AAECC (2018) 29:59–76
https://doi.org/10.1007/s00200-017-0329-8

ORIGINAL PAPER

Weight enumerators of a class of linear codes

Jaehyun Ahn[1] · Dongseok Ka[1]

Received: 4 November 2016 / Revised: 12 May 2017 / Accepted: 21 May 2017 /
Published online: 26 May 2017

Abstract Recently, linear codes constructed from defining sets have been studied widely and they have many applications. For an odd prime p, let $q = p^m$ for a positive integer m and Tr_m the trace function from $\mathbb{F}_q$ onto $\mathbb{F}_p$. In this paper, for a positive integer t, let $D \subset \mathbb{F}_q^t$ and $D = \{(x_1, x_2) \in (\mathbb{F}_q^*)^2 : \mathrm{Tr}_m(x_1 + x_2) = 0\}$, we define a p-ary linear code $\mathcal{C}_D$ by

$$\mathcal{C}_D = \left\{\mathbf{c}(a_1, a_2) : (a_1, a_2) \in \mathbb{F}_q^2\right\},$$

where

$$\mathbf{c}(a_1, a_2) = \left(\mathrm{Tr}_m\left(a_1 x_1^2 + a_2 x_2^2\right)\right)_{(x_1, x_2) \in D}.$$

We compute the weight enumerators of the punctured codes $\mathcal{C}_D$.

Keywords Linear codes · Weight distribution · Gauss sums

Mathematics Subject Classification 94B05 · 11T23 · 11T71

J. Ahn was financially supported by a research fund of Chungnam National University in 2015.

✉ Dongseok Ka
dska@cnu.ac.kr

Jaehyun Ahn
jhahn@cnu.ac.kr

1 Chungnam National University, Daejeon 305-764, Korea

1 Introduction

Let $\mathbb{F}_p$ be the finite field with p elements, where p is an odd prime. An $[n, k, d]$ *linear code* $\mathcal{C}$ over $\mathbb{F}_p$ is a k-dimensional subspace of $\mathbb{F}_p^n$ with *minimum distance* d. Let A_i denote the number of codewords with Hamming weight i the code $\mathcal{C}$ of length n. The weight enumerator of $\mathcal{C}$ is defined by $1 + A_1 z + A_2 z^2 + \cdots + A_n z^n$. The sequence $(1, A_1, A_2, \ldots, A_n)$ is called the *weight distribution* of the code $\mathcal{C}$. The weight distribution of the linear code is an important subject in coding theory. However, it is difficult to compute the weight distribution of a linear code in general.

Recently, the weight enumerators of linear codes were studied in [1,2,4–6,9–12,15–18] with the help of exponential sums in some cases. Ahn, Ka and Li [1] defined a class of linear codes as follows. Let $D' = \{(x_1, x_2, \ldots, x_t) \in \mathbb{F}_q^t \backslash \{(0, 0, \ldots, 0)\} : \mathrm{Tr}_m(x_1 + x_2 + \cdots + x_t) = 0\}$. A p-ary linear code $\mathcal{C}_{D'}$ is defined by

$$\mathcal{C}_{D'} = \left\{\mathbf{c}(a_1, a_2, \ldots, a_t) : (a_1, a_2, \ldots, a_t) \in \mathbb{F}_q^t\right\},$$

where

$$\mathbf{c}(a_1, a_2, \ldots, a_t) = \left(\mathrm{Tr}_m\left(a_1 x_1^2 + a_2 x_2^2 + \cdots + a_t x_t^2\right)\right)_{(x_1, x_2, \cdots, x_t) \in D'}.$$

They determined the complete weight enumerators of $\mathcal{C}_{D'}$. Yang and Yao [17] generalized the results of Ahn, Ka and Li [1]. They defined $D_b = \{(x_1, x_2, \ldots, x_t) \in \mathbb{F}_q^t : \mathrm{Tr}_m(x_1 + x_2 + \cdots + x_t) = b\}$ for any $b \in \mathbb{F}_p^*$ and determined the complete weight enumerator of a class of p-ary linear codes given by

$$\mathcal{C}_{D_b} = \left\{\mathbf{c}(a_1, a_2, \ldots, a_t) : (a_1, a_2, \ldots, a_t) \in \mathbb{F}_q^t\right\},$$

where

$$\mathbf{c}(a_1, a_2, \ldots, a_t) = \left(\mathrm{Tr}_m\left(a_1 x_1^2 + a_2 x_2^2 + \cdots + a_t x_t^2\right)\right)_{(x_1, x_2, \cdots, x_t) \in D_b}.$$

In this paper, we define

$$D = \left\{(x_1, x_2) \in \left(\mathbb{F}_q^*\right)^2 : \mathrm{Tr}_m(x_1 + x_2) = 0\right\} \tag{1}$$

and a p-ary linear code $\mathcal{C}_D$ by

$$\mathcal{C}_D = \left\{\mathbf{c}(a_1, a_2) : (a_1, a_2) \in \mathbb{F}_q^2\right\}, \tag{2}$$

where

$$\mathbf{c}(a_1, a_2) = \left(\mathrm{Tr}_m\left(a_1 x_1^2 + a_2 x_2^2\right)\right)_{(x_1, x_2) \in D}.$$

The purpose of this paper is to compute the weight enumerators of the punctured codes $\mathcal{C}_D$.

Minimal linear codes can be used to construct secret sharing schemes with interesting access structures [7,8]. The codes presented in this paper are minimal in the sense of Ding and Yuan [7,8]. We shall explain it at the end of this paper in detail.

2 Preliminaries

Let p be an odd prime and $q = p^m$ for a positive integer m. For any $a \in \mathbb{F}_q$, we can define an additive character of the finite field $\mathbb{F}_q$ as follows:

$$\psi_a : \mathbb{F}_q \longrightarrow \mathbb{C}^*, \psi_a(x) = \zeta_p^{\mathrm{Tr}_m(ax)},$$

where $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$ is a p-th primitive root of unity and Tr_m denotes the trace function from $\mathbb{F}_q$ onto $\mathbb{F}_p$. It is clear that $\psi_0(x) = 1$ for all $x \in \mathbb{F}_q$. Then ψ_0 is called the trivial additive character of $\mathbb{F}_q$. If $a = 1$, we call $\psi := \psi_1$ the canonical additive character of $\mathbb{F}_q$. It is easy to see that $\psi_a(x) = \psi(ax)$ for all $a, x \in \mathbb{F}_q$. The orthogonal property of additive characters is given by

$$\sum_{x \in \mathbb{F}_q} \psi_a(x) = \begin{cases} q, & \text{if } a = 0, \\ 0, & \text{if } a \in \mathbb{F}_q^*. \end{cases}$$

Let $\lambda : \mathbb{F}_q^* \to \mathbb{C}^*$ be a multiplicative character of $\mathbb{F}_q^*$. Now we define the Gauss sum over $\mathbb{F}_q$ by

$$G(\lambda) = \sum_{x \in \mathbb{F}_q^*} \lambda(x)\psi(x).$$

Let $q - 1 = sN$ for two positive integers $s > 1$, $N > 1$ and α be a fixed primitive element of $\mathbb{F}_q$. Let $\langle\alpha^N\rangle$ denote the subgroup of $\mathbb{F}_q^*$ generated by α^N. The *cyclotomic classes* of order N in $\mathbb{F}_q$ are the cosets $C_i^{(N,q)} = \alpha^i\langle\alpha^N\rangle$ for $i = 0, 1, \ldots, N-1$. We know that $|C_i^{(N,q)}| = \frac{q-1}{N}$. The Gaussian periods of order N are defined by

$$\eta_i^{(N,q)} = \sum_{x \in C_i^{(N,q)}} \psi(x).$$

Suppose that η is the quadratic character of $\mathbb{F}_q^*$ and η_p is the quadratic character of $\mathbb{F}_p^*$. For $z \in \mathbb{F}_p^*$, it is easily checked that

$$\eta(z) = \begin{cases} 1, & \text{if } m \text{ is even}, \\ \eta_p(z), & \text{if } m \text{ is odd}. \end{cases} \tag{3}$$

Lemma 1 [3,13] *Suppose that $q = p^m$ where p is an odd prime and $m \geq 1$. Then*

$$G(\eta) = (-1)^{m-1}\sqrt{(p^*)^m} = \begin{cases} (-1)^{m-1}\sqrt{q}, & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{m-1}(\sqrt{-1})^m\sqrt{q}, & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

where $p^* = \left(\frac{-1}{p}\right)p = (-1)^{\frac{p-1}{2}}p$.

Lemma 2 [13] *If q is odd and $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_q[x]$ with $a_2 \neq 0$, then*

$$\sum_{x \in \mathbb{F}_q} \zeta_p^{\mathrm{Tr}_m(f(x))} = \zeta_p^{\mathrm{Tr}_m(a_0 - a_1^2(4a_2)^{-1})}\eta(a_2)G(\eta).$$

Lemma 3 [14] *When $N = 2$, the Gaussian periods are given by*

$$\eta_0^{(2,q)} = \begin{cases} \frac{-1+(-1)^{m-1}\sqrt{q}}{2}, & \text{if } p \equiv 1 \pmod 4, \\ \frac{-1+(-1)^{m-1}(\sqrt{-1})^m\sqrt{q}}{2}, & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

and $\eta_1^{(2,q)} = -1 - \eta_0^{(2,q)}$.

3 Weight enumerators of the linear codes of $\mathcal{C}_D$

In this section, we present the weight distribution of the linear code $\mathcal{C}_D$ defined by (1) and (2), where

$$D = \left\{(x_1, x_2) \in \left(\mathbb{F}_q^*\right)^2 : \mathrm{Tr}_m(x_1 + x_2) = 0\right\}.$$

To get the length of $\mathcal{C}_D$, we need the following lemma.

Lemma 4 *Denote $n_c = |\{x_1, x_2, \in \mathbb{F}_q^* : \mathrm{Tr}_m(x_1 + x_2) = c\}|$ for each $c \in \mathbb{F}_p$. Then*

$$n_c = \begin{cases} \frac{(p^m-1)^2+p-1}{p}, & \text{if } c = 0, \\ \frac{(p^m-1)^2-1}{p}, & \text{if } c \neq 0. \end{cases}$$

Proof By the orthogonal property of additive characters, we have

$$\begin{aligned} n_c &= \sum_{x_1, x_2, \in \mathbb{F}_q^*} \frac{1}{p} \sum_{y \in \mathbb{F}_p} \zeta_p^{y\left(\mathrm{Tr}_m(x_1+x_2)-c\right)} \\ &= \frac{(q-1)^2}{p} + \frac{1}{p} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-yc} \sum_{x_1 \in \mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}_m(yx_1)} \sum_{x_2 \in \mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}_m(yx_2)}. \end{aligned}$$

Thus, we get the desired results. □

By Lemma 4 it is easy to see that the length of $\mathcal{C}_D$ is $n_0 = \frac{(p^m-1)^2+p-1}{p}$.

For a codeword $\mathbf{c}(a_1, a_2)$ of $\mathcal{C}_D$ and $\rho \in \mathbb{F}_p^*$, let $N_0 := N(a_1, a_2)$ be the number of components $\mathrm{Tr}_m(a_1 x_1^2 + a_2 x_2^2)$ of $\mathbf{c}(a_1, a_2)$ which are equal to 0. Then

$$\begin{aligned} N_0 &= \sum_{x_1, x_2 \in \mathbb{F}_q^*} \left(\frac{1}{p} \sum_{y \in \mathbb{F}_p} \zeta_p^{y\mathrm{Tr}_m(x_1+x_2)} \right) \left(\frac{1}{p} \sum_{z \in \mathbb{F}_p} \zeta_p^{z\mathrm{Tr}_m(a_1 x_1^2 + a_2 x_2^2)} \right) \\ &= \frac{1}{p^2} \sum_{x_1, x_2 \in \mathbb{F}_q^*} \left(1 + \sum_{y \in \mathbb{F}_p^*} \zeta_p^{y\mathrm{Tr}_m(x_1+x_2)} \right) \left(1 + \sum_{z \in \mathbb{F}_p^*} \zeta_p^{z\mathrm{Tr}_m(a_1 x_1^2 + a_2 x_2^2)} \right) \\ &= \frac{(p^m-1)^2}{p^2} + \frac{1}{p^2}(\Omega_1 + \Omega_2 + \Omega_3), \end{aligned} \tag{4}$$

where

$$\Omega_1 = \sum_{y \in \mathbb{F}_p^*} \sum_{x_1 \in \mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}_m(yx_1)} \sum_{x_2 \in \mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}_m(yx_2)} = p - 1,$$

$$\Omega_2 = \sum_{z \in \mathbb{F}_p^*} \sum_{x_1 \in \mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}_m(za_1 x_1^2)} \sum_{x_2 \in \mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}_m(za_2 x_2^2)},$$

and

$$\Omega_3 = \sum_{y, z \in \mathbb{F}_p^*} \sum_{x_1 \in \mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}_m(za_1 x_1^2 + yx_1)} \sum_{x_2 \in \mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}_m(za_2 x_2^2 + yx_2)}.$$

We are going to determine the values of Ω_2 and Ω_3 in Lemmas 5 and 6. To simplify formulas, denote $G_i = G(\eta)\eta(a_i)$ for $i \in \{1, 2\}$.

Lemma 5 *If $a_1 = 0$ and $a_2 = 0$, then*

$$\Omega_2 = (p^m - 1)^2 (p - 1).$$

(1) *If m is even, then*

$$\Omega_2 = \begin{cases} (p^m - 1)(p - 1)(G_1 - 1), & \text{if } a_1 \neq 0, a_2 = 0, \\ (p^m - 1)(p - 1)(G_2 - 1), & \text{if } a_1 = 0, a_2 \neq 0, \\ (p - 1)(G_1 G_2 - G_1 - G_2 + 1), & \text{if } a_1 \neq 0, a_2 \neq 0. \end{cases}$$

(2) *If m is odd, then*

$$\Omega_2 = \begin{cases} -(p^m - 1)(p - 1), & \text{if } a_1 \neq 0, a_2 = 0 \text{ or if } a_1 = 0, a_2 \neq 0, \\ (p - 1)(G_1 G_2 + 1), & \text{if } a_1 \neq 0, a_2 \neq 0. \end{cases}$$

Proof When $a_1 = 0$ and $a_2 = 0$, it is obvious that Ω_2 is equal to $(p^m - 1)^2 (p - 1)$.

If $a_1 \neq 0$ and $a_2 = 0$, then by the orthogonal property of additive characters, we have

$$\begin{aligned}\Omega_2 &= \sum_{z\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q^*}\zeta_p^{\mathrm{Tr}_m(za_1x_1^2)}\sum_{x_2\in\mathbb{F}_q^*}1\\ &= (q-1)\sum_{z\in\mathbb{F}_p^*}\left(\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_1x_1^2)}-1\right).\end{aligned}$$

By Lemma 2, we obtain

$$\begin{aligned}\Omega_2 &= (q-1)\sum_{z\in\mathbb{F}_p^*}(G(\eta)\eta(za_1)-1)\\ &= (p^m-1)G_1\sum_{z\in\mathbb{F}_p^*}\eta(z)-(p^m-1)(p-1).\end{aligned}$$

By (3), we get the results. Similarly, we compute the value of Ω_2 when $a_1 = 0$ and $a_2 \neq 0$.

If $a_1 \neq 0$ and $a_2 \neq 0$, then by Lemma 2, we get

$$\begin{aligned}\Omega_2 &= \sum_{z\in\mathbb{F}_p^*}\left(\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_1x_1^2)}-1\right)\left(\sum_{x_2\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_2x_1^2)}-1\right)\\ &= \sum_{z\in\mathbb{F}_p^*}(G(\eta)\eta(za_1)-1)(G(\eta)\eta(za_2)-1)\\ &= G_1G_2\sum_{z\in\mathbb{F}_p^*}\eta(z^2)-G_1\sum_{z\in\mathbb{F}_p^*}\eta(z)-G_2\sum_{z\in\mathbb{F}_p^*}\eta(z)+(p-1).\end{aligned}$$

By (3), we get the results. □

To simplify results, we denote $G(\eta)G(\eta_p)$ by G and $A_i = \eta(a_i)\eta_p(-\mathrm{Tr}_m(a_i^{-1}))$ for $i \in \{1, 2\}$.

Lemma 6 *If $a_1 = 0$ and $a_2 = 0$, then*

$$\Omega_3 = (p-1)^2.$$

Suppose that m is even.
(1) *If $a_1 \neq 0$ and $a_2 = 0$, then*

$$\Omega_3 = \begin{cases} -(p-1)^2G_1+(p-1)^2, & \text{if } \mathrm{Tr}_m(a_1^{-1})=0,\\ (p-1)G_1+(p-1)^2, & \text{if } \mathrm{Tr}_m(a_1^{-1})\neq 0.\end{cases}$$

(2) *If* $a_1 = 0$ *and* $a_2 \neq 0$*, then*

$$\Omega_3 = \begin{cases} -(p-1)^2 G_2 + (p-1)^2, & \text{if } \mathrm{Tr}_m(a_2^{-1}) = 0, \\ (p-1)G_2 + (p-1)^2, & \text{if } \mathrm{Tr}_m(a_2^{-1}) \neq 0. \end{cases}$$

(3) *If* $a_1 \neq 0$ *and* $a_2 \neq 0$*, then*

$$\Omega_3 = \begin{cases} (p-1)^2(G_1G_2 - G_1 - G_2 + 1), \\ \qquad \text{if } \mathrm{Tr}_m(a_1^{-1}) = 0 \text{ and } \mathrm{Tr}_m(a_2^{-1}) = 0, \\ (p-1)(-G_1G_2 + G_1 - (p-1)G_2 + (p-1)), \\ \qquad \text{if } \mathrm{Tr}_m(a_1^{-1}) \neq 0 \text{ and } \mathrm{Tr}_m(a_2^{-1}) = 0, \\ (p-1)(-G_1G_2 + G_2 - (p-1)G_1 + (p-1)), \\ \qquad \text{if } \mathrm{Tr}_m(a_1^{-1}) = 0 \text{ and } \mathrm{Tr}_m(a_2^{-1}) \neq 0, \\ (p-1)((p-1)G_1G_2 + G_1 + G_2 + (p-1)), \\ \qquad \text{if } \mathrm{Tr}_m(a_1^{-1}) \neq 0,\ \mathrm{Tr}_m(a_2^{-1}) \neq 0 \text{ and } \mathrm{Tr}_m(a_1^{-1} + a_2^{-1}) = 0, \\ (p-1)(-G_1G_2 + G_1 + G_2 + (p-1)), \\ \qquad \text{if } \mathrm{Tr}_m(a_1^{-1}) \neq 0,\ \mathrm{Tr}_m(a_2^{-1}) \neq 0 \text{ and } \mathrm{Tr}_m(a_1^{-1} + a_2^{-1}) \neq 0. \end{cases}$$

Suppose that m is odd.
(1) *If* $a_1 \neq 0$ *and* $a_2 = 0$*, then*

$$\Omega_3 = \begin{cases} (p-1)^2, & \text{if } \mathrm{Tr}_m(a_1^{-1}) = 0, \\ -(p-1)GA_1 + (p-1)^2, & \text{if } \mathrm{Tr}_m(a_1^{-1}) \neq 0. \end{cases}$$

(2) *If* $a_1 = 0$ *and* $a_2 \neq 0$*, then*

$$\Omega_3 = \begin{cases} (p-1)^2, & \text{if } \mathrm{Tr}_m(a_2^{-1}) = 0, \\ -(p-1)GA_2 + (p-1)^2, & \text{if } \mathrm{Tr}_m(a_2^{-1}) \neq 0. \end{cases}$$

(3) *If* $a_1 \neq 0$ *and* $a_2 \neq 0$*, then*

$$\Omega_3 = \begin{cases} (p-1)((p-1)G_1G_2 + (p-1)), \\ \qquad \text{if } \mathrm{Tr}_m(a_1^{-1}) = 0 \text{ and } \mathrm{Tr}_m(a_2^{-1}) = 0, \\ (p-1)(-G_1G_2 - GA_1 + (p-1)), \\ \qquad \text{if } \mathrm{Tr}_m(a_1^{-1}) \neq 0 \text{ and } \mathrm{Tr}_m(a_2^{-1}) = 0, \\ (p-1)(-G_1G_2 - GA_2 + (p-1)), \\ \qquad \text{if } \mathrm{Tr}_m(a_1^{-1}) = 0 \text{ and } \mathrm{Tr}_m(a_2^{-1}) \neq 0, \\ (p-1)((p-1)G_1G_2 - GA_1 - GA_2 + (p-1)), \\ \qquad \text{if } \mathrm{Tr}_m(a_1^{-1}) \neq 0,\ \mathrm{Tr}_m(a_2^{-1}) \neq 0 \text{ and } \mathrm{Tr}_m(a_1^{-1} + a_2^{-1}) = 0, \\ (p-1)(-G_1G_2 - GA_1 - GA_2 + (p-1)), \\ \qquad \text{if } \mathrm{Tr}_m(a_1^{-1}) \neq 0,\ \mathrm{Tr}_m(a_2^{-1}) \neq 0 \text{ and } \mathrm{Tr}_m(a_1^{-1} + a_2^{-1}) \neq 0. \end{cases}$$

Proof We only compute the value of Ω_3 for the case $a_1 \neq 0$ and $a_2 \neq 0$. One can compute the other cases similarly. By the orthogonal property of additive characters, we have

$$\begin{aligned}\Omega_3 &= \sum_{y,z\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q^*}\zeta_p^{\mathrm{Tr}_m(za_1x_1^2+yx_1)}\sum_{x_2\in\mathbb{F}_q^*}\zeta_p^{\mathrm{Tr}_m(za_2x_2^2+yx_2)}\\ &= \sum_{y,z\in\mathbb{F}_p^*}\left(\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_1x_1^2+yx_1)}-1\right)\left(\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}_m(za_1x_1^2+yx_1)}-1\right).\end{aligned}$$

By Lemma 2, we obtain

$$\begin{aligned}\Omega_3 = &\sum_{y,z\in\mathbb{F}_p^*}\Big(\zeta_p^{\mathrm{Tr}_m(-y^2((4a_1)^{-1}+(4a_2)^{-1}))}G(\eta)^2\eta(a_1a_2)-\zeta_p^{\mathrm{Tr}_m(-y^2(4a_1)^{-1})}G(\eta)\eta(za_1)\\ &-\zeta_p^{\mathrm{Tr}_m(-y^2(4a_2)^{-1})}G(\eta)\eta(za_2)+1\Big)\\ =& \, G_1G_2\sum_{z\in\mathbb{F}_p^*}\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y^2(4z)^{-1}\mathrm{Tr}_m(a_1^{-1}+a_2^{-1})}-G_1\sum_{z\in\mathbb{F}_p^*}\eta(z)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y^2(4z)^{-1}\mathrm{Tr}_m(a_1^{-1})}\\ &-G_2\sum_{z\in\mathbb{F}_p^*}\eta(z)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y^2(4z)^{-1}\mathrm{Tr}_m(a_2^{-1})}+(p-1)^2. \qquad (5)\end{aligned}$$

If one of $\mathrm{Tr}_m(a_1^{-1})$, $\mathrm{Tr}_m(a_2^{-1})$ and $\mathrm{Tr}_m(a_1^{-1}+a_2^{-1})$ is zero, then it is easy to compute the term corresponding to it. We only consider the case of $\mathrm{Tr}_m(a_1^{-1})\neq 0$, $\mathrm{Tr}_m(a_2^{-1})\neq 0$ and $\mathrm{Tr}_m(a_1^{-1}+a_2^{-1})\neq 0$. Other cases can be computed similarly. From (5) we have

$$\begin{aligned}\Omega_3 = G_1G_2&\sum_{z\in\mathbb{F}_p^*}\left(\sum_{y\in\mathbb{F}_p}\zeta_p^{-y^2(4z)^{-1}\mathrm{Tr}_m(a_1^{-1}+a_2^{-1})}-1\right)\\ &-G_1\sum_{z\in\mathbb{F}_p^*}\eta(z)\left(\sum_{y\in\mathbb{F}_p}\zeta_p^{-y^2(4z)^{-1}\mathrm{Tr}_m(a_1^{-1})}-1\right)\\ &-G_2\sum_{z\in\mathbb{F}_p^*}\eta(z)\left(\sum_{y\in\mathbb{F}_p}\zeta_p^{-y^2(4z)^{-1}\mathrm{Tr}_m(a_2^{-1})}-1\right)+(p-1)^2.\end{aligned}$$

By Lemma 2, we obtain

$$\begin{aligned}\Omega_3 = G_1G_2&\sum_{z\in\mathbb{F}_p^*}(\eta_p(-(4z)^{-1})\eta_p(\mathrm{Tr}_m(a_1^{-1}+a_2^{-1})G(\eta_p)-1)\\ &-G_1\sum_{z\in\mathbb{F}_p^*}\eta(z)(\eta_p(-(4z)^{-1})\eta_p(\mathrm{Tr}_m(a_1^{-1})G(\eta_p)-1)\\ &-G_2\sum_{z\in\mathbb{F}_p^*}\eta(z)(\eta_p(-(4z)^{-1})\eta_p(\mathrm{Tr}_m(a_2^{-1})G(\eta_p)-1)+(p-1)^2\end{aligned}$$

$$
\begin{aligned}
&= G_1 G_2 \sum_{z \in \mathbb{F}_p^*} (\eta_p(z)\eta_p(-\mathrm{Tr}_m(a_1^{-1} + a_2^{-1})G(\eta_p) - 1) \\
&\quad - G_1 \sum_{z \in \mathbb{F}_p^*} \eta(z)(\eta_p(z)\eta_p(-\mathrm{Tr}_m(a_1^{-1})G(\eta_p) - 1) \\
&\quad - G_2 \sum_{z \in \mathbb{F}_p^*} \eta(z)(\eta_p(z)\eta_p(-\mathrm{Tr}_m(a_2^{-1})G(\eta_p) - 1) + (p-1)^2.
\end{aligned}
$$

By (3), we get the result. □

Then by Lemmas 5 and 6, we obtain the values of N_0. To get the frequency of each composition, we need the following lemmas.

Lemma 7 [1, Lemma 3.4] *For any $c \in \mathbb{F}_p$, let*

$$m_c = |\{a \in \mathbb{F}_q^* : \mathrm{Tr}_m(a^{-1}) = c\}|.$$

Then we have

$$m_c = \begin{cases} p^{m-1} - 1, & \text{if } c = 0, \\ p^{m-1}, & \text{if } c \neq 0. \end{cases}$$

Lemma 8 [1, Lemma 3.5] *For any $c \in \mathbb{F}_p$, let*

$$n_{i,c} = |\{a \in \mathbb{F}_q^* : \eta(a) = i \text{ and } \mathrm{Tr}_m(a^{-1}) = c\}|, \quad i \in \{-1, 1\}.$$

(1) *If m is even, then*

$$n_{i,c} = \begin{cases} \frac{1}{2p}(q - p + i(p-1)G(\eta)), & \text{if } c = 0, \\ \frac{1}{2p}(q - iG(\eta)), & \text{if } c \neq 0. \end{cases}$$

(2) *If m is odd, then*

$$n_{i,c} = \begin{cases} \frac{1}{2p}(q - p), & \text{if } c = 0, \\ \frac{1}{2p}(q + i\eta_p(-c)G), & \text{if } c \neq 0. \end{cases}$$

Proof If $c = 0$, then we get the result from [1, Lemma 3.5] with $t = 1$. If $c \neq 0$, then by the orthogonal property of additive characters, we have

$$
\begin{aligned}
n_{1,c} &= \sum_{a \in C_0^{(2,q)}} \frac{1}{p} \sum_{x \in \mathbb{F}_p} \zeta_p^{x(\mathrm{Tr}_m(a^{-1}) - c)} \\
&= \sum_{a \in C_0^{(2,q)}} \frac{1}{p} \left(\sum_{x \in \mathbb{F}_p^*} \zeta_p^{x(\mathrm{Tr}_m(a^{-1}) - c)} + 1 \right)
\end{aligned}
$$

$$= \frac{1}{p}\left(\sum_{x\in\mathbb{F}_p^*}\zeta_p^{-cx}\sum_{a\in C_0^{(2,q)}}\zeta_p^{\mathrm{Tr}_m(a^{-1}x)}+\frac{q-1}{2}\right). \tag{6}$$

Assume that m is even, then 2 divides $\frac{q-1}{p-1}$ and so $\mathbb{F}_p^* \subseteq C_0^{(2,q)}$. By (6) we obtain

$$n_{1,c} = \frac{1}{p}\left(\sum_{x\in\mathbb{F}_p^*}\zeta_p^{-cx}\eta_0^{(2,q)}+\frac{q-1}{2}\right).$$

Thus, we get the results. Also the case of $n_{-1,c}$ is proved similarly.

Now suppose that m is odd, then $\mathbb{F}_p^* = \{\mathbb{F}_p^* \cap C_0^{(2,q)}\} \cup \{\mathbb{F}_p^* \cap C_1^{(2,q)}\}$. i.e.,$|\mathbb{F}_p^* \cap C_0^{(2,q)}| = |\mathbb{F}_p^* \cap C_1^{(2,q)}| = \frac{p-1}{2}$. By (6) we obtain

$$\begin{aligned} n_{1,c} &= \frac{1}{p}\left(\sum_{x\in\mathbb{F}_p^*\cap C_0^{(2,q)}}\zeta_p^{-cx}\sum_{a\in C_0^{(2,q)}}\zeta_p^{\mathrm{Tr}_m(a^{-1}x)}+\sum_{x\in\mathbb{F}_p^*\cap C_1^{(2,q)}}\zeta_p^{-cx}\sum_{a\in C_0^{(2,q)}}\zeta_p^{\mathrm{Tr}_m(a^{-1}x)}+\frac{q-1}{2}\right)\\ &= \frac{1}{p}\left(\sum_{x\in\mathbb{F}_p^*\cap C_0^{(2,q)}}\zeta_p^{-cx}\eta_0^{(2,p)}+\sum_{x\in\mathbb{F}_p^*\cap C_1^{(2,q)}}\zeta_p^{-cx}\eta_1^{(2,p)}+\frac{q-1}{2}\right). \end{aligned}$$

If $-c \in C_0^{(2,p)}$, then we have

$$n_{1,c} = \frac{1}{p}\left(\eta_0^{(2,p)}\eta_0^{(2,q)}+\eta_1^{(2,p)}\eta_1^{(2,q)}+\frac{q-1}{2}\right).$$

If $-c \in C_1^{(2,p)}$, then we have

$$n_{1,c} = \frac{1}{p}\left(\eta_1^{(2,p)}\eta_0^{(2,q)}+\eta_0^{(2,p)}\eta_1^{(2,q)}+\frac{q-1}{2}\right).$$

It is easily checked that $\eta_0^{(2,p)}\eta_0^{(2,q)}+\eta_1^{(2,p)}\eta_1^{(2,q)} = \frac{G+1}{2}$ and $\eta_1^{(2,p)}\eta_0^{(2,q)}+\eta_0^{(2,p)}\eta_1^{(2,q)} = \frac{-G+1}{2}$. Thus, we get the results. Also $n_{-1,c}$ is computed similarly. This completes the proof. □

Lemma 9 [1, Lemma 3.7] *Suppose that m is odd, let*

$$n'_{i,j} = |\{a\in\mathbb{F}_q^* : \eta(a)=i \text{ and } \eta_p(-\mathrm{Tr}_m(a^{-1}))=j\}|,\ i,j\in\{-1,1\}.$$

Then we have

$$n'_{i,j} = \frac{p-1}{4p}(q+ijG).$$

Table 1 The weight distribution of C_D for $m = 2$

Weight	Frequency
0	1
$p(p-1)^2$	$(p+1)(p-1)$
$(p-1)(p^2 \pm p - 1)$	$(p \pm 1)(p-1)$
$(p-1)(p^2 \pm p - 2)$	$\frac{(p^2 \mp 1)(p-1)}{2}$
$(p-1)^2(p+1)$	$2p(p-1)^2$
$(p-1)(p^2-2)$	$p^2(p-1)(p-2)$

Table 2 The weight distribution of C_D for even $m \geq 4$

Weight	Frequency
0	1
$(p-1)\left(p^{2(m-1)} - p^{m-2} \pm p^{\frac{m-4}{2}}(p^m - p)\right)$	$p^m \pm p^{\frac{m-2}{2}}(p-1)$
$(p-1)\left(p^{2(m-1)} - p^{m-2} \pm p^{\frac{3m-4}{2}}\right)$	$(p-1)\left(p^{m-1} \pm p^{\frac{m-2}{2}}\right)$
$(p-1)\left(p^{2(m-1)} - 2p^{m-2} - p^{m-1} \pm 2p^{\frac{m-2}{2}}\right)$	$\frac{\left(p^{m-1}-1 \pm p^{\frac{m-2}{2}}(p-1)\right)^2}{4}$
$(p-1)\left(p^{2(m-1)} - 2p^{m-2} \pm p^{\frac{m-4}{2}}\right)$	$(p-1)\left(p^{2(m-1)} - p^{m-1} \pm p^{\frac{3m-4}{2}}(p-1)\right)$
$(p-1)(p^{2(m-1)} - 2p^{m-2} + p^{m-1})$	$\frac{(p^{m-1}-1)(p^m-1)}{2}$
$(p-1)(p^{2(m-1)} - 2p^{m-2} - p^{m-1})$	$\frac{p^{m-2}(p-1)(p^m+1)}{2}$
$(p-1)(p^{2(m-1)} - 2p^{m-2})$	$p^{2(m-1)}(p-1)(p-2)$

Theorem 1 *Let $\mathcal{C}_D$ be a linear code defined by (1) and (2) where $D = \{(x_1, x_2) \in (\mathbb{F}_q^*)^2 : \mathrm{Tr}_m(x_1 + x_2) = 0\}$. Suppose that m is even. If $m = 2$, then the weight distribution of $\mathcal{C}_D$ is given by Table 1 and the code $\mathcal{C}_D$ has parameters $[\frac{(p^2-1)^2+p-1}{p}, 4, (p-1)(p^2-p-2)]$. If $m \geq 4$, then the weight distribution of $\mathcal{C}_D$ is given by Table 2 and the code $\mathcal{C}_D$ has parameters $[\frac{(p^m-1)^2+p-1}{p}, 2m, (p-1)\big(p^{2(m-1)} - p^{m-2} - p^{\frac{3m-4}{2}}\big)]$.*

Proof Recall that $N_0 = \frac{(q-1)^2}{p^2} + \frac{1}{p^2}(\Omega_1 + \Omega_2 + \Omega_3)$. We employ Lemmas 5 and 6 to compute N_0.

Assume that $a_1 \neq 0$ and $a_2 = 0$.
If $\mathrm{Tr}_m(a_1^{-1}) = 0$, then we obtain

$$N_0 = \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}((q-p)G(\eta)\eta(a_1) + p - q + 1).$$

Thus,

$$N_0 = \begin{cases} \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}((q-p)(G(\eta)-1)+1), & \text{if } \eta(a_1) = 1, \\ \frac{(q-1)^2}{p^2} - \frac{(p-1)}{p^2}((q-p)(G(\eta)+1)-1), & \text{if } \eta(a_1) = -1. \end{cases}$$

Now the frequencies are $n_{1,0}$ and $n_{-1,0}$ in Lemma 8, respectively.
If $\mathrm{Tr}_m(a_1^{-1}) \neq 0$, then we obtain

$$N_0 = \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}(qG(\eta)\eta(a_1) + p - q + 1).$$

Thus,

$$N_0 = \begin{cases} \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}(q\big(G(\eta)-1) + p + 1), & \text{if } \eta(a_1) = 1, \\ \frac{(q-1)^2}{p^2} - \frac{(p-1)}{p^2}(q\big(G(\eta)+1) - p - 1), & \text{if } \eta(a_1) = -1. \end{cases}$$

Now the frequencies are $\sum_{c\in\mathbb{F}_p^*} n_{1,c}$ and $\sum_{c\in\mathbb{F}_p^*} n_{-1,c}$, respectively. If $a_1 = 0$ and $a_2 \neq 0$, then we also have the same weights and the same frequencies with the case of $a_1 \neq 0$ and $a_2 = 0$.

Now, assume that $a_1 \neq 0$ and $a_2 \neq 0$.
If $\mathrm{Tr}_m(a_1^{-1}) = \mathrm{Tr}_m(a_2^{-1}) = 0$, then we obtain

$$N_0 = \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}(pG(\eta)^2\eta(a_1a_2) - pG(\eta)\eta(a_1) - pG(\eta)\eta(a_2) + p + 1).$$

Thus,

$$N_0 = \begin{cases} \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}(pG(\eta)^2 & \\ -2pG(\eta) + p + 1), & \text{if } \eta(a_1) = 1 \text{ and } \eta(a_2) = 1, \\ \frac{(q-1)^2}{p^2} - \frac{(p-1)}{p^2}(pG(\eta)^2 - p - 1), & \text{if } \eta(a_1a_2) = -1, \\ \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}(pG(\eta)^2 & \\ +2pG(\eta) + p + 1), & \text{if } \eta(a_1) = -1 \text{ and } \eta(a_2) = -1. \end{cases}$$

Now the frequencies are $(n_{1,0})^2$, $2n_{1,0}n_{-1,0}$, $(n_{-1,0})^2$, respectively.
If $\mathrm{Tr}_m(a_1^{-1}) \neq 0$ and $\mathrm{Tr}_m(a_2^{-1}) = 0$, then we have

$$N_0 = \frac{(q-1)^2}{p^2} - \frac{(p-1)}{p^2}\big(pG(\eta)\eta(a_2) - p - 1).$$

Thus,

$$N_0 = \begin{cases} \frac{(q-1)^2}{p^2} - \frac{(p-1)}{p^2}\big(pG(\eta) - p - 1\big), & \text{if } \eta(a_2) = 1, \\ \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(pG(\eta) + p + 1\big), & \text{if } \eta(a_2) = -1. \end{cases}$$

Now the frequencies are $\left(\sum_{c\in\mathbb{F}_p^*} m_c\right) n_{1,0}$ and $\left(\sum_{c\in\mathbb{F}_p^*} m_c\right) n_{-1,0}$, in Lemmas 7 and 8, respectively.
If $\mathrm{Tr}_m(a_1^{-1}) = 0$ and $\mathrm{Tr}_m(a_2^{-1}) \neq 0$, then we have the same weights and the same frequencies with the case of $\mathrm{Tr}_m(a_1^{-1}) \neq 0$ and $\mathrm{Tr}_m(a_2^{-1}) = 0$.

If $\mathrm{Tr}_m(a_1^{-1}) \neq 0$, $\mathrm{Tr}_m(a_2^{-1}) \neq 0$ and $\mathrm{Tr}_m(a_1^{-1} + a_2^{-1}) = 0$, then we have

$$N = \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\left(pG(\eta)^2\eta(a_1a_2) + p + 1\right).$$

Thus,

$$N_0 = \begin{cases} \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\left(pG(\eta)^2 + p + 1\right), & \text{if } \eta(a_1a_2) = 1, \\ \frac{(q-1)^2}{p^2} - \frac{(p-1)}{p^2}\left(pG(\eta)^2 - p - 1\right), & \text{if } \eta(a_1a_2) = -1. \end{cases}$$

Now the frequencies are $\sum_{c\in\mathbb{F}_p^*} n_{1,c}^2 + n_{-1,c}^2$ and $2\sum_{c\in\mathbb{F}_p^*} n_{1,c}n_{-1,c}$, respectively.
If $\mathrm{Tr}_m(a_1^{-1}) \neq 0$, $\mathrm{Tr}_m(a_2^{-1}) \neq 0$ and $\mathrm{Tr}_m(a_1^{-1} + a_2^{-1}) \neq 0$, then we have

$$N_0 = \frac{(q-1)}{p^2} + \frac{(p-1)}{p^2}\big(p+1\big).$$

And the frequency is

$$|\{a_1, a_2 \in \mathbb{F}_q^* : \mathrm{Tr}_m(a_1^{-1}) \neq 0,\ \mathrm{Tr}_m(a_2^{-1}) \neq 0,\ \text{and } \mathrm{Tr}_m(a_1^{-1} + a_2^{-1}) \neq 0\}|.$$

It is equal to $T - \sum_{c\in\mathbb{F}_p^*} \big(m_c m_{-c}\big)$, where $T = |\{(a_1, a_2) \in (\mathbb{F}_q^*)^2 : \mathrm{Tr}_m(a_1^{-1}) \neq 0$ and $\mathrm{Tr}_m(a_2^{-1}) \neq 0\}|$. By Lemmas 4, 7 we get

$$T = \left(\sum_{c\in\mathbb{F}_q^*} m_c\right)^2 = \frac{(p-1)^2q^2}{p^2} \text{ and } \sum_{c\in\mathbb{F}_p^*} \big(m_c m_{-c}\big) = \frac{(p-1)q^2}{p^2}.$$

Thus we compute the frequency.

Since the Hamming weight of $\mathbf{c}(a_1, a_2)$ is equal to $W_H(\mathbf{c}(a_1, a_2)) = n_0 - N_0$, we immediately have the desired results. □

Example 1 (1) Let $p = 3$ and $m = 2$. Then $q = 9$ and $n = 22$. By Theorem 1, the code C_D is a [22, 4, 8] linear code. Its weight enumerator is

$$1 + 10x^8 + 4x^{10} + 18x^{14} + 8x^{12} + 24x^{16} + 8x^{20} + 8x^{22},$$

which is checked by Magma.
(2) Let $p = 5$ and $m = 2$. Then $q = 25$ and $n = 116$. By Theorem 1, the code C_D is a [116, 4, 72] linear code. Its weight enumerator is

$$1 + 52x^{72} + 16x^{76} + 24x^{80} + 300x^{92} + 160x^{96} + 48x^{112} + 24x^{116},$$

which is checked by Magma.
(3) Let $p = 3$ and $m = 4$. Then $q = 81$ and $n = 2134$. By Theorem 1, the code C_D is a [2134, 8, 1278] linear code. Its weight enumerator is

$$\begin{aligned} &1 + 48x^{1278} + 32x^{1284} + 100x^{1356} + 738x^{1368} \\ &+ 256x^{1380} + 1080x^{1416} + 1458x^{1422} \\ &+ 1728x^{1428} + 1040x^{1476} + 20x^{1596} + 60x^{1602}, \end{aligned}$$

which is checked by Magma.

Theorem 2 *Let $\mathcal{C}_D$ be a linear code defined by (1) and (2) where $D = \{(x_1, x_2) \in (\mathbb{F}_q^*)^2 : \mathrm{Tr}_m(x_1 + x_2) = 0\}$. Suppose that m is odd and $m \geq 3$. Then the weight distribution of $\mathcal{C}_D$ is given by Table 3 and the code $\mathcal{C}_D$ has parameters* $\left[\frac{(p^m-1)^2+p-1}{p},\ 2m,\ (p-1)\big(p^{2(m-1)} - 2p^{m-2} - p^{m-1} - 2p^{\frac{m-3}{2}}\big)\right]$.

Proof Recall that $N_0 = \frac{(q-1)^2}{p^2} + \frac{1}{p^2}(\Omega_1 + \Omega_2 + \Omega_3)$. We employ Lemmas 5 and 6 to compute N_0.

Suppose that $a_1 \neq 0$ and $a_2 = 0$.
If $\mathrm{Tr}_m(a_1^{-1}) = 0$, then we obtain

$$N_0 = \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(p - q + 1\big).$$

Now the frequency is m_0 in Lemma 7.
If $\mathrm{Tr}_m(a_1^{-1}) \neq 0$, then we obtain

$$N_0 = \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(-GA_1 + p - q + 1\big).$$

Thus,

$$N_0 = \begin{cases} \frac{(q-1)^2}{p^2} - \frac{(p-1)}{p^2}\big(G - p + q - 1\big), & \text{if } A_1 = 1, \\ \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(G + p - q + 1\big), & \text{if } A_1 = -1. \end{cases}$$

Table 3 The weight distribution of C_D for odd $m \geq 3$

Weight	Frequency
0	1
$(p-1)(p^{2(m-1)} - p^{m-2})$	$2(p^{m-1} - 1)$
$(p-1)\left(p^{2(m-1)} - p^{m-2} \pm p^{\frac{m-3}{2}}\right)$	$(p-1)\left(p^{m-1} \pm p^{\frac{m-1}{2}}\right)$
$(p-1)(p^{2(m-1)} - 2p^{m-2} + p^{m-1})$	$\frac{\left(p^{2m-1} - p^{m-1} - p^m + 1\right)}{2}$
$(p-1)(p^{2(m-1)} - 2p^{m-2} - p^{m-1})$	$\frac{(p^{m-1}-1)^2}{2}$
$(p-1)\left(p^{2(m-1)} - 2p^{m-2} \pm p^{\frac{m-3}{2}}\right)$	$(p-1)\left(p^{m-1} \pm p^{\frac{m-1}{2}}\right)(p^{m-1} - 1)$
$(p-1)\left(p^{2(m-1)} - 2p^{m-2} - p^{m-1} \pm 2p^{\frac{m-3}{2}}\right)$	$\frac{p^{m-1}(p-1)\left(p^{\frac{m-1}{2}} \pm 1\right)^2}{4}$
$(p-1)\left(p^{2(m-1)} - 2p^{m-2} \pm 2p^{\frac{m-3}{2}}\right)$	$\frac{p^{m-1}(p-1)(p-2)\left(p^{\frac{m-1}{2}} \pm 1\right)^2}{4}$
$(p-1)(p^{2(m-1)} - 2p^{m-2})$	$\frac{p^{m-1}(p-1)(p-2)(p^{m-1}-1)}{2}$

Now the frequencies are $n'_{1,1} + n'_{-1,-1}$ and $n'_{1,-1} + n'_{-1,1}$ in Lemma 9, respectively. If $a_1 = 0$ and $a_2 \neq 0$, then we have the same values and the same frequencies with the case of $a_1 \neq 0$ and $a_2 = 0$.

Now, assume that $a_1 \neq 0$ and $a_2 \neq 0$.
If $\mathrm{Tr}_m(a_1^{-1}) = \mathrm{Tr}_m(a_2^{-1}) = 0$, then we obtain

$$N_0 = \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\left(pG(\eta)^2\eta(a_1a_2) + p + 1\right).$$

Thus,

$$N_0 = \begin{cases} \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\left(pG(\eta)^2 + p + 1\right), & \text{if } \eta(a_1a_2) = 1, \\ \frac{(q-1)^2}{p^2} - \frac{(p-1)}{p^2}\left(pG(\eta)^2 - p - 1\right), & \text{if } \eta(a_1a_2) = -1. \end{cases}$$

Now the frequencies are $n_{1,0}^2 + n_{-1,0}^2$ and $2n_{1,0}n_{-1,0}$ in Lemma 8, respectively.
If $\mathrm{Tr}_m(a_1^{-1}) \neq 0$ and $\mathrm{Tr}_m(a_2^{-1}) = 0$, then we have

$$N_0 = \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\left(-GA_1 + p + 1\right).$$

Thus,

$$N_0 = \begin{cases} \frac{(q-1)^2}{p^2} - \frac{(p-1)}{p^2}\big(G - p - 1\big), & \text{if } A_1 = 1, \\ \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(G + p + 1\big), & \text{if } A_1 = -1. \end{cases}$$

Now the frequencies are $m_0(n'_{1,1} + n'_{-1,-1})$ and $m_0(n'_{1,-1} + n'_{-1,1})$, respectively. If $\mathrm{Tr}_m(a_1^{-1}) = 0$ and $\mathrm{Tr}_m(a_2^{-1}) \neq 0$, then we have the same values and the same frequencies with the case of $\mathrm{Tr}_m(a_1^{-1}) \neq 0$ and $\mathrm{Tr}_m(a_2^{-1}) = 0$.
If $\mathrm{Tr}_m(a_1^{-1}) \neq 0$, $\mathrm{Tr}_m(a_2^{-1}) \neq 0$ and $\mathrm{Tr}_m(a_1^{-1} + a_2^{-1}) = 0$, then we have

$$N_0 = \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(pG(\eta)^2\eta(a_1a_2) - GA_1 - GA_2 + p + 1\big).$$

Since $\mathrm{Tr}_m(a_1^{-1} + a_2^{-1}) = 0$, we have $A_1A_2 = \eta(a_1a_2)\eta_p(-1)$.
Assume that $p \equiv 1 \pmod 4$. Then,

$$N_0 = \begin{cases} \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(pG(\eta)^2 - 2G + p + 1\big), & \text{if } \eta(a_1a_2) = A_1 = A_2 = 1, \\ \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(pG(\eta)^2 + 2G + p + 1\big), & \text{if } \eta(a_1a_2) = 1 \text{ and } A_1 = A_2 = -1, \\ \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(-pG(\eta)^2 + p + 1\big), & \text{if } \eta(a_1a_2) = A_1A_2 = -1. \end{cases}$$

Now the frequencies are $\sum_{c\in C_0^{(2,p)}} n_{1,c}n_{1,-c} + \sum_{c\in C_1^{(2,p)}} n_{-1,c}n_{-1,-c}$, $\sum_{c\in C_1^{(2,p)}} n_{1,c}n_{1,-c} + \sum_{c\in C_0^{(2,p)}} n_{-1,c}n_{-1,-c}$, $\sum_{c\in C_0^{(2,p)}} n_{1,c}n_{-1,-c} + \sum_{c\in C_1^{(2,p)}} n_{1,c}n_{-1,-c} + \sum_{c\in C_0^{(2,p)}} n_{-1,c}n_{1,-c} + \sum_{c\in C_1^{(2,p)}} n_{-1,c}n_{1,-c}$, in Lemma 8, respectively.
In the case of $p \equiv 3 \pmod 4$, we compute similarly.
If $\mathrm{Tr}_m(a_1^{-1}) \neq 0$, $\mathrm{Tr}_m(a_2^{-1}) \neq 0$ and $\mathrm{Tr}_m(a_1^{-1} + a_2^{-1}) \neq 0$, then we have

$$N_0 = \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(-GA_1 - GA_2 + p + 1\big).$$

Thus,

$$N_0 = \begin{cases} \frac{(q-1)^2}{p^2} - \frac{(p-1)}{p^2}\big(2G - p - 1\big), & \text{if } A_1 = A_2 = 1, \\ \frac{(q-1)^2}{p^2} + \frac{(p-1)}{p^2}\big(2G + p + 1\big), & \text{if } A_1 = A_2 = -1, \\ \frac{(q-1)^2}{p^2} + \frac{(p-1)(p+1)}{p^2}, & \text{if } A_1A_2 = -1. \end{cases}$$

We compute the frequency for the case of $A_1A_2 = -1$. We compute similarly for the other cases. From Lemma 8, the frequency is

$$\begin{aligned}
&\sum_{c\in C_0^{(2,p)}} \frac{1}{2p}\big(\eta_p(c)G+q\big) \sum_{\substack{d\in C_0^{(2,p)}\\ d\neq -c}} \frac{1}{2p}\big(\eta_p(d)G+q\big)\\
&\quad+\sum_{c\in C_0^{(2,p)}} \frac{1}{2p}\big(\eta_p(c)G+q\big) \sum_{\substack{d\in C_1^{(2,p)}\\ d\neq -c}} \frac{1}{2p}\big(-\eta_p(d)G+q\big)\\
&\quad+\sum_{c\in C_1^{(2,p)}} \frac{1}{2p}\big(-\eta_p(c)G+q\big) \sum_{\substack{d\in C_0^{(2,p)}\\ d\neq -c}} \frac{1}{2p}\big(\eta_p(d)G+q\big)\\
&\quad+\sum_{c\in C_1^{(2,p)}} \frac{1}{2p}\big(-\eta_p(c)G+q\big) \sum_{\substack{d\in C_1^{(2,p)}\\ d\neq -c}} \frac{1}{2p}\big(-\eta_p(d)G+q\big),\\
&=\frac{1}{4p^2}(G+q)^2\frac{p-1}{4}(p-1+p-1+p-3+p-3),\\
&=\frac{(p-1)(p-2)}{4p^2}(G+q)^2.
\end{aligned}$$

□

Example 2 (1) Let $p=3$ and $m=3$. Then $q=27$ and $n=226$. By Theorem 2, the code C_D is a [226, 6, 128] linear code. Its weight enumerator is

$$1+18x^{128}+32x^{132}+72x^{136}+18x^{146}+96x^{148}+72x^{150}+192x^{152}$$
$$+84x^{154}+16x^{156}+24x^{158}+104x^{168},$$

which is checked by Magma.
(2) Let $p=5$ and $m=3$. Then $q=125$ and $n=3076$. By Theorem 2, the code C_D is a [3076, 6, 2352] linear code. Its weight enumerator is

$$1+400x^{2352}+288x^{2360}+900x^{2368}+1200x^{2452}+1920x^{2456}+3600x^{2460}$$
$$+2880x^{2464}+2700x^{2468}+80x^{2476}+48x^{2480}+120x^{2484}+1488x^{2560},$$

which is checked by Magma.

4 Concluding remarks

Let w_{min} and w_{max} be the minimum and maximum nonzero weight of linear code C_D, respectively. We recall that if

$$\frac{w_{min}}{w_{max}}>\frac{p-1}{p},$$

then all nonzero codewords of code C_D are minimal (see [8]).

By Theorem 1, we easily check

$$\frac{w_{min}}{w_{max}} = \frac{(p-1)\left(p^{2(m-1)} - p^{m-2} - p^{\frac{3m-4}{2}}\right)}{(p-1)\left(p^{2(m-1)} - p^{m-2} + p^{\frac{3m-4}{2}}\right)} > \frac{p-1}{p},$$

where even $m \geq 4$. Moreover, by Theorem 2 we easily check

$$\frac{w_{min}}{w_{max}} = \frac{(p-1)\left(p^{2(m-1)} - 2p^{m-2} - p^{m-1} - 2p^{\frac{m-3}{2}}\right)}{(p-1)\left(p^{2(m-1)} - 2p^{m-1} + p^{m-1}\right)} > \frac{p-1}{p},$$

where odd $m \geq 3$.
Hence, the linear codes in this paper satisfy $w_{min}/w_{max} > (p-1)/p$ for $m \geq 3$, and can be used to get secret sharing schemes with interesting access structures.

Acknowledgements The authors are very grateful to the editor and the anonymous reviewers for their valuable comments and suggestions to improve the quality of this paper.

References

1. Ahn, J., Ka, D., Li, C.: Complete weight enumerators of a class of linear codes. Des. Codes Cryptogr. **83**, 83–99 (2017)
2. Bae, S., Li, C., Yue, Q.: Some results on two-weight and three-weight linear codes. preprint (2015)
3. Berndt, B., Evans, R., Williams, K.: Gauss and Jacobi Sums. Wiley, New York (1997)
4. Ding, C.: Codes from Difference Sets. World Scientific, Singapore (2014)
5. Ding, C.: Linear codes from some 2-designs. IEEE Trans. Inf. Theory **61**(61), 3265–3275 (2015)
6. Ding, C., Luo, J., Niederreiter, H.: Two-weight codes punctured from irreducible cyclic codes. In: Li, Y., et al. (eds.) Proceedings of the First Worshop on Coding and Cryptography, pp. 119–124. World Scientific, Singapore (2008)
7. Ding, C., Yuan, J.: Covering and secret sharing with linear codes. In: Calude, C.S., Dinneen, M.J., Vajnovszki, V. (eds.) Discrete Mathematics and Theoretical Computer Science, pp. 11–25, Springer, Berlin (2003)
8. Ding, C., Yuan, J.: Secret sharing schemes from three classes of linear codes. IEEE Trans. Inf. Theory **52**(1), 206–212 (2006)
9. Ding, C., Niederreiter, H.: Cyclotomic linear codes of order 3. IEEE Trans. Inf. Theory **53**(6), 2274–2277 (2007)
10. Ding, K., Ding, C.: Binary linear codes with three weights. IEEE Commun. Letters **18**(11), 1879–1882 (2014)
11. Ding, K., Ding, C.: A class of two-weight and three-weight codes and their applications in secret sharing. IEEE Trans. Inf. Theory **61**(11), 5835–5842 (2015)
12. Li, C., Bae, S., Ahn, J., Yang, S., Yao, Z.: Complete weight enumerators some linear codes and their application. Des. Codes Cryptogr. **81**, 153–168 (2016)
13. Lidl, R., Niederreiter, H.: Finite Fields. Addison-Wesley Publishing Inc, Boston (1983)
14. Myerson, G.: Period polynomials and Gauss sums for finite fields. Acta Arith. **39**(3), 251–264 (1981)
15. Tang, C., Li, N., Qi, Y., Zhou, Z., Helleseth, T.: Linear codes with two or three weights from weakly regular bent functions. IEEE Trans. Inf. Theory **62**(3), 1166–1176 (2016)
16. Yang, S., Yao, Z.: Complete weight enumerators of a family of three-weight liner codes. Des. Codes Cryptogr. **82**, 663–674 (2017)
17. Yang, S., Yao, Z.: Complete weight enumerators of a class of linear codes. Discrete Math. **340**, 729–739 (2017)
18. Zhou, Z., Li, N., Fan, C., Helleseth, T.: Linear codes with two or three weights from quadratic Bent functions. Des. Codes Cryptogr. **81**, 283–295 (2016)