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Reconstructing masks from markers in non-distributive lattices

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Abstract In mathematical morphology for sets (binary images), the *geodesic reconstruction* associates to a set called *mask* and a subset of it called *marker*, the union of all connected components of the mask intersected by the marker. It is obtained by iteration of a *geodesic dilation* applied to the marker inside the mask. This operation extends naturally to numerical functions (grey-level images), where it allows us to reconstruct *flat zones*. Considering that the family of images constitutes a complete lattice under some partial order relation, a general theory of geodesic dilations and reconstructions in a complete lattice was given in Ronse and Serra [Fundam Inf 46(4):349–395, 2001]. It relies on the assumption that the lattice is *infinitely supremum distributive*, and it fails for pictorial objects forming a non-distributive lattice. In this paper we give a more general theory of geodesic operations, that can be applied to non-distributive lattices; it is compatible with the previous theory when the lattice is infinitely supremum distributive. We study one particular form of geodesic operator, the *impulsive geodesic dilation*, which gives good results for images with values in a *bundle lattice* (e.g., the lattice of labels and the reference lattice). We also briefly discuss geodesy on the lattice of partitions.

Keywords Mathematical morphology · Geodesic dilation and reconstruction · Non-distributive lattices

1 Introduction

A well-known tool in raster graphics is the so-called "seed-fill": selecting a pixel in a connected zone, then the entire zone marked by that pixel is reconstructed. In

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image processing, mathematical morphology provides a similar (but more elaborate) operation called *geodesic reconstruction*.

Let E be a space provided with a connectivity; for example we can take the Euclidean space $E = \mathbf{R}^n$ with the usual topological connectivity or the connectivity by arcs, or the digital space $E = \mathbf{Z}^n$ provided with an adjacency graph (such as the classical four- and eight-adjacencies on $\mathbf{Z}^2$), and the connectivity by paths arising from that graph. Let $R, S \in \mathscr{P}(E)$ such that $R \subseteq S$; S is called a *mask* and R is called a *marker*. Then the *geodesic reconstruction by dilation* (from marker R in the mask S) is the union of all connected components of S having a non-empty intersection with R.

When the connectivity arises from a graph (for example in the digital case), this mathematical definition admits a constructive version. Let us first recall some terminology. A *dilation* [15,32] is an operator $\delta : \mathscr{P}(E) \to \mathscr{P}(E)$ that commutes with the union: for $X_i \in \mathscr{P}(E)$ $(i \in I)$, we have $\delta\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} \delta(X_i)$; equivalently, for $X \in \mathscr{P}(X)$ we have $\delta(X) = \bigcup_{x\in X} \delta(x)$, where we write $\delta(x)$ for $\delta(\{x\})$. For any dilation δ, the *transpose* of δ is the dilation $\check{\delta}$ defined by:

$$\forall x, y \in E, \qquad y \in \check{\delta}(x) \iff x \in \delta(y). \tag{1}$$

For example, with $E = \mathbf{R}^n$ or $\mathbf{Z}^n$, if δ is the dilation $X \mapsto X \oplus A$ by a structuring element $A \in \mathscr{P}(E)$, $\check{\delta}$ is the dilation $X \mapsto X \oplus \check{A}$ by the transposed structuring element $\check{A} = \{-a \mid a \in A\}$. We say that δ is *symmetrical* if $\check{\delta} = \delta$. Let $\sim$ be a symmetrical (and irreflexive) adjacency relation defining the adjacency graph and the corresponding connectivity. Then every point $p \in E$ has a *neighbourhood*

$$N(p) = \{p\} \cup \{q \in E \mid p \sim q\}. \tag{2}$$

Let $\delta_N : \mathscr{P}(E) \to \mathscr{P}(E)$ be the neighbourhood dilation given by $\delta_N(p) = N(p)$ for each $p \in E$; thus δ_N adds to a set all its neighbouring points:

$$\forall X \in \mathscr{P}(E), \qquad \delta_N(X) = \bigcup_{x\in X} N(x) = X \cup \{y \in E \mid \exists x \in X,\ x \sim y\}. \tag{3}$$

Since the relation $\sim$ is symmetrical, the dilation δ_N is symmetrical. It is also *extensive*: $X \subseteq \delta_N(X)$. Conversely, any extensive and symmetrical dilation δ takes this form δ_N, for the symmetrical and irreflexive adjacency relation $\sim$ given by $x \sim y$ iff $y \in \delta(x)$ and $y \neq x$.

Now for a mask $S \in \mathscr{P}(E)$ and a marker $R \subseteq S$, the geodesic reconstruction by dilation from R in S is the limit

$$\rho_{\delta_N}(S, R) = \bigcup_{n\in\mathbf{N}} R_n$$

of the increasing sequence of sets R_n, $n \in \mathbf{N}$, defined recursively as follows:

$$R_0 = R \quad \text{and} \quad \forall n \in \mathbf{N}, \qquad R_{n+1} = \delta_N(R_n) \cap S.$$

Indeed, the marker is grown iteratively by repeatedly adding its neighbouring points in the mask, until the mask is filled.

