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Convergence of Solutions to the Boltzmann Equation in the Incompressible Euler Limit

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Abstract

We consider here the problem of deriving rigorously, for well-prepared initial data and without any additional assumption, dissipative or smooth solutions of the incompressible Euler equations from renormalized solutions of the Boltzmann equation. This completes the partial results obtained by GOLSE [B. Perthame and L. Desvillettes eds., *Series in Applied Mathematics 4* (2000), Gauthier-Villars, Paris] and LIONS & MASMOUDI [Arch. Rational Mech. Anal. **158** (2001), 195–211].

1. Introduction

The present work establishes the convergence of appropriately scaled families of DiPerna-Lions renormalized solutions of the Boltzmann equation towards solutions of the incompressible Euler equations for well-prepared initial data. In [6] or [13], this was done by using an energy method and assuming:

(i) the local conservation of momentum, which is not guaranteed for the renormalized solutions of the Boltzmann equation;
(ii) some control on large velocities.

In [17], LIONS & MASMOUDI have developed an argument based on the study of a defect measure governed by a transport equation, in order to remove completely assumption (i).

In the present paper, we show how to circumvent the need for assumption (ii): the new estimates on large velocities use in a crucial way the dissipation control given by the H Theorem. They come from a combination of the arguments of [18] used to derive the incompressible Euler limit from the BGK (Bhatnager Gross Krock) Boltzmann model, and the ideas of [14] used to study the Navier-Stokes asymptotic of the Boltzmann equation.

1.1. The Boltzmann equation

In kinetic theory, a gas is described by a function $F \equiv F(t,x,v) \geqq 0$, usually called the "distribution function" or the "number density", measuring the density of gas molecules which at time $t \in \mathbf{R}_+$ are located at $x \in \Omega \subset \mathbf{R}^d$ and have instantaneous velocity $v \in \mathbf{R}^d$. If the dimension of the ambient space $d = 3$, the collisional model most commonly accepted is the Boltzmann one, which has been rigorously derived by LANFORD for large systems of particles [9]:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F,F) \tag{1}$$

where $\mathcal{B}(F,F)$ is the Boltzmann collision integral. This collision integral acts only on the v-argument of the number density F and is given by the expression

$$\mathcal{B}(F,F)(t,x,v) = \iint_{\mathbf{S}^2\times\mathbf{R}^3} (F'F_1' - FF_1)b(v-v_1,\sigma)\,d\sigma\,dv_1, \tag{2}$$

where the terms F_1, F' and F_1' designate respectively the values $F(t,x,v_1)$, $F(t,x,v')$ and $F(t,x,v_1')$, with v' and v_1' given in terms of $v_1 \in \mathbf{R}^3$ and $\sigma \in \mathbf{S}^2$ by the formulas

$$v' = \frac{v+v_1}{2} + \frac{|v-v_1|}{2}\sigma, \quad v_1' = \frac{v+v_1}{2} - \frac{|v-v_1|}{2}\sigma, \tag{3}$$

which ensure the conservation of momentum and kinetic energy for each binary collision between gas molecules (of like mass).

The collision kernel $b \equiv b(z,\sigma)$ is in general an a.e. positive function defined on $\mathbf{R}^3 \times \mathbf{S}^2$ which encodes whichever features of the molecular interaction are relevant in kinetic theory; it depends only on $|z|$ and $|z\cdot\sigma|$. Moreover, it is supposed to satisfy the weak cut-off condition of DIPERNA & LIONS [10],

$$(1+|v|)^{-2}\int_{|z|<R}\int_{\mathbf{S}^2} b(z+v,\sigma)\,d\sigma\,dz \to 0 \text{ as } |v| \to +\infty \text{ for all } R>0 \tag{H0}$$

as well as

$$\int_{\mathbf{S}^2} b(z,\sigma)\,d\sigma \leqq k_b(1+|z|^2)\,,\ \ z\in\mathbf{R}^3 \quad \text{for some } k_b > 0. \tag{H1}$$

This second condition holds for all hard cut-off potentials in the sense of Grad (see [8, 15]), in particular for cut-off Maxwell molecules and for hard spheres.

The solutions of the functional equation $\mathcal{B}(F,F) = 0$ are Maxwellians, i.e., functions of the form

$$M_{(\rho,u,\theta)}(v) = \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}} \tag{4}$$

for some $\rho > 0$, $\theta > 0$ and $u \in \mathbf{R}^3$. In particular, global Maxwellians are equilibrium states for the inhomogeneous Boltzmann equation. Below, we shall always use the notation M as an abbreviation for $M_{(1,0,1)}$.

From now on, we are concerned with the Cauchy problem on $\mathbf{R}_t^+ \times \mathbf{T}_x^3 \times \mathbf{R}_v^3$ (where the spatial domain $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$ has no boundary), for a scaled variant of (1):

$$\begin{aligned} &\varepsilon\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon^q}\mathcal{B}(F, F)\,, \quad t > 0\,,\ (x, v) \in \mathbf{T}^3 \times \mathbf{R}^3\,, \\ &F_\varepsilon(0, x, v) = F_\varepsilon^{\text{in}}(x, v)\,, \quad (x, v) \in \mathbf{T}^3 \times \mathbf{R}^3\,, \end{aligned} \tag{5}$$

where $\varepsilon > 0$ designates the order of magnitude of the Mach number and ε^q ($q > 1$) the order of the Knudsen number (see [1, 2] for a detailed discussion on these questions of scalings), and where $F_\varepsilon^{\text{in}} \geqq 0$ a.e. is a family of measurable functions such that

$$\sup_{\varepsilon>0} \frac{1}{\varepsilon^2} \iint \left[F_\varepsilon^{\text{in}} \log\left(\frac{F_\varepsilon^{\text{in}}}{M}\right) - F_\varepsilon^{\text{in}} + M\right] dv\, dx < +\infty. \tag{6}$$

Define $h : z \in]-1, +\infty[\mapsto h(z) = (1+z)\log(1+z) - z$. For any pair of measurable functions f and g defined a.e. and nonnegative on $\mathbf{T}^3 \times \mathbf{R}^3$, we use the following notation for the relative entropy:

$$H(f|g) = \iint g\, h\left(\frac{f}{g} - 1\right) dx\, dv \in [0, +\infty]. \tag{7}$$

A *renormalized solution* of (5) is a nonnegative function F_ε which belongs to $C(\mathbf{R}_+; w\text{-}L^1(\mathbf{T}^3; L^1(\mathbf{R}^3)))$, satisfies

$$\Gamma'(F_\varepsilon)\mathcal{B}(F_\varepsilon, F_\varepsilon) \in L^1_{\text{loc}}(\mathbf{R}_+; L^1(\mathbf{T}^3 \times \mathbf{R}^3))$$

for all $\Gamma \in C^1(\mathbf{R}_+)$ such that

$$\Gamma(0) = 0 \text{ and } z \mapsto (1+z)\Gamma'(z) \text{ is bounded on } \mathbf{R}_+, \tag{8}$$

has finite relative entropy for all positive times:

$$H(F_\varepsilon(t, \cdot, \cdot)|M) < +\infty \quad \forall t > 0, \tag{9}$$

and finally satisfies

$$\begin{aligned} &\int_0^{+\infty} \iint \Gamma(F_\varepsilon)\left(\partial_t\chi + \frac{1}{\varepsilon}v \cdot \nabla_x\chi\right) dx\, dv\, dt \\ &\quad + \iint \Gamma(F_\varepsilon^{\text{in}}(x, v))\chi(0, x, v)\, dx\, dv \\ &\quad + \frac{1}{\varepsilon^2}\int_0^{+\infty} \iint \Gamma'(F_\varepsilon)\mathcal{B}(F_\varepsilon, F_\varepsilon)\chi\, dx\, dv\, dt = 0 \end{aligned} \tag{10}$$

for all $t > 0$ and each test function $\chi \in C_c^\infty(\mathbf{R}_+ \times \mathbf{T}^3 \times \mathbf{R}^3)$.

The global existence of such renormalized solutions, as well as the local conservation of mass, are established in [10], while [11] provides the entropy inequality

$$\iint F_\varepsilon \log F_\varepsilon(t)\, dx\, dv + \frac{1}{\varepsilon^{q+1}}\int_0^t \iint D(F_\varepsilon)(s)\, ds\, dx\, dv \leqq \iint F_\varepsilon^{\text{in}} \log F_\varepsilon^{\text{in}}\, dx\, dv \tag{11}$$

for all $t > 0$, where the dissipation term $D(f)$ is defined for all nonnegative measurable function $f \equiv f(v)$ by

$$D(f) = \frac{1}{4} \iint f f_1 \, r\left(\frac{f' f_1'}{f f_1} - 1\right) b(v - v_1, \sigma)\, d\sigma\, dv_1, \tag{12}$$

with $r : z \in]-1, +\infty[\mapsto r(z) = z \log(1 + z)$. Whether the local conservation of momentum holds in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{T}^3$ is still unknown; this is one of the difficulties in rigorously deriving hydrodynamic models from the Boltzmann equation. A remark by LIONS & MASMOUDI [17] shows actually that the construction of [10] yields a solution which satisfies in addition

$$\partial_t \int v F_\varepsilon \, dv + \nabla_x \cdot \frac{1}{\varepsilon} \int v \otimes v F_\varepsilon \, dv + \nabla_x \cdot m_\varepsilon = 0,$$

$$\iint |v|^2 F_\varepsilon(t)\, dx\, dv + \varepsilon \iint \mathrm{tr}(m_\varepsilon)(t) = \iint |v|^2 F_\varepsilon^{\mathrm{in}}\, dx\, dv,$$

where the matrix-valued measure m_ε comes from a possible lack of compactness of the sequence of approximating solutions.

Theorem 1 (DiPerna-Lions-Masmoudi)**.** *For fixed $\varepsilon > 0$, let $F_\varepsilon^{\mathrm{in}} \equiv F_\varepsilon^{\mathrm{in}}(x, v)$ be an a.e. nonnegative, measurable function defined on $\mathbf{T}^3 \times \mathbf{R}^3$ such that $H(F_\varepsilon^{\mathrm{in}}|M) < +\infty$. Then there exists a renormalized solution to* (5) *which satisfies*

– *the local conservation of mass in the sense of distributions*

$$\partial_t \int F_\varepsilon \, dv + \nabla_x \cdot \frac{1}{\varepsilon} \int v F_\varepsilon \, dv = 0, \quad t > 0, \ x \in \mathbf{T}^3, \tag{13}$$

– *the local conservation of momentum with a symmetric nonnegative matrix-valued defect measure $m_\varepsilon \in L^\infty(\mathbf{R}^+, \mathcal{M}(\mathbf{T}^3, M_3(\mathbf{R})))$* (*coming from the approximation scheme of the Boltzmann equation*)

$$\partial_t \int v F_\varepsilon \, dv + \nabla_x \cdot \frac{1}{\varepsilon} \int v \otimes v F_\varepsilon \, dv + \nabla_x \cdot m_\varepsilon = 0, \quad t > 0, \ x \in \mathbf{T}^3, \tag{14}$$

– *the global conservation of energy with the defect measure m_ε*

$$\iint |v|^2 F_\varepsilon(t)\, dx\, dv + \varepsilon \iint \mathrm{tr}(m_\varepsilon)(t) = \iint |v|^2 F_\varepsilon^{\mathrm{in}}\, dx\, dv, \quad t > 0, \tag{15}$$

– *and, by* (11) *and* (15)*, the relative entropy inequality with the defect measure m_ε*

$$H(F_\varepsilon(t)|M) + \varepsilon \int \mathrm{tr}(m_\varepsilon)(t) + \frac{1}{\varepsilon^{q+1}} \int_0^t \iint D(F_\varepsilon)(s)\, ds\, dx\ dv \leqq H(F_\varepsilon^{\mathrm{in}}|M) \tag{16}$$

for all $t > 0$, where the dissipation term $D(F_\varepsilon)$ is defined by (12).

1.2. The incompressible Euler equations

The Euler equations govern the velocity field $u \equiv u(t, x)$ of a non-viscous incompressible fluid. In the three-dimensional case, they are

$$
\begin{aligned}
\nabla_x \cdot u = 0, \quad & t > 0, \ x \in \mathbf{T}^3, \\
\partial_t u + u \cdot \nabla_x u + \nabla_x p = 0, \quad & t > 0, \ x \in \mathbf{T}^3.
\end{aligned} \tag{17}
$$

The first equality in (17) states that the fluid motion preserves the volume, and is referred to as the incompressibility condition; the second equality expresses Newton's law of dynamics for any infinitesimal volume of fluid.

Consider the function space

$$\mathcal{H}^s = \{u \in H^s(\mathbf{T}^3; \mathbf{R}^3) \mid \nabla_x \cdot u = 0\}.$$

Let $u^{\text{in}} \in \mathcal{H}^s$, and consider the Cauchy problem for (17) with initial data

$$u(0, x) = u^{\text{in}}(x), \quad x \in \mathbf{T}^3. \tag{18}$$

We recall that the only existence theorem that is known to hold for the Cauchy problem (17), (18) is the following [5].

Theorem 2 (Beale-Kato-Majda). *For each* $u^{\text{in}} \in \mathcal{H}^s$ $\left(s > 1 + \frac{3}{2}\right)$, *there exist a unique* $T^* \in]0, +\infty]$ *and a unique* $u \in L^\infty_{\text{loc}}([0, T^*[, \mathcal{H}^s)$ *solution of* (17), (18) *which satisfies in addition*

$$\int_0^{T^*} \|\nabla_x \wedge u(t, x)\|_{L^\infty(\mathbf{T}^3)} \, dt = +\infty. \tag{19}$$

As the global existence of solutions of (17), (18) (even of weak solutions) is not known, LIONS has proposed the following very weak notion of solution [16]:

A *dissipative solution* of (17), (18) on $[0, T)$ is a function

$$u \in L^\infty([0, T), L^2(\mathbf{T}^3)) \cap C^0([0, T), w - L^2(\mathbf{T}^3))$$

satisfying $\nabla_x \cdot u = 0$, $u(0, .) = u^{\text{in}}$ in the sense of distributions and such that

$$
\begin{aligned}
&\int |w - u|^2(t, x) \, dx \\
&\quad \leqq \int |w^{\text{in}} - u^{\text{in}}|^2(x) \, dx \exp\left(\int_0^t 2\|X(w)(\tau)\|_\infty \, d\tau\right) \\
&\qquad + 2 \int_0^t \exp\left(\int_\tau^t 2\|X(w)(s)\|_\infty \, ds\right) \int E(w).(w - u)(\tau, x) \, dx \, d\tau
\end{aligned} \tag{20}
$$

for all $t \in [0, T)$ and all $w \in C^0([0, T] \times \mathbf{T}^3)$ satisfying

$$
\begin{aligned}
&\nabla_x \cdot w = 0, \\
&X(w) = \frac{1}{2}(\nabla_x w + (\nabla_x w)^T) \in L^1([0, T], L^\infty(\mathbf{T}^3)), \\
&E(w) = \partial_t w + w \cdot \nabla_x w \in L^1([0, T], L^2(\mathbf{T}^3)).
\end{aligned}
$$

Such solutions always exist, they are not weak solutions of (17), (18) in conservative form, but they coincide with the unique smooth solution with the same initial data as long as the latter does exist.

Proposition 1 (Lions). *If there exists $u \in C([0,T], L^2(\mathbf{T}^3))$ solution of* (17), (18) *on* $[0,T] \times \mathbf{T}^3$ *such that* $X(u) \in L^1([0,T], L^\infty(\mathbf{T}^3))$ *and* $E(u) \in L^1([0,T], L^2(\mathbf{T}^3))$, *then any dissipative solution u of* (17), (18) *is equal to u on* $[0,T] \times \mathbf{T}^3$.

