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The Singular Set of Solutions to Non-Differentiable Elliptic Systems

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Abstract

We estimate the Hausdorff dimension of the singular set of solutions to elliptic systems of the type

$$- \operatorname{div} a(x, Du) = b(x, Du).$$

If the vector fields a and b are Hölder continuous with respect to the variable x with exponent α, then the Hausdorff dimension of the singular set of any weak solution is at most $n - 2\alpha$.

1. Introduction

Let us consider the following elliptic system in divergence form:

$$- \operatorname{div} a(x, Du) = b(x, Du) \tag{1.1}$$

in a bounded open subset $\Omega \subset \mathbb{R}^n$, $n \geqq 2$. Suppose that the continuous vector fields $a : \Omega \times \mathbb{M}^{N\times n} \to \mathbb{M}^{N\times n}$ and $b : \Omega \times \mathbb{M}^{N\times n} \to \mathbb{R}^N$ satisfy the following growth, ellipticity and continuity assumptions:

$$|D_z a(x, z)| \leqq L(1 + |z|^2)^{\frac{p-2}{2}}, \qquad |b(x, z)| \leqq L(1 + |z|^2)^{\frac{p-1}{2}}, \tag{1.2}$$

$$L^{-1}(1 + |z|^2)^{\frac{p-2}{2}}|\lambda|^2 \leqq \frac{\partial a_i^k}{\partial z_j^h}(x, z)\lambda_i^k\lambda_j^h, \tag{1.3}$$

$$|a(x, z) - a(x_0, z)| \leqq L|x - x_0|^\alpha(1 + |z|^2)^{\frac{p-1}{2}}, \tag{1.4}$$

for any $z, \lambda \in \mathbb{M}^{N\times n}$ and $x, x_0 \in \Omega$, where $p \geqq 2$, $L \in (1, +\infty)$ and $\alpha \in (0, 1)$. Then it is well know that any local weak solution $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$ is partially

$C^{1,\tilde{\alpha}}$-regular, that is, there exists an open subset of full measure $\Omega_0 \subseteq \Omega$ such that Du is Hölder continuous in Ω_0:

$$u \in C^{1,\tilde{\alpha}}(\Omega_0; \mathbb{R}^N), \qquad |\Omega \setminus \Omega_0| = 0, \qquad \tilde{\alpha} \equiv \tilde{\alpha}(\alpha, n, p) \in (0, \alpha]. \quad (1.5)$$

The study of partial regularity of solutions to elliptic systems started with papers by MORREY and GIUSTI & MIRANDA (see [19] and [14]), inspired by the works of Almgren and De Giorgi on Minimal Surfaces, and it has continued up to now, gaining contributions valid in more general settings (see [10, 8, 9]).

The reason for the interest in partial regularity lies in the fact that, except for very peculiar cases in which the vector field a has a special structure (see for instance [22]), local weak solutions are not everywhere regular. This fact is known since the counterexamples of DE GIORGI and MAZ'YA [6] and [18]. In particular, JOHN, MALÝ & STARÁ [17] show that solutions to linear elliptic systems can be nowhere continuous and, as recently discovered by SVERAK & YAN [21], minimizers of smooth convex variational integrals can be even unbounded.

In this situation we try to prove that $\Omega \setminus \Omega_0$ is "reasonably small" in the sense that it is not only negligible but it has also a low Hausdorff dimension. We consider for simplicity the case when $b(x, z) = 0$. If we work with a differentiable system, that is with the additional assumption

$$\left|\frac{\partial a}{\partial x}(x, z)\right| \leqq L(1 + |z|^2)^{\frac{p-1}{2}}, \qquad (1.6)$$

corresponding to Lipschitz continuity with respect to the variable x, then (see for instance [5],[11],[16])

$$\dim_{\mathcal{H}}(\Omega \setminus \Omega_0) \leqq n - 2 \qquad (1.7)$$

with $\dim_{\mathcal{H}}(A)$ denoting the Hausdorff dimension of a set $A \subset \mathbb{R}^n$. This latter fact relies on the possibility of differentiating the system (this is why (1.6) is then introduced) and obtaining the existence (in suitable Sobolev spaces) of second-order derivatives of the solution. In turn, this implies a better control on certain integral quantities measuring the oscillations of the gradient Du and thus controlling the Hölder continuity of Du itself. So, in general, what is known is either (1.5) if $a(\cdot, z)$ is Hölder continuous or (1.7) if $a(\cdot, z)$ is Lipschitz continuous, (1.6). In this paper we build a bridge between Hölder and Lipschitz continuity, trying to fill such a gap. Indeed, since partial $C^{1,\tilde{\alpha}}$ regularity of solutions (that is, (1.5)) holds under the only Hölder continuity condition (1.4), it is natural to wonder whether it is still possible to give an estimate on the dimension of the singular set $\Omega \setminus \Omega_0$ of local weak solutions under such a natural assumption, i.e., without passing through the second derivatives $D^2 u$ and, in other words, without the possibility of differentiating the system. Our aim is to show that this is actually the case; indeed under appropriate assumptions on b, as for example,

$$\begin{aligned} |b(x, z) - b(x_0, z)| &\leqq L|x - x_0|^{\alpha}(1 + |z|^2)^{\frac{p-1}{2}}, \\ |b(x, z) - b(x, z_0)| &\leqq L|z - z_0|^{\alpha}(1 + |z_0|^2 + |z|^2)^{\frac{p-1-\alpha}{2}}, \end{aligned} \qquad (1.8)$$

we have:

