

A class of cyclotomic linear codes and their generalized Hamming weights

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Received: 18 June 2017 / Revised: 27 March 2018 / Accepted: 29 March 2018 /
Published online: 2 April 2018
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Abstract Firstly, we give a formula on the generalized Hamming weights of linear codes constructed generically by defining sets. Secondly, by choosing properly the defining set we obtain a class of cyclotomic linear codes and then present two alternative formulas for calculating their generalized Hamming weights. Lastly, we determine their weight distributions and generalized Hamming weights partially. Especially, we solve the generalized Hamming weights completely in one case.

Keywords Cyclotomic linear code · Generalized Hamming weight · Weight distribution · Gauss sum · Gaussian period

Mathematics Subject Classification 94B05 · 11T22 · 11T23

1 Introduction

Let $q = p^e$ for a prime p . Denote by $\mathbb{F}_Q = \mathbb{F}_{q^m}$ the finite field with Q elements and $\mathbb{F}_{q^m}^*$ the multiplicative group of \mathbb{F}_{q^m} .

If C is a k -dimensional \mathbb{F}_q -vector subspace of \mathbb{F}_q^n , then it is called an $[n, k, d]$ linear code with length n and minimum Hamming distance d over \mathbb{F}_q . Here the Hamming distance $d(x, y)$ between two codewords $x, y \in C$ is defined as the numbers of places in which x is different from y . And $d = \min\{d(x, 0) | x \in C, x \neq 0\}$ since

This research is supported in part by National Natural Science Foundation of China (61602342).

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C is linear. Denote by A_i the number of codewords with Hamming weight i in C . If $\{|i : A_i \neq 0, 1 \leq i \leq n|\} = t$, then C is called a t -weight code. The readers are referred to [12] for more details and general theory of linear codes.

A generic construction of linear code as below was proposed by Ding et al. [5, 6]. Let $D = \{d_1, d_2, \dots, d_n\}$ be a subset of \mathbb{F}_Q^* . Define a linear code C_D of length n over \mathbb{F}_q as follows:

$$C_D = \{(\text{Tr}_{Q/q}(xd_1), \text{Tr}_{Q/q}(xd_2), \dots, \text{Tr}_{Q/q}(xd_n)) : x \in \mathbb{F}_Q\}, \tag{1}$$

where $\text{Tr}_{Q/q}$ is the standard trace map from \mathbb{F}_Q to \mathbb{F}_q and D is called the defining set. The method is used in a lot of references to get linear codes with a few weights [9, 17, 23, 24] by choosing properly defining sets.

For an $[n, k, d]$ linear code C , we could extend Hamming weight to obtain the concept of the generalized Hamming weight (GHW) $d_r(C)$ ($0 < r \leq k$) (see [15, 20]). It is defined as follows. Denote by $[C, r]_q$ the set of the r -dimensional \mathbb{F}_q -vector subspaces of C . For $V \in [C, r]_q$, let $\text{Supp}(V)$ be the set of positions i where there exists a codeword $x = (x_1, x_2, \dots, x_n) \in V$ with $x_i \neq 0$. Then the r th generalized Hamming weight (GHW) $d_r(C)$ of the linear code C is defined by

$$d_r(C) = \min\{|\text{Supp}(V)| : V \in [C, r]_q\},$$

and $\{d_i(C) : 1 \leq i \leq k\}$ is defined to be the weight hierarchy of C . In particular, the GHW $d_1(C)$ is just the usual minimum distance d . Since the classic results of Wei in the paper [20] in 1991, many people researched into the generalized Hamming weight. A survey on known results on this topic up to 1995 was done in [19]. Afterwards there have been a number of studies on the generalized Hamming weight of some particular families of codes [1–3, 7, 11, 13, 14, 21, 22]. It is worth mentioning that the recent work in [22] gave a very instructive approach to calculating the GHWs of irreducible cyclic codes. Generally, it is not easy to determine the weight hierarchy.