Remark. In the two-dimensional case, the vorticity $\omega = \nabla_x \wedge u$ (which is a scalar) satisfies

$$\partial_t \omega + u \cdot \nabla_x \omega = 0$$

so that Theorem 2 provides a global existence result for regular initial data [20]. Moreover, the comparison principle stated in Proposition 1 provides uniqueness in the class of dissipative solutions. Nevertheless the two-dimensional case will not be considered in the limiting process, since the Boltzmann equation seems not to be relevant, and the Caflisch estimates – which are a crucial argument in the proof – are not established in this case.

1.3. Main results

The incompressible Euler limit of the Boltzmann equation considers fluctuations of the number density about an absolute Maxwellian. We shall need the linearized collision operator

$$\mathcal{L}g = \iint (g + g_1 - g' - g'_1) b(v - v_1, \sigma)\, d\sigma\, M_1\, dv_1; \tag{21}$$

For all interaction potentials considered in the present paper, it was proved by GRAD [15] that $\mathcal{L}$ is a possibly unbounded, self-adjoint Fredholm operator on $L^2(M dv)$ with the null space

$$\operatorname{Ker} \mathcal{L} = \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}. \tag{22}$$

In particular, each component of the tensor $v^{\otimes 2} - \frac{1}{3}|v|^2 \mathrm{Id}$ is orthogonal to Ker$\mathcal{L}$, which guarantees the existence and uniqueness of a tensor A such that

$$\mathcal{L}A = v^{\otimes 2} - \tfrac{1}{3}|v|^2 \mathrm{Id}, \quad A \perp \operatorname{Ker} \mathcal{L}. \tag{23}$$

For the sake of simplicity we will assume from now on that the Boltzmann collision kernel b satisfies

$$\iiint M M_1 (A + A_1 - A' - A'_1)^2 b(v - v_1, \sigma)\, dv_1\, dv\, d\sigma < +\infty \tag{H2}$$

which is guaranteed for all hard cut-off potentials in the sense of GRAD [15], and

$$\frac{1}{k_b} \leqq b(z, \sigma), \; z \in \mathbf{R}^3, \; \omega \in \mathbf{S}^2, \quad \text{for some } k_b > 0. \tag{H3}$$

Theorem 3 (Dissipative Euler Limit)**.** *Let b satisfy* (H0)–(H3)*, and let $F_\varepsilon^{\mathrm{in}}$ be a family of nonnegative, measurable functions on $\mathbf{T}^3 \times \mathbf{R}^3$ such that there exists a divergence-free vector field $u^{\mathrm{in}} \in L^2(\mathbf{T}^3)$ for which*

$$\frac{1}{\varepsilon^2} H(F_\varepsilon^{\mathrm{in}} | M_{1,\varepsilon u^{\mathrm{in}},1}) \to 0 \text{ as } \varepsilon \to 0. \tag{24}$$

Let F_ε be a family of renormalized solutions to (5)*. Then,*

$$\left(\frac{1}{\varepsilon}\int v F_\varepsilon \, dv\right)_{\varepsilon>0} \text{ is relatively compact in } w\text{-}L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^3))$$

and each of its limit points as $\varepsilon \to 0$ is a dissipative solution of (17)*,* (18).

If the limiting initial data u^{in} is smooth and such that (17), (18) has a (unique) smooth solution u, the stability result above can be strengthened: the convenient notion of convergence has been introduced in [3].

Definition 1. A family $g_\varepsilon \equiv g_\varepsilon(x, v)$ of $L^1_{\mathrm{loc}}(M dv\, dx)$ converges to $g \equiv g(x, v)$ entropically as $\varepsilon \to 0$ if

- for all ε, $1 + \varepsilon g_\varepsilon \geqq 0$ a.e. on $\mathbf{T}^3 \times \mathbf{R}^3$,
- $g_\varepsilon \rightharpoonup g$ in w-$L^1_{\mathrm{loc}}(M dv\, dx)$ as $\varepsilon \to 0$,
- and

$$\frac{1}{\varepsilon^2} H(M(1+\varepsilon g_\varepsilon)|M) \to \frac{1}{2}\iint g^2 M dv\, dx$$

as $\varepsilon \to 0$.

Theorem 4 (Strong Euler Limit)**.** *Under the same assumptions as in Theorem 3, assume that*

$$\frac{1}{\varepsilon}\frac{F_\varepsilon^{\mathrm{in}}(x, v) - M(v)}{M(v)} \to u^{\mathrm{in}}(x) \cdot v$$

entropically as $\varepsilon \to 0$, where u^{in} is a smooth divergence-free vector field such that the Euler equations (17)*,* (18) *have a strong solution u on* $[0, T]$*. Then, for almost every $t \in [0, T]$,*

$$\frac{1}{\varepsilon}\frac{F_\varepsilon(t, x, v) - M(v)}{M(v)} \to u(t, x) \cdot v$$

entropically as $\varepsilon \to 0$.

This theorem follows immediatly from Theorem 3 by the strong-weak stability argument stated in Proposition 1.

Remark. Both convergence results hold only for "well-prepared" initial data in the sense of (24), i.e., for data satisfying the incompressibility condition and having no density or temperature fluctuation. If these conditions are not satisfied, we expect formally that high-frequency acoustic oscillations occur, which do not modify the weak limit but cannot be taken into account in the framework of our study. Indeed, acoustic oscillations introduce a coupling between the density, the momentum and the temperature, for which no local conservation holds.

The second restriction in our convergence results is about collision kernels which are supposed to be bounded from below (satisfying (H3)): for such collision kernels, the entropy dissipation gives a good control on the integrand $|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|$. This assumption can be relaxed if we keep some control on the size of the sets where b is close to 0. For instance, Theorems 1.3 and 1.4 can be extended if $b(z,\sigma) \geqq C|z|^\gamma$ for some $\gamma > 0$, and in particular for hard spheres ($b(z,\sigma) = |z|$).

2. Sketch of the proof

The main results of this paper are obtained by a classical energy method. The principle is to modulate a Lyapunov functional of the system by test functions, and to obtain a stability inequality of the same type as (20). The idea to use the relative entropy for this type of problem comes from the notion of entropic convergence defined in [3] and from the method used by YAU [19] to derive hydrodynamic limits from the Ginzburg-Landau models.

The first step consists in computing the variation in time of the relative entropy $H(F_\varepsilon|M_{(1,\varepsilon w,1)})(t)$ where w is any divergence-free test function.

Theorem 5. *Consider $w \in C^1([0,T]\times \mathbf{T}^3)$ satisfying $\nabla_x \cdot w = 0$. Define the family F_ε as in Theorem 3. Then,*

$$\begin{aligned}
&\frac{1}{\varepsilon^2} H(F_\varepsilon|M_{(1,\varepsilon w,1)})(t) - \frac{1}{\varepsilon^2} H(F_\varepsilon^{\text{in}}|M_{(1,\varepsilon w^{\text{in}},1)}) + \frac{1}{\varepsilon}\int \operatorname{tr}(m_\varepsilon)(t)\\
&\quad \leqq -\frac{1}{\varepsilon}\int_0^t \int E(w).\int (v-\varepsilon w) F_\varepsilon(s,x,v)\,dx\,dv\,ds\\
&\qquad -\int_0^t \int X(w) : \frac{1}{\varepsilon^2}\int (v-\varepsilon w)^{\otimes 2} F_\varepsilon(s,x,v)\,dv\,dx\,ds\\
&\qquad -\int_0^t \int X(w) : \frac{1}{\varepsilon} m_\varepsilon(s,x)\,ds
\end{aligned}$$

Proof. The relative entropy inequality (16) implies

$$H(F_\varepsilon(t)|M) + \varepsilon \int \operatorname{tr}\,(m_\varepsilon)(t) \leqq H(F_\varepsilon^{\text{in}}|M).$$

It is easy to check that

$$H(F_\varepsilon|M_{(1,\varepsilon w,1)})(t) = H(F_\varepsilon(t)|M) + \tfrac{1}{2}\iint (\varepsilon^2 w^2 - 2v\cdot \varepsilon w) F_\varepsilon(t,x,v)\,dx\,dv$$

from which we deduce that

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}H(F_\varepsilon|M_{(1,\varepsilon w,1)})(t) - \frac{1}{\varepsilon^2}H(F_\varepsilon^{\mathrm{in}}|M_{(1,\varepsilon w^{in},1)}) + \frac{1}{\varepsilon}\int \mathrm{tr}(m_\varepsilon)(t) \\
&\quad \leqq \tfrac{1}{2\varepsilon^2}\int_0^t ds \tfrac{d}{ds}\iint (\varepsilon^2 w^2 - 2v\cdot\varepsilon w)F_\varepsilon(s,x,v)\,dx\,dv
\end{aligned}
\tag{25}
$$

Using the local conservation laws (13), (14) to compute the time derivative in (25) leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\iint F_\varepsilon(t,x,v)(\varepsilon w^2 - 2v\cdot w)\,dx\,dv \\
&\quad = \iint \partial_t w\cdot(\varepsilon w - v)F_\varepsilon(t,x,v)\,dx\,dv \\
&\qquad + \int\left(\frac{\varepsilon}{2}w^2\partial_t\int F_\varepsilon(t,x,v)\,dv - w\cdot\partial_t\int F_\varepsilon(t,x,v)v\,dv\right)dx \\
&\quad = \int \partial_t w\cdot\int(\varepsilon w - v)F_\varepsilon(t,x,v)\,dv\,dx - \int\frac{1}{2}w^2\nabla_x\cdot\int F_\varepsilon(t,x,v)\,vdv\,dx \\
&\qquad + \int w\cdot(\nabla_x\cdot m_\varepsilon)(t,x) + \int\frac{1}{\varepsilon}w\cdot\left(\nabla_x\cdot\int F_\varepsilon(t,x,v)v\otimes v\,dv\right)dx \\
&\quad = \iint \partial_t w\cdot(\varepsilon w - v)F_\varepsilon(t,x,v)dx\,dv + \iint(v\cdot\nabla_x)w\cdot wF_\varepsilon(t,x,v)\,dv\,dx \\
&\qquad + \int w\cdot(\nabla_x\cdot m_\varepsilon)(t,x) - \frac{1}{\varepsilon}\iint(v\cdot\nabla_x)w\cdot vF_\varepsilon(t,x,v)\,dv\,dx \\
&\quad = \iint(\partial_t w + (w\cdot\nabla_x)w)\cdot(\varepsilon w - v)F_\varepsilon(t,x,v)dx\,dv + \int w\cdot(\nabla_x\cdot m_\varepsilon)(t,x) \\
&\qquad - \frac{1}{\varepsilon}\iint((v-\varepsilon w)\cdot\nabla_x)w\cdot(v-\varepsilon w)F_\varepsilon(t,x,v)\,dv\,dx.
\end{aligned}
\tag{26}
$$

As m_ε is a symmetric matrix-valued measure,

$$
\int w.(\nabla_x\cdot m_\varepsilon) = -\int \nabla_x w : m_\varepsilon = -\int X(w) : m_\varepsilon, \tag{27}
$$

and

$$
\begin{aligned}
&\iint((v-\varepsilon w)\cdot\nabla_x)w\cdot(v-\varepsilon w)F_\varepsilon(t,x,v)\,dv\,dx \\
&\quad = \iint \nabla_x w : (v-\varepsilon w)^{\otimes 2}F_\varepsilon(t,x,v)\,dv\,dx \\
&\quad = \iint X(w) : (v-\varepsilon w)^{\otimes 2}F_\varepsilon(t,x,v)\,dv\,dx.
\end{aligned}
\tag{28}
$$

Replacing the three formulas (26), (27) and (28) in (25) gives the expected inequality. □

The second step consists in establishing an inequality of the same type as (20). In previous works [6, 17], the main arguments are approximatively the following:

$$
\left\|\frac{1}{\varepsilon}\int\tilde{F}_\varepsilon v\,dv - w\right\|^2_{L^2(\mathbf{T}^3)} \leqq \frac{1}{\varepsilon^2}H(F_\varepsilon|M_{(1,\varepsilon w,1)}) \tag{29}
$$

where $\tilde{F}_\varepsilon$ designates some convenient truncation of F_ε, and

$$\begin{aligned}&\left\| X(w) : \frac{1}{\varepsilon^2}\int (v-\varepsilon w)^{\otimes 2} F_\varepsilon(s,x,v)\,dv \right\|_{L^1(\mathbf{T}^3)}\\ &\qquad \leqq \|X(w)\|_{L^\infty(\mathbf{T}^3)} \left\| \frac{1}{\varepsilon}\int \tilde{F}_\varepsilon v\,dv - w\right\|^2_{L^2(\mathbf{T}^3)} + r_\varepsilon\ ,\end{aligned} \tag{30}$$

where r_ε converges to 0 in an appropriate sense. This last property comes from the assumption on large velocities which ensures that

$$\tfrac{1}{\varepsilon^2}(F_\varepsilon - \tilde{F}_\varepsilon)(1+|v|^2) \text{ is equi-integrable in } L^1([0,T]\times\mathbf{T}^3\times\mathbf{R}^3).$$

We first remark that there is little hope of establishing such a claim: with this type of scaling, we are not able to obtain *a priori* compactness with respect to space variables. In particular, in the limit case, we do not have compactness on the average velocity, which is why we do not have global weak solutions.

The key idea here is to estimate the remainder $(F_\varepsilon - \tilde{F}_\varepsilon)$ in terms of the relative entropy $H(F_\varepsilon|M_{(1,\varepsilon w,1)})$ and of the entropy dissipation. More precisely, we will not use here the lower bound (29) which looses any control on large tails in the distribution F_ε. The crucial argument in our proof is adapted from a similar work on the BGK model [18], it consists in establishing a Gronwall inequality for the relative entropy, rather than for the relative mean velocity. Indeed we replace the estimate (30) by the following

Theorem 6. *Consider $T > 0$ and $w \in C^1([0,T]\times\mathbf{T}^3)$ satisfying $\nabla_x \cdot w = 0$. Define the family F_ε as in Theorem 3. Then, there exists a nonnegative constant C such that*

$$\begin{aligned}&\int_0^t \int X(w) : \frac{1}{\varepsilon^2}\int (v-\varepsilon w)^{\otimes 2} F_\varepsilon(s,x,v)\,dv\,dx\,ds\\ &\qquad \leqq \frac{C}{\varepsilon^2}\int_0^t \|X(w)\|_{L^\infty(\mathbf{T}^3)} H(F_\varepsilon|M_{(1,\varepsilon w,1)})(s)ds + o(1) \text{ as } \varepsilon\to 0.\end{aligned}$$

The proof of this estimate will be given in detail in Section 4. It relies on proper decompositions of $(F_\varepsilon - M_{(1,\varepsilon w,1)})$ given in Section 3 and on the fact that $\varepsilon^{-2}\mathcal{B}(F_\varepsilon,F_\varepsilon)\Gamma'(F_\varepsilon)$ converges to 0 for all $\Gamma \in C^1(\mathbf{R}^+)$ satisfying (8).

