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Theory of Extended Solutions for Fast-Diffusion Equations in Optimal Classes of Data. Radiation from Singularities

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Dedicated to the memory of Philippe Bénilan (*1940–2001*)

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Abstract

This paper is devoted to constructing a general theory of nonnegative solutions for the equation

$$u_t = \Delta(u^m), \quad 0 < m < 1,$$

called "the fast-diffusion equation" in the literature. We consider the Cauchy problem taking initial data in the set $\mathcal{B}^+$ of all nonnegative Borel measures, which forces us to work with singular solutions which are not locally bounded, not even locally integrable. A satisfactory theory can be formulated in this generality in the range $1 > m > m_c = \max\{(N-2)/N, 0\}$, in which the limits of classical solutions are also continuous in $\mathbb{R}^N$ as extended functions with values in $\mathbb{R}_+ \cup \{\infty\}$. We introduce a precise class of extended continuous solutions $\mathcal{E}_c$ and prove (i) that the initial-value problem is well posed in this class, (ii) that every solution $u(x,t)$ in $\mathcal{E}_c$ has an initial trace in $\mathcal{B}^+$, and (iii) that the solutions in $\mathcal{E}_c$ are limits of classical solutions.

Our results settle the well-posedness of two other related problems. On the one hand, they solve the initial-and-boundary-value problem in $\mathcal{R} \times (0,\infty)$ in the class of large solutions which take the value $u = \infty$ on the lateral boundary $x \in \partial\mathcal{R}$, $t > 0$. Well-posedness is established for this problem for $m_c < m < 1$ when $\mathcal{R}$ is any open subset of $\mathbb{R}^N$ and the restriction of the initial data to $\mathcal{R}$ is any locally finite nonnegative measure in $\mathcal{R}$. On the other hand, by using the special solutions which have the separate-variables form, our results apply to the elliptic problem $\Delta f = f^q$ posed in any open set $\mathcal{R}$. For $1 < q < N/(N-2)_+$ this problem is well posed in the class of large solutions which tend to infinity on the boundary in a strong sense.

As is well known, initial data with such a generality are not allowed for $m \geqq 1$. On the other hand, the present theory fails in several aspects in the subcritical range $0 < m \leqq m_c$, where the limits of smooth solutions need not be extended-continuously.

0. Introduction and statement of the main results

In this paper we extend the existing theory of nonnegative solutions of the nonlinear evolution equation

$$u_t = \Delta(u^m), \qquad 0 < m < 1, \tag{0.1}$$

which serves as a mathematical model for the interplay of fast and slow propagation speeds in an evolution process of diffusive type. Indeed, equation (0.1) has been extensively studied in the literature under the name of *fast-diffusion equation* (FDE), cf. for instance [7, 11, 16, 34–36], because it can be written in the form

$$u_t = \operatorname{div}(D(u)\nabla u), \qquad \text{with} \quad D(u) = m\, u^{m-1},$$

so that the diffusion coefficient $D(u) \to \infty$ as $u \to 0$ if $m < 1$. This is reflected in the well-known property of infinite speed of propagation of small disturbances with respect to the zero level: a continuous and nonnegative weak solution of the equation defined in an open cylinder $\Omega \times (0, T) \subset \mathbb{R}^{N+1}$ is necessarily positive for all $x \in \Omega$ at time $t > 0$ unless it is identically zero at this time. This is a property shared by the *heat equation*, $u_t = \Delta u$, but not by the equation with $m > 1$ (the *porous-medium equation*). However, the previous approach to equation (0.1) overlooks the fact that for high values of u the diffusion coefficient $D(u)$ decreases and $D(u) \to 0$ as $u \to \infty$, opening up a perspective of *slow propagation* for $u \gg 1$, or even no propagation at the level $u = \infty$, that is absent in the equations with $m \geqq 1$. To describe our results it will be convenient to use a terminology taken from the theory of thermal propagation and think of (0.1) as a nonlinear heat equation for the temperature distribution $u(x, t)$, and then $D(u)$ is the thermal diffusivity. Moreover, the space integral of u over a set $E \subset \mathbb{R}^N$ is taken as a measure of the thermal energy contained in E. However, in problems of diffusion u is a concentration or a density, and in that case the term energy is replaced by mass, as is often used in properties like conservation of mass. Examples of this character appear in plasma physics where u is the particle density and $m = 1/2$ (Okuda-Dawson law) [11] and in the description of the diffusion of impurities in silicon, where u stands for the concentration of impurities [36].

It is the purpose of this work to formulate a theory of existence, uniqueness and continuous dependence of solutions for (0.1) with arbitrarily large data in a suitable class of large solutions, as well as the inverse problem of assigning an initial trace to any given solution. The project has a successful and simple answer in the so-called *supercritical exponent range*, $m_c < m < 1$, with $m_c = \max\{(N-2)/N, 0\}$. We show that in that range three closely related problems can be solved in an optimal sense which is described next. A fourth problem, the well-posedness of the so-called pressure equation, will be studied in [21]. Before we discuss the results and the proofs, let us point out the main novelty of our study, namely the existence of strong singularities which behave like permanent *sources of radiation*. Actually, they radiate into the surrounding space an infinite amount of energy, which is then spread according to the diffusion law.

(I) Theory of the Cauchy problem. Borel measures and very hot spots. The Cauchy problem, posed for $x \in \mathbb{R}^N$ and $t > 0$, admits a classical solution $u(x, t) >$

0 for any smooth and bounded initial data $u_0(x) \geqq 0$. This result was extended to locally integrable data by Herrero & Pierre [35], to finite Radon measures in a bounded domain by Brezis & Friedman [16] and to locally finite measures in the whole space by Pierre [42] and Dahlberg & Kenig [29]. The purpose of this work is to extend as far as possible the class of data and solve (0.1) for all nonnegative Borel measures as initial data, $u_0 = \nu \in \mathcal{B}^+(\mathbb{R}^N)$. Let us recall that every nonnegative Borel measure $\nu \in \mathcal{B}^+(\mathbb{R}^N)$ can be described by a pair $(\mathcal{S}, \mu)$, where $\mathcal{S}$ is the set of *strongly singular points* of the measure ν, defined as

$$\mathcal{S} = \{x \in \mathbb{R}^N : \nu(B_r(x)) = +\infty \quad \forall r > 0\}. \tag{0.2}$$

and μ, the restriction of ν to $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$, is a locally finite Radon measure, cf. [31]. The strongly singular set $\mathcal{S}$, abbreviated to SSS, can be any arbitrary closed subset of $\mathbb{R}^N$. We shall think of its points as *very hot spots,* since there is an infinite thermal energy in every neighborhood, as small as we please, of a very hot spot. Observe that μ need not be locally finite in $\mathbb{R}^N$ since it may blow up on the boundary of $\mathcal{R}$. As an example, every measurable function $f(x)$, $x \in \mathbb{R}^N$, with values in $\mathbb{R}_+ \cup \{\infty\}$, induces a Borel measure $(\mathcal{S}, \mu)$, where $\mathcal{S}$ is the complement of the maximal open set $\mathcal{R}$ where f is locally integrable and $d\mu(x) = f(x)\,dx$ on $\mathcal{R}$. While $\mathcal{S}$ contains all strongly singular points of ν, weak singularities like Dirac deltas are contained in μ, i.e., they are not considered very hot. Similar notation applies for the solutions $u(x,t)$ at $t > 0$. By a singularity of the temperature distribution $u(x,t)$ at time $t = t_0$, we mean a point x_0 near which $u(t_0)$ is not locally bounded. A weak singularity is a singularity such that $u(\cdot, t_0)$ is locally integrable near x_0, otherwise the singularity is strong.

By taking limits of classical solutions, Dahlberg & Kenig [29] considered initial data in the class of locally finite measures and arrived at the class $\mathcal{C}$ of continuous weak solutions. In the present generality, $\nu \in \mathcal{B}^+(\mathbb{R}^N)$, we arrive at the class $\mathcal{E}_c$ of *extended continuous solutions with constant singular set,* which is precisely described as follows.

Definition 1. A solution $u \in \mathcal{E}_c$ is a nonnegative and measurable function defined in $Q = \mathbb{R}^N \times (0, \infty)$, possibly infinite-valued, and satisfying the following conditions:

(i) It is continuous as an extended function: $u \in C(Q, \overline{\mathbb{R}_+})$, where $\overline{\mathbb{R}_+} = [0, \infty]$.
(ii) It is a classical solution of (0.1) in the regular set $\Omega = \{(x,t) \mid u(x,t) < \infty\} \subset Q_T$.
(iii) For every $t > 0$ the infinite level-set $\mathcal{S}(t) = \{x : u(x,t) = \infty\}$ consists of strong singularities and is constant in time.

A solution $u \in \mathcal{E}_c$ of the Cauchy problem with initial data $\nu \in \mathcal{B}^+(\mathbb{R}^N)$ is a solution which takes on the initial value ν in the sense of Borel trace, *i.e.*,

(iv) For any compactly supported test function $\varphi \in C_0(\mathbb{R}^N)$, $\varphi \geqq 0$,

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\varphi(x)\,dx = \int_{\mathbb{R}^N} \varphi(x)\,d\nu(x) \in \mathbb{R}_+ \cup \{+\infty\}. \tag{0.3}$$

Conditions (i) and (ii) are the definition of extended continuous solutions, which form a larger class $\mathcal{E}$; (iii) restricts this class to a constant and strongly singular set, because this restriction is naturally found when we obtain solutions as limits of continuous weak solutions. This is a nontrivial restriction, since solutions with expanding SSS can be constructed cf. Section 8, but they are *not* the limits of the classical theory of the Cauchy problem. Finally, condition (iv) is not necessarily imposed in the definition of the class since every solution satisfying (i)–(iii) will be shown to have an initial trace. Note that the Borel trace means the standard trace in $\mathcal{R}$ and $\int_U u(x,t)\,dx \to \infty$ for every neighborhood U of any point of $\mathcal{S}$, cf. (2.6). Because of (iii) the regular set Ω of a solution $\mathcal{E}_c$ is the cylinder

$$Q_{\mathcal{R}} = \mathcal{R} \times (0,\infty), \qquad \mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}.$$

Since the solutions are strictly positive on $\mathcal{R}$ and infinite on $\mathcal{S}$, it turns out that (i) is equivalent to the stronger assertion that $1/u$ (or equivalently, $D(u)$) is continuous in Q. The solutions are also smooth in $\mathcal{R} \times (0,\infty)$. The extended continuity of u near the set $\mathcal{S}$ is important to eliminate false solutions of the type $u = +\infty$ on $B \times (0,T)$, where B is a ball and $u = 0$ on the complement.

The main results of this paper establish that the Cauchy problem is well posed in the class $\mathcal{E}_c$ and that the resulting theory is the unique extension of the classical theory to general initial data. We sum up next the precise results which are proved in the paper.

Theorem. *The following holds for the set of nonnegative limit solutions of the Cauchy problem for* (0.1) *in the range* $m_c < m < 1$*:*

(a) *The map* $\nu \mapsto u$ *from* $\mathcal{B}^+(\mathbb{R}^N)$ *into* $\mathcal{E}_c$ *is one-to-one, monotone and continuous with respect to the convergence of data in the sense of Borel measures.*
(b) *Conversely, every solution defined for* $t > 0$ *admits an initial trace in* $\mathcal{B}^+(\mathbb{R}^N)$.
(c) *For locally finite initial masses, i.e., when* $\mathcal{S} = \emptyset$, *we recover the set* $\mathcal{C}$ *of continuous weak solutions.*
(d) *All the solutions in* $\mathcal{E}_c$ *are limits of classical solutions with smooth initial data.*

In terms of semigroup theory, we may say that (0.1) generates a semigroup of maps $S_t : u_0 \mapsto u(t)$ in $\mathcal{B}^+(\mathbb{R}^N)$ which is the closure of the classical one. As usual, we write $u(t)$ to denote the function $x \mapsto u(x,t)$. Let us remark that extended continuous solutions with non-empty SSS *are not* distributional solutions since the functions involved are not locally integrable. The theory is, however, an extension of the distributional theory since the extended solutions are limits of distributional solutions. A natural *renormalization* of the form $v = 1/u^{\alpha}$ for a convenient value of $\alpha > 0$ may allow us to fall into a distributional theory, see Final Remark at the end of the Introduction.

As a reference to results of a similar generality, the use of Borel measures as data appears in the recent work of Marcus and Véron. The well-posedness of the Cauchy problem in the class of Borel measures has been established by these authors in [37] for the semi-linear heat equation $u_t = \Delta u - u^q$, with $1 < q < 1 + 2/N$ and we use their outline for the uniqueness proof although the technical aspects are different. Concerning equations of the type (0.1), a theory with this generality is typical of the fast-diffusion range $m < 1$, and does not apply to the heat equation, $m = 1$, where the initial data accepted in the limit theory must be locally finite Radon measures with an exponential growth rate at infinity. More precisely, for any nonnegative solution of the heat equation existing in a time interval $0 < t < T$ the initial trace is a measure ν such that

$$\int_{\mathbb{R}^N} e^{-c|x|^2} d\nu < \infty$$

for some $c > 0$, and then $T \geqq 1/4c$, cf. [53]. For this equation the presence of a very hot spot forces the function obtained as the limit of classical solutions to blow up in the whole space-time domain, $u \equiv +\infty$ in Q. The same applies to the porous-medium rage $m > 1$ with the only proviso that the allowed growth rate for the optimal class of initial data is power-like, $u_0(x) \sim O(|x|^{2/(m-1)})$. In a more precise form, the initial trace of a nonnegative solution existing for a time interval $0 < t < T$ is a Radon measure $u_0 = \nu$ such that

$$\int_{|x|\leqq R} d\nu = O(R^{N+(2/(m-1)}) \quad \text{as } R \to \infty,$$

cf. [4,9]. Again, failure to satisfy this condition produces instantaneous and global blow-up. On the other hand, while the existence of nontrivial solutions with very hot spots can be extended to the subcritical range $0 < m \leqq m_c$, the simple characterization of well-posedness fails, and a number of new properties arise from the even slower rate of propagation for large u, the most typical being the absence of a point-source solution, [16]. A complete theory is still missing in this case.

(I') Special solutions. Radiation from very hot spots. Let us turn to the question of qualitative behavior of the solutions in the good range $m_c < m < 1$. In dealing with general initial data we have extended the theory by introducing the subclass of solutions with a nontrivial SSS, $\mathcal{E}_c' = \mathcal{E}_c \setminus \mathcal{C}$. Such solutions have many peculiar features which separate them from the class of continuous weak solutions $\mathcal{C}$. A representative example of the relationship and differences between the two kinds of solutions occurs in the passage from a finite point source to an infinite point source. Thus, let us consider the family of special solutions $U_c(x,t)$ of (0.1) with initial mass a Dirac delta, $U_c(x,0) = c\delta(x)$. These solutions, known as *source-type solutions*, Barenblatt solutions or fundamental solutions, exist in the classical sense for all $m > m_c$ and are given by the explicit formula

$$U_c(x,t) = \left(\frac{Ct}{x^2 + At^{2\theta/N}}\right)^{1/(1-m)} \tag{0.4}$$

where $\theta = N/(2 - N(1-m))$, $C = 2mN/(\theta(1-m))$ is a fixed constant depending only on m and N, and $A > 0$ is a decreasing function of the mass c, $A = k(m,N)c^{-\alpha}$. If we take the limit of the fundamental solutions with increasing masses, we get the formula

$$U_\infty(x,t) = \left[\frac{Ct}{|x|^2}\right]^{1/(1-m)}, \quad C = \frac{2m}{1-m}(2 - N(1-m)) > 0. \tag{0.5}$$

Notice that $C > 0$ precisely for $m_c < m < 1$. This is an extended continuous solution of (0.1) that has the simplest singular initial trace with an *infinite source*, of the form $\mathcal{S} = \{0\}$, $\mu = 0$. Accordingly, we shall call U_∞ the infinite point-source solution, IPSS. We see that while the solutions U_c become bounded and smooth for $t > 0$ (the so-called regularizing effect, the hot spot disappears), U_∞ keeps its strong singularity at $x = 0$, $\int_B U_\infty(x,t)\,dx = \infty$ for any $t > 0$ and any ball $B_R(0)$ with $r > 0$. Solutions like this one with a standing singularity were called "razor blades" in the classification of the types of singular solutions that appear as limits of fundamental solutions performed in [51]. The IPSS can be considered as a relative of another well-known type, the very singular solutions, introduced in [17, 33] for the study of the nonlinear heat equation $u_t = \Delta u - u^p$ and subsequently found in other models. They have in common the strong singularity at $t = 0$ but they differ in the fact that the VSS is bounded for all $t > 0$ while the "razor blade" is not.

Next we remark that the strong singularity is not passive, since the solution becomes positive everywhere for positive time, and moreover, it increases and tends to $+\infty$ everywhere when $t \to \infty$. We say that the singularity *radiates* energy into the surrounding space with a rate that can be calculated as

$$\frac{d}{dt}\int_{|x|>r} U_\infty(x,t)\,dx = -\int_{|x|=r} \frac{\partial u^m}{\partial r}\,dS = K\,\frac{t^{m/(1-m)}}{r^{N/\theta(1-m)}},$$

which goes to infinity as $r \to 0$, to zero as $r \to \infty$. Thus, the very hot spot acts as a permanent source of radiation, and the same property holds for all solutions with a SSS $\mathcal{S} \neq \emptyset$. For them the above theory implies that $u > 0$ everywhere in $\mathcal{R}$, which means that even though the radiating hot set $\mathcal{S}$ (which has diffusivity $D(u) = 0$) stays very hot for ever, it also radiates into the surrounding space, and the radiation arrives up to infinity. Actually, the IPSS is probably the single most important solution of the equation since many of the qualitative properties, and even quantitative estimates, of the general class of solutions are modeled on its behaviour. Thus, we prove that any extended solution diverges in the neighborhood of very hot spots at least like the IPSS,

$$u(x,t) \geqq U_\infty(x-y,t) \quad \forall y \in \mathcal{S}, \tag{0.6}$$

and the estimate holds for all $x \in \mathbb{R}^N$, and gives a lower bound of the asymptotic behavior as $|x| \to \infty$ for fixed $t > 0$, which is exact for solutions with compactly supported u_0. This "radiation lemma" is one of the key estimates on which the theory is based. Also of interest is the long-time behavior of solutions with strong singularities. It turns out that this behavior is governed only by the singular set in first approximation. Indeed, even when the locally finite part of the initial trace does not vanish, its effect becomes negligible compared with the radiation of the singular part when time goes to infinity, and this happens independently of the rate of growth of μ as $x \to \infty$, even if this rate can be arbitrarily large.

The phenomenon of radiation will not be always true for general initial data if $m \leqq m_c$, being the source of new complexities in the theory.

(II) Evolution problem with infinite boundary data. A consequence of the above results is the well-posedness of the initial-and-boundary value for (0.1) posed in an arbitrary open set $\mathcal{R} \subset \mathbb{R}^N$, with infinite boundary data

$$u(x,t) \to \infty \quad \text{as} \quad (x,t) \to \Sigma = \partial\mathcal{R} \times (0,\infty), \tag{0.7}$$

and initial data

$$u_0 = \mu \in \mathcal{M}^+(\mathcal{R}), \tag{0.8}$$

the set of locally finite nonnegative measures. By a solution of this problem we mean a continuous function $u : Q_{\mathcal{R}} = \mathcal{R} \times (0,\infty) \to \mathbb{R}_+$ with solves the equation in $Q_{\mathcal{R}}$, diverges at the lateral boundary Σ locally uniformly, and takes the initial data μ in the sense of trace. We show that there exists a unique maximal solution in this class, which is at the same time the maximal element in the whole class of continuous solutions with initial data μ and arbitrary boundary behaviour. It is therefore a *universal upper barrier*. It can be constructed by considering the Cauchy problem with initial data the measure ν which is strongly singular on $\mathcal{S} = \mathbb{R}^N \setminus \mathcal{R}$ and equals μ on $\mathcal{R}$. Uniqueness follows from the uniqueness of the Cauchy problem if we add to the definition of solution the condition of *strong singularity* in $\Gamma = \partial\mathcal{R}$ as $t \to 0$. This condition can be stated in this setting as

$$\lim_{t\to 0} \int_{U(y)\cap\mathcal{R}} u(x,t)\,dx = \infty \tag{0.9}$$

for every $y \in \Gamma$ and every U neighborhood of y. We also show that this condition need only be checked on a certain subset Γ_0 of Γ consisting of points of zero density in $\mathcal{S}$. Thus, when every point of Γ is a point of density of $\mathcal{S}$, uniqueness holds without the strong singularity condition. On the other hand, we construct examples of non-uniqueness when Γ has isolated points in the form of solutions which do not satisfy the divergence condition (0.9), i.e., they exhibit weak singularities.

Note the complete generality *both of the open set* $\mathcal{R}$, which is allowed to be unbounded and have irregular boundary Γ, and of *the locally finite measure* μ, which may diverge at any rate near Γ. The properties derived for the Cauchy problem can be translated here. In particular, the map: $(\mathcal{S},\mu) \to u$ is continuous in both arguments. The solutions are limits of classical solutions in the same domain. The results cannot be extended to $m \geqq 1$ and only partially to $0 < m \leqq m_c$.