From this constructive approach, two generalizations are straightforward: first we can apply it to any dilation on $\mathscr{P}(E)$, next we can generalize it to grey-level images.

Given a dilation δ on $\mathscr{P}(E)$, although the geodesic reconstruction operator ρ_δ cannot in general be expressed in terms of connected components, it has some specific algebraic properties (such as: $\rho_\delta\,(S, \rho_\delta(S, R)) = \rho_\delta\,(\rho_\delta(S, R), R) = \rho_\delta(S, R)$), which we will discuss in detail in the next section. This general point of view led to a new understanding of the notions of connectivity and connected components, which were formalized in the axioms of a *connection* on sets [32] (also called *connectivity class*).

Grey-level images are numerical functions $E \to T$, where T is a set of values representing grey-level intensities. Usually one takes $T = \overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$, $\overline{\mathbf{Z}} = \mathbf{Z} \cup \{-\infty, +\infty\}$, $[a, b] = \{x \in \overline{\mathbf{R}} \mid a \leq x \leq b\}$ (with $a, b \in \overline{\mathbf{R}}$ and $a < b$), or $[a \ldots b] = [a, b] \cap \overline{\mathbf{Z}}$ (with $a, b \in \overline{\mathbf{Z}}$ and $a < b$). Then the set T^E of numerical functions is a complete lattice with the pointwise ordering:

$$F \leq G \quad \iff \quad \forall p \in E, \qquad F(p) \leq G(p),$$

and the pointwise supremum and infimum operations:

$$\bigvee_{i\in I} F_i : E \to T : p \mapsto \sup_{i\in I} F_i(p), \quad \bigwedge_{i\in I} F_i : E \to T : p \mapsto \inf_{i\in I} F_i(p).$$

One calls a *flat zone* of a function $F \in T^E$ a connected component of $F^{-1}(t)$ for some $t \in \{F(p) \mid p \in E\}$, in other words a flat zone of F is a maximal connected subset of E on which F has constant value; then the flat zones of F constitute a partition of E [28,30,38].

Every set dilation $\delta : \mathscr{P}(E) \to \mathscr{P}(E)$ extends to a *flat dilation* [15,22,23] on numerical functions, $\delta^T : T^E \to T^E$ given by:

$$\forall F \in T^E, \qquad \delta^T(F) : E \to T : p \mapsto \delta^T(F)(p) = \sup\{F(q) \mid q \in \check{\delta}(p)\}.$$

Then for a *mask* S and a *marker* R (where $R, S \in T^E$ and $R \leq T$), we define the *geodesic reconstruction by dilation from* R *under* S as

$$\rho_{\delta^T}(S, R) = \bigvee_{n\in\mathbf{N}} R_n, \quad \text{where} \quad \begin{cases} R_0 = R \quad \text{and} \\ \forall n \in \mathbf{N}, \quad R_{n+1} = \delta^T(R_n) \wedge S. \end{cases} \tag{4}$$

When δ is the neighbourhood dilation δ_N, cf. (2,3), $\rho_{\delta^T}(S, R)$ will extend the grey-levels of peaks of R to their neighbourhoods, and so on, with the constraint that the grey-levels must be maintained under those of S, see Fig. 1.

Here $\rho_{\delta_N^T}(S, R)$ does not reconstruct some "connected components" of S, but we have the following behaviour: given a connected set $C \in \mathscr{P}(E)$ on which S has constant value, then $\rho_{\delta_N^T}(S, R)$ will have constant value on C. Thus every flat zone of

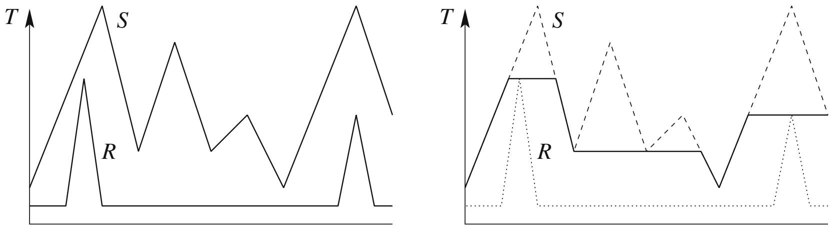


Fig. 1 *Left* the mask S and marker R, where $R \leq S$. *Right* between the mask S (*dashed*) and the marker R (*dotted*), the geodesic reconstruction $\rho_{\delta_N^T}(S, R)$

S is included in a flat zone of $\rho_{\delta_N^T}(S, R)$, in other words the partition made of the flat zones of $\rho_{\delta_N^T}(S, R)$ is coarser than the one of S.

Geodesic reconstruction (for sets and for functions) is used in many image processing tasks, see for example [32,40]. It is in particular at the basis of the theory of *connected operators* (operators which coarsen the flat zone partition) [8–11,29,30,38,39]. It has thus become a fundamental tool of mathematical morphology.