The last step in the convergence proof is to deduce (20) from the Gronwall inequality implied by Theorems 5 and 6

$$\begin{aligned}&\frac{1}{\varepsilon^2}H(F_\varepsilon|M_{(1,\varepsilon w,1)})(t) - \frac{1}{\varepsilon^2}H(F_\varepsilon^{\text{in}}|M_{(1,\varepsilon w^{\text{in}},1)}) + \frac{1}{\varepsilon}\int \operatorname{tr}(m_\varepsilon)(t)\,dx\\ &\quad\leqq -\frac{1}{\varepsilon}\int_0^t\iint E(w).(v-\varepsilon w)F_\varepsilon(s,x,v)\,dx\,dv\,ds\\ &\qquad + C\int_0^t \|X(w)\|_{L^\infty(\mathbf{T}^3)}\left(\frac{1}{\varepsilon^2}H(F_\varepsilon|M_{(1,\varepsilon w,1)})(s) + \frac{1}{\varepsilon}\int \operatorname{tr}\,(m_\varepsilon)(s)\right)ds + o(1).\end{aligned} \tag{31}$$

Then we have to take weak limits (up to extraction of a subsequence $\varepsilon \to 0$)

$$\int F_\varepsilon\,dv \to 1, \quad \frac{1}{\varepsilon}\int F_\varepsilon v\,dv \rightharpoonup u,$$

and to prove that

$$\nabla_x \cdot u = 0, \quad \|u - w\|^2_{L^2(\mathbf{T}^3)} \leqq \liminf_{\varepsilon \to 0} \left(\frac{1}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}) \right).$$

This last step will be detailed in Section 5.

3. A priori estimates

The goal of this section is to establish the following

Theorem 7. *Let $\gamma : \mathbf{R}_+ \to [0, 1]$ be a C^∞ function supported in $[\frac{1}{2}, \frac{3}{2}]$ such that $\gamma([\frac{3}{4}, \frac{5}{4}]) = \{1\}$. Consider a test divergence-free vector field w as in Theorem 6. Define the family F_ε as in Theorem 3. Then, there exists $C > 0$ such that the following bounds hold:*

$$\left\| \frac{(F_\varepsilon - M_{(1,\varepsilon w,1)})^2}{\varepsilon^2 M_{(1,\varepsilon w,1)}} \gamma^2 \left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}} \right) (1 + v^2) \right\|_{L^1(dv\,dx)} \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon / M_{(1,\varepsilon w,1)}) + r_\varepsilon, \tag{32}$$

$$\left\| \frac{(F_\varepsilon - M_{(1,\varepsilon w,1)})}{\varepsilon^2} (1 - \gamma) \left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}} \right) (1 + v^2) \right\|_{L^1(dv\,dx)} \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon / M_{(1,\varepsilon w,1)}) + r_\varepsilon, \tag{33}$$

where r_ε converges to 0 in $L^1([0, T])$.

This theorem provides a decomposition of the fluctuation $\frac{1}{\varepsilon}(F_\varepsilon - M_{(1,\varepsilon w,1)})$ into two parts: the first one is controlled in L^2 norm, while the second one is of order ε in L^1 norm, with two moments in v. This decomposition is a variant of the so-called "Flat-Sharp decomposition" introduced in [3] and used in almost all works concerning the hydrodynamic limits of the Boltzmann equation (for example [6, 13, 17]). The main difference is that we consider here fluctuations with respect to the local Maxwellian $M_{(1,\varepsilon w,1)}$ instead of the global Maxwellian M: this kind of estimate, already used in [18] and [13] in the case of the BGK model, allows us to establish the inequality of Gronwall's type (31). The second particularity of this result is the control on large velocities: we are able to obtain estimates with two moments in v. This comes from dissipation estimates, and from regularizing properties of the gain part of the collision operator established by Caflisch [7]. This method has already been used successfully in [14] to obtain estimates for the Boltzmann equation with the Navier-Stokes scaling.

The proof of Theorem 7 is rather technical and will be divided into several steps. We start with the standard decomposition associated with the entropy bound $H(f|g)$: it consists in splitting $(f - g)$ into two parts depending on the size of f/g.

Lemma 1 (Usual decomposition)**.** *Consider a test divergence-free vector field w as in Theorem 6. Define the family F_ε as in Theorem 3. Then, if γ is a truncation function as in Theorem 7, there exists $C > 0$ such that the following bounds hold:*

$$\left\| \frac{(F_\varepsilon - M_{(1,\varepsilon w,1)})}{\varepsilon\sqrt{M_{(1,\varepsilon w,1)}}}\gamma\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}}\right)\right\|^2_{L^2(dv\,dx)} \leqq \frac{C}{\varepsilon^2}H(F_\varepsilon/M_{(1,\varepsilon w,1)}),$$

$$\left\| \frac{F_\varepsilon + M_{(1,\varepsilon w,1)}}{\varepsilon^2}(1-\gamma)\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}}\right)\right\|_{L^1(dv\,dx)} \leqq \frac{C}{\varepsilon^2}H(F_\varepsilon/M_{(1,\varepsilon w,1)}). \quad (34)$$

Proof. From the elementary inequalities

$$z^2 \leqq Ch(z) \text{ for } |z| \leq \tfrac{1}{2}, \quad 1+|z| \leqq Ch(z) \quad \text{for } z \in [-1,-\tfrac{1}{4}] \cup [\tfrac{1}{4},+\infty[, \quad (35)$$

which hold for some constant $C > 0$, and

$$\begin{aligned}(2+z)(1-\gamma(1+z)) &\leqq (2\times 4|z| + |z|)(1-\gamma(1+z))\\ &\leqq 9|z|(1-\gamma(1+z)) \quad \text{for } z > -1, \end{aligned} \quad (36)$$

we deduce

$$\begin{aligned}&\left\| \frac{(F_\varepsilon - M_{(1,\varepsilon w,1)})}{\varepsilon\sqrt{M_{(1,\varepsilon w,1)}}}\gamma\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}}\right)\right\|^2_{L^2(dv\,dx)}\\ &\quad\leqq \left\| \frac{M_{(1,\varepsilon w,1)}}{\varepsilon^2}\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}}-1\right)^2\gamma^2\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}}\right)\right\|_{L^1(dv\,dx)}\\ &\quad\leqq \frac{C}{\varepsilon^2}\|\gamma\|^2_\infty H(F_\varepsilon/M_{(1,\varepsilon w,1)}),\end{aligned}$$

and

$$\begin{aligned}&\left\| \frac{F_\varepsilon + M_{(1,\varepsilon w,1)}}{\varepsilon^2}(1-\gamma)\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}}\right)\right\|_{L^1(dv\,dx)}\\ &\quad\leqq 9\left\| \frac{M_{(1,\varepsilon w,1)}}{\varepsilon^2}\left|\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}}-1\right|(1-\gamma)\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}}\right)\right\|_{L^1(dv\,dx)}\\ &\quad\leqq \frac{9C}{\varepsilon^2}\|1-\gamma\|_\infty H(F_\varepsilon/M_{(1,\varepsilon w,1)}),\end{aligned}$$

which are the expected estimates. □

In order to establish Theorem 7, the main difficulty is in adding a $|v|^2$ weight in the estimates in Lemma 1. It will be obtained essentially by using the decomposition

$$F_\varepsilon - M_{(1,\varepsilon w,1)} = \big(F_\varepsilon - \mathcal{A}^+(F_\varepsilon, F_\varepsilon)\big) + \mathcal{A}^+(F_\varepsilon - M_{(1,\varepsilon w,1)}, F_\varepsilon + M_{(1,\varepsilon w,1)}), \quad (37)$$

where

$$\mathcal{A}^+(f,g) = \frac{1}{8\pi}\iint (f_1'g' + f'g_1')\, dv_1 d\sigma. \tag{38}$$

The first term is then controlled by the entropy dissipation, while the second term is a little more regular with respect to v than $(F_\varepsilon - M_{(1,\varepsilon w,1)})$ by CAFLISCH estimates [7]

$$\left\| M^{-1/2}\,\mathcal{A}^+\left(\sqrt{M}g, M\right)(1+|v|^{3/2})\right\|_{L^\infty_v} \leqq C\|g\|_{L^2_v}\ ,$$
$$\left\| M^{-1/2}\,\mathcal{A}^+(\sqrt{M}g, M)(1+|v|^2)\right\|_{L^\infty_v} \leqq C\|g\|_{L^\infty_v}\ ,$$

from which we deduce that

$$\begin{aligned}
&\left\| M^{-1/2}_{(1,\varepsilon w,1)}\,\mathcal{A}^+(\sqrt{M_{(1,\varepsilon w,1)}}\,g, M_{(1,\varepsilon w,1)})(1+|v|^{3/2})\right\|_{L^\infty_v}\\
&\quad\leqq \left\| M^{-1/2}\,\mathcal{A}^+(\sqrt{M}\,g(.+\varepsilon w), M)(1+|v+\varepsilon w|^{3/2})\right\|_{L^\infty_v}\\
&\quad\leqq C_{\|w\|_\infty}\,\|g(.+\varepsilon w)\|_{L^2_v} \leqq C_{\|w\|_\infty}\,\|g\|_{L^2_v},\\
&\left\| M^{-1/2}_{(1,\varepsilon w,1)}\,\mathcal{A}^+(\sqrt{M_{(1,\varepsilon w,1)}}\,g, M_{(1,\varepsilon w,1)})(1+|v|^{2})\right\|_{L^\infty_v}\\
&\quad\leqq \left\| M^{-1/2}\,\mathcal{A}^+(\sqrt{M}\,g(.+\varepsilon w), M)(1+|v+\varepsilon w|^{2})\right\|_{L^\infty_v}\\
&\quad\leqq C_{\|w\|_\infty}\,\|g(.+\varepsilon w)\|_{L^\infty_v} \leqq C_{\|w\|_\infty}\,\|g\|_{L^\infty_v},
\end{aligned} \tag{39}$$

because $\varepsilon \in [0,1]$ and $w \in C^1([0,T]\times\mathbf{T}^3) \subset L^\infty([0,T]\times\mathbf{T}^3)$.

Actually this decomposition will give good estimates under two conditions. The first one is linked to the fact that the operator $\mathcal{A}^+$ is quadratic: thus, in order to define correctly all the terms in the decomposition, we will need a $L^\infty_{t,x}$ bound on $\int M_{(1,\varepsilon w,1)} h\big((F_\varepsilon/M_{(1,\varepsilon w,1)}) - 1\big)\, dv$. The second condition is linked to the form of the entropy dissipation which gives a control in $L\log L$ and not in L^2: then it provides only a control for moderate velocities. Large values of the local relative entropy $\int M_{(1,\varepsilon w,1)} h\big((F_\varepsilon/M_{(1,\varepsilon w,1)}) - 1\big)\, dv$ and very large velocities have to be treated separately. This is the substance of the two following lemmas.

Lemma 2 (Macroscopic truncation). *Consider a test divergence-free vector field w as in Theorem 6. Define the family F_ε as in Theorem 3. Denote by $\chi_\varepsilon \equiv \chi_\varepsilon(t,x)$ the indicator function of the set*

$$\left\{(t,x)\in[0,T]\times\mathbf{T}^3 \mid \int M_{(1,\varepsilon w,1)} h\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}} - 1\right) dv \leqq 1\right\}.$$

Then, the following estimate holds:

$$\left\|(1-\chi_\varepsilon)\frac{F_\varepsilon + M_{(1,\varepsilon w,1)}}{\varepsilon^2}(1+|v|^2)\right\|_{L^1(dv\,dx)} \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon/M_{(1,\varepsilon w,1)}) \tag{40}$$

for all $\varepsilon \in [0,1]$ and for some constant $C > 0$ depending only on $\|w\|_\infty$.

Proof. By Young's inequality (91) with $p = (1+|v|^2)/8$ and $z = (F_\varepsilon/M_{(1,\varepsilon w,1)}) - 1$, and (89),

$$\begin{aligned}
&\frac{1}{\varepsilon^2}|F_\varepsilon - M_{(1,\varepsilon w,1)}|(1+|v|^2) \\
&\quad \leqq \frac{8}{\varepsilon^2} M_{(1,\varepsilon w,1)} \left(h\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}} - 1 \right) + e^{(1+|v|^2)/8} \right).
\end{aligned} \tag{41}$$

In order to estimate the second term in the right-hand side of (41), we need the following decay estimate

$$\begin{aligned}
&M_{(1,\varepsilon w,1)} \exp\left(\tfrac{1}{8}(1+|v|^2)\right) = (2\pi)^{-3/2} \exp(-\tfrac{1}{2}|v-\varepsilon w|^2) \exp\left(\tfrac{1}{8}(1+|v|^2)\right) \\
&\quad \leqq (2\pi)^{-3/2} \exp(-\tfrac{1}{4}|v|^2 + \tfrac{1}{2}|\varepsilon w|^2) \exp\left(\tfrac{1}{8} + \tfrac{1}{8}|v|^2\right) \\
&\quad \leqq C_{\|w\|_\infty} \exp(-\tfrac{1}{8}|v|^2)
\end{aligned} \tag{42}$$

Then, by (41) and (42),

$$\begin{aligned}
&\frac{1}{\varepsilon^2} \int (F_\varepsilon + M_{(1,\varepsilon w,1)})(1+|v|^2)\, dv \\
&\quad \leqq \frac{1}{\varepsilon^2} \int \left(|F_\varepsilon - M_{(1,\varepsilon w,1)}| + 2M_{(1,\varepsilon w,1)} \right) (1+|v|^2)\, dv \\
&\quad \leqq \frac{8}{\varepsilon^2} \int M_{(1,\varepsilon w,1)} h\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}} - 1 \right) dv \\
&\qquad + \frac{1}{\varepsilon^2} \int M_{(1,\varepsilon w,1)} \left(8e^{(1+|v|^2)/8} + 2(1+|v|^2) \right) dv \\
&\quad \leqq \frac{8}{\varepsilon^2} \int M_{(1,\varepsilon w,1)} h\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}} - 1 \right) dv + \frac{1}{\varepsilon^2} C_{\|w\|_\infty}
\end{aligned} \tag{43}$$

for some $C_{\|w\|_\infty} > 0$ depending only on $\|w\|_\infty$. By the definition of χ_ε,

$$\frac{1-\chi_\varepsilon}{\varepsilon^2} \leqq \frac{1-\chi_\varepsilon}{\varepsilon^2} \int M_{(1,\varepsilon w,1)} h\left(F_\varepsilon / M_{(1,\varepsilon w,1)} - 1 \right) dv. \tag{44}$$

Then, multiply (43) by $(1-\chi_\varepsilon)$, use (44), and integrate with respect to x to obtain the expected estimate. □

Lemma 3 (Control of very large velocities)**.** *Consider a test divergence-free vector field w as in Theorem 6. Define the family F_ε as in Theorem 3. Then, the following estimate holds*

$$\begin{aligned}
&\left\| \mathbf{1}_{|v|^2 \geqq 100|\log \varepsilon|} \frac{F_\varepsilon + M_{(1,\varepsilon w,1)}}{\varepsilon^2} (1+|v|^2) \right\|_{L^1(dv\,dx)} \\
&\quad \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon / M_{(1,\varepsilon w,1)}) + o(1)
\end{aligned} \tag{45}$$

as $\varepsilon \to 0$, for some $C > 0$ depending on $\|w\|_\infty$.

