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ORIGINAL PAPER

# Gaussian sums, hyper Eisenstein sums and Jacobi sums over a local ring and their applications

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## Abstract
It is well known that any finite commutative ring is isomorphic to a direct product of local rings via the Chinese remainder theorem. Hence, there is a great significance to the study of character sums over local rings. Character sums over finite rings have applications that are analogous to the applications of character sums over finite fields. In particular, character sums over local rings have many applications in algebraic coding theory. In this paper, we firstly present an explicit description on additive characters and multiplicative characters over a certain local ring. Then we study Gaussian sums, hyper Eisenstein sums and Jacobi sums over a certain local ring and explore their properties. It is worth mentioning that we are the first to define Eisenstein sums and Jacobi sums over this local ring. Moreover, we present a connection between hyper Eisenstein sums over this local ring and Gaussian sums over finite fields, which allows us to give the absolute value of hyper Eisenstein sums over this local ring. As an application, several classes of codebooks with new parameters are presented.

**Keywords** Local ring · Gaussian sum · Hyper Eisenstein sum · Jacobi sum · Codebook · Welch bound

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## 1 Introduction

Exponential sums are important tools in number theory and arithmetic geometry for solving problems involving integers and real numbers. It has been well known for a long time that Gaussian sums, Jacobi sums and Eisenstein sums over finite fields, as special cases of general exponential sums, have many remarkable applications in combinatorics, coding theory and cryptography. Whereafter, exponential sums over Galois rings have become very important tools to construct good error-correcting codes, sequences and combinatorial designs (see, for example, [10, 17]). In [30], Oh et al. investigated Gaussian sums over the Galois ring $GR(p^t, r)$ with $t = 2$. For the general case $t \geq 2$, Gaussian sums were studied by Kwon and Yoo [18] and used to construct difference sets in [43]. In 2013, the reference [28] presented more explicit computations on Gaussian sums and Jacobi sums over the Galois ring $GR(p^2, r)$ and showed that they can simply be reduced to Gaussian sums and Jacobi sums over the finite field $\mathbb{F}_{p^r}$. A recent book [33] by Shi et al. is entirely dedicated to character sums over rings. Afterwards, in [22], Luo and Cao proposed a construction of complex codebooks from Gaussian sums over the Galois ring $GR(p^2, r)$. In addition, they were the first to define the Eisenstein sums over this Galois ring and were able to produce some asymptotically optimal codebooks.

Let $C = \{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{N-1}\}$ be a set of $N$ unit-norm complex vectors $\mathbf{c}_l \in \mathbb{C}^K$ over an alphabet $A$, where $l = 0, 1, \ldots, N-1$. The size of $A$ is called the alphabet size of $C$. Such a set $C$ is called an $(N, K)$ codebook (also called a signal set), where $N$ is the number of elements of the codebook $C$ and $K$ is the length of the codebook $C$. The maximum cross-correlation amplitude, which is a performance measure of a codebook in practical applications, of the $(N, K)$ codebook $C$ is defined as

$$I_{\max}(C) = \max_{0 \leq i < j \leq N-1} |\mathbf{c}_i \mathbf{c}_j^H|,$$

where $\mathbf{c}_j^H$ denotes the conjugate transpose of the complex vector $\mathbf{c}_j$. For a certain length $K$, it is desirable to design a codebook such that the number $N$ of codewords is as large as possible and the maximum cross-correlation amplitude $I_{\max}(C)$ is as small as possible. To evaluate a codebook $C$ with parameters $(N, K)$, it is important to find the minimum achievable $I_{\max}(C)$. The following result, which is known as the Welch bound, gives a lower bound for $I_{\max}(C)$.

**Lemma 1** [41] *For any* $(N, K)$ *codebook* $C$ with $N \geq K$,

$$I_{\max}(C) \geq I_w = \sqrt{\frac{N-K}{(N-1)K}}. \tag{1}$$

*Furthermore, the equality in* (1) *is achieved if and only if*

$$|\mathbf{c}_i \mathbf{c}_j^H| = \sqrt{\frac{N-K}{(N-1)K}}$$

*for all pairs* $(i, j)$ with $i \neq j$.

A codebook is referred to as a maximum-Welch-bound-equality (MWBE) codebook [37] or an equiangular tight frame [16] if it meets the Welch bound equality in (1). Codebooks meeting the Welch bound are used to distinguish among the signals of different users in code-division multiple-access (CDMA) systems [29]. Furthermore, MWBE codebooks have been used in a wide range of applications, such as multiple description coding over erasure channels [38], communications [37], compressed sensing [3], space-time codes [39], coding theory [8] and quantum computing [32] etc. In general, it is very difficult to construct optimal codebooks achieving the Welch bound (i.e. to construct MWBE codebooks). There are many results on optimal or almost optimal codebooks with respect to the Welch bound: interested readers may refer to [2–4, 6, 7, 11–14, 20–22, 25, 27, 44–46]. It is worth mentioning that character sums over finite fields are extremely useful tools for constructing codebooks [1, 26]. In [13, 14, 20, 21, 44], the authors constructed codebooks using character sums over finite fields.

In fact, we know that many scholars have studied character sums over local rings and their applications in coding theory [9, 23, 34–36] etc. Luo and Cao established Eisenstein sums over the Galois ring $GR(p^2, r)$ in [22]. Recently, we have studied the character sums over a finite non-chain ring and their applications to the constructions of codebooks in [31]. One purpose of this paper is to investigate Gaussian sums, hyper Eisenstein sums and Jacobi sums over the local ring $R = \mathbb{F}_q + u\mathbb{F}_q$ $(u^2 = 0)$ and present some properties of these character sums. Furthermore, we establish a connection between these character sums and character sums over finite fields. Another purpose of this paper is to present constructions of codebooks via Gaussian sums, Eisenstein sums and Jacobi sums over the local ring $R$ and show that these codebooks asymptotically meet the Welch bound.

The rest of this paper is arranged as follows. Section 2 presents some notation and basic results. In Sect. 3, we give an explicit description of additive characters and multiplicative characters over the finite local ring $R$. In Sect. 4, we define Gaussian sums, hyper Eisenstein sums and Jacobi sums over the finite local ring $R$ and present some computational results about these character sums. Moreover, we establish a relationship between character sums over $R$ and character sums over $\mathbb{F}_q$. Four generic constructions of asymptotically optimal codebooks and a specific construction of optimal codebooks associated with these character sums over $R$ are presented in Sect. 5. In Sect. 6, we present our concluding remarks.

## 2 Preliminaries

Let $q$ be a prime power, and $\mathbb{F}_q$ denote the finite field with $q$ elements. We consider the chain ring $R = \mathbb{F}_q + u\mathbb{F}_q = \{\alpha + \beta u : \alpha, \beta \in \mathbb{F}_q\}(u^2 = 0)$ having the unique maximal ideal $M = \langle u \rangle$. In fact, $R = \mathbb{F}_q \oplus u\mathbb{F}_q \simeq \mathbb{F}_q^2$ is a two-dimensional vector space over $\mathbb{F}_q$ and $|R| = q^2$. The invertible elements of $R$ are

$$R^* = R \backslash M = \mathbb{F}_q^* + u\mathbb{F}_q = \{\alpha + \beta u : \alpha \in \mathbb{F}_q^*, \beta \in \mathbb{F}_q\}.$$

It is easy to know that $|R^*| = q(q-1)$. $R^*$ can also be represented as $\mathbb{F}_q^* \times (1+M)$ (direct product).

We next begin to introduce some basic results on characters and character sums over finite fields, which will be useful for our subsequent discussion. We first give some notation valid for the whole paper.

### 2.1 Some notation fixed throughout this paper

- Let $r = p^l$ and $q = r^m$, where $l(\geq 1)$ and $m(\geq 1)$ are positive integers. $\mathbb{F}_p$, $\mathbb{F}_r$ and $\mathbb{F}_q$ denote finite fields, and $\mathbb{F}_p \subseteq \mathbb{F}_r \subseteq \mathbb{F}_q$.
- Let $R_r = \mathbb{F}_r + u\mathbb{F}_r$ $(u^2 = 0)$.
- $\mathrm{Tr}_p^r(\cdot)$ is the trace function from $\mathbb{F}_r$ to $\mathbb{F}_p$.
- $\mathrm{Tr}_r^q(\cdot)$ is the trace function from $\mathbb{F}_q$ to $\mathbb{F}_r$.
- $\mathrm{Tr}_p^q(\cdot)$ is the trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$.
- $\mathrm{Tr}_{R_r}^R(\cdot)$ is the trace function from $R$ to $R_r$.

### 2.2 Characters over finite fields

In this subsection, we will recall the definitions of the additive and multiplicative characters of $\mathbb{F}_q$ (see, for example, [26]).

- The additive character $\chi_a$ of $\mathbb{F}_q$ is defined by

$$\chi_a(x) = \zeta_p^{\mathrm{Tr}_p^q(ax)}$$

  for each $a \in \mathbb{F}_q$, where $\zeta_p = e^{\frac{2\pi i}{p}}$ and $x \in \mathbb{F}_q$. If $a = 1$, then $\chi_1(x) = \chi(x)$ denotes the canonical additive character of $\mathbb{F}_q$. If $a = 0$, then $\chi_0(x)$ denotes the trivial additive character of $\mathbb{F}_q$ and $\chi_0(x) = 1$ for all $x \in \mathbb{F}_q$; all other additive characters of $\mathbb{F}_q$ are called nontrivial. Moreover, the group that consists of all additive characters of $\mathbb{F}_q$ is denoted by $\widehat{\mathbb{F}}_q$. The group of characters is isomorphic to $(\mathbb{F}_q, +)$. With each additive character $\chi_a(x)$ of $\mathbb{F}_q$, there is an associated conjugate character $\overline{\chi_a}(x)$ defined by $\overline{\chi_a}(x) = \overline{\chi_a(x)} = \chi_a(-x)$ for all $x \in \mathbb{F}_q$. In addition, $\chi_a(0) = 1$ for all $a \in \mathbb{F}_q$.
- The multiplicative character $\psi_j$ of $\mathbb{F}_q$ is defined by

$$\psi_j(g^k) = \zeta_{q-1}^{jk}$$

  for each $j = 0, 1, \dots, q-2$, where $\zeta_{q-1} = e^{\frac{2\pi i}{q-1}}$, $k = 0, 1, \dots, q-2$ and $g$ is a fixed primitive element of $\mathbb{F}_q$. If $j = 0$, then $\psi_0$ denotes the trivial multiplicative character of $\mathbb{F}_q$. Moreover, the group that consists of all multiplicative characters of $\mathbb{F}_q$ is denoted by $\widehat{\mathbb{F}}_q^*$. The group of characters is isomorphic to $(\mathbb{F}_q^*, *)$. With each multiplicative character $\psi$ of $\mathbb{F}_q$, there is an associated conjugate character $\overline{\psi}$ defined by $\overline{\psi} = \psi^{-1}$. If $\psi$ is trivial, then $\psi(0) = 1$; if $\psi$ is nontrivial, then we define $\psi(0) = 0$.

### 2.3 Character sums over finite fields

Firstly, we recall the definition of Gaussian sums over finite fields.

- **Gaussian sums**

**Definition 1** Let $\psi$ be a multiplicative and $\chi_a$ an additive character of $\mathbb{F}_q$, where $a \in \mathbb{F}_q$. Then the Gaussian sum $G(\psi, \chi_a)$ over $\mathbb{F}_q$ is defined by

$$G(\psi, \chi_a) = \sum_{x \in \mathbb{F}_q^*} \psi(x)\chi_a(x).$$

The absolute value of $G(\psi, \chi_a)$ is at most $q-1$, but is in general much smaller, as the following lemma shows.

**Lemma 2** [26, Theorem 5.11] *Let $\psi$ be a multiplicative and $\chi_a$ an additive character of $\mathbb{F}_q$. Then the Gaussian sum $G(\psi, \chi)$ satisfies*

$$G(\psi, \chi) = \begin{cases} q-1, & \text{if } \psi = \psi_0 \text{ and } \chi_a = \chi_0; \\ -1, & \text{if } \psi = \psi_0 \text{ and } \chi_a \neq \chi_0; \\ 0, & \text{if } \psi \neq \psi_0 \text{ and } \chi_a = \chi_0. \end{cases}$$

*If $\psi \neq \psi_0$ and $\chi_a \neq \chi_0$, then $|G(\psi, \chi_a)| = q^{\frac{1}{2}}$.*

Now, we let $\mu_b$ denote an additive character of $\mathbb{F}_r$ and $\phi$ a multiplicative character of $\mathbb{F}_r$. In particular, $\mu = \mu_1$ denotes the canonical additive character of $\mathbb{F}_r$. We can define the Gaussian sum $G(\phi, \mu_b)$ on $\mathbb{F}_r$ similarly. For convenience, we usually write $G(\psi, \chi_1)$ and $G(\phi, \mu_1)$ simply as $G(\psi)$ and $G(\phi)$, respectively.

Next, we introduce hyper Eisenstein sums over finite fields.

- **Hyper Eisenstein sums**

  Let $\psi_1, \psi_2, \ldots, \psi_n$ be multiplicative characters of $\mathbb{F}_q$. For $1 \leq i \leq n$, the restriction of $\psi_i$ to $\mathbb{F}_r$ will be denoted by $\psi_i^*$. In particular, if $\psi_i$ is a trivial character on $\mathbb{F}_q$, then $\psi_i^*$ is a trivial character on $\mathbb{F}_r$. Now, we give the definition of hyper Eisenstein sums over the finite field $\mathbb{F}_q$ as follows.

**Definition 2** [21] The hyper Eisenstein sum $E_{\mathbb{F}_q}(\psi_1, \ldots, \psi_n;1)$ is defined by

$$E_{\mathbb{F}_q}(\psi_1, \ldots, \psi_n) := E_{\mathbb{F}_q}(\psi_1, \ldots, \psi_n;1) = \sum_{\substack{x_1,\ldots,x_n \in \mathbb{F}_q^*, \\ \mathrm{Tr}_r^q(x_1+\cdots+x_n)=1}} \psi_1(x_1) \cdots \psi_n(x_n),$$

where $\psi_1, \psi_2, \ldots, \psi_n$ are multiplicative characters of $\mathbb{F}_q$. Moreover, we define

$$E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; s) = \sum_{x_1,\dots,x_n \in \mathbb{F}_q^*, \mathrm{Tr}_r^q(x_1+\cdots+x_n)=s} \psi_1(x_1)\cdots\psi_n(x_n)$$

for all $s \in \mathbb{F}_r$.