The rest of this paper is organized as follows: in Sect. 2, we review basic concepts and results on Gauss sum and exponential sums which are needed in this paper; in Sect. 3, we follow the work of Ding et al. [8, 10] to construct a class of cyclotomic linear codes and give general formulas on $d_r(C)$. Meanwhile, we determine their weight distribution under certain conditions; in Sect. 4, we give the conclusion of this paper.

2 Preliminaries

We assume that h is a positive divisor of $Q - 1$ and $1 < h < \sqrt{Q} + 1$. And θ is a fixed primitive element of $\mathbb{F}_Q = \mathbb{F}_{q^m}$. We start with the additive character. Let $b \in \mathbb{F}_Q$, the mapping

$$\chi_b(c) = \zeta_p^{\text{Tr}_{Q/p}(bc)} \text{ for all } c \in \mathbb{F}_Q,$$

defines an additive character of \mathbb{F}_Q , where $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$. Particularly, the character χ_1 is called the *canonical additive character* of \mathbb{F}_Q . The *multiplicative characters* of \mathbb{F}_Q are defined by

$$\psi_j(\theta^k) = e^{2\pi\sqrt{-1}jk/(Q-1)} \text{ for } k = 0, 1, \dots, Q - 2, 0 \leq j \leq Q - 2.$$

For each additive χ and multiplicative character ψ , we define the *Gauss sum* $G_Q(\psi, \chi)$ over \mathbb{F}_Q by

$$G_Q(\psi, \chi) = \sum_{x \in \mathbb{F}_Q^*} \psi(x)\chi(x).$$

The reader can refer to [16] for more information about the explicit values of *Gauss sums*.

For each $\alpha \in \mathbb{F}_Q$, an *exponential sum* $S(\alpha)$ is defined as follows.

$$S(\alpha) = \sum_{x \in \mathbb{F}_Q} \chi_1(\alpha x^h).$$

For an integer i , define

$$C_i = \left\{ \theta^i(\theta^h)^j : 0 \leq j < \frac{Q-1}{h} \right\}, \eta_i = \sum_{x \in C_i} \chi_1(x).$$

It is easy to see $C_u = C_v$ if and only if $u \equiv v \pmod{h}$. These sets C_i and numbers η_i are called the *cyclotomic classes* and *Gaussian periods* (see [4]) of order h in \mathbb{F}_Q^* , respectively. By definition, it is not hard to get $S(\theta^i) = h\eta_i + 1$.

The following lemma is about the explicit values of the *exponential sum* $S(\alpha)$. It will be used later.

Lemma 1 ([18]) *Assume $m = 2lk, h|(q^k + 1)$. Then for any $\alpha \in \mathbb{F}_Q^*$,*

$$S(\alpha) = \begin{cases} (-1)^l\sqrt{Q}, & \text{if } \alpha \notin C_{h_0}, \\ (-1)^{l-1}(h-1)\sqrt{Q}, & \text{if } \alpha \in C_{h_0}, \end{cases}$$

where

$$h_0 = \begin{cases} \frac{h}{2}, & \text{if } p > 2, l \text{ odd, and } \frac{q^k+1}{h} \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Here we present three bounds on GHWs of linear codes. The reader may refer to the literature [19] for them.

Lemma 2 *Let C be a linear code over \mathbb{F}_q with parameters $[n, m]$. For $1 \leq r \leq m$,*

1. (Singleton type bound) $r \leq d_r(C) \leq n - m + r$. And C is called an r -MDS code if $d_r(C) = n - m + r$.
2. (Griesmer-like bound)

$$d_r(C) \geq \sum_{i=0}^{r-1} \left\lceil \frac{d_1(C)}{q^i} \right\rceil.$$

3. (Plotkin-like bound)

$$d_r(C) \leq \left\lfloor \frac{n(q^r - 1)q^{m-r}}{q^m - 1} \right\rfloor.$$

3 Main results and proofs

First of all, we give a general formula for computing the GHWs of the linear code defined by the generic method in (1) with the defining set D .

Theorem 1 For each $r(1 \leq r \leq m)$, if the dimension of C_D is m , then $d_r(C_D) = n - \max\{|D \cap H| : H \in [\mathbb{F}_Q, m - r]_q\}$.