Proof. By (41) and (42), we have

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int (F_\varepsilon + M_{(1,\varepsilon w,1)})(1+|v|^2)\mathbf{1}_{|v|^2\geqq 100|\log\varepsilon|}\,dv\\
&\quad\leqq \frac{1}{\varepsilon^2}\int \left(|F_\varepsilon - M_{(1,\varepsilon w,1)}| + 2M_{(1,\varepsilon w,1)}\right)(1+|v|^2)\mathbf{1}_{|v|^2\geqq 100|\log\varepsilon|}\,dv\\
&\quad\leqq \frac{8}{\varepsilon^2}\int M_{(1,\varepsilon w,1)} h\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}}-1\right)dv\\
&\qquad + \frac{1}{\varepsilon^2}\int M_{(1,\varepsilon w,1)}\big(8e^{(1+|v|^2)/8} + 2(1+|v|^2)\big)\mathbf{1}_{|v|^2\geqq 100|\log\varepsilon|}\,dv\\
&\quad\leqq \frac{8}{\varepsilon^2}\int M_{(1,\varepsilon w,1)} h\left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}}-1\right)dv\\
&\qquad + \frac{1}{\varepsilon^2}C_{\|w\|_\infty}\int \exp\left(-\frac{1}{8}|v|^2\right)\mathbf{1}_{|v|^2\geqq 100|\log\varepsilon|}\,dv
\end{aligned}\tag{46}$$

for some $C_{\|w\|_\infty} > 0$ depending on $\|w\|_\infty$. Then, check that

$$\begin{aligned}
\int \exp\left(-\frac{1}{8}|v|^2\right)\mathbf{1}_{|v|^2\geqq 100|\log\varepsilon|}\,dv &= 4\pi\int_{\sqrt{100|\log\varepsilon|}}^{+\infty}\exp(-\frac{1}{8}r^2)r^2dr\\
&\leqq C\varepsilon^{100/8}|\log\varepsilon|^{1/2}
\end{aligned}$$

for some $C > 0$, and integrate (46) on $\mathbf{T}^3$ to get the expected estimate. □

Equipped with these preliminary results, we can now restrict our attention to the situation where we can use decomposition (37) to obtain further estimates on the L^1 part of $(F_\varepsilon - M_{(1,\varepsilon w,1)})$. We start by giving the precise form of the decomposition we will use.

Lemma 4 (Decomposition by $\mathcal{A}^+$)**.** *Consider a test divergence-free vector field w as in Theorem 6. Define the family F_ε as in Theorem 3. Then, if γ is a truncation function as in Theorem 7, the following estimate holds for some nonnegative constant C:*

$$\begin{aligned}
&\frac{1}{\varepsilon}|F_\varepsilon - M_\varepsilon|\\
&\quad\leqq C\mathcal{A}^+(\sqrt{M_\varepsilon}g_\varepsilon, M_\varepsilon) + C\sqrt{1+\frac{F_\varepsilon}{M_\varepsilon}}\mathcal{A}^+(\sqrt{M_\varepsilon}k_\varepsilon, M_\varepsilon)\\
&\qquad + C\varepsilon^3\mathcal{A}^+(k_\varepsilon^2, k_\varepsilon^2) + C\varepsilon^{(q+1)/2}\left(\frac{D(F_\varepsilon)}{\varepsilon^{q+3}} + F_\varepsilon\right)\\
&\qquad + C(F_\varepsilon + M_\varepsilon)\left(\frac{1}{\varepsilon^2}\int M_\varepsilon h\left(\frac{F_\varepsilon}{M_\varepsilon}-1\right)dv\right)^{1/2},
\end{aligned}\tag{47}$$

where M_ε is an abbreviation for $M_{(1,\varepsilon w,1)}$, and g_ε and k_ε are defined by

$$g_\varepsilon = \frac{|F_\varepsilon - M_\varepsilon|}{\varepsilon\sqrt{M_\varepsilon}}\gamma\left(\frac{F_\varepsilon}{M_\varepsilon}\right),\quad k_\varepsilon = \left(\frac{F_\varepsilon + M_\varepsilon}{\varepsilon^2}(1-\gamma)\left(\frac{F_\varepsilon}{M_\varepsilon}\right)\right)^{1/2}.\tag{48}$$

Proof. Define $\tilde F_\varepsilon$ by

$$\tilde F_\varepsilon = F_\varepsilon \gamma\left(\frac{F_\varepsilon}{M_\varepsilon}\right) + M_\varepsilon(1-\gamma)\left(\frac{F_\varepsilon}{M_\varepsilon}\right),$$

and $\tilde R_\varepsilon = \int \tilde F_\varepsilon \, dv$. By the definition of γ,

$$\tfrac{1}{2}M_\varepsilon \leqq \tilde F_\varepsilon \leqq \tfrac{3}{2}M_\varepsilon, \quad \tfrac{1}{2} \leqq \tilde R_\varepsilon \leqq \tfrac{3}{2}. \tag{49}$$

A slighty modified version of (37) gives

$$\begin{aligned} \tfrac{1}{\varepsilon}|F_\varepsilon - M_\varepsilon| &\leqq \tfrac{1}{\varepsilon}\left|F_\varepsilon - \tfrac{\mathcal{A}^+(\tilde F_\varepsilon, \tilde F_\varepsilon)}{\tilde R_\varepsilon}\right| + \tfrac{1}{\varepsilon}\left|\tfrac{\mathcal{A}^+(\tilde F_\varepsilon, \tilde F_\varepsilon)}{\tilde R_\varepsilon} - \mathcal{A}^+(M_\varepsilon, M_\varepsilon)\right| \\ &= J_1 + J_2. \end{aligned}$$

Then J_2 can be decomposed into two terms as follows:

$$J_2 \leqq \frac{1}{\tilde R_\varepsilon}\left|\mathcal{A}^+\left(\frac{\tilde F_\varepsilon - M_\varepsilon}{\varepsilon}, \tilde F_\varepsilon + M_\varepsilon\right)\right| + \frac{1}{\tilde R_\varepsilon}\left|\frac{\tilde R_\varepsilon - 1}{\varepsilon}\right| M_\varepsilon = J_2^1 + J_2^2.$$

From (49) we deduce that

$$J_2^1 \leqq 5\mathcal{A}^+\left(\frac{|F_\varepsilon - M_\varepsilon|}{\varepsilon}\gamma\left(\frac{F_\varepsilon}{M_\varepsilon}\right), M_\varepsilon\right) \leqq 5\mathcal{A}^+(\sqrt{M_\varepsilon} g_\varepsilon, M_\varepsilon). \tag{50}$$

On the other hand, by (35), there exists a nonnegative constant C such that

$$\begin{aligned} |\tilde R_\varepsilon - 1| &= \left|\int (F_\varepsilon - M_\varepsilon)\gamma\left(\frac{F_\varepsilon}{M_\varepsilon}\right) dv\right| \\ &\leqq \left(\int M_\varepsilon \, dv\right)^{1/2}\left(\int \frac{(F_\varepsilon - M_\varepsilon)^2}{M_\varepsilon}\gamma^2\left(\frac{F_\varepsilon}{M_\varepsilon}\right) dv\right)^{1/2} \\ &\leqq C\|\gamma\|_\infty \left(\int M_\varepsilon h\left(\frac{F_\varepsilon}{M_\varepsilon} - 1\right) dv\right)^{1/2}, \end{aligned}$$

from which we deduce that

$$J_2^2 \leqq 2C\|\gamma\|_\infty M_\varepsilon \left(\frac{1}{\varepsilon^2}\int M_\varepsilon h\left(\frac{F_\varepsilon}{M_\varepsilon} - 1\right) dv\right)^{1/2}. \tag{51}$$

As we have replaced F_ε by its truncation $\tilde F_\varepsilon$ inside the bilinear form $\mathcal{A}^+$, J_1 cannot be directly estimated by the entropy dissipation: we have to decompose it into several terms using the basic identity

$$\tilde F_\varepsilon = \gamma_\varepsilon F_\varepsilon + (1-\gamma_\varepsilon)M_\varepsilon,$$

where $\gamma_\varepsilon = \gamma(F_\varepsilon/M_\varepsilon)$. If we denote respectively by $\gamma_{\varepsilon 1}$, γ'_ε and $\gamma'_{\varepsilon 1}$ the values of γ_ε at points v_1, v' and v'_1,

$$\begin{aligned}
J_1 &= \frac{1}{\varepsilon \tilde{R}_\varepsilon} \left| F_\varepsilon \int \tilde{F}_{\varepsilon 1}\, dv_1 - \frac{1}{4\pi} \iint \tilde{F}'_\varepsilon \tilde{F}'_{\varepsilon 1}\, dv_1 d\sigma \right| \\
&= \frac{1}{4\pi \varepsilon \tilde{R}_\varepsilon} \left| \iint (F_\varepsilon \tilde{F}_{\varepsilon 1} - \tilde{F}'_\varepsilon \tilde{F}'_{\varepsilon 1})\, dv_1 d\sigma \right| \\
&\leqq \frac{1}{4\pi \varepsilon \tilde{R}_\varepsilon} \iint (1-\gamma_{\varepsilon 1}) |F_\varepsilon M_{\varepsilon 1} - \tilde{F}'_\varepsilon \tilde{F}'_{\varepsilon 1}|\, dv_1 d\sigma \\
&\quad + \frac{1}{4\pi \varepsilon \tilde{R}_\varepsilon} \iint \gamma_{\varepsilon 1} (1-\gamma'_\varepsilon)(1-\gamma'_{\varepsilon 1}) |F_\varepsilon F_{\varepsilon 1} - M'_\varepsilon M'_{\varepsilon 1}|\, dv_1 d\sigma \\
&\quad + \frac{1}{4\pi \varepsilon \tilde{R}_\varepsilon} \iint (1-\gamma'_\varepsilon) \gamma_{\varepsilon 1} \gamma'_{\varepsilon 1} |F_\varepsilon F_{\varepsilon 1} - M'_\varepsilon F'_{\varepsilon 1}|\, dv_1 d\sigma \\
&\quad + \frac{1}{4\pi \varepsilon \tilde{R}_\varepsilon} \iint (1-\gamma'_{\varepsilon 1}) \gamma_{\varepsilon 1} \gamma'_\varepsilon |F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon M'_{\varepsilon 1}|\, dv_1 d\sigma \\
&\quad + \frac{1}{4\pi \varepsilon \tilde{R}_\varepsilon} \iint \gamma'_\varepsilon \gamma_{\varepsilon 1} \gamma'_{\varepsilon 1} |F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|\, dv_1 d\sigma \\
&= J_1^1 + J_1^2 + J_1^3 + J_1^4 + J_1^5.
\end{aligned}$$

By (49) and the identity $M'_\varepsilon M'_{\varepsilon 1} = M_\varepsilon M_{\varepsilon 1}$,

$$\begin{aligned}
J_1^1 &\leqq \frac{1}{2\pi \varepsilon} \iint (1-\gamma_{\varepsilon 1})(F_\varepsilon M_{\varepsilon 1} + \tilde{F}'_\varepsilon \tilde{F}'_{\varepsilon 1})\, dv_1 d\sigma \\
&\leqq \frac{1}{2\pi \varepsilon} \iint (1-\gamma_{\varepsilon 1}) \left(F_\varepsilon M_{\varepsilon 1} + \tfrac{9}{4} M'_\varepsilon M'_{\varepsilon 1} \right) dv_1 d\sigma \\
&\leqq \frac{9}{8\pi} (F_\varepsilon + M_\varepsilon) \iint \frac{(1-\gamma_{\varepsilon 1})}{\varepsilon} M_{\varepsilon 1}\, dv_1 d\sigma.
\end{aligned}$$

As the support of $(1-\gamma)$ is a subset of $[0, \frac{3}{4}] \cup [\frac{5}{4}, +\infty[$, by (35), there exists a nonnegative constant C such that

$$\begin{aligned}
&\int M_\varepsilon (1-\gamma) \left(\frac{F_\varepsilon}{M_\varepsilon} \right) dv \\
&\leqq \|1-\gamma\|_\infty^{1/2} \left(\int M_\varepsilon (1-\gamma) \left(\frac{F_\varepsilon}{M_\varepsilon} \right) dv \right)^{1/2} \left(\int M_\varepsilon\, dv \right)^{1/2} \\
&\leqq \|1-\gamma\|_\infty \left(C \int M_\varepsilon h \left(\frac{F_\varepsilon}{M_\varepsilon} - 1 \right) dv \right)^{1/2}.
\end{aligned}$$

Since $\|1-\gamma\|_\infty = 1$,

$$J_1^1 \leqq C (F_\varepsilon + M_\varepsilon) \left(\frac{1}{\varepsilon^2} \int M_\varepsilon h \left(\frac{F_\varepsilon}{M_\varepsilon} - 1 \right) dv \right)^{1/2}. \tag{52}$$

In order to estimate the other terms in J_1, we will combine the control on the entropy dissipation with bounds on k_ε. For instance, by (49),

$$\begin{aligned}
J_1^2 &\leqq \frac{1}{2\pi\varepsilon}\iint \gamma_{\varepsilon 1}(1-\gamma'_\varepsilon)(1-\gamma'_{\varepsilon 1})|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|\,dv_1 d\sigma\\
&\quad + \frac{1}{2\pi\varepsilon}\iint \gamma_{\varepsilon 1}(1-\gamma'_\varepsilon)(1-\gamma'_{\varepsilon 1})|F'_\varepsilon F'_{\varepsilon 1} - M'_\varepsilon M'_{\varepsilon 1}|\,dv_1 d\sigma\\
&\leqq \frac{1}{2\pi\varepsilon}\iint \gamma_{\varepsilon 1}(1-\gamma'_\varepsilon)(1-\gamma'_{\varepsilon 1})|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|\,dv_1 d\sigma\\
&\quad + \frac{\|\gamma\|_\infty}{2\pi\varepsilon}\iint (1-\gamma'_\varepsilon)(1-\gamma'_{\varepsilon 1})(F'_\varepsilon + M'_\varepsilon)(F'_{\varepsilon 1} + M'_{\varepsilon 1})\,dv_1 d\sigma,
\end{aligned}$$

which, coupled with the definitions (38) and (48) of $\mathcal{A}^+$ and k_ε, and the assumption $\gamma(\mathbf{R}^+) \subset [0, 1]$, implies

$$J_1^2 \leqq \frac{1}{2\pi\varepsilon}\iint \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|\,dv_1 d\sigma + 2\varepsilon^3\mathcal{A}^+(k_\varepsilon^2, k_\varepsilon^2). \tag{53}$$

Elementary computations give

$$\begin{aligned}
&\gamma_{\varepsilon 1}\gamma'_{\varepsilon 1}(1-\gamma'_\varepsilon)|F_\varepsilon F_{\varepsilon 1} - M'_\varepsilon F'_{\varepsilon 1}|\\
&\quad\leqq \gamma_{\varepsilon 1}\gamma'_{\varepsilon 1}(1-\gamma'_\varepsilon)\inf\big(|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}| + |F'_\varepsilon F'_{\varepsilon 1} - M'_\varepsilon F'_{\varepsilon 1}|, |F_\varepsilon F_{\varepsilon 1} - M'_\varepsilon F'_{\varepsilon 1}|\big)\\
&\quad\leqq \gamma_{\varepsilon 1}\gamma'_{\varepsilon 1}(1-\gamma'_\varepsilon)\inf\big(|F'_\varepsilon F'_{\varepsilon 1} - M'_\varepsilon F'_{\varepsilon 1}|, |F_\varepsilon F_{\varepsilon 1} - M'_\varepsilon F'_{\varepsilon 1}|\big)\\
&\qquad + \gamma_{\varepsilon 1}\gamma'_{\varepsilon 1}(1-\gamma'_\varepsilon)|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|\\
&\quad\leqq \gamma_{\varepsilon 1}\gamma'_{\varepsilon 1}(1-\gamma'_\varepsilon)|F'_\varepsilon F'_{\varepsilon 1} - M'_\varepsilon F'_{\varepsilon 1}|^{1/2}|F_\varepsilon F_{\varepsilon 1} - M'_\varepsilon F'_{\varepsilon 1}|^{1/2}\\
&\qquad + \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|\\
&\quad\leqq \gamma_{\varepsilon 1}\gamma'_{\varepsilon 1}(1-\gamma'_\varepsilon)(F'_\varepsilon F'_{\varepsilon 1} + M'_\varepsilon F'_{\varepsilon 1})^{1/2}(F_\varepsilon F_{\varepsilon 1} + M'_\varepsilon F'_{\varepsilon 1})^{1/2}\\
&\qquad + \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|\\
&\quad\leqq (1-\gamma'_\varepsilon)(F'_\varepsilon F'_{\varepsilon 1}\gamma'_{\varepsilon 1}\gamma_{\varepsilon 1} + M'_\varepsilon F'_{\varepsilon 1}\gamma'_{\varepsilon 1}\gamma_{\varepsilon 1})^{1/2}(F_\varepsilon F_{\varepsilon 1}\gamma_{\varepsilon 1}\gamma'_{\varepsilon 1} + M'_\varepsilon F'_{\varepsilon 1}\gamma'_{\varepsilon 1}\gamma_{\varepsilon 1})^{1/2}\\
&\qquad + \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|.
\end{aligned}$$