On the other hand, let us point out that the generality of domain $\mathcal{R}$ is a characteristic of the problem with infinite boundary values which is not allowed for other initial and boundary-value problems, like the homogeneous Dirichlet problem, where the zero data cannot be prescribed on isolated parts of $\Gamma = \partial\mathcal{R}$. For the Cauchy-Dirichlet problem, we refer to Section 6 and the forthcoming papers [22, 23].

(III) The Elliptic Problem with Infinite Data. Another consequence of the results for the Cauchy problem is obtained when we consider data of the form $\nu = (\mathcal{S}, 0)$, i.e., a pure SSS with cold surrounding space. Then it can be proved that the solution takes the separate-variables form

$$u(x,t) = t^{1/(1-m)} f(x) \tag{0.10}$$

where f is a solution of the elliptic equation $\Delta f^m = (1/(1-m))f$ in $\mathcal{R}$. Taking the more convenient variable $\psi = c\,f^m$, it satisfies

$$-\Delta\psi + \psi^q = 0 \quad \text{in} \quad \mathcal{R}, \tag{0.11}$$

with $q = 1/m$ in the interval $1 < q < N/(N-2)$ $(1 < q < \infty$ if $N = 1, 2)$, and takes infinite boundary values

$$\psi(x) \to \infty \quad \text{as} \quad x \to \partial\mathcal{R}. \tag{0.12}$$

The results of (I), (II) imply that this problem, called the *problem of large solutions*, is well posed for any open set $\mathcal{R}$ if we add a condition of strong divergence at special points of Γ. In fact, problem (III) is completely equivalent in this setting to problem (II) with $\mu = 0$. Our results complete the deep study of MARCUS & VÉRON [38, 40], from which we draw a number of basic techniques.

As in the problem (II), the full generality of domain $\mathcal{R}$ is a characteristic of the class of large solutions and is not allowed for other boundary values, like the homogeneous Dirichlet data, which cannot be prescribed on isolated parts of $\Gamma = \partial\mathcal{R}$. The results cannot be extended to $q \leqq 1$ and only partially to $q \geqq N/(N-2)$.

Distribution. The results are organized in ten sections as follows: Section 1 contains the preliminary information on smooth and weak solutions for the Cauchy problem, references and the main local estimates.

Section 2 contains a study of the properties of extended solutions $\mathcal{E}$ without the restriction of aconstant singular set. The radiation lemma is proved, the expanding character of the SSS follows, and the existence of the initial trace is proved in a class $\mathcal{E}_s$ containing $\mathcal{E}_c$. As a consequence, a first existence result is proved, and extended solutions with constant singular set (ECS) are obtained as limits of continuous weak solutions.

The following three sections establish the well-posedness for the Cauchy problem. Existence of the minimal and maximal ECS is shown in Section 3. Separation of variables leads to the elliptic problem, studied in Section 4, and uniqueness follows for data of the form $(\mathcal{S}, 0)$. The uniqueness of the general evolution problem is settled in Section 5, as well as continuous dependence and several other properties of the solutions. In Section 6 we go back to the Dirichlet problem which is shown to be well posed in the class of large solutions.

Section 7 investigates the asymptotic behavior, both for large x and large t, where the existence of very hot spots is shown to have an important influence.

Two further sections are devoted to presenting the starting aspects of theories that run parallel to the main theme of this paper. Thus, the theory of extended solutions with expanding singular sets and with weak singularities is pursued in Section 8. Section 9 contains a partial analysis of the fast-diffusion equation with subcritical exponent, $0 < m \leqq m_c$. Basic known results are reviewed, new directions stated, similarities and differences with the supercritical theory are discussed. A final section contains comments, extensions and open problems.

A number of auxiliary results are collected in an Appendix, among them a concentrated summary on self-similar solutions. It concludes by a terminology list.

Final Remark. We want to point out that the change of variables $v = m\,u^{m-1}$, defining the so-called pressure variable, allows us to pass from the fast-diffusion equation (1.1) to the equation

$$v_t = v\Delta v + \gamma|\nabla v|^2, \tag{0.13}$$

with $\gamma = 1/(m-1) < 0$. This equation is a very interesting example of degenerate parabolic equation in non-divergence form. Nonnegative solutions are considered. In the case $\gamma > 0$ it turns out to be equivalent to the porous medium equation [2]. The less-known case $\gamma < 0$ has been studied by several authors and interesting phenomena of non-uniqueness have been described, cf. [1, 6, 12–14]. Its degeneracy makes it a good benchmark for current theories of classical, weak or viscosity solutions, see [18, 19, 24–26]. We devote a separate paper [21] to investigating the consequences of the present theory of fast diffusion for the solutions of (0.13) for $\gamma < -N/2$. It will become apparent that the consideration of solutions of (1.1) with hot spots is important for understanding the behavior of the solutions of (0.13) which take on zero values, and it is in fact the key to establishing the well-posedness of the Cauchy problem in an optimal class of initial data that turns out to include *all nonnegative and measurable functions* $v_0 : \mathbb{R}^N \mapsto \mathbb{R}_+ \cup \{\infty\}$, a quite infrequent situation.

1. The Cauchy problem. Preliminaries

We begin our detailed study by the Cauchy problem posed in $Q_T = \mathbb{R}^N \times (0, T)$ with initial data

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.1}$$

in the *supercritical range* of exponents $m \in (m_c, 1)$. We want to take as initial data *any nonnegative Borel measure*. Before introducing the precise definitions and results we will briefly review what is known and how the extension arises. The Cauchy problem (0.1), (1.1) has been studied by a number of authors. One of the simplest results states that a nonnegative and bounded initial datum u_0 gives rise to a unique smooth and positive solution. A result valid for all $0 < m < \infty$ says that for every initial data $u_0 \in L^1(\mathbb{R})$, $u_0 \geqq 0$, there exists a unique so-called mild solution $u \in C([0; \infty] : L^1(\mathbb{R}^N))$ and $u \geqq 0$, cf. [7]. The critical exponent m_c appears for the first time to mark the interval $m \geqq m_c$ where the integral $\int u(x, t)\, dx$ is a conserved quantity in time; this law is usually called conservation of energy or conservation of mass. A general existence result for functions in the range $0 < m < 1$ is obtained by Herrero & Pierre [35], who show that for every nonnegative function $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ there exists a nonnegative weak solution $u \in C(0, T; L^1_{\mathrm{loc}}(\mathbb{R}^N))$; it solves (0.1) in a distribution sense and takes on the initial value in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. The exponent m_c appears in this framework to mark the limit of the range where weak solutions become locally bounded as a consequence of the following $L^1_{\mathrm{loc}} \to L^\infty_{\mathrm{loc}}$ regularizing effect: for every solution u, every point $x \in \mathbb{R}^N$ and every time $t > 0$ the value $u(x, t)$ can be estimated uniformly in terms of $t, r > 0$ and the integral of u_0 in the ball of radius r around x. This L^∞_{loc} regularity is false for $0 < m < m_c$, as can be seen on the explicit solution

$$u(x,t) = \left(C\frac{T-t}{|x|^2}\right)^{\frac{1}{1-m}}, \quad C = \frac{2m}{1-m}(N(1-m)-2), \tag{1.2}$$

which has initial data $u(x,0) \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for all $p < N(1-m)/2$. A counterexample for $m = m_c$ and $N \geqq 3$ is constructed in the Appendix. On the other hand, BREZIS & FRIEDMAN [16] showed that when we consider as initial data a Dirac mass (in a bounded domain) and we try to obtain a limit solution by approximation with smooth data, the process fails for $0 < m \leqq m_c$ because the limit is constant in time: $u(x,t) = \delta_0(x) \otimes 1(t)$, hence the usual concept of solution is lost by lack of radiation. The same happens when the domain is the whole space.

We quote next the two main estimates from [35] which will be of great use in the sequel. First, the control of the local energy in time:

Lemma 1.1. *Let $0 < m < 1$, and $u \in C(0,T; L^1(\Omega))$ be a solution of $u_t = \Delta u^m$ in $\Omega \times (0,T)$ in the sense of distributions. Then for every $\varphi \in C^2_0(\Omega)$ and every $0 < s, t < T$, the following estimate holds:*

$$\left| \left(\int_\Omega u(t)\varphi\right)^{1-m} - \left(\int_\Omega u(s)\varphi\right)^{1-m} \right| \leqq C(\varphi)|t-s|. \tag{1.3}$$

The precise form of the L^1_{loc}-L^∞_{loc} regularizing effect is as follows.

Lemma 1.2. *For every $m_c < m < 1$ and every nonnegative weak solution of the Cauchy problem u, the following estimate holds:*

$$u(x,t) \leqq C(m,N)\left[t^{-\theta}\left(\int_{B_r(x)} u_0(y)\,dy\right)^{2\theta/N} + (t/r^2)^{1/(1-m)}\right], \tag{1.4}$$

with $\theta = (m-1+(2/N))^{-1}$, $x \in \mathbb{R}^N$ and $t > 0$.

PIERRE [42] showed existence of a weak solution for all nonnegative Radon measures as initial data in the supercritical range $m_c < m < 1$, and under a necessary capacity condition on μ when $0 < m \leqq m_c$. In the latter case the solutions need not be locally bounded, only $L^1_{\mathrm{loc}}(Q_T)$. Subsequently, DAHLBERG & KENIG [28] prove the uniqueness of continuous weak solutions for this supercritical range of m allowing the initial data to be a nonnegative Radon (i.e., locally finite) measure, $\mu \in \mathcal{M}_+(\mathbb{R}^N)$. Note that locally bounded solutions are continuous and even C^∞ smooth by standard regularity theory.

We will also need a uniqueness result for finite measures in cylinders.

Proposition 1.1. *Let $0 < m < 1$, $\Omega \subset \mathbb{R}^N$ regular and bounded, and $\mu \in \mathcal{M}^+(\Omega)$ of finite mass. Then there exist at most a weak solution $u \in \mathcal{C}^0(\overline{\Omega} \times (0,T)) \cap L^\infty(0,T; L^1(\Omega))$ such that*

$$\begin{aligned} u_t &= \Delta u^m && \text{in } \mathcal{D}'(\Omega \times (0,T)),\\ u(x,t) &= 0 && \text{on } \partial\Omega \times (0,T),\\ u(x,0) &= \mu && \text{in } \Omega. \end{aligned}$$

This uniqueness result is a direct consequence of the fact that Lemmas 2 and 3 of [28] remain valid even in the fast- diffusion case, although in this reference they are stated in the case of slow diffusion $m > 1$. A study of the homogeneous Dirichlet problem is made in [22]. We recall that, according to PIERRE's results [42], existence can be shown only under a capacity condition in the subcritical case.

Next, we need a comparison result for the Cauchy-Dirichlet problem in a possibly unbounded domain.

Proposition 1.2. *Let $m > m_c$ and Ω be a regular open subset of $\mathbb{R}^N$ (not necessarily bounded). Let u and v be two smooth solutions of*

$$u_t - \Delta u^m = 0 \quad in \quad \Omega \times (t_1, t_2),$$

such that u and v are continuous in $\overline{\Omega} \times [t_1, t_2]$ and $u(x,t) \leqq v(x,t)$ on the lateral boundary $\partial\Omega \times (t_1, t_2)$. Note that $\Omega_R = \Omega \cap B_R(0)$. Then for every $t \in (t_1, t_2)$, and every $R > 0$, the following estimate holds:

$$\int_{\Omega_R} (u-v)_+(t) \leqq C \int_{\Omega_{2R}} (u-v)_+(t_1) + C(t-t_1)^{1/(1-m)} R^{N-2/(1-m)}, \quad (1.5)$$

where C depends only on m and N. In particular if $u(t_1) \leqq v(t_1)$,

$$u \leqq v \quad in \quad \overline{\Omega} \times [t_1, t_2]. \quad (1.6)$$

Proof. We recall that the Heaviside or sign$^+$ function is defined as $H(s) = 1$ for $s > 0$, $H(s) = 0$ for $s \leqq 0$. Let p_k be an approximation of $H(s)$ such that $p_k \in C^2(\mathbb{R})$, $p_k(0) = 0$ and $p_k' \geqq 0$. We also note $j_k(\alpha) = \int_0^\alpha p_k(\sigma)d\sigma$. For $\varphi \in C_0^\infty(\mathbb{R}^N)$, we use $p_k(u^m - v^m)\varphi$ as the test function, which is allowed since both u and v are smooth solutions in $\Omega \times (t_1, t_2)$. Here, p stands for $p_k(u^m - v^m)$, and $t \in (t_1, t_2)$:

$$\begin{aligned}\int_\Omega [(u-v)_t\, p\,\varphi](t) = &\int_{\partial\Omega} \frac{\partial(u^m - v^m)}{\partial\nu} p\varphi \\ &- \int_\Omega \nabla j(u^m - v^m)\nabla\varphi - \int_\Omega p'|\nabla(u^m - v^m)|^2\varphi.\end{aligned}$$

Hence if φ is nonnegative, with another integration by parts, we find that

$$\begin{aligned}\int_\Omega [(u-v)_t\, p\,\varphi](t) \leqq &\int_{\partial\Omega} \frac{\partial(u^m - v^m)}{\partial\nu} p\varphi - \int_{\partial\Omega} j(u^m - v^m)\frac{\partial\varphi}{\partial\nu} \\ &+ \int_\Omega j(u^m - v^m)\Delta\varphi.\end{aligned}$$

The surface integrals are zero since $u \leqq v$ on $\partial\Omega$ and since v and u are smooth solutions on $\Omega \times (t_1, t_2)$, we can pass to the limit as $k \to \infty$, which yields:

$$\frac{d}{dt}\int_\Omega (u-v)_+(t)\varphi \leqq \int_\Omega (u^m - v^m)_+\Delta\varphi.$$

Finally, since Ω is not bounded, we apply the techniques of [35, p. 155]: taking test functions $\varphi_R \in C_0^\infty(\mathbb{R}^N)$ of the form

$$\varphi_R(x) = \varphi_1\left(\frac{x}{R}\right),$$

where φ_1 has compact support in $B_2(0)$ and $0 \leqq \varphi \leqq 1$, $\varphi = 1$ on $B_1(0)$, we find that

$$\left|\frac{d}{dt}\int_\Omega (u-v)_+(t)\varphi_R\right| \leqq C(\varphi_R)\left[\int_\Omega (u-v)_+\varphi_R\right]^m,$$

and by the estimate on $C(\varphi_R)$ (same reference), this implies that for some other $C = C(m, N)$,

$$\int_\Omega (u-v)_+(t)\varphi_R \leqq C\int_\Omega (u-v)_+(t_1)\varphi_R + C(t-t_1)^{1/(1-m)}R^{N-2/(1-m)}. \tag{1.7}$$

Inequality (1.5) follows easily from our assumptions on φ_R. In the case when $u(t_1) \leqq v(t_1)$, letting R increase to infinity, since $m > m_c$ yields

$$\int_\Omega (u-v)_+(t) = 0$$

for every $t \in (t_1, t_2)$, hence $u \leqq v$ in $\overline{\Omega} \times [t_1, t_2]$. □

We use the notation f_+ to denote the positive part of a measurable function, $f_+ = \max\{f, 0\}$ almost everywhere. To end these preliminaries, let us mention that by well-known properties of the fast-diffusion equation when $m > m_c$, every extended solution $u \in \mathcal{E}$ with non-zero initial trace will be positive where it is finite, and thus it will be smooth in this set. We shall show this property in the following section thanks to the radiation lemma.

2. Limit solutions and extended solutions

We shall see in this section that taking limits of weak solutions leads to the wider class of extended continuous solutions with singular sets which are preserved in time. Before we prove the convergence result, Theorem 2.2, we will establish some basic properties of the classes of extended solutions. We point out that, though the condition of constancy of the singular set will be satisfied by the limits of smooth solutions, it is not a condition that appears as necessary in the definition of the extended solution. Actually, properties like the persistence of strong singularities, cf. Lemma 2.1, the lower estimate near a strong singularity, (2.3), and the existence of an initial trace, Theorem 2.1, which we show below, are shared by larger classes of extended solutions. Among them, we may find solutions with weak singularities. Even if we consider the class of solutions whose singularities are strong, there are solutions with expanding singular sets, hence not in $\mathcal{E}_c$.

We start by the important property of persistence of the strongly singular set for all solutions in $\mathcal{E}$, that we call the *radiation lemma*.

Lemma 2.1. *Let $u \in \mathcal{E}$ and define its initial singular set as*

$$\mathcal{S} = \left\{ y \in \mathbb{R}^N \mid \int_{B_r(y)} u(x,t)dx \underset{t\to 0}{\longrightarrow} \infty \quad \forall r > 0 \right\}. \tag{2.1}$$

Then for C defined in (0.5),

$$u(x,t) \geqq \left[\frac{Ct}{\operatorname{dist}(x,\mathcal{S})^2} \right]^{\frac{1}{1-m}}. \tag{2.2}$$

In particular, $\mathcal{S}$ remains a singular set for u at all later times, and for every $t > 0$, $u(x,t) \to +\infty$ as $x \to \mathcal{S}$.

Proof. (i) Let u and $\mathcal{S}$ be as above and take $y \in \mathcal{S}$. Let us fix $c > 0$ and $0 < r < R$, and let us recall that, by standard arguments, u is positive and smooth in $\Omega = \{(x,t) \in Q_T \mid u(x,t) < \infty\}$ (see Appendix). We choose $\tau > 0$ small enough such that

$$\int_{B_r(y)} u(x,\tau)dx \geqq 2c.$$

Since this integral may be infinite, we then define the solution v_r^c in $B \times (\tau, T)$, where $B = B_R(y)$, by the initial data

$$v_r^c(x,\tau) = f(x),$$

where $f(x) \in C(B)$ with compact support in $B_r(y)$ is chosen so that

$$0 \leqq f(x) < u(x,\tau), \qquad \int_{B_r(y)} f(x)dx = c,$$

which is always possible if τ is small enough. Moreover, we put zero lateral data for v_r^c on $\partial B \times (\tau, T)$. Now we will compare u and v_r^c in $\Omega_R = (B_R(y) \times (\tau, T)) \cap \Omega$. At time $t = \tau$, it is clear that $v_r^c < u$, and

$$0 = v_r^c(x,t) < u(x,t) \quad \text{on} \quad \partial B \times (\tau, T),$$

so that by continuity of both solutions, they remain strictly ordered in Ω_R up to a time greater than τ. Thus we can define the first moment where v_r^c and u "touch":

$$t_0 = \sup\{t \in (\tau, T) \mid u(x,t) > v_r^c(x,t) \quad \forall x \in B\} > \tau,$$

and let us assume that t_0 is finite. Then consider a point x_0 such that $u(x_0,t_0) = v_r^c(x_0,t_0)$. It is clear that x_0 cannot belong to $\partial\Omega$ since $u(t_0) = +\infty$ on this set, hence $x_0 \in \Omega$. By continuity of u there exists a small cylinder

$$\overline{B}_\eta(x_0) \times [t_1, t_0] \subset B \times (\tau, t_0],$$

which is strictly included in the set Ω where u is finite. In this cylinder, by continuity and positivity of both u and v_r^c, there exist constants $\alpha, \beta > 0$ such that

$$0 < \alpha \leqq u, v_r^c \leqq \beta < \infty \quad \text{in} \quad B_\eta(x_0) \times (t_1, t_0).$$

Thus setting $w = u - v_r^c$, we have

$$w_t - a(x,t)\Delta w = 0 \quad \text{in} \quad B_\eta(x_0) \times (t_1, t_0),$$

where $a \in C(B_\eta(x_0) \times (t_1, t_0))$ is such that $0 < \alpha' \leqq a \leqq \beta' < \infty$ for some other constants α' and β'. Since w is positive on $B_\eta(x_0) \times \{t_1\}$ and on $\partial B_\eta(x_0) \times (t_1, t_0)$, because $u > v_r^c$ in this domain, we reach a contradiction by applying the strong maximum principle to w, which should be positive in $B_\eta(x_0) \times \{t_0\}$. Hence, u and v_r^c never touch, which means that

$$u(x,t) > v_r^c(x,t) \quad \text{in} \quad B \times (\tau, T).$$

(ii) By letting r decrease to zero, τ also decreases to zero, and thus v_r^c converges to the fundamental solution $v_{c\delta_y}^B$ in $B \times (\tau, T)$. Hence we obtain

$$u(x,t) \geqq v_{c\delta_y}^B(x,t) \quad \text{in} \quad B \times (0, T).$$

By letting the radius of B increase to infinity (recall that $B = B_R(y)$), we find that the fundamental solution $v_{c\delta_y}(x,t) = U_c(x-y,t)$ (this time defined in $\mathbb{R}^N \times (0,T)$) minorizes u in $\mathbb{R}^N \times (0,T)$. Finally, we let c increase to $+\infty$, and we find the comparison with the self-similar IPSS:

$$u(x,t) \geqq \left[\frac{Ct}{|x-y|^2}\right]^{\frac{1}{1-m}}. \tag{2.3}$$

The Proposition follows since $y \in \mathcal{S}$ was chosen arbitrary. □

Remark. A first consequence of this estimate is the everywhere positivity of extended solutions with nontrivial initial singular set. Furthermore, by local regularity, they will be C^∞ smooth on the regular set $\Omega = \{u < \infty\}$. Another consequence of the lower estimate is that we are able to show that any extended solution has a strong blow-up in $\mathcal{S}$ for all times $t > 0$. Even more, the divergence of the integral can be computed in $\mathcal{R}$ for points of $\partial\mathcal{S}$.