Proof The proof is similar to that of Theorem 6 in [22]. But for the convenience of the reader, we provide the proof. Let ϕ be a mapping from \mathbb{F}_Q to \mathbb{F}_q^m defined by

$$\phi(x) = (\text{Tr}_{Q/q}(xd_1), \text{Tr}_{Q/q}(xd_2), \dots, \text{Tr}_{Q/q}(xd_n))$$

for each $x \in \mathbb{F}_Q$. Obviously, ϕ is an \mathbb{F}_q -linear mapping and the image of ϕ is C_D . And ϕ is injective since the dimension of C_D is m . For an r -dimension subspace $U_r \in [C_D, r]_q$, denote by H_r the pre-image $\phi^{-1}(U_r)$ in \mathbb{F}_Q . Also H_r is an r -dimension subspace of \mathbb{F}_Q . By definition, $d_r(C_D) = n - \max\{N(U_r) : U_r \in [C_D, r]_q\}$, where

$$\begin{aligned} N(U_r) &= \#\{i : 1 \leq i \leq n, c_i = 0 \text{ for each } c = (c_1, c_2, \dots, c_n) \in U_r\} \\ &= \#\{i : 1 \leq i \leq n, \text{Tr}_{Q/q}(\beta d_i) = 0 \text{ for each } \beta \in H_r\}. \end{aligned}$$

Let $\{\beta_1, \beta_2, \dots, \beta_r\}$ be an \mathbb{F}_q -basis of H_r . Then

$$\begin{aligned} N(U_r) &= \frac{1}{q^r} \sum_{i=1}^n \sum_{x_1 \in \mathbb{F}_q} \zeta_p^{\text{Tr}_{q/p}(\text{Tr}_{Q/q}(\beta_1 d_i) x_1)} \dots \sum_{x_r \in \mathbb{F}_q} \zeta_p^{\text{Tr}_{q/p}(\text{Tr}_{Q/q}(\beta_r d_i) x_r)} \\ &= \frac{1}{q^r} \sum_{i=1}^n \sum_{x_1, \dots, x_r \in \mathbb{F}_q} \zeta_p^{\text{Tr}_{Q/p}(d_i(\beta_1 x_1 + \dots + \beta_r x_r))} \\ &= \frac{1}{q^r} \sum_{i=1}^n \sum_{\beta \in H_r} \zeta_p^{\text{Tr}_{Q/p}(\beta d_i)}. \end{aligned}$$

Let $H^\perp = \{v \in \mathbb{F}_Q : \text{Tr}_{Q/q}(uv) = 0 \text{ for any } u \in H\}$. It is called the dual of H . We know that $\dim_{\mathbb{F}_q}(H) + \dim_{\mathbb{F}_q}(H^\perp) = m$.

For $y \in \mathbb{F}_Q$,

$$\sum_{\beta \in H_r} \zeta_p^{\text{Tr}_{Q/p}(\beta y)} = \begin{cases} |H_r|, & \text{if } y \in H_r^\perp, \\ 0, & \text{otherwise.} \end{cases}$$

By the above equation, we have

$$N(U_r) = \frac{1}{q^r} \sum_{y \in D \cap H_r^\perp} |H_r| = |D \cap H_r^\perp|.$$

So the desired result follows from the fact that there is a bijection between $[\mathbb{F}_Q, r]_q$ and $[\mathbb{F}_Q, m - r]_q$. We complete the proof.

From now on, we suppose $h(q - 1)$ is also a divisor of $Q - 1$. In [10], C. Ding and H. Niederreiter presented two classes of cyclotomic linear codes of order 3 and determined their weight distributions. Inspired by their work, we construct linear codes by choosing the defining set to be

$$\overline{D} = \{\theta^{t_1} d_1, \dots, \theta^{t_1} d_{n_0}, \theta^{t_2} d_1, \dots, \theta^{t_2} d_{n_0}, \dots, \theta^{t_s} d_1, \dots, \theta^{t_s} d_{n_0}\},$$