Because of the properties of the support of γ,

$$\gamma_\varepsilon F_\varepsilon \leqq \tfrac{3}{2}M_\varepsilon.$$

Then, by the identity $M'_\varepsilon M'_{\varepsilon 1} = M_\varepsilon M_{\varepsilon 1}$ and the definition (48) of k_ε,

$$\begin{aligned}
&\gamma_{\varepsilon 1}\gamma'_{\varepsilon 1}(1-\gamma'_\varepsilon)|F_\varepsilon F_{\varepsilon 1} - M'_\varepsilon F'_{\varepsilon 1}| \\
&\quad \leqq (1-\gamma'_\varepsilon)\left(\tfrac{3}{2}F'_\varepsilon M'_{\varepsilon 1} + \tfrac{3}{2}M'_\varepsilon M'_{\varepsilon 1}\right)^{1/2}\left(\tfrac{3}{2}F_\varepsilon M_{\varepsilon 1} + \tfrac{3}{2}M'_\varepsilon M'_{\varepsilon 1}\right)^{1/2} \\
&\qquad + \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}| \\
&\quad \leqq \frac{3}{2}(1-\gamma'_\varepsilon)(F'_\varepsilon M'_{\varepsilon 1} + M'_\varepsilon M'_{\varepsilon 1})^{1/2}(F_\varepsilon M_{\varepsilon 1} + M_\varepsilon M_{\varepsilon 1})^{1/2} \\
&\qquad + \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}| \\
&\quad \leqq \frac{3}{2}(1-\gamma'_\varepsilon)(M'_\varepsilon M'_{\varepsilon 1} M_\varepsilon M_{\varepsilon 1})^{1/2}\left(\frac{F'_\varepsilon}{M'_\varepsilon}+1\right)^{1/2}\left(\frac{F_\varepsilon}{M_\varepsilon}+1\right)^{1/2} \\
&\qquad + \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}| \\
&\quad \leqq \frac{3}{2}(1-\gamma'_\varepsilon)^{1/2} M'_\varepsilon M'_{\varepsilon 1}(1-\gamma'_\varepsilon)^{1/2}\left(\frac{F'_\varepsilon}{M'_\varepsilon}+1\right)^{1/2}\left(\frac{F_\varepsilon}{M_\varepsilon}+1\right)^{1/2} \\
&\qquad + \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}| \\
&\quad \leqq \frac{3}{2}(M'_\varepsilon)^{1/2}\|1-\gamma\|_\infty^{1/2}\varepsilon k'_\varepsilon M'_{\varepsilon 1}\left(\frac{F_\varepsilon}{M_\varepsilon}+1\right)^{1/2} + \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|,
\end{aligned} \tag{54}$$

and in the same way

$$\begin{aligned}
&\gamma_{\varepsilon 1}\gamma'_{\varepsilon 1}(1-\gamma'_{\varepsilon 1})|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon M'_{\varepsilon 1}| \\
&\quad \leqq \tfrac{3}{2}(M'_{\varepsilon 1})^{1/2}\|1-\gamma\|_\infty^{1/2}\varepsilon k'_{\varepsilon 1} M'_\varepsilon\left(\tfrac{F_\varepsilon}{M_\varepsilon}+1\right)^{1/2} \\
&\qquad + \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|.
\end{aligned} \tag{55}$$

Then, by (54), (55) and the definition of $\mathcal{A}^+$,

$$J_1^3 + J_1^4 \leqq \frac{1}{\pi\varepsilon}\iint \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|\,dv_1 d\sigma + 6\sqrt{1+\frac{F_\varepsilon}{M_\varepsilon}}\mathcal{A}^+(M_\varepsilon^{1/2}k_\varepsilon, M_\varepsilon). \tag{56}$$

It remains to estimate the dissipation terms. Apply Young's inequality (91) with $z = (F'_\varepsilon F'_{\varepsilon 1} - F_\varepsilon F_{\varepsilon 1})/F_\varepsilon F_{\varepsilon 1}$, $p = 1$, and $\eta = \varepsilon^{(q+3)/2}$ and use (H3) to get

$$\begin{aligned}
&\iint \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|\,dv_1 d\sigma \\
&\quad \leqq \frac{4k_b}{\varepsilon^{(q+3)/2}}D(F_\varepsilon) + \varepsilon^{(q+3)/2}h^*(1)F_\varepsilon\iint F_{\varepsilon 1}\gamma_{\varepsilon 1}\,dv_1 d\sigma.
\end{aligned}$$

Thus

$$\frac{1}{\varepsilon}\iint \gamma_{\varepsilon 1}|F_\varepsilon F_{\varepsilon 1} - F'_\varepsilon F'_{\varepsilon 1}|\,dv_1 d\sigma \leqq \varepsilon^{(q+1)/2}\left(4k_b\frac{D(F_\varepsilon)}{\varepsilon^{q+3}} + 6\pi F_\varepsilon\right). \tag{57}$$

Combining (50)–(53), and (56),(57) leads to the decomposition (47). □

Applying Caflisch estimates [7] (see also Appendix A) to the previous decomposition leads to

Lemma 5 (Decay estimates)**.** *Consider a test divergence-free vector field w as in Theorem 6. Define the family F_ε as in Theorem 3. Then, if γ is a truncation function as in Theorem 7, there exists $C > 0$ such that the following bounds hold:*

$$\left\| \frac{(F_\varepsilon - M_{(1,\varepsilon w,1)})^2}{\varepsilon^2 M_{(1,\varepsilon w,1)}} \gamma^2 \left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}} \right) |v|^2 \right\|_{L^1(dv\,dx)} \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon / M_{(1,\varepsilon w,1)}) + r_\varepsilon \tag{58}$$

$$\left\| \frac{F_\varepsilon + M_{(1,\varepsilon w,1)}}{\varepsilon^2} (1-\gamma) \left(\frac{F_\varepsilon}{M_{(1,\varepsilon w,1)}} \right) |v|^2 \right\|_{L^1(dv\,dx)} \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon / M_{(1,\varepsilon w,1)}) + r_\varepsilon \tag{59}$$

where r_ε converges to 0 in $L^1([0,T])$ as $\varepsilon \to 0$.

Proof. Lemma 5 gives bounds on the second moments of the functions g_ε^2 and k_ε^2 defined by (48).

The first step of the proof consists in applying decomposition (47) to estimate both functions. Recall that the definition of γ, coupled with the elementary inequality (36), implies

$$\begin{aligned} &\tfrac{1}{2} M_\varepsilon \gamma_\varepsilon \leqq F_\varepsilon \gamma_\varepsilon \leqq \tfrac{3}{2} M_\varepsilon \gamma_\varepsilon \quad \text{thus} \quad |g_\varepsilon \gamma_\varepsilon| \leqq \tfrac{\sqrt{M_\varepsilon}}{2\varepsilon} \text{ and } \left(1 + \tfrac{F_\varepsilon}{M_\varepsilon}\right) \gamma_\varepsilon \leqq \tfrac{5}{2}, \\ &(F_\varepsilon + M_\varepsilon)(1-\gamma_\varepsilon) \leqq 9|F_\varepsilon - M_\varepsilon|(1-\gamma_\varepsilon) \leqq 9|F_\varepsilon - M_\varepsilon|(1-\gamma_\varepsilon)^{1/2}. \end{aligned} \tag{60}$$

Then, multiplying (47) by $\frac{|F_\varepsilon - M_\varepsilon|}{\varepsilon M_\varepsilon} \gamma_\varepsilon^2 = \frac{g_\varepsilon}{\sqrt{M_\varepsilon}} \gamma_\varepsilon$ leads to the estimate

$$\begin{aligned} g_\varepsilon^2 \leqq{}& C M_\varepsilon^{-1/2} \mathcal{A}^+(\sqrt{M_\varepsilon} g_\varepsilon, M_\varepsilon) g_\varepsilon + \sqrt{\frac{5}{2}} C M_\varepsilon^{-1/2} \mathcal{A}^+(\sqrt{M_\varepsilon} k_\varepsilon, M_\varepsilon) g_\varepsilon \\ &+ \frac{1}{2\varepsilon} \left(C\varepsilon^3 \mathcal{A}^+(k_\varepsilon^2, k_\varepsilon^2) + C\varepsilon^{(q+1)/2} \left(\frac{D(F_\varepsilon)}{\varepsilon^{q+3}} + F_\varepsilon \right) \right) \\ &+ \frac{5}{2} C M_\varepsilon^{1/2} g_\varepsilon \left(\frac{1}{\varepsilon^2} \int M_\varepsilon h \left(\frac{F_\varepsilon}{M_\varepsilon} - 1 \right) dv \right)^{1/2} \end{aligned} \tag{61}$$

for some $C > 0$, which can be rewritten

$$g_\varepsilon^2 \leqq X g_\varepsilon + Y,$$

with

$$\begin{aligned} X ={}& C M_\varepsilon^{-1/2} \mathcal{A}^+ \left(\sqrt{M_\varepsilon} g_\varepsilon, M_\varepsilon \right) + C \sqrt{\frac{5}{2}} M_\varepsilon^{-1/2} \mathcal{A}^+ \left(\sqrt{M_\varepsilon} k_\varepsilon, M_\varepsilon \right) \\ &+ \frac{5}{2} C M_\varepsilon^{1/2} \left(\frac{1}{\varepsilon^2} \int M_\varepsilon h \left(\frac{F_\varepsilon}{M_\varepsilon} - 1 \right) dv \right)^{1/2}, \\ Y ={}& \frac{1}{2} \left(C\varepsilon^2 \mathcal{A}^+(k_\varepsilon^2, k_\varepsilon^2) + C\varepsilon^{(q-1)/2} \left(\frac{D(F_\varepsilon)}{\varepsilon^{q+3}} + F_\varepsilon \right) \right). \end{aligned}$$

Then, from the trivial inequality

$$Xg_\varepsilon \leqq \tfrac{1}{2}X^2 + \tfrac{1}{2}g_\varepsilon^2,$$

we deduce that

$$g_\varepsilon^2 \leqq X^2 + 2Y,$$

and we get

$$\begin{aligned} g_\varepsilon^2 \leqq\ & C^2 M_\varepsilon^{-1}\mathcal{A}^+(\sqrt{M_\varepsilon}g_\varepsilon, M_\varepsilon)^2 + \tfrac{5}{2}C^2 M_\varepsilon^{-1}\mathcal{A}^+(\sqrt{M_\varepsilon}k_\varepsilon, M_\varepsilon)^2 \\ & + \tfrac{25}{4}C^2 M_\varepsilon \left(\tfrac{1}{\varepsilon^2}\!\int M_\varepsilon h\left(\tfrac{F_\varepsilon}{M_\varepsilon} - 1\right) dv\right) \\ & + C\varepsilon^2\mathcal{A}^+(k_\varepsilon^2, k_\varepsilon^2) + C\varepsilon^{(q-1)/2}\left(\frac{D(F_\varepsilon)}{\varepsilon^{q+3}} + F_\varepsilon\right). \end{aligned} \tag{62}$$

In the same way, multiplying (47) by $\frac{(1-\gamma_\varepsilon)^{1/2}}{\varepsilon}$ and using (60), we obtain

$$\begin{aligned} k_\varepsilon^2 \leqq\ & \frac{9}{\varepsilon^2}|F_\varepsilon - M_\varepsilon|(1-\gamma_\varepsilon)^{1/2} \\ \leqq\ & 9C\mathcal{A}^+\left(\sqrt{M_\varepsilon}g_\varepsilon, M_\varepsilon\right)\tfrac{(1-\gamma_\varepsilon)^{1/2}}{\varepsilon} + 9CM_\varepsilon^{-1/2}k_\varepsilon\mathcal{A}^+\left(\sqrt{M_\varepsilon}k_\varepsilon, M_\varepsilon\right) \\ & + 9C\varepsilon^2(1-\gamma_\varepsilon)^{1/2}\mathcal{A}^+(k_\varepsilon^2, k_\varepsilon^2) + 9C\varepsilon^{(q-1)/2}(1-\gamma_\varepsilon)^{1/2}\left(\frac{D(F_\varepsilon)}{\varepsilon^{q+3}} + F_\varepsilon\right) \\ & + 9C(F_\varepsilon + M_\varepsilon)^{1/2}k_\varepsilon\left(\frac{1}{\varepsilon^2}\int M_\varepsilon h\left(\frac{F_\varepsilon}{M_\varepsilon} - 1\right) dv\right)^{1/2}. \end{aligned}$$

Remarking that

$$\frac{1}{\varepsilon}M_\varepsilon^{1/2}(1-\gamma_\varepsilon)^{1/2} \leqq k_\varepsilon,$$

we obtain

$$k_\varepsilon^2 \leqq Xk_\varepsilon + Y,$$

with

$$\begin{aligned} X =\ & 9CM_\varepsilon^{-1/2}\mathcal{A}^+\left(\sqrt{M_\varepsilon}g_\varepsilon, M_\varepsilon\right) + 9CM_\varepsilon^{-1/2}\mathcal{A}^+\left(\sqrt{M_\varepsilon}k_\varepsilon, M_\varepsilon\right) \\ & + 9C(F_\varepsilon + M_\varepsilon)^{1/2}\left(\frac{1}{\varepsilon^2}\int M_\varepsilon h\left(\frac{F_\varepsilon}{M_\varepsilon} - 1\right) dv\right)^{1/2}, \\ Y =\ & 9C\varepsilon^2\mathcal{A}^+(k_\varepsilon^2, k_\varepsilon^2) + 9C\varepsilon^{(q-1)/2}\left(\frac{D(F_\varepsilon)}{\varepsilon^{q+3}} + F_\varepsilon\right). \end{aligned}$$

Then,

$$\begin{aligned} k_\varepsilon^2 \leqq\ & X^2 + 2Y \\ \leqq\ & 81C^2M_\varepsilon^{-1}\mathcal{A}^+\left(\sqrt{M_\varepsilon}g_\varepsilon, M_\varepsilon\right)^2 + 81C^2M_\varepsilon^{-1}\mathcal{A}^+\left(\sqrt{M_\varepsilon}k_\varepsilon, M_\varepsilon\right)^2 \\ & + 81C^2(F_\varepsilon + M_\varepsilon)\left(\frac{1}{\varepsilon^2}\int M_\varepsilon h\left(\frac{F_\varepsilon}{M_\varepsilon} - 1\right) dv\right) \\ & + 18C\varepsilon^2\mathcal{A}^+(k_\varepsilon^2, k_\varepsilon^2) + 18C\varepsilon^{(q-1)/2}\left(\frac{D(F_\varepsilon)}{\varepsilon^{q+3}} + F_\varepsilon\right). \end{aligned} \tag{63}$$