Theorem 1.1. *Let $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$ be a local weak solution to the system* (1.1), *under the assumptions* (1.2)–(1.4) *and* (1.8). *Then:*

$$\dim_{\mathcal{H}}(\Omega \setminus \Omega_0) \leqq n - 2\alpha. \tag{1.9}$$

As far as we know, this result is the first of general type. The proof of Theorem 1.1 rests basically on the simple observation that the Hölder continuity can be read as a fractional differentiability and thus the system can be still differentiated, but in a fractional sense. Therefore, rather than proving the existence of weak second derivatives, we come up with the fact that the gradient is in a certain fractional Sobolev space $W^{\theta,q}$. This suggests the idea that the estimate of the singular set can be performed via a suitable result on the pointwise behavior of functions in this space. This is exactly what we do, but following only elementary and short arguments, without appealing to heavy tools from Potential Theory: spaces of Bessel potentials, $L^{\theta,q}$, the theorems describing the pointwise behavior of functions from $L^{\theta,q}$ via Bessel (θ, q)-capacities and so on (see [4]). Instead we just use an elementary Poincaré-type inequality valid in $W^{\theta,q}$ and simple measure density results (Lemmata 4.1, 4.2) that rely on Vitali covering argument only.

2. Preliminaries and statements

Further Notation and Statements. In the following we shall define $B_R \equiv B(x_0, R) := \{x \in \mathbb{R}^n : |x - x_0| < R\}$. When not differently specified, all the balls considered will be concentric; c will denote a constant not necessarily the same in any two occurrences. As usual $\{e_s\}_{1 \leqq s \leqq n}$ stands for the standard basis of $\mathbb{R}^n$ while, if $v, w \in \mathbb{R}^k$, the tensor product $v \otimes w \in \mathbb{R}^{k^2}$ is defined by $(v \otimes w)_{i,j} := v_i w_j$. If $B(x_0, R) \subset\subset \Omega$ and v is a locally integrable function on Ω, we put

$$(v)_R \equiv (v)_{x_0,R} := \fint_{B_R} v \, dx \equiv \frac{1}{|B_R|} \int_{B_R} v \, dx.$$

A function $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$ is a local weak solution to the system (1.1) if and only, if, for any $\varphi \in W^{1,p}(\Omega; \mathbb{R}^N)$ such that supp $\varphi \subset \Omega$,

$$\int_\Omega a(x, Du) D\varphi \, dx = \int_\Omega b(x, Du)\varphi \, dx. \tag{2.1}$$

If $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$ is a local weak solution to the system (1.1), then we shall put

$$\Sigma := \Omega \setminus \Omega_0 := \text{“the singular set of } u\text{”}$$

with the convention that $\Omega_0 \subseteq \Omega$ denotes an open subset, with full measure, where Du is Hölder continuous with some exponent (note, anyway, that $\tilde{\alpha} = \alpha$ in (1.5), for instance when $p = 2$ and $b(x, z) = 0$, with generalizations to elliptic systems with nonlinear growth ($p \neq 2$) being possible as well; see [7]).

Under the previous assumptions we shall first prove Theorem 1.1 and subsequently we shall perform a sharp and significant refinement of the result that perfectly parallels the usual situation. Indeed, it is known that for certain classes of elliptic differentiable systems the Hausdorff dimension of Σ can be further reduced: $\dim_{\mathcal{H}}(\Sigma) < n-2$ (see [5]). Here we prove an analogous result; in order to avoid technicalities cropping up, we shall confine ourself to the model case of homogeneous systems with linear growth, i.e., $p = 2$ and $b(x, z) = 0$.

Theorem 2.1. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ be a local weak solution to the system* (1.1) *under the assumptions in* (1.2)–(1.4) *with $p = 2$ and $b(x, z) = 0$. Then there exists $t \equiv t(n, L) > 1$ such that $\mathcal{H}^{n-2t\alpha+\varepsilon}(\Sigma) = 0$ for any $\varepsilon > 0$, i.e.,*

$$\dim_{\mathcal{H}}(\Sigma) < n - 2\alpha.$$

Some Preliminary Material. We shall use fractional Sobolev spaces; given a smooth, bounded open subset $A \subset \mathbb{R}^n$ and $\theta \in (0, 1)$, $1 \leqq q < +\infty$, a function $u \in W^{\theta,q}(A; \mathbb{R}^N)$ if and only if

$$||u||_{W^{\theta,q}} := \left(\int_A |u(x)|^q \, dx\right)^{\frac{1}{q}} + \left(\int_A \int_A \frac{|u(x) - u(y)|^q}{|x-y|^{n+q\theta}} \, dx \, dy\right)^{\frac{1}{q}}$$

with the previous quantity being a norm making $W^{\theta,q}(A; \mathbb{R}^N)$ a Banach space, the local variant $W^{\theta,q}_{\mathrm{loc}}(A; \mathbb{R}^N)$ is then defined in the usual way. For the properties of fractional Sobolev spaces the reader is referred to [3], Chapter 7. In a standard notation, for a vector-valued function $G : \mathbb{R}^n \to \mathbb{R}^k$ we define

$$\tau_{s,h} G(x) = G(x + he_s) - G(x).$$

The following are simple properties of Sobolev functions:

Lemma 2.2. *If $0 < \rho < R$, $|h| < R - \rho$, $1 \leqq q < \infty$, $s \in \{1, \ldots, n\}$ and $G \in L^q(B_R)$, then*

$$\int_{B_\rho} |G(x + he_s)|^q \, dx \leqq c(n, q) \int_{B_R} |G(x)|^q \, dx;$$

moreover, if $D_s G \in L^q(B_R)$ then

$$\int_{B_\rho} |\tau_{s,h} G(x)|^q \, dx \leqq |h|^q \int_{B_R} |D_s G(x)|^q \, dx.$$

The next result exploits the relations between fractional Sobolev spaces $W^{\theta,q}$ and Nikolskii spaces $\mathcal{H}^{\theta,q}$. See [3], 7.73.

