v = U_*$ is our desired solution. □

Appendix A. Construction of *h*, ϕ and φ

Here we construct our test functions in Section 1 and Section 2, based on a space-time ball B. For simplicity we set B: a unit space ball centered at the origin. First we construct ϕ in the proof of Lemma 2.4 based on $B = B_1$: a space-time exterior ball of $\partial\{u > 0\}$ at P_1. We will specify later that we also construct h in the proof of Lemma 1.3 with a slight modification. For convenience we may assume that $B_1 = B_1(0,0)$ is a unit space-time ball centered at the origin, with $P_1 = (\cos\alpha e_1, \sin\alpha) \in \partial B_1$ for some $0 < \alpha < \pi/2$. Note that in this case we have $m = \tan\alpha > 0$ as the advancing speed of $\partial\{u > 0\}$ at P_1.

At first we solve the ordinary differential equation

$$\phi_{rr} + \frac{n-1}{r}\phi_r = 0, \quad \text{where } r > 0.$$

Then we get $\phi_0(r) = -\ln r$ if $n = 2$ or $\phi_0(r) = r^{2-n}$ if $n > 2$. For $r > 0$, let

$$\phi_1(r) = \begin{cases} \ln r - r^{-1} & \text{if n=2}, \\ \\ 2 - r^{2-n} - r^{1-n} & \text{if n>2}. \end{cases}$$

For $x \in \mathbb{R}^n$, let

$$\phi_2(x) = \varphi_1((x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}) \quad \text{outside } \tfrac{1}{4}B_1(0).$$

then for $r \geqq 1/4$, $\phi_{2,r} > 0$ and

$$-\Delta\phi_2 = -\phi_{1,rr} - \frac{n-1}{r}\phi_{1,r} = (n-1)r^{-1-n} > 0.$$

Since $\phi_{2,r} > 0$ on $\partial\frac{1}{4}B_1(0)$, we can extend ϕ_2 to $\frac{1}{4}B_1(0)$ so that

$$\phi_2(x) \quad \begin{cases} > 0 & \text{outside } B_1(0), \\ = 0 & \text{on } \partial B_1(0), \\ < 0 & \text{inside } B_1(0). \end{cases}$$

Now to extend it to the space-time ball $B_1(0,0)/\{t = \pm 1\}$, define

$$\phi(x,t) = \phi_2(\frac{x}{\sqrt{1-t^2}}), -1 < t < 1.$$

After multiplying by a constant depending on m, for given $\varepsilon > 0$ we get

$$\begin{aligned} &\phi(x,t) \quad \begin{cases} > 0 & \text{outside } B_1, \\ = 0 & \text{on } \partial B_1, \\ < 0 & \text{inside } B_1, \end{cases} \\ &\phi(x,t) = \phi(r,t), \qquad \text{where } r = |x|, \\ &-\Delta\phi > 0 \qquad \text{outside } \tfrac{1}{4}B_v, \\ &\phi_r = m(1-\varepsilon/3) \quad \text{on } \partial B_1 \cap \{t = \sin\alpha\}. \end{aligned} \tag{A.1}$$

In particular at P_1,

$$\frac{\phi_t}{\phi_r} = \frac{\phi_t}{|D\phi|} = m > \phi_r = |D\phi| > 0.$$

Thus ϕ is a local supersolution of $[HS]$ near P_1. For constuction of h in Lemma 1.3, instead of the last condition in (A.1) we put

$$0 < \phi < \inf_{\frac{1}{4}\bar{B}} v \quad \text{on } \tfrac{1}{4}\bar{B}.$$

(This is possible since $v > 0$ in B.)