The second step is to obtain estimates on the terms in (62) and (63) which are not in the form $M_\varepsilon^{-1/2}\mathcal{A}^+(\sqrt{M_\varepsilon}..., M_\varepsilon)$. By (41), (42), and the definition of χ_ε,

$$\begin{aligned}
&\chi_\varepsilon \int (F_\varepsilon + M_\varepsilon)(1+|v|^2)\,dv \\
&\leqq \chi_\varepsilon \int (|F_\varepsilon - M_\varepsilon| + 2M_\varepsilon)(1+|v|^2)\,dv \\
&\leqq 8\chi_\varepsilon \int M_\varepsilon h\left(\frac{F_\varepsilon}{M_\varepsilon} - 1\right) dv + \int M_\varepsilon (8e^{(1+|v|^2)/8} + 2(1+|v|^2))\,dv \\
&\leqq 8 + C_{\|w\|_\infty}
\end{aligned} \tag{64}$$

for some $C_{\|w\|_\infty} > 0$ depending on $\|w\|_\infty$. By definition of $\mathcal{A}^+$, and because $dv\,dv_1 d\sigma = dv' dv_1' d\sigma$,

$$\begin{aligned}
&\|\mathcal{A}^+(k_\varepsilon^2, k_\varepsilon^2)(1+|v|^2)\|_{L^1_v} \\
&\leqq \frac{1}{4\pi} \iiint (k_\varepsilon')^2 (k_{\varepsilon 1}')^2 (1 + |v'|^2 + |v_1'|^2)\,dv' dv_1' d\sigma \\
&\leqq 2\|k_\varepsilon^2\|_{L^1_v} \|k_\varepsilon^2 (1+|v|^2)\|_{L^1_v},
\end{aligned}$$

from which we deduce by (48), Lemma 1 and (64), that

$$\begin{aligned}
&\left\|\varepsilon^2 \chi_\varepsilon \mathcal{A}^+(k_\varepsilon^2, k_\varepsilon^2)|v|^2\right\|_{L^1_{x,v}} \\
&\leqq 2\left\|k_\varepsilon^2\right\|_{L^1_{x,v}} \left\|\varepsilon^2 \chi_\varepsilon k_\varepsilon^2 (1+|v|^2)\right\|_{L^\infty_x(L^1_v)} \\
&\leqq 2\left\|k_\varepsilon^2\right\|_{L^1_{x,v}} \left\|\chi_\varepsilon (F_\varepsilon + M_\varepsilon)(1+|v|^2)\right\|_{L^\infty_x(L^1_v)} \\
&\leqq \frac{C}{\varepsilon^2} H(F_\varepsilon | M_\varepsilon)(8 + C_{\|w\|_\infty}).
\end{aligned} \tag{65}$$

In the same way, by (64),

$$\begin{aligned}
&\left\|\chi_\varepsilon (F_\varepsilon + M_\varepsilon)(1+|v|^2)\left(\frac{1}{\varepsilon^2}\int M_\varepsilon h(\frac{F_\varepsilon}{M_\varepsilon} - 1)\,dv\right)\right\|_{L^1_{x,v}} \\
&\leqq \|\chi_\varepsilon (F_\varepsilon + M_\varepsilon)(1+|v|^2)\|_{L^\infty_x(L^1_v)} \left\|\frac{1}{\varepsilon^2}\int M_\varepsilon h(\frac{F_\varepsilon}{M_\varepsilon} - 1)\,dv\right\|_{L^1_x} \\
&\leqq \frac{(8 + C_{\|w\|_\infty})}{\varepsilon^2} H(F_\varepsilon | M_\varepsilon).
\end{aligned} \tag{66}$$

On the other hand, the dissipation bound and the global energy inequality provide

$$\begin{aligned}
&\left\|\varepsilon^{(q-1)/2} \chi_\varepsilon \mathbf{1}_{|v|^2 \leqq 100|\log\varepsilon|} \left(\tfrac{D(F_\varepsilon)}{\varepsilon^{q+3}} + F_\varepsilon\right)(1+|v|^2)\right\|_{L^1([0,T], L^1_{x,v})} \\
&\leqq C\varepsilon^{(q-1)/2} |\log\varepsilon|.
\end{aligned} \tag{67}$$

Replacing (65), (66) and (67) in (62) and (63) leads to

$$g_\varepsilon^2 + k_\varepsilon^2 \leqq C M_\varepsilon^{-1} \mathcal{A}^+ \left(\sqrt{M_\varepsilon}\sqrt{g_\varepsilon^2 + k_\varepsilon^2}, M_\varepsilon\right)^2 + I,$$

$$\left\| \chi_\varepsilon \mathbf{1}_{|v|^2 \leqq 100|\log\varepsilon|} I(1+|v|^2) \right\|_{L^1_{x,v}} \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon|M_\varepsilon) + r_\varepsilon \ ,$$

for some $C > 0$ depending only on $\|w\|_\infty$, where $r_\varepsilon \to 0$ in $L^1([0,T])$.

In order to get rid of the truncation in the previous estimate of I, we now use the preliminary results giving some control on very large velocities and on the macroscopic truncation χ_ε. Using the definition (48) of g_ε and k_ε, and the pointwise estimate (49) coming from the properties of γ, we get

$$\begin{aligned} g_\varepsilon^2 + k_\varepsilon^2 &\leqq \frac{1\sqrt{M_\varepsilon}}{2\varepsilon} \frac{|F_\varepsilon - M_\varepsilon|}{\varepsilon\sqrt{M_\varepsilon}} \gamma_\varepsilon + \frac{F_\varepsilon + M_\varepsilon}{\varepsilon^2}(1-\gamma_\varepsilon) \\ &\leqq \frac{1}{\varepsilon^2}(F_\varepsilon + M_\varepsilon)\gamma_\varepsilon + \frac{1}{\varepsilon^2}(F_\varepsilon + M_\varepsilon)(1-\gamma_\varepsilon) \\ &\leqq \frac{1}{\varepsilon^2}(F_\varepsilon + M_\varepsilon). \end{aligned}$$

Then, by Lemmas 2 and 3, we have

$$\begin{aligned} &\left\| (1 - \chi_\varepsilon \mathbf{1}_{|v|^2 \leqq 100|\log\varepsilon|})(g_\varepsilon^2 + k_\varepsilon^2)(1+|v|^2) \right\|_{L^1_{x,v}} \\ &\quad \leqq \left\| \tfrac{1}{\varepsilon^2}(1 - \chi_\varepsilon \mathbf{1}_{|v|^2 \leqq 100|\log\varepsilon|})(F_\varepsilon + M_\varepsilon)(1+|v|^2) \right\|_{L^1_{x,v}} \\ &\quad \leqq \left\| \tfrac{1}{\varepsilon^2} \mathbf{1}_{|v|^2 \geqq 100|\log\varepsilon|}(F_\varepsilon + M_\varepsilon)(1+|v|^2) \right\|_{L^1_{x,v}} \\ &\qquad + \left\| \tfrac{1}{\varepsilon^2}(1-\chi_\varepsilon)(F_\varepsilon + M_\varepsilon)(1+|v|^2) \right\|_{L^1_{x,v}} \\ &\quad \leqq \tfrac{C}{\varepsilon^2} H(F_\varepsilon|M_\varepsilon) + o(1). \end{aligned}$$

Finally, from the trivial decomposition

$$\begin{aligned} g_\varepsilon^2 + k_\varepsilon^2 &= (1 - \chi_\varepsilon \mathbf{1}_{|v|^2 \leqq 100|\log\varepsilon|})(g_\varepsilon^2 + k_\varepsilon^2) + \chi_\varepsilon \mathbf{1}_{|v|^2 \leqq 100|\log\varepsilon|}(g_\varepsilon^2 + k_\varepsilon^2) \\ &\leqq (1 - \chi_\varepsilon \mathbf{1}_{|v|^2 \leqq 100|\log\varepsilon|})(g_\varepsilon^2 + k_\varepsilon^2) \\ &\quad + \chi_\varepsilon \mathbf{1}_{|v|^2 \leqq 100|\log\varepsilon|} \left(C M_\varepsilon^{-1} \mathcal{A}^+ \left(\sqrt{M_\varepsilon}\sqrt{g_\varepsilon^2 + k_\varepsilon^2}, M_\varepsilon\right)^2 + I \right) \\ &\leqq C M_\varepsilon^{-1} \mathcal{A}^+ \left(\sqrt{M_\varepsilon}\sqrt{g_\varepsilon^2 + k_\varepsilon^2}, M_\varepsilon\right)^2 \\ &\quad + (1 - \chi_\varepsilon \mathbf{1}_{|v|^2 \leqq 100|\log\varepsilon|})(g_\varepsilon^2 + k_\varepsilon^2) + \chi_\varepsilon \mathbf{1}_{|v|^2 \leqq 100|\log\varepsilon|} I, \end{aligned}$$

we deduce that

$$g_\varepsilon^2 + k_\varepsilon^2 \leqq C M_\varepsilon^{-1} \mathcal{A}^+ \left(\sqrt{M_\varepsilon}\sqrt{g_\varepsilon^2 + k_\varepsilon^2}, M_\varepsilon\right)^2 + I_2 \ ,$$

$$\left\| I_2(1+|v|^2) \right\|_{L^1_{x,v}} \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon|M_\varepsilon) + r_\varepsilon \ , \tag{68}$$

where $r_\varepsilon \to 0$ in $L^1([0,T])$.

The third step of the proof uses in a crucial way the Caflisch estimates (39) on the operator $g \mapsto M_\varepsilon^{-1/2}\mathcal{A}^+\left(\sqrt{M_\varepsilon}g, M_\varepsilon\right)$. By (68),

$$g_\varepsilon^2 + k_\varepsilon^2 \leqq I_1 + I_2 \ ,$$

with

$$I_1 = CM_\varepsilon^{-1}\mathcal{A}^+(\sqrt{M_\varepsilon}\sqrt{g_\varepsilon^2 + k_\varepsilon^2}, M_\varepsilon)^2,$$

and

$$\left\| I_2(1+|v|^2)\right\|_{L^1_{x,v}} \leqq \frac{C}{\varepsilon^2}H(F_\varepsilon|M_\varepsilon) + r_\varepsilon \ .$$

By Lemma 1, for all $t \in [0, T]$,

$$\|g_\varepsilon\|^2_{L^2_{x,v}} \leqq \frac{C}{\varepsilon^2}H(F_\varepsilon|M_\varepsilon), \quad \|k_\varepsilon\|^2_{L^2_{x,v}} \leqq \frac{C}{\varepsilon^2}H(F_\varepsilon|M_\varepsilon).$$

Then, by (39),

$$\|M_\varepsilon^{-1}\mathcal{A}^+(\sqrt{M_\varepsilon}\sqrt{g_\varepsilon^2 + k_\varepsilon^2}, M_\varepsilon)^2(1+|v|^3)\|_{L^1_x(L^\infty_v)} \leqq \frac{C_{\|w\|_\infty}}{\varepsilon^2}H(F_\varepsilon|M_\varepsilon).$$

Finally this first iteration leads to

$$\begin{aligned} g_\varepsilon^2 + k_\varepsilon^2 &\leqq I_1 + I_2,\\ \left\| I_1(1+|v|^3)\right\|_{L^1_x(L^\infty_v)} &\leqq \frac{C_{\|w\|_\infty}}{\varepsilon^2}H(F_\varepsilon|M_\varepsilon),\\ \left\| I_2(1+|v|^2)\right\|_{L^1_{x,v}} &\leqq \frac{C}{\varepsilon^2}H(F_\varepsilon|M_\varepsilon) + r_\varepsilon. \end{aligned} \tag{69}$$

The idea is then to iterate $\mathcal{A}^+$ on I_1, the piece in $L^1_x(L^\infty_v)$, to improve its decay in v by the Caflisch estimate, while the stability in $L^1_{x,v}$ – implied by Proposition 2 (in Appendix A) – controls the other piece. From (87) we deduce that

$$\left\| M_\varepsilon^{-1}\,\mathcal{A}^+(\sqrt{M_\varepsilon}\sqrt{g}, M_\varepsilon)^2(1+|v|^2)\right\|_{L^1_v} \leqq C_{\|w\|_\infty}\|g(1+|v|^2)\|_{L^1_v} \tag{70}$$

for some $C_{\|w\|_\infty} > 0$ depending on $\|w\|_\infty$. Then, by (69) and (39), (70),

$$\begin{aligned} I_1 &= CM_\varepsilon^{-1}\mathcal{A}^+(\sqrt{M_\varepsilon}\sqrt{g_\varepsilon^2 + k_\varepsilon^2}, M_\varepsilon)^2\\ &\leqq 2CM_\varepsilon^{-1}\mathcal{A}^+(\sqrt{M_\varepsilon}\sqrt{I_1}, M_\varepsilon)^2 + 2CM_\varepsilon^{-1}\mathcal{A}^+(\sqrt{M_\varepsilon}\sqrt{I_2}, M_\varepsilon)^2\\ &= I_1^1 + I_1^2, \end{aligned}$$

with

$$\begin{aligned} \|I_1^2(1+|v|^2)\|_{L^1_{x,v}} &\leqq C_{\|w\|_\infty}\|I_2(1+|v|^2)\|_{L^1_{x,v}}\\ &\leqq \tfrac{C_{\|w\|_\infty}}{\varepsilon^2}H(F_\varepsilon|M_\varepsilon) + r_\varepsilon,\\ \|I_1^1(1+|v|^7)\|_{L^1_x(L^\infty_v)} &\leqq C_{\|w\|_\infty}\left\|I_1(1+|v|^3)\right\|_{L^1_x(L^\infty_v)}\\ &\leqq \tfrac{C_{\|w\|_\infty}}{\varepsilon^2}H(F_\varepsilon|M_\varepsilon), \end{aligned} \tag{71}$$

and $r_\varepsilon \to 0$ in $L^1([0, T])$.