Lemma 2.3. *If $G : \mathbb{R}^n \to \mathbb{M}^{N\times n}$, $G \in L^q(B_{4R}; \mathbb{M}^{N\times n})$, $1 < q < +\infty$ and, for some $\rho \in (0, R]$ and $\theta \in (0, 1)$,*

$$\int_{B_\rho} \sum_{s=1}^{n} |\tau_{s,h} G(x)|^q \, dx \leqq M^q |h|^{q\theta}$$

for every h with $|h| < R/A$ with $M \geqq 0$, $A \geqq 1$, then $G \in W^{b,q}(B_\rho; \mathbb{M}^{N\times n})$ for every $b \in (0, \theta)$, and there exists $c \equiv c(n, k, q, b, \theta, R, A)$ such that

$$\| G \|_{W^{b,q}(B_\rho)} \leqq c \left(M + \| G \|_{L^q(B_{4R})}\right).$$

The following uniform version of Gehring's lemma can be inferred from [20]:

Lemma 2.4. *Let* $\{v_h\}_{h>0} \subset L^2_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$, $\{w_h\}_{h>0} \subset L^{2(1+\delta_1)}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ *be two families of functions such that*

$$\fint_{B_{R/2}} |v_h|^2\, dx \leqq c_1 \left(\fint_{B_R} |v_h|^{2\sigma}\, dx \right)^{\frac{1}{\sigma}} + c_1 \fint_{B_R} |w_h|^2\, dx$$

for any $B_R \subset\subset \Omega$, *where* $c_1 \in (1, +\infty)$, $\delta_1 > 0$ *and* $\sigma \in (0, 1)$ *are independent of* h. *Then there exist* $\tilde{c} \equiv \tilde{c}(n, c_1, \sigma, \delta_1)$ *and* $1 < t \equiv t(n, c_1, \sigma, \delta_1) < (1 + \delta_1)$, *independent of* h, *such that, for any* $B_R \subset\subset \Omega$ *and* $h > 0$,

$$\left(\fint_{B_{R/2}} |v_h|^{2t}\, dx \right)^{\frac{1}{t}} \leqq \tilde{c} \fint_{B_R} |v_h|^2\, dx + \tilde{c} \left(\fint_{B_R} |w_h|^{2(1+\delta_1)}\, dx \right)^{\frac{1}{1+\delta_1}}.$$

3. Fractional estimates

Proposition 3.1. *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ *be a local weak solution to the system* (1.1) *under the assumptions* (1.2)–(1.4), (1.8). *Then* $Du \in W^{2\beta/p,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{N\times n})$ *for any* $\beta < \alpha$.

Proof. Since we are going to prove a local result, we shall assume without loss of generality that $u \in W^{1,p}(\Omega; \mathbb{R}^N)$. We fix a ball $B(x_0, 4R) \equiv B_{4R} \subset\subset \Omega$ and, in the weak formulation (2.1), we pick the test function $\varphi := \tau_{s,-h}(\eta^2 \tau_{s,h} u)$, where $0 < |h| \leqq \min\{R/1000, 1\}$ and $\eta \in C^\infty_0(B_{3R/2})$ is a cut-off function such that $\eta \equiv 1$ on B_R, $|D\eta| \leqq c(n)/R$ and $0 \leqq \eta \leqq 1$. We obtain

$$\begin{aligned}
&\int_\Omega \eta^2 \tau_{s,h}(a(x, Du(x)))\tau_{s,h} Du\, dx \\
&\quad = -\int_\Omega 2\eta \tau_{s,h}(a(x, Du(x))) D\eta \otimes \tau_{s,h} u\, dx \\
&\quad\quad + \int_\Omega \eta^2 \tau_{s,h}(b(x, Du(x)))\tau_{s,h} u\, dx.
\end{aligned} \tag{3.1}$$

Now, let us write

$$\begin{aligned}
\tau_{s,h}(a(x, Du(x))) &= a(x + he_s, Du(x + he_s)) - a(x, Du(x + he_s)) \\
&\quad + a(x, Du(x + he_s)) - a(x, Du(x)) \\
&=: \mathcal{A}(h) + \mathcal{B}(h), \\
\tau_{s,h}(b(x, Du(x))) &= b(x + he_s, Du(x + he_s)) - b(x, Du(x + he_s)) \\
&\quad + b(x, Du(x + he_s)) - b(x, Du(x)) \\
&=: \mathcal{C}(h) + \mathcal{D}(h).
\end{aligned}$$

With such notation (3.1) becomes:

$$\begin{aligned}\int_\Omega \eta^2\,[\mathcal{A}(h)+\mathcal{B}(h)]\,\tau_{s,h}Du\;dx \\ &= -\int_\Omega 2\eta\,[\mathcal{A}(h)+\mathcal{B}(h)]\,D\eta\otimes\tau_{s,h}u\;dx \\ &\quad+\int_\Omega \eta^2\,[\mathcal{C}(h)+\mathcal{D}(h)]\,\tau_{s,h}u\;dx.\end{aligned}\tag{3.2}$$

We proceed to estimate the various terms arising from (3.2).