Corollary 2.1. *Let u be an extended solution with initial singular set $\mathcal{S}$ defined above in* (2.1). *Then for every $y \in \partial\mathcal{S}$, every $t > 0$ and $r > 0$,*

$$\int_{B_r(y)\cap\mathcal{R}} u(x,t)dx = +\infty.$$

Proof. Let $x \in B_{r/2}(y)$ and let $y' \in \mathcal{S}$ be a point in $\partial\mathcal{S}$ that realizes the distance $d(x,\mathcal{S}) = r' \leqq r/2$. Let $B = B_{r'}(x) \subset B_r(y)$. Since u satisfies the lower estimate (2.2), then

$$\int_{B_r(y)\cap\mathcal{R}} u(x,t)dx \geqq \int_B U_\infty(x-y',t)dx.$$

But we can compute the last integral: since $m > m_c$,

$$\int_B U_\infty(x-y',t)dx = C' t^{\frac{1}{1-m}} \int_0^{r'} r^{-\frac{2}{1-m}} r^{N-1} dr = +\infty,$$

which proves the integral property of u. □

This estimate will have an interest in the study of the Cauchy-Dirichlet problem and the elliptic problem in Sections 4 and 6. By applying the result of Lemma 2.1 to solutions starting at times $t_1 > 0$ we conclude the monotonicity of strongly singular sets of extended solutions at different times.

Corollary 2.2. *If u is an extended solution and the strongly singular set $\mathcal{S}(t)$ at a time t is defined as the set of points $y \in \mathbb{R}^N$ at which $u(\cdot\,, t)$ is not locally integrable, then*

$$\mathcal{S}(t_1) \subset \mathcal{S}(t_2) \quad \textit{for all} \quad 0 \leqq t_1 < t_2.$$

The rate of divergence (2.3) is characteristic of all strongly singular points of extended continuous solutions. To get this lower bound it is in fact sufficient to check condition (2.1) only on the points of zero density in $\mathcal{S}$. More precisely, we have the following technical result that can be skipped at this stage but will be useful later.

Proposition 2.1. *Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^N$ with frontier $\Gamma = \partial\mathcal{S}$ and let the set Γ_0 be such that*

$$\Gamma_0 = \{y \in \Gamma \mid \exists r_0 > 0,\ \text{meas}\{B_r(y) \cap \mathcal{S}\} = 0\ \text{ if } r < r_0\}. \tag{2.4}$$

Let $u \in \mathcal{E}$ be any extended solution such that $u(x, t) = \infty$ on $\mathcal{S}$ for any $t > 0$ and such that for every $y \in \Gamma_0$, $r > 0$,

$$\lim_{t\to 0} \int_{B_r(y)} u(x, t) = +\infty.$$

Then u satisfies the lower bound (2.2) *for every $(x, t) \in Q_T$.*

Proof. We observe that Γ_0 is a subset of points of Γ with zero Lebesgue density in $\mathcal{S}$. For the points $y \in \Gamma_0$, the proof is the same as above, and we find that

$$u(x, t) \geqq \left[\frac{Ct}{|x - y|^2}\right]^{\frac{1}{1-m}}.$$

So we consider a point $y \in \mathcal{S} \setminus \Gamma_0$: for any $r > 0$, the measure of the set $B_r(y) \cap \mathcal{S}$ is positive. Let $\tau > 0$, consider the weak solution U_r^c in $\mathbb{R}^N \times (\tau, T)$ defined by its initial data

$$U_r^c(\tau) = \frac{c}{|B_r(y) \cap \mathcal{S}|}\chi_r,$$

χ_r being the characteristic function of $B_r(y) \cap \mathcal{S}$. Then outside $\mathcal{S}$, $U_r^c(\tau)$ is zero, and on $\partial\mathcal{S} \times (\tau, T)$, U_r^c remains bounded while there u blows up. We can then apply our comparison result on $\mathcal{R}^\varepsilon \times (\tau, T)$, where $\mathcal{R}^\varepsilon = \mathbb{R}^N \setminus \mathcal{S}^\varepsilon$, $\mathcal{S}^\varepsilon$ being the ε-neighborhood of the singular set

$$\mathcal{S}^\varepsilon = \{x \in \mathbb{R}^N \mid \text{dist}(x, \mathcal{S}) \leqq \varepsilon\},$$

and pass to the limit when $\varepsilon \to 0$, which gives

$$u(x,t) \geqq U_r^c(x,t) \quad \text{in} \quad \mathcal{R} \times (\tau, T).$$

Now we let r decrease to zero. It is always possible to define U_r^c because the measure of $B_r(y) \cap \mathcal{S}$ is always positive, so that in the limit, we concentrate the mass c at the point y. Hence by uniqueness of the fundamental solution $U_{c\delta_y} = U_c(x-y,t)$ in $\mathbb{R}^N$, we get, in the limit,

$$u(x,t) \geqq U_{c\delta_y}(x,t) \quad \text{in} \quad \mathcal{R} \times (\tau, T).$$

We end the proof as in the previous theorem by letting c increase to infinity, and then τ decrease to zero, which gives the estimate we were looking for:

$$u(x,t) \geqq \left[\frac{Ct}{|x-y|^2}\right]^{\frac{1}{1-m}} \qquad \forall y \in \mathcal{S} \setminus \Gamma_0.$$

Hence (2.2) holds since this estimate holds for any $y \in \mathcal{S}$. $\square$

We will construct in Section 8 extended solutions having *weak singularities* which do not satisfy the conditions of the last Proposition and the divergence of the solution near them proceeds at a lower rate than (2.3). Moreover, these weak singularities may appear or disappear in time, so that the monotonicity result of Corollary 2.2 is false for the complete singular set. It is therefore advisable to introduce the intermediate class $\mathcal{E}_s$ of extended solutions $u \in \mathcal{E}$ having only strong singularities for $t > 0$: for every space neighborhood U of a point $(x,t) \notin \Omega$, $t > 0$ we have

$$\int_U u(x,t)\,dx = \infty.$$

It is clear that $\mathcal{E}_c \subset \mathcal{E}_s \subset \mathcal{E}$. An example of an extended solution in $\mathcal{E}_s$ with expanding singular set is given by the explicit solution

$$v(x,t) = \left(\frac{Ct}{x^2 - At^{2\theta/N}}\right)^{1/(1-m)}, \quad A > 0, \tag{2.5}$$

with C and θ as in (0.4), which is a variant of the fundamental solutions and has an expanding singular set of the form $\{(x,t) : t > 0, |x| \leqq A^{1/2} t^{\theta/N}\}$.

Thanks to the non-shrinking property of the strong singular set we will deduce two important results. The first one establishes the existence of an initial trace for solutions in $\mathcal{E}_s$. The Borel trace can be expressed as follows: for every nonnegative and compactly supported test function $\varphi \in C_0(\mathbb{R}^N)$,

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\varphi(x)\,dx = \int_{\mathbb{R}^N} \varphi(x)\,d\nu(x), \tag{2.6}$$

the limit being finite or infinite.

Theorem 2.1. *Every extended solution $u \in \mathcal{E}_s$ of* (0.1) *defined in a strip* $\mathbb{R}^N \times (0, T)$ *possesses an initial trace which is a Borel measure $\nu = (\mathcal{S}, \mu)$ with initial singular set*

$$\mathcal{S} = \bigcap_{t>0} \mathcal{S}(t).$$

Proof. We define the initial singular set $\mathcal{S}$ as above and put $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$. Existence of the initial trace on $\mathcal{R}$ is a consequence of (1.3) which we can write as

$$\int_{\mathbb{R}^N} u(x, s)\psi(x)dx \leqq \int_{\mathbb{R}^N} u(x, t)\psi(x)dx + C(\psi)|t - s|^{\frac{1}{1-m}} \tag{2.7}$$

for every $\psi \in C_0^\infty(\mathcal{R})$ and $0 < s, t$ small enough. Indeed, by definition of $\mathcal{R}$, for every open $U \subset\subset \mathcal{R}$, there exists $t(U) > 0$ and $C > 0$ such that

$$\int_U u(x, t)dx \leqq C \quad \forall\, 0 < t < t(U).$$

Thus if t and s are small enough, (2.7) proves first that $\int u(s)\psi$ is bounded when s goes to zero, hence there exists a sequence $s_n \to 0$ such that $u(s_n)$ converges weakly to a measure μ in $\mathcal{R}$ (because ψ is arbitrary). Now if there exists another sequence $t_n \to 0$ such that $u(t_n)$ converges to some other measure μ' in $\mathcal{R}$, then letting s_n decrease first in (2.7), and then $t_n \to 0$, we find that $\mu \leqq \mu'$. But reverting the roles of s_n and t_n, we get the other inequality, so that $\mu = \mu'$, and finally the whole sequence converges to μ in $\mathcal{R}$. Thus,

$$u(t) \underset{t\to 0}{\longrightarrow} (\mathcal{S}, \mu) \quad \text{in the sense of Borel measures,}$$

since by definition of $\mathcal{S}$, $u(t) \to \infty$ in Borel measure on $\mathcal{S}$. □

The second result gives the existence of extended solutions as limits of continuous weak solutions.

Theorem 2.2. *Let μ_n be a sequence of nonnegative Radon measures which converge to $\nu = (\mathcal{S}, \mu)$ in the sense of Borel measures. Let $u_n \geqq 0$ the continuous weak solution of Problem* (0.1),(1.1) *with initial data μ_n. Then u_n converges along a subsequence to an extended solution $u \in \mathcal{E}$ of the problem with constant singular set $\mathcal{S}$ and initial data ν.*

Proof. We recall that a sequence of Radon measures μ_n converges to a measure ν in the sense of Borel measures if, for every smooth test function $\varphi \geqq 0$ with compact support in $\mathbb{R}^N$, we have

$$\lim_{n\to\infty} \int \varphi \, d\mu_n = \int \varphi \, d\nu \tag{2.8}$$

If as usual we note $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$, then for every compact $K \subset \mathcal{R}$, the sequence $\mu_n(K)$ remains bounded as a consequence of (1.3), so that using (1.4) in $\mathcal{R}$, we see that u_n is locally bounded and will converge along a subsequence locally uniformly in $\mathcal{R} \times (0, T)$ to a solution u of the equation with initial trace μ' on $\mathcal{R}$. The fact that $\mu' = \mu$ is a consequence of passing to the limit in (1.3) with $s = 0$ and $t > 0$ when $n \to \infty$, and then letting $t \to 0$. Now for every $y \in \mathcal{S}$, estimate (1.3) with $s = 0$, $t > 0$ and ψ with support intersecting $\mathcal{S}$ shows that for every neighborhood U of y and every $t > 0$,

$$\int_U u(x,t)dx = +\infty,$$

hence $\mathcal{S}$ is preserved in the strong sense for every $t > 0$. It is also clear that $u \in C(\mathcal{R} \times (0,T))$, and that $u(t) \to +\infty$ on $\mathcal{S}$ for every $t > 0$ by Lemma 2.1. Hence $u \in \mathcal{E}_c$ and has initial trace ν. $\square$

The existence of limits of solutions which are singular in some constant set $\mathcal{S}$ and are solutions of (0.1) in the complement of $\mathcal{S}$, has been the motivation to introduce the new concept of an extended continuous solution with non-expanding singular set when the initial data is not a locally finite measure. These solutions will be smooth in $Q_{\mathcal{R}} = \mathcal{R} \times (0,T)$. In the following sections we will use the abbreviation ECS to denote a solution in the class $\mathcal{E}_c$ as defined in the Introduction by properties (i)–(iv).

3. Existence of minimal and maximal ECS

This section is devoted to the construction of extremal ECS with a prescribed initial trace. To begin with, the following result gives existence of a minimal solution with respect to a locally finite μ on $\mathcal{R}$. Note that this solution is only defined in $Q_{\mathcal{R}}$, and not the whole Q_T (in fact it is not an ECS).

Lemma 3.1. *Let $m > m_c$, $\mathcal{R}$ be an open subset of $\mathbb{R}^N$ and $\mu \in \mathcal{M}^+(\mathcal{R})$. Then there exists a minimal solution $\underline{u}_\mu$ of $u_t = \Delta u^m$ in $Q_{\mathcal{R}}$ such that $\underline{u}_\mu(0) = \mu$, in the sense that for any ECS u with initial trace μ on $\mathcal{R}$,*

$$u(x,t) \geqq \underline{u}_\mu(x,t) \quad \text{in} \quad Q_{\mathcal{R}}.$$

Proof. Let u be as above any ECS with initial trace μ on $\mathcal{R}$, and let K_n be an increasing sequence of compact subsets of $\mathcal{R}$, with regular boundary and such that $\cup K_n = \mathcal{R}$. We define u_n^ε as the solution of the Cauchy-Dirichlet problem:

$$\begin{aligned} \partial_t u_n^\varepsilon &= \Delta(u_n^\varepsilon)^m && \text{in } \mathcal{D}'(K_n \times (\varepsilon, T)), \\ u_n^\varepsilon(x,t) &= 0 && \text{on } \partial K_n \times (\varepsilon, T), \\ u_n^\varepsilon(x,\varepsilon) &= u(x,\varepsilon)\chi_n(x) && \text{in } K_n, \end{aligned}$$

χ_n being the characteristic function of K_n. By construction, since u_n^ε and u are bounded in $K_n \times (\varepsilon, T)$,

$$u_n^\varepsilon(x,t) \leqq u(x,t) \quad \text{in} \quad K_n \times (\varepsilon, T).$$

Moreover, when ε decreases to zero, u_n^ε converges locally uniformly in $K_n \times (0, T)$ to some solution u_n with initial data $u_n(0) = \mu\chi_n$. But since $\mu\chi_n$ is finite, by Proposition 1.1 we know that u_n is the unique strong solution of the problem

$$\begin{aligned} \partial_t u_n &= \Delta u_n^m \ \text{ in } \mathcal{D}'(K_n \times (0, T)), \\ u_n(x, t) &= 0 \quad\ \text{ on } \partial K_n \times (0, T), \\ u_n(x, 0) &= \mu\chi_n \ \text{ in } K_n. \end{aligned}$$

In particular, u_n can be constructed independently of any solution. Now we let n go to infinity, then u_n converges locally uniformly in $Q_{\mathcal{R}}$ to a solution $\underline{u}_\mu$, independent of any solution in $Q_{\mathcal{R}}$, with initial data μ and such that

$$\underline{u}_\mu(x, t) \leqq u(x, t) \quad \text{in} \quad Q_{\mathcal{R}},$$

for any solution u. Hence it is the announced minimal solution. $\square$

It might be thought that $\underline{u}_\mu$ is the minimal solution we are looking for, but the problem is that this solution does not necessarily blow up near $\mathcal{S}$ (by the way, this implies that $\underline{u}_\mu$, extended by $+\infty$ on $\mathcal{S}$, is not necessarily a limit solution). In the following Theorem, we impose the blow-up on $\mathcal{S}$:

Theorem 3.1. *Let $m > m_c$ and $\nu = (\mathcal{S}, \mu) \in \mathcal{B}^+(\mathbb{R}^N)$. Then there exists a minimal solution $\underline{u}$ of* (0.1) *with singular set $\mathcal{S}$. In other words, for any ECS u such that* $\operatorname{tr}_{\mathbb{R}^N}(u) = \nu$,

$$\underline{u}(x, t) \leqq u(x, t) \quad \textit{in} \quad \mathcal{R} \times (0, T).$$

Moreover, $\underline{u}$ is a limit of weak solutions, and thus of smooth solutions.

Proof. For $\varepsilon > 0$, let $\mathcal{S}^\varepsilon$ be an ε-neighborhood of $\mathcal{S}$, with regular boundary and $\mathcal{R}^\varepsilon$ be its complement. Let us take for $v_n^{\varepsilon,c}$ a limit solution on $\mathbb{R}^N \times (1/n, T)$ with initial data:

$$v_n^{\varepsilon,c}(x, 1/n) = \begin{cases} U_{c\delta_y}(x, 1/n) & \text{if } x \in \mathcal{S}^{\varepsilon/2}, \\ \underline{u}_\mu(x, 1/n) & \text{if } x \in \mathcal{R}^{3\varepsilon/2}, \\ 0 & \text{if } x \in \mathcal{S}^{3\varepsilon/2} \setminus \mathcal{S}^{\varepsilon/2}, \end{cases}$$

where $y(x) \in \mathcal{S}$ is a point such that $\operatorname{dist}(x, \mathcal{S}) = \operatorname{dist}(x, y)$, $U_{c\delta_y}$ being the fundamental solution with mass $c > 0$ placed at y. We will first compare $v_n^{\varepsilon,c}$ with any ECS u on $\mathcal{R}^\varepsilon \times (1/n, T)$. First, we know from Lemmas 2.1 and 3.1, that

$$u(x, 1/n) \geqq v_n^{\varepsilon,c}(x, 1/n) \quad \text{on} \quad \mathbb{R}^N.$$

We also know that u blows up on $\mathcal{S}$ locally uniformly in time:

$$u(x, t) \geqq \left[\frac{Ct}{d(x, \mathcal{S})^2}\right]^{\frac{1}{1-m}},$$

while $v_n^{\varepsilon,c}$ is bounded. We will compare both solutions in the domain $D = \mathcal{R}^\delta \times (1/n, T)$. The comparison has been made at the initial time. Besides, we can choose

$\delta > 0$ small, depending on u, ε and c, such that on the lateral boundary $\partial\mathcal{R}^\delta \times (1/n, T)$

$$v_n^{\varepsilon,c} \leqq u.$$

We also take $\delta < \varepsilon/2$. Applying Proposition 1.2 we conclude that

$$v_n^{\varepsilon,c}(x,t) \leqq u(x,t) \quad \text{in} \quad \mathcal{R}^\delta \times (1/n, T). \tag{3.1}$$

We observe next that the same inequality is true in the smaller set $\mathcal{R}^\varepsilon \times (1/n, T)$.

Now we let c increase to infinity: $v_n^{\varepsilon,c}$ converges monotonically to a solution v_n^ε with singular set $\mathcal{S}$ (because of Lemma 2.1 and the fact that points $y(x)$ remain in $\mathcal{S}$), and inequality (3.1) is preserved at the limit. Now we let n increase to infinity, and in the limit, we get a solution v^ε with singular set $\mathcal{S}$ and the initial trace of v^ε is μ on $\mathcal{R}^{3\varepsilon/2}$. In fact the initial trace of v^ε is zero on $\mathcal{S}^{3\varepsilon/2} \setminus \mathcal{S}$, but this is not important here since when ε decreases to zero, v^ε converges to a solution $\underline{u}$ with initial trace exactly

$$\operatorname{tr}_{\mathbb{R}^N}(\underline{u}) = (\mathcal{S}, \mu),$$

and with singular set $\mathcal{S}$ for every $t > 0$ (because it is a limit solution – see Theorem 2.2). Moreover, this solution is minimal since by passing to the limit in the different variables, we get

$$u(x,t) \geqq \underline{u}(x,t) \quad \text{in} \quad \mathcal{R} \times (0, T). \qquad \square$$

Remark. For the minimal solution, as well as for the maximal solution as we shall see below, it is not sufficient to assign the infinite value on $\mathcal{S}$ since $\mathcal{S}$ may be of zero measure in $\mathbb{R}^N$. By first taking the $\mathcal{S}^\varepsilon$, we make sure that $\mathcal{S}^\varepsilon$ is “viewed”. But we need the lower bound near $\mathcal{S}$ to prevent the possibility that the singular set of v^ε shrinks to the empty set when ε decreases. It is important that the lower comparison holds thanks to the existence of solutions with Dirac masses, which is not the case when $m \leqq (N-2)_+/N$. Actually, in this range of parameters, when v^ε decreases, we “lose” some points of $\mathcal{S}$, so that existence holds only for a class of singular sets, and a class of measures on its complement (μ has to satisfy a capacity condition – see [42] and Section 9).

We next show that there exists a maximal ECS with a given initial trace by modifying a little the construction of the minimal ECS.

Theorem 3.2. *Let $m > m_c$ and $\nu = (\mathcal{S}, \mu) \in \mathcal{B}^+(\mathbb{R}^N)$. Then there exists a maximal ECS solution $\overline{u}$ of* (0.1) *with initial trace*

$$\operatorname{tr}_{\mathbb{R}^N}(\overline{u}) = \nu,$$

which has a constant blow-up set $\mathcal{S}$. As for the minimal ECS, it is a limit of smooth solutions.