It is easy to see that

$$E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; s) = (\psi_1 \cdots \psi_n)(s) E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; 1) \tag{2}$$

for each $s \in \mathbb{F}_r^*$. If $\psi_1, \dots, \psi_n$ are all trivial, then

$$E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; 1) = \frac{(q-1)^n + (-1)^{n+1}}{r} \tag{3}$$

by [21, Lemma 5]. If some, but not all, of the $\psi_i$ are trivial, without loss of generality, we assume that $\psi_1, \dots, \psi_h$ are nontrivial and $\psi_{h+1}, \dots, \psi_n$ are trivial, where $1 \le h \le n-1$. Then (see [21, Theorem 1])

$$E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; 1) = (-1)^{n-h} E_{\mathbb{F}_q}(\psi_1, \dots, \psi_h; 1). \tag{4}$$

In the following, we describe a relationship between hyper Eisenstein sums and Gaussian sums over $\mathbb{F}_q$.

**Lemma 3** [21, Theorem 3] *Let $\psi_1, \psi_2, \dots, \psi_n$ be nontrivial multiplicative characters on $\mathbb{F}_q$. Let $(\psi_1 \cdots \psi_n)^*$ be the restriction of $\psi_1 \cdots \psi_n$ to $\mathbb{F}_r$. Then*

$$E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; 1) = \begin{cases} \frac{G_{\mathbb{F}_q}(\psi_1)\cdots G_{\mathbb{F}_q}(\psi_n)}{G_{\mathbb{F}_r}((\psi_1\cdots\psi_n)^*)}, & \text{if } (\psi_1 \cdots \psi_n)^* \text{ is nontrivial;} \\ -\frac{G_{\mathbb{F}_q}(\psi_1)\cdots G_{\mathbb{F}_q}(\psi_n)}{r}, & \text{if } (\psi_1 \cdots \psi_n)^* \text{ is trivial.} \end{cases}$$

From Lemma 3 and Eq. (2), we can determine the absolute value of the sum $E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; s)$ for each $s \in \mathbb{F}_r^*$.

**Lemma 4** [21, Corollary 1] Let $\psi_1, \psi_2, \dots, \psi_n$ *be nontrivial multiplicative characters on $\mathbb{F}_q$. Let $(\psi_1 \cdots \psi_n)^*$ be the restriction of $\psi_1 \cdots \psi_n$ to $\mathbb{F}_r$. Then*

$$|E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; s)| = \begin{cases} r^{\frac{mn-1}{2}}, \text{if } (\psi_1 \cdots \psi_n)^* \text{ is nontrivial;} \\ r^{\frac{mn-2}{2}}, \text{if } (\psi_1 \cdots \psi_n)^* \text{ is trivial,} \end{cases}$$

*for each $s \in \mathbb{F}_r^*$.*

The following result relates the sum $E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; 0)$ to the hyper Eisenstein sum $E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; 1)$.

**Lemma 5** [21, Theorem 2] Let $\psi_1, \psi_2, \dots, \psi_n$ *be multiplicative characters on $\mathbb{F}_q$. Let $(\psi_1 \cdots \psi_n)^*$ be the restriction of $\psi_1 \cdots \psi_n$ to $\mathbb{F}_r$. Then* $E_{\mathbb{F}_q}(\psi_1, \dots, \psi_n; 0)$

$$= \begin{cases} \frac{(q-1)^n+(-1)^n(r-1)}{r}, & \text{if } \psi_1, \ldots, \psi_n \text{ are all trivial;} \\ 0, & \text{if } (\psi_1 \cdots \psi_n)^* \text{ is nontrivial;} \\ -(r-1)E_{\mathbb{F}_q}(\psi_1, \ldots, \psi_n; 1), & \text{if } \psi_1, \ldots, \psi_n \text{ are not all trivial} \\ & \text{and } (\psi_1 \cdots \psi_n)^* \text{ is trivial.} \end{cases}$$

## 3 Characters over $R = \mathbb{F}_q + u\mathbb{F}_q$

In this section, we will describe the additive and multiplicative characters of the local ring $R = \mathbb{F}_q + u\mathbb{F}_q$.

▲ **Additive characters of** $R$

The group of additive characters of $(R, +)$ is

$$\widehat{R} := \{\lambda : R \longrightarrow \mathbb{C}^* | \lambda(\alpha + \beta) = \lambda(\alpha)\lambda(\beta), \alpha, \beta \in R\}.$$

For any additive character $\lambda$ of $R$,

$$\lambda : R \longrightarrow \mathbb{C}^*.$$

Since $\lambda(a_0 + ua_1) = \lambda(a_0)\lambda(ua_1)$ for any $a_0, a_1 \in \mathbb{F}_q$, we define the two mappings $\lambda'$ and $\lambda''$ as follows. The mapping $\lambda' : \mathbb{F}_q \longrightarrow \mathbb{C}^*$ is defined as

$$\lambda'(c) := \lambda(c)$$

for $c \in \mathbb{F}_q$. And the mapping $\lambda'' : \mathbb{F}_q \longrightarrow \mathbb{C}^*$ is defined by

$$\lambda''(c) := \lambda(uc)$$

for $c \in \mathbb{F}_q$. It is easy to check that $\lambda'(c_1 + c_2) = \lambda'(c_1)\lambda'(c_2)$ and $\lambda''(c_1 + c_2) = \lambda''(c_1)\lambda''(c_2)$ for $c_1, c_2 \in \mathbb{F}_q$. We know that $\lambda'$ and $\lambda''$ are both additive characters of $(\mathbb{F}_q, +)$. Hence, there exist $b, c \in \mathbb{F}_q$ such that

$$\lambda'(x) = \zeta_p^{\mathrm{Tr}_p^q(bx)} = \chi_b(x) \text{ and } \lambda''(x) = \zeta_p^{\mathrm{Tr}_p^q(cx)} = \chi_c(x)$$

for all $x \in \mathbb{F}_q$, where $\zeta_p = e^{\frac{2\pi i}{p}}$ is a primitive $p$th root of unity over $\mathbb{F}_q$. Therefore, we can express an additive character of $R$ as follows.

$$\begin{aligned} \lambda(a_0 + ua_1) =& \lambda'(a_0)\lambda''(a_1) \\ =& \chi_b(a_0)\chi_c(a_1). \end{aligned}$$

Thus, there is an one-to-one correspondence:

$$\begin{aligned} \tau : \widehat{(R,+)} \longrightarrow & \widehat{(\mathbb{F}_q,+)} \times \widehat{(\mathbb{F}_q,+)}, \\ \lambda \longmapsto & (\chi_b, \chi_c). \end{aligned}$$

It is easy to prove that the mapping $\tau$ is an isomorphism.

▲ **Multiplicative characters of** $R$

Now, we have

$$\begin{aligned} R^* =&\{a_0 + ua_1 : a_0 \in \mathbb{F}_q^*, a_1 \in \mathbb{F}_q\} \\ =&\{b_0(1 + ub_1) : b_0 \in \mathbb{F}_q^*, b_1 \in \mathbb{F}_q\}. \end{aligned}$$

The group of multiplicative characters of $(R^*, *)$ is

$$\widehat{R}^* := \{\varphi : R^* \longrightarrow \mathbb{C}^* | \varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta), \alpha, \beta \in R\}.$$

For any multiplicative character $\varphi$ of $R$,

$$\varphi : R^* \longrightarrow \mathbb{C}^*.$$

Since $\varphi(b_0(1 + ub_1)) = \varphi(b_0)\varphi(1 + ub_1)$ for any $b_0 \in \mathbb{F}_q^*, b_1 \in \mathbb{F}_q$, we define the two mappings $\varphi'$ and $\varphi''$ as follows. The mapping $\varphi' : \mathbb{F}_q^* \longrightarrow \mathbb{C}^*$ is defined as

$$\varphi'(c) := \varphi(c)$$

for $c \in \mathbb{F}_q$. And the mapping $\varphi'' : \mathbb{F}_q \longrightarrow \mathbb{C}^*$ is defined by

$$\varphi''(c) := \varphi(1 + uc)$$

for $c \in \mathbb{F}_q$. For any $c_1, c_2 \in \mathbb{F}_q^*$, we have $\varphi'(c_1c_2) = \varphi'(c_1)\varphi'(c_2)$ and

$$\begin{aligned} \varphi''(c_1 + c_2) &= \varphi(1 + u(c_1 + c_2)) \\ &= \varphi((1 + uc_1)(1 + uc_2)) \\ &= \varphi(1 + uc_1)\varphi(1 + uc_2) \\ &= \varphi''(c_1)\varphi''(c_2). \end{aligned}$$

It follows that $\varphi'$ is a multiplicative character of $\mathbb{F}_q$ and $\varphi''$ is an additive character of $\mathbb{F}_q$. Hence, we can represent a multiplicative character of $R$ as a product

$$\varphi(b_0(1 + ub_1)) = \varphi'(b_0)\varphi''(b_1),$$

where $\varphi' \in \widehat{\mathbb{F}}_q^*$ and $\varphi'' \in \widehat{\mathbb{F}}_q$. Since $\varphi''$ is an additive character of $\mathbb{F}_q$, there exists $a \in \mathbb{F}_q$ such that $\varphi'' = \chi_a$. Moreover, we have

$$\begin{aligned} \sigma : \widehat{(R^*, *)} &\longrightarrow \widehat{(\mathbb{F}_q^*, *)} \times \widehat{(\mathbb{F}_q, +)}, \\ \varphi &\longmapsto (\psi, \chi_a), \end{aligned}$$

where $\psi = \varphi'$ is a multiplicative character of $\mathbb{F}_q$. One can show that the mapping $\sigma$ is an isomorphism.

## 4 Gaussian sums, hyper Eisenstein sums and Jacobi sums over $R = \mathbb{F}_q + u\mathbb{F}_q$

In this section, we introduce Gaussian sums, hyper Eisenstein sums and Jacobi sums over $R$ and present some fundamental properties of these character sums.

Let $R = \mathbb{F}_q + u\mathbb{F}_q$ and $R_r = \mathbb{F}_r + u\mathbb{F}_r$, where $u^2 = 0$ and $q = r^m$. Then $R/R_r$ is a Galois extension of rings and the Galois group $\mathrm{Gal}(R/R_r) = \langle \sigma_r \rangle$, where $\sigma_r$ is the $R_r$-automorphism of $R$ defined by

$$\sigma_r(\alpha + u\beta) = \alpha^r + u\beta^r \ (\alpha, \beta \in \mathbb{F}_q).$$

Then, we can define the trace mapping:

$$\begin{aligned} \mathrm{Tr}^R_{R_r} \ : \ \ & R \longrightarrow R_r, \\ \mathrm{Tr}^R_{R_r}(\alpha + u\beta) =& \mathrm{Tr}^q_r(\alpha) + u\mathrm{Tr}^q_r(\beta) \\ =& \sum_{i=0}^{m-1} \sigma_r^i(\alpha + u\beta). \end{aligned}$$

Moreover, it is easy to show that $\mathrm{Tr}^R_{R_r}(\mathfrak{s}t) = \mathfrak{s}\mathrm{Tr}^R_{R_r}(t)$ for each $\mathfrak{s} \in R_r$ and $t \in R$. For convenience, $\mathrm{Tr}^R_{R_r}$ is abbreviated as Tr.

From Sect. 3, for $a, b, c \in \mathbb{F}_q$, $\chi_a, \chi_b, \chi_c \in \widehat{\mathbb{F}}_q$ and $\psi \in \widehat{\mathbb{F}}_q^*$, we denote $\varphi := \psi \star \chi_a$ and $\lambda := \chi_b \star \chi_c$. Then, for any $t = t_0(1 + ut_1) \in R$, $\varphi(t) = (\psi \star \chi_a)(t) = \psi(t_0)\chi_a(t_1)$ and $\lambda(t) = (\chi_b \star \chi_c)(t) = \chi_b(t_0)\chi_c(t_0 t_1)$.

### 4.1 Gaussian sums over *R*

Let $\lambda$ and $\varphi$ be an additive character and a multiplicative character of $R$, respectively. The Gaussian sum for $\lambda$ and $\varphi$ over $R = \mathbb{F}_q + u\mathbb{F}_q$ $(u^2 = 0)$ is defined by

$$G_R(\varphi, \lambda) = \sum_{t \in R^*} \varphi(t)\lambda(t).$$

**Theorem 1** *Let $\varphi$ be a multiplicative character and $\lambda$ be an additive character of $R$, where $\varphi := \psi \star \chi_a$, $\lambda := \chi_b \star \chi_c$, $\psi \in \widehat{\mathbb{F}}_q^*$ and $a, b, c \in \mathbb{F}_q$. Then the Gaussian sum $G_R(\varphi, \lambda)$ satisfies*

$$G_R(\varphi, \lambda) = \begin{cases} qG_{\mathbb{F}_q}(\psi, \chi_b), & \text{if } a = 0 \ \textit{and} \ c = 0; \\ 0, & \text{if } a = 0 \ \textit{and} \ c \neq 0; \\ 0, & \text{if } a \neq 0 \ \textit{and} \ c = 0; \\ q\psi\left(-\frac{a}{c}\right)\chi\left(-\frac{ab}{c}\right), & \text{if } a \neq 0 \ \textit{and} \ c \neq 0, \end{cases}$$

*where $G_{\mathbb{F}_q}(\psi, \chi_b)$ denotes the Gaussian sum over $\mathbb{F}_q$.*

***Proof*** Assume that $t = t_0(1 + ut_1)$, where $t_0 \in \mathbb{F}_q^*$ and $t_1 \in \mathbb{F}_q$.

$$\begin{aligned}
G_R(\varphi, \lambda) &= \sum_{t \in R^*} \varphi(t)\lambda(t) \\
&= \sum_{t_0 \in \mathbb{F}_q^*, t_1 \in \mathbb{F}_q} \varphi(t_0(1 + ut_1))\lambda(t_0(1 + ut_1)) \\
&= \sum_{t_0 \in \mathbb{F}_q^*, t_1 \in \mathbb{F}_q} \psi(t_0)\chi_a(t_1)\chi_b(t_0)\chi_c(t_0t_1) \\
&= \sum_{t_0 \in \mathbb{F}_q^*, t_1 \in \mathbb{F}_q} \psi(t_0)\chi(at_1 + bt_0 + ct_0t_1) \\
&= \sum_{t_0 \in \mathbb{F}_q^*} \psi(t_0)\chi(bt_0) \sum_{t_1 \in \mathbb{F}_q} \chi((a + ct_0)t_1) \\
&= q \sum_{t_0 \in \mathbb{F}_q^*, a + ct_0 = 0} \psi(t_0)\chi(bt_0) = \begin{cases} qG_{\mathbb{F}_q}(\psi, \chi_b), & \text{if } a = 0 \ \ and \ \ c = 0; \\ 0, & \text{if } a = 0 \ \ and \ \ c \neq 0; \\ 0, & \text{if } a \neq 0 \ \ and \ \ c = 0; \\ q\psi(-\frac{a}{c})\chi(-\frac{ab}{c}), & \text{if } a \neq 0 \ \ and \ \ c \neq 0, \end{cases}
\end{aligned}$$

where $G_{\mathbb{F}_q}(\psi, \chi_b)$ is a Gaussian sum over $\mathbb{F}_q$. □

***Remark 1***

1. Although Gaussian sums over finite commutative rings have been studied in [24, 40] and the ring $R$ in this paper is a special finite commutative ring, our results are not completely covered. In [24, 40], the authors give additive and multiplicative characters over finite commutative rings and define the Gaussian sum related to these characters. Our contributions are as follows. We present an explicit description on the additive and multiplicative characters over the special finite commutative ring $R$ in Sect. 3. In addition, we establish a relationship between Gaussian sums over the finite ring $R$ and Gaussian sums over the finite field $\mathbb{F}_q$ in one case of Theorem 1, which helps one to calculate the exact value of certain Gaussian sums over the ring $R$ (by making use of known formulae for Gaussian sums over $\mathbb{F}_q$).
2. Comparing with [28, Theorem 3.3], it is easy to see that a similar result was proven by Li, Zhu and Feng for Gaussian sums over $GR(p^2, r)$ using similar techniques. Both results show that Gaussian sums over certain finite local rings can be expressed in terms of Gaussian sums over finite fields.