where $d_i = \theta^{h(i-1)}$, $n_0 = \frac{q^m - 1}{h(q-1)}$, $0 \leq t_1 < t_2 < \dots < t_s \leq h - 1$, $1 \leq s \leq h$. The code $C_{\overline{D}}$ is closely related to irreducible cyclic codes. Note that $\{d_1, d_2, \dots, d_{n_0}\}$ is a complete set of coset representatives of the quotient group C_0/\mathbb{F}_q^* since $h(q - 1)$ divides $Q - 1$. If $s = 1$, $t_1 = 0$, then $C_{\overline{D}}$ is a linear code punctured from the code C_{C_0} (see [8]). Here C_{C_0} is the code defined in (1) with the defining set $D = C_0$. It is known as the irreducible cyclic code. Thus we also call $C_{\overline{D}}$ a cyclotomic linear code since \overline{D} has relation to the *cyclotomic classes* of order h in \mathbb{F}_Q^* .

In addition to Theorem 1, we give alternative formulas for calculating the GHWs of the cyclotomic linear code $C_{\overline{D}}$.

Theorem 2 For each $r(1 \leq r \leq m)$, if $\dim(C_{\overline{D}}) = m$, then $d_r(C_{\overline{D}}) = sn_0 - N_r$, where

1. $N_r = \frac{s(q^m - q^r)}{hq^r(q-1)} + \frac{1}{hq^r(q-1)} \max\{A_{H_r} : H_r \in [\mathbb{F}_Q, r]_q\}$, and $A_{H_r} = \sum_{j=1}^s \sum_{\lambda=1}^{h-1} \sum_{\beta \in H_r^*} \overline{\phi}^\lambda(\beta \theta^{t_j}) G_Q(\phi^\lambda)$, or
2. $N_r = \frac{sn_0}{q^r} + \frac{1}{q^r(q-1)} \max\{\sum_{i=0}^{h-1} |H_r \cap (\cup_{j=1}^s C_{i-t_j})| \eta_i : H_r \in [\mathbb{F}_Q, r]_q\}$.

Proof 1. By definition, $d_r(C_{\overline{D}}) = sn_0 - N_r$, $N_r = \max\{N(U_r) : U_r \in [C_{\overline{D}}, r]_q\}$. Let $\{\beta_1, \beta_2, \dots, \beta_r\}$ be an \mathbb{F}_q -basis of H_r . Here $\phi(H_r) = U_r$. See the proof of Theorem 1 for the definitions of $N(U_r)$ and ϕ . Set $H_r^* = H_r \setminus \{0\}$. Then

$$\begin{aligned}
 N(U_r) &= \frac{1}{q^r} \sum_{u_i \in \overline{D}} \left(\sum_{x_1 \in \mathbb{F}_q} \zeta_p^{\text{Tr}_{Q/p}(\text{Tr}_{Q/q}(\beta_1 u_i) x_1)} \right) \cdots \left(\sum_{x_r \in \mathbb{F}_q} \zeta_p^{\text{Tr}_{Q/p}(\text{Tr}_{Q/q}(\beta_r u_i) x_r)} \right) \\
 &= \frac{1}{q^r} \sum_{\beta \in H_r} \sum_{u_i \in \overline{D}} \zeta_p^{\text{Tr}_{Q/p}(\beta u_i)} = \frac{sn_0}{q^r} + \frac{1}{q^r} \sum_{\beta \in H_r^*} \sum_{u_i \in \overline{D}} \zeta_p^{\text{Tr}_{Q/p}(\beta u_i)} \\
 &= \frac{sn_0}{q^r} + \frac{1}{q^r(q-1)} \sum_{\beta \in H_r^*} \sum_{u_i \in \mathbb{F}_q^* \overline{D}} \zeta_p^{\text{Tr}_{Q/p}(\beta u_i)},
 \end{aligned}$$

where $\mathbb{F}_q^* \overline{D} = \{xy | x \in \mathbb{F}_q^*, y \in \overline{D}\}$. So