Using Young's inequality with $\varepsilon\in(0,1)$, (1.4) and Lemma 2.2, we have

$$\begin{aligned}&\int_\Omega \eta^2\mathcal{A}(h)\tau_{s,h}Du\;dx\\ &\leqq c\int_\Omega \eta^2|h|^\alpha(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-1}{2}}|\tau_{s,h}Du|\;dx\\ &\leqq \varepsilon\int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_{s,h}Du|^2\;dx\\ &\quad+C_\varepsilon\left(\int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p}{2}}\;dx\right)|h|^{2\alpha}\\ &\leqq \varepsilon\int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_{s,h}Du|^2\;dx\\ &\quad+C_\varepsilon\left(\int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\;dx\right)|h|^{2\alpha}.\end{aligned}$$

Using (1.4), the Hölder inequality and Lemma 2.2 (recall that supp $\eta\subset\subset B_{3R/2}$, $|h|\leqq R/1000$ and $0\leqq\eta\leqq 1$), we get,

$$\begin{aligned}&\int_\Omega \eta\mathcal{A}(h)D\eta\otimes\tau_{s,h}u\;dx\\ &\leqq c\left(\int_\Omega \eta||D\eta||_\infty(1+|Du(x+he_s)|^2)^{\frac{p-1}{2}}|\tau_{s,h}u|\;dx\right)|h|^\alpha\\ &\leqq c\left(\int_\Omega \eta(1+|Du(x+he_s)|^2)^{\frac{p}{2}}\;dx\right)^{1-\frac{1}{p}}\times\left(\int_{B_R}\eta|\tau_{s,h}u|^p\;dx\right)^{\frac{1}{p}}|h|^\alpha\\ &\leqq c\left(\int_{B_{2R}}(1+|Du(x)|^2)^{\frac{p}{2}}\;dx\right)|h|^{2\alpha},\end{aligned}$$

where we have estimated $|h|^{1+\alpha}\leqq|h|^{2\alpha}$ and $c\equiv c(||D\eta||_\infty)$.

In order to estimate the terms containing $\mathcal{B}(h)$ we write

$$\begin{aligned}\mathcal{B}(h) &= \int_0^1 D_za(x,Du(x)+t\tau_{s,h}(Du(x)))\;dt\;\tau_{s,h}Du\\ &=: \tilde{\mathcal{B}}(h)\tau_{s,h}Du.\end{aligned}\tag{3.3}$$

Moreover the following estimate is a standard consequence of (1.2) and (1.3) via Lemmata 2.1 and 2.2 from [1] (which hold for any $p > 1$):

$$
\begin{aligned}
&c^{-1}(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_{s,h}Du|^2 \\
&\leqq \tilde{\mathcal{B}}(h)\tau_{s,h}Du \otimes \tau_{s,h}Du,
\end{aligned} \tag{3.4}
$$

where $c \equiv c(n, p, L) > 0$. Therefore, combining (3.3) and (3.4) in a suitable way, we may write:

$$
\begin{aligned}
&\int_\Omega \eta^2 \mathcal{B}(h)\tau_{s,h}Du\, dx \\
&\geqq \frac{1}{2}\int_\Omega \eta^2 \tilde{\mathcal{B}}(h)\tau_{s,h}Du \otimes \tau_{s,h}Du\, dx \\
&\quad + c^{-1}\int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_{s,h}Du|^2\, dx.
\end{aligned} \tag{3.5}
$$

Using the notation introduced before and, in a standard way, Cauchy-Schwartz and Young inequalities with $\varepsilon \in (0, 1)$, we obtain, as for the previous integral:

$$
\begin{aligned}
&\int_\Omega 2\eta \mathcal{B}(h) D\eta \otimes \tau_{s,h}u\, dx \\
&= \int_\Omega 2\eta \tilde{\mathcal{B}}(h)\tau_{s,h}Du \otimes (D\eta \otimes \tau_{s,h}u)\, dx \\
&\leqq \varepsilon \int_\Omega \eta^2 \tilde{\mathcal{B}}(h)\tau_{s,h}Du \otimes \tau_{s,h}Du\, dx \\
&\quad + C_\varepsilon ||D\eta||_\infty^2 \int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_{s,h}u|^2\, dx \\
&\leqq \varepsilon \int_\Omega \eta^2 \tilde{\mathcal{B}}(h)\tau_{s,h}Du \otimes \tau_{s,h}Du\, dx \\
&\quad + C_\varepsilon \left(\int_\Omega \eta(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p}{2}}\, dx\right)^{1-\frac{2}{p}} \\
&\qquad\qquad \times \left(\int_\Omega \eta|\tau_{s,h}u(x)|^p\, dx\right)^{\frac{2}{p}} \\
&\leqq \varepsilon \int_\Omega \eta^2 \tilde{\mathcal{B}}(h)\tau_{s,h}Du \otimes \tau_{s,h}Du\, dx \\
&\quad + C_\varepsilon \left(\int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\, dx\right)|h|^{2\alpha},
\end{aligned}
$$

where $C_\varepsilon \equiv C_\varepsilon(||D\eta||_\infty)$ and we have used Lemma 2.2.

Proceeding as for the term $\int_\Omega 2\eta \mathcal{A}(h) D\eta \otimes \tau_{s,h}u\, dx$ we gain, via $(1.8)_1$,

$$
\begin{aligned}
\int_\Omega \eta^2 \mathcal{C}(h)\tau_{s,h}u\, dx &\leqq c\left(\int_\Omega \eta^2(1+|Du(x+he_s)|^2)^{\frac{p-1}{2}}|\tau_{s,h}u|\, dx\right)|h|^\alpha \\
&\leqq c\left(\int_{B_{2R}} (1+|Du(x)|^2)^{\frac{p}{2}}\, dx\right)|h|^{2\alpha}.
\end{aligned}
$$

Again using Young and Hölder inequalities with $(1.8)_2$, we estimate

$$
\begin{aligned}
&\int_\Omega \eta^2 \mathcal{D}(h)\tau_{s,h}u\,dx \\
&\leqq c\int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-1-\alpha}{2}}|\tau_{s,h}Du|^\alpha|\tau_{s,h}u|\,dx \\
&\leqq \varepsilon\int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_{s,h}Du|^2\,dx \\
&\quad +C_\varepsilon\int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p}{2}-\frac{1}{2-\alpha}}|\tau_{s,h}u|^{\frac{2}{2-\alpha}}\,dx \\
&\leqq \varepsilon\int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_{s,h}Du|^2\,dx \\
&\quad +C_\varepsilon\left(\int_{B_R}(1+|Du(x)|^2)^{\frac{p}{2}}\,dx\right)^{p-\frac{2}{(2-\alpha)p}}\times\left(\int_\Omega \eta|\tau_{s,h}u|^p\,dx\right)^{\frac{2}{(2-\alpha)p}} \\
&\leqq \varepsilon\int_\Omega \eta^2(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\tau_{s,h}Du|^2\,dx \\
&\quad +C_\varepsilon\left(\int_{B_{2R}}(1+|Du(x)|^2)^{\frac{p}{2}}\,dx\right)|h|^{2\alpha}, \qquad (3.6)
\end{aligned}
$$

where in the last estimate we have used Lemma 2.2, estimating $|h|^{2/(2-\alpha)} \leqq |h|^{2\alpha}$ in view of the elementary inequality $2\alpha \leqq 2/(2-\alpha)$.