Next, we introduce the definition of quadratic characters over $R$.

**Definition 3** Let $\varphi$ be a multiplicative character of $R$. If $(\varphi(t))^2 = 1$ for any $t \in R^*$, then $\varphi$ is called the quadratic character of $R$, denoted by $\rho$. Moreover, $G_R(\rho, \lambda)$ denotes the quadratic Gaussian sum over $R$, where $\lambda$ is an additive character of $R$.

In the following, we determine the form of the quadratic character $\rho$ of $R$. Let $\eta, \psi_0$ and $\chi_0$ denote the quadratic character, the trivial multiplicative character and the trivial additive character of the finite field $\mathbb{F}_q$, respectively. We use the convention that $\psi(0) = 0$ for a nontrivial multiplicative character $\psi$ of $\mathbb{F}_q$. For any $t = t_0(1 + ut_1) \in R^*$, if the multiplicative character $\varphi$ of $R$ is a quadratic character, then we need $(\varphi(t))^2 = (\psi(t_0)\chi_a(t_1))^2 = 1$. However,

$$\begin{aligned}(\varphi(t))^2 &= (\psi(t_0)\chi_a(t_1))^2 \\ &= (\psi(t_0))^2\chi(2at_1) \\ &= (\psi(t_0))^2\zeta_p^{\mathrm{Tr}_p^q(2at_1)},\end{aligned}$$

where $\zeta_p = e^{\frac{2\pi i}{p}}$ is a primitive $p$th root of unity over $\mathbb{F}_q$.

- If $p = 2$, there is no quadratic character $\eta$ of $\mathbb{F}_q$ since $2 \nmid (q-1)$ and $\zeta_p^{\mathrm{Tr}_p^q(2at_1)} = 1$. Hence, when $\psi$ is a trivial character and $a \neq 0$, we obtain that $\varphi$ is a quadratic character $\rho$ of $R$, denoted by $\rho := \psi_0 \star \chi_a$.
- If $p \neq 2$, then $\varphi$ is a quadratic character $\rho$ of $R$ when $\psi = \eta$ and $a = 0$, denoted by $\rho := \eta \star \chi_0$.

Based on Theorem 1, we have the following corollary.

**Corollary 1** *Let $\rho$ be a quadratic character and $\lambda$ be an additive character of $R$.* Let $\chi_a, \chi_b, \chi_c \in \widehat{\mathbb{F}}_q$, *$\eta$ denote the quadratic character of $\mathbb{F}_q$ and $\chi_0$ denote the trivial additive character of $\mathbb{F}_q$.*

1. If $p = 2$, then $G_R(\rho, \lambda) = q\chi(-\frac{ab}{c})$ if $c \neq 0$ and $G_R(\rho, \lambda) = 0$ if $c = 0$, where $\rho := \psi_0 \star \chi_a, \lambda := \chi_b \star \chi_c$ and $a \in \mathbb{F}_q^*, b, c \in \mathbb{F}_q$.
2. If $p \neq 2$, then $|G_R(\rho, \lambda)| = q^{\frac{1}{2}}$ if $b \neq 0$ and $G_R(\rho, \lambda) = 0$ otherwise, where $\rho := \eta \star \chi_0, \lambda := \chi_b \star \chi_c$ and $b, c \in \mathbb{F}_q$.

***Proof*** The proof is obvious by Theorem 1, so we omit it here. □

## 4.2 Hyper Eisenstein sums over *R*

Now, we give the definition of hyper Eisenstein sums over $R = \mathbb{F}_q + u\mathbb{F}_q$ $(u^2 = 0)$.

**Definition 4** Let $n$ be a positive integer and $\varphi_1, \varphi_2, \dots, \varphi_n$ multiplicative characters of $R$. Then the hyper Eisenstein sum for $\varphi_1, \varphi_2, \dots, \varphi_n$ over $R$ is defined by

$$E_R(\varphi_1, \varphi_2, \dots, \varphi_n; 1) = \sum_{\substack{t_1, t_2, \dots, t_n \in R^*, \\ \mathrm{Tr}(t_1 + t_2 + \dots + t_n) = 1}} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n). \tag{5}$$

Moreover, we can define $E_R(\varphi_1, \varphi_2, \dots, \varphi_n; \mathfrak{z})$ as follows: for each $\mathfrak{z} \in R_r$,

$$E_R(\varphi_1,\varphi_2,\ldots,\varphi_n;\mathfrak{s}) = \sum_{t_1,t_2,\ldots,t_n\in R^*,\mathrm{Tr}(t_1+t_2+\cdots+t_n)=\mathfrak{s}} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n).$$

In this section, we calculate the value of hyper Eisenstein sums over $R$.
If $\mathfrak{s}\in R_r^*$, then

$$\begin{aligned}
&E_R(\varphi_1,\varphi_2,\ldots,\varphi_n;\mathfrak{s})\\
&= \sum_{t_1,t_2,\ldots,t_n\in R^*,\mathrm{Tr}(t_1+t_2+\cdots+t_n)=\mathfrak{s}} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n) \overset{t_i\to\mathfrak{s}t_i}{=} \sum_{\substack{\mathfrak{s}t_1,\mathfrak{s}t_2,\ldots,\mathfrak{s}t_n\in R^*,\\ \mathrm{Tr}(\mathfrak{s}t_1+\mathfrak{s}t_2+\cdots+\mathfrak{s}t_n)=\mathfrak{s}}} \varphi_1(\mathfrak{s}t_1)\varphi_2(\mathfrak{s}t_2)\cdots\varphi_n(\mathfrak{s}t_n)\\
&= \varphi_1\cdots\varphi_n(\mathfrak{s}) \sum_{\substack{t_1,t_2,\cdots,t_n\in R^*,\\ \mathrm{Tr}(t_1+t_2+\cdots+t_n)=1}} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n) = \varphi_1\cdots\varphi_n(\mathfrak{s})E_R(\varphi_1,\varphi_2,\ldots,\varphi_n;1).
\end{aligned}$$

If $\mathfrak{s}=ub\in u\mathbb{F}_r^*$ ($b\in\mathbb{F}_r^*\subset R^*$), then

$$\begin{aligned}
&E_R(\varphi_1,\varphi_2,\ldots,\varphi_n;\mathfrak{s})\\
&= \sum_{t_1,t_2,\ldots,t_n\in R^*,\mathrm{Tr}(t_1+t_2+\cdots+t_n)=\mathfrak{s}} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n) \overset{t_i\to bt_i}{=} \sum_{\substack{bt_1,bt_2,\ldots,bt_n\in R^*,\\ \mathrm{Tr}(bt_1+bt_2+\cdots+bt_n)=ub}} \varphi_1(bt_1)\varphi_2(bt_2)\cdots\varphi_n(bt_n)\\
&= \varphi_1\cdots\varphi_n(b) \sum_{\substack{t_1,t_2,\ldots,t_n\in R^*,\\ \mathrm{Tr}(t_1+t_2+\cdots+t_n)=u}} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n) = \varphi_1\cdots\varphi_n(b)E_R(\varphi_1,\varphi_2,\ldots,\varphi_n;u).
\end{aligned}$$

Thus, it is sufficient to compute

$$\begin{aligned}
E_R(\varphi_1,\varphi_2,\cdots,\varphi_n;0) &= \sum_{t_1,t_2,\ldots,t_n\in R^*,\mathrm{Tr}(t_1+t_2+\cdots+t_n)=0} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n),\\
E_R(\varphi_1,\varphi_2,\ldots,\varphi_n) &= E_R(\varphi_1,\varphi_2,\ldots,\varphi_n;1)\\
&= \sum_{\substack{t_1,t_2,\ldots,t_n\in R^*,\\ \mathrm{Tr}(t_1+t_2+\cdots+t_n)=1}} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n), \text{ and } E_R(\varphi_1,\varphi_2,\ldots,\varphi_n;u)\\
&= \sum_{t_1,t_2,\ldots,t_n\in R^*,\mathrm{Tr}(t_1+t_2+\cdots+t_n)=u} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n).
\end{aligned}$$

Before calculating the sums $E_R(\varphi_1,\varphi_2,\ldots,\varphi_n;0), E_R(\varphi_1,\varphi_2,\ldots,\varphi_n;1)$ and $E_R(\varphi_1,\varphi_2,\ldots,\varphi_n;u)$, we need to establish some preliminary results.

**Lemma 6** *Let $a\in\mathbb{F}_q$, $y\in\mathbb{F}_r$ and $t'\in\mathbb{F}_q^*$. Then*

$$\sum_{t''\in\mathbb{F}_q}\chi((a+yt')t'') = \begin{cases} q, & \text{if } a=0, \forall\, t'\in\mathbb{F}_q^* \text{ and } y=0;\\ 0, & \text{if } a=0, \forall\, t'\in\mathbb{F}_q^* \text{ and } y\neq 0;\\ q, & \text{if } a\neq 0, t'\in a\mathbb{F}_r^* \text{ and } y=-\frac{a}{t'};\\ 0, & \text{if } a\neq 0, t'\in a\mathbb{F}_r^* \text{ and } y\neq -\frac{a}{t'};\\ 0, & \text{if } a\neq 0, t'\notin a\mathbb{F}_r^* \text{ and } \forall\, y\in\mathbb{F}_r. \end{cases}$$

***Proof*** The proof of the result is easy, so we omit it here. □

**Lemma 7** *Let* $t'_1, t'_2, \dots, t'_n \in \mathbb{F}_q^*$ *and* $a_1, a_2, \dots, a_n \in \mathbb{F}_q$.

1. If $A := A(a_1, \dots, a_n; t'_1, \dots, t'_n) = \sum_{t''_1,\dots,t''_n \in \mathbb{F}_q,} \chi_{a_1}(t''_1) \cdots \chi_{a_n}(t''_n)$, then
$$A = \begin{cases} \frac{q^n}{r}, & \text{if } a_1 = a_2 = \cdots = a_n = 0; \\ \frac{q^n}{r}, & \text{if } a_1 \cdots a_n \neq 0 \text{ and } \frac{a_1}{t'_1} = \cdots = \frac{a_n}{t'_n} \in \mathbb{F}_r^*; \\ 0, & \text{otherwise.} \end{cases}$$

2. If $B := B(a_1, \dots, a_n; t'_1, \dots, t'_n) = \sum_{\substack{t''_1,\dots,t''_n \in \mathbb{F}_q, \\ \mathrm{Tr}_r^q(t'_1 t''_1 + \cdots + t'_n t''_n) = 1}} \chi_{a_1}(t''_1) \cdots \chi_{a_n}(t''_n)$,

   then
$$B = \begin{cases} \frac{q^n}{r}, & \text{if } a_1 = a_2 = \cdots = a_n = 0; \\ \frac{q^n}{r}\lambda(z), & \text{if } a_1 \cdots a_n \neq 0 \text{ and } \frac{a_1}{t'_1} = \cdots = \frac{a_n}{t'_n} = z \in \mathbb{F}_r^*; \\ 0, & \text{otherwise.} \end{cases}$$

***Proof*** Since $t'_1, t'_2, \dots, t'_n \in \mathbb{F}_q^*$, we have

1. 
$$\begin{aligned} A &= \sum_{\substack{t''_1,\dots,t''_n \in \mathbb{F}_q, \\ \mathrm{Tr}_r^q(t'_1 t''_1 + \cdots + t'_n t''_n) = 0}} \chi_{a_1}(t''_1) \cdots \chi_{a_n}(t''_n) \\ &= \sum_{t''_1,\dots,t''_n \in \mathbb{F}_q} \chi(a_1 t''_1 + \cdots + a_n t''_n) \frac{1}{r} \sum_{y \in \mathbb{F}_r} \mu(y \mathrm{Tr}_r^q(t'_1 t''_1 + \cdots + t'_n t''_n)) \\ &= \frac{1}{r} \sum_{y \in \mathbb{F}_r} \sum_{t''_1,\dots,t''_n \in \mathbb{F}_q} \chi(a_1 t''_1 + \cdots + a_n t''_n + y(t'_1 t''_1 + \cdots + t'_n t''_n)) \\ &= \frac{1}{r} \sum_{y \in \mathbb{F}_r} \sum_{t''_1 \in \mathbb{F}_q} \chi((a_1 + y t'_1) t''_1) \cdots \sum_{t''_n \in \mathbb{F}_q} \chi((a_n + y t'_n) t''_n) \\ &= \frac{1}{r} \Big( \sum_{t''_1 \in \mathbb{F}_q} \chi(a_1 t''_1) \cdots \sum_{t''_n \in \mathbb{F}_q} \chi(a_n t''_n) + \sum_{y \in \mathbb{F}_r^*} \sum_{t''_1 \in \mathbb{F}_q} \chi((a_1 + y t'_1) t''_1) \cdots \sum_{t''_n \in \mathbb{F}_q} \chi((a_n + y t'_n) t''_n) \Big). \end{aligned}$$

   It is obvious that
$$\sum_{t''_1 \in \mathbb{F}_q} \chi(a_1 t''_1) \cdots \sum_{t''_n \in \mathbb{F}_q} \chi(a_n t''_n) = \begin{cases} q^n, & \text{if } a_1 = a_2 = \cdots = a_n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

   Let $T = \sum_{y \in \mathbb{F}_r^*} \sum_{t''_1 \in \mathbb{F}_q} \chi((a_1 + y t'_1) t''_1) \cdots \sum_{t''_n \in \mathbb{F}_q} \chi((a_n + y t'_n) t''_n)$. We divide the rest of the proof into two cases according to Lemma 6.

- Assume that $a_1 \cdots a_n = 0$. Then $T = 0$.
- Assume that $a_1 \cdots a_n \neq 0$, so that $a_1 \neq 0, \ldots, a_n \neq 0$.
  - If there exists $t'_i$ such that $t'_i \notin a_i\mathbb{F}_r^*$, then $T = 0$.
  - If $t'_1 \in a_1\mathbb{F}_r^*$ and $\cdots$ and $t'_n \in a_n\mathbb{F}_r^*$, so that $\frac{a_1}{t'_1}, \ldots, \frac{a_n}{t'_n} \in \mathbb{F}_r^*$, then

$$T = \begin{cases} q^n, & \text{if } \frac{a_1}{t'_1} = \cdots = \frac{a_n}{t'_n}; \\ 0, & \text{otherwise.} \end{cases}$$

To sum up, we can get the desired result.