$$\begin{aligned}
 N(U_r) &= \frac{sn_0}{q^r} + \frac{1}{hq^r(q-1)} \sum_{j=1}^s \sum_{\beta \in H_r^*} \sum_{x \in \mathbb{F}_Q^*} \zeta_p^{\text{Tr}_{Q/p}(\beta x)} \sum_{\lambda=0}^{h-1} \varphi^\lambda(\theta^{-t_j} x) \\
 &= \frac{s(q^m - q^r)}{hq^r(q-1)} + \frac{1}{hq^r(q-1)} \sum_{j=1}^s \sum_{\lambda=1}^{h-1} \sum_{\beta \in H_r^*} \sum_{x \in \mathbb{F}_Q^*} \zeta_p^{\text{Tr}_{Q/p}(\beta x)} \varphi^\lambda(\theta^{-t_j} x) \\
 &= \frac{s(q^m - q^r)}{hq^r(q-1)} + \frac{1}{hq^r(q-1)} \sum_{j=1}^s \sum_{\lambda=1}^{h-1} G_Q(\varphi^\lambda) \sum_{\beta \in H_r^*} \overline{\varphi^\lambda}(\theta^{t_j} \beta) \\
 &= \frac{s(q^m - q^r)}{hq^r(q-1)} + \frac{1}{hq^r(q-1)} \sum_{j=1}^s \sum_{\lambda=1}^{h-1} \overline{\varphi^\lambda}(\theta^{t_j}) G_Q(\varphi^\lambda) \sum_{\beta \in H_r^*} \overline{\varphi^\lambda}(\beta).
 \end{aligned}$$

For simplicity, we set $A_{H_r} = \sum_{j=1}^s \sum_{\lambda=1}^{h-1} \overline{\varphi^\lambda}(\theta^{t_j}) G_Q(\varphi^\lambda) \sum_{\beta \in H_r^*} \overline{\varphi^\lambda}(\beta)$. So

$$N(U_r) = \frac{s(q^m - q^r)}{hq^r(q-1)} + \frac{A_{H_r}}{hq^r(q-1)}.$$

2. By the proof of Part 1, we have

$$\begin{aligned}
 N(U_r) &= \frac{sn_0}{q^r} + \frac{1}{q^r(q-1)} \sum_{\beta \in H_r^*} \sum_{u \in \mathbb{F}_q^* \overline{D}} \zeta_p^{\text{Tr}_{Q/p}(\beta u)} \\
 &= \frac{sn_0}{q^r} + \frac{1}{q^r(q-1)} \sum_{\beta \in H_r^*} \sum_{j=1}^s \sum_{u \in C_{t_j}} \zeta_p^{\text{Tr}_{Q/p}(\beta u)} \\
 &= \frac{sn_0}{q^r} + \frac{1}{q^r(q-1)} \sum_{j=1}^s \sum_{\beta \in H_r^*} \sum_{u \in C_{t_j}} \chi_1(\beta u).
 \end{aligned}$$

Table 1 The weight distribution of the codes of Theorem 3

Weight w	Multiplicity A
0	1
$\frac{1}{qh}(sQ + (-1)^l(h-s)\sqrt{Q})$	$\frac{s(Q-1)}{h}$
$\frac{1}{qh}(sQ - s(-1)^l\sqrt{Q})$	$\frac{(h-s)(Q-1)}{h}$

By definition, $\eta_i = \sum_{x \in C_i} \chi_1(x)$. So we have

$$\sum_{j=1}^s \sum_{\beta \in H_r^*} \sum_{u \in C_{t_j}} \chi_1(\beta u) = \sum_{i=0}^{h-1} \sum_{j=1}^s |H_r \cap C_{i-t_j}| \eta_i = \sum_{i=0}^{h-1} |H_r \cap \left(\bigcup_{j=1}^s C_{i-t_j} \right)| \eta_i.$$

Then the desired result follows and the proof is completed.

Remarks (1) If $s = h$, then $C_{\overline{D}}$ is a $[\frac{q^m-1}{q-1}, m, q^{m-1}]$ code, the nonzero elements of which all have weights q^{m-1} . It is a simplex code. By Theorem 1 or Theorem 2(2), it is easy to get $d_r(C_{\overline{D}}) = \frac{q^m - q^{m-r}}{q-1}$.