Next we set $B = B_2$ as in the proof of Theorem 2.2, where B_2 is the interior ball of $\{v > 0\}$ at $P_2 \in \partial\{v > 0\}$. Recall that at $t = t_2$, B_2 propagates with normal velocity $m(1-\delta)$. By letting $\varphi = -\phi$ and multiplying by a constant, we can construct φ such that

$$\begin{aligned} &\varphi(x,t) \quad \begin{cases} > 0 & \text{inside B}, \\ = 0 & \text{on } \partial B, \\ < 0 & \text{outside } B. \end{cases} \\ &-\Delta\varphi < 0 \qquad \text{outside } \tfrac{1}{4}B, \\ &\varphi(x,t) = \varphi(r,t), \qquad \text{where } r = |x|, \\ &-\varphi_r = m(1-\varepsilon_0), \quad \varepsilon_0 \leqq \delta/4 \quad \text{on } \partial[B \cap \{t = t_2\}]. \end{aligned} \tag{A.2}$$

Observe that at $t = t_0 = \sin\alpha$,

$$-\frac{\varphi_t}{\varphi_r} = \frac{\varphi_t}{|D\varphi|} = m(1-\delta/2).$$

Thus for $\tau > 0$ small enough and $0 < \varepsilon_0 < \delta/2$ we get

$$-\frac{\varphi_t}{\varphi_r} \leqq m(1-\delta/2) < -\varphi_r \quad \text{on } \partial B \cap \{t_2 - \tau \leqq t \leqq t_2\}.$$

Appendix B. Test functions for the Stefan problem

Here the test functions used in Section 4 are constructed. Observe that for the supersolution part, for $m > 0$, the advancing speed of the free boundary at P_1, we can simply use the same function ϕ constructed in the previous section, since we have $\phi_t = m|D\phi| > 0$ at P_1 and thus $\phi_t - \Delta\phi > 0$ in $\{\phi \geqq 0\}$ in a neighborhood of P_1.

Therefore we only have to construct a local subsolution φ in a neighborhood of $P_2 = (x_2, t_2) \in \partial\{v > 0\}$ for the proof of Theorem 4.2. Recall that at P_2 $\{v > 0\}$ has an interior space-time ball B_2 with its advancing speed $m - \delta$. We consider S such that

$$S \cap \{t = t_2 + s\} = \frac{r_1 + m_1 s}{r_1} B,$$

where r_1 is the radius of B, $m_1 = m - \delta/2$ and B, where $B = B_2 \cap \{t = t_2\}$. Note that $S \cap [t_2 - \tau, t_2] \subseteq B_2$ for small $\tau > 0$. Let $\varphi(gx, t) = \varphi(r - m_1 t)$, where $r = |x - c|$, c is the center of B, and solve for $\varphi_t - \Delta\varphi < 0$ in $S - 1/4S$. For convenience, we take $c = 0$, $r_1 = 1$ and $t_2 = 0$. Then we get

$$\varphi(x, t) = \varphi_0(r - m_1 t) + \varepsilon(r^2 - (m_1 t + 1)^2),$$

where $\varepsilon > 0$ and

$$\varphi_0(r) = \int_r^1 s^{1-n} e^{-s} ds.$$

After multiplying by a constant, for small ε and for $t \in [t_2 - \tau, t_2]$, we get

$$\begin{array}{lll} \varphi(x,t) & \begin{cases} > 0 & \text{inside } S - \frac{1}{4}S, \\ = 0 & \text{on } \partial S. \end{cases} & \\ \varphi_t - \Delta\varphi < 0 & & \text{outside } \frac{1}{4}S, \\ \varphi(x,t) = \varphi(r,t), & & \text{where } r = |x|, \\ -\varphi_r = m(1 - \varepsilon_0), \quad \varepsilon_0 \leqq \delta/4 & & \text{on } \partial S \cap \{t = t_2\}. \end{array} \tag{B.1}$$

Note that for small τ we have

$$\varphi_t/|D\varphi| < m_1 < -\varphi_r \quad \text{on } \partial S \cap \{t_2 - \tau \leqq t \leqq t_2\}.$$

Now we proceed the same as in the proof of Theorem 2.2 to show that for small r_1 we have $\varphi \leqq v$ on $(S - rS) \cap \{t_2 - \tau \leqq t \leqq t_2\}$. Thus $\tilde{\varphi}$ crosses v from below at P_2, and this leads to a contradiction since φ is a (strict) local subsolution of (4.1). □

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