From (69), (71) and the decomposition

$$g_\varepsilon^2 + k_\varepsilon^2 \leqq I_1 + I_2 \leqq I_1^1 + I_1^2 + I_2 \ ,$$

we deduce the bounds (58) and (59). □

4. Estimate of the flux term

Equipped with the previous *a priori* estimates, we can now perform the

Proof of Theorem 6. Define F_ε^w, f_ε^w and γ_ε^w by

$$F_\varepsilon^w(t,x,v) = F_\varepsilon(t,x,v+\varepsilon w), \quad f_\varepsilon^w = \frac{F_\varepsilon^w - M}{\varepsilon M}, \quad \gamma_\varepsilon^w = \gamma\left(\frac{F_\varepsilon^w}{M}\right).$$

With this notation, Theorem 7 can be recast as

$$\begin{aligned} \|f_\varepsilon^w \gamma_\varepsilon^w\|^2_{L^2(dxM(1+v^2)\,dv)} &\leqq \tfrac{C}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}) + r_\varepsilon, \\ \|\tfrac{1}{\varepsilon} f_\varepsilon^w (1-\gamma_\varepsilon^w)\|_{L^1(dxM(1+v^2)\,dv)} &\leqq \tfrac{C}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}) + r_\varepsilon, \end{aligned} \tag{72}$$

with $r_\varepsilon \to 0$ in $L^1([0,T])$. Following Golse & Levermore (see [12]), we also define

$$N_\varepsilon^w = 1 + \frac{\varepsilon}{3} f_\varepsilon^w,$$

so that

$$N_\varepsilon^w \geqq \frac{2}{3}, \quad N_\varepsilon^w \geqq \frac{1}{3}\frac{F_\varepsilon^w}{M}. \tag{73}$$

Then the fluctuation f_ε^w can be decomposed as

$$f_\varepsilon^w = \frac{f_\varepsilon^w}{N_\varepsilon^w} + \frac{\varepsilon}{3}\frac{(f_\varepsilon^w)^2}{N_\varepsilon^w} \tag{74}$$

with

$$\begin{aligned} \frac{2}{3}\left(\frac{f_\varepsilon^w}{N_\varepsilon^w}\right)^2 \leqq \frac{(f_\varepsilon^w)^2}{N_\varepsilon^w} &= \frac{1}{N_\varepsilon^w}(f_\varepsilon^w\gamma_\varepsilon^w)^2 + \frac{\varepsilon|f_\varepsilon^w|}{N_\varepsilon^w}(1+\gamma_\varepsilon^w)\frac{(1-\gamma_\varepsilon^w)|f_\varepsilon^w|}{\varepsilon} \\ &\leqq \frac{3}{2}(f_\varepsilon^w\gamma_\varepsilon^w)^2 + 6\frac{(1-\gamma_\varepsilon^w)|f_\varepsilon^w|}{\varepsilon}. \end{aligned}$$

Estimates (72) imply then

$$\begin{aligned} \left\|\frac{f_\varepsilon^w}{N_\varepsilon^w}\right\|^2_{L^2(dxM(1+v^2)\,dv)} &\leqq \tfrac{C}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}) + r_\varepsilon, \\ \left\|\frac{(f_\varepsilon^w)^2}{N_\varepsilon^w}\right\|_{L^1(dxM(1+v^2)\,dv)} &\leqq \tfrac{C}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}) + r_\varepsilon, \end{aligned} \tag{75}$$

with $r_\varepsilon \to 0$ in $L^1([0,T])$. Using the new decomposition (74), we rewrite the flux term

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\int \left((v-\varepsilon w)^{\otimes 2} - \frac{|v-\varepsilon w|^2}{3}\mathrm{Id}\right) F_\varepsilon\, dv\\
&= \frac{1}{\varepsilon^2}\int \left(v^{\otimes 2} - \frac{|v|^2}{3}\,\mathrm{Id}\right)(F_\varepsilon^w - M)\, dv\\
&= \frac{1}{\varepsilon}\int M(\mathcal{L}A)\frac{f_\varepsilon^w}{N_\varepsilon^w}\, dv + \frac{1}{3}\int M\left(v^{\otimes 2} - \frac{|v|^2}{3}\,\mathrm{Id}\right)\frac{(f_\varepsilon^w)^2}{N_\varepsilon^w}\, dv\\
&= \frac{1}{4\varepsilon}\iiint MM_1\overline{A}\left(\frac{f_\varepsilon^w}{N_\varepsilon^w} + \frac{f_{\varepsilon 1}^w}{N_{\varepsilon 1}^w} - \frac{f_\varepsilon^{w\prime}}{N_\varepsilon^{w\prime}} - \frac{f_{\varepsilon 1}^{w\prime}}{N_{\varepsilon 1}^{w\prime}}\right) b\, dv_1 d\sigma\, dv\\
&\quad + \frac{1}{3}\int M\left(v^{\otimes 2} - \frac{|v|^2}{3}\,\mathrm{Id}\right)\frac{(f_\varepsilon^w)^2}{N_\varepsilon^w}\, dv,
\end{aligned}
$$

where A is defined by (23), and $\overline{A} = (A + A_1 - A' - A_1')$. Then we introduce the scaled collision integrand

$$
q_\varepsilon^w = -\frac{1}{\varepsilon^{(q+1)/2}}\left(f_\varepsilon^w + f_{\varepsilon 1}^w - f_\varepsilon^{w\prime} - f_{\varepsilon 1}^{w\prime}\right) + \frac{1}{\varepsilon^{(q-1)/2}}\left(f_\varepsilon^{w\prime} f_{\varepsilon 1}^{w\prime} - f_{\varepsilon 1}^w f_\varepsilon^w\right).
$$

A computation given in [12] (formula (10.6)) shows that

$$
\begin{aligned}
&\frac{\varepsilon^{(q-1)/2} q_\varepsilon^w}{N_\varepsilon^w N_{\varepsilon 1}^w N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime}} + \frac{1}{\varepsilon}\left(\frac{f_\varepsilon^w}{N_\varepsilon^w} + \frac{f_{\varepsilon 1}^w}{N_{\varepsilon 1}^w} - \frac{f_\varepsilon^{w\prime}}{N_\varepsilon^{w\prime}} - \frac{f_{\varepsilon 1}^{w\prime}}{N_{\varepsilon 1}^{w\prime}}\right)\\
&= \frac{1}{3}\left(\frac{1}{N_\varepsilon^w N_{\varepsilon 1}^w} + \frac{1}{N_\varepsilon^w} + \frac{1}{N_{\varepsilon 1}^w} - 2\right)\frac{f_\varepsilon^{w\prime} f_{\varepsilon 1}^{w\prime}}{N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime}}\\
&\quad - \frac{1}{3}\left(\frac{1}{N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime}} + \frac{1}{N_\varepsilon^{w\prime}} + \frac{1}{N_{\varepsilon 1}^{w\prime}} - 2\right)\frac{f_\varepsilon^w f_{\varepsilon 1}^w}{N_\varepsilon^w N_{\varepsilon 1}^w}.
\end{aligned}
\tag{76}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\int \left((v-\varepsilon w)^{\otimes 2} - \frac{|v-\varepsilon w|^2}{3}\,\mathrm{Id}\right) F_\varepsilon\, dv\\
&= -\frac{1}{4}\varepsilon^{(q-1)/2}\iiint MM_1\overline{A}\frac{q_\varepsilon^w}{N_\varepsilon^w N_{\varepsilon 1}^w N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime}} b\, dv_1 d\sigma\, dv\\
&\quad + \frac{1}{12}\iiint MM_1\overline{A}\left(\frac{1}{N_\varepsilon^w N_{\varepsilon 1}^w} + \frac{1}{N_\varepsilon^w} + \frac{1}{N_{\varepsilon 1}^w} - 2\right)\frac{f_\varepsilon^{w\prime} f_{\varepsilon 1}^{w\prime}}{N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime}} b\, dv_1 d\sigma\, dv\\
&\quad - \frac{1}{12}\iiint MM_1\overline{A}\left(\frac{1}{N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime}} + \frac{1}{N_\varepsilon^{w\prime}} + \frac{1}{N_{\varepsilon 1}^{w\prime}} - 2\right)\frac{f_\varepsilon^w f_{\varepsilon 1}^w}{N_\varepsilon^w N_{\varepsilon 1}^w} b\, dv_1 d\sigma\, dv\\
&\quad + \frac{1}{3}\int M\left(v^{\otimes 2} - \frac{|v|^2}{3}\,\mathrm{Id}\right)\frac{(f_\varepsilon^w)^2}{N_\varepsilon^w}\, dv\\
&= I_1 + I_2 + I_3 + I_4
\end{aligned}
$$

Estimates (75) will provide the convenient bounds on I_2, I_3 and I_4, while I_1 will be proved to converge to 0. The first term is estimated by means of the entropy dissipation in the following way. By Cauchy-Schwarz inequality and (H2),

$$\begin{aligned}
|I_1| &\leqq \frac{\varepsilon^{(q-1)/2}}{4}\left(\iiint MM_1\left(\frac{q_\varepsilon^w}{N_\varepsilon^w N_{\varepsilon 1}^w N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime}}\right)^2 b\,dv\,dv_1 d\sigma\right)^{1/2} \\
&\qquad \left(\iiint MM_1(\overline{A})^2 b\,dv\,dv_1 d\sigma\right)^{1/2} \\
&\leqq \frac{C\varepsilon^{(q-1)/2}}{4}\left(\iiint MM_1\left(\frac{q_\varepsilon^w}{N_\varepsilon^w N_{\varepsilon 1}^w N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime}}\right)^2 b\,dv\,dv_1 d\sigma\right)^{1/2}.
\end{aligned} \tag{77}$$

Define γ as in Theorem 7. Then decompose the previous integrand as follows:

$$\begin{aligned}
&MM_1\frac{(q_\varepsilon^w)^2}{(N_\varepsilon^w N_{\varepsilon 1}^w N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime})^2} \\
&\quad = \frac{1}{\varepsilon^{q+3}}\frac{|F_\varepsilon^w F_{\varepsilon 1}^w - F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}|^2}{MM_1(N_\varepsilon^w N_{\varepsilon 1}^w N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime})^2} \\
&\quad = \frac{1}{\varepsilon^{q+3}}\frac{|F_\varepsilon^w F_{\varepsilon 1}^w - F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}|^2}{F_\varepsilon^w F_{\varepsilon 1}^w}\gamma\left(\frac{F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}}{F_\varepsilon^w F_{\varepsilon 1}^w}\right)\frac{F_\varepsilon^w F_{\varepsilon 1}^w}{MM_1(N_\varepsilon^w N_{\varepsilon 1}^w N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime})^2} \\
&\qquad + \frac{1}{\varepsilon^{q+3}}|F_\varepsilon^w F_{\varepsilon 1}^w - F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}|\left(1-\gamma\left(\frac{F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}}{F_\varepsilon^w F_{\varepsilon 1}^w}\right)\right)\frac{|F_\varepsilon^w F_{\varepsilon 1}^w - F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}|}{MM_1(N_\varepsilon^w N_{\varepsilon 1}^w N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime})^2}.
\end{aligned}$$

By (73) and the elementary inequalities (35), we have

$$\begin{aligned}
&MM_1\frac{(q_\varepsilon^w)^2}{(N_\varepsilon^w N_{\varepsilon 1}^w N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime})^2} \\
&\quad \leqq \frac{3^8}{2^6}\frac{1}{\varepsilon^{q+3}}F_\varepsilon^w F_{\varepsilon 1}^w\left|\frac{F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}}{F_\varepsilon^w F_{\varepsilon 1}^w}-1\right|^2\gamma\left(\frac{F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}}{F_\varepsilon^w F_{\varepsilon 1}^w}\right) \\
&\qquad + \frac{3^8}{2^6}\frac{1}{\varepsilon^{q+3}}F_\varepsilon^w F_{\varepsilon 1}^w\left|\frac{F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}}{F_\varepsilon^w F_{\varepsilon 1}^w}-1\right|\left(1-\gamma\left(\frac{F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}}{F_\varepsilon^w F_{\varepsilon 1}^w}\right)\right) \\
&\quad \leqq \frac{C\|\gamma\|_\infty}{\varepsilon^{q+3}}F_\varepsilon^w F_{\varepsilon 1}^w h\left(\frac{F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}}{F_\varepsilon^w F_{\varepsilon 1}^w}-1\right) \\
&\qquad + \frac{C\|1-\gamma\|_\infty}{\varepsilon^{q+3}}F_\varepsilon^w F_{\varepsilon 1}^w h\left(\frac{F_\varepsilon^{w\prime} F_{\varepsilon 1}^{w\prime}}{F_\varepsilon^w F_{\varepsilon 1}^w}-1\right).
\end{aligned}$$

Using (88), we deduce that

$$\iiint MM_1\frac{(q_\varepsilon^w)^2}{(N_\varepsilon^w N_{\varepsilon 1}^w N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime})^2}b\,dv\,dv_1 d\sigma \leqq \frac{C}{\varepsilon^{q+3}}\int D(F_\varepsilon)\,dv \tag{78}$$

for some $C > 0$. Plugging (78) into (77) and using the dissipation bound, we get

$$\|I_1\|_{L^1_{t,x,v}} \leqq C\varepsilon^{(q-1)/2}. \tag{79}$$

By (73), we have

$$\left| \frac{1}{N_\varepsilon^w N_{\varepsilon 1}^w} + \frac{1}{N_\varepsilon^w} + \frac{1}{N_{\varepsilon 1}^w} - 2 \right| \leqq \frac{13}{4},$$

and then, by Cauchy-Schwarz inequality,

$$\begin{aligned}
|I_2| &\leqq C \iiint MM_1 |\overline{A}| \left| \frac{f_\varepsilon^{w\prime} f_{\varepsilon 1}^{w\prime}}{N_\varepsilon^{w\prime} N_{\varepsilon 1}^{w\prime}} \right| b\, dv_1 d\sigma\, dv \\
&\leqq C \left(\iiint MM_1 |\overline{A}|^2 b\, dv_1 d\sigma\, dv \right)^{1/2} \\
&\qquad \left(\iiint M \left(\frac{f_\varepsilon^{w\prime}}{N_\varepsilon^{w\prime}} \right)^2 M_1 \left(\frac{f_{\varepsilon 1}^{w\prime}}{N_{\varepsilon 1}^{w\prime}} \right)^2 b\, dv_1 d\sigma\, dv \right)^{1/2} \\
&\leqq C \left(\iiint MM_1 |\overline{A}|^2 b\, dv_1 d\sigma\, dv \right)^{1/2} \\
&\qquad \left(\iiint M' \left(\frac{f_\varepsilon^{w\prime}}{N_\varepsilon^{w\prime}} \right)^2 M_1' \left(\frac{f_{\varepsilon 1}^{w\prime}}{N_{\varepsilon 1}^{w\prime}} \right)^2 b dv_1' d\sigma\, dv' \right)^{1/2},
\end{aligned}$$

which coupled with (H2), the first estimate in (75) and (H1) gives

$$\|I_2\|_{L^1_{x,v}} \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}) + r_\varepsilon, \tag{80}$$

with $r_\varepsilon \to 0$ in $L^1([0, T])$. In the same way,

$$\|I_3\|_{L^1_{x,v}} \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}) + r_\varepsilon. \tag{81}$$

And the second estimate in (75) gives directly

$$\|I_4\|_{L^1_{x,v}} \leqq \frac{C}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}) + r_\varepsilon. \tag{82}$$

Combining (79), (80), (81) and (82) gives

$$\begin{aligned}
&\left\| X(w) : \tfrac{1}{\varepsilon^2} \left((v - \varepsilon w)^{\otimes 2} - \tfrac{|v-\varepsilon w|^2}{3} \operatorname{Id} \right) F_\varepsilon \right\|_{L^1_{t,x,v}} \\
&\quad \leqq C \tfrac{1}{\varepsilon^2} \textstyle\int_0^T \|X(w)\|_{L^\infty_x} H(F_\varepsilon | M_{(1,\varepsilon w,1)})(t)\, dt + o(1).
\end{aligned}$$

As w is divergence-free,

$$X(w) : \frac{1}{\varepsilon^2} \int \left(\frac{|v - \varepsilon w|^2}{3} \operatorname{Id} \right) F_\varepsilon\, dv = 0,$$

and Theorem 6 is established. $\square$

5. Convergence proof

From Theorems 5 and 6 we deduce that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}H(F_\varepsilon|M_{(1,\varepsilon w,1)})(t) - \frac{1}{\varepsilon^2}H(F_\varepsilon^{\text{in}}|M_{(1,\varepsilon w^{\text{in}},1)}) + \frac{1}{\varepsilon}\int \operatorname{tr}(m_\varepsilon)(t)\\
&\quad \leqq -\frac{1}{\varepsilon}\int_0^t \iint E(w).(v-\varepsilon w)F_\varepsilon(s,x,v)\,dx\,dv\,ds\\
&\qquad + C\int_0^t \|X(w)\|_{L^\infty(\mathbf{T}^3)}\left(\frac{1}{\varepsilon^2}H(F_\varepsilon|M_{(1,\varepsilon w,1)})(s) + \frac{1}{\varepsilon}\int \operatorname{tr}(m_\varepsilon)(s)\right)ds + o(1),
\end{aligned}$$

which implies that for all $w \in C^1([0,T]\times\mathbf{T}^3)$ satisfying $\nabla_x\cdot w = 0$, and for all $t \leqq T$,

$$\begin{aligned}
&\frac{1}{\varepsilon^2}H(F_\varepsilon|M_{(1,\varepsilon w,1)})(t)\\
&\quad \leqq \frac{1}{\varepsilon^2}H(F_\varepsilon^{\text{in}}|M_{(1,\varepsilon w^{\text{in}},1)})\exp\Big(C\int_0^t \|X(w)(s)\|_{L^\infty(\mathbf{T}^3)}ds\Big)\\
&\quad -\int_0^t \exp\Big(C\int_s^t \|X(w)(\tau)\|_{L^\infty(\mathbf{T}^3)}d\tau\Big)\\
&\quad \times \iint E(w).\frac{v-\varepsilon w}{\varepsilon}F_\varepsilon(s,x,v)dx\,dv\,ds + o(1).
\end{aligned} \tag{83}$$

In order to see that any limit point u in the limit $\varepsilon \to 0$ of the sequence $(\frac{1}{\varepsilon}\int F_\varepsilon v\,dv)$ is a dissipative solution of the incompressible Euler equation, it remains to prove that

$$\begin{aligned}
&\nabla_x\cdot u = 0,\\
&\|u-w\|^2_{L^2(\mathbf{T}^3)} \leqq 2\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}H(F_\varepsilon|M_{(1,\varepsilon w,1)}),
\end{aligned}$$

and that

$$\int \frac{v-\varepsilon w}{\varepsilon}F_\varepsilon\,dv \rightharpoonup u - w,$$

up to extraction of a subsequence.