- (2) By the construction of \overline{D} , for any two elements α and β in \overline{D} , we have $(\frac{\alpha}{\beta})^{q-1} \neq 1$. This means that $\frac{\alpha}{\beta} \notin \mathbb{F}_q^*$. So $\max\{|\overline{D} \cap H| : H \in [\mathbb{F}_Q, 1]_q\} = 1$. By Theorem 1, if $\dim(C_{\overline{D}}) = m$, then $d_{m-1}(C_{\overline{D}}) = \frac{s(q^m-1)}{h(q-1)} - 1$. By the Singleton type bound in Lemma 2, $C_{\overline{D}}$ is an $(m-1)$ -MDS code [19] over \mathbb{F}_q . Especially, if $m = 2$, then the code $C_{\overline{D}}$ is an $[\frac{s(q+1)}{h}, 2, \frac{s(q+1)}{h} - 1]$ MDS code [12] over \mathbb{F}_q .
- (3) Generally, it is difficult to establish linkage between the additive properties and the multiplicative ones of a field. So Theorems 1 and 2 indicate that it is difficult to give the explicit values of the generalized Hamming weights of $C_{\overline{D}}$ in other cases.

Next under certain conditions, we give the weight distributions of the cyclotomic linear codes $C_{\overline{D}}$ in the following theorem.

Theorem 3 Assume $m = 2lk$ and $h|(q^k + 1)$. Then the code $C_{\overline{D}}$ is an $[\frac{s(Q-1)}{h(q-1)}, m]$ linear code over \mathbb{F}_q with the weight distribution in Table 1. And the dual code $C_{\overline{D}}^\perp$ of $C_{\overline{D}}$ is an $[\frac{s(Q-1)}{h(q-1)}, \frac{s(Q-1)}{h(q-1)} - m, d^\perp]$ linear code with minimum distance $d^\perp \geq 3$.

Proof For $x \in \mathbb{F}_q^*$, let $c_x = (\text{Tr}_{Q/q}(xd))_{d \in \overline{D}}$ and $w(c_x)$ denote the Hamming weight of the codeword c_x , then we have

$$\begin{aligned} w(c_x) &= sn_0 - \sum_{j=1}^s \{ |i : 1 \leq i \leq n_0, \text{Tr}_{Q/q}(x\theta^{t_j} d_i) = 0| \} \\ &= sn_0 - \frac{1}{q} \sum_{j=1}^s \sum_{i=1}^{n_0} \sum_{u \in \mathbb{F}_q} \zeta_p^{\text{Tr}_{q/p}(u \text{Tr}_{Q/q}(x\theta^{t_j} d_i))} \end{aligned}$$

$$\begin{aligned}
 &= \frac{s(Q-1)}{hq} - \frac{1}{q} \sum_{j=1}^s \sum_{i=1}^{n_0} \sum_{u \in \mathbb{F}_q^*} \chi_1(ux\theta^{t_j} d_i) \\
 &= \frac{s(Q-1)}{hq} - \frac{1}{q} \sum_{j=1}^s \sum_{k=1}^{\frac{Q-1}{h}} \chi_1(x\theta^{t_j+hk}) \\
 &= \frac{s(Q-1)}{hq} - \frac{1}{qh} \sum_{j=1}^s \sum_{k=1}^{Q-1} \chi_1(x\theta^{t_j+hk}) \\
 &= \frac{s(Q-1)}{hq} - \frac{1}{qh} \sum_{j=1}^s ((S(x\theta^{t_j}) - 1)) \\
 &= \frac{1}{qh} (s(Q-1) + s - \sum_{j=1}^s S(x\theta^{t_j})).
 \end{aligned}$$

By Lemma 1, we have

$$w(c_x) = \begin{cases} \frac{1}{qh}(s(Q-1) + s + (-1)^l(h-s)\sqrt{Q}), & \text{if one of } x\theta^{t_j} \in C_{h_0}, \\ \frac{1}{qh}(s(Q-1) + s - s(-1)^l\sqrt{Q}), & \text{otherwise.} \end{cases}$$

As for the parameters of the dual code, it is enough to prove $d^\perp \geq 3$. It is easy to show that any two elements in \bar{D} are linearly independent over \mathbb{F}_q . Then the desired results follow and we complete the proof.