Since $p \geqq 2$, we can estimate

$$
\begin{aligned}
|\tau_{s,h}(Du(x))|^p &= |\tau_{s,h}(Du(x))|^{p-2}|\tau_{s,h}(Du(x))|^2 \\
&\leqq c(n,p)(1+|Du(x)|^2+|Du(x+he_s)|^2)^{\frac{p-2}{2}}|\,\tau_{s,h}Du(x)|^2\,.
\end{aligned}
$$

Combining the last inequality with (3.5) and the other ones found for the terms arising from (3.2) and developed up to (3.6), choosing ε suitably small in a standard way, we finally get, summing up on $s \in \{1,\dots,n\}$:

$$
\int_{B_R}\sum_{s=1}^n |\tau_{s,h}(Du(x))|^p\,dx \leqq c\left(\int_{B_{2R}} 1+|Du(x)|^p\,dx\right)|h|^{p(2\alpha/p)} \qquad (3.7)
$$

where the constant c depends on n, p, L, R and is independent of the particular ball considered B_R, α and even of the particular vector fields a and b. Using Lemma 2.3 (with $\rho := R$) it follows that $Du \in W^{2\beta/p,p}(B_R;\mathbb{M}^{N\times n})$ for any $\beta < \alpha$ with the quantity $||Du||_{W^{2\beta/p,p}(B_R;\,\mathbb{M}^{N\times n})}$ being bounded by a constant $c \equiv c(n,p,L,R,\beta,||Du||_{L^p(\Omega)})$ independent of the ball considered: now the assertion follows via a standard covering argument. □

Proposition 3.2. *Let* $u \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *be a local weak solution to the system* (1.1) *under the assumptions in* (1.2)–(1.4) *with* $p = 2$ *and* $b(x,z) = 0$. *Then there exists* $t \equiv t(n,L) > 1$ *such that* $Du \in W^{\beta,2t}_{\mathrm{loc}}(\Omega;\mathbb{M}^{N\times n})$ *for any* $\beta < \alpha$.

Proof. Let us observe that, being a local weak solution to (1.1), the function u is higher integrable: $u \in W^{1,2(1+\delta_1)}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for some $\delta_1 \equiv \delta_1(n,L) > 0$ (see [12]). In particular, for any ball $B_{4R} \subset\subset \Omega$, we have the following reverse Hölder inequality:

$$\left(\fint_{B_{2R}} (1+|Du(x)|^2)^{(1+\delta_1)}\,dx\right)^{\frac{1}{1+\delta_1}} \leqq c \fint_{B_{4R}} 1+|Du(x)|^2\,dx \tag{3.8}$$

with $c \equiv c(n,L)$. In (2.1) we can replace the test function φ with the translated one $\tau_{s,-h}\varphi$ for $0<|h| \leqq \operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega)/1000$; we find that

$$\int_\Omega \tau_{s,h}(a(x,Du))D\varphi\,dx = 0.$$

With the notation of Proposition 3.1, we get

$$\int_\Omega \tilde{\mathcal{B}}(h)D(|h|^{-\alpha}\tau_{s,h}u)D\varphi\,dx = -\int_\Omega |h|^{-\alpha}\mathcal{A}(h)D\varphi\,dx;$$

therefore, if we introduce

$$v_h := \frac{\tau_{s,h}u}{|h|^\alpha}, \qquad W(h) := -\frac{\mathcal{A}(h)}{|h|^\alpha}, \tag{3.9}$$

then $v_h \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a local weak solution to the elliptic system

$$\int_\Omega \tilde{\mathcal{B}}(h)Dv_h D\varphi\,dx = \int_\Omega W(h)D\varphi\,dx \tag{3.10}$$

with growth and ellipticity bounds dictated by the ones in (1.2), (1.3) (recall that $p=2$); and for any $\lambda \in \mathbb{M}^{N\times n}$

$$L^{-1}|\lambda|^2 \leqq \tilde{\mathcal{B}}(h)\lambda\otimes\lambda \leqq L|\lambda|^2. \tag{3.11}$$

Our aim now is to derive a family of uniform (in h) reverse Hölder inequalities for the functions Dv_h. We fix $B_R \subset\subset \Omega$ and a non-negative cut-off function $\eta \in C^\infty_0(B_R)$ such that $\eta \equiv 1$ on $B_{R/2}$ and $|D\eta| \leqq c(n)/R$ and we test (3.10) with $\varphi := \eta^2(v_h - (v_h)_R)$. Using in a standard way Young's inequality and (3.11), we finally get the following Caccioppoli-type inequality:

$$\begin{aligned}&\int_{B_{R/2}} |Dv_h|^2\,dx\\ &\quad\leqq \frac{c(n,L)}{R^2}\int_{B_R} |v_h-(v_h)_R|^2\,dx + c(n,L)\int_{B_R}|W(h)|^2\,dx.\end{aligned} \tag{3.12}$$