Proof. Let u be any solution. We use the same notation as in the previous Theorem, and we define $v_n^{\varepsilon,c}(x,t)$ as the solution in $\mathbb{R}^N \times (1/n, T)$ with initial data

$$v_n^{\varepsilon,c}(1/n) = \begin{cases} c & \text{if } x \in \mathcal{S}^\varepsilon, \\ \underline{u}_\mu(x, 1/n) & \text{if } x \in \mathcal{R}^\varepsilon. \end{cases}$$

As for the minimal ECS, it is clear that if we let c go to infinity, then n go to infinity and finally ε decrease to zero, we have in the limit a solution v with initial trace $(\mathcal{S}, \mu) = \nu$, thus we have only to check the maximality of v. So let u be any ECS, and notice that u is bounded on $\mathcal{R}^\varepsilon \times (1/n, T)$. But if we let c increase to infinity, $v_n^{\varepsilon,c}$ will converge to a solution v_n^ε such that $v_n^\varepsilon = +\infty$ on $\mathcal{S}^\varepsilon$ and by comparison with IPSS, we have

$$v_n^\varepsilon(x,s) \geqq \left[\frac{Ct}{d(x,\mathcal{S}^\varepsilon)^2}\right]^{\frac{1}{1-m}},$$

hence v_n^ε will be greater than u on $\partial\mathcal{R}^{\varepsilon'} \times (1/n, \infty)$ for some $\varepsilon' > \varepsilon$ small enough. Moreover, by Lemma 3.1,

$$u(x, 1/n) \geqq v_n^\varepsilon(x, 1/n) = \underline{u}_\mu(x, 1/n) \quad \forall x \in \mathcal{R}^{\varepsilon'}.$$

So it is possible to apply Proposition 1.2 on $\Omega = \mathcal{R}^{\varepsilon'}$ and $t_1 = 1/n$, $t_2 = T$. Inequality (1.5) becomes here:

$$\int_{\mathcal{R}^{\varepsilon'} \cap B_R} (u - v_n^\varepsilon)_+(t) \leqq C \int_{\mathcal{R}^{\varepsilon'} \cap B_{2R}} (u - \underline{u}_\mu)_+(1/n) + CR^{N-2/(1-m)}.$$

But since $u \geqq \underline{u}_\mu$, we can let n go to infinity: by weak convergence in measure on $\mathcal{R}$,

$$\int_{\mathcal{R}^{\varepsilon'} \cap B_{2R}} (u - \underline{u}_\mu)_+(1/n) = \int_{\mathcal{R}^{\varepsilon'} \cap B_{2R}} u(1/n) - \int_{\mathcal{R}^{\varepsilon'} \cap B_{2R}} \underline{u}_\mu(1/n) \underset{n\to\infty}{\longrightarrow} 0.$$

This gives

$$\int_{\mathcal{R}^{\varepsilon'} \cap B_{2R}} (u - v^\varepsilon)_+(t) \leqq CR^{N-2/(1-m)},$$

where $v^\varepsilon = \lim_{n\to\infty} v_n^\varepsilon$, and letting R go to infinity, we obtain

$$u(x,t) \leqq v^\varepsilon(x,t) \quad \text{in} \quad \mathcal{R}^{\varepsilon'} \times (0, T).$$

We can let ε' decrease to ε with no problem, so that the same inequality holds on $\mathcal{R}^\varepsilon \times (0, T)$. By passing to the limit when ε goes to zero, we find

$$u(x,t) \leqq v(x,t) \quad \text{in} \quad Q_{\mathcal{R}}.$$

Hence $\overline{u} \equiv v$ is the announced maximal solution. □

Remark. In proving maximality we do not use the continuity property at $u = \infty$, i.e., that $u(x,t) \to \infty$ as $\operatorname{dist}(x, \mathcal{S}) \to 0$. This property is however needed in the proof of minimality. Hence, $\overline{u}$ is maximal in a larger class of solutions with constant singular set.

4. The elliptic problem

In this section, we relate the solutions of the Cauchy problem (0.1), (1.1) with special initial data $(\mathcal{S}, 0)$ to the elliptic equations $-\Delta f^m + \frac{1}{1-m} f = 0$, and $-\Delta\psi + \psi^q = 0$. Passing from one of the elliptic equations to the other is just a matter of changing functions and variables. Let us mention that although we use here some estimates and results for the extended solutions of the fast-diffusion equation, the results of this section can be completely handled by elliptic techniques, which constitute in fact the elliptic version of this work. The complete elliptic theory was already done in [38, 40, 52] except for Theorem 4.1. Our improvement comes from the fact that we only deal with strong singularities here (see however Section 8 for more general singularities).

If u and v are two nonnegative functions defined near a point x_0, we note $u \approx v$ near x_0 if there exist two constants $C_1, C_2 > 0$ such that

$$C_1 v \leqq u \leqq C_2 v.$$

We begin with the following Lemma which explains the link with the elliptic problem.

Lemma 4.1. *Let $m_c < m < 1$ and $\mathcal{S}$ be a closed subset of $\mathbb{R}^N$. Then the minimal and maximal ECS $\underline{u}$ and $\overline{u}$ with initial trace $(\mathcal{S}, 0)$ have the form*

$$\underline{u}(x,t) = t^{\frac{1}{1-m}} f_1(x), \quad \overline{u}(x,t) = t^{\frac{1}{1-m}} f_2(x),$$

where f_1 and f_2 are classical positive solutions of the problem

$$\begin{aligned} -\Delta f^m + \tfrac{1}{1-m} f &= 0 \quad && \text{in } \mathbb{R}^N \setminus \mathcal{S}, \\ f &= +\infty && \text{on } \partial\mathcal{S}. \end{aligned} \tag{4.1}$$

Proof. We show the result for the maximal solution, the method being the same for the minimal solution. (i) If we put

$$v_\lambda(x,t) = \lambda^{-\frac{1}{1-m}} \overline{u}(x, \lambda t),$$

then v_λ is again an ECS for any $\lambda > 0$, and it has the same initial trace, $(\mathcal{S}, 0)$. Indeed, v_λ satisfies the equation in $\mathcal{R} \times (0,\infty)$ and as $t \to 0$, $v_\lambda \to 0$ locally uniformly in $\mathcal{R}$. On $\mathcal{S}$, the integrals of v_λ are always infinite since $\mathcal{S}$ is the strongly singular set of $\overline{u}(\cdot, \lambda)$, so that the initial trace of v_λ is exactly $(\mathcal{S}, 0)$.

(ii) Let us prove next that the property of maximality implies that $v_\lambda(x,t)$ must equal $\overline{u}$, and that this one must have the self-similar form $t^{1/(1-m)} f(x)$ with $f(x) = \overline{u}(x, 1)$. Note that $f \in C^\infty(\mathbb{R}^N \setminus \mathcal{S})$, $f(x) \to \infty$ when $\operatorname{dist}(x, \partial\mathcal{S}) \to 0$. By maximality of $\overline{u}$, we have $v_\lambda(x,t) \leqq \overline{u}(x,t)$ in $\mathbb{R}^N \times (0,T)$, hence, putting $\lambda = 1/t$,

$$v_{1/t}(x,t) = t^{\frac{1}{1-m}} f(x) \leqq \overline{u}(x,t) \quad \text{in} \quad \mathbb{R}^N \times (0,T).$$

Moreover, taking $t = 1$, we get, again from the maximality of $\overline{u}$,

$$v_\lambda(x,1) = \lambda^{-\frac{1}{1-m}}\overline{u}(x,\lambda) \leqq f(x)$$

for every $\lambda > 0$. Putting both inequalities together we get

$$\overline{u}(x,t) = t^{\frac{1}{1-m}} f(x),$$

hence $\overline{u}$ is self-similar, and thus necessarily f satisfies the elliptic problem (4.1). □

Here is now our main result concerning the elliptic problem. Existence and uniqueness under suitable assumptions on $\partial\Omega$ were shown by Marcus & Véron [40] (no conditions for existence). But here we prove uniqueness in the class of solutions which take on the infinite value on $\partial\Omega$ in the strong sense, avoiding the possibility of weak singularities at isolated points:

Definition 1. By a *very large solution* of the problem

$$\begin{aligned} -\Delta\psi + \psi^q &= 0 \quad \text{ in } \Omega,\\ \psi &= +\infty \text{ on } \partial\Omega, \end{aligned} \tag{4.2}$$

we mean a positive function $\psi \in C^2(\Omega)$, satisfying the equation in the classical sense, such that $\psi(x) \to \infty$ as $x \to y \in \partial\Omega,\ x \in \Omega$, and for every $y \in \partial\Omega$, and every $r > 0$,

$$\int_{\Omega\cap B_r(y)} \psi^q(x)dx = +\infty. \tag{4.3}$$

The class of such solutions is called $\mathcal{E}_L$.

In fact, less regularity on ψ can be required, since by standard arguments, ψ will be automatically smooth if it satisfies the equation in the distributional sense.

The solutions constructed above by separation of variables belong to the class $\mathcal{E}_L$, in particular, the strong blow-up condition (4.3) was proved in Corollary 2.1. In fact, there is one-to-one correspondence between separate-variable solutions $u \in \mathcal{E}_c$ with initial data $(\mathcal{S}, 0)$ and solutions $\psi \in \mathcal{E}_L$.

Theorem 4.1. *Let* $1 < q < \frac{N}{(N-2)}$ *if* $N > 2$, $q > 1$ *if* $N = 1, 2$ *and* Ω *be an open subset of* $\mathbb{R}^N$. *Then there exists a unique solution* ψ *of problem* (4.2) *in the class* $\mathcal{E}_L$. *Moreover,*

$$\psi(x) \approx \text{dist}(x,\partial\Omega)^{-\frac{2}{q-1}}. \tag{4.4}$$

Proof. We first apply the preceding result to prove the existence of a minimal and a maximal solution to (4.2): we can construct a minimal and a maximal ECS with initial trace $(\mathcal{S}, 0)$, where $\mathcal{S} = \mathbb{R}^N \setminus \Omega$, and these solutions have the separation-of-variables form. Hence it yields two solutions $\underline{f}$ and $\overline{f}$ of the elliptic problem (4.1). By using the transformation $\psi = f^m$, we get indeed two solutions of problem (4.2) satisfying (4.3) with $q = 1/m \in (1, N/(N-2)_+)$. Moreover, if ψ is a solution of (4.2), satisfying (4.3), then

$$u(x,t) = t^{\frac{1}{1-m}} \psi^q(x)$$

is obviously an ECS with initial trace $(\mathcal{S}, 0)$, hence it can be compared with $\overline{u}$ and $\underline{u}$, which proves the maximality and minimality of $\overline{\psi}$ and $\underline{\psi}$. Finally, the behavior (4.4) of all solutions ψ comes from the behavior of the minimal ECS near $\mathcal{S}$: for any ECS with initial, trace $(\mathcal{S}, 0)$,

$$u(x,t) \geqq \left(\frac{Ct}{\operatorname{dist}(x, \partial\mathcal{S})^2} \right)^{\frac{1}{1-m}},$$

so that using the variable separation for $\overline{u}_{(\mathcal{S},0)}$ and $\underline{u}_{(\mathcal{S},0)}$, and the fact that $q = 1/m$, we find exactly the required behavior from below for any ψ. The behavior from above is well known, and comes from comparison with explicit super-solutions [38].

Now thanks to estimate (4.4), we will prove uniqueness of ψ by using the methods of [38]. Noting as above $\overline{\psi}$ and $\underline{\psi}$ the maximal and minimal solutions for the elliptic problem, we know by (4.4) that there exists a constant $C = C(q, N) \geqq 1$ such that

$$\overline{\psi} \leqq C \cdot \underline{\psi} \quad \text{in} \quad \Omega = \mathbb{R}^N \setminus \mathcal{S}.$$

Now if we assume that $\underline{\psi} \neq \overline{\psi}$, then by the strong maximum principle,

$$\underline{\psi} < \overline{\psi} \quad \text{in} \quad \Omega.$$

So let $\alpha \in (0, 1/C)$, and put

$$V = \underline{\psi} - \alpha(\overline{\psi} - \underline{\psi}) = (1+\alpha)\underline{\psi} - \alpha\overline{\psi}.$$

Since our choice for α, V is nonnegative, $V < \underline{\psi}$, and $(q = 1/m)$

$$\Delta V = (1+\alpha)\underline{\psi}^{1/m} - \alpha\overline{\psi}^{1/m}.$$

By an easy convexity argument, it follows that $\Delta V < V^{1/m}$, that is, V is a super-solution of the elliptic problem, and moreover, for every $\beta \in (0, \alpha)$, it is clear that $\beta\underline{\psi}$ is a sub-solution and

$$\beta\underline{\psi} < V < \underline{\psi}.$$

To construct a solution which lies between $\beta\underline{\psi}$ and V, we use here the parabolic fast-diffusion problem: if

$$U(x,t) = t^{\frac{1}{1-m}} V^{1/m}(x,t),$$

then clearly U is a super-solution of the fast-diffusion problem with initial trace $(\mathcal{S}, 0)$. Let $n \in \mathbb{N}$ and let u_n be the minimal solution with initial trace

$$\operatorname{tr}_{\mathbb{R}^N}(u_n) = (\mathcal{S}, U(1/n)).$$

Then by comparison on $\Omega \times (1/n, T)$, since $U \to \infty$ on $\partial\Omega$ (because $V > \beta\underline{\psi}$),

$$u_n(x,t) \leqq U(x,t) \quad \text{in} \quad \Omega \times (1/n, T).$$

Letting n go to infinity, we find that (up to extraction), u_n will converge to an ECS u with initial trace $(\mathcal{S}, 0)$. Indeed, the singular set $\mathcal{S}$ is preserved thanks to lower estimates for the u_n, and on Ω, since $u \leqq U$ in the limit, the initial trace of u on Ω is zero, but since

$$u(x,t) \leqq U < \underline{u}_{(\mathcal{S},0)} = t^{\frac{1}{1-m}}\underline{\psi}(x)^{1/m},$$

we contradict the minimality of $\underline{u}_{(\mathcal{S},0)}$. Thus necessarily $\underline{\psi} \equiv \overline{\psi}$, which proves uniqueness. □

Remark. Some form of the L^1 divergence condition (4.3) is necessary to obtain uniqueness for general Ω. If, for instance, $\partial\Omega$ contains an isolated point, we can construct a solution with a weak singularity at this point, of the form

$$u(x) \sim c|x|^{-\frac{N-2}{m}} \quad \text{if} \quad N \geqq 3, \qquad u(x) \sim c(\log|x|)^{\frac{1}{m}} \quad \text{if} \quad N = 2,$$

which will be of course different form the very large solution, which has a strong singularity at this point. Solutions with weak singularities will be studied in Section 8.

In order to weaken condition (4.3), we observe that at any point $y \in \partial\Omega$ such that $\text{meas}\{B_r(y)\setminus\Omega\} > 0$ for every $r > 0$, the solution u of the associated evolution problem will have a strong singular point, as we have shown in Lemma 2.1. This implies that ψ will automatically satisfy (4.3) at y. Thus we only need to check (4.3) on points of the set

$$\Gamma_0 = \{y \in \partial\Omega \mid \exists r > 0,\ \text{meas}(B_r(y) \setminus \Omega) = 0\}.$$

Marcus & Véron [40] prove uniqueness without (4.3) for domains such that

$$\partial\Omega = \partial\overline{\Omega}^c,$$

because they show that, then, all $\partial\Omega$ is strongly singular. Their result differs from ours in the fact that they impose a condition on Ω for uniqueness, while we impose a local behavior on the class of solutions.

Corollary 4.1. *Let $m_c < m < 1$ and $\mathcal{S}$ be a closed subset of $\mathbb{R}^N$. Then there exists a unique ECS with initial trace $(\mathcal{S}, 0)$.*

Proof. Since $\overline{u}_{(\mathcal{S},0)}$ and $\underline{u}_{(\mathcal{S},0)}$ have the separation-of-variables form and the elliptic problem has a unique solution, they are equal, hence the solution is unique. □

Estimate (4.4) gives the behavior of ψ as x approaches $\partial\Omega$. When we consider a particular point $x_0 \in \partial\Omega$, the lower estimate

$$\psi(x) \geqq C_1|x - x_0|^{-\frac{2}{q-1}}, \quad x \in \Omega$$

is still true but an upper estimate with this rate may be false when Ω forms a spine near x_0, so that $\operatorname{dist}(x, \partial\Omega)$ is not comparable with $|x - x_0|$. Indeed, any higher rate may be allowed, if the spine is slender enough, as the following result shows. This Proposition extends results form [43] and shows that we can obtain arbitrarily large rates of divergence near special points of $\partial\Omega$ (thin spines).

Proposition 4.1. *Let* $N \geqq 2$, $1 < q < N/(N-2)$ *and* Ω *be an open subset of* $\mathbb{R}^N$. *Assume that* $0 \in \partial\Omega$ *and there exist* $\delta > 0$ *such that*

$$B_\delta(0) \cap \Omega = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}^{N-1} \mid |y| < f(x)\},$$

where f *is a convex function such that* $f \in C^1([0, \delta]; \mathbb{R}_+)$, $f(0) = f'(0) = 0$. *Then the solution* ψ *of the elliptic problem* (4.2) *satisfies*

$$\psi(x, 0) \approx f(x)^{-\frac{2}{q-1}}.$$

Proof. Let $A(x, 0)$ for $0 < x < \delta$. Since f is convex,

$$\operatorname{dist}(A, \partial\Omega) = \operatorname{dist}(A, B),$$

where $B(x_B, y_B)$ is the orthogonal projection of A on the graph of f. More precisely, if T is the tangent to the graph of f at the point B, we have necessarily $(AB) \perp T$, which gives $x_B - x = -f'(x_B)|y_B|$. Hence,

$$\operatorname{dist}(A, B) = \left(f'^2(x_B)|y_B| + |y_B|^2\right)^{\frac{1}{2}} = f(x_B)\left(1 + f'^2(x_B)\right)^{\frac{1}{2}}.$$

On the other hand, as $x \to 0$ we have $y_B = f(x_B) = o(x_B) + o(x)$ $(0 \leqq x_B \leqq x)$, so that $x_B = x + o(x)$. Hence,

$$\operatorname{dist}(A, B) = [f(x) + o(x)][1 + o(1)]^{\frac{1}{2}},$$

which in turn implies that $\operatorname{dist}(A, \partial\Omega) = f(x) + o(x)$. Now, by our estimate (4.4), it follows that for every solution ψ,

$$\psi(x, 0) \approx f(x)^{-\frac{2}{q-1}}. \qquad \square$$

5. Well-posedness of the Cauchy problem in $\mathcal{E}_c$

We proceed now with the central results of our paper. We have seen that uniqueness of ECS holds when $\mu = 0$. We show next uniqueness of the ECS for any initial trace $(\mathcal{S}, \mu) \in \mathcal{B}^+(\mathbb{R}^N)$. We first show the monotone convergence of extremal solutions.

Lemma 5.1. *Let $\nu = (\mathcal{S}, \mu) \in \mathcal{B}^+(\mathbb{R}^N)$ and, for every $R > 0$, let $\nu_R = \nu\chi_R$, where χ_R is the characteristic function of the ball $B_R(0)$. Then when R increases to infinity,*

$$\underline{u}_{\nu_R} \nearrow \underline{u}_{\nu} \quad \textit{and} \quad \overline{u}_{\nu_R} \nearrow \overline{u}_{\nu}.$$

Proof. The convergence of the minimal solutions is obvious since we approach the minimal solution from below. It can be also proved by the method we use below to handle the convergence of the maximal solutions, but this one is not so obvious. Recall that $\overline{u}_{\nu}$ and $\overline{u}_{\nu_R}$ are constructed as the limits of the weak solutions with initial data at $t = 1/n$:

$$(u_\nu)_n^{\varepsilon,c}(1/n) = \begin{cases} c & \text{on } \mathcal{S}^\varepsilon, \\ \underline{u}_\mu & \text{on } \mathcal{R}^\varepsilon, \end{cases}$$

and

$$(u_{\nu_R})_n^{\varepsilon,c}(1/n) = \begin{cases} c & \text{on } \ \mathcal{S}^\varepsilon \cap B_R, \\ \underline{u}_{\mu_R} & \text{on } \ \mathcal{R}^\varepsilon \cup \{|x| \geqq R\}, \end{cases}$$

where $\mu_R = \mu$ on $B_R \cap \mathcal{R}$ and 0 on $\mathbb{R}^N \setminus B_R$. Now we use the techniques of Herrero and Pierre to compare the approximations on $\mathbb{R}^N \times (1/n, T)$, which are ordered: for every $\varphi \in C_0^\infty(\mathbb{R}^N)$ nonnegative,

$$\left(\int [(u_\nu)_n^{\varepsilon,c} - (u_{\nu_R})_n^{\varepsilon,c}](t)\varphi\right)^{1-m} \leqq \left(\int [(u_\nu)_n^{\varepsilon,c} - (u_{\nu_R})_n^{\varepsilon,c}](1/n)\varphi\right)^{1-m} + C(\varphi)|t - 1/n|.$$

We take now the special test functions

$$\varphi_R(x) = \varphi_1(x/R),$$

where $\varphi_1 \in C_0^\infty(\mathbb{R}^N)$, with compact support in $B_2(0)$, and $\varphi = 1$ on $B_1(0)$. Then it follows that

$$\int [(u_\nu)_n^{\varepsilon,c} - (u_{\nu_R})_n^{\varepsilon,c}](1/n)\varphi_{R/2} = \int_{\mathcal{R}^\varepsilon} [\underline{u}_\mu - \underline{u}_{\mu_R}](1/n)\varphi_{R/2},$$

because $\varphi_{R/2}$ has compact support in B_R. Thus when c increases to infinity, and then n increases to infinity, we obtain in the limit

$$\left(\int [u_\nu^\varepsilon - u_{\nu_R}^\varepsilon](t)\varphi_{R/2}\right)^{1-m} \leqq C(\varphi_{R/2})t.$$

Indeed, $\underline{u}_\mu(1/n)$ and $\underline{u}_{\mu_R}(1/n)$ converge weakly in measure to μ and μ_R respectively when n increases to infinity, and these two measures are equal on B_R. Finally we let ε decrease to zero and get

$$0 \leqq \int [\underline{u}_\nu - \underline{u}_{\nu_R}](t)\varphi_{R/2} \leqq C(\varphi_{R/2})^{\frac{1}{1-m}} t^{\frac{1}{1-m}} \leqq CT^{\frac{1}{1-m}} R^{N-2/(1-m)},$$

by our choice for φ. Fixing $t > 0$ and letting R increase to infinity, we get the result:

$$\underline{u}_\nu(t) = \lim_{R\to\infty} \underline{u}_{\nu_R}(t) \quad \forall t > 0. \qquad \square$$

Now we can establish our general uniqueness result.

Theorem 5.1. *Let $m_c < m < 1$ and $\nu = (\mathcal{S}, \mu) \in \mathcal{B}^+(\mathbb{R}^N)$. Then there exists a unique ECS $u_{(\mathcal{S},\mu)}$ with initial trace ν.*

Proof. *Step 1.* We will first assume that $\nu = (\mathcal{S}, \mu)$ has compact support. Let $Z_\mu = \overline{u}^m_{(\mathcal{S},\mu)} - \underline{u}^m_{(\mathcal{S},\mu)}$, where we keep the same notation for the approximations of extremal solutions. Let ϕ_m be the function defined by

$$\phi_m(r,s) = \begin{cases} \frac{r-s}{r^m - s^m} & \text{if } r \neq s, \\ 0 & \text{if } r = s. \end{cases}$$

Then by convexity of ϕ_m (since $0 < m < 1$), it is clear that if $r_1 \geqq r_0$, $s_1 \geqq s_0$ and $r_0 \geqq s_0$, $r_1 \geqq s_1$, we have

$$\phi_m(r_1, s_1) \geqq \phi_m(r_0, s_0).$$

Then writing the equation satisfied for each solution, we get

$$\partial_t(\lambda_\mu Z_\mu - \lambda_0 Z_0) = \Delta(Z_\mu - Z_0),$$

where $\lambda_\mu = \phi_m(\overline{u}_{(\mathcal{S},\mu)}, \underline{u}_{(\mathcal{S},\mu)})$. Integrating the equation in time, and using the fact that $Z_\mu = Z_0$ at $t = 0$, we get

$$\Delta \int_0^t (Z_\mu - Z_0) = \lambda_\mu Z_\mu - \lambda_0 Z_0 \geqq \lambda_\mu(Z_\mu - Z_0).$$

Thus defining $W(t) = \int_0^t (Z_\mu - Z_0)$, we get

$$\begin{aligned} W_t &\leqq \frac{1}{\lambda_\mu}\Delta W \ \text{in } \mathbb{R}^N \times (0,T), \\ W(0) &= 0 \qquad \text{in } \mathbb{R}^N. \end{aligned}$$

Now we use the fact that ν has compact support, which implies that $W(t)$ goes to zero at infinity for every $t > 0$. This is a consequence of the L^∞ estimates of Herrero and Pierre on the complement of the support of ν. Thus we define the function

$$H(t) = W(t)e^{-t} \in C(\mathbb{R}^N) \quad \forall t > 0.$$

Moreover, $Z_\mu - Z_0$ is bounded in $\mathbb{R}^N \times (0, T)$ (because we work with the approximations), W and H are in fact continuous in $\mathbb{R}^N \times [0, T]$. Assume now that there exists a point $(x, t) \in \mathbb{R}^N \times (0, T)$ such that $H(x, t) > 0$. Then the maximum of H is positive and attained at some point $(x_0, t_0) \in \mathbb{R}^N \times (0, T]$. Indeed, H goes to zero when $t = 0$ and when $|x| \to \infty$. Thus we have

$$H_t(x_{0,}t_0) \geqq 0 \quad \text{and} \quad \Delta H(x_0, t_0) = e^{-t}\Delta W(x_0, t_0) \leqq 0,$$

which implies that at this point,

$$0 \leqq H_t = e^{-t}W_t - e^{-t}W \leqq \frac{1}{\lambda_\mu}\Delta W - e^{-t}W \leqq -e^{-t}W = -H,$$

and we reach a contradiction since $H(x_0, t_0) > 0$. Thus

$$W \leqq 0 \quad \text{in} \quad \mathbb{R}^N \times (0, T).$$

This holds for W constructed from the approximate solutions. But by uniqueness in the case $\mu = 0$, we know that when passing to the limit on the approximations, Z_0 goes to zero, hence

$$W = \int_0^t (Z_\mu - Z_0) \to \int_0^t Z_\mu \leqq 0.$$

Since $Z_\mu \geqq 0$ in the limit, this implies that $W = 0$, hence $\overline{u}_{(\mathcal{S},\mu)} = \underline{u}_{(\mathcal{S},\mu)}$.

Step 2. Now, for general Borel measures ν, we use the sequence ν_R of Borel measures defined by

$$\nu_R = \nu\chi_R,$$

where χ_R is the characteristic function of $B_R(0)$. By Lemma 5.1 we know that $\underline{u}_{\nu_R}$ and $\overline{u}_{\nu_R}$ converge monotonically to $\underline{u}_\nu$ and $\overline{u}_\nu$ respectively. But since $\underline{u}_{\nu_R} \equiv \overline{u}_{\nu_R}$ for every $R > 0$, the same holds in the limit:

$$\underline{u}_\nu \equiv \overline{u}_\nu,$$

and uniqueness is proved. □

We can now prove the well-posedness of the Cauchy problem (0.1), (1.1) in the class $\mathcal{E}_c$ together with the converse problem of initial traces. The set of extended solutions $\mathcal{E}_c$ is equipped with the local uniform convergence in $\overline{\mathbb{R}_+}$, while the convergence in the sense of Borel measures has already been defined.

Theorem 5.2. *Let $m_c < m < 1$. Then the mapping*

$$\begin{gathered} \mathrm{tr}_{\mathbb{R}^N} : \mathcal{E}_c \to \mathcal{B}^+(\mathbb{R}^N), \\ u \mapsto u(0) \end{gathered}$$

is a bicontinuous, nondecreasing one-to-one correspondence.

Proof. The fact that the trace application is invertible is a direct consequence of existence. Now by uniqueness, it is clear that the mapping is monotone. Indeed, the solutions are ordered by construction. Finally let us assume that ν_n is a sequence of Borel measures converging to ν in the Borel sense. Then we have already seen in Theorem 2.2 that up to extraction, the associated sequences of solutions converge locally uniformly in $\overline{\mathbb{R}_+}$ to the unique solution u with initial trace ν. In fact, in this Theorem, we also assumed that the ν_n were in locally finite (Radon) measures, but the same proof works also here with no changes. Now if there exists another subsequence which converges, then by uniqueness, the limit is again u, so that finally all the sequences of solutions converge to u. The continuity is thus proved. Obviously, the inverse mapping is also continuous; this can be done essentially in the same way. $\square$

We collect here the most important properties of the ECS. Most of them have been already shown, or they are consequences of former results.

Proposition 5.1. *Let u be the ECS with initial trace $\nu = (\mathcal{S}, \mu) \in \mathcal{B}^+(\mathbb{R}^N)$. Then*

- $u > 0$ *in* $Q_{\mathcal{R}}$,
- $u \in C^\infty(Q_{\mathcal{R}})$,
- $D(u) = mu^{m-1} \in C(Q_T; \overline{\mathbb{R}}_+)$,
- $u_t = \Delta u^m$ *in the classical sense in* $Q_{\mathcal{R}}$.

Proof. In the case of a non-empty singular set $\mathcal{S}$, the positivity property comes from estimate (2.2). Then, since u is continuous, it is automatically smooth in the regular set $Q_{\mathcal{R}}$, since the equation is locally not degenerate. In the case of an empty singular set, this property is well known (see for instance [35] or [3]). The continuity of $D(u)$ is a consequence of the fact that $u > 0$ and regular in $Q_{\mathcal{R}}$, and that $u \to \infty$ as $x \to S$. $\square$

The following derivative estimates are also valid for ECS since they are limit solutions.

Proposition 5.2. *For every ECS in the range $m_c < m < 1$, the following estimates hold:*

(i) *The ratio u_t/u is bounded for any $t > 0$. More precisely,*

$$-\frac{\theta u}{t} \leqq u_t \leqq \frac{u}{(1-m)t}, \quad \theta = (m-1+2/N)^{-1}. \tag{5.1}$$

(ii) *The bounds for spatial derivatives of u^{m-1} are*

$$-\frac{1}{t} \leqq \Delta(mu^{m-1}) \leqq \frac{C_1}{t}, \quad C_1 = \theta(1-m), \tag{5.2}$$

and

$$|\nabla(mu^{m-1})|^2 \leqq C_2 \frac{u^{m-1}}{t}, \quad C_2 = 2\theta(1-m)/N. \tag{5.3}$$

(iii) *Moreover, when the initial trace is* $(\mathcal{S}, 0)$*, then* $u_t > 0$ *in* $Q_{\mathcal{R}}$.

Proof. Inequalities (5.1) and the right-hand one in (5.2) are due to ARONSON & BÉNILAN [3] for regular solutions. We pass to the local uniform limit in $Q_{\mathcal{R}}$ to obtain the same bounds for ECS. Concerning the lower estimate for Δv, $v = mu^{m-1}$, writing the fast-diffusion equation for classical solutions in the form

$$v\Delta v = v_t + |\gamma||\nabla v|^2,$$

it turns out that $v\Delta v \geqq v_t \geqq -v/t$. Dividing by v, we obtain the lower bound for Δv. The estimate for ∇v is then immediate.

When $\mu = 0$, it is clear that $u(t) > u(t+\tau)$ for any $t, \tau > 0$. Indeed, $u(t+\tau)$ can be viewed as an ECS with initial trace $(\mathcal{S}, u(\tau))$, and thus by uniqueness, it is greater than $u(t)$ which has initial trace $(\mathcal{S}, 0)$. Since u_t is defined in $Q_{\mathcal{R}}$ because u is smooth there, we see that indeed $u_t \geqq 0$ in $Q_{\mathcal{R}}$. Since $v = u_t$ satisfies a classical parabolic equation in $Q_{\mathcal{R}}$, $u_t > 0$ in this set. □

The two-sided estimates on u_t and on Δv are typical of the range $m_c < m < 1$ and do not hold for $m > 1$ or $m < m_c$. An estimate from below for Δu^{m-1} works for $m > 1$ as well as an estimate from below for u_t/u; the estimate from above for u_t/u holds for all $m < 1$. Estimates (5.2) and (5.3) are an indication of the interest of reviewing the whole theory in terms of the pressure variable v. We will devote paper [21] to such an analysis.

Let us remark that, but for (5.2)-left, the constants in the estimates are sharp, as can be checked by inspection of the IPSS and the fundamental solutions (take the limit $x \to \infty$ or $x = 0$).

We conclude this section with a simple continuity result at $t = 0$.

Proposition 5.3. *Let* $u \in \mathcal{E}_c$ *and assume that the initial trace* ν *is given in a neighborhood* U *of a point* $x_0 \in \mathbb{R}^N$ *as a function* $d\nu = f(x)dx$ *which is continuous at* x_0*. Then*

$$\lim_{t\to 0, x\to x_0} u(x,t) = f(x_0). \tag{5.4}$$

Proof. Since ν is continuous at x_0, for any $\varepsilon > 0$, there exists a ball $V \subset U$ centered at x_0 such that

$$c - \varepsilon \leqq f(x) \leqq c + \varepsilon \quad \forall x \in V, \quad c = f(x_0).$$

If $c > 0$, we construct a lower barrier by considering any smooth bounded function $g(x)$ supported in V such that $g(x) \leqq f(x)$ in V and $g(x_0) \geqq c - 2\varepsilon$. By classical theory, the corresponding solution u_1 is continuous down to $t = 0$ in $\mathbb{R}^N$, and $u_1(x,t) \leqq u(x,t)$ in Q. Hence we obtain the lower estimate

$$\lim_{t\to 0, x\to x_0} u(x,t) \geqq f(x_0).$$

The barrier for the upper bound is constructed by separation of variables in the form

$$u_2(x,t) = (t+a)^{\frac{1}{1-m}} F(x),$$

where $F(x)$ solves the elliptic problem (4.1) in V, and we select a so that $F(x_0)$ $a^{1/1-m} = c + \varepsilon$. Comparison shows again that $u_2(x,t) \geqq u(x,t)$ in $V \times (0,\infty)$. Hence the upper limit since F is a continuous function. □

6. The Dirichlet problem

Our results prove directly that the singular Dirichlet problem posed in an open domain $\Omega \subset \mathbb{R}^N$ has a unique solution, which is a universal barrier in Ω. Here, Ω need be neither bounded nor regular.

Theorem 6.1. *Let $m_c < m < 1$ and μ be a nonnegative Radon measure in Ω, open subset of $\mathbb{R}^N$. Then there exists a unique extended solution $u \in C(\overline{\Omega} \times (0,\infty); \overline{\mathbb{R}_+})$ of the following problem:*

$$(\mathrm{CD})_\infty \quad \begin{cases} u_t = \Delta u^m & \text{in } \Omega \times (0,\infty), \\ u(x,t) = +\infty & \text{on } \partial\Omega \times (0,\infty), \\ u(0) = \mu & \text{in } \Omega. \end{cases}$$

In fact this solution is the unique ECS in $\mathbb{R}^N \times (0,\infty)$ with initial trace $(\mathcal{S},\mu)$, where $\mathcal{S} = \mathbb{R}^N \setminus \Omega$.

On the contrary, the homogeneous Dirichlet problem in an open set Ω does not necessarily have a solution for two reasons. The first one is well known and is due to the geometry of Ω. For instance, if Ω is a punctured ball, it is clear that it is not possible to construct a positive solution of the elliptic problem

$$\begin{aligned} -\Delta\psi + \psi^q &= 0 \text{ in } \Omega, \\ \psi &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Hence for similar reasons, it is not possible in this case to construct a solution of the parabolic Dirichlet problem

$$(\mathrm{CD})_0 \quad \begin{cases} u_t - \Delta u^m = 0 & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \neq 0 & \text{in } \Omega. \end{cases}$$

The optimal condition on $\partial\Omega$ is that it satisfies a Wiener criterion, which can be expressed as an H^1-capacity density.

Now, even in the case of regular, bounded subsets Ω of $\mathbb{R}^N$, it turns out that there is no solution to $(\mathrm{CD})_0$ if

$$\int_\Omega \mathrm{dist}(x,\partial\Omega)\,d\mu(x) = +\infty, \tag{6.1}$$

since in this case the zero lateral data is lost not only pointwise but also in a strong sense on some part of the boundary. We derive a theory of extended solutions for the homogeneous Cauchy-Dirichlet problem $(\mathrm{CD})_0$ in [22] when $(N-2)_+/N < m < 1$, where the following facts are established:

- If $\nu = (\mathcal{S},\mu)$ is a Borel measure in Ω such that $\mathrm{dist}(\mathcal{S},\partial\Omega) > 0$ and $\int_\Omega \mathrm{dist}(x,\partial\Omega)\,d\mu(x) < \infty$, there exists a unique extended solution u in the sense of the present paper, which takes on the zero lateral data on $\partial\Omega \times (0,\infty)$ in the weak sense (i.e., in the sense if integration by parts).

- If $\nu = (\mathcal{S}, \mu)$ is such that $\overline{\mathcal{S}} \cap \partial\Omega \neq \emptyset$ or (6.1) holds, then the lateral data is lost somewhere in $\partial\Omega$ for any positive time.

We refer to [22] for a more detailed study, including results concerning the long-time behavior of the solutions, the presence of an extra measure term on $\partial\Omega \times \{0\}$, as well as new regularizing effects. A new critical exponent appears in that study, $m_1 = (N-1)/(N+1)$.

7. Asymptotic behavior for the Cauchy problem

We discuss in this section the behavior of the solutions as $|x| \to \infty$ or $t \to \infty$. We start with a general result on space asymptotics, valid for all nontrivial ECS. For weak solutions, this result was proved by HERRERO & PIERRE [35].

Proposition 7.1. *For every ECS u,*

$$\lim_{|x|\to\infty} |x|^{2/(1-m)} u(x,t) \geqq (Ct)^{\frac{1}{1-m}}, \tag{7.1}$$

where $C = C(m, N)$ is defined in (0.5). *This rate is the optimal minimal rate and, moreover, the limit is exact in the case of compactly supported initial traces.*

Proof. The lower rate (7.1) comes from the lower estimates for u: if we first assume that $\nu = (\mathcal{S}, \mu)$ has compact support, then it is clear that

$$\frac{\operatorname{dist}(x, \mathcal{S})}{|x|} \underset{|x|\to\infty}{\longrightarrow} 1,$$

so that (2.2) implies (7.1). For general ν we take an increasing sequence of compactly supported Borel measures ν_n. As in Lemma 5.1, by uniqueness, the solution u_n with initial trace ν_n converges monotonically to u, and since u_n satisfies (7.1), u also in the limit. Now let us assume that $\nu = (\mathcal{S}, \mu)$ is compactly supported. Consider the pseudo-Barenblatt solutions

$$U_c(x,t) = \left(\frac{Ct}{(x^2 - At^{2\theta/N})_+}\right)^{1/1-m}, \tag{7.2}$$

where $\theta = N/(2 - N(1-m))$, $A = k(m, N)c^{-\alpha}$ where c is the mass carried by U_c. Then $U_c(x,t) = \infty$ on the set $\{(x,t) \in Q_T \mid |x| \leqq \sqrt{A} \cdot t^{\theta/N}\}$, so for $\tau > 0$ fixed, there exists c big enough such that A is small and the support of ν is contained in the set $\{x \in \mathbb{R}^N \mid \sqrt{A} \cdot \tau^{\theta/N}\}$. Thus we can easily compare the ECS u with U_c, since u is obtained as a limit of smooth solutions with initial data in the same support as ν. This proves that for any $t > 0$,

$$\lim_{|x|\to\infty} |x|^{\frac{2}{1-m}} u(x,t) \leqq \lim_{|x|\to\infty} |x|^{\frac{2}{1-m}} U_c(t) = (Ct)^{\frac{1}{1-m}}.$$

Thus we have the exact rate at infinity for such solutions. □

Such a lower bound holds also near the singularities and is typical of the singularities that remain constant in time, i.e., it characterizes our class $\mathcal{E}_c$. In fact, we will see in Section 8 solutions with expanding strong singularities whose behavior is like $|x|^{1/(1-m)}$, and weak singularities with a rate $|x|^{-(N-2)/m}$.

There are three kinds of results that can be obtained for the large-time behaviour of the ECS with non-empty singular sets. They describe the way in which the singular set $\mathcal{S}$ radiates into the surrounding space. The first one concerns the asymptotics for fixed x. We show convergence to the stationary problem. As announced, only the radiation of the singular set determines the limit, the locally finite part of the measure being negligible for large t.

Theorem 7.1. *Let $u_{(\mathcal{S},\mu)}$ be an ECS. with initial trace $(\mathcal{S},\mu) \in \mathcal{B}^+(\mathbb{R}^N)$, and assume that $\mathcal{S}$ is non-empty. Then*

$$\lim_{t\to\infty} t^{-\frac{1}{1-m}} |u_{(\mathcal{S},\mu)} - u_{(\mathcal{S},0)}|(x,t) = 0 \quad \textit{locally uniformly in } \mathcal{R},$$

or in other words,

$$\lim_{t\to\infty} t^{-\frac{1}{1-m}} u_{(\mathcal{S},\mu)}(x,t) = f(x) \quad \textit{locally uniformly in } \mathcal{R},$$

where f is the unique solution of problem (4.1) (*with strong singularities on the boundary*).

Proof. We will first assume that $\mathcal{S}$ contains the complement of a ball B_R. Thus the regular set $\mathcal{R}$ is included in B_R. Let $\varepsilon > 0$, and $t_0 > 0$. Then $u_{(\mathcal{S},\mu)}(t_0)$ is bounded on $\mathcal{R}^\varepsilon = \mathbb{R}^N \setminus \mathcal{S}^\varepsilon$, where as usual $\mathcal{S}^\varepsilon = \{x \in \mathbb{R}^N \mid \mathrm{dist}(x,\mathcal{S}) \leqq \varepsilon\}$ (recall that $\mathcal{R}^\varepsilon$ is bounded). Since $u_{(\mathcal{S},0)}$ satisfies the lower bound (2.2), there exists $\tau > 0$ such that

$$u_{(\mathcal{S},\mu)}(t) \leqq u_{(\mathcal{S},0)}(t+\tau) \quad \text{on} \quad \mathcal{R}^\varepsilon \times (t_0,\infty).$$

Now we use the fact that $u_{(\mathcal{S},0)}$ has the separation-of-variable form: if f is as above, then

$$t^{-\frac{1}{1-m}} u_{(\mathcal{S},\mu)}(t) \leqq t^{-\frac{1}{1-m}} (t+\tau)^{\frac{1}{1-m}} f(x),$$

and thus when $t \to +\infty$,

$$\lim_{t\to\infty} t^{-\frac{1}{1-m}} u_{(\mathcal{S},\mu)}(x,t) \leqq f(x) \quad \text{in} \quad \mathcal{R}^\varepsilon.$$

Now since $\varepsilon > 0$ is arbitrary, we obtain the upper estimate pointwise in $\mathcal{R}$. The lower estimate comes from the fact that $u_{(\mathcal{S},\mu)} \geqq u_{(\mathcal{S},0)}$ in $\mathcal{R} \times (0,T)$. Moreover, if x stays in a fixed compact $K \subset \mathcal{R}$, then there exists $\tau > 0$ as above such that

$$t^{-\frac{1}{1-m}} f(x) \leqq t^{-\frac{1}{1-m}} u_{(\mathcal{S},\mu)}(t) \leqq t^{-\frac{1}{1-m}} (t+\tau)^{\frac{1}{1-m}} f(x) \quad \text{in} \quad K \times (t_0,\infty), \tag{7.3}$$

and thus the limit is uniform in K, which proves the result.

Now for general $\mathcal{S}$, we have only to find an upper estimate since the lower one comes from the fact that $u_{(\mathcal{S},\mu)} \geqq u_{(\mathcal{S},0)}$, as above. Let us consider a sequence $\mathcal{S}_n$ of singular sets such that

$$\mathcal{S}_n = \mathcal{S} \cup \{\mathbb{R}^N \setminus B_n(0)\}.$$

Then the unique ECS u_n with initial trace $(\mathcal{S}_n, \mu_n)$, where $\mu_n = \mu/_{\mathcal{R}_n}$ converges monotonically to the unique ECS $u_{(\mathcal{S},\mu)}$ with initial trace $(\mathcal{S}, \mu)$. Thus defining f_n as the unique solution of (4.1) in $\Omega_n = \mathcal{R}_n$, (7.3) gives, for some $\tau = \tau(n, \varepsilon)$,

$$t^{-\frac{1}{1-m}} u_{(\mathcal{S},\mu)}(t) \leqq t^{-\frac{1}{1-m}} u_n(t) \leqq t^{-\frac{1}{1-m}} (t+\tau)^{\frac{1}{1-m}} f_n(x) \quad \text{in} \quad \mathcal{R}_n^{\varepsilon}.$$

Moreover, it is clear that if K is a fixed compact in $\mathcal{R}$, then τ can be chosen independently of n and ε. Thus letting n increase to infinity, by monotonicity of the f_n, we get

$$t^{-\frac{1}{1-m}} u_{(\mathcal{S},\mu)}(t) \leqq t^{-\frac{1}{1-m}} (t+\tau)^{\frac{1}{1-m}} f(x) \quad \text{in} \quad K,$$

where f is the unique solution of (4.1) in $\Omega = \mathcal{R}^{\varepsilon}$. We end as above by letting t increase to infinity, which gives the local uniform convergence. $\square$

Remark. More precisely, it is obvious by (7.3) that if x remains in a compact set $K \subset \mathcal{R}$,

$$u_{(\mathcal{S},\mu)}(x,t) = t^{\frac{1}{1-m}} f(x)(1 + O(1/t)). \tag{7.4}$$

When $\mathcal{S} = \emptyset$, the case of classical weak solutions, the long-time behavior is given by a smaller rate than (7.4) (such a rate characterizes then the presence of singularities).

Theorem 7.2. *Let u be a continuous distributional solution of $u_t = \Delta u^m$. Then as t goes to infinity,*

$$u(x,t) = o(t^{\frac{1}{1-m}}) \tag{7.5}$$

locally uniformly in $\mathbb{R}^N$.

Proof. Let $u_R \in \mathcal{E}_c$ be the unique ECS with initial data ν_R, the Borel measure defined as

$$\nu_R(E) = \begin{cases} \mu(E) & \text{if } E \subset B_R(0), \\ +\infty & \text{otherwise}, \end{cases}$$

where μ is the initial trace of u (it is a Radon measure in $\mathbb{R}^N$ since u does not have singularities in the cylinder $\mathbb{R}^N \times (0, T)$). Then obviously,

$$u(x,t) \leqq u_R(x,t) \quad \text{in} \quad B_R(0) \times (0,T),$$

and by 7.4, since in our case the singular set is exactly $\mathbb{R}^N \setminus B_R(0)$, we have

$$u_R(x,t) = C(m,N) t^{\frac{1}{1-m}} (R - |x|)^{-\frac{2}{1-m}} [1 + O_R(1/t)].$$

Thus if x remains in a compact set K, for any $\varepsilon > 0$, there exists R big enough such that when t goes to infinity,

$$u(x,t) = \varepsilon \cdot t^{\frac{1}{1-m}}[1 + O_\varepsilon(1/t)],$$

which proves that $u(x,t) = o(t^{\frac{1}{1-m}})$ locally uniformly in $\mathbb{R}^N$. $\square$

Remark. We can obtain continuous weak solutions having as time growth rate any power less than $1/(1-m)$ by means of the family of self-similar solutions of the form

$$u(x,t) = t^\alpha f(xt^{-\beta}),$$

which solve the Cauchy problem with initial data $u_0(x) = c|x|^\gamma$, with $\gamma(1-m) + 2 \neq 0$. The relation between α, β and γ is then given by

$$\alpha = \frac{\gamma}{2 + \gamma(1-m)}, \quad \beta = \frac{1}{2 + \gamma(1-m)}. \tag{7.6}$$

It is clear that $u(0,t) = O(t^\alpha)$, and as $\gamma \to \infty$, we see that $\alpha \to 1/(1-m)$, a rate that is never attained according to (7.5). For a more detailed description of the self-similar solutions cf. Appendix, Subsection A4.

The second asymptotic result states the *intermediate asymptotics* in expanding sets. A first result in the direction of time asymptotics in expanding sets was proved by FRIEDMAN & KAMIN [32], who show that the source-type solutions, which are self-similar solutions formally corresponding to $\gamma = -N$ in (7.6), represent the asymptotic behaviour of all solutions with integrable initial data, $u_0 \in L^1(\mathbb{R}^N)$. We show that for a large class of ECS, the intermediate behavior is given by the IPSS.

Theorem 7.3. *Let $\beta > 0$ and assume that u is an ECS with initial trace $\nu = (\mathcal{S}, \mu) \in \mathcal{B}^+(\mathbb{R}^N)$, where $\mathcal{S}$ is bounded and non-empty, and $d\mu = f\,dx$. Under the condition*

$$f(x) = o(|x|^\gamma), \qquad \gamma = \frac{1-2\beta}{\beta(1-m)}, \tag{7.7}$$

the asymptotic formula is

$$\lim_{t\to\infty} t^{-\alpha}|u(y,t) - U_\infty(y,t)| = 0, \quad \alpha = \frac{1-2\beta}{1-m} > 0, \tag{7.8}$$

uniformly on sets of the form $C_1 t^\beta \leqq |y| \leqq C_2 t^\beta$.

Proof. Note that $\gamma > -\frac{2}{1-m}$. Given β as (7.6), we perform the change of variables

$$u_\lambda(x,t) = \lambda^{-\alpha} u(\lambda^\beta x, \lambda t), \quad \lambda > 1,$$

which is again a solution since $\alpha(1-m)+2\beta = 1$. The initial trace of u_λ satisfies the following estimate as $\lambda \to \infty$: for every $\varepsilon, c > 0$ there exist λ_0 such that for $\lambda > \lambda_0$ and $|x| > c$

$$u_{\lambda 0}(x) = \lambda^{-\alpha} u_0(\lambda^\beta x) \leqq \varepsilon |x|^\gamma \lambda^{\beta\gamma-\alpha}.$$

By our assumptions on α and γ, the exponent of λ vanishes so that

$$\lim_{\lambda\to\infty} u_{\lambda 0}(x) = 0$$

uniformly away from zero. On the other hand, all the u_λ's are strongly singular at zero, hence $u_\lambda(x,t)$ converges to the IPSS on compact sets away from $|x| = 0$. This happens in particular for $t = 1$, which gives

$$\lim_{\lambda\to\infty} \lambda^{-\alpha} u(\lambda^\beta x, \lambda) = U_\infty(x,1) \quad \text{for} \quad C_1 \leqq |x| \leqq C_2.$$

Writing $\lambda = t$ and $\lambda^\beta x = y$, we get the desired formula (7.8). $\square$

Remark. Actually, the same proof works if μ is a Radon measure and condition (7.7) is satisfied in integral average,

$$\frac{1}{R^N}\int_{B_R} d\mu = o(R^\gamma), \qquad R \to \infty.$$

The condition is optimal since for $f(x) = O(|x|^\gamma)$ the behavior is not given by U_∞. This can be also checked on the family of self-similar solutions $u(x,t) = t^\alpha f(xt^{-\beta})$. These solutions behave like $t^\alpha f(\xi_0)$ on the set $|x| = \xi_0 t^\beta$. Actually, they give the intermediate asymptotics for a larger class of data

Theorem 7.4. *Under the conditions of Theorem 7.3, with the assumption of f replaced by*

$$f(x) = c|x|^\gamma + o(|x|^\gamma) \quad as \quad |x| \to \infty,$$

the asymptotic formula is

$$\lim_{t\to\infty} t^{-\alpha} |u(y,t) - U_\gamma(y,t)| = 0,$$

where U_γ is the self-similar solution with initial data $(\{0\}, c|x|^\gamma)$, and the convergence is uniform on the same sets $C_1 t^\beta \leqq |y| \leqq C_2 t^\beta$. If $\mathcal{S} = \emptyset$, the same result is true taking U_γ as the self-similar solution with initial data $(\emptyset, c|x|^\gamma)$, and convergence is uniform in $|y| \leqq C_2 t^\beta$.

The proof is essentially the same as in Theorem 7.3 and we leave it to the reader.

The term “intermediate asymptotics” refers to the fact that this limit does not happen either for fixed x, or for far-away regions of the form $|y|t^{-\beta} \to \infty$ as $t \to \infty$. In this *far-field region* the behavior depends on the asymptotics of the initial data. This third behavior can be checked on the same family of self-similar solutions For $\gamma > -2/(1-m)$, i.e., $\beta > 0$, the initial data imply that $f(\xi) \sim c|\xi|^\gamma$ as $|\xi| \to \infty$ whenever $\gamma > -2/(1-m)$. This implies that u behaves like $|x|^\gamma t^{\alpha-\beta\gamma}$ on sets of the form $|x|t^{-\beta} \gg 1$.

8. Extended solutions with weak singularities. Expanding singular sets

Up to now, we have only considered strong singularities in $\mathbb{R}^N \times (0, T)$, that is, the points y where the solution u is infinite and such that, at some time $t > 0$, for every $r > 0$,

$$\int_{B_r(y)} u(x,t)dx = +\infty.$$

In this section, we first investigate the possibility that solutions develop weak singularities. That is, some points where u is infinite, but where the above integral is finite for some $r_0 > 0$. Our previous study has proved that the theory of initial trace for solutions having only non-expanding strong singularities is complete, but this is not the case for solutions presenting weak singularities. Indeed, we shall see below that the initial trace does not characterize these solutions, since weak singularities may appear only after some positive time, and then possibly increase to form a strong singularity, or even disappear in finite time.

Proposition 8.1. *Let $(N-2)_+/N < m < 1$, and $f \in C([0,T])$, nonnegative. Then the following problem has a weak solution:*

$$\begin{aligned} u_t - \Delta u^m &= \delta_0(x) \otimes f(t) \ \ \textit{in } \mathcal{D}'(Q_T), \\ u(0) &= 0 \qquad\qquad\quad \textit{in } \mathbb{R}^N, \end{aligned}$$

and $u \in L^1(Q_T)$.

Proof. We follow the proof in [44]. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be nonnegative and supported in $B_1(0)$ such that $\int \eta = c$, and moreover assume that η is radially symmetric and decreasing. Let $\eta_k(x) = k^N \eta(kx)$, then η_k converges to δ_0 weakly in measure in $\mathbb{R}^N$, and according to standard theory, there exists a unique bounded solution u_k of the following problem:

$$\begin{aligned} u_t - \Delta u^m &= \eta_k(x) \otimes f(t) \ \ \text{in } \mathcal{D}'(Q_T), \\ u(0) &= 0 \qquad\qquad\quad \text{in } \mathbb{R}^N. \end{aligned}$$

It is obvious that the u_k are radially symmetric, non-increasing in $|x|$, and that they are bounded in $L^\infty(0,T; L^1(\mathbb{R}^N))$ since $\int u_k(t) = f(t)$ is bounded in $[0,T]$. As in [44], we have the following uniform bound: for some constant $K > 0$,

$$u_k(x,t) \leqq K|x|^{-\frac{N-2}{m}}. \tag{8.1}$$

This allows us to pass to the limit in the equation, and we find that (up to extraction) the sequence $\{u_k\}$ converges to a function u which is solution of our problem with the singular right-hand side. Moreover, estimate (8.1) still holds in the limit for u, so that $u \in L^1(Q_T)$. $\square$

Consequences of this result.

- We can construct the IPSS by solving the above problem with $f = c$ and letting c increase to infinity. Indeed, it is clear that in this case the solution u_k satisfies the scaling property

$$u_k(x,t) = k^{\frac{N-2}{m}} u_1\Big(kx, k^{\frac{N(m-1)+2}{m}} t\Big),$$

hence when $c \to \infty$, we find a self-similar solution which is the IPSS.
- We can construct solutions with standing weak singularities, by taking $f = 1$ in the above example, although it is not seen on the initial trace which is zero.
- We can construct solutions with singularities which appear only after a certain time, and which disappear in finite time. This is achieved by taking f with compact support in $(t_1, t_2) \subset (0, T)$.
- We can create a strong singularity after some positive time thanks to weak singularities (see the theorem just below). In this case, the strong singularity is not seen on the initial trace.

Theorem 8.1. *Let $(N-2)_+/N < m < 1$ and $f \geqq 0$, continuous in $[0, T)$ with values in $\overline{\mathbb{R}_+}$ We assume that $f \nearrow +\infty$ when $t \nearrow \tau$ for some $\tau \in (0, T)$. Then there exists a solution u to the following problem:*

$$\begin{aligned} u_t - \Delta u^m &= \delta_0(x) \otimes f(t) \ \textit{in}\ \mathcal{D}'(Q_T), \\ u(0) &= 0 \qquad\qquad\quad\ \textit{in}\ \mathbb{R}^N. \end{aligned}$$

Such a solution u has a weak singularity at $x = 0$ for every $t \in (0, \tau)$, and it has a strong singularity at $x = 0$ on $[\tau, T)$ if for instance $f = \infty$ on $[\tau, \tau + \varepsilon]$.

Proof. The construction on $(0, \tau)$ has been already made above. Moreover, it is easy to see that for any $c > 0$, there exists $t_c \in (0, \tau)$ sufficiently close to τ such that $f(t_c) > c$. Hence on (t_c, T),

$$u(x,t) \geqq u_c(x, t - t_c),$$

where u_c is the solution with $f = c$. We have seen that letting c increase to infinity yields the IPSS v_∞, and thus

$$u(x,t) \geqq v_\infty(t - \tau) \quad \text{on} \quad \mathbb{R}^N \times (\tau, T).$$

Thus u has a strong singularity at $x = 0$ for any $t \geqq \tau$ (at time $t = \tau$, we take the trace). $\square$

The previous examples show that the set of weak singularities does not enjoy any monotonicity properties, which is not the case for strong singularities, as proved by Lemma 2.1: once a strong singularity is created, it remains as such for any larger time. We give below a construction of solutions with general expanding strong singular sets, which proves in particular that strong singularities may develop in finite time although there are none in the initial trace. So in this case also, the initial trace does not characterize the solutions.

Theorem 8.2. *Let $m_c < m < 1$ and, for every $t \in (0, T)$, let $\mathcal{S}(t) \in \mathbb{R}^N$ be a closed set such that the mapping $t \mapsto \mathcal{S}(t)$ is nondecreasing. Then there exists an extended solution with strong singular set $\mathcal{S}(t)$ at time $t \in (0, T)$ and initial trace $(\mathcal{S}(0), 0)$, where*

$$\mathcal{S}(0) = \bigcap_{t>0} \mathcal{S}(t).$$

In particular, it is possible to start with $\mathcal{S} = \emptyset$. If moreover the mapping $t \mapsto \mathcal{S}(t)$ is continuous, then the solution constructed is continuous with values in $\mathbb{R}_+ \cup \{+\infty\}$.

Proof. We define $\mathcal{S}_0 = \cap_{t>0}\mathcal{S}(t)$. Let χ_t be the characteristic function of $\mathcal{S}(t)$. Then by our assumptions, $t \mapsto \chi_t$ is nondecreasing. We solve the problem

$$\begin{aligned} u_t - \Delta u^m &= f_n^{\varepsilon,c}(x,t) \text{ in } \mathcal{D}'(Q_T), \\ u(0) &= 0 \qquad\qquad \text{ in } \mathbb{R}^N, \end{aligned}$$

where $f_n^{\varepsilon,c}$ is continuous and such that

$$f_n^{\varepsilon,c}(x,t) \underset{n\to\infty}{\longrightarrow} c \cdot \chi_t^{\varepsilon}(x,t),$$

χ_t^{ε} being the characteristic function of $\mathcal{S}^{\varepsilon}(t) = \{x \in \mathbb{R}^N \mid \operatorname{dist}(x, \mathcal{S}) \leqq \varepsilon\}$. By arguments similar to those in the case of point singularities, we can solve the problem and pass to the limit which yields a solution $u^{\varepsilon,c}$ with right-hand side $c \cdot \chi_t^{\varepsilon,c}$. Now letting c increase to infinity, the $u^{\varepsilon,c}$ converge monotonically to a solution u^{ε} possessing a strong singularity on $\mathcal{S}^{\varepsilon}(t)$ for every $t > 0$. Indeed, it is clear that we can compare u^{ε} with any solution with right-hand side $c\delta_y(x) \times 1(t)$, y being any point in $\mathcal{S}^{\varepsilon}$, and we can let c increase to infinity, which yields an IPSS at y. The same being true for $y \in \mathcal{S}$, when we let ε decrease to zero, the u^{ε} decreases to a solution u which has strong singularities on $\mathcal{S}$, and we find for any $t > t_0 \geqq 0$,

$$u(x,t) \geqq \left(\frac{C(t-t_0)}{\operatorname{dist}(x, \mathcal{S}(t_0))^2} \right)^{\frac{1}{1-m}}. \tag{8.2}$$

It is moreover clear that if $t \mapsto \mathcal{S}(t)$ is continuous, the solution is continuous with extended values in $\mathbb{R}_+ \cup \{+\infty\}$. □

Remark. It is important to recall that limit solutions of the fast-diffusion equation $u_t = \Delta u^m$ have a constant singular set: these are in fact what we call ECS. Thus, solutions with a strictly expanding singular set are not limit solutions of this equation, but they are if we add a right-hand side to the equation. It seems that a general theory for such solutions can be drawn. At least, we have just constructed a maximal solution, and a minimal solution can be constructed by using the same method as for ECS. However, the question of uniqueness (given the $\mathcal{S}(t)$) is not clear.

Asymptotic behavior near singularities. We finally give three examples which prove that the space behavior near the singular set given in (2.2) does not hold in the case of an expanding singular set. In fact the blow-up rate (exponent) is divided by two in this case, and can be explained by means of the Darcy Law, which says that the velocity of normal movement of an interface is given by a multiple of the pressure gradient, cf. [2].

Example 1. The travelling wave with speed $c > 0,$ given by

$$U_c(x,t) = \left[\frac{m}{(1-m)\,c\,(x_1 - ct)_+}\right]^{\frac{1}{1-m}}.$$

Then U_c behaves like

$$C(m,c)\mathrm{dist}(x, \mathcal{S}(t))^{-\frac{1}{1-m}}$$

as x goes to $\partial\mathcal{S}(t) = \{x_1 = ct\}$ with $x_1 > ct$. The interface is $x_1 = ct$, its velocity c, and Darcy's Law reads

$$\lim_{x_1 \to ct,\ x_1 < ct} \frac{m}{1-m} \nabla(u^{m-1})(x,t) = (c, 0, \dots, 0).$$

Example 2: The pseudo-Barenblatt solution v given in (2.5). Clearly if t is fixed, $v(x,t)$ behaves like

$$\left(\frac{C}{2A^{1/2}t^{(\theta/N)-1}}\,\frac{1}{\|x\| - r_0(t)}\right)^{\frac{1}{1-m}},$$

as x goes to $\partial\mathcal{S}(t) = \{|x| = A^{1/2}t^{\theta/N} = r_0(t)\}$ with $|x| > r_0(t)$. Now the interface is $|x| = r_0(t)$ and Darcy's Law says

$$\lim_{|x| \to r_0(t),\ |x| > r_0(t)} \frac{m}{1-m}\frac{\partial}{\partial r}(u^{m-1})(x,t) = r_0'(t).$$

Example 3: The pseudo-Barenblatt solution with complete blow-up in finite time. It is another variation of the source-type solution which is obtained by replacing t by $(T-t)$ and changing accordingly the signs of the profile. This reads

$$U_T(x,t) = \left(\frac{C(T-t)}{A(T-t)^{2\theta/N} - |x|^2}\right)^{1/(1-m)}, \quad A > 0,$$

C defined in (0.5) and $|x| < A^{1/2}(T-t)^{\theta/N}$, the solution being defined as infinite for $|x| \geqq A^{1/2}(T-t)^{\theta/N}$. Indeed, U_T has a complete blow-up at time $t = T$. The asymptotic behavior near the singular set is as above.

To conclude the section, let us recall that ECS solutions with expanding singular sets translate into continuous weak solutions of the pressure equation (0.13) with shrinking support. In particular, the last example becomes a model for extinction in finite time which has been studied by Barenblatt *et al.* [6].

9. Fast diffusion with subcritical exponents

As a complement to the study of (0.1) in the supercritical range $m_c < m < 1$ we discuss in this section several aspects of subcritical diffusion, $0 < m \leqq m_c$. A new phenomenon occurs for Dirac masses [16]. In fact, the Dirac mass does not radiate, and moreover, the fundamental solutions understood as limits of classical solutions in the usual way turn out to be stationary masses. The following result from BRÉZIS & FRIEDMAN [16] explains the phenomenon:

Proposition 9.1 (Brezis and Friedman**).** *Let* $0 < m \leqq m_c$ *and* $\eta_n \in C^\infty(\mathbb{R}^N)$, $\eta_n \geqq 0$ *and* $\eta_n \to c\delta_0$ *weakly in measure. If* u_n *is the associated sequence of solutions, then*

$$u_n(x,t) \to c\delta_0(x) \otimes 1(t) \quad \textit{in} \quad \mathcal{D}'(\mathbb{R}^N \times (0,T)).$$

Hence, when taking the limit of those singular fundamental solutions with increasing masses, we find an IPSS which is also stationary, hence it is not continuous, not even a function. Consequently, there is no hope of finding a theory of continuous extended solutions for general initial data. These difficulties are related to the fact that there is no L^∞-regularizing effect from L^1, as can be seen on the explicit solution (1.2). On the other hand, it was proved by PIERRE [42] that the Radon measures admissible as initial data are exactly those which satisfy the following condition:

$$C_{2,\frac{1}{1-m}}(E) = 0 \Rightarrow \mu(E) = 0, \tag{9.1}$$

$C_{2,1/(1-m)}$ being the capacity associated with the Sobolev space $W^{2,\frac{1}{1-m}}$. Under this condition, he proves that the constructed solution is a locally integrable function, not just a measure.

We can generalize Brezis and Friedman's result to the case of general measures: if μ is an arbitrary nonnegative Radon measure charging some set of zero capacity, then it will not be regularized and will remain fixed on this set (we refer to [20] for more details). Moreover, the same phenomenon happens for singular sets as the following result shows:

Lemma 9.1. *Let* $0 < m \leqq m_c$ *and* $\mathcal{S}$ *be closed such that*

$$C_{2,\frac{1}{1-m}}(\mathcal{S}) = 0.$$

Then the maximal solution $\overline{u}_{(\mathcal{S},0)}(t)$ *is zero outside* $\mathcal{S}$ *for any* $t > 0$.

Proof. We construct the maximal solution as usual by constructing a solution u^ε with singular set $\mathcal{S}^\varepsilon$, which is an ε-neighborhood of $\mathcal{S}$. Since $u(0) = 0$ outside $\mathcal{S}$, we know that u remains bounded outside $\mathcal{S}^\varepsilon$, so that when ε decreases, u^ε also and there exist local uniform bounds for the u^ε on the complement of $\mathcal{S}$. The local estimates outside $\mathcal{S}$ are obtained as in [16] by comparison with a super-solution in $B_R(x_0) \times (0,T)$ of the form

$$V(x,t) = \frac{Ct^{\frac{1}{1-m}}}{(R^2 - |x-x_0|^2)^{\frac{N-2}{m}}}.$$

The limit solution u is obviously the maximal solution we are looking for. We will prove that outside $\mathcal{S}$, u is zero. Of course, $u(0)$ is zero for $x \notin \mathcal{S}$.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, nonnegative. Then $\mathcal{S} \cap \operatorname{supp}(\varphi)$ is also compact (because $\mathcal{S}$ is closed) and it has zero capacity, thus there exists a sequence $v_n \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leqq v_n \leqq 1$, $v_n = 1$ on a neighborhood of $\mathcal{S} \cap \operatorname{supp}(\varphi)$ and $v_n \to 0$ in $W^{2,1/(1-m)}(\mathbb{R}^N)$. For $\alpha > \frac{2}{1-m}$, we use the test function $\zeta_n^\alpha = [\varphi(1-v_n)]^\alpha \in C_0^\infty(\mathbb{R}^N \setminus \mathcal{S})$:

$$\int u(t)\zeta_n^\alpha - \int_0^t \int u^m \Delta\zeta_n^\alpha = 0, \tag{9.2}$$

since the support of ζ_n is outside $\mathcal{S}$. Then we estimate

$$\int_0^t \int u^m |\Delta\zeta_n^\alpha| \leqq C(\zeta_n^\alpha) \left[\int_0^T \int u\zeta_n^\alpha\right]^m,$$

with

$$C(\zeta_n^\alpha) = T^{1-m} \left[\int |\Delta\zeta_n^\alpha|^{\frac{1}{1-m}} \zeta_n^{-\frac{\alpha m}{1-m}}\right]^{1-m} \leqq C(m,N)\|\varphi\|_{W^{2,\frac{1}{1-m}}(\mathbb{R}^N)}.$$

The estimate of $C(\zeta_n^\alpha)$ comes from easy computations and the fact that $(1-v_n)$ remains bounded in $W^{2,\frac{1}{1-m}}(\mathbb{R}^N)$. Thus integrating (9.2) on $(0,T)$, we get

$$\int_0^T \int u\zeta_n^\alpha \leqq C' \left[\int_0^T \int u\zeta_n^\alpha\right]^m. \tag{9.3}$$

Hence it follows that for some constant C'' depending only on m, N and $\|\varphi\|_{W^{2,\frac{1}{1-m}}(\mathbb{R}^N)}$,

$$\int_0^T \int u\zeta_n^\alpha \leqq C''\Big(m, N, \|\varphi\|_{W^{2,\frac{1}{1-m}}(\mathbb{R}^N)}\Big).$$

Passing to the limit when n goes to infinity, and using the fact that φ is arbitrary, we obtain

$$u \in L^1(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^N)). \tag{9.4}$$

Now take $\varphi \in C_0^\infty(\mathbb{R}^N \times [0,T))$, and put $\zeta_n = \varphi(1-v_n)$ as the test function, where v_n is as above:

$$\int u(t)\zeta_n - \int_0^t \int u\partial_t\zeta_n - \int_0^t \int u^m \Delta\zeta_n = 0.$$

When n goes to infinity, the two first terms converge by dominated convergence and for the last one, we use both the fact that

$$\zeta_n \underset{n\to\infty}{\longrightarrow} \varphi \quad \text{in} \quad W^{2,\frac{1}{1-m}}(\mathbb{R}^N),$$

and that $u^m \in L^{1/m}(0,T;L^{1/m}_{\mathrm{loc}}(\mathbb{R}^N))$ (by (9.4)), so that

$$\int_0^t \int u^m \Delta\zeta_n \underset{n\to\infty}{\longrightarrow} \int_0^t \int u^m \Delta\varphi.$$

Hence we find that u satisfies

$$\int u(t)\varphi - \int_0^t \int u\varphi_t - \int_0^t \int u^m \Delta\varphi = 0,$$

which means that $u \in L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N))$ is a solution of

$$u_t - \Delta u^m = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N \times (0,T)),$$

with zero initial data. Thus by Lemma 3.1 of [35], we have

$$\int_{B_R} u(t) \leqq Ct^{1/(1-m)} R^{N-2/(1-m)}.$$

(In fact, in [35, Lemma 3.1], the result is stated under the extra assumption that $u \in C([0,T);L^1_{\mathrm{loc}}(\mathbb{R}^N))$, but it is easy to see that if u is only assumed to satisfy (9.4), and $u(t) \to 0$ in measure when $t \to 0$, then the same result holds, for almost every $t > 0$).

The problem here is that since $m \leqq m_c$, letting R go to infinity does not yield $u(t) = 0$. However, using the same technique as in the proof of [35, Theorem 2.3], we easily show that $u \equiv 0$ almost everywhere. Indeed, if $w(x) = \int_0^t u^m(x,\sigma)d\sigma$, which is defined almost everywhere in $\mathbb{R}^N$, then w is sub-harmonic, and thus

$$\begin{aligned} w(\xi) &\leqq \frac{C}{R^N}\int_{B_R(\xi)} w(x)dx \\ &\leqq \frac{C}{R^N}\int_0^t R^{N(1-m)}\left[\int_{B_R(\xi)} u(s)\right]^m ds \\ &\leqq Ct^{1/(1-m)}R^{-2m/(1-m)}, \end{aligned}$$

which goes to zero when R goes to infinity. Note that here we do not need to assume that u is a strong solution (in [35], this assumption is needed because they use Kato's inequality). $\square$

As a consequence, the IPSS, *does not exist* for $0 < m < m_c$. More precisely, the limit solution corresponding to an infinite mass located at $x = 0$ is constant in time. This result does not imply however that solutions with strong singularities cannot exist. In the simplest case, the solution corresponding to data $\nu = (\mathcal{S}, 0)$, where $\mathcal{S} = B_r(0)$, $r > 0$ is a ball, is easily shown to exist as an extended continuous solution and to have the standard separation-of-variables form discussed in Section 4. More generally, it is possible to construct solutions which will preserve the singular set if either $\mathcal{S}$ is created by the measure μ, or it is dense enough. More precisely, the "bad" points of $\mathcal{S}$, i.e., the points that do not radiate, are those which have zero density in the sense of (2.4), with the Lebesgue measure replaced by the $C_{2,\frac{1}{1-m}}$

capacity. Following Marcus & Véron [39], we can thus define the singular interior $\mathcal{S}^*$ of $\mathcal{S}$, which consists of density points of $\mathcal{S}$, and the singular part $\partial_s[\mu]$ created by μ:

$$\mathcal{S}^* = \{y \in \mathcal{S} \mid C_{2,1/(1-m)}(B_r(y) \cap \mathcal{S}) > 0 \quad \forall r > 0\},$$
$$\partial_s[\mu] = \{y \in \partial\mathcal{S} \mid \mu(B_r(y) \cap \mathcal{R}) = +\infty \quad \forall r > 0\}.$$

Then the result is that there exists a non-stationary solution which preserves $\mathcal{S}$ as a strong singular set if and only if $\mathcal{S} = \partial_s[\mu] \cup \mathcal{S}^*$ and μ satisfies the capacity condition (9.1) (see [20] for a proof). However, the continuity property of such solutions is not preserved unless $\mu = 0$. See also Section A.4 for the study of self-similar solutions in this range.

Another interesting aspect of the subcritical case is the property of *extinction in finite time* which has been proved by Bénilan & Crandall [7] for all solutions in $L^p(\mathbb{R}^N)$, $p = N(1-m)/2$, $0 < m < m_c$. The result can be extended to the Marcinkiewicz space $M^p(\mathbb{R}^N)$, the space to which the explicit solution (1.2) belongs. This property does not hold for $m = m_c$ since then conservation of mass is true, $\int u(x,t)\,dx = \int u_0(x)\,dx$.

10. Comments, extensions and open problems

We have constructed in this paper a complete theory of existence and uniqueness of nonnegative solutions of the Cauchy problem for equation $u_t = \Delta(u^m)$ in the range $m_c < m < 1$, posed in the whole space $\Omega = \mathbb{R}^N$. We have also solved the initial-and-boundary value problem posed in an arbitrary open set $\Omega \subset \mathbb{R}^N$ with infinite boundary values. However, the same problem with finite boundary values cannot be solved with the same generality, and the theory offers a number of new qualitative aspects. Thus, there is a new critical exponent $m_1 = (N-1)/(N+1)$. We study this problem in [22,23].

There is a remarkable novelty in the class of extended continuous solutions with strong singularities that we have already noted in the Introduction, namely the radiation of energy from the singularities. This phenomenon has nothing to do with the diffusion process that takes place in the regular set $\mathcal{R}$. Actually, the equation does not hold in $\mathcal{S}$, even though the radiative solutions arise as limits of purely diffusive solutions (the subclass $\mathcal{C}$). We want to emphasize at this moment that the radiation is a consequence of the presence of strongly singular initial data and cannot be stopped later at any given moment (for instance by trying to enforce the equation in $\mathbb{R}^N$ for $t \geqq T > 0$) without breaking the maximum principle, since the radiation lemma says that all singular solutions lie above any Barenblatt solution (which is a classical solution for $t > 0$), hence above the IPSS in the limit, thus they must be singular for all t at the very hot spots of the initial data. It would be very interesting to find more related models of radiation-diffusion equations, maybe systems, in particular systems with quenching mechanisms for the radiation process.

As we have pointed out, the fast-diffusion equation can be transformed into the so-called pressure equation (0.13) by means of the transformation $v = mu^{m-1}$, which maps $u = \infty$ into $v = 0$. Therefore, problems concerning singularities transform into problems concerning zero values. The investigation of the correspondence between the two equations for general classes of solutions is taken up in [21]. In particular, we prove that *a bounded and continuous viscosity solution of* (0.13) *in the range* $\gamma < -N/2$ *is not uniquely determined by prescribing continuous initial data if these data have a nontrivial zero set.*

We list below a number of other problems that represent natural extensions of the above theory. Since none of the applications is immediate and they are not essential at this point, we will only present the main features, results and open questions as far as we know. We begin with the extensions which offer larger similarities. A first extension concerns very fast diffusion in one dimension. We note that, though we have taken the critical exponent to be $m_c = 0$ in this case, it is formally given by $m_c = (N-2)/N = -1$. Similarities and some differences appear in the remaining range $-1 < m \leqq 0$ where the equation is written in the form $u_t = (u^{m-1}u_x)_x$ which preserves the parabolicity. Actually, a theory of existence has been developed in this range and applies to all $u_0 \in L^1_{\text{loc}}(\mathbb{R}),\ u_0 \geqq 0$, while the peculiar feature is the non-uniqueness of solutions of the Cauchy problem for integrable data [30, 47]. The present theory of ECS, based on local estimates and strong singularities of the IPSS type, can be extended to this range with few differences aside from the non-uniqueness phenomenon. The detailed analysis will appear elsewhere.

A second extension concerns the filtration equation, $u_t = \Delta\Phi(u)$, under suitable conditions on the monotone increasing function Φ. Following DAHLBERG & KENIG [29] we assume the power-growth condition

$$0 < c_1 \leqq \frac{s\Phi'(s)}{\Phi(s)} \leqq c_2$$

holds for all large $s \gg 1$, where $m_c < c_1 \leqq c_2 < 1$. Examples of such equations have been proposed by KING [36] in the study of diffusion of impurities in silicon subject to cluster formation, where Φ is linear near zero and of the form $\Phi(u) \sim u^m$, $0 < m < 1$ for large u. The Okuda-Dawson law $D(u) = \Phi'(u) = u^{-1/2}$ proposed in plasma physics is based on experimental evidence which for other density and field regimes can take the more general form $D(u) \sim u^\delta$, $-1 \leqq \delta < 0$ [11].

More work is needed to extend the results to reaction-diffusion equations of the form $u_t = \Delta\Phi(u) + F(u, \nabla u)$ with Φ as above. An interesting aspect of these equations is the possibility of having a theory of limit solutions with *moving* strongly singular sets. Let us advance the simplest example: the diffusion-convection equation

$$u_t = \Delta_x u^m - \mathbf{a} \cdot \nabla_x u, \tag{10.1}$$

with $\mathbf{a} \in \mathbb{R}^N \setminus \{0\}$ and m as before, can be transformed into (0.1) by the change of variables $x = y + \mathbf{a}\,t$. Translating the solutions of (0.1) with space variable $y : u_t = \Delta_y u^m$ into solutions of (10.1), we obtain a class of extended continuous solutions of (10.1) with a strongly singular set which moves in the $\mathbf{a}$ direction with speed $c = |\mathbf{a}|$. It will be interesting to investigate the motion and behaviour of the strongly singular sets for more general convection terms like in $u_t = \Delta_x u^m - \mathbf{a} \cdot \nabla_x f(u)$.

Next is the extension to the so-called "p-Laplacian heat equation"

$$u_t = \Delta_p(u) = \nabla \cdot (|\nabla u|^{p-2}\nabla u).$$

This equation is quite similar to (0.1) in the sense that it has scaling invariance, which leads to the existence of self-similar solutions. Thus, for $p_c = 2N/(N+1) < p < 2$ there exists an IPSS which now takes the form

$$U_\infty(x,t) = \left(\frac{Ct}{|x|^p}\right)^{\frac{1}{2-p}}, \quad C(p,N) > 0.$$

The basic lemmas hold and allow for a similar theory of extended solutions. However, the uniqueness proofs do not apply and actually uniqueness when the initial data is a measure is an important open problem.

A further line of extension concerns the nonlinear heat flow on Riemannian manifolds. Thus, we can consider a Riemannian manifold, $(\mathcal{M}, g)$, say, without boundary, and pose the Cauchy problem

$$u_t = \Delta_g(u^m), \quad u(x,0) = u_0(x), \quad (x,t) \in \mathcal{M} \times (0,T),$$

where $m > 0$, Δ_g is the Laplacian operator with respect to the metric g, cf. [10]. Since the basic estimates that we have used in the range $m_c < m < 1$ are local, they can be extended to this framework and the theory of extended continuous solutions can be developed.

We consider next problems with markedly different qualitative aspects. As we have pointed out, the theory of the Cauchy problem has many new features and difficulties in the subcritical range $m \leqq m_c$: non-existence, lack of continuity, extinction in finite time, ... Partial results of our current research on this subject are given in the last section. The range can be also extended to $m \leqq 0$ in the form $u_t = \nabla \cdot (u^{m-1}\nabla u)$, but less is known in that range, cf. [36,48].

There is also an interest in better understanding the theory of solutions with expanding singularities, either strong or weak. Connected with it is the study of solutions with shrinking supports for the pressure equation. Both subjects have been briefly discussed above.

Finally, the theory can be considered for solutions with changing sign. Existence and uniqueness of solutions is known for $m > 0$ when the data are integrable, $u_0 \in L^1(\mathbb{R}^N)$, since the solutions form a semigroup of ordered contractions in that space [8]. A theory for measures or for data which are large at infinity has not yet been developed. For $m \leqq 0$ there can be no solution with changing sign. This has been proved in the one-dimensional setting in [46].

A. Appendix

We collect here some preliminary results used in the text, a construction of unbounded solutions, an overview of self-similar solutions and a terminology list.

A.1. Comparison of weak solutions

We give a basic comparison result in the case of bounded solutions, which comes from [15].

Lemma A.1. *Assume that Ω is a bounded open subset with smooth boundary. Let $0 \leqq g \leqq g^* \in L^\infty(\Omega \times (0,T))$ and $u, v \in C(\overline{\Omega} \times [0,T])$, all nonnegative such that u is a weak solution of $u_t - \Delta u^m = g$ in $\Omega \times (0,T)$, and v is a weak solution with right-hand side g^*. Then if $u \leqq v$ on $\partial\Omega \times (0,T)$ and on $\Omega \times \{0\}$,*

$$u \leqq v \quad in \quad \overline{\Omega} \times [0,T].$$

Proof. This is Lemma 2.2. of [15] applied to our case: the lemma states that for every $\lambda \geqq 0$ and $0 \leqq t \leqq T$,

$$e^{\lambda t} \int_\Omega [u(t) - v(t)]_+ \leqq \int_\Omega [u(0) - v(0)]_+ + \int_0^t \int_\Omega e^{\lambda s} [g - g^* + \lambda(u - v)]_+.$$

Hence taking $\lambda = 0$, since $u(0) \leqq v(0)$ and $g \leqq g^*$, we find that

$$\int_\Omega [u(t) - v(t)]_+ \leqq 0,$$

so that $u \leqq v$ since t is arbitrary. □

A.2. Regularity

We show an adaptation of the regularity results of DAHLBERG & KENIG [27] in the case of fast diffusion. Since their methods apply here with no changes (in fact it is even easier in the L^∞_{loc} case), we only give a sketch of the proof:

Proposition A.1. *Let $u \in L^\infty_{\mathrm{loc}}(\Omega \times (0,T))$ be a nonnegative solution of $u_t = \Delta u^m$ in the sense of distributions in $\Omega \times (0,T)$. Then $u = u^*$ almost everywhere in $\Omega \times (0,T)$, where $u^* \in C(\Omega \times (0,T))$ is also a solution in the sense of distributions.*

Proof. Let $B = B_r(x_0)$ be a ball in Ω and $0 < a < b < T$. The first step consists in proving that u has traces on $B \times \{a\}$ and $\partial B \times (a,b)$, which are nonnegative bounded measures ν_a and μ respectively on these sets. Then we can write, for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{N+1})$ such that $\varphi = 0$ on $\partial B \times (a,b)$, the following integral version (where $\partial/\partial n$ denotes the derivative with respect to the inward normal):

$$\int_B \psi(x,t) u(x,t) dx - \int_a^t \int_B \{u \partial_t \psi + u^m \Delta \psi\} = \int_a^t \int_{\partial B} \frac{\partial \psi}{\partial n} d\mu + \int_B \psi(a) d\nu_a.$$

This is exactly Lemma 3.4 of [27]. No use has been made of the fact that in their case $m > 1$.

Then the construction of u^* is made by approximation in $B_\varepsilon \times (a, b)$, $B_\varepsilon = B_{r-\varepsilon}(x_0)$ for $\varepsilon > 0$ sufficiently small. If T_ε is a convolution kernel in $\mathbb{R}^{N+1}$, let u_ε^* be the solution of the following problem:

$$\begin{aligned} \partial_t u_\varepsilon^* - \Delta(u_\varepsilon^*)^m &= 0 && \text{in } B_\varepsilon \times (a, b), \\ u_\varepsilon^* &= (T_\varepsilon u^m)^{1/m} && \text{on } \partial B_\varepsilon \times (a, b), \\ u_\varepsilon^*(a) &= T_\varepsilon u(a) && \text{in } B_\varepsilon. \end{aligned}$$

Then there exists a constant $C = C(\|u\|_{L^\infty_{\text{loc}}})$ such that

$$u_\varepsilon^* \leqq C \quad \text{in} \quad B_\varepsilon \times (a, b),$$

hence we can extract a subsequence still denoted u_ε^* converging locally uniformly to a solution u^* in $B \times (a, b)$. Moreover, as in Lemma 5.1 of [27], we can pass to the limit in the weak formulation and see that the boundary trace of u^* is μ, and its trace at $t = a$ equals ν_a.

Finally we have to show that $u = u^*$ almost everywhere. This is done in Lemma 5.1 of [27] thanks to the extra estimate that $u \in L^{m+1}_{\text{loc}}(\Omega \times (0, T))$. But here, since we assume that $u \in L^\infty_{\text{loc}}(\Omega \times (0, T))$, the proof is made easier. The adaptations are straightforward. □

A.3. Unbounded solutions

We give here the construction of an unbounded solution with initial data in $L^1(\mathbb{R}^N)$ for the critical case $m = m_c$ announced in Section 1. We start with a collection of smooth and compactly supported initial data $\varphi_n(x)$ such that $\|\varphi_n\|_{L^1} = 1/n^2$. Let u_n be the corresponding weak solution. Using the scaling properties, we construct solutions

$$v_n(x, t) = k_n^N u_n(kx, t),$$

with k_n chosen so large that $\|v_n(x, 1)\|_{L^\infty} = k^N \|u_n(x, 1)\|_{L^\infty} = n$. Besides, $\|v_n(t)\|_{L^1} = \|v_n(0)\|_{L^1} = \|\varphi_n\|_{L^1} = 1/n^2$. Next, we consider the continuous weak solution u with initial data

$$u(x, 0) = \sum_{n=1}^{\infty} k_n^N \varphi_n(k_n(x - y_n), 0),$$

where y_n is an arbitrary sequence of points in $\mathbb{R}^N$. It is clear that $u(x, 0)$ is integrable in $\mathbb{R}^N$. Also by comparison $u(x, t) \geqq v_n(x, t)$ for every $n \in \mathbb{N}_*$. Therefore, $\|u(1)\|_{L^\infty} = \infty$. Using the inequality $u_t \leqq u/[(1 - m)t]$, we see that $\|u(t)\|_{L^\infty} = \infty$ for any $t \in (0, 1)$ since $t \mapsto u(t)t^{-1/(1-m)}$ is non-increasing. Let us remark that for $N = 2$, the critical exponent is $m_c = 0$ and the equation is usually written in the form

$$u_t = \Delta \ln(u) = \nabla \cdot (u^{-1} \nabla u).$$

In this case it has been shown in [45] that initial data in $L^1(\mathbb{R}^N)$ implies that $u(t)$ is bounded for every $t > 0$. The difference can be explained as a consequence of the property of finite-time extinction of the last equation, which implies that the solutions for the small masses $\phi_n(x)$ that we use in the above construction will vanish in times that go to zero with n, and this invalidates the conclusion of the scaling and addition performed later. However, adapting the above proof, we can show that this regularizing effect is not local: the fact that $\int_B u_0(x)\,dx$ is uniformly bounded in all balls B of radius R does not imply that $u(t) \in L^\infty(\mathbb{R}^2)$.

A.4. Self-similar solutions

We consider the solutions of (0.1) with initial data

$$u_0(x) = c|x|^\gamma, \quad x \neq 0, \tag{A.1}$$

with $c > 0$ and $\gamma \in \mathbb{R}$. For $\gamma \neq -2/(1-m)$, both the equation and the initial data are invariant under the family of transformations

$$u_k(x,t) = k^{-\alpha} u(k^\beta x, kt), \quad k > 0, \tag{A.2}$$

where $\alpha = \gamma/[2+\gamma(1-m)]$, $\beta = 1/[2+\gamma(1-m)]$. Let us first discuss the main features of the case $m_c < m < 1$. By uniqueness of the continuous weak solution in that range, we have $u \equiv u_k$, so that putting $k = 1/t$, we get the representation of the solutions in the form

$$u(x,t) = t^\alpha f(xt^{-\beta}). \tag{A.3}$$

We notice that for $\gamma > -N$, u_0 is locally integrable, so that f is a locally bounded function. On the other hand, taking initial data means that $f(\xi) \sim c|\xi|^\gamma$ as $|\xi| \to \infty$, since $\beta > 0$. This gives the asymptotic behavior of the solutions as $|x| \to \infty$. In this range, we can still consider the same initial data plus a strong singularity at $|x| = 0$, i.e., the initial data is $(\mathcal{S}, \mu)$, with $\mathcal{S} = \{0\}$ and $d\mu = c|x|^\gamma dx$. Then the representation is valid with the same α and β but now f diverges as $|\xi| \to 0$. According to the formula, this is the same behavior of u when $t \to \infty$, and we get

$$f(\xi) \sim C|\xi|^{-\frac{2}{1-m}} \quad \text{as} \quad |\xi| \to 0,$$

with C as in the IPSS. In the range $-2/(1-m) < \gamma < -N$, the initial data are not integrable at zero, and only the solution of the second kind exists. In the critical case $\gamma = -N$, there are solutions with a strong singularity at $|x| = 0$, corresponding to $u_0(x) = c|x|^{-N}$ plus solutions with a bounded profile at zero and correspond to a Dirac mass as initial data $u_0(x) = c\delta_0(x)$. Finally, when $\gamma < -2/(1-m)$, then $\beta < 0$ and the initial behavior of u is equivalent to the behavior of f as $|\xi| \to 0$, which has to be singular of the form

$$f(\xi) \sim c|\xi|^\gamma \quad \text{as} \quad |\xi| \to 0.$$

We are left with the case $\gamma = -2/(1-m)$. Then, the equation and the data are invariant under the scaling

$$u_k(x,t) = k^{\frac{2}{1-m}} u(kx,t), \quad k > 0.$$

Therefore the solution takes the form $u(x,t) = |x|^{-\frac{2}{1-m}} f(t)$, and we finally get the IPSS delayed in time since $f(0) = c$. We can do a similar analysis in the more general case where $u_0 = |x|^\gamma f(\sigma)$, with $\sigma = x/|x|$, hence considering non-radial solutions. The exponents and the representation formulas are similar, but now the profile f depends also on the angular variable.

Next, we give a quick glimpse of the case $0 < m \leqq m_c$. We also take initial data of the form (A.1) and we construct by approximation an extended continuous solution which has the self-similar form (A.3) with α and β as above for $\gamma \neq -2/(1-m)$. In this case, the value of the profile $f(0)$ is bounded for $\gamma > -2/(1-m)$ and infinite for $\gamma < -2/(1-m)$. It is easy to prove that in the latter case, $f(\xi) \sim c|\xi|^\gamma$ when $\xi \to 0$ so that these self-similar solutions are unbounded. There are two subranges. For $\gamma \leqq -N$, the solution $u(x,t)$ has a *strong singularity* at $|x| = 0$ for all times. On the other hand, the singularity at zero is integrable for $-N < \gamma < -2/(1-m)$ thus proving that in the subcritical range, the $L^1_{\text{loc}} \to L^\infty_{\text{loc}}$ regularizing effect fails. The corresponding self-similar solutions exhibit a *weak singularity* at $|x| = 0$ for all times, something that did not happen for $m > m_c$. Both strong and weak singularities may have different divergence rates. Finally, when $\gamma = -2/(1-m)$, we get for $0 < m < m_c$ the explicit solution

$$u(x,t) = c|x|^{-\frac{2}{1-m}} (T-t)^{\frac{1}{1-m}}.$$

Using this solution and symmetrization [49], we can prove that solutions with initial data in the Marcinkiewicz space $M^p(\mathbb{R}^N)$, $p = N(1-m)/2$, $0 < m < m_c$, vanish in finite time, thus improving the result of BÉNILAN & CRANDALL [8]. Proofs of all these facts have been omitted here due to lack of space, but they will appear elsewhere.

Remark. The nonlinear elliptic equation satisfied by f in the radial case can be translated into an autonomous system which is then studied by phase-plane techniques, starting with the early papers of the 50's, cf. [5, 54]. There is a huge literature for $m > 1$, see for instance [44] and the references therein for the study of self-similar solutions with weak singularities which give the asymptotic behaviour of the Cauchy-Dirichlet problem in an exterior domain. For $0 < m < 1$, we refer to [36, 41]. There are other types of self-similar solutions that can be considered, and many interesting properties have been observed (singular behaviour, asymptotic attraction, stability, ...) but we refrain in this paper from further details on the subject.

A.5. Terminology

We end the appendix with the following brief terminology for convenience of the reader.

Barenblatt solution:	see source-type solution.
$\mathcal{E}$:	The class of extended continuous solutions.
$\mathcal{E}_s$:	Subclass of $\mathcal{E}$ with only strong singularities.
$\mathcal{E}_c$:	Subclass of $\mathcal{E}_s$ with only constant strong singularities.
ECS:	Extended continuous solution in $\mathcal{E}_c$.
Hot spot:	Singularity, a point x_0 where $u(x_0, t) = \infty$ for $t > 0$ (or where u_0 is not locally bounded).
IPSS:	Infinite point source solution, see (0.5).
Pressure:	The function $v = mu^{m-1}$ where u satisfies (0.1).
Self-similar solution:	Special solution of the form $u(x, t) = t^{\alpha} f(xt^{-\beta})$.
Source-type solution:	Solution with a Dirac mass as initial data, more precisely finite point-source solution see (0.4),
Strong singularity:	A singular point where f is not locally integrable.
SSS:	Strongly singular set, consisting of the strong singularities.
Travelling wave:	A special solution of the form $u(x, t) = f(x - ct)$.
Very hot spot:	Strong singularity.
Weak singularity:	A singular point of f where it is locally integrable.

Note. As an answer to one of the referees' questions, we would like to add the following comment: the class of extended continuous solutions described here offers a new perspective of combining diffusion and radiation in a compact form and we conjecture its usefulness in models of reaction-diffusion and other fields. However, two years after first announcing this mechanism, the possibility of real-world applications is still the main open issue and we would like to draw the attention of potential readers to it. On the other hand, there has been rapid progress in the study of solutions with expanding singularities (studied in Section 8) in the work of BARENBLATT *et al.*, cf. [6, 50].

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References

1. S. B. ANGENENT, Local existence and regularity for a class of degenerate parabolic equations, *Math. Ann.* **280** (1988), 465–482.
2. D. G. ARONSON, The Porous Medium Equation, *Some problems of Nonlinear Diffusion, Lecture Notes in Mathematics 1224,* Springer-Verlag, New York, 1986.
3. D. G. ARONSON & P. BÉNILAN, Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^N$, *C. R. Acad. Sci. Paris Sér. I Math.* **288** (1979), 103–105.
4. D. G. ARONSON & L. A. CAFFARELLI, The initial trace of a solution of the porous medium equation, *Trans. A.M.S.* **280** (1983), 351–366.

5. G. I. Barenblatt, On self-similar motion of compressible fluids in porous media, *Prikl. Mat. Mekh.* **16** (1952), 679–698 (in Russian).
6. G. I. Barenblatt & M. Bertsch & A.E. Chertock & V. M. Prostokishin, Self-similar asymptotics for a degenerate parabolic filtration-absorption equation, Univ. of California Preprint PAM-782 (2000).
7. P. Bénilan & M. G. Crandall, Regularizing effects of homogeneous evolution equations, *Contribution to Analysis and Geometry* (D. N. Clark *et al.*, eds.), John Hopkins Univ. Press, Baltimore, Md. (1981), 23–30.
8. P. Bénilan & M. G. Crandall, The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$, *Indiana Univ. Math. Jl.* **30** (1991), 161–177.
9. P. Bénilan, M. G. Crandall & M. Pierre, Solutions of the Porous Medium Equation in $\mathbb{R}^N$ under optimal conditions on the initial data, *Indiana Univ. Math. Jl.* **33** (1984), 51–87.
10. M. Berger, P. Gauduchon & E. Mazet, *Le Spectre d'une Variété Riemmanienne*, Lecture Notes in Mathematics **194**, Springer Verlag, Berlin, 1970.
11. J. G. Berryman & C. J. Holland, Stability of the separable solution for fast diffusion equation, *Arch. Rational Mech. Anal.* **74** (1980), 379–388.
12. M. Bertsch & M. Ughi, Positivity properties of viscosity solutions of a degenerate parabolic equation, *Nonlinear Anal. T.M.A.* **14** (1990), 571–592.
13. M. Bertsch, R. Dal Passo & M. Ughi, Discontinuous viscosity solutions of a degenerate parabolic equation, *Trans. A.M.S.* **320** (1990), 779–798.
14. M. Bertsch, R. Dal Passo & M. Ughi, Nonuniqueness of solutions of a degenerate parabolic equation, *Annali Mat. Pura Appl.* **161** (1992), 57–81.
15. M. Borelli & M. Ughi, The fast diffusion equation with absorption: the instantaneous shrinking phenomenon, *Rendiconti Ist. Mat. Trieste* **26** (1994), 109-140.
16. H. Brezis & A. Friedman, Nonlinear parabolic equations involving measures as initial conditions, *Jl. Math. Pures Appl.* **62** (1983), 73–97.
17. H. Brezis, L. A. Peletier & D. Terman, A very singular solution of the heat equation with absorption, *Arch. Rational Mech. Anal.* **99** (1986), 185–209.
18. L. A. Caffarelli & X. Cabré, *Fully nonlinear elliptic equations*, Coll. Publ. **43**, Amer. Math. Soc., Providence, 1995.
19. L. A. Caffarelli & J. L. Vázquez, Viscosity solutions for the porous medium equation, *Proc. Symposia in Pure Mathematics* volume **65**, in honor of Profs. P. Lax and L. Nirenberg, M. Giaquinta *et al.* eds, 1999, 13–26.
20. E. Chasseigne, *Thesis*, Univ. Tours, France. December 2000.
21. E. Chasseigne & J. L. Vázquez, The pressure equation in the fast diffusion case, *Rev. Mat. Iberoamericana*, to appear.
22. E. Chasseigne & J. L. Vázquez, Theory of Weak and Limit Solutions for Fast Diffusion Equations in Optimal Classes of Data. The Cauchy-Dirichlet Problem, Preprint UAM 2000, submitted.
23. E. Chasseigne & J.L. Vázquez, Theory of Extended Solutions for Fast Diffusion Equations in Optimal Classes of Data. The Generalized Initial-Boundary Value Problem, in preparation.
24. M. G. Crandall, L. C. Evans & P. L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **282** (1994), 487–502.
25. M. G. Crandall, H. Ishii & P. L. Lions, User's guide to viscosity solutions for second-order partial differential equations, *Bull. Amer. Math. Soc.* **27** (1992), 1–67.
26. M.G. Crandall & P.L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **277** (1993), 1–42.
27. B. E. J. Dahlberg & C. Kenig, Weak solutions to the Porous Medium Equation, *Trans. A.M.S.* **336** (1993), 710–725.
28. B. E. J. Dahlberg & C. Kenig, Nonnegative solutions the of the initial-Dirichlet problem for generalized porous medium equation in cylinders, *Jl. Amer. Math. Soc.* **1** (1988), 401–412.
29. B. E. J. Dahlberg & C. Kenig, Nonnegative solutions to fast diffusion equations, *Rev. Mat. Iberoamericana* **4** (1988), 11–29.

30. J. R. ESTEBAN, A. RODRÍGUEZ & J. L. VAZQUEZ, A nonlinear heat equation with singular diffusivity, *Comm. Partial Diff. Eqs.* **13** (1988), 985-1039.
31. L. C. EVANS & R. F. GARIEPY, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
32. A. FRIEDMAN & S. KAMIN, The asymptotic behavior of a gas in an n-dimensional porous medium, *Trans. A.M.S.* **262** (1980), 373–401.
33. V. A. GALAKTIONOV, S. P. KURDYUMOV & A. A. SAMARSKII, On the asymptotic "eigenfunctions" of the Cauchy problem for some nonlinear parabolic equation, *Mat. Sb.* **126** (1985), 435–472 (in Russian), English translation Mat. USSR Sbornik **54** (1986), 421–455.
34. V. A. GALAKTIONOV, L. A. PELETIER & J. L. VAZQUEZ, Asymptotics of the fast diffusion equation with critical exponent, *SIAM J. Math. Anal.* **31** (2000), 1157–1174.
35. M. A. HERRERO & M. PIERRE, The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$, *Trans. A.M.S.* **291** (1985), 145–158.
36. J. R. KING, Self-similar behavior for the equation of fast nonlinear diffusion, *Phil. Trans. Roy. Soc. London A* **343** (1993), 337–375.
37. M. MARCUS & L. VÉRON, Initial trace of positive solutions of some nonlinear parabolic equations, *Comm. P.D.E.* **24** (1999), 1445–1499.
38. M. MARCUS, L. VÉRON, The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case, *Arch. Rational Mech. An.* **144** (1998), 201–231.
39. M. MARCUS & L. VÉRON, The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case, *J. Math. Pures Appl.* **77** (1998), 481–524.
40. M. MARCUS & L. VÉRON, Uniqueness and asymptotic behaviour of solutions with boundary blow-up for a class of nonlinear elliptic equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), 237–274.
41. M. A. PELETIER & H. F. ZHANG, Self-similar solutions of fast diffusion equations that do not conserve mass, *Diff. Int. Eq.* **8** (1995), 2045–2064.
42. M. PIERRE, Nonlinear fast diffusion with measures as data, *Pitman Research Notes in Math.* **149** (1987), 179–188.
43. M. DEL PINO & R. LETELIER, The influence of domain geometry in boundary blow-up elliptic problems, Preprint (2000).
44. F. QUIRÓS, J. L. VAZQUEZ, Asymptotic behavior of the porous media equation in an exterior domain, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4), **28** (1999), 183–227.
45. A. RODRÍGUEZ, J. R. ESTEBAN & J. L. VAZQUEZ, The maximal solution of the logarithmic fast diffusion equation in two space dimensions, *Advances Diff. Eqns.* **2** (1997), 867–894.
46. A. RODRÍGUEZ, J. L. VAZQUEZ, Obstructions to existence in fast-diffusion equations, *J. Diff. Eqns.*, to appear. Preprint U.A.M. (1999).
47. A. RODRÍGUEZ & J. L. VAZQUEZ, Nonuniqueness of solutions of nonlinear heat equations of fast diffusion type, *Ann. Inst. H. Poincaré, Anal. non Linéaire* **12** (1995), 173–200.
48. J. L. VAZQUEZ, Nonexistence of solutions for nonlinear heat equations of fast- diffusion type, *Journal Math. Pures Appl.* **71** (1992), 503–526.
49. J. L. VAZQUEZ, Symétrisation pour $u_t = \Delta\phi(u)$ et applications, *C. R. Acad. Sci. Paris Sér. I Math.* **295** (1982), 71–74.
50. J. L. VAZQUEZ, Darcy's Law and the theory of shrinking solutions of fast diffusion equations, *TICAM Report 01-18*, Univ. of Texas at Austin, 2001.
51. J. L. VAZQUEZ & L. VÉRON, Different kinds of singular solutions of nonlinear parabolic equations, *Nonlinear Problems in Applied Mathematics* (volume in honor of Ivar Stakgold on his 70th birthday), T.S. ANGELL *et al.* eds., SIAM, Philadelphia, 1996, pp. 240–249.
52. L. VÉRON, Generalized boundary value problems for nonlinear elliptic equations: an introduction, Course in Roma, Feb. 1999.
53. D. V. WIDDER, *The Heat Equation*, Academic Press, New York (1975).

54. I. B. ZELDOVICH & A. S. KOMAPANEETS, On the theory of heat conduction depending on the temperature, *Lectures dedicated on the 70th Anniversary of A.F. Joffe*, Akad. Nauk. SSSR. (1950), 61–71 (in Russian).

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