2.
$$\begin{aligned} B &= \sum_{\substack{t''_1,\ldots,t''_n \in \mathbb{F}_q, \\ \mathrm{Tr}_r^q(t'_1t''_1+\cdots+t'_nt''_n)=1}} \chi_{a_1}(t''_1)\cdots\chi_{a_n}(t''_n) \\ &= \sum_{t''_1,\ldots,t''_n \in \mathbb{F}_q} \chi(a_1t''_1+\cdots+a_nt''_n)\frac{1}{r}\sum_{y\in\mathbb{F}_r}\mu(y(\mathrm{Tr}_r^q(t'_1t''_1+\cdots+t'_nt''_n)-1)) \\ &= \frac{1}{r}\sum_{y\in\mathbb{F}_r}\mu(-y)\sum_{t''_1,\ldots,t''_n \in \mathbb{F}_q}\chi(a_1t''_1+\cdots+a_nt''_n+y(t'_1t''_1+\cdots+t'_nt''_n)) \\ &= \frac{1}{r}\sum_{y\in\mathbb{F}_r}\mu(-y)\sum_{t''_1\in\mathbb{F}_q}\chi((a_1+yt'_1)t''_1)\cdots\sum_{t''_n\in\mathbb{F}_q}\chi((a_n+yt'_n)t''_n) \\ &= \frac{1}{r}\Big(\sum_{t''_1\in\mathbb{F}_q}\chi(a_1t''_1)\cdots\sum_{t''_n\in\mathbb{F}_q}\chi(a_nt''_n)+\sum_{y\in\mathbb{F}_r^*}\mu(-y)\sum_{t''_1\in\mathbb{F}_q}\chi((a_1+yt'_1)t''_1)\cdots\sum_{t''_n\in\mathbb{F}_q}\chi((a_n+yt'_n)t''_n)\Big). \end{aligned}$$

It is easy to check that

$$\sum_{t''_1\in\mathbb{F}_q}\chi(a_1t''_1)\cdots\sum_{t''_n\in\mathbb{F}_q}\chi(a_nt''_n) = \begin{cases} q^n, & \text{if } a_1 = a_2 = \cdots = a_n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let $T = \sum_{y\in\mathbb{F}_r^*}\mu(-y)\sum_{t''_1\in\mathbb{F}_q}\chi((a_1+yt'_1)t''_1)\cdots\sum_{t''_n\in\mathbb{F}_q}\chi((a_n+yt'_n)t''_n)$. We will calculate $T$ in the following two cases according to Lemma 6.

- Assume that $a_1 \cdots a_n = 0$. Then $T = 0$.
- Assume that $a_1 \cdots a_n \neq 0$, so that $a_1 \neq 0, \ldots, a_n \neq 0$.
  - If there exists $t'_i$ such that $t'_i \notin a_i\mathbb{F}_r^*$, then $T = 0$.
  - If $t'_1 \in a_1\mathbb{F}_r^*$ and $\cdots$ and $t'_n \in a_n\mathbb{F}_r^*$, so that $\frac{a_1}{t'_1}, \ldots, \frac{a_n}{t'_n} \in \mathbb{F}_r^*$, then

$$T = \begin{cases} q^n\mu(z), & \text{if } \frac{a_1}{t'_1} = \cdots = \frac{a_n}{t'_n} = z; \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof. □

Our next result relates the sums $E_R(\varphi_1, \varphi_2, \cdots, \varphi_n;0), E_R(\varphi_1, \varphi_2, \ldots, \varphi_n;1)$ and $E_R(\varphi_1, \varphi_2, \ldots, \varphi_n;u)$ to the sums $E_{\mathbb{F}_q}(\psi_1, \psi_2, \ldots, \psi_n;0)$ and $E_{\mathbb{F}_q}(\psi_1, \psi_2, \cdots, \psi_n;1)$.

**Theorem 2** *Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be multiplicative characters of $R$ and $\varphi_i := \psi_i \star \chi_{a_i} (1 \le i \le n)$, where $\psi_i$ and $\chi_{a_i}$ are multiplicative and additive characters of $\mathbb{F}_q$, respectively. Then*

1. $E_R(\varphi_1, \varphi_2, \ldots, \varphi_n;0)$

$$
= \begin{cases}
\frac{q^n}{r} E_{\mathbb{F}_q}(\psi_1, \psi_2, \ldots, \psi_n;0), & \text{if } a_1 = \cdots = a_n = 0; \\
0, & \text{if } a_1 \cdots a_n = 0 \text{ but not all of them are zero;} \\
0, & \text{if } a_1 \cdots a_n \neq 0 \text{ and } \mathrm{Tr}_r^q(a_1 + \cdots + a_n) \neq 0; \\
\frac{q^{n(r-1)}}{r} \psi_1(a_1) \cdots \psi_n(a_n), & \text{if } a_1 \cdots a_n \neq 0, \mathrm{Tr}_r^q(a_1 + \cdots + a_n) = 0 \text{ and} \\
& (\psi_1 \cdots \psi_n)^* \text{ is trivial;} \\
0, & \text{if } a_1 \cdots a_n \neq 0, \mathrm{Tr}_r^q(a_1 + \cdots + a_n) = 0 \text{ and} \\
& (\psi_1 \cdots \psi_n)^* \text{ is nontrivial,}
\end{cases}
$$

where $(\psi_1 \cdots \psi_n)^*$ is the restriction of $\psi_1 \cdots \psi_n$ to $\mathbb{F}_r$.

2. $E_R(\varphi_1, \varphi_2, \ldots, \varphi_n;1)$

$$
= \begin{cases}
\frac{q^n}{r} E_{\mathbb{F}_q}(\psi_1, \psi_2, \ldots, \psi_n;1), & \text{if } a_1 = \cdots = a_n = 0; \\
0, & \text{if } a_1 \cdots a_n = 0 \text{ but not all of them are zero;} \\
\frac{q^n}{r} \psi_1\left(\frac{a_1}{\mathrm{Tr}_r^q(a_1+\cdots+a_n)}\right) \cdots \psi_n\left(\frac{a_n}{\mathrm{Tr}_r^q(a_1+\cdots+a_n)}\right), & \text{if } a_1 \cdots a_n \neq 0 \text{ and } \mathrm{Tr}_r^q(a_1 + \cdots + a_n) \neq 0; \\
0, & \text{if } a_1 \cdots a_n \neq 0 \text{ and } \mathrm{Tr}_r^q(a_1 + \cdots + a_n) = 0.
\end{cases}
$$

where $E_{\mathbb{F}_q}(\psi_1, \psi_2, \ldots, \psi_n;1)$ denotes the hyper Eisenstein sum of $\mathbb{F}_q$.

3. $E_R(\varphi_1, \varphi_2, \ldots, \varphi_n;u)$

$$
= \begin{cases}
\frac{q^n}{r} E_{\mathbb{F}_q}(\psi_1, \psi_2, \ldots, \psi_n;0), & \text{if } a_1 = \cdots = a_n = 0; \\
0, & \text{if } a_1 \cdots a_n = 0 \text{ but not all of them are zero;} \\
0, & \text{if } a_1 \cdots a_n \neq 0 \text{ and } \mathrm{Tr}_r^q(a_1 + \cdots + a_n) \neq 0; \\
\frac{q^n}{r} \psi_1(a_1) \cdots \psi_n(a_n) G_{\mathbb{F}_r}((\overline{\psi_1 \cdots \psi_n})^*), & \text{if } a_1 \cdots a_n \neq 0 \text{ and } \mathrm{Tr}_r^q(a_1 + \cdots + a_n) = 0,
\end{cases}
$$

where $(\psi_1 \cdots \psi_n)^*$ is the restriction of $\psi_1 \cdots \psi_n$ to $\mathbb{F}_r$.

***Proof*** Let $t_1, t_2, \dots, t_n \in R^*$, where $t_i = t_i'(1 + ut_i'')$ for $1 \le i \le n$. Then

1. $$\begin{aligned} & E_R(\varphi_1, \varphi_2, \dots, \varphi_n;0) \\ &= \sum_{t_1,t_2,\dots,t_n \in R^*, \mathrm{Tr}(t_1+t_2+\cdots+t_n)=0} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n) \\ &= \sum_{\substack{t_1',\dots,t_n' \in \mathbb{F}_q^*, t_1'',\dots,t_n'' \in \mathbb{F}_q, \\ \mathrm{Tr}_r^q(t_1'+\cdots+t_n')=0, \mathrm{Tr}_r^q(t_1't_1''+\cdots+t_n't_n'')=0}} \psi_1(t_1')\chi_{a_1}(t_1'')\cdots\psi_n(t_n')\chi_{a_n}(t_n'') \\ &= \sum_{\substack{t_1',\dots,t_n' \in \mathbb{F}_q^*, \\ \mathrm{Tr}_r^q(t_1'+\cdots+t_n')=0}} \psi_1(t_1')\cdots\psi_n(t_n')\psi_1(t_1')\cdots\psi_n(t_n') \sum_{\substack{t_1'',\dots,t_n'' \in \mathbb{F}_q, \\ \mathrm{Tr}_r^q(t_1't_1''+\cdots+t_n't_n'')=0}} \chi_{a_1}(t_1'')\cdots\chi_{a_n}(t_n'') \\ &= \sum_{\substack{t_1',\dots,t_n' \in \mathbb{F}_q^*, \\ \mathrm{Tr}_r^q(t_1'+\cdots+t_n')=0}} \psi_1(t_1')\cdots\psi_n(t_n')\, A \text{ (By Lemma 7 (1)).} \end{aligned}$$

- If $a_1 = \cdots = a_n = 0$, then

$$E_R(\varphi_1, \varphi_2, \dots, \varphi_n;0) = \frac{q^n}{r} \sum_{\substack{t_1',\dots,t_n' \in \mathbb{F}_q^*, \\ \mathrm{Tr}_r^q(t_1'+\cdots+t_n')=0}} \psi_1(t_1')\cdots\psi_n(t_n') = \frac{q^n}{r} E_{\mathbb{F}_q}(\psi_1, \psi_2, \dots, \psi_n;0).$$

- If $a_1 \cdots a_n = 0$ but not all of them are zero, then $E_R(\psi_1, \psi_2, \dots, \psi_n;0) = 0$.
- If $a_1 \cdots a_n \neq 0$, then

$$\begin{aligned} E_R(\varphi_1, \varphi_2, \dots, \varphi_n;0) &= \frac{q^n}{r} \sum_{\substack{t_1',\dots,t_n' \in \mathbb{F}_q^*, \mathrm{Tr}_r^q(t_1'+\cdots+t_n')=0, \\ \frac{a_1}{t_1'}=\cdots=\frac{a_n}{t_n'} \in \mathbb{F}_r^*}} \psi_1(t_1')\cdots\psi_n(t_n') \\ &= \frac{q^n}{r} \sum_{\substack{z \in \mathbb{F}_r^*, \\ z\mathrm{Tr}_r^q(a_1+\cdots+a_n)=0}} \psi_1(a_1 z)\cdots\psi_n(a_n z) \left( \text{Let } z = \frac{t_1'}{a_1} = \cdots = \frac{t_n'}{a_n} \right) \\ &= \frac{q^n}{r} \psi_1(a_1)\cdots\psi_n(a_n) \sum_{\substack{z \in \mathbb{F}_r^*, \\ z\mathrm{Tr}_r^q(a_1+\cdots+a_n)=0}} (\psi_1 \cdots \psi_n)^*(z) \\ &= \begin{cases} 0, & \text{if } \mathrm{Tr}_r^q(a_1+\cdots+a_n) \neq 0; \\ \frac{q^n(r-1)}{r} \psi_1(a_1)\cdots\psi_n(a_n), & \text{if } \mathrm{Tr}_r^q(a_1+\cdots+a_n) = 0 \text{ and} \\ & (\psi_1 \cdots \psi_n)^* \text{ is trivial;} \\ 0, & \text{if } \mathrm{Tr}_r^q(a_1+\cdots+a_n) = 0 \text{ and} \\ & (\psi_1 \cdots \psi_n)^* \text{ is nontrivial.} \end{cases} \end{aligned}$$

2. $$\begin{aligned} E_R(\varphi_1,\varphi_2,\dots,\varphi_n;1) &= \sum_{t_1,t_2,\dots,t_n\in R^*,\mathrm{Tr}(t_1+t_2+\cdots+t_n)=1} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n) \\ &= \sum_{\substack{t'_1,\dots,t'_n\in\mathbb{F}_q^*,t''_1,\dots,t''_n\in\mathbb{F}_q,\\ \mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=1,\mathrm{Tr}_r^q(t'_1t''_1+\cdots+t'_nt''_n)=0}} \psi_1(t'_1)\chi_{a_1}(t''_1)\cdots\psi_n(t'_n)\chi_{a_n}(t''_n) \\ &= \sum_{\substack{t'_1,\dots,t'_n\in\mathbb{F}_q^*,\\ \mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=1\mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=1}} \psi_1(t'_1)\cdots\psi_n(t'_n) \sum_{\substack{t''_1,\dots,t''_n\in\mathbb{F}_q,\\ \mathrm{Tr}_r^q(t'_1t''_1+\cdots+t'_nt''_n)=0}} \chi_{a_1}(t''_1)\cdots\chi_{a_n}(t''_n) \\ &= \sum_{\substack{t'_1,\dots,t'_n\in\mathbb{F}_q^*,\\ \mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=1}} \psi_1(t'_1)\cdots\psi_n(t'_n)A \text{ (By Lemma 7 (1)).} \end{aligned}$$

- If $a_1=\cdots=a_n=0$, then

$$E_R(\varphi_1,\varphi_2,\dots,\varphi_n;1) = \frac{q^n}{r}\sum_{\substack{t'_1,\dots,t'_n\in\mathbb{F}_q^*,\\ \mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=1}} \psi_1(t'_1)\cdots\psi_n(t'_n) = \frac{q^n}{r}E_{\mathbb{F}_q}(\psi_1,\psi_2,\dots,\psi_n;1).$$

- If $a_1\cdots a_n=0$ but not all of them are zero, then $E_R(\psi_1,\psi_2,\dots,\psi_n;1)=0$.
- If $a_1\cdots a_n\neq 0$, then $E_R(\varphi_1,\varphi_2,\dots,\varphi_n;1)$

$$\begin{aligned} &= \frac{q^n}{r}\sum_{\substack{t'_1,\dots,t'_n\in\mathbb{F}_q^*,\mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=1,\\ \frac{a_1}{t'_1}=\cdots=\frac{a_n}{t'_n}\in\mathbb{F}_r^*}} \psi_1(t'_1)\cdots\psi_n(t'_n) \\ &= \frac{q^n}{r}\sum_{\substack{z\in\mathbb{F}_r^*,\\ z\mathrm{Tr}_r^q(a_1+\cdots+a_n)=1}} \psi_1(a_1z)\cdots\psi_n(a_nz) \left(\text{Let } z=\frac{t'_1}{a_1}=\cdots=\frac{t'_n}{a_n}\right) \\ &= \frac{q^n}{r}\psi_1(a_1)\cdots\psi_n(a_n)\sum_{\substack{z\in\mathbb{F}_r^*,\\ z\mathrm{Tr}_r^q(a_1+\cdots+a_n)=1}} (\psi_1\cdots\psi_n)(z) \\ &= \begin{cases} 0, & \text{if } \mathrm{Tr}_r^q(a_1+\cdots+a_n)=0; \\ \frac{q^n}{r}\psi_1\left(\frac{a_1}{\mathrm{Tr}_r^q(a_1+\cdots+a_n)}\right)\cdots\psi_n\left(\frac{a_n}{\mathrm{Tr}_r^q(a_1+\cdots+a_n)}\right), & \text{if } \mathrm{Tr}_r^q(a_1+\cdots+a_n)\neq 0. \end{cases} \end{aligned}$$

3. $E_R(\varphi_1, \varphi_2, \ldots, \varphi_n; u)$

$$= \sum_{t_1,t_2,\ldots,t_n \in R^*, \mathrm{Tr}(t_1+t_2+\cdots+t_n)=u} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n)$$

$$= \sum_{\substack{t'_1,\ldots,t'_n \in \mathbb{F}_q^*, t''_1,\ldots,t''_n \in \mathbb{F}_q, \\ \mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=0, \mathrm{Tr}_r^q(t'_1 t''_1+\cdots+t'_n t''_n)=1}} \psi_1(t'_1)\chi_{a_1}(t''_1)\cdots\psi_n(t'_n)\chi_{a_n}(t''_n)$$

$$= \sum_{\substack{t'_1,\ldots,t'_n \in \mathbb{F}_q^*, \\ \mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=0}} \psi_1(t'_1)\cdots\psi_n(t'_n) \sum_{\substack{t''_1,\ldots,t''_n \in \mathbb{F}_q, \\ \mathrm{Tr}_r^q(t'_1 t''_1+\cdots+t'_n t''_n)=1}} \chi_{a_1}(t''_1)\cdots\chi_{a_n}(t''_n)$$

$$= \sum_{\substack{t'_1,\ldots,t'_n \in \mathbb{F}_q^*, \\ \mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=1}} \psi_1(t'_1)\cdots\psi_n(t'_n) B \text{ (By Lemma 7 (2)).}$$

- If $a_1 = \cdots = a_n = 0$, then

$$E_R(\varphi_1, \varphi_2, \ldots, \varphi_n; u) = \frac{q^n}{r} \sum_{\substack{t'_1,\ldots,t'_n \in \mathbb{F}_q^*, \\ \mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=0}} \psi_1(t'_1)\cdots\psi_n(t'_n) = \frac{q^n}{r} E_{\mathbb{F}_q}(\psi_1, \psi_2, \ldots, \psi_n; 0).$$

- If $a_1 \cdots a_n = 0$ but not all of them are zero, then $E_R(\psi_1, \psi_2, \ldots, \psi_n; u) = 0$.
- If $a_1 \cdots a_n \neq 0$, then $E_R(\varphi_1, \varphi_2, \cdots, \varphi_n; u)$

$$= \frac{q^n}{r} \sum_{\substack{t'_1,\ldots,t'_n \in \mathbb{F}_q^*, \mathrm{Tr}_r^q(t'_1+\cdots+t'_n)=0, \\ z=\frac{a_1}{t'_1}=\cdots=\frac{a_n}{t'_n} \in \mathbb{F}_r^*}} \psi_1(t'_1)\cdots\psi_n(t'_n)\lambda(z)$$

$$= \frac{q^n}{r} \sum_{\substack{z \in \mathbb{F}_r^*, \\ \frac{1}{z}\mathrm{Tr}_r^q(a_1+\cdots+a_n)=0}} \psi_1\left(\frac{a_1}{z}\right)\cdots\psi_n\left(\frac{a_n}{z}\right)\lambda(z)$$

$$= \frac{q^n}{r}\psi_1(a_1)\cdots\psi_n(a_n) \sum_{\substack{z \in \mathbb{F}_r^*, \\ \frac{1}{z}\mathrm{Tr}_r^q(a_1+\cdots+a_n)=0}} (\overline{\psi_1\cdots\psi_n})^*(z)\lambda(z)$$

$$= \begin{cases} 0, & \text{if } \mathrm{Tr}_r^q(a_1+\cdots+a_n) \neq 0; \\ \frac{q^n}{r}\psi_1(a_1)\cdots\psi_n(a_n) G_{\mathbb{F}_r}((\overline{\psi_1\cdots\psi_n})^*), & \text{if } \mathrm{Tr}_r^q(a_1+\cdots+a_n) = 0. \end{cases}$$

This completes the proof of this theorem. □

From the above theorem, we combine Eqs. (3), (4) with Lemma 3. Then we can calculate the exact value of the hyper Eisenstein sum $E_R(\varphi_1, \varphi_2, \cdots, \varphi_n; 1)$ over $R$. It is worth mentioning that we obtain a connection between hyper Eisenstein sums of $R$ and Gaussian sums of $\mathbb{F}_q$ when $\psi_1, \psi_2, \ldots, \psi_n$ are not all trivial by [21, Theorem 3]. Therefore, we obtain the following corollary.

**Corollary 2** *Let* $\varphi_1, \varphi_2, \ldots, \varphi_n$ *be multiplicative characters of* $R$ and $\varphi_i := \psi_i \star \chi_{a_i}$ $(1 \le i \le n)$, *where* $\psi_i$ *is a multiplicative character of* $\mathbb{F}_q$ and $\chi_{a_i}$ *is an additive character of* $\mathbb{F}_q$ *with* $a_i \in \mathbb{F}_q$. *We obtain the following three direct consequences.*

1. If $\psi_1, \psi_2, \ldots, \psi_n$ *are all trivial, then*

$$E_R(\varphi_1, \ldots, \varphi_n;1) = \begin{cases} \frac{q^n((q-1)^n+(-1)^{n+1})}{r^2}, & \text{if } a_1 = \cdots = a_n = 0; \\ \frac{q^n}{r}, & \text{if } a_1 \cdots a_n \ne 0 \text{ and } \mathrm{Tr}_r^q(a_1 + \cdots \\ & + a_n) \ne 0; \\ 0, & \text{otherwise.} \end{cases}$$

2. If $\psi_1, \ldots, \psi_h$ *are all nontrivial and* $\psi_{h+1}, \ldots, \psi_n$ *are all trivial for* $1 \le h \le n-1$, *then* $E_R(\varphi_1, \ldots, \varphi_n;1)$

$$= \begin{cases} \frac{(-1)^{n-h}q^n G_{\mathbb{F}_q}(\psi_1)\cdots G_{\mathbb{F}_q}(\psi_h)}{rG_{\mathbb{F}_r}((\psi_1\cdots\psi_h)^*)}, & \text{if } a_1 = \cdots = a_n = 0 \text{ and} \\ & (\psi_1 \cdots \psi_h)^* \text{ is nontrivial;} \\ \frac{(-1)^{n-h+1}q^n G_{\mathbb{F}_q}(\psi_1)\cdots G_{\mathbb{F}_q}(\psi_h)}{r^2}, & \text{if } a_1 = \cdots = a_n = 0 \text{ and} \\ & (\psi_1 \cdots \psi_h)^* \text{ is trivial;} \\ \frac{q^n}{r}\psi_1\left(\frac{a_1}{\mathrm{Tr}_r^q(a_1+\cdots+a_n)}\right)\cdots\psi_h\left(\frac{a_h}{\mathrm{Tr}_r^q(a_1+\cdots+a_n)}\right), & \text{if } a_1 \cdots a_n \ne 0 \text{ and} \\ & \mathrm{Tr}_r^q(a_1 + \cdots + a_n) \ne 0; \\ 0, & \text{otherwise.} \end{cases}$$

3. If $\psi_1, \psi_2, \ldots, \psi_n$ *are all nontrivial, then* $E_R(\varphi_1, \ldots, \varphi_n;1)$

$$= \begin{cases} \frac{q^n G_{\mathbb{F}_q}(\psi_1)\cdots G_{\mathbb{F}_q}(\psi_n)}{rG_{\mathbb{F}_r}((\psi_1\cdots\psi_n)^*)}, & \text{if } a_1 = \cdots = a_n = 0 \text{ and} \\ & (\psi_1 \cdots \psi_n)^* \text{ is nontrivial;} \\ -\frac{q^n G_{\mathbb{F}_q}(\psi_1)\cdots G_{\mathbb{F}_q}(\psi_n)}{r^2}, & \text{if } a_1 = \cdots = a_n = 0 \text{ and} \\ & (\psi_1 \cdots \psi_n)^* \text{ is trivial;} \\ \frac{q^n}{r}\psi_1\left(\frac{a_1}{\mathrm{Tr}_r^q(a_1+\cdots+a_n)}\right)\cdots\psi_n\left(\frac{a_n}{\mathrm{Tr}_r^q(a_1+\cdots+a_n)}\right), & \text{if } a_1 \cdots a_n \ne 0 \text{ and} \\ & \mathrm{Tr}_r^q(a_1 + \cdots + a_n) \ne 0; \\ 0, & \text{otherwise.} \end{cases}$$

***Remark 2*** Similarly, we can also calculate the exact value of the sums $E_R(\varphi_1, \varphi_2, \ldots, \varphi_n;0)$ and $E_R(\varphi_1, \varphi_2, \ldots, \varphi_n;u)$ over $R$ using Lemma 5.

In view of the fact that $E_R(\varphi_1, \ldots, \varphi_n;\mathfrak{s}) = \varphi_1 \cdots \varphi_n(\mathfrak{s})E_R(\varphi_1, \cdots, \varphi_n;1)$ and Corollary 2(3), we can determine the absolute value of $E_R(\varphi_1, \ldots, \varphi_n;\mathfrak{s})$ for all $\mathfrak{s} \in R_r^*$.

**Corollary 3** *Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be multiplicative characters of $R$* and *$\varphi_i := \psi_i \star \chi_{a_i}$ $(1 \le i \le n)$, where $\psi_i$ is a nontrivial multiplicative character of $\mathbb{F}_q$ and $\chi_{a_i}$ is an additive character of $\mathbb{F}_q$ with $a_i \in \mathbb{F}_q$. Assume that $(\psi_1 \cdots \psi_n)^*$ is the restriction of $\psi_1 \cdots \psi_n$ to $\mathbb{F}_r$. Then*

$$|E_R(\varphi_1, \ldots, \varphi_n; \mathfrak{s})| = \begin{cases} r^{\frac{3}{2}(mn-1)}, & \text{if } a_1 = \cdots = a_n = 0 \text{ and } (\psi_1 \cdots \psi_n)^* \text{ is nontrivial;} \\ r^{\frac{3mn-4}{2}}, & \text{if } a_1 = \cdots = a_n = 0 \text{ and } (\psi_1 \cdots \psi_n)^* \text{ is trivial;} \\ r^{mn-1}, & \text{if } a_1 \cdots a_n \neq 0 \text{ and } \mathrm{Tr}_r^q(a_1 + \cdots + a_n) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

In fact, we can get the value of the sum $E_R(\varphi_1, \varphi_2, \ldots, \varphi_n; \mathfrak{s})$ when $n = 1$ in Theorem 2. If $\mathfrak{s} = 1$ and $n = 1$, then the sum $E_R(\varphi; 1)$ is usually called the Eisenstein sum over $R$, where $\varphi$ is a multiplicative character of $R$. Hence, we have the following corollary as a special case of Theorem 2.

**Corollary 4** *Let $\varphi$* be a multiplicative character of $R$ and $\varphi := \psi \star \chi_a$, where $\psi$ is a multiplicative character of $\mathbb{F}_q$ and $\chi_a$ is an additive character of $\mathbb{F}_q$ with $a \in \mathbb{F}_q$. Then

1.
$$E_R(\varphi; 0) = \begin{cases} \frac{q}{r} E_{\mathbb{F}_q}(\psi; 0), & \text{if } a = 0; \\ 0, & \text{if } a \neq 0 \text{ and } \mathrm{Tr}_r^q(a) \neq 0; \\ \frac{q(r-1)}{r} \psi(a), & \text{if } a \neq 0, \mathrm{Tr}_r^q(a) = 0 \text{ and } \psi^* \text{ is trivial;} \\ 0, & \text{if } a \neq 0, \mathrm{Tr}_r^q(a) = 0 \text{ and } \psi^* \text{ is nontrivial,} \end{cases}$$

where $E_{\mathbb{F}_q}(\psi; 0)$ *denotes the sum* $E_{\mathbb{F}_q}(\psi; s)$ *over* $\mathbb{F}_q$ *with* $s = 0$.

2.
$$E_R(\varphi; 1) = \begin{cases} \frac{q}{r} E_{\mathbb{F}_q}(\psi; 1), & \text{if } a = 0; \\ \frac{q}{r} \psi\left(\frac{a}{\mathrm{Tr}_r^q(a)}\right), & \text{if } a \neq 0 \text{ and } \mathrm{Tr}_r^q(a) \neq 0; \\ 0, & \text{if } a \neq 0 \text{ and } \mathrm{Tr}_r^q(a) = 0, \end{cases}$$

*where $E_{\mathbb{F}_q}(\psi; 1)$ denotes the Eisenstein sum over $\mathbb{F}_q$.*

3.
$$E_R(\varphi; u) = \begin{cases} \frac{q}{r} E_{\mathbb{F}_q}(\psi; 0), & \text{if } a = 0; \\ 0, & \text{if } a \neq 0 \text{ and } \mathrm{Tr}_r^q(a) \neq 0; \\ \frac{q}{r} \psi(a) G_{\mathbb{F}_r}(\overline{\psi^*}), & \text{if } a \neq 0 \text{ and } \mathrm{Tr}_r^q(a) = 0, \end{cases}$$

*where $E_{\mathbb{F}_q}(\psi; 0)$ is the sum $E_{\mathbb{F}_q}(\psi; s)$ over $\mathbb{F}_q$ with $s = 0$ and $G_{\mathbb{F}_r}(\overline{\psi^*})$ is a Gaussian sum over $\mathbb{F}_r$.*

***Remark 3*** In view of the definition of Jacobi sums over $\mathbb{F}_q$ in [44], we have the Jacobi sum $J_{\mathbb{F}_q}(\varphi_1, \varphi_2, \dots, \varphi_n;1)$ defined by

$$J_{\mathbb{F}_q}(\varphi_1, \varphi_2, \dots, \varphi_n;1) = \sum_{x_1,x_2,\dots,x_n \in \mathbb{F}_q^*, x_1+x_2+\cdots+x_n=1} \varphi_1(x_1)\varphi_2(x_2)\cdots\varphi_n(x_n).$$

Similarly, we can define Jacobi sums over the ring $R$ as follows:

$$J_R(\varphi_1, \varphi_2, \dots, \varphi_n;1) = \sum_{t_1,t_2,\dots,t_n \in R^*, t_1+t_2+\cdots+t_n=1} \varphi_1(t_1)\varphi_2(t_2)\cdots\varphi_n(t_n).$$

Let $q = r^m$ and $r = p^l$. If $m = 1$ in (5) of Definition 4, Jacobi sums over $R$ are special types of the hyper Eisenstein sums. Therefore, we have the following corollary, which relates the Jacobi sum $J_R(\varphi_1, \varphi_2, \dots, \varphi_n;1)$ over the ring $R$.

**Corollary 5** *Let $\varphi_1, \varphi_2, \dots, \varphi_n$ be multiplicative characters of $R$ and $\varphi_i := \psi_i \star \chi_{a_i} (1 \le i \le n)$, where $\psi_i$ and $\chi_{a_i}$ are multiplicative and additive characters of $\mathbb{F}_q$, respectively. Then $J_R(\varphi_1, \varphi_2, \dots, \varphi_n;1)$*

$$= \begin{cases} q^{n-1} J_{\mathbb{F}_q}(\psi_1, \psi_2, \dots, \psi_n;1), & \text{if } a_1 = \cdots = a_n = 0; \\ 0, & \text{if } a_1 \cdots a_n = 0 \text{ but not all of them are zero;} \\ q^{n-1}\psi_1(\frac{a_1}{a_1+\cdots+a_n})\cdots\psi_n(\frac{a_n}{a_1+\cdots+a_n}), & \text{if } a_1 \cdots a_n \neq 0 \text{ and } a_1 + \cdots + a_n \neq 0; \\ 0, & \text{if } a_1 \cdots a_n \neq 0 \text{ and } a_1 + \cdots + a_n = 0. \end{cases}$$

*Here, the Jacobi sum $J_{\mathbb{F}_q}(\psi_1, \psi_2, \dots, \psi_n;1)$*

$$= \begin{cases} \frac{(q-1)^n + (-1)^{n+1}}{q}, & \text{if } \psi_1, \dots, \psi_n \text{ are trivial;} \\ (-1)^{n-h}\frac{(q-1)^h + (-1)^{h+1}}{q}, & \text{if } \psi_1, \dots, \psi_h \text{ are nontrivial and } \psi_{h+1}, \dots, \psi_n \text{ are trivial;} \\ \frac{G_{\mathbb{F}_q}(\psi_1)\cdots G_{\mathbb{F}_q}(\psi_n)}{G_{\mathbb{F}_q}(\psi_1\cdots\psi_n)}, & \text{if } \psi_1, \dots, \psi_n \text{ and } \psi_1\cdots\psi_n \text{ are nontrivial;} \\ -\frac{G_{\mathbb{F}_q}(\psi_1)\cdots G_{\mathbb{F}_q}(\psi_n)}{q}, & \text{if } \psi_1, \dots, \psi_n \text{ are nontrivial and } \psi_1\cdots\psi_n \text{ are trivial.} \end{cases}$$

## 5 Applications

In this section, we mainly study the applications of character sums over the local ring $R = \mathbb{F}_q + u\mathbb{F}_q$ $(u^2 = 0)$ to the construction of codebooks.

### 5.1 The generic constructions of asymptotically optimal codebooks

This subsection presents several families of asymptotically optimal codebooks constructed using Gaussian sums, hyper Eisenstein sums and Jacobi sums over $R$.

### 5.1.1 The constructions of codebooks via Gaussian sums over $R$

Note that $|R^*| = q(q-1)$. Let $\varphi := \psi \star \chi_a$ and $\lambda := \chi_b \star \chi_c$, where $a, b, c \in \mathbb{F}_q$, $\chi_a, \chi_b, \chi_c \in \widehat{\mathbb{F}}_q$ and $\psi \in \widehat{\mathbb{F}}_q^*$. Assume that $t = t_0(1 + ut_1)$, where $t_0 \in \mathbb{F}_q^*$ and $t_1 \in \mathbb{F}_q$. Then we can define a set $C_0(R^*, \widehat{R}^* \times \widehat{R})$ as

$$C_0(R^*, \widehat{R}^* \times \widehat{R}) = \left\{ \frac{1}{\sqrt{K}}(\varphi(t)\lambda(t))_{t\in R^*}, \varphi \in \widehat{R}^*, \lambda \in \widehat{R} \right\}$$
$$= \left\{ \frac{1}{\sqrt{K}}(\psi(t_0)\chi_a(t_1)\chi_b(t_0)\chi_c(t_0t_1))_{t_0\in\mathbb{F}_q^*, t_1\in\mathbb{F}_q}, \psi \in \widehat{\mathbb{F}}_q^*, \chi_a, \chi_b, \chi_c \in \widehat{\mathbb{F}}_q \right\},$$

where $K = |R^*| = q(q-1)$.

Next, we will give two constructions of codebooks over the ring $R$.

***A. The first construction of codebooks***

The codebook $C_1 := C_1(R^*, \widehat{R}^* \times \widehat{R})$ of length $K_1 = |R^*| = q(q-1)$ over $R$ is constructed as

$$C_1 = \left\{ \frac{1}{\sqrt{K_1}}(\psi(t_0)\chi_a(t_1)\chi_b(t_0)\chi_c(t_0t_1))_{t_0\in\mathbb{F}_q^*, t_1\in\mathbb{F}_q}, \right.$$
$$\left. \psi \text{ is a fixed multiplicative character over } \mathbb{F}_q, \chi_a, \chi_b, \chi_c \in \widehat{\mathbb{F}}_q \right\}.$$

Based on this construction of the codebook $C_1$, we have the following theorem.

***Theorem 3*** *Let* $C_1$ be a codebook defined as above. Then $C_1$ *is a* $(q^3, q(q-1))$ *codebook having maximum cross-correlation amplitude* $I_{\max}(C_1) = \frac{1}{q-1}$. *Moreover, the codebook* $C_1$ *asymptotically meets the Welch bound.*

***Proof*** By the definition of $C_1$, it is obvious that $C_1$ has $N_1 = q^3$ codewords of length $K_1 = q(q-1)$. Let $\mathbf{c}_1 = \frac{1}{\sqrt{K_1}}(\psi(t_0)\chi_{a_1}(t_1)\chi_{b_1}(t_0)\chi_{c_1}(t_0t_1))_{t_0\in\mathbb{F}_q^*, t_1\in\mathbb{F}_q}$ and $\mathbf{c}_2 = \frac{1}{\sqrt{K_1}}(\psi(t_0)\chi_{a_2}(t_1)\chi_{b_2}(t_0)\chi_{c_2}(t_0t_1))_{t_0\in\mathbb{F}_q^*, t_1\in\mathbb{F}_q}$ be any two distinct codewords in $C_1$. Denote the trivial multiplicative character of $\mathbb{F}_q$ by $\psi_0$. Let $a = a_1 - a_2, b = b_1 - b_2$ and $c = c_1 - c_2$. Set $\varphi := \psi_0 \star \chi_a$ and $\lambda := \chi_b \star \chi_c$. Then the correlation of $\mathbf{c}_1$ and $\mathbf{c}_2$ is as follows.

$$\begin{aligned}
K_1\mathbf{c}_1\mathbf{c}_2^H &= \sum_{t_0\in\mathbb{F}_q^*,t_1\in\mathbb{F}_q} \psi(t_0)\chi_{a_1}(t_1)\chi_{b_1}(t_0)\chi_{c_1}(t_0t_1)\overline{\psi(t_0)\chi_{a_2}(t_1)\chi_{b_2}(t_0)\chi_{c_2}(t_0t_1)} \\
&= \sum_{t_0\in\mathbb{F}_q^*,t_1\in\mathbb{F}_q} \psi_0(t_0)\chi((a_1-a_2)t_1+(b_1-b_2)t_0+(c_1-c_2)t_0t_1) \\
&= \sum_{t_0\in\mathbb{F}_q^*} \psi_0(t_0)\chi((b_1-b_2)t_0)\sum_{t_1\in\mathbb{F}_q}\chi((a_1-a_2)t_1+(c_1-c_2)t_0t_1) \\
&= \sum_{t_0\in\mathbb{F}_q^*} \psi_0(t_0)\chi(bt_0)\sum_{t_1\in\mathbb{F}_q}\chi((a+ct_0)t_1) \\
&= \sum_{t_0\in\mathbb{F}_q^*,a+ct_0=0} \psi_0(t_0)\chi_b(t_0) \\
&= G_R(\varphi,\lambda).
\end{aligned}$$

Since $\mathbf{c}_1 \neq \mathbf{c}_2$, $a$, $b$ and $c$ are not all equal to 0. In view of Theorem 1, we have

$$K_1\mathbf{c}_1\mathbf{c}_2^H = \begin{cases} -q, & \text{if } a=0,\ c=0 \text{ and } b\neq 0; \\ q\chi\left(-\frac{ab}{c}\right), & \text{if } a\neq 0 \text{ and } c\neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we infer that $|\mathbf{c}_1\mathbf{c}_2^H| \in \left\{0, \frac{1}{q-1}\right\}$ for any two distinct codewords $\mathbf{c}_1, \mathbf{c}_2$ in $C_1$. Hence, $I_{\max}(C_1) = \frac{1}{q-1}$.

Next, we show that the codebook $C_1$ asymptotically meets the Welch bound. The corresponding Welch bound of the codebook $C_1$ is

$$I_w = \sqrt{\frac{N_1-K_1}{(N_1-1)K_1}} = \sqrt{\frac{q^3-q(q-1)}{(q^3-1)q(q-1)}} = \sqrt{\frac{q^2-q+1}{q^4-q^3-q+1}}.$$

From $\frac{I_{\max}(C_1)}{I_w} = \sqrt{\frac{q^4-q^3-q+1}{(q^2-q+1)(q-1)^2}}$, we have $\lim_{q\longrightarrow\infty}\frac{I_{\max}(C_1)}{I_w} = 1$, which implies that $C_1$ asymptotically meets the Welch bound. □

***B. The second construction of codebooks***

The codebook $C_2 := C_2(R^*, \widehat{R}^*\times\widehat{R})$ of length $K_2 = |R^*| = q(q-1)$ over $R$ is defined by

$$C_2 = \left\{ \frac{1}{\sqrt{K_2}}(\psi(t_0)\chi_a(t_1)\chi_b(t_0)\chi_c(t_0t_1))_{t_0\in\mathbb{F}_q^*,t_1\in\mathbb{F}_q}, \right.$$
$$\left. \psi\in\widehat{\mathbb{F}}_q^*, \chi_b \text{ is a fixed additive character over } \mathbb{F}_q, \chi_a,\chi_c\in\widehat{\mathbb{F}}_q \right\}.$$

With this construction, we can derive the following theorem.

**Theorem 4** *Let $C_2$ be a codebook defined as above. Then $C_2$ is a $(q^2(q-1), q(q-1))$ codebook having maximum cross-correlation amplitude $I_{\max}(C_2) = \frac{1}{q-1}$. Moreover, the codebook $C_2$ asymptotically meets the Welch bound.*

***Proof*** According to the definition of $C_2$, it is easy to see that $C_2$ has $N_2 = q^2(q-1)$ codewords of length $K_2 = q(q-1)$. Let $\mathbf{c}_1 = \frac{1}{\sqrt{K_2}}(\psi_1(t_0)\chi_{a_1}(t_1)\chi_b(t_0)\chi_{c_1}(t_0t_1))_{t_0\in\mathbb{F}_q^*,t_1\in\mathbb{F}_q}$ and $\mathbf{c}_2 = \frac{1}{\sqrt{K_2}}(\psi_2(t_0)\chi_{a_2}(t_1)\chi_b(t_0)\chi_{c_2}(t_0t_1))_{t_0\in\mathbb{F}_q^*,t_1\in\mathbb{F}_q}$ be any two distinct codewords in $C_2$. Set $\psi = \psi_1\overline{\psi}_2, a = a_1 - a_2$ and $c = c_1 - c_2$. Then the correlation of $\mathbf{c}_1$ and $\mathbf{c}_2$ is as follows.

$$\begin{aligned}
K_2\mathbf{c}_1\mathbf{c}_2^H &= \sum_{t_0\in\mathbb{F}_q^*,t_1\in\mathbb{F}_q} \psi_1(t_0)\chi_{a_1}(t_1)\chi_b(t_0)\chi_{c_1}(t_0t_1)\overline{\psi_2(t_0)\chi_{a_2}(t_1)\chi_b(t_0)\chi_{c_2}(t_0t_1)} \\
&= \sum_{t_0\in\mathbb{F}_q^*,t_1\in\mathbb{F}_q} \psi_1\overline{\psi}_2(t_0)\chi((a_1-a_2)t_1 + (c_1-c_2)t_0t_1) \\
&= \sum_{t_0\in\mathbb{F}_q^*} \psi(t_0) \sum_{t_1\in\mathbb{F}_q} \chi((a+ct_0)t_1) \\
&= q \sum_{t_0\in\mathbb{F}_q^*, a+ct_0=0} \psi(t_0).
\end{aligned}$$

- If $a = c = 0$, since $\mathbf{c}_1 \neq \mathbf{c}_2$, it follows that $\psi$ is nontrivial. Then we have

$$K_2\mathbf{c}_1\mathbf{c}_2^H = q \sum_{t_0\in\mathbb{F}_q^*} \psi(t_0) = 0;$$

- If $a = 0, c \neq 0$ or $a \neq 0, c = 0$, then $K_2\mathbf{c}_1\mathbf{c}_2^H = 0$;
- If $a \neq 0$ and $c \neq 0$, then $K_2\mathbf{c}_1\mathbf{c}_2^H = q\psi(-\frac{a}{c})$.

$$\mathbf{c}_1\mathbf{c}_2^H = \begin{cases} \frac{q}{K_2}\psi(-\frac{a}{c}), & \text{if } a \neq 0 \text{ and } c \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we infer that $|\mathbf{c}_1\mathbf{c}_2^H| \in \left\{0, \frac{1}{q-1}\right\}$ for any two distinct codewords $\mathbf{c}_1, \mathbf{c}_2$ in $C_2$. Therefore, $I_{\max}(C_2) = \frac{1}{q-1}$.

Finally, we show that the codebook $C_2$ asymptotically meets the Welch bound. The proof is similar to the proof of Theorem 3, and by calculating, we have

$$I_w = \sqrt{\frac{N_2 - K_2}{(N_2-1)K_2}} = \sqrt{\frac{q^2(q-1) - q(q-1)}{(q^3-q^2-1)q(q-1)}} = \sqrt{\frac{q-1}{q^3-q^2-1}}.$$

Apparently, we get $\lim_{q\longrightarrow\infty}\frac{I_{\max}(C_2)}{I_w}=\lim_{q\longrightarrow\infty}\sqrt{\frac{q^3-q^2-1}{(q-1)(q-1)^2}}=1$. This completes the proof. □

### 5.1.2 The constructions of codebooks via Eisenstein sums over *R*

Next, we present the asymptotically optimal codebooks which are constructed by Eisenstein sums over $R$. Based on this, we first give the following lemma.

**Lemma 8** *Let* $G:=\{\phi_j \mid (r-1)|j\}\subseteq\widehat{\mathbb{F}}_q^*$, *where* $\phi_j=\phi_1^j$ *and* $\phi_1$ *is a generator of* $\widehat{\mathbb{F}}_q^*$ *with* $0\le j\le q-2$. *Then* $G$ is a subgroup of $\mathbb{F}_q^*$ *and* $|G|=\frac{q-1}{r-1}$. *Moreover, for every* $\psi\in\widehat{\mathbb{F}}_q^*$, $\psi^*$ *is trivial if and only if* $\psi\in G$, *where* $\psi^*$ *denotes the restriction of* $\psi$ *to* $\mathbb{F}_r$.

***Proof*** Assume that $\mathbb{F}_q^*=\langle\theta\rangle$, i.e, let $\theta$ be a primitive element of $\mathbb{F}_q$. Then $\mathbb{F}_r^*=\langle\theta^{\frac{q-1}{r-1}}\rangle$. We can further assume that $\phi_1(\theta)=\zeta_{q-1}$. Then $\psi^*$ is trivial $\Longleftrightarrow\psi(\theta^{\frac{q-1}{r-1}})=1$ $\Longleftrightarrow\psi(\theta)^{\frac{q-1}{r-1}}=1\Longleftrightarrow(\zeta_{q-1}^j)^{\frac{q-1}{r-1}}=1, 0\le j\le q-2\Longleftrightarrow(q-1)|j\frac{q-1}{r-1}\Longleftrightarrow(r-1)|j$. □

***C. The third construction of codebooks***

Let

$$D=\{t\in R^*|\mathrm{Tr}(t)=1\}\text{ and }K_3:=|D|.$$

Here, we consider the case that $m=2$ and $q=r^2$. Hence, it is easy to check that $K_3=r^2$. Assume that $H$ is a subgroup of $G:=\{\phi_j\mid(r-1)|j\}\subseteq\widehat{\mathbb{F}}_q^*$ and $k=|H|$. Then $k\mid(r+1)$ since $|G|=\frac{q-1}{r-1}=r+1$.

The codebook $C_3:=C_3(D,H\times\widehat{\mathbb{F}}_q)$ of length $K_3=r^2$ over $R$ is built as

$$C_3:=\left\{\frac{1}{\sqrt{K_3}}((\psi\star\chi_a)(t))_{t\in D},\psi\in H,\chi_a\in\widehat{\mathbb{F}}_q\right\}.$$

Based on this construction of the codebook $C_3$, we get the following theorem.

**Theorem 5** *Let* $C_3$ *be the codebook defined as above. Then* $C_3$ *is a* $(kr^2,r^2)$ *codebook having maximum cross-correlation amplitude* $I_{\max}(C_3)=\frac{1}{r}$. *Moreover, the codebook* $C_3$ *asymptotically meets the Welch bound.*

***Proof*** According to the definition of $C_3$, it is obvious that $C_3$ has $N_3=kr^2$ codewords of length $K_3=r^2$. Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be any two distinct codewords in $C_3$, where $\mathbf{c}_1=\frac{1}{\sqrt{K_3}}((\psi_1\star\chi_{a_1})(t))_{t\in D}$ and $\mathbf{c}_2=\frac{1}{\sqrt{K_3}}((\psi_2\star\chi_{a_2})(t))_{t\in D}$. Let

$\varphi_1 := \psi_1 \star \chi_{a_1}$ and $\varphi_2 := \psi_2 \star \chi_{a_2}$. Set $\varphi = \varphi_1 \overline{\varphi_2}$ and $\varphi := \psi \star \chi_a$. Then the correlation of $\mathbf{c}_1$ and $\mathbf{c}_2$ is as follows.

$$\begin{aligned} K_3 \mathbf{c}_1 \mathbf{c}_2^H &= \sum_{t \in D} (\psi_1 \star \chi_{a_1})(t) \overline{(\psi_2 \star \chi_{a_2})(t)} \\ &= \sum_{t \in R^*, \mathrm{Tr(t)}=1} \varphi_1(t) \overline{\varphi_2(t)} \\ &= E_R(\varphi; 1) \\ &= \begin{cases} \frac{q}{p} E_{\mathbb{F}_q}(\psi; 1), & \text{if } a = 0; \\ \frac{q}{p} \psi\left(\frac{a}{\mathrm{Tr}_r^q(a)}\right), & \text{if } a \neq 0 \text{ and } \mathrm{Tr}(a) \neq 0; \text{ (By Corollary 4 (2))} \\ 0, & \text{if } a \neq 0 \text{ and } \mathrm{Tr}(a) = 0. \end{cases} \end{aligned}$$

Since $\mathbf{c}_1 \neq \mathbf{c}_2$, it follows that $\psi$ and $\chi_a$ are not all trivial. In view of Corollary 3 $(n = 1, m = 2)$, we have

$$K_3 |\mathbf{c}_1 \mathbf{c}_2^H| = \begin{cases} r^{\frac{3}{2}}, & \text{if } a = 0, \psi \text{ and } \psi^* \text{ are nontrivial;} \\ r, & \text{if } a = 0, \psi \text{ is nontrivial and } \psi^* \text{ is trivial;} \\ r, & \text{if } a \neq 0, \mathrm{Tr}_r^q(a) \neq 0 \text{ and } \psi \text{ is an arbitrary multiplicative} \\ & \text{character of } \mathbb{F}_q; \\ 0, & \text{if } a \neq 0, \mathrm{Tr}_r^q(a) = 0 \text{ and } \psi \text{ is an arbitrary multiplicative} \\ & \text{character of } \mathbb{F}_q. \end{cases}$$

Since $H \leq G$, which implies that $\psi^*$ is trivial (by Lemma 8), we infer that $|\mathbf{c}_1 \mathbf{c}_2^H| \in \left\{0, \frac{1}{r}\right\}$ for any $\mathbf{c}_1, \mathbf{c}_2 \in C_3$. Hence, $I_{\max}(C_3) = \frac{1}{r}$.

Next, we prove that the codebook $C_3$ asymptotically meets the Welch bound. An argument analogous to the one given in the proof of Theorem 3 establishes that

$$I_w = \sqrt{\frac{N_3 - K_3}{(N_3 - 1)K_3}} = \sqrt{\frac{kr^2 - r^2}{(kr^2 - 1)r^2}} = \sqrt{\frac{k - 1}{kr^2 - 1}}.$$

Obviously, we have $\lim\limits_{q \longrightarrow \infty} \frac{I_{\max}(C_3)}{I_w} = \lim\limits_{q \longrightarrow \infty} \sqrt{\frac{kq-1}{q(k-1)}} = 1$, which implies that $C_3$ asymptotically meets the Welch bound. □

### 5.1.3 The constructions of codebooks via Jacobi sums over *R*

In the following, we present the asymptotically optimal codebooks which are constructed using Jacobi sums over $R$.

***D. The fourth construction of codebooks***

Now, we consider the case that $n = 2$ and $m = 1$. Let $t_1 = t_1'(1 + ut_1'') \in R^*$ and $t_2 = t_2'(1 + ut_2'') \in R^*$. We define

$$\begin{aligned} D' =& \{t_1, t_2 \in R^* | t_1 + t_2 = 1\} \\ =& \{t'_1, t'_2 \in \mathbb{F}_q^*, t''_1, t''_2 \in \mathbb{F}_q | t'_1 + t'_2 = 1, t'_1 t''_1 + t'_2 t''_2 = 0\} \text{ and } K_4 := |D'|. \end{aligned}$$

The codebook $C_4 := C_4(D', \widehat{R}^* \times \widehat{R}^*)$ of length $K_4$ over $R$ is assembled as

$$C_4 = \left\{ \frac{1}{\sqrt{K_4}} (\varphi_1(t_1)\varphi_2(t_2))_{t_1,t_2 \in D'}, \varphi_1 = \psi_1 \star \chi_{a_1}, \varphi_2 = \psi_2 \star \chi_{a_2}, \right.$$
$$\left. \psi_1 \text{ is a fixed multiplicative character over } \mathbb{F}_q, \psi_2 \in \widehat{\mathbb{F}}_q^*, \chi_{a_1}, \chi_{a_2} \in \widehat{\mathbb{F}}_q \right\}.$$

With this construction, we can derive the following theorem.

**Theorem 6** *Let $C_4$ be the codebook defined as above. Then $C_4$ is a $(q^2(q-1), q(q-2))$ codebook having maximum cross-correlation amplitude $I_{\max}(C_4) = \frac{1}{q-2}$. Moreover, the codebook $C_4$ asymptotically meets the Welch bound.*

***Proof*** By the definition of $C_4$, it is obvious that $C_4$ has $N_4 = q^2(q-1)$ codewords of length $K_4 = q(q-2)$. Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be any two distinct codewords in $C_4$, where $\mathbf{c}_1 = \frac{1}{\sqrt{K_4}}(\psi_1(t'_1)\chi_{a_1}(t''_1)\psi_2(t'_2)\chi_{a_2}(t''_2))_{t'_1,t'_2 \in \mathbb{F}_q^*, t''_1, t''_2 \in \mathbb{F}_q}$ and $\mathbf{c}_2 = \frac{1}{\sqrt{K_4}}(\psi_1(t'_1)\chi_{b_1}(t''_1)\psi_3(t'_2)$ $\chi_{b_2}(t''_2))_{t'_1,t'_2 \in \mathbb{F}_q^*, t''_1, t''_2 \in \mathbb{F}_q}$. Denote the trivial multiplicative character of $\mathbb{F}_q$ by $\psi_0$. Let $a = a_1 - b_1$ and $b = a_2 - b_2$. Set $\varphi_1 = \psi_0 \star \chi_a$ and $\varphi_2 = \psi_2\overline{\psi_3} \star \chi_b$. Then the correlation of $\mathbf{c}_1$ and $\mathbf{c}_2$ is as follows.

$$\begin{aligned} K_4 \mathbf{c}_1 \mathbf{c}_2^H =& \sum_{\substack{t'_1,t'_2 \in \mathbb{F}_q^*, t''_1, t''_2 \in \mathbb{F}_q, \\ t'_1 + t'_2 = 1, t'_1 t''_1 + t'_2 t''_2 = 0 t'_1 + t'_2 = 1, t'_1 t''_1 + t'_2 t''_2 = 0}} \psi_1(t'_1)\chi_{a_1}(t''_1)\psi_2(t'_2)\chi_{a_2}(t''_2)\overline{\psi_1(t'_1)\chi_{b_1}(t''_1)\psi_3(t'_2)\chi_{b_2}(t''_2)} \\ =& \sum_{\substack{t'_1,t'_2 \in \mathbb{F}_q^*, t''_1, t''_2 \in \mathbb{F}_q, \\ t'_1 + t'_2 = 1, t'_1 t''_1 + t'_2 t''_2 = 0}} \psi_0(t'_1)\chi((a_1 - b_1)t''_1)\psi_2\overline{\psi_3}(t'_2)\chi((a_2 - b_2)t''_2) \\ =& \sum_{\substack{t_1, t_2 \in R^*, \\ t_1 + t_2 = 1 t_1 + t_2 = 1}} \varphi_1(t_1)\varphi_2(t_2) \\ =& J_R(\varphi_1, \varphi_2). \end{aligned}$$

According to Corollary 5 ($n = 2$), we have

$$K_4 \mathbf{c}_1 \mathbf{c}_2^H = \begin{cases} -q, & \text{if } a = b = 0; \text{ (since } \mathbf{c}_1 \neq \mathbf{c}_2, \ \psi_2\overline{\psi_3} \text{ is nontrivial)} \\ 0, & \text{if } a = 0 \text{ and } b \neq 0; \\ q\psi_2\overline{\psi_3}\left(\frac{a}{a+b}\right), & \text{if } a \neq 0, \ b \neq 0 \text{ and } a \neq -b \\ 0, & \text{if } a \neq 0, \ b \neq 0 \text{ and } a = -b. \end{cases}$$

Consequently, we infer that $|\mathbf{c}_1 \mathbf{c}_2^H| \in \{0, \frac{1}{q-2}\}$ for any two distinct codewords $\mathbf{c}_1, \mathbf{c}_2$ in $C_4$. Hence, $I_{\max}(C_4) = \frac{1}{q-2}$.

Finally, we prove that the codebook $C_4$ asymptotically meets the Welch bound. The corresponding Welch bound of the codebook $C_4$ is

$$I_w = \sqrt{\frac{N_4 - K_4}{(N_4 - 1)K_4}} = \sqrt{\frac{q^2(q-1) - q(q-2)}{(q^3 - q^2 - 1)q(q-2)}} = \sqrt{\frac{q^2 - 2q + 2}{(q^3 - q^2 - 1)(q-2)}}.$$

It follows that $\lim_{q \longrightarrow \infty} \frac{I_{\max}(C_4)}{I_w} = \lim_{q \longrightarrow \infty} \sqrt{\frac{q^3 - q^2 - 1}{(q-2)(q^2-2q+2)}} = 1$. This completes the proof. □

### 5.2 The specific constructions of optimal codebooks

In this subsection, we study a class of codebooks achieving the Welch bound that can be constructed using quadratic character sums over the local ring $R = \mathbb{F}_q + u\mathbb{F}_q$ $(u^2 = 0)$, where $q = 2^m$.

***E. The fifth construction of codebooks***

Note that $|R^*| = q(q-1)$ and the quadratic character $\rho = \psi_0 \star \chi_a$ $(a \in \mathbb{F}_q^*)$ for $p = 2$. Assume that $\lambda := \chi_b \star \chi_c$ and $t = t_0(1 + ut_1)$, where $b, c, t_1 \in \mathbb{F}_q$ and $t_0 \in \mathbb{F}_q^*$. Let

$$D'' = \{t \in R^* | \rho(t) = -1\} \text{ and } K_5 = |D''|,$$

where $\rho := \psi_0 \star \chi_a$ is the quadratic multiplicative character of $R$ with $a \in \mathbb{F}_q^*$ and $\eta(0)$ is defined as 0 for convenience.

Then the codebook $C_5 := C_5(D'', \widehat{R})$ of length $K_5$ over $R$ is defined by

$$C_5 = \left\{ \frac{1}{\sqrt{K_5}}(\lambda(t))_{t \in D''}, \lambda \in \widehat{R} \right\}.$$

Based on this construction of the codebook $C_5$, we have the following result.

**Theorem 7** *Let $C_5$ be a codebook defined as above. Then $C_5$ is a $(q^2, \frac{q(q-1)}{2})$ codebook having maximum cross-correlation amplitude $I_{\max}(C_5) = \frac{1}{q-1}$. Moreover, the codebook $C_5$ meets the Welch bound.*

***Proof*** In the light of the definition of $C_5$, it is easy to see that $C_5$ has $N_5 = q^2$ codewords of length $K_5 = |D''| = \frac{q(q-1)}{2}$. Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be any two distinct codewords in $C_5$, where $\mathbf{c}_1 = \frac{1}{\sqrt{K_5}}(\lambda_1(t))_{t \in D''}$ and $\mathbf{c}_2 = \frac{1}{\sqrt{K_5}}(\lambda_2(t))_{t \in D''}$. Denote the trivial multiplicative character of $\mathbb{F}_q$ by $\psi_0$. Let $b = b_1 - b_2$ and $c = c_1 - c_2$. Set $\rho := \psi_0 \star \chi_a$ and $\lambda := \chi_b \star \chi_c$. Then the correlation of $\mathbf{c}_1$ and $\mathbf{c}_2$ is as follows.

$$\begin{aligned}
K_5\mathbf{c}_1\mathbf{c}_2^H &= \sum_{t\in D''} \lambda_1(t)\overline{\lambda_2(t)} \\
&= \sum_{t_0\in\mathbb{F}_q^*, t_1\in\mathbb{F}_q} \chi_{b_1}(t_0)\chi_{c_1}(t_0t_1)\overline{\chi_{b_2}(t_0)\chi_{c_2}(t_0t_1)}\frac{1-\psi_0(t_0)\chi_a(t_1)}{2} \\
&= \sum_{t_0\in\mathbb{F}_q^*, t_1\in\mathbb{F}_q} \chi((b_1-b_2)t_0)\chi((c_1-c_2)t_0t_1)\frac{1-\psi_0(t_0)\chi_a(t_1)}{2} \\
&= \sum_{t_0\in\mathbb{F}_q^*, t_1\in\mathbb{F}_q} \chi_b(t_0)\chi_c(t_0t_1)\frac{1-\psi_0(t_0)\chi_a(t_1)}{2} \\
&=\frac{1}{2}\sum_{t_0\in\mathbb{F}_q^*} \chi_b(t_0)\sum_{t_1\in\mathbb{F}_q}\chi_c(t_0t_1) - \frac{1}{2K}G_R(\rho,\lambda).
\end{aligned}$$

Since $\mathbf{c}_1 \neq \mathbf{c}_2$, $b$ and $c$ are not both equal to 0. Then we have

$$\sum_{t_0\in\mathbb{F}_q^*} \chi_b(t_0)\sum_{t_1\in\mathbb{F}_q}\chi_c(t_0t_1) = \begin{cases} -q, & \text{if } b\neq 0 \text{ and } c=0; \\ 0, & \text{if } c\neq 0. \end{cases}$$

In view of Corollary 1, we have

$$G_R(\rho,\lambda) = \begin{cases} q\chi(-\frac{ab}{c}), & \text{if } c\neq 0; \\ 0, & \text{if } c=0. \end{cases}$$

Hence,

$$K_5\mathbf{c}_1\mathbf{c}_2^H = \begin{cases} -\frac{1}{2}q, & \text{if } c=0; \\ -\frac{1}{2}q\chi(-\frac{ab}{c}), & \text{if } c\neq 0. \end{cases}$$

Therefore, we get $|\mathbf{c}_1\mathbf{c}_2^H| = \frac{1}{q-1}$ for any two distinct codewords $\mathbf{c}_1, \mathbf{c}_2$ in $C_5$. Hence, $I_{\max}(C_5) = \frac{1}{q-1}$.

Next, we prove that the codebook $C_5$ asymptotically meets the Welch bound. The corresponding Welch bound of the codebook $C_5$ is

$$I_w = \sqrt{\frac{N_5-K_5}{(N_5-1)K_5}} = \sqrt{\frac{q^2-\frac{1}{2}q(q-1)}{(q^2-1)\frac{1}{2}q(q-1)}} = \frac{1}{q-1}.$$

It follows that $\frac{I_{\max}(C_5)}{I_w} = 1$. Obviously, $C_5$ meets the Welch bound. □

***Remark 4*** In fact, the set $D'' = \{t\in R^* | \rho(t) = -1\}$ is a difference set in $(R,+)$ with parameters $(q^2, \frac{q(q-1)}{2}, \frac{q(q-2)}{4})$, where $q=2^m$. We can easily prove this result by the definition of difference sets. In addition, we will show another way to prove this result by defining the bent function over the ring $R$ as follows.

Firstly, we define the function

$$f: R = \mathbb{F}_{2^m} + u\mathbb{F}_{2^m} \longrightarrow \mathbb{F}_2,$$

$$f(r) = f(r_0 + ur_1) = \begin{cases} 0, & \text{if } r_0 = 0, \\ \mathrm{Tr}_2^{2^m}(\frac{r_1}{r_0}), & \text{if } r_0 \neq 0, \end{cases}$$

for any $r \in R$, then $D''$ as defined above is actually the support of the function $f$ (simply, suppt($f$)), namely, $D'' = \{r \in R | f(r) = 1\} = \mathrm{suppt}(f)$.

It is easy to prove that the function $f$ is bent by the definition of bent functions. Moreover, since [5, Theorem 6.3] says that a function $f$ from $\mathbb{F}_{2^m}$ to $\mathbb{F}_2$ is bent if and only if the support of $f$ is a difference set in $(\mathbb{F}_{2^m}, +)$ with $(2^m, 2^{m-1} \pm 2^{\frac{m-2}{2}}, 2^{m-2} \pm 2^{\frac{m-2}{2}})$. Hence, $D''$ is a difference set in $(R, +)$ with parameters

$$\left(q^2, \frac{q(q-1)}{2}, \frac{q(q-2)}{4}\right). \tag{6}$$

Difference sets with parameters given in (6) are examples of Hadamard difference sets (see [5, Section 6.2.1]).

It is worth noting that Ding and Feng [6, Section A] obtained optimal codebooks from the difference set with parameters $(2^m, 2^{m-1} \pm 2^{\frac{m-2}{2}}, 2^{m-2} \pm 2^{\frac{m-2}{2}})$. The optimal codebook we constructed by using quadratic Gaussian sums of $R$ corresponds to a difference set.

***Remark 5***

1. Let the set $\xi_n$ be the standard basis of the $n$-dimensional Hilbert space which is given by the rows of the identity matrix $I_n$. Let $\widetilde{C}_i = C_i \cup \xi_{K_i}$, where $i = 1, 2, 3, 4$. Then the codebooks $\widetilde{C}_i$ are also asymptotically optimal and their parameters are as follows.

    (i) $\widetilde{N}_1 = N_1 + K_1 = q(q^2 + q + 1), \widetilde{K}_1 = K_1 = q(q-1)$ and $I_{\max}(\widetilde{C}_1) = I_{\max}(C_1) = \frac{1}{q-1}$.
    (ii) $\widetilde{N}_2 = N_2 + K_2 = q(q^2 - 1), \widetilde{K}_2 = K_2 = q(q-1)$ and $I_{\max}(\widetilde{C}_2) = I_{\max}(C_2) = \frac{1}{q-1}$.
    (iii) $\widetilde{N}_3 = N_3 + K_3 = kr^2 + r^2, \widetilde{K}_3 = K_3 = r^2$ and $I_{\max}(\widetilde{C}_3) = I_{\max}(C_3) = \frac{1}{r}$.
    (iv) $\widetilde{N}_4 = N_4 + K_4 = q(q^2 - 2), \widetilde{K}_4 = K_4 = q(q-2)$ and $I_{\max}(\widetilde{C}_4) = I_{\max}(C_4) = \frac{1}{q-2}$.

    The parameters of the codebooks $\widetilde{C}_1, \widetilde{C}_3, \widetilde{C}_4$ are new. The proof of this result is similar to the proof of [27, Theorem 4.1], so we omit the detail here.
2. In Table 1, we list the parameters of some known classes of asymptotically optimal codebooks with respect to the Welch bound. By a comparison, we find that the parameters of codebooks obtained in Theorems 3, 5 and 6 are new.

**Table 1** The parameters of codebooks asymptotically meeting the Welch bound

| $(N, K)$ | $I_{\max}$ | References |
|---|---|---|
| $(N_1N_2, \frac{N_1N_2-1}{2})$, where $N_i \equiv 3 \pmod 4$ for $i=1,2$ | $\frac{\sqrt{(N_1+1)(N_2+1)}}{N_1N_2-1}$ | [11] |
| $(N_1\cdots N_l, \frac{N_1\cdots N_l-1}{2})$, where $N_i \equiv 3 \pmod 4$ for any $l>1$ | $\frac{\sqrt{(N_1+1)\cdots(N_l+1)}}{N_1\cdots N_l-1}$ | [11] |
| $(p^n, \frac{p-1}{2p}(p^n+p^{\frac{n}{2}})+1)$, where $p$ is an odd prime | $\frac{(p+1)p^{\frac{n}{2}}}{2pK}$ | [12] |
| $((q-1)^k+q^{k-1}, q^{k-1})$ for any $k>2$ and $q\ge 4$ | $\frac{\sqrt{q^{k+1}}}{(q-1)^k+(-1)^{k+1}}$ | [13] |
| $((q-1)^k+K, K)$ for any $k>2$, where $K=\frac{(q-1)^k+(-1)^{k+1}}{q}$ | $\frac{\sqrt{q^{k-1}}}{K}$ | [13] |
| $(2K+1, K)$, where $K=\frac{(2^{s_1}-1)^n(2^{s_2}-1)^n-1}{2}, n\ge 1, s_1, s_2>1$ | $\frac{2^{\frac{s_1n+s_2n}{2}}}{2K}$ | [20] |
| $(2K+(-1)^{ln}, K)$, where $K=\frac{(2^{s_1}-1)^n\cdots(2^{s_l}-(-1)^{ln})^n-1}{2}, n\ge 1, l>1, s_i>1$ for any $1\le i\le l$ | $\frac{2^{\frac{s_1n+s_2n+\cdots+s_ln}{2}}}{2K}$ | [20] |
| $((q^s-1)^n+K, K)$ for any $s>1$ and $n>1$, where $K=\frac{(q^s-1)^n+(-1)^{n+1}}{q}$ | $\frac{\sqrt{q^{sn+1}}}{(q^s-1)^n+(-1)^{n+1}}$ | [21] |
| $((q^s-1)^n+q^{sn-1}, q^{sn-1})$ for any $s>1$ and $n>1$ | $\frac{\sqrt{q}+1}{q-1}$ | [21] |
| $(q^3+q^2-q, q^2-q)$ | $\frac{1}{q-1}$ | [22] |
| $(kp^2+p^2, p^2)$, where $k\|(p+1)$ | $\frac{1}{p}$ | [22] |
| $(q(q+4), \frac{q+1}{2})$ | $\frac{\sqrt{q}+1}{q-1}$ | [25] |
| $(q, \frac{(q+3)(q+1)}{2})$ | $\frac{1}{q+1}$ | [25] |
| $(q^3, q^2)$ and $(q^3+q^2, q^2)$ | $\frac{1}{q}$ | [27] |
| $((q-1)q^2, (q-1)q)$ and $(q^2-1, (q-1)q)$ | $\frac{1}{q-1}$ | [27] |
| $((q-1)q^2, (q-1)^2)$ and $(q^3-2q+1, (q-1)^2)$ | $\frac{q}{(q-1)^2}$ | [27] |
| $((q-1)^2q, (q-1)^2)$ and $(q^3-q^2-q+1, (q-1)^2)$ | $\frac{q}{(q-1)^2}$ | [27] |
| $((q-1)^2q, (q-1)(q-2))$ and $(q^3-q^2-2q+2, (q-1)(q-2))$ | $\frac{q}{(q-1)(q-2)}$ | [27] |
| $((q-1)^3, (q-2)^2)$ and $(q^3-2q^2-q+3, (q-2)^2)$ | $\frac{q}{(q-2)^2}$ | [27] |
| $(p^n-1, \frac{p^n-1}{2})$, where $p$ is an odd prime | $\frac{\sqrt{p^n}+1}{p^n-1}$ | [43] |
| $(q^2, \frac{(q-1)^2}{2})$, where $q=p^s$ and $p$ is an odd prime | $\frac{q+1}{(q-1)^2}$ | [44] |
| $(q^l+q^{l-1}-1, q^{l-1})$ for any $l>2$ | $\frac{1}{\sqrt{q^l-1}}$ | [46] |
| $(q^3, q(q-1))$ | $\frac{1}{q-1}$ | Theorem 3 |
| $(q^2(q-1), q(q-1))$ | $\frac{1}{q-1}$ | Theorem 4 |
| $(kr^2, r^2)$, where $q=r^2, k\|(r+1)$ | $\frac{1}{r}$ | Theorem 5 |
| $(q^2(q-1), q(q-2))$ | $\frac{1}{q-2}$ | Theorem 6 |

## 6 Conclusions

In this paper, we describe the additive and multiplicative characters over the finite chain ring $R=\mathbb{F}_q+u\mathbb{F}_q$ ($u^2=0$). We present Gaussian sums, hyper Eisenstein sums and Jacobi sums of $R$ and their applications to the problem of constructing codebooks. The main contributions of this paper are the following:

1. An explicit description on additive characters and multiplicative characters over $R = \mathbb{F}_q + u\mathbb{F}_q$ ($u^2 = 0$) is given in Sect.. 3.
2. Gaussian sums (including quadratic Gaussian sums), hyper Eisenstein sums and Jacobi sums over $R$ are defined in Sect. 4 and some good properties with respect to these character sums are investigated.
3. We firstly establish a relationship between Gaussian sums (resp. Eisenstein sums and Jacobi sums) over $R$ and Gaussian sums (resp. Eisenstein sums and Jacobi sums) over $\mathbb{F}_q$ (see Theorems 1, Theorem 2 and Corollary 5). Moreover, we explore a connection between hyper Eisenstein sums over $R$ and Gaussian sums over $\mathbb{F}_q$ under certain conditions (see Corollary 2).
4. We propose five constructions of codebooks and obtain four families of asymptotically optimal codebooks (see **Constructions *A*, *B*, *C*** and ***D***) and a family of MWBE codebooks (see **Construction *E***). The parameters of codebooks obtained from Constructions $A$, $C$ and $D$ are new.

The codebooks constructed in this paper always have the parameter $N$ less than $K^2$, so the codebooks we constructed can nearly achieve the Welch bound. When $N$ is large, there is no codebook can meet the Welch bound. A new bound, called the Levenshtein bound, is better than the Welch bound when $N$ is large (see, for example, [15, 19, 42]). In [13], Heng et al. obtained asymptotically optimal codebooks with respect to the Levenshtein bound, which are constructed by Jacobi sums over finite fields. In further research, it would be interesting to investigate the applications of new families of asymptotically optimal codebooks meeting the Levenshtein bound by using character sums over finite commutative rings.

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