We start with the weak compactness results.

Lemma 6. *Under the assumptions of Theorem 3, for all $T > 0$, any sequence of fluctuations $\frac{1}{\varepsilon_n}(F_{\varepsilon_n} - M)$ with $\varepsilon_n \to 0$ is weakly compact in $L^\infty([0,T], L^1(dx(1+|v|^2)\,dv))$. In particular,*

$$\int F_\varepsilon\,dv \to 1 \text{ in } L^\infty([0,T],L^1(\mathbf{T}^3)),$$

and there exists $u \in L^\infty([0,T],L^1(\mathbf{T}^3))$ such that, up to extraction of a subsequence,

$$\frac{1}{\varepsilon}\int F_\varepsilon v\,dv \rightharpoonup u \text{ in } w - L^\infty([0,T],L^1(\mathbf{T}^3)).$$

Proof. From the entropy bound

$$
\begin{aligned}
\frac{1}{\varepsilon^2} H(F_\varepsilon(t)|M) &\leqq \frac{1}{\varepsilon^2} H(F_\varepsilon^{\mathrm{in}}|M)\\
&\leqq \frac{1}{\varepsilon^2} H(F_\varepsilon^{\mathrm{in}}|M_{(1,\varepsilon u^{\mathrm{in}},1)}) + \frac{1}{2\varepsilon}\iint F_\varepsilon^{\mathrm{in}} u^{\mathrm{in}}.(2v-\varepsilon u^{\mathrm{in}})\, dv\, dx\\
&\leqq \frac{1}{\varepsilon^2} H(F_\varepsilon^{\mathrm{in}}|M_{(1,\varepsilon u^{\mathrm{in}},1)}) + \|u^{\mathrm{in}}\|^2_{L^2(\mathbf{T}^3)}\\
&\quad + \frac{1}{2\varepsilon}\iint (F_\varepsilon^{\mathrm{in}} - M_{(1,\varepsilon u^{\mathrm{in}},1)}) u^{\mathrm{in}}.(2v-\varepsilon u^{\mathrm{in}})\, dv\, dx\\
&\leqq \frac{C_{\|u^{\mathrm{in}}\|_\infty}}{\varepsilon^2} H(F_\varepsilon^{\mathrm{in}}|M_{(1,\varepsilon u^{\mathrm{in}},1)}) + C\|u^{\mathrm{in}}\|^2_{L^2(\mathbf{T}^3)},
\end{aligned}
$$

and Young's inequality (91) with $p=(1+|v|^2)/4$, $z=(F_\varepsilon - M)/M$ and $\eta = 4\varepsilon/\alpha$

$$\frac{1}{\varepsilon}|F_\varepsilon - M|(1+|v|^2) \leqq \frac{\alpha}{\varepsilon^2} Mh\left(\frac{F_\varepsilon}{M} - 1\right) + \frac{16}{\alpha} Me^{(1+|v|^2)/4}, \tag{84}$$

we deduce (see [4], Proposition 3.2) that any subsequence of

$$\frac{1}{\varepsilon}(F_\varepsilon - M)(1+|v|^2)$$

is bounded and uniformly equi-integrable in $L^\infty([0,T], L^1(\mathbf{T}^3\times\mathbf{R}^3))$, and thus relatively weakly compact by the Dunford-Pettis theorem. Then, up to extraction of a subsequence,

$$\frac{1}{\varepsilon}(F_\varepsilon - M) \rightharpoonup f \text{ in } w - L^\infty([0,T], L^1(\mathbf{T}^3\times\mathbf{R}^3)).$$

The convergence of the density of mass $\int F_\varepsilon\, dv$ follows immediately:

$$\int F_\varepsilon\, dv = \int M\, dv + \varepsilon \int \frac{1}{\varepsilon}(F_\varepsilon - M)\, dv = 1 + O(\varepsilon)$$

in $L^\infty([0,T], L^1(\mathbf{T}^3))$. Moreover, as $v = o(|v|^2)$ as $|v|\to\infty$,

$$\int \frac{1}{\varepsilon}(F_\varepsilon - M) v\, dv \rightharpoonup \int f v\, dv \equiv u \text{ in } w - L^\infty([0,T], L^1(\mathbf{T}^3)).$$

In particular, the weak convergence

$$\int \frac{v-\varepsilon w}{\varepsilon} F_\varepsilon\, dv \rightharpoonup u - w$$

holds in $L^\infty([0,T], L^1(\mathbf{T}^3))$. □

In the second step, we establish the incompressibility relation taking limits in the local conservation of mass (13) in the sense of distributions (see [4]).

Lemma 7. *Under the assumptions of Theorem 3, any limit point u of the sequence $(\frac{1}{\varepsilon}\int F_\varepsilon v\, dv)$ satisfies the incompressibility relation*

$$\nabla_x \cdot u = 0$$

in the sense of distributions.

It remains then to estimate the L^2 norm of $(u - w)$ for all divergence-free test vector fields w.

Lemma 8. *Under the assumptions of Theorem 3, any limit point u of the sequence $(\frac{1}{\varepsilon}\int F_\varepsilon v\,dv)$ satisfies*

$$\|u - w\|^2_{L^2(\mathbf{T}^3)} \leqq 2 \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}).$$

Proof. A direct computation shows that

$$H(F_\varepsilon | M_{(1,\varepsilon w,1)}) = H(M_{F_\varepsilon} | M_{(1,\varepsilon w,1)}) + H(F_\varepsilon | M_{F_\varepsilon}) \geqq H(M_{F_\varepsilon} | M_{(1,\varepsilon w,1)}),$$

where M_{F_ε} denotes the local Maxwellian having the same moments as F_ε. If we define R_ε, J_ε and T_ε by

$$R_\varepsilon = \int F_\varepsilon\,dv, \quad J_\varepsilon = \frac{1}{\varepsilon}\int F_\varepsilon v\,dv, \quad R_\varepsilon T_\varepsilon + \varepsilon^2 \frac{J_\varepsilon^2}{R_\varepsilon} = \int F_\varepsilon v^2\,dv,$$

it is easy to check that

$$\begin{aligned} H(M_{F_\varepsilon} | M_{(1,\varepsilon w,1)}) = {} & \textstyle\int (R_\varepsilon \log R_\varepsilon - R_\varepsilon + 1)\,dx + \dfrac{\varepsilon^2}{2} \displaystyle\int \frac{(J_\varepsilon - R_\varepsilon w)^2}{R_\varepsilon} dx \\ & + \tfrac{3}{2} \textstyle\int R_\varepsilon (T_\varepsilon - \log T_\varepsilon - 1)\,dx. \end{aligned}$$

In particular, we have

$$\int \frac{(J_\varepsilon - R_\varepsilon w)^2}{R_\varepsilon} dx \leqq \frac{2}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}).$$

By convexity of the functional $(R, J) \mapsto (J - Rw)^2/R$, since $R_\varepsilon \to 1$ and $J_\varepsilon \to u$ in the vague sense of measure,

$$\|u - w\|^2_{L^2(\mathbf{T}^3)} \leqq 2 \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} H(F_\varepsilon | M_{(1,\varepsilon w,1)}),$$

which is the expected estimate. □

From (24) and the identity

$$\begin{aligned} & \frac{1}{\varepsilon^2} H(F_\varepsilon^{\text{in}} | M_{(1,\varepsilon w^{\text{in}},1)}) \\ & \quad = \frac{1}{\varepsilon^2} H(F_\varepsilon^{\text{in}} | M_{(1,\varepsilon u^{\text{in}},1)}) + \frac{1}{2} \iint F_\varepsilon^{\text{in}}\,dv (u^{\text{in}} - w^{\text{in}})^2 dx \\ & \qquad + \frac{1}{\varepsilon} \iint F_\varepsilon^{\text{in}} (v - \varepsilon u^{\text{in}})\,dv . (u^{\text{in}} - w^{\text{in}})\,dx \\ & \quad = \frac{1}{\varepsilon^2} H(F_\varepsilon^{\text{in}} | M_{(1,\varepsilon u^{\text{in}},1)}) + \frac{1}{2} \|u^{\text{in}} - w^{\text{in}}\|^2_{L^2(\mathbf{T}^3)} \\ & \qquad + \frac{1}{2} \iint (F_\varepsilon^{\text{in}} - M_{(1,\varepsilon u^{\text{in}},1)})\,dv (u^{\text{in}} - w^{\text{in}})^2\,dx \\ & \qquad + \frac{1}{\varepsilon} \iint (F_\varepsilon^{\text{in}} - M_{(1,\varepsilon u^{\text{in}},1)})(v - \varepsilon u^{\text{in}})\,dv . (u^{\text{in}} - w^{\text{in}})\,dx, \end{aligned}$$

we deduce that

$$\frac{1}{\varepsilon^2} H(F_\varepsilon^{\text{in}} | M_{(1,\varepsilon w^{\text{in}},1)}) = \|u^{\text{in}} - w^{\text{in}}\|^2_{L^2(\mathbf{T}^3)} + o(1). \tag{85}$$

Combining Lemmas 6, 7 and 8 with (83) and (85) shows that any limit point u of the sequence $(\frac{1}{\varepsilon}\int F_\varepsilon v\, dv)$ satisfies the incompressibility relation as well as a stability inequality similar to (20) for all $w \in C^1([0,T]\times\mathbf{T}^3)$ such that $\nabla_x\cdot w = 0$, $X(w) \in L^\infty([0,T], L^\infty(\mathbf{T}^3))$, $E(w) \in L^1([0,T], L^2(\mathbf{T}^3))$. A standard argument of density shows that (20) holds actually for all $w \in C^0([0,T]\times\mathbf{T}^3)$ satisfying $\nabla_x\cdot w = 0$, $X(w) \in L^1([0,T], L^\infty(\mathbf{T}^3))$, $E(w) \in L^1([0,T], L^2(\mathbf{T}^3))$. This means exactly that u is a dissipative solution of the incompressible Euler equation.

6. Appendix A. Caflisch estimates

Continuity results regarding the operator $g \mapsto M^{-1/2}\mathcal{A}^+(M^{1/2}g, M)$ are the key arguments of the estimates established in this article: we record them below for the sake of completeness.

Theorem 8 (Caflisch Theorem)**.** *Define* $\mathcal{A}^+(f,g) = \frac{1}{8\pi}\iint (f_1'g' + g_1'f')\, dv_1 d\omega$ *with the notation defined in* (3)*, and denote by* M *any Maxwellian. Then, for all measurable function* g*,*

$$\begin{aligned} &\left\| M^{-1/2}\mathcal{A}^+(\sqrt{M}g, M)(1+|v|^{3/2})\right\|_{L^\infty_v} \leqq C\|g\|_{L^2_v},\\ &\left\| M^{-1/2}\mathcal{A}^+(\sqrt{M}g, M)(1+|v|^{r+2})\right\|_{L^\infty_v} \leqq C\|g(1+|v|^r)\|_{L^\infty_v}. \end{aligned} \tag{86}$$

We complete this result by the following

Proposition 2. *With the same notation as in Theorem 8, for all measurable non-negative function* g*,*

$$\left\| M^{-1}\mathcal{A}^+(\sqrt{M}\sqrt{g}, M)^2(1+|v|^2)\right\|_{L^1_v} \leqq C\|g(1+|v|^2)\|_{L^1_v}. \tag{87}$$

Proof. The identity $M'M_1' = MM_1$ implies

$$M^{-1}\mathcal{A}^+(\sqrt{M}\sqrt{g}, M)^2 = \left(\frac{1}{4\pi}\iint \sqrt{g'}\sqrt{M_1'}\sqrt{M_1}\, dv_1 d\sigma\right)^2.$$

As $|v|^2 \leqq |v_1'|^2 + |v'|^2$, by Cauchy-Schwarz inequality,

$$\begin{aligned} &M^{-1}\mathcal{A}^+(\sqrt{M}\sqrt{g}, M)^2(1+|v|^2)\\ &\quad\leqq \left(\frac{1}{4\pi}\iint M_1\, dv_1 d\sigma\right)\left(\frac{1}{4\pi}\iint g'M_1'(1+|v'|^2+|v_1'|^2)\, dv_1 d\sigma\right) \end{aligned}$$

As $dv\, dv_1 d\sigma = dv' dv_1' d\sigma$, integrating with respect to v leads to the expected inequality. □

7. Appendix B. Young's inequality

The functions $h : [-1, +\infty[\to \mathbf{R}_+$ and $r :]-1, +\infty[\to \mathbf{R}_+$ are both strictly convex, and satisfy, for all $z > -1$,

$$h(|z|) \leqq h(z), \quad r(|z|) \leqq r(z), \quad h(z) \leqq r(z). \tag{88}$$

The Legendre transform of h is defined for all $p \in \mathbf{R}$ by

$$h^*(p) = \sup_{z>-1} (pz - h(z)) = e^p - p - 1 \leqq e^p\,; \tag{89}$$

that of r is also defined for all $p \in \mathbf{R}$ by the implicit relation

$$r^*(p) = \sup_{z>-1} (pz - r(z)) = \frac{z^2}{1+z}, \text{ with } \log(1+z) + \frac{z}{1+z} = p.$$

Further, the Legendre transform h^* is super-quadratic, i.e.,

$$h^*(\eta p) \leq \eta^2 h^*(p), \quad p \in \mathbf{R}, \ \eta \in [0,1]. \tag{90}$$

Finally, Young's inequality states that, for all $p \in \mathbf{R}$, $z > -1$ and $\eta \in [0,1]$,

$$p|z| \leqq \frac{1}{\eta} h(z) + \eta h^*(p) \leq \frac{1}{\eta} r(z) + \eta h^*(p). \tag{91}$$

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