Therefore, using Poincaré inequality in (3.12), we come up with:

$$\begin{aligned}&\fint_{B_{R/2}} |Dv_h|^2\,dx\\ &\quad\leqq c(n,L)\left(\fint_{B_R}|Dv_h|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}} + c(n,L)\fint_{B_R}|W(h)|^2\,dx.\end{aligned} \tag{3.13}$$

Let us observe that by (1.4) and (3.8) it follows that $|W(h)|^2 \in L^{1+\delta_1}$; moreover, if $0 < |h| \leqq \min\{R/1000, 1\}$, the following bound takes place using the definition of $W(h)$ (see (3.9)) and again (1.4) together with Lemma 2.2:

$$\int_{B_R} |W(h)|^{2(1+\delta_1)}\, dx \leqq c(n, L) \int_{B_{2R}} 1 + |Du(x)|^{2(1+\delta_1)}\, dx. \tag{3.14}$$

Taking into account that the constants in the previous inequalities are independent of h, and since B_R is arbitrary, by (3.13) we can apply Lemma 2.4 with a standard covering argument and with $\sigma = n/(n+2)$, to show that for a fixed exponent $1 < t \equiv t(n, L) < 1 + \delta_1$, and constant $c \equiv c(n, L)$, both independent of h, we have, for any $B_{4R} \subset\subset \Omega$,

$$\begin{aligned} &\left(\fint_{B_{R/2}} |Dv_h|^{2t}\, dx\right)^{\frac{1}{t}} \\ &\quad \leqq c \fint_{B_R} |Dv_h|^2\, dx + c\left(\fint_{B_{2R}} (1 + |Du(x)|^2)^{(1+\delta_1)}\, dx\right)^{\frac{1}{1+\delta_1}} \\ &\quad \leqq c \fint_{B_{2R}} 1 + |Du(x)|^2\, dx + c\left(\fint_{B_{2R}} (1 + |Du(x)|^2)^{(1+\delta_1)}\, dx\right)^{\frac{1}{1+\delta_1}} \\ &\quad \leqq c \fint_{B_{4R}} 1 + |Du(x)|^2\, dx, \end{aligned} \tag{3.15}$$

where we also used, in order: (3.14), (3.7) (we are assuming that $p = 2$) and finally (3.8).

Now, taking into account (3.8), from the very definition of v_h (see (3.9)) and using (3.15) for every $s \in \{1, \dots, n\}$, it follows that

$$\int_{B_{R/2}} \sum_{s=1}^{n} |\tau_{s,h} Du|^{2t}\, dx \leqq c(n, L) R^{n(1-t)} \left(\int_{B_{4R}} 1 + |Du(x)|^2\, dx\right)^t |h|^{2t\alpha},$$

which is analogous to (3.7) when $p = 2$. The proof finally follows as for the previous Proposition, via Lemma 2.3 (with $\rho := R/2$) and a standard covering argument. □

4. Proof of the theorems

We only give the proof of Theorem 1.1, the proof of Theorem 2.1 being the same but using Proposition 3.2 instead of Proposition 3.1. Before starting, we observe that since we are going to show a local partial regularity result, up to passing to open subsets $\Omega_h \subset\subset \Omega$ such that $\Omega_h \uparrow \Omega$, and in view of the local result of Proposition 3.1, without loss of generality, we shall assume that the fundamental fractional differentiablity result of Du holds globally in Ω:

$$Du \in W^{2\beta/p,p}(\Omega; \mathbb{M}^{N\times n}) \qquad \forall\, \beta < \alpha. \tag{4.1}$$

Step 1. A fractional Poincaré inequality

Here we want to state a Poincaré-type inequality valid for functions belonging to fractional Sobolev spaces. We are interested in the following statement: if $B_R \equiv B(x_0, R)$, then

$$\int_{B_R} |v(x) - (v)_R|^q \, dx \leqq c(n,q) R^{q\theta} \int_{B_R} \int_{B_R} \frac{|v(x) - v(y)|^q}{|x-y|^{n+q\theta}} \, dx \, dy \tag{4.2}$$

whenever $v \in W^{\theta,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$, where $q \geqq 1$, $\theta \in (0,1)$ and $B_R \subset\subset \Omega$.

It is difficult to trace back in the literature an explicit statement of such a fact, therefore we give here a very simple and elementary proof. Clearly, the previous inequality can be obtained via a standard scaling and translation argument from the following one (here $B_1 \equiv B(0,1)$):

$$\int_{B_1} |v(x) - (v)_1|^q \, dx \leqq c(n,q) \int_{B_1} \int_{B_1} \frac{|v(x) - v(y)|^q}{|x-y|^{n+q\theta}} \, dx \, dy \tag{4.3}$$

for any $v \in W^{\theta,q}(B_1; \mathbb{R}^N)$. To prove (4.3), fix $x \in B_1$. Then, by Jensen's inequality, we trivially have

$$\begin{aligned} |v(x) - (v)_1|^q &\leqq c(n,q) \fint_{B_1} |v(x) - v(y)|^q \, dy \\ &\leqq c(n,q) \fint_{B_1} |v(x) - v(y)|^q G_\varepsilon(|x-y|) \, dy, \end{aligned}$$

where, for $\varepsilon \in (0,1)$, $G_\varepsilon(t) := \min\{t^{-(n+q\theta)}, \varepsilon^{-1}\}$. Integrating, we get

$$\int_{B_1} |v(x) - (v)_1|^q \, dx \leqq c(n,q) \int_{B_1} \int_{B_1} |v(x) - v(y)|^q G_\varepsilon(|x-y|) \, dx \, dy$$

and (4.3) follows letting $\varepsilon \to 0^+$.