A dilation on T^E [15,32] is an operator $\Delta : T^E \to T^E$ which commutes with the supremum: for $F_i \in T^E$ $(i \in I)$, we have $\Delta\left(\bigvee_{i\in I} F_i\right) = \bigvee_{i\in I} \Delta(F_i)$. For example the flat dilation δ^T (derived from a set dilation δ) is a dilation. One can thus generalize the above method (4) for the geodesic reconstruction to any dilation on T^E. Again, we obtain specific properties for the reconstruction operator ρ_Δ. This led to a general study of geodesic reconstruction for grey-level images [27], in particular on the relation between reconstruction and the connected components of a numerical function (this relies on the notion of connectivity on functions, what one calls a *connection* [27,33]). The theory is easily extended to multivalued functions $E \to T^m$, where one associates to each point a vector of values, with such vectors ordered componentwise:

$$(v_1, \ldots, v_m) \leq (w_1, \ldots, w_m) \iff \forall i = 1, \ldots, m, \quad v_i \leq w_i.$$

In fact, the theory of [27] (independently studied also in [7]) assumes that the images are elements of an arbitrary complete lattice. However the most important results require this lattice to satisfy the *infinite supremum distributivity* law, that the binary infimum distributes arbitrary suprema: $a \wedge \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a \wedge b_i)$. This property is verified by the lattice of subsets of E, the one of numerical functions $E \to T$, and the one of multivalued functions $E \to T^m$.

However there are some types of pictorial objects whose ordering leads to a complete lattice which is not distributive. We give here two examples:

- Partitions are models of image segmentation, and segmentation algorithms can be studied in terms of the operations combining several partitions [37]. Partitions (of a space E) admit a fine-to-coarse ordering: given two partitions π_1 and π_2, we have $\pi_1 \leq \pi_2$ iff every class of π_1 is included in a class of π_2 [32,34]. This ordering leads to a complete lattice which is not distributive.
- Images $E \to V$, where V is a non-distributive complete lattice of values. We will see several examples for V (which are particular cases of what we will call *bundle*

lattices): the lattice of labels [24], the reference ordering on numbers [16,18,19], polar representations of colour, etc.

Both examples deserve a detailed study in separate works, but we will discuss them briefly in this paper. We will see that for such lattices, the standard theory of geodesy [7,27] fails, the properties are not preserved, but also the standard operations lead to practically meaningless results. We are thus facing two tasks:

1. To give a more general theory of geodesic operations, which allows us to obtain the standard properties, even in the non-distributive case.
2. To provide some methods for constructing specific types of geodesic operations, adapted to some types of lattices.

The paper is organized as follows. Section 2 recalls the standard theory of geodesy [7,27], and shows its failure on images with label values. Section 3 is devoted to our first task: introducing a more general theory of geodesy. A key notion is that of a *geodesic map system* (a generalization of the conditional dilation $R \mapsto \delta(R) \cap S$); we also refine the notion of a *geodesic reconstruction system* introduced in [27] (a generalization of the reconstruction operator ρ_δ). Section 4 describes a general method for devising a geodesic map system, the *generated geodesic dilation*, which in the case of images $E \to V$, takes the form of the *impulsive geodesic dilation*. Section 5 discusses briefly geodesic map systems on two examples of non-distributive lattices:

1. The lattice of images $E \to V$, where V is a *bundle lattice*: here the impulsive geodesic dilation leads to meaningful reconstructions.
2. The lattice of partitions: a peculiar form of geodesic dilation is introduced, it gives good results.

Thus we have gone some way in our second task. Section 6 summarizes our results, and links them with other works.

Due to lack of space, we have not presented the complete theory of morphology and geodesy for the above two examples (images with values in a bundle lattice and partitions). This will be the subject further papers.

2 Standard theory of connections, geodesic dilation and reconstruction

We will recall here the main issues in the theory of geodesy on arbitrary complete lattices, given in [27], and independently in [7]. Then we show how it is inadapted to the case of non-distributive lattices.

From now on we assume that the reader is familiar with the rudiments of lattice theory used in mathematical morphology [15,17,26,32]. Fundamental books on lattice theory are [5,14]. We adopt the following notation. We will write elements of a lattice L by lower-case letters $a, \dots, z$, and subsets of that lattice by upper-case letters $A, \dots, Z$; the *least* and *greatest* elements of L will be written $\mathbf{0}$ and $\mathbf{1}$, respectively. We make an exception for the lattice $\mathscr{P}(E)$ of subsets of a set E, and the lattice V^E of functions defined on a set E with values in a set V: here subsets of E and functions $E \to V$ will be written by upper-case letters $A, \dots, Z$, while families of sets or of functions will be written by upper-case calligraphic letters $\mathscr{A}, \dots, \mathscr{Z}$; then lower-case letters $a, \dots, z$ will denote points in E or values in V. The least and greatest elements of $\mathscr{P}(E)$ are $\emptyset$ and E, those of V are written $\perp$ and $\top$, and those of V^E are the constant functions with values $\perp$ and $\top$, written $C_\perp$ and $C_\top$.

We recall first some standard notions. Let L be a complete lattice (with order $\leq$, supremum and infimum operations $\bigