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$\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes are asymptotically good

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Abstract
In this paper, we construct a class of $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes generated by 3-tuples of polynomials. We discuss their algebraic structure and show that generator matrices can be constructed for all codes in this class. We study asymptotic properties of this class of codes by using a Bernoulli random variable. Moreover, let $0 < \delta < 1$ be a real number such that the entropy $h_4(\frac{(k+l+t)\delta}{6}) < \frac{1}{4}$, we show that the relative minimum distance converges to δ and the rate of the random codes converges to $\frac{1}{k+l+t}$, where k, l, and t are pairwise co-prime positive odd integers. Finally, we conclude that the $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes are asymptotically good.

Keywords $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes · Relative minimum distance · Asymptotically good code

1 Introduction

Codes over finite rings gained researchers interest after Hammons et al. developed binary images under a Gray map of linear cyclic codes over $\mathbb{Z}_4$ in [23]. For instance, the class of finite rings of the form $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ has been widely used as alphabets of certain constacyclic codes. In 2010, Dinh [10] determined the algebraic structures of constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ and their dual codes. In 2012, Dinh et al. [8] gave the algebraic structures of constacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ and their dual codes. In 2018, Dinh et al. [11] investigated the algebraic structures of negacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ and their dual codes. In addition, constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ are investigated in [12] and [13]. Moreover, Dinh et al. [14] provided all constacyclic codes of length $3p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$.

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It is well known that the ideals of $\frac{\mathbb{Z}_q[x]}{\langle x^n-1\rangle}$ are same as the cyclic codes over $\mathbb{Z}_q$ (see, for example, [29]). The researchers in [21, 28] introduced the additive cyclic codes, which are a special case of generalized quasi-cyclic codes. Moreover, Borges et al. [6] investigated $\mathbb{Z}_2\mathbb{Z}_4$-additive codes which were later extended by Abualrub et al. for additive cyclic codes in [1] and Gao et al. for double cyclic codes over $\mathbb{Z}_4$ [20]. These works were further extended to $\mathbb{Z}_2\mathbb{Z}_2\mathbb{Z}_4$ by Wu et al. [30] and $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$ -additive cyclic codes by Aydogdu and Gursoy [3].

From the application point of view, it is necessary to study the asymptotic properties of these cyclic codes, because the rate of cyclic codes is used to measure the proportion of the number of information coordinates of a family of cyclic codes to the total number of coordinates, and the relative minimum distance of cyclic codes is used to measure error-correcting capability. In particular, it would be interesting to find out whether cyclic codes are asymptotically good, i.e., whether the rate and the relative minimum distance of cyclic codes are both positively bounded from below when the length of the code goes to infinity. This has been an open problem for quite fifty-five years as can be seen in [2]. In 2006, Martínez-Pérez and Willems, discussed in [25] whether the class of cyclic codes is asymptotically good. In 2015, Fan et al. showed that there exist numerous asymptotically good quasi-abelian codes attaining the GV-bound in [17], and in [15], they proved that quasi-cyclic codes of index $1\frac{1}{2}$ are asymptotically good. Moreover, in 2016, they also showed that the quasi-cyclic codes of index $1\frac{1}{3}$ are asymptotically good in [16]. Further, in [27], Shi et al. proved that there are additive cyclic codes that are asymptotically good.

In 2019, Fan and Liu proved that $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes are asymptotically good by using a Bernoulli random variable in [18]. Few other works such as [31, 32] generalised [18] for $\mathbb{Z}_p\mathbb{Z}_{p^s}$ and $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ where p is any prime number and $1 \le r < s$. Recently, Gao et al. [19] investigated the $\mathbb{Z}_4$-double cyclic codes and found them asymptotically good.

The above mentioned literature is concerned with doubly additive cyclic codes. In this paper, we work on $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes; we show that these codes are asymptotically good.

The paper is organized as follows: In Sect. 2, we discuss the algebraic structure of $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive codes over $\mathbb{Z}_4$-module. Then, we identify $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes of length $n = \alpha + \beta + \gamma$ with $\mathbb{Z}_4[x]$-submodules of $\mathbb{R}_\alpha \times \mathbb{R}_\beta \times \mathbb{R}_\gamma$, where $\mathbb{R}_\alpha = \frac{\mathbb{Z}_4[x]}{\langle x^\alpha-1\rangle}$, $\mathbb{R}_\beta = \frac{\mathbb{Z}_4[x]}{\langle x^\beta-1\rangle}$ and $\mathbb{R}_\gamma = \frac{\mathbb{Z}_4[x]}{\langle x^\gamma-1\rangle}$. In Sect. 3, we define a class of cyclic codes C_{abc} as $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes in $\mathbb{Z}_4^{km} \times \mathbb{Z}_4^{lm} \times \mathbb{Z}_4^{tm}$ as

$$C_{abc} = \{(f(x)a(x), f(x)b(x), f(x)c(x)) \in \mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm} \mid f(x) \in \mathbb{R}_{kltm}\},$$

which can be seen as $\mathbb{Z}_4[x]$-submodules of $\mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$, for $(a(x), b(x), c(x)) \in \mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$. Then we proved that C_{abc} is an $\mathbb{R}_{kltm}$-submodule of $\mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$ generated by $(a(x), b(x), c(x))$. In Sect. 4, we study the asymptotic properties of this class of cyclic codes using a Bernoulli random variable Y_f, which implies that $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes are asymptotically good. In Sect. 5, we conclude the paper with some open directions for future work.

2 Preliminary

Consider the quaternary ring $\mathbb{Z}_4$ and define a Gray map $\psi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2^2$ given as $\psi(0) = (0,0)$, $\psi(1) = (0,1)$, $\psi(2) = (1,1)$, $\psi(3) = (1,0)$. It can also be extended for $\mathbb{Z}_4^n$ to $\mathbb{Z}_2^{2n}$, where n is an odd positive integer, given by

$$(x_0, x_1, \dots, x_{n-1}) \longmapsto (\psi(x_0), \psi(x_1), \dots, \psi(x_{n-1})).$$

$\mathbb{Z}_4$ is equipped the Lee weight and the Gray image is equipped the Hamming weight. The Hamming weight is the number of non zero coordinates of the Gray image. The relation between the Lee weight wt_L and the Hamming weight wt_H for each element $x_i \in \mathbb{Z}_4,\ i = 0, \dots, 3$ is given by

$$wt_L(x_i) = wt_H(\psi(x_i)).$$

For example, $wt_L(0) = 0$, $wt_L(1) = 1$, $wt_L(2) = 2$, $wt_L(3) = 1$. Therefore, for $x = (x_0, x_1, \dots, x_{n-1}) \in \mathbb{Z}_4^n$, the Lee weight $wt_L(x)$ can be defined as

$$wt_L(x) = wt_H(\psi(x)) = \sum_{j=0}^{n-1} wt_L(x_j).$$

The Lee distance between any two elements $x = (x_0, x_1, \dots, x_{n-1})$ and $y = (y_0, y_1, \dots, y_{n-1})$ in $\mathbb{Z}_4^n$ is defined as

$$d_L(x, y) = \sum_{j=0}^{n-1} wt_L(x_j - y_j).$$

Now, it can be seen that ψ is a distance preserving map from $(\mathbb{Z}_4^n, d_L)$ to $(\mathbb{Z}_2^{2n}, d_H)$. Let C be a nonzero code of length n in $\mathbb{Z}_4^n$ then the minimum Lee weight $wt_L(C)$ is defined as

$$wt_L(C) = \min\{wt_L(x) \mid x \in C, x \neq 0\}.$$

The minimum Lee distance of the code C is defined as

$$d_L(C) = \min\{wt_L(x - y) \mid x, y \in C, x \neq y\}.$$

Define

$$\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4 = \{(\mu, \nu, \rho) \mid \mu, \nu, \rho \in \mathbb{Z}_4\},$$
$$\mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma = \{(a, b, c) \in \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma \mid a \in \mathbb{Z}_4^\alpha, b \in \mathbb{Z}_4^\beta, c \in \mathbb{Z}_4^\gamma\},$$

where α, β and γ are positive integers. Thus, the set $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma$ is an abelian group. For $(a, b, c) \in \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma$ and $d \in \mathbb{Z}_4$, we define a multiplication operation $\cdot$ as

$$d \cdot (a, b, c) = (da \pmod 4,\ db \pmod 4,\ dc \pmod 4).$$

So, the set $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma$ is closed with respect to multiplication for any $d \in \mathbb{Z}_4$. Hence the abelian group $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma$ is a $\mathbb{Z}_4$-module. We now present some definitions related to this module $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma$.

Definition 2.1 A subset C of $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma$ is called a $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive code of length $n = \alpha + \beta + \gamma$, if C is a subgroup of $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma$, where the first α coordinates of C are entries from $\mathbb{Z}_4$, which is also true for the next β and the last γ coordinates.

Definition 2.2 Let $C \subseteq \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma$ be a $\mathbb{Z}_4$-additive code then C is called a $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$ -additive cyclic code of block length (α, β, γ), if whenever $(a_0, \dots, a_{\alpha-1}, b_0, \dots, b_{\beta-1}, c_0, \dots, c_{\gamma-1})$ is in C, then $(a_{\alpha-1}, a_0, \dots, a_{\alpha-2}, b_{\beta-1}, b_0, \dots, b_{\beta-2}, c_{\gamma-1}, c_0, \dots, c_{\gamma-2})$ is also in C.

Let $\mathbb{R}_\alpha = \frac{\mathbb{Z}_4[x]}{\langle x^\alpha - 1\rangle}$, $\mathbb{R}_\beta = \frac{\mathbb{Z}_4[x]}{\langle x^\beta - 1\rangle}$, $\mathbb{R}_\gamma = \frac{\mathbb{Z}_4[x]}{\langle x^\gamma - 1\rangle}$ and define a map

$$\phi : \mathbb{Z}_4^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Z}_4^\gamma \longrightarrow \mathbb{R}_\alpha \times \mathbb{R}_\beta \times \mathbb{R}_\gamma$$

given by

$$(a, b, c) \longmapsto (a(x), b(x), c(x))$$

where $a(x) = a_0 + a_1x + \cdots + a_{\alpha-1}x^{\alpha-1}$, $b(x) = b_0 + b_1x + \cdots + b_{\beta-1}x^{\beta-1}$, $c(x) = c_0 + c_1x + \cdots + c_{\gamma-1}x^{\gamma-1}$. Thus, using the map ϕ it can be seen clearly that $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$ -additive cyclic codes are $\mathbb{Z}_4[x]$-submodules of $\mathbb{R}_\alpha \times \mathbb{R}_\beta \times \mathbb{R}_\gamma$.

Note that if C is a $\mathbb{Z}_4$-free, then there exists $\mathbb{Z}_4$-free basis for C. If cardinality of a $\mathbb{Z}_4$-free basis set is r then the rank of C is r. The rate of C is defined as $R(C) = \frac{\text{rank}(C)}{n}$ and the relative distance of C is defined as $\Delta(C) = \frac{d_L(C)}{n}$.

Definition 2.3 [25] If there exists a sequences of $\mathbb{Z}_4$-free $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes $\{C_i\}_{i=0}^{\infty}$ of length n_i, where $n_i \to \infty$ and if the relative distance and rate of C_i are positively bounded from below, then these class of $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes are said to be asymptotically good.

3 A class of $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes

Let $\mathbb{R}_{km} = \frac{\mathbb{Z}_4[x]}{\langle x^{km}-1\rangle}$, $\mathbb{R}_{lm} = \frac{\mathbb{Z}_4[x]}{\langle x^{lm}-1\rangle}$, $\mathbb{R}_{tm} = \frac{\mathbb{Z}_4[x]}{\langle x^{tm}-1\rangle}$ and $\mathbb{R}_{kltm} = \frac{\mathbb{Z}_4[x]}{\langle x^{kltm}-1\rangle}$, where m, k, l, t are positive integers such that $\gcd(m, 4) = 1$ and k, l, t, 4 are pairwise co-prime positive integers. It is easy to see that $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes in $\mathbb{Z}_4^{km} \times \mathbb{Z}_4^{lm} \times \mathbb{Z}_4^{tm}$ are $\mathbb{Z}_4[x]$-submodules of $\mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$, for $(a(x), b(x), c(x)) \in \mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$.

For any $f(x) \in \mathbb{Z}_4[x]$ and $(a(x), b(x), c(x)) \in \mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$, we define the scalar multiplication, denoted by $\star$, as follows $f(x) \star (a(x), b(x), c(x)) = (f(x)a(x)$

mod $(x^{km}-1), f(x)b(x) \mod (x^{lm}-1), f(x)c(x) \mod (x^{tm}-1))$. For convenience, we write it as

$$f(x) \star (a(x), b(x), c(x)) = (f(x)a(x), f(x)b(x), f(x)c(x)).$$

Clearly, $\mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$ is closed under the usual addition and scalar multiplication $\star$ of $\mathbb{R}_{kltm} = \mathbb{Z}_4[x]/\langle x^{kltm}-1\rangle$. Let

$$C_{abc} = \{(f(x)a(x), f(x)b(x), f(x)c(x)) \in \mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm} \mid f(x) \in \mathbb{R}_{kltm}\},$$

then C_{abc} is an $\mathbb{R}_{kltm}$-submodule of $\mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$ generated by $(a(x), b(x), c(x))$, i.e., C_{abc} is a $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic code generated by $(a(x), b(x), c(x))$.

We have the following lemma.

Lemma 3.1 *Let C_{abc} be a $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic code with the generator polynomial $F(x) = (a(x), b(x), c(x)) \in \mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$, where $a(x)$, $b(x)$ and $c(x)$ are $\mathbb{Z}_4[x]$-monic polynomials. Let*

$$h(x) = \operatorname{lcm}\left\{\frac{x^{km}-1}{g_1(x)}, \frac{x^{lm}-1}{g_2(x)}, \frac{x^{tm}-1}{g_3(x)}\right\}$$

be a monic parity-check polynomial of C_{abc} with degree h_0, where $g_1(x) = \gcd(a(x), x^{km}-1)$, $g_2(x) = \gcd(a(x), x^{lm}-1)$, and $g_3(x) = \gcd(a(x), x^{tm}-1)$, *then C_{abc} can be generated by the set $\{F(x), xF(x), \ldots, x^{h_0-1}F(x)\}$.*

Proof Let $f(x) \in C_{abc}$, i.e., $f(x) = v(x)F(x)$, where $v(x) \in \mathbb{Z}_4[x]$. Since $h(x)$ is monic, there exist polynomials $p(x), r(x) \in \mathbb{Z}_4[x]$ such that

$$v(x) = p(x)h(x) + r(x),$$

where $\deg r(x) < \deg h(x)$ or $r(x) = 0$. Therefore,

$$f(x) = v(x)F(x) = (p(x)h(x) + r(x))F(x) = p(x)h(x)F(x) + r(x)F(x).$$

Now since,

$$h(x) = \operatorname{lcm}\left\{\frac{x^{km}-1}{g_1(x)}, \frac{x^{lm}-1}{g_2(x)}, \frac{x^{tm}-1}{g_3(x)}\right\},$$

then there exist three polynomials $d_1(x)$, $d_2(x)$ and $d_3(x)$ such that

$$h(x) = d_1(x)\left(\frac{x^{km}-1}{g_1(x)}\right) \text{ or } h(x) = d_2(x)\left(\frac{x^{lm}-1}{g_2(x)}\right) \text{ or } h(x) = d_3(x)\left(\frac{x^{tm}-1}{g_3(x)}\right).$$

It is also given that $g_1(x) = \gcd(a(x), x^{km}-1)$, $g_2(x) = \gcd(a(x), x^{lm}-1)$ and $g_3(x) = \gcd(a(x), x^{tm}-1)$, there exist three polynomials $e_1(x)$, $e_2(x)$ and $e_3(x)$ such that $a(x) = e_1(x)g_1(x)$, $b(x) = e_2(x)g_2(x)$ and $c(x) = e_3(x)g_3(x)$. Therefore, $h(x)F(x) = 0$ in $\mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$. Consequently, $f(x) = r(x)F(x)$. Let

$$f(x) = (r_0 + r_1x + \cdots + r_{h_0-1}x^{h_0-1})F(x)$$
$$= r_0F(x) + r_1xF(x) + \cdots + r_{h_0-1}x^{h_0-1}F(x),$$

which implies that $f(x)$ can be expressed as a $\mathbb{Z}_4$-linear combination of the elements $F(x),\ xF(x), \dots, x^{h_0-1}F(x)$. This proves the lemma. □

Lemma 3.2 [5] *Let $C = \langle f(x)\rangle$ be a $\mathbb{Z}_4$-cyclic code of length m. Then C is $\mathbb{Z}_4$-free if and only if there exists a polynomial $q(x) \in \mathbb{Z}_4[x]$ such that $f(x) = q(x)g(x)$ and $C = \langle g(x)\rangle$, where $g(x)|(x^m - 1)$ and $\gcd\left(q(x),\ \frac{x^m-1}{g(x)}\right) = 1$.*

Now by Lemmas 3.1 and 3.2, we get the following result.

Proposition 3.3 *Let C_{abc} be a $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic code with the generator polynomial $F(x) = (a(x), b(x), c(x)) \in \mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$, where $a(x)$, $b(x)$, $c(x)$ are $\mathbb{Z}_4[x]$ monic polynomials. Let $C_1 = \langle a(x)\rangle, C_2 = \langle b(x)\rangle$ and $C_3 = \langle c(x)\rangle$ be $\mathbb{Z}_4$-free cyclic codes and*

$$g_1(x) = \gcd(a(x), x^{km} - 1), g_2(x)$$
$$= \gcd(b(x), x^{lm} - 1), g_3(x) = \gcd(c(x), x^{tm} - 1).$$

If $h(x) = \operatorname{lcm}\left\{\frac{x^{km}-1}{g_1(x)}, \frac{x^{lm}-1}{g_2(x)}, \frac{x^{tm}-1}{g_3(x)}\right\}$ is a monic parity-check polynomial of C_{abc} with degree h_0, then C_{abc} is a $\mathbb{Z}_4$-free module of rank h_0. Moreover, the set $\{F(x), xF(x), \dots, x^{h_0-1}F(x)\}$ is a basis of C_{abc}.

Proof By Lemma 3.1, we can see that C_{abc} can be generated by the set $\{F(x), xF(x), \dots, x^{h_0-1}F(x)\}$. Therefore, it is sufficient to show that $\{F(x), xF(x), \dots, x^{h_0-1}F(x)\}$ is linearly independent over $\mathbb{Z}_4$. Now, suppose that there exist $k_0, k_1, \dots, k_{h_0-1} \in \mathbb{Z}_4$ such that

$$k_0F(x) + xF(x) + \cdots + x^{h_0-1}F(x) = \sum_{i=0}^{h_0-1} k_ix^iF(x) = 0.$$

Let $k(x) = \sum_{i=0}^{h_0-1} k_ix^i$, then $k(x)F(x) = 0$ if and only if $k(x)a(x) = 0$, $k(x)b(x) = 0$ and $k(x)c(x) = 0$ in R_{kltm}. In other words, we can say that $(x^{km} - 1)|k(x)a(x)$, $(x^{lm} - 1)|k(x)b(x)$ and $(x^{tm} - 1)|k(x)c(x)$, also that $g_1|(x^{km} - 1)$, $g_2|(x^{lm} - 1)$ and $g_3|(x^{tm} - 1)$. Now using Lemma 3.2, there exist $q_1(x),\ q_2(x),\ q_3(x) \in \mathbb{Z}_4[x]$ such that

1. $a(x) = q_1(x)g_1(x)$ and $\gcd\left(q_1(x), \frac{x^{km}-1}{g_1(x)}\right) = 1$,
2. $b(x) = q_2(x)g_2(x)$ and $\gcd\left(q_2(x), \frac{x^{lm}-1}{g_2(x)}\right) = 1$,
3. $c(x) = q_3(x)g_3(x)$ and $\gcd\left(q_3(x), \frac{x^{tm}-1}{g_3(x)}\right) = 1$.

Since $(x^{km}-1)|k(x)a(x)$, $(x^{lm}-1)|k(x)b(x)$ and $(x^{tm}-1)|k(x)c(x)$. Therefore, $(x^{km}-1)|k(x)q_1(x)g_1(x)$, $(x^{lm}-1)|k(x)q_2(x)g_2(x)$ and $(x^{tm}-1)|k(x)q_3(x)g_3(x)$ which implies

$$\left(\frac{x^{km}-1}{g_1(x)}\right)|k(x)q_1(x),\ \left(\frac{x^{lm}-1}{g_2(x)}\right)|k(x)q_2(x) \text{ and } \left(\frac{x^{tm}-1}{g_3(x)}\right)|k(x)q_3(x).$$

Also, since

$$\gcd\left(q_1(x), \frac{x^{km}-1}{g_1(x)}\right)=1,\ \gcd\left(q_2(x), \frac{x^{lm}-1}{g_2(x)}\right)=1 \text{ and } \gcd\left(q_3(x), \frac{x^{tm}-1}{g_3(x)}\right)=1.$$

So $\left(\frac{x^{km}-1}{g_1(x)}\right)|k(x)$, $\left(\frac{x^{lm}-1}{g_2(x)}\right)|k(x)$ and $\left(\frac{x^{tm}-1}{g_3(x)}\right)|k(x)$.

Therefore, $\operatorname{lcm}\left\{\frac{x^{km}-1}{g_1(x)}, \frac{x^{lm}-1}{g_2(x)}, \frac{x^{tm}-1}{g_3(x)}\right\}|k(x)$, i.e., $h(x)|k(x)$ and the monic polynomial $h(x)$ has deg h_0. Now, if $\deg(h(x)) \le (h_0-1)$, then $k(x)=0$. This implies that $F(x), xF(x), \dots, x^{h_0-1}F(x)$ are linearly independent over $\mathbb{Z}_4$. So the set $\{F(x), xF(x), \dots, x^{h_0-1}F(x)\}$ is a basis of C_{abc}. □

Note that $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes are $\mathbb{Z}_4$-free. Now, using the results of a Proposition 3.3, we shown a method to determine a generator matrix of the code C_{abc}. For the polynomials $a(x)=a_0+a_1x+\cdots+a_{km-1}x^{km-1}$, $b(x)=b_0+b_1x+\cdots+b_{lm-1}x^{lm-1}$ and $c(x)=c_0+c_1x+\cdots+c_{tm-1}x^{tm-1}$, the circulant matrices A, B and C are defined as follows:

$$A=\begin{pmatrix} a_0 & a_1 & \dots & a_{km-1} \\ a_{km-1} & a_0 & \dots & a_{km-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & a_0 \end{pmatrix},$$

$$B=\begin{pmatrix} b_0 & b_1 & \dots & b_{lm-1} \\ b_{lm-1} & b_0 & \dots & b_{lm-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \dots & c_0 \end{pmatrix},$$

$$C=\begin{pmatrix} c_0 & c_1 & \dots & c_{tm-1} \\ c_{tm-1} & c_0 & \dots & c_{tm-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \dots & c_0 \end{pmatrix},$$

thus the circulant matrix for $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$ over $\mathbb{Z}_4$ can be constructed as

$$M=\begin{pmatrix} A & B & C \\ A & B & C \\ \vdots & \vdots & \vdots \\ A & B & C \end{pmatrix}_{kltm\times(k+l+t)m}. \tag{1}$$

Thus,

$$C_{abc} = \{(x_0, x_1, \dots, x_{kltm-1})M \in \mathbb{Z}_4^{km} \times \mathbb{Z}_4^{lm} \times \mathbb{Z}_4^{tm} \mid (x_0, x_1, \dots, x_{kltm-1}) \in \mathbb{Z}_4^{kltm}\}.$$

If the parity-check polynomial of C_{abc}, $h(x) = \text{lcm}\left\{\frac{x^{km}-1}{g_1(x)}, \frac{x^{lm}-1}{g_2(x)}, \frac{x^{tm}-1}{g_3(x)}\right\}$ has deg h_0, then rank$(C_{abc}) = h_0$. Therefore, the first h_0 rows of M form a generator matrix of C_{abc}. Now, we present an example to illustrate the method discussed above.

Example 3.4 Let $m = 9,\ k = l = t = 1$, $a(x) = x^2 + x + 1,\ b(x) = x^6 + x^3 + 1,\ c(x) = x^2 + x + 1$, we find rank$(C_{abc})$.

At first, we find that $g_1(x) = \gcd(a(x), x^9 - 1) = x^2 + x + 1$, $g_2(x) = \gcd(b(x), x^9 - 1) = x^6 + x^3 + 1$, $g_3(x) = \gcd(c(x), x^9 - 1) = x^2 + x + 1$. Therefore, $h(x) = \text{lcm}$
$h(x) = \text{lcm}\{\frac{x^9-1}{g_1(x)}, \frac{x^9-1}{g_2(x)}, \frac{x^9-1}{g_3(x)}\} = (x-1)(x^2+x+1)(x^6+x^3+1)$.

The circulant matrices corresponding to the polynomials $a(x)$, $b(x)$ and $c(x)$ are

$$A = C = \begin{pmatrix} 1&1&1&0&0&0&0&0&0\\ 0&1&1&1&0&0&0&0&0\\ 0&0&1&1&1&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&1&1&1&0&0\\ 0&0&0&0&0&1&1&1&0\\ 0&0&0&0&0&0&1&1&1\\ 1&0&0&0&0&0&0&1&1\\ 1&1&0&0&0&0&0&0&1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1 \end{pmatrix}.$$

Therefore, from (1), we have

$$M = \begin{pmatrix} A & B & C \end{pmatrix}_{9\times 27.}$$

Hence, the first 9 rows of M form a generator matrix for C_{abc}. So, by Proposition 3.3, we have rank$(C_{abc}) = \deg(h(x)) = 9$.

4 Asymptotically good $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes

There is a long standing question whether the class of cyclic codes is asymptotically good. This has been an open problem for more than half a century as can be seen in [2]. Important research has been done related to this question by many researchers (see [15–18, 24] etc). To consider this question, entropy function has an important role (see [9]). Define a forth order entropy function $h_4(x)$ as follows,

$$h_4(x) = x\log_4 3 - x\log_4 x - (1-x)\log_4(1-x),$$

where, $0 \le x \le 1$. Further, let δ be a real number such that $0 < \delta < 1$ and $h_4(\frac{\delta}{2}) < \frac{1}{4}$.

We can see that $x^m - 1 = (x-1)(x^{m-1} + x^{m-2} + \cdots + 1)$, and using the Chinese Remainder Theorem (CRT), we have

$$\frac{\mathbb{Z}_4[x]}{\langle x^m - 1\rangle} = \frac{\mathbb{Z}_4[x]}{\langle x^{m-1} + x^{m-2} + \cdots + 1\rangle} \oplus \frac{\mathbb{Z}_4[x]}{\langle x-1\rangle}.$$

The cyclic code generated by $x^{m-1} + x^{m-2} + \cdots + 1$ is just the code consisting of multiple of the all-one vector, and then we only consider the cyclic codes generated by $x-1$ which are defined as,

$$\mathbb{J}_m = \langle x-1\rangle_{\mathbb{R}_m},\ \ \mathbb{J}_{kltm} = \left\langle \frac{x^{kltm}-1}{x^m-1}(x-1)\right\rangle_{\mathbb{R}_{kltm}},$$
$$\mathbb{J}_{km} = \left\langle \frac{x^{km}-1}{x^m-1}(x-1)\right\rangle_{\mathbb{R}_{km}},\ \mathbb{J}_{lm} = \left\langle \frac{x^{lm}-1}{x^m-1}(x-1)\right\rangle_{\mathbb{R}_{lm}},\ \mathbb{J}_{tm} = \left\langle \frac{x^{tm}-1}{x^m-1}(x-1)\right\rangle_{\mathbb{R}_{tm}}.$$

Now, for $(a(x), b(x), c(x)) \in \mathbb{J}_{km} \times \mathbb{J}_{lm} \times \mathbb{J}_{tm}$, let

$$C_{abc} = \{(f(x)a(x), f(x)b(x), f(x)c(x)) \in \mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm} \mid f(x) \in \mathbb{J}_{kltm}\}.$$

Then reformulating C_{abc} as a $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic code, we want to discuss the asymptotic properties of the rate $R(C_{abc})$ and the relative distance $\Delta(C_{abc})$ of C_{abc}. First, we will have discussion on the asymptotic properties of

$$C_{a'b'c'} = \{(f(x)(a'(x), f(x)b'(x), f(x)c'(x)) \in \mathbb{R}_m \times \mathbb{R}_m \times \mathbb{R}_m \mid f(x) \in \mathbb{J}_m\},$$

where $(a'(x), b'(x), c'(x)) \in \mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m$.

Thus $\mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m$ and $\mathbb{J}_{km} \times \mathbb{J}_{lm} \times \mathbb{J}_{tm}$ can be viewed as probability spaces of $\mathbb{R}_m \times \mathbb{R}_m \times \mathbb{R}_m$ and $\mathbb{R}_{km} \times \mathbb{R}_{lm} \times \mathbb{R}_{tm}$, respectively. Moreover, let C_{abc} be a random code of the probability space $\mathbb{J}_{km} \times \mathbb{J}_{lm} \times \mathbb{J}_{tm}$ with random variable $R(C_{abc})$ and $\Delta(C_{abc})$. Also, let $C_{a'b'c'}$ be a random code of the probability space $\mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m$ with random variable $R(C_{a'b'c'})$ and $\Delta(C_{a'b'c'})$. Clearly, if we are using $R(C_{abc})$ and $\Delta(C_{abc})$ as random variables on the probability space $\mathbb{J}_{km} \times \mathbb{J}_{lm} \times \mathbb{J}_{tm}$, then by the definition of asymptotically good codes, the problem has been transformed into studying of probabilities of $\mathbb{P}_r(\Delta(C_{abc}) \ge \delta)$ and $\mathbb{P}_r(\text{rank}(C_{abc}) = m-1)$, where δ is a real number such that $0 < \delta < 1$ and $\mathbb{P}_r$ denotes the probabilities of random variables $R(C_{abc})$ and $\Delta(C_{abc})$.

To see the relation between $R(C_{abc})$ and $R(C_{a'b'c'})$, we define a map ψ' as

$$\begin{aligned}\psi' \,:\, \mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m &\longrightarrow \mathbb{J}_{km} \times \mathbb{J}_{lm} \times \mathbb{J}_{tm}\\ (a'(x), b'(x), c'(x)) &\longmapsto (a(x), b(x), c(x))\end{aligned}$$

where $(a(x), b(x), c(x)) = \left(a'(x)\frac{x^{km}-1}{x^m-1}, b'(x)\frac{x^{lm}-1}{x^m-1}, c'(x)\frac{x^{tm}-1}{x^m-1}\right)$. Clearly, ψ' is a $\mathbb{R}_{kltm}$ -isomorphism and

$$(a(x), b(x), c(x)) = \psi'(a'(x), b'(x), c'(x)),\ C_{abc} = \psi'(C_{a'b'c'}).$$

Moreover, this also implies

$$\begin{aligned} wt_L(a(x), b(x), c(x)) &= wt_L(a(x)) + wt_L(b(x)) + wt_L(c(x))\\ &= kwt_L(a'(x)) + lwt_L(b'(x)) + twt_L(c'(x))\\ &\geq wt_L(a'(x), b'(x), c'(x)).\end{aligned}$$

By using the definition of relative distance, define,

$$\Delta(C_{abc}) = \frac{d_L(C_{abc})}{(k+l+t)m} = \frac{wt_L(C_{abc})}{(k+l+t)m}$$

and

$$\Delta(C_{a'b'c'}) = \frac{d_L(C_{a'b'c'})}{3m} = \frac{wt_L(C_{a'b'c'})}{3m}.$$

Now, if $\Delta(C_{abc}) \geq \Delta(C_{a'b'c'})$ then

$$(k+l+t)m\Delta(C_{abc}) \geq 3m\Delta(C_{a'b'c'}),\ \text{i.e.,}\ \Delta(C_{abc}) \geq \frac{3}{k+l+t}\Delta(C_{a'b'c'}).$$

Lemma 4.1 $\mathbb{P}_r(\Delta(C_{abc}) \geq \delta) \geq \mathbb{P}_r(\Delta(C_{a'b'c'}) \geq \frac{k+l+t}{3}\delta)$.

Proof Let $\Delta(C_{a'b'c'}) \geq \frac{k+l+t}{3}\delta$ and $\Delta(C_{abc}) \geq \frac{3}{k+l+t}\Delta(C_{a'b'c'})$ then $\Delta(C_{abc}) \geq \delta$. Thus,

$$|\Delta(C_{abc}) \geq \delta| \geq \left|\Delta(C_{a'b'c'}) \geq \frac{k+l+t}{3}\delta\right|.$$

Since ψ' is an isomorphism, we have $|\mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m| = |\mathbb{J}_{km} \times \mathbb{J}_{lm} \times \mathbb{J}_{tm}|$. So, we get

$$\begin{aligned}\mathbb{P}_r(\Delta(C_{abc}) \geq \delta) = \frac{|(\Delta(C_{abc}) \geq \delta)|}{|\mathbb{J}_{km} \times \mathbb{J}_{lm} \times \mathbb{J}_{tm}|} &\geq \frac{|\Delta(C_{a'b'c'}) \geq \frac{k+l+t}{3}\delta|}{|\mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m|}\\ &= \mathbb{P}_r\left(\Delta(C_{a'b'c'}) \geq \frac{k+l+t}{3}\delta\right).\end{aligned}$$

□

Now, in order to study the asymptotic properties of $\mathbb{P}_r(\Delta(C_{abc}) \geq \delta)$ using Lemma 4.1, we need to study the asymptotic properties of $\mathbb{P}_r(\Delta(C_{a'b'c'}) \geq \frac{k+l+t}{3}\delta)$. For that we need the following definition. For any $f(x) \in \mathbb{J}_m$ and $(a'(x), b'(x), c'(x)) \in \mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m$ over the probability space $\mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m$. We have

Definition 4.2 *The Bernoulli random variable* Y_f is defined as

$$Y_f = \begin{cases} 1 & 1 \leq wt_L(a'(x), b'(x), c'(x)) \leq 3m\delta \\ 0 & otherwise. \end{cases}$$

Given that $f(x) \in \mathbb{J}_m$, consider the set $\{f(x)a'(x) \in \mathbb{R}_m \mid a'(x) \in \mathbb{J}_m\}$. It can be inferred that this set is an ideal of $\mathbb{R}_m$ generated by $f(x)$. Let $\mathbb{I}_f = \langle f(x) \rangle \subseteq \mathbb{J}_m$ and $|\mathbb{I}_f| = 2^{d_f}$.

We have the following:

Lemma 4.3 *If* $\mathbb{I}_f \times \mathbb{I}_f \times \mathbb{I}_f \subseteq \mathbb{R}_m \times \mathbb{R}_m \times \mathbb{R}_m$, *and*

$$(\mathbb{I}_f \times \mathbb{I}_f \times \mathbb{I}_f)^{\leq 3m\delta} = \{(f_1(x), f_2(x), f_3(x)) \in \mathbb{I}_f \times \mathbb{I}_f \times \mathbb{I}_f \mid wt_L(f_1(x), f_2(x), f_3(x)) \leq 3m\delta\},$$

then

$$|(\mathbb{I}_f \times \mathbb{I}_f \times \mathbb{I}_f)^{\leq 3m\delta}| \leq 4^{3d_f h_4(\frac{\delta}{2})} = 2^{6d_f h_4(\frac{\delta}{2})}.$$

Proof Since $|\mathbb{R}_m \times \mathbb{R}_m \times \mathbb{R}_m| = 4^{3m} = 2^{6m}$ and $|\mathbb{I}_f \times \mathbb{I}_f \times \mathbb{I}_f| = 2^{3d_f}$ then the fraction of $3m\delta$ over the length $6m$ is $\frac{3m\delta}{6m} = \frac{\delta}{2}$. Additionally $0 < \delta < 1$, so, $0 < \frac{\delta}{2} < \frac{1}{2} < \frac{3}{4}$. Therefore, by extending the results in [[17], Corollary 3.5, Remark 3.2] for $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$, we have

$$|(\mathbb{I}_f \times \mathbb{I}_f \times \mathbb{I}_f)^{\leq 3m\delta}| \leq 4^{3d_f h_4(\frac{\delta}{2})} = 2^{6d_f h_4(\frac{\delta}{2})}.$$

□

Now, by Lemma 4.3 we have the following:

Lemma 4.4 $\mathbb{E}(Y_f) \leq 4^{3d_f h_4(\frac{\delta}{2}) - \frac{3d_f}{2}}$, *where* $\mathbb{E}$ *denotes the expectation of a random variable.*

Proof From Lemma 4.3, $|(\mathbb{I}_f \times \mathbb{I}_f \times \mathbb{I}_f)^{\leq 3m\delta}| \leq 4^{3d_f h_4(\frac{\delta}{2})}$. So

$$\mathbb{E}(Y_f) = \mathbb{P}_r(Y_f = 1) = \frac{|(\mathbb{I}_f \times \mathbb{I}_f \times \mathbb{I}_f)^{\leq 3m\delta}| - 1}{|\mathbb{I}_f \times \mathbb{I}_f \times \mathbb{I}_f|}$$
$$\leq \frac{4^{3d_f h_4(\frac{\delta}{2})}}{2^{3d_f} = 4^{\frac{3d_f}{2}}}$$
$$= 4^{3d_f h_4(\frac{\delta}{2})} 4^{-\frac{3d_f}{2}}$$
$$= 4^{3d_f h_4(\frac{\delta}{2}) - \frac{3d_f}{2}}.$$

□

By CRT, we have

$$\mathbb{J}_m = \langle x-1 \rangle_{R_m} \cong \frac{\mathbb{Z}_4[x]}{\langle x^{m-1} + x^{m-2} + \cdots + 1 \rangle}$$
$$= \frac{\mathbb{Z}_4[x]}{\langle q_1(x) \rangle} \times \frac{\mathbb{Z}_4[x]}{\langle q_2(x) \rangle} \times \cdots \times \frac{\mathbb{Z}_4[x]}{\langle q_r(x) \rangle},$$

where $q_1(x), q_2(x), \ldots, q_r(x)$ are monic basic irreducible factors of $x^{m-1} + x^{m-2} + \cdots + 1 \in \mathbb{Z}_4[x]$. Let $q_k(x)$, for $1 \leq k \leq r$, be a polynomial lowest degree among $q_1(x), q_2(x), \ldots, q_r(x)$. Then the minimal Galois ring among them is $\frac{\mathbb{Z}_4[x]}{\langle q_k(x) \rangle}$ and it contains a non-zero ring of least size 2^{k_m}. By CRT, the ideals in $\mathbb{J}_m$ correspond to the ideals in $\frac{\mathbb{Z}_4[x]}{\langle q_1(x) \rangle} \times \frac{\mathbb{Z}_4[x]}{\langle q_2(x) \rangle} \times \cdots \times \frac{\mathbb{Z}_4[x]}{\langle q_r(x) \rangle}$ (see [19, Lemma 9] and [7]). So, the minimal size of the non-zero ideal contained in $\mathbb{J}_m$ is equal to 2^{k_m}.

Lemma 4.5 [19] *The number of non-zero ideals of size 2^d contained in $\mathbb{J}_m$ is at most $(2m)^{\frac{d}{k_m}}$, where $k_m \leq d \leq 2(m-1)$.*

Now, we will show that $\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{a'b'c'}) \geq \delta) = 1$. For that, by Lemmas 4.4 and 4.5, we prove an useful lemma:

Lemma 4.6 *Let $0 < \delta < 1$ be a real number and $h_4(\frac{\delta}{2}) < \frac{1}{4}$, then*

$$\mathbb{P}_r(\Delta(C_{a'b'c'}) \leq \delta) \leq \sum_{i=k_m}^{2(m-1)} 4^{-3j(\frac{1}{3} - h_4(\delta_2) - \frac{\log_4 2m}{3k_m})}.$$

Proof Let Y_f for $f(x) \in J_m$ be a Bernoulli variable with a value 0 or 1. Let $Y = \sum_{f(x)\in J_m} Y_f$, then Y is a non-negative integer random variable over the probability space $\mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m$. Y stands for the cardinality of $f(x) \in \mathbb{J}_m$ such that the weight of the codewords is at most $3m\delta$ and $\Delta(C_{a'b'c'}) = \frac{wt_L(C_{a'b'c'})}{3m}$, we get $\mathbb{P}_r(\Delta(C_{a'b'c'}) \leq \delta) = \mathbb{P}_r(Y > 0)$. By Markov's inequality [26, Theorem 3.1], $\mathbb{P}_r(Y > 0) \leq \mathbb{E}(Y)$. So, we only need to find the value of $\mathbb{E}(Y)$. From [22], we have

$$\mathbb{E}(\alpha Y_1 + Y_2) = \alpha \mathbb{E}(Y_1) + \mathbb{E}(Y_2).$$

So, $\mathbb{E}(Y) = \mathbb{E}(\sum_{f(x)\in\mathbb{J}_m} Y_f)$, for any ideal $\mathbb{I}$ of $\mathbb{J}_m$, denoted as $(\mathbb{I} \le \mathbb{J}_m)$. Let $\mathbb{I}^* = \{f(x) \in \mathbb{I} \mid \mathbb{I}_f = \mathbb{I}\}$, where $\mathbb{I}_f = \langle f(x)\rangle_{\mathbb{R}_m} \subseteq \mathbb{J}_m$. Since $d_f = \mathrm{rank}(\mathbb{I}_f)$ then $\mathbb{I}^* = \{f(x) \in \mathbb{I} \mid d_f = \mathrm{rank}(\mathbb{I})\}$. Therefore,

$$\mathbb{J}_m = \bigcup_{\mathbb{I}\subseteq\mathbb{J}_m} \mathbb{I}^*$$

and $0 \ne \mathbb{I} \le \mathbb{J}_m$ then $k_m \le \mathrm{rank}(\mathbb{I}) = d \le 2(m-1)$. So,

$$\mathbb{E}(Y) = \sum_{\mathbb{I}\le\mathbb{J}_m}\sum_{f(x)\in\mathbb{I}^*} \mathbb{E}(Y_f) = \sum_{i=k_m}^{2(m-1)} \sum_{\substack{\mathbb{I}\le\mathbb{J}_m \\ \mathrm{rank}\mathbb{I} = j}} \sum_{f(x)\in\mathbb{I}^*} \mathbb{E}(Y_f).$$

For $\mathbb{I} \le \mathbb{J}_m$ with $\mathrm{rank}(\mathbb{I}) = j$ & $|\mathbb{I}^*| \le |\mathbb{I}| = 2^i$. Using Lemma 4.4, we have

$$\sum_{f\in\mathbb{I}^*} \mathbb{E}(Y_f) \le \sum_{f\in\mathbb{I}^*} 4^{3d_f h_4(\frac{\delta}{2}) - \frac{3d_f}{2}}$$
$$= \sum_{f\in\mathbb{I}^*} 4^{3jh_4(\frac{\delta}{2}) - \frac{3j}{2}}.$$

By Lemma 4.5, for $\mathbb{I} \le \mathbb{J}_m$ with $\mathrm{rank}\mathbb{I} = j$ which is less than $(2m)^{\frac{j}{k_m}}$ and we know that $\log_4 2m \le \frac{j\log_4 2m}{k_m}$ as $k_m \le j$, so

$$\mathbb{E}(Y) \le \sum_{j=k_m}^{2(m-1)} (2m)^{\frac{j}{k_m}} 4^{-j+3jh_4(\frac{\delta}{2})}$$
$$= \sum_{j=k_m}^{2(m-1)} 4^{\frac{3j}{3k_m}\log_4 2m} 4^{\frac{-3j}{3}+3jh_4(\frac{\delta}{2})}$$

$$\mathbb{E}(Y) \le \sum_{j=k_m}^{2(m-1)} 4^{-3j(\frac{1}{3} - h_4(\delta_2) - \frac{\log_4 2m}{3k_m})}.$$

Thus, we have

$$\mathbb{P}_r(\Delta(C_{a'b'c'}) \le \delta) \le \sum_{i=k_m}^{2(m-1)} 4^{-3j(\frac{1}{3} - h_4(\delta_2) - \frac{\log_4 2m}{3k_m})}.$$

□

Remark 4.7 By [4, Lemma 2.6] there exist positive integers $m_1, m_2, \dots$ such that $\gcd(m_i, 4) = 1, m_i \to \infty, \lim_{i\to\infty} \frac{\log_4 m_i}{k_{m_i}} = 0$ where k_{m_i} are defined as in Lemma 4.5.

Let

$$C^i_{a'b'c'} = \{f(x)a'(x), f(x)b'(x), f(x)c'(x) \in \mathbb{R}_{m_i} \times \mathbb{R}_{m_i} \times \mathbb{R}_{m_i} | f(x) \in \mathbb{J}_{m_i}\}$$

be a random $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-cyclic code of length $3m_i$, where $(a'(x), b'(x), c'(x)) \in \mathbb{J}_{m_i} \times \mathbb{J}_{m_i} \times \mathbb{J}_{m_i}$.

Now, by using Lemma 4.6, we have one of the main results of the paper in the following proposition.

Proposition 4.8 *Let* $0 < \delta < 1$ *be a real number and* $h_4(\frac{\delta}{2}) < \frac{1}{4}$ *then*

$$\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{a'b'c'}) \geq \delta) = 1.$$

Proof From the assumptions on δ and h_4, we have $h_4(\frac{\delta}{2}) < \frac{1}{4} < \frac{1}{3}$ which implies that $\frac{1}{3} - h_4(\frac{\delta}{2}) > 0$. Since $\lim_{i\to\infty} \frac{\log_4 m_i}{k_{m_i}} = 0$, then $\lim_{i\to\infty} \frac{\log_4 2m_i}{k_{m_i}} = 0$. Therefore, for a given $\epsilon > 0$ there exists a non-negative integer N such that for $i > N$, we have $\frac{1}{3} - h_4(\frac{\delta}{2}) - \frac{\log_4 2m_i}{3k_{m_i}} \geq \epsilon > 0$. From Lemma 4.6, we have

$$\begin{aligned}
\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{a'b'c'}) \leq \delta) &\leq \lim_{i\to\infty} \sum_{j=k_{m_i}}^{2(m-1)} 4^{-3j(\frac{1}{3}-h_4(\frac{\delta}{2})-\frac{\log_4 2m_i}{3k_{m_i}})} \\
&\leq \lim_{i\to\infty} \sum_{j=k_{m_i}}^{2(m-1)} 4^{-3j\epsilon} \\
&\leq \lim_{i\to\infty} \sum_{j=k_{m_i}}^{2(m-1)} 4^{-3k_{m_i}\epsilon} \\
&\leq \lim_{i\to\infty} 2m_i 4^{-3k_{m_i}\epsilon} \\
&= \lim_{i\to\infty} 4^{-3k_{m_i}(\epsilon-\frac{\log_4 2m_i}{3k_{m_i}})}.
\end{aligned}$$

Also, since $\lim_{i\to\infty} \frac{\log_4 m_i}{k_{m_i}} = 0$, then $\lim_{i\to\infty} \frac{\log_4 2m_i}{3k_{m_i}} = 0$ which yields $\lim_{i\to\infty} 3m_i \to \infty$. Therefore, $\lim_{i\to\infty} 4^{-3k_{m_i}(\epsilon-\frac{\log_4 2m_i}{3k_{m_i}})} = 0$, i.e., $\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{a'b'c'}) \leq \delta) = 0$ which implies that

$$\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{a'b'c'}) \geq \delta) = 1.$$

□

From Proposition 4.8, $0 < \delta < 1$ and $h_4(\frac{\delta}{2}) < \frac{1}{4}$ it can be seen that,

$$\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{a'b'c'}) \geq \delta) = 1.$$

In other words, we can say that if $0 < \delta < 1$ and $h_4(\frac{1}{2}\frac{k+l+t}{3}\delta) < \frac{1}{4}$, i.e., $h_4(\frac{k+l+t}{6}\delta) < \frac{1}{4}$, then we have

$$\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{a'b'c'}) \geq \frac{k+l+t}{3}\delta) = 1.$$

Now, by Proposition 4.8 and Lemma 4.1, we have one of the main results of the paper in the following proposition.

Proposition 4.9 *If* $h_4(\frac{k+l+t}{6}\delta) < \frac{1}{4}$ *then* $\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{abc}) \geq \delta) = 1$.

Proof By Proposition 4.8, $0 < \delta < 1$ and $h_4(\frac{k+l+t}{6}\delta) < \frac{1}{4}$, we have

$$\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{a'b'c'}) \geq \frac{k+l+t}{3}\delta) = 1.$$

From Lemma 4.1, we have

$$\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{abc}) \geq \delta) \geq \lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{a'b'c'}) \geq \frac{k+l+t}{3}\delta) = 1.$$

So $\lim_{i\to\infty} \mathbb{P}_r(\Delta(C^i_{abc}) \geq \delta) = 1$. □

Now, we will prove that $\lim_{i\to\infty} \mathbb{P}_r(\mathrm{rank}(C^i_{abc}) = m_i - 1) = 1$. For that, we need the following lemma:

Lemma 4.10 *Let*

$$C_{a'b'c'} = \{(f(x)a'(x), f(x)b'(x), f(x)c'(x)) \in \mathbb{R}_m \times \mathbb{R}_m \times \mathbb{R}_m \mid f(x) \in \mathbb{J}_m\},$$

where $(a'(x), b'(x), c'(x)) \in \mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m$. *Then* $\mathrm{rank}(C_{a'b'c'}) \leq m-1$. *Note that* $\mathrm{rank}(C_{a'b'c'}) = m-1$ *if and only if there is no basic irreducible factor* $q(x)$ *of* $\frac{x^m-1}{x-1}$ *in* $\mathbb{Z}_4[x]$ *such that*

$$q(x)|a'(x), q(x)|b'(x) \text{ and } q(x)|c'(x).$$

Proof Suppose $g_{a'b'c'}(x) = \gcd(a'(x), b'(x), c'(x), x^m - 1)$ and consider

$$(a'(x), b'(x), c'(x)) \in \mathbb{J}_m \times \mathbb{J}_m \times \mathbb{J}_m.$$

We have $(x-1)|g_{a'b'c'}(x)$, i.e., $\langle g_{a'b'c'}(x)\rangle \subseteq \langle x-1\rangle = \mathbb{J}_m$, which implies that

$$\mathrm{rank}(C_{a'b'c'}) = \deg(\frac{x^m-1}{g_{a'b'c'}(x)}) \leq m-1.$$

Clearly, $\mathrm{rank}(C_{a'b'c'}) < m-1$ if and only if $\deg(g_{a'b'c'}(x) > 1)$ if and only if there is a basic irreducible factor $q(x)$ of $\frac{x^m-1}{x-1}$ in $\mathbb{Z}_4[x]$ such that

$$q(x)|a'(x), q(x)|b'(x) \text{ and } q(x)|c'(x).$$

Therefore, it is easy to see that $\mathrm{rank}(C_{a'b'c'}) = m-1$ if and only if $g_{a'b'c'}(x) = x-1$ if and only if there is no basic irreducible factor $q(x)$ of $\frac{x^m-1}{x-1}$ in $\mathbb{Z}_4[x]$ such that

$$q(x)|a'(x), q(x)|b'(x) \text{ and } q(x)|c'(x).$$

□

Proposition 4.11 *Let* $m_1, m_2, \dots$ *be positive integers such that* $\gcd(m_i, 4) = 1$ *and* $\lim_{i\to\infty} \frac{\log_4 m_i}{k_{m_i}} = 0$, *for* $m_i \to \infty$ *where* k_{m_i} *are as defined in Lemma 4.5. Let*

$$C^i_{a'b'c'} = \{(f(x)a'(x), f(x)b'(x), f(x)c'(x)) \in \mathbb{R}_{m_i} \times \mathbb{R}_{m_i} \times \mathbb{R}_{m_i} | f(x) \in \mathbb{J}_{m_i}\}$$

then $\lim_{i\to\infty} \mathbb{P}_r(rank(C^i_{a'b'c'}) = m_i - 1) = 1$.

Proof For any i, suppose that

$$\begin{aligned} x^{m_i} - 1 &= (x-1)(x^{m_i-1} + x^{m_i-2} + \cdots + 1) \\ &= (x-1)q_1(x), q_2(x), \dots, q_{r_i}(x). \end{aligned}$$

where $q_1(x), q_2(x), \dots, q_{r_i}(x)$ are monic basic irreducible factors of $x^{m-1} + x^{m-2} + \cdots + 1 \in \mathbb{Z}_4[x]$. Using CRT, we have

$$\begin{aligned} \mathbb{J}_{m_i} = \langle x-1 \rangle_{R_{m_i}} &\cong \frac{\mathbb{Z}_4[x]}{\langle x^{m_i-1} + x^{m_i-2} + \cdots + 1 \rangle} \\ &= \frac{\mathbb{Z}_4[x]}{\langle q_1(x) \rangle} \times \frac{\mathbb{Z}_4[x]}{\langle q_2(x) \rangle} \times \cdots \times \frac{\mathbb{Z}_4[x]}{\langle q_{r_i}(x) \rangle}, \end{aligned}$$

define a function

$$(a'(x)) \longmapsto (a'_1(x), a'_2(x), \dots, a'_{r_i}(x))$$

where $a'_j(x) = a'(x)(\mod q_j)$, $j = 1, 2, \dots, r_i$ for $(a'(x), b'(x), c'(x)) \in \mathbb{J}_{m_i} \times \mathbb{J}_{m_i} \times \mathbb{J}_{m_i}$. By Lemma 4.10, we have $\text{rank}(C^i_{a'b'c'}) \le m_i - 1$ and $\text{rank}(C^i_{a'b'c'}) < m_i - 1$ if and only if there is basic irreducible factor $q_j(x), j = 1, 2, \dots, r_i$ of $\frac{x^{m_i}-1}{x-1}$ in $\mathbb{Z}_4[x]$ such that $q_j(x)|a'(x)$, $q_j(x)|b'(x)$ and $q_j(x)|c'(x)$ which can only defined when $a'_j(x) = b'_j(x) = c'_j(x) = 0$. In other words, $\text{rank}(C^i_{a'b'c'}) = m_i - 1$ if and only if $(a'_j(x), b'_j(x), c'_j(x)) \ne (0,0,0)$. Let $k_j = \deg q_j(x)$ then $|\frac{\mathbb{Z}_4[x]}{\langle q_j(x) \rangle}| = 4^{k_j}$. Since there is a surjective homomorphism

$$\mathbb{J}_{m_i} \longrightarrow \frac{\mathbb{Z}_4[x]}{\langle q_j(x) \rangle},$$

so there are $4^{3k_j} - 1$ polynomial triples $(a'_j(x), b'_j(x), c'_j(x)) \ne (0,0,0)$. i.e., $\mathbb{P}_r((a'_j(x), b'_j(x), c'_j(x)) \ne (0,0,0)) = \frac{4^{3k_j}-1}{4^{3k_j}} = 1 - 4^{-3k_j}$ which yields,

$$\mathbb{P}_r(\text{rank}(C^i_{a'b'c'}) = m_i - 1) = \prod_{j=1}^{r_i} (1 - 4^{-3k_j}).$$

Since $k_{m_i} \le k_j$ then $r_i \le \frac{m_i - 1}{k_{m_i}} \le \frac{m_i}{k_{m_i}}$ (Lemma 4.5).

Therefore,

$$\mathbb{P}_r(\mathrm{rank}(C^i_{a'b'c'}) = m_i - 1) \geq (1 - 4^{-3k_{m_i}})^{\frac{m_i}{k_{m_i}}}$$
$$= (1 - 4^{-3k_{m_i}})^{4^{3k_{m_i}} \frac{m_i}{k_{m_i}4^{3k_{m_i}}}}.$$

Since $\lim_{i\to\infty} \frac{m_i}{k_{m_i}4^{3k_{m_i}}} = 0$ and $\lim_{i\to\infty}(1 - 4^{-3k_{m_i}})^{4^{3k_{m_i}}} = \frac{1}{e}$, therefore

$$\lim_{i\to\infty}(1 - 4^{-3k_{m_i}})^{4^{3k_{m_i}} \frac{m_i}{k_{m_i}4^{3k_{m_i}}}} = (\frac{1}{e})^0 = 1.$$

Thus, $\lim_{i\to\infty} \mathbb{P}_r(\mathrm{rank}(C^i_{a'b'c'}) = m_i - 1) \geq 1$, i.e., $\lim_{i\to\infty} \mathbb{P}_r(\mathrm{rank}(C^i_{a'b'c'}) = m_i - 1) = 1$. □

By the isomorphism ψ', it gives us $C^i_{abc} = \psi'(C^i_{a'b'c'})$ and using Proposition 4.11, we have one of the main results of the paper in the following proposition.

Proposition 4.12 $\lim_{i\to\infty} \mathbb{P}_r(rank(C^i_{abc}) = m_i - 1) = 1.$

Proof From isomorphism ψ', $C^i_{abc} = \psi'(C^i_{a'b'c'})$ and $\mathrm{rank}(C^i_{abc}) = \mathrm{rank}(\psi'(C^i_{a'b'c'})) = \mathrm{rank}(C^i_{a'b'c'})$ and using Proposition 4.11 we have $\lim_{i\to\infty} \mathbb{P}_r(\mathrm{rank}(C^i_{abc}) = m_i - 1) = 1.$ □

Now, by using Propositions 4.9 and 4.12 we get the asymptotic properties of $\mathbb{P}_r(\Delta(C^i_{abc}) \geq \delta)$ and $\mathbb{P}_r(\mathrm{rank}(C^i_{abc}) = m_i - 1)$ as follows.

Corollary 4.13 *Let* $C^i_{abc} = \{(f(x)a(x), f(x)b(x), f(x)c(x)) \in \mathbb{R}_{km_i} \times \mathbb{R}_{lm_i} \times \mathbb{R}_{tm_i} | f(x) \in \mathbb{J}_{kltm_i}\}$ *and* $m_1, m_2, \ldots$ *such that* $\gcd(m_i, 4) = 1$ *and* $\lim_{i\to\infty} \frac{\log_4 m_i}{k_{m_i}} = 0$ *for* $m_i \to \infty$.

- If $h_4(\frac{k+l+t}{6}\delta) < \frac{1}{4}$, then $\lim_{i\to\infty} \mathbb{P}_r(\Delta C^i_{abc} \geq \delta) = 1.$
- $\lim_{i\to\infty} \mathbb{P}_r(\mathrm{rank}(C^i_{abc}) = m_i - 1) = 1.$

Considering all the results mentioned above, a main result of this paper can be stated in the following theorem.

Theorem 4.14 *Let* $0 < \delta < 1$ *be a real number and* $h_4(\frac{k+l+t}{6}\delta) < \frac{1}{4}$ *then there exists a sequence of* $\mathbb{Z}_4$*-free* $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$*-additive cyclic codes* $\{C_i\}_{i=0}^{\infty}$ *of block length* (km_i, lm_i, tm_i), *when* $m_i \to \infty$, *such that*

- $\lim_{i\to\infty} R(C_i) = \frac{1}{k+l+t}$

- $\Delta(C_i) \geq \delta$

Consequently, $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes are asymptotically good.

Proof By Corollary 4.13, if $h_4(\frac{k+l+t}{6}\delta) < \frac{1}{4}$ then $\lim_{i\to\infty} \mathbb{P}_r(\Delta C_i \geq \delta) = 1$ and $\lim_{i\to\infty} \mathbb{P}_r(\text{rank}(C_i) = m_i - 1) = 1$. It implies that, there exists an integer $N > 0$ such that for $i > N$, we have $\text{rank}(C_i) = m_i - 1$ and $\Delta(C_i) \geq \delta$. Thus, if we delete the first N codes and then for the remaining codes we have $\text{rank}(C_i) = m_i - 1$ and $\Delta(C_i) \geq \delta$. The asymptotic rate of C_i is

$$\lim_{i\to\infty} R(C_i) = \lim_{i\to\infty} \frac{\text{rank}(C_i)}{km_i + lm_i + tm_i} = \lim_{i\to\infty} \frac{m_i - 1}{(k+l+t)m_i} = \frac{1}{k+l+t}$$

and the asymptotic relative distance of C_i is $\Delta(C_i) \geq \delta$. Now, it can be seen that the relative distance and the rate of C_i are positively bounded from below. So, by definition, $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes are asymptotically good. □

Example 4.15 We find a sequence of codes $\{C_i\}_{i=0}^{\infty}$ of $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes and their rate converges to $\frac{1}{3}$ and relative distance greater than or equal to $\frac{1}{8}$, and to show they are asymptotically good.

Assume that $k = l = t = 1$, let $\delta = \frac{1}{8}$ and $h_4(\frac{1}{16}) = .21817511 < .25$. So, $\mathbb{R}_{km} = \mathbb{R}_m = \frac{\mathbb{Z}_4[x]}{\langle x^m - 1\rangle} = \mathbb{R}_{lm} = \mathbb{R}_{tm} = \mathbb{R}_{kltlm}$, where m, k, l and t are positive integers such that $\gcd(m, 4) = 1$ and k, l, t and 4 are pairwise co-prime. Therefore, it is easy to see that $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes in $\mathbb{Z}_4^m \times \mathbb{Z}_4^m \times \mathbb{Z}_4^m$ are $\mathbb{Z}_4[x]$-submodules of $\mathbb{R}_m \times \mathbb{R}_m \times \mathbb{R}_m$, for $(a(x), b(x), c(x)) \in \mathbb{R}_m \times \mathbb{R}_m \times \mathbb{R}_m$. Hence, consider a sequence of codes $\{C_i\}_{i=0}^{\infty}$ of $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes as follows.

Let $C_{abc}^i = \{(f(x)a(x), f(x)b(x), f(x)c(x)) \in \mathbb{R}_{m_i} \times \mathbb{R}_{m_i} \times \mathbb{R}_{m_i} | f(x) \in \mathbb{J}_{m_i}\}$ and m_i be the positive integers such that $\gcd(m_i, 4) = 1$. Further, $\lim_{i\to\infty} \frac{\log_4 m_i}{k_{m_i}} = 0$ for $m_i \to \infty$, where k_{m_i} is as defined in Lemma 4.5. Now by Corollary 4.13, we get $\lim_{i\to\infty} \mathbb{P}_r(\Delta C_{abc}^i \geq \frac{1}{8}) = 1$ and $\lim_{i\to\infty} \mathbb{P}_r(\text{rank}(C_{abc}^i) = m_i - 1) = 1$. Therefor, by Theorem 4.14

- $\lim_{i\to\infty} R(C_i) = \frac{1}{3}$
- $\Delta(C_i) \geq \frac{1}{8}$

Now, it can be seen that the relative distance and the rate of C_i are positively bounded from below. Hence, the sequence of codes $\{C_i\}_{i=0}^{\infty}$ of $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes is asymptotically good.

5 Conclusion

In this paper, we have discussed $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes of different component lengths and constructed a class of $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes C_{abc}. Moreover, we have found a basis set for C_{abc} and presented a method to determine a generator matrix for the code C_{abc}. By using a probabilistic method, we have constructed a random sequence of codes C^i_{abc} of $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$-additive cyclic codes. Moreover, we have studied the asymptotic properties of these classes of $\mathbb{Z}_4\mathbb{Z}_4\mathbb{Z}_4$ -additive cyclic codes and then we proved $\lim_{i\to\infty} \mathbb{P}_r(\Delta C^i_{abc} \geq \delta) = 1$ and $\lim_{i\to\infty} \mathbb{P}_r(\text{rank}(C^i_{abc}) = m_i - 1) = 1$. Additionally, we have determined the asymptotic rates and relative distances of these classes of codes using probabilistic methods and found that they are asymptotically good. Also, we have presented a supporting example for these classes of codes.

In the future, it would be interesting to study the asymptotic properties of other families of codes, such as other additive cyclic codes generated by 3-tuples of polynomials of different code lengths.

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References

1. Abualrub, T., Siap, I., Aydin, N.: $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. IEEE Trans. Inf. Theory **60**, 1508–1514 (2014)
2. Assmus, E.F., Mattson, H.F., Turyn, R.: Cyclic codes, AF Cambridge Research Labs, Bedford, AFCRL, 66-348 (1966)
3. Aydogdu, I., Gursoy, F.: $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-cyclic codes. J. Appl. Math. Comput. **60**, 327–341 (2019)
4. Bazzi, L.M.J., Mitter, S.K.: Some randomized code constructions from group actions. IEEE Trans. Inf. Theory **52**, 3210–3219 (2006)
5. Bhaintwal, M., Wasan, S.K.: On quasi-cyclic codes over $\mathbb{Z}_q$. Appl. Algebra Eng. Commun. Comput. **20**, 459–480 (2009)
6. Borges, J., Fernandez-Cordoba, C., Pujol, J., Rifa, J., Villanueva, M.: $\mathbb{Z}_2\mathbb{Z}_4$-linear codes: generator matrices and duality. Designs Codes Cryptogr. **54**, 167–179 (2010)
7. Cao, Y.: Generalized quasi-cyclic codes over Galois rings; structural properties and enumeration. Appl. Algebra Eng. Commun. Comput. **22**, 219–233 (2011)
8. Chen, B., Dinh, H.Q., Liu, H., Wang, L.: Constacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. Finite Fields Appl. **36**, 108–130 (2016)
9. Cover, T.M., Thomas, J.A.: Elements of Information Theory. Wiley, New York (1991)
10. Dinh, H.Q.: Constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. J. Algebra **324**, 940–950 (2010)
11. Dinh, H.Q., Nguyen, B.T., Sriboonchitta, S.: Negacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ and their duals. Discrete Math. **341**, 1055–1071 (2018)
12. Dinh, H.Q., Nguyen, B.T., Sriboonchitta, S., Vo, T.M.: Constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. J. Algebra Appl. **18**, 1950022 (2019)
13. Dinh, H.Q., Nguyen, B.T., Sriboonchitta, S., Vo, T.M.: $(\alpha + u\beta)$-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. J. Algebra Appl. **18**, 1950023 (2019)
14. Dinh, H.Q., Nguyen, B.T., Yamaka, W.: Constacyclic codes of length $3p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ and their application in various distance distributions. IEEE Access **8**, 204031–204056 (2020)
15. Fan, Y., Liu, H.: Quasi-cyclic codes of index $1\frac{1}{2}$. arXiv:1505.02252 (2015)

16. Fan, Y., Liu, H.: Quasi-cyclic codes of index $1\frac{1}{3}$. IEEE Trans. Inf. Theory **60**, 6342–6347 (2016)
17. Fan, Y., Lin, L.: Thresholds of random quasi-abelian codes. IEEE Trans. Inf. Theory **62**, 82–90 (2015)
18. Fan, Y., Liu, H.: $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes are asymptotically good. arxiv:1911.09350 (2019)
19. Gao, J., Hou, X.: $\mathbb{Z}_4$-double cyclic codes are asymptotically good. IEEE Commun. Lett. **24**, 1593–1597 (2020)
20. Gao, J., Shi, M., Wu, T., Fu, F.: On double cyclic codes over $\mathbb{Z}_4$. Finite Fields Appl. **39**, 233–250 (2016)
21. Güneri, C., Özbudak, F., Özkaya, B., Saçıkara, E., Sepasdar, Z., Solé, P.: Structure and performance of generalized quasi-cyclic codes. Finite Fields Appl. **47**, 183–202 (2017)
22. Gupta, S.C., Kapoor, V.K.: Fundamental of Mathematical Statistics. Sultan Chand and Sons, Delhi (1970)
23. Hammons, A., Kumar, P.V., Calderbank, A.R., Sloane, N.J.A., Solè, P.: The $\mathbb{Z}_4$ linearity of kerdock, preparata, goethals and related codes. IEEE Trans. Inf. Theory **40**, 301–319 (1994)
24. Huffman, W.C., Pless, V.: Fundamentals of Error Correcting Codes. Cambridge university Press, Cambridge (2003)
25. Martinez-Perez, C., Willems, W.: Is the class of cyclic codes asymptotically good? IEEE Trans. Inf. Theory **52**, 696–700 (2006)
26. Mitzenmacher, M., Upfal, E.: Probability and Computing, Randomized Algorithm and Probabilistic Analysis. Cambridge University Press, Cambridge (2005)
27. Shi, M., Wu, R., Solè, P.: Asymptotically good additive cyclic codes exist. IEEE Commun. Lett. **22**, 1980–1983 (2018)
28. Siap, I., Kulhan, N.: The structure of generalized quasi cyclic codes. Appl. Math. E-Notes **5**, 24–30 (2005)
29. Wesley, W.: Peterson and E. J. Weldon, Error Correcting Codes, MIT Press, Cambridge (1972)
30. Wu, T., Gao, J., Gao, Y., Fu, F.: $\mathbb{Z}_2\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. Adv. Math. Commun. **12**, 641–657 (2018)
31. Yao, T., Zhu, S.: $\mathbb{Z}_p\mathbb{Z}_{p^s}$-additive cyclic codes are asymptotically good. Cryptogr. Commun. **12**, 253–264 (2019)
32. Yao, T., Zhu, S., Kai, X.: $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes are asymptotically good. Finite Fields Appl. **63**, 101633 (2020)

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