Step 2. An idea of Giusti, revisited

After Giusti ([13]), it is standard to use the following measure density result to estimate the Hausdorff dimension of singular sets of solutions to elliptic systems:

Lemma 4.1. *Let $A \subset \mathbb{R}^n$ be an open subset and μ be a Radon measure on A such that $\mu(A) < +\infty$. If $0 < t < n$. Then* $\dim_{\mathcal{H}}(E^t) \leqq t$ *where*

$$E^t := \left\{ x \in A : \limsup_{\rho \to 0^+} \rho^{-t} \mu(B(x,\rho)) > 0 \right\}.$$

We make a simple observation. As an inspection of the proof easily reveals, the previous result still holds if the Radon measure μ is replaced by a non-negative and increasing, finite-set function λ defined on the family of open subsets of A, which is also countably superadditive, the latter meaning that whenever $\{\Pi_i\}_{i \in \mathbb{N}}$ is a family of disjoint open subsets of A, then

$$\sum_{i \in \mathbb{N}} \lambda(\Pi_i) \leqq \lambda\left(\bigcup_{i \in \mathbb{N}} \Pi_i \right).$$

Now we observe that if $v \in W^{\theta,q}(\Omega;\mathbb{R}^n)$, then the function

$$\lambda(B) \equiv \lambda_v(B) := \int_B \int_B \frac{|v(x)-v(y)|^q}{|x-y|^{n+q\theta}}\,dx\,dy, \tag{4.4}$$

defined for any open subset $B \subseteq \Omega$, actually meets all these requirements. Therefore Lemma 4.1 can be recast as follows:

Lemma 4.2. *Let* $v \in W^{\theta,q}(\Omega;\mathbb{R}^n)$ *and* $0 < t < n$. *If* $\lambda \equiv \lambda_v$ *is as in* (4.4), *then* $\dim_{\mathcal{H}}(E^t) \leqq t$ *where*

$$E^t := \left\{ x \in \Omega : \limsup_{\rho\to 0^+} \rho^{-t}\lambda(B(x,\rho)) > 0 \right\}.$$

Step 3. The structure of the singular set Σ

Here we recall a characterization of singular set Σ of a local weak solution u to (1.1). It is very well known that this set is included in the set of non-Lebesgue points of Du. Indeed, if we let

$$\Sigma_0 := \left\{ x \in \Omega : \liminf_{\rho\to 0^+} \fint_{B(x,\rho)} |Du(y) - (Du)_{x,\rho}|^p\,dy > 0 \right\}$$

$$\Sigma_1 := \left\{ x \in \Omega : \limsup_{\rho\to 0^+} |(Du)_{x,\rho}| = +\infty \right\},$$

it follows that

$$\Sigma \subseteq \Sigma_0 \cup \Sigma_1. \tag{4.5}$$

This fact is standard and it can be easily inferred, for instance, from [15] or [2] (see also [11] and [7] for the case of systems with linear growth $p = 2$; see [8] and [9] for similar results in the variational setting and for quasiconvex integrals). Such a type of characterization (that is, via non-Lebesgue points) has been known since the original paper [14] (see also [19]).

Step 4. Conclusion

By the inclusion stated in (4.5), in order to prove (1.9) it suffices to show that $\mathcal{H}^{n-2\alpha+\varepsilon}(\Sigma_0) = \mathcal{H}^{n-2\alpha+\varepsilon}(\Sigma_1) = 0$ for any $\varepsilon > 0$. We do this by proving that for every $\beta < \alpha$ and for any $\varepsilon > 0$ it follows that $\mathcal{H}^{n-2\beta+\varepsilon}(\Sigma_0) = \mathcal{H}^{n-2\beta+\varepsilon}(\Sigma_1) = 0$.

By (4.1) we apply Lemma 4.2 with $t := n - 2\beta$, $q := p$, $\theta := 2\beta/p$, $v := Du$ and

$$\lambda(B) \equiv \lambda_v(B) := \int_B \int_B \frac{|Du(x)-Du(y)|^p}{|x-y|^{n+2\beta}}\,dx\,dy \qquad B \text{ open subset of } \Omega.$$

Therefore, if we let

$$S_0 := \left\{ x \in \Omega : \limsup_{\rho\to 0^+} \rho^{2\beta-n}\lambda(B(x,\rho)) > 0 \right\}$$

we end up with $\mathcal{H}^{n-2\beta+\varepsilon}(S_0) = 0$. On the other hand, applying (4.2) gives

$$\begin{aligned}
&\fint_{B(x_0,\rho)} |Du(x) - (Du)_{x_0,\rho}|^p \, dx \\
&\qquad \leqq c(n,p)\rho^{2\beta-n} \int_{B(x_0,\rho)} \int_{B(x_0,\rho)} \frac{|Du(x) - Du(y)|^p}{|x-y|^{n+2\beta}} \, dx \, dy.
\end{aligned}$$

So we conclude that if $x_0 \in \Sigma_0$, then $x_0 \in S_0$, so that $\Sigma_0 \subseteq S_0$ and therefore $\mathcal{H}^{n-2\beta+\varepsilon}(\Sigma_0) = 0$.

Now we prove that $\mathcal{H}^{n-2\beta+\varepsilon}(\Sigma_1) = 0$. To this end, fix $0 < \varepsilon_0 < \varepsilon$ and consider

$$S_1 := \left\{ x \in \Omega : \limsup_{\rho \to 0^+} \rho^{2\beta-n-\varepsilon_0} \lambda(B(x,\rho)) > 0 \right\}.$$