Example 1 Let $(q, m, l, k, h, s) = (3, 4, 1, 2, 5, 3)$ and $(t_1, t_2, t_3) = (1, 2, 3)$. Then, the corresponding code $C_{\bar{D}}$ has parameters $[24, 4, 15]$, weight enumerator $1 + 48x^{15} + 32x^{18}$ and its dual code has parameters $[24, 20, 3]$.

Example 2 Let $(q, m, l, k, h, s) = (5, 4, 2, 1, 6, 2)$ and $(t_1, t_2) = (0, 1)$. Then, the corresponding code $C_{\bar{D}}$ has parameters $[52, 4, 40]$, weight enumerator $1 + 416x^{40} + 208x^{45}$ and its dual code has parameters $[52, 48, 3]$.

Corollary 1 Assume $m = 2lk$ and $h|(q^k + 1)$. If $(l, 2) = 1$, then

$$d_r(C_{\bar{D}}) = \begin{cases} \frac{s(q^m - q^{m-r}) + (s-h)q^{\frac{m}{2}-r}(q^r - 1)}{h(q-1)}, & \text{if } 1 \leq r \leq \frac{m}{2}, \\ \frac{s(q^m - 1) - h(q^{m-r} - 1)}{h(q-1)}, & \text{if } \frac{m}{2} \leq r \leq m. \end{cases}$$

Proof By Lemma 1, we have $\eta_i = \frac{(h-1)\sqrt{Q}-1}{h}$ if $i = h_0$, otherwise $\eta_i = \frac{-\sqrt{Q}-1}{h}$. So by Theorem 2(2), we get $\sum_{j=1}^s \sum_{i=0}^{h-1} |H_r \cap C_{i-t_j}| \eta_i$

$$\begin{aligned}
 &= \sum_{j=1}^s \sum_{i=0}^{h-1} |H_r \cap C_{i-t_j}| \frac{-\sqrt{Q}-1}{h} + \sum_{j=1}^s |H_r \cap C_{h_0-t_j}| \left(\eta_{h_0} - \frac{-\sqrt{Q}-1}{h} \right) \\
 &= s(q^r - 1) \frac{-\sqrt{Q}-1}{h} + \sqrt{Q} \sum_{j=1}^s |H_r \cap C_{h_0-t_j}|.
 \end{aligned}$$

If $(l, 2) = 1$, then $\mathbb{F}_{q^{lk}} \subset C_0$. Notice that $C_i = \theta^i C_0$ and $\theta^i H_r$ is also an r -dimensional subspace. So we have

$$\max \left\{ |H_r \cap \left(\bigcup_{j=1}^s C_{h_0-t_j} \right)| : H_r \in [\mathbb{F}_Q, r]_q \right\} = q^r - 1$$

for each r with $1 \leq r \leq \frac{m}{2}$. By Theorem 2, we get the first part of the corollary. If $\frac{m}{2} \leq r \leq m$, then $0 \leq m - r \leq \frac{m}{2}$. So we know that there is an $(m - r)$ -dimensional subspace $H_{m-r} \subset \mathbb{F}_{q^{lk}} \subset C_0$. Therefore, $\max\{ |(\bigcup_{j=1}^s C_{t_j}) \cap H| : H \in [\mathbb{F}_Q, m - r]_q \} = q^{m-r} - 1$. Note that $\mathbb{F}_q^* \bar{D} = \bigcup_{j=1}^s C_{t_j}$ and $\frac{\alpha}{\beta} \notin \mathbb{F}_q^*$ for any two elements α and β in \bar{D} . So $|(\bigcup_{j=1}^s C_{t_j}) \cap H| = (q - 1)|H \cap \bar{D}|$ for any subspace H . Therefore, $\max\{ |\bar{D} \cap H| : H \in [\mathbb{F}_Q, m - r]_q \} = \frac{q^{m-r}-1}{q-1}$. By Theorem 1, we get the second part of this corollary. The proof is completed.

Example 3 For the code in Example 1, its weight hierarchy is $d_1 = 15, d_2 = 20, d_3 = 23, d_4 = 24$.

Corollary 2 Also assume $m = 2lk$ and $h|(q^k + 1)$. If $l = 2^u l'$ with $u > 0, (l', 2) = 1$, and $s < h$, then

$$d_r(C_{\bar{D}}) = \begin{cases} \frac{sq \frac{m}{2}^{-r} (q^r - 1)(q^{\frac{m}{2}} - 1)}{h(q-1)}, & \text{if } 1 \leq r \leq l'k, \\ \frac{s(q^m - 1) - h(q^{m-r} - 1)}{h(q-1)}, & \text{if } m - l'k \leq r \leq m. \end{cases}$$

Proof Also by Lemma 1, we have $\eta_i = -\frac{(h-1)\sqrt{Q}+1}{h}$ if $i = h_0$, otherwise $\eta_i = \frac{\sqrt{Q}-1}{h}$. For an r -dimensional subspace H_r ,

$$\begin{aligned}
 \sum_{i=0}^{h-1} |H_r \cap \left(\bigcup_{j=1}^s C_{i-t_j} \right)| \eta_i &= \sum_{j=1}^s \sum_{i=0}^{h-1} |H_r \cap C_{i-t_j}| \eta_i \\
 &= \sum_{j=1}^s \sum_{i=0}^{h-1} |H_r \cap C_{i-t_j}| \frac{\sqrt{Q}-1}{h} + \sum_{j=1}^s |H_r \cap C_{h_0-t_j}| \left(\eta_{h_0} - \frac{\sqrt{Q}-1}{h} \right) \\
 &= s(q^r - 1) \frac{\sqrt{Q}-1}{h} - \sqrt{Q} \sum_{j=1}^s |H_r \cap C_{h_0-t_j}|.
 \end{aligned}$$

By assumption, we have $\mathbb{F}_{q^{l'k}} \subset C_0$. Notice that $\bigcup_{j=1}^s C_{h_0-t_j} \neq \overline{D}$ since $s < h$. So for each r with $1 \leq r \leq l'k$, we have

$$\min \left\{ \left| H_r \cap \left(\bigcup_{j=1}^s C_{h_0-t_j} \right) \right| : H_r \in [\mathbb{F}_Q, r]_q \right\} = 0.$$

Then the desired result of the first part follows directly from Theorem 2(2). For $m - l'k \leq r \leq m$, we know that there is an $(m - r)$ -dimensional subspace $H_{m-r} \subset \mathbb{F}_{q^{l'k}} \subset C_0$. So $\max\{|\left(\bigcup_{j=1}^s C_{t_j}\right) \cap H| : H \in [\mathbb{F}_Q, m - r]_q\} = q^{m-r} - 1$ and $\max\{|\overline{D} \cap H| : H \in [\mathbb{F}_Q, m - r]_q\} = \frac{q^{m-r}-1}{q-1}$. By Theorems 1 and 3, we get the second part of this corollary. The proof is completed.

Example 4 For the code in Example 2, its weight hierarchy is $d_1 = 40, d_2 = 48, d_3 = 51, d_4 = 52$.

The above four examples have been verified by Magma.

4 Concluding remarks

In this paper, we gave a formula for computing the generalized Hamming weights of linear code C_D , which is constructed by the generic method proposed by Ding et al. By choosing properly the defining set, we presented a class of cyclotomic linear codes $C_{\overline{D}}$. We gave two alternative formulas about their generalized Hamming weights in terms of Gauss sums and Gaussian periods. Under certain conditions, we solved the weight distribution of $C_{\overline{D}}$ and proved that it is a two-weight linear code. We determined completely the generalized Hamming weights of $C_{\overline{D}}$ in one case.

Acknowledgements I explicitly acknowledge anonymous reviewers for their valuable suggestions and comments, which have helped improve the quality of the paper. I am also extremely grateful to the editors for their careful considerations and kind help.

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