Then applying Lemma 4.2 it follows that $\mathcal{H}^{n-2\beta+\varepsilon}(S_1) = 0$. We shall prove that $\Sigma_1 \subset S_1$ and the proof will be finished. In turn this will be achieved as follows. We take $x_0 \in \Omega \setminus S_1$ and $B(x_0, R) \subset\subset \Omega$ and we prove that the following limit exists and is finite:

$$\lim_{k \to +\infty} |(Du)_{x_0,2^{-k}R}| < +\infty. \tag{4.6}$$

Then a simple interpolation argument (see Remark 1 below) shows that, for any sequence $a_k \to 0$, $\limsup_k |(Du)_{x_0,Ra_k}| < +\infty$, therefore $x_0 \in \Omega \setminus \Sigma_1$ and $\Sigma_1 \subseteq S_1$. It remains to prove (4.6). Arguing as in Step 1, using Jensen's inequality and the fractional Poincaré inequality in (4.2), we estimate:

$$\begin{aligned}
&|(Du)_{x_0,2^{-(k+1)}R} - (Du)_{x_0,2^{-k}R}|^p \\
&\qquad \leqq \fint_{B_{2^{-(k+1)}R}} |Du(x) - (Du)_{2^{-k}R}|^p \, dx \\
&\qquad \leqq 2^n \fint_{B_{2^{-k}R}} |Du(x) - (Du)_{2^{-k}R}|^p \, dx \\
&\qquad \leqq c(n,p) \left(\frac{R}{2^k}\right)^{2\beta-n} \lambda(B(x_0, 2^{-k}R)) \\
&\qquad = c(n,p) \left(\frac{R}{2^k}\right)^{\varepsilon_0} \left(\frac{R}{2^k}\right)^{2\beta-n-\varepsilon_0} \lambda(B(x_0, 2^{-k}R)) \\
&\qquad \leqq \tilde{c} \left(\frac{1}{2^k}\right)^{\varepsilon_0}
\end{aligned} \tag{4.7}$$

and (4.6) easily follows. □

Remark 1. The interpolation argument goes as follows: observe that the estimate in (4.7) works if we replace $\{R/2^k\}$ with any decreasing sequence $\{Rb_k\}$ such that $b_k/b_{k+1} \leqq c$ and $\sum b_k^{\varepsilon_0/p}$ converges for any $\varepsilon_0 > 0$. Then when considering an arbitrary sequence $\{a_k\}$, we first observe, eventually passing to a subsequence, that we can assume it to be decreasing and $a_k \leqq 2^{-k}$; then it suffices to nest it in a dyadic one (for example considering the decreasing rearrangement of $\{a_k\} \cup \{2^{-k}\}$) to view it as a subsequence of the type $\{b_k\}$.

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References

1. ACERBI, E., FUSCO, N.: Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$. *J. Math. Anal. Appl.* **140**, 115–135 (1989)
2. ACERBI, E., MINGIONE, G.: Regularity results for stationary electro-rheological fluids. *Arch. Rational Mech. Anal.* **164**, 213–259 (2002)
3. ADAMS, R.A.: *Sobolev Spaces*. New York: Academic Press, 1975
4. ADAMS, D.R., HEDBERG, L.I.: *Function spaces and potential theory*. Berlin: Springer-Verlag, 1996
5. CAMPANATO, S.: Hölder continuity of the solutions of some nonlinear elliptic systems. *Adv. in Math.* **48**, 16–43 (1983)
6. DE GIORGI, E.: Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. *Boll. Un. Mat. Ital.* **1**, 135–137 (1968)
7. DUZAAR, F., GROTOWSKI, J.: Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation. *Manuscripta Math.* **103**, 267–298 (2000)
8. EVANS, L.C.: Quasiconvexity and partial regularity in the Calculus of Variations. *Arch. Rational Mech. Anal.* **95**, 227–252 (1984)
9. EVANS, L.C., GARIEPY, R.: Blow-up, compactness and partial regularity in the Calculus of Variations. *Indiana Univ. Math. J.* **36**, 361–371 (1987)
10. GIAQUINTA, M.: *Multiple integrals in the Calculus of Variations and nonlinear elliptic systems*. Annals of Mathematics Studies, 105 Princeton: Princeton University Press, 1983
11. GIAQUINTA, M., MODICA, G.: Almost-everywhere regularity for solutions of nonlinear elliptic systems. *Manuscripta Math.* **28**, 109–158 (1979)
12. GIUSTI, E.: *Metodi Diretti nel Calcolo delle Variazioni*. UMI, Bologna, 1994
13. GIUSTI, E.: Precisazione delle funzioni di $H^{1,p}$ e singolarità delle soluzioni deboli di sistemi ellittici non lineari. *Boll. Un. Mat. Ital.* **2**, 71–76 (1969)
14. GIUSTI, E., MIRANDA, M.: Sulla regolarità delle soluzioni deboli di una classe di sistemi ellittici quasi-lineari. *Arch. Rational Mech. Anal.* **31**, 173–184 (1968/1969)
15. HAMBURGER, C.: A new partial regularity proof for solutions of nonlinear elliptic systems. *Manuscripta Math.* **95**, 11–31 (1998)
16. IVERT, P.A.: Regularitätsuntersuchungen von lösungen elliptiscer systeme von quasilinearen differetialgléichungen zweiter ordnung. *Manuscripta Math.* **30**, 53–88 (1979)
17. JOHN, O., MALÝ, J., STARÁ, J.: Nowhere continuous solutions to elliptic systems. *Comment. Math. Univ. Carolin.* **30**, 33–43 (1989)
18. MAZÝA, V.G.: Examples of irregular solutions of quasilinear elliptic equations with analytic coefficients. *Funct. Anal. Appl.* **2**, 230–234 (1968)
19. MORREY JR. C.B.: Partial regularity results for non-linear elliptic systems. *J. Math. Mech.* **17**, 649–670 (1967/1968)
20. STREDULINSKY, E.W.: Higher integrability from reverse Hölder inequalities. *Indiana Univ. Math. J.* **29**, 407–413 (1980)

21. SVERÀK, V., YAN, X.: Non Lipschitz minimizers of smooth uniformly convex variational integrals. *Proc. Nat. Ac. Sci. USA*, to appear
22. UHLENBECK, K.: Regularity for a class of nonlinear elliptic systems. *Acta Math.* **138**, 219–240 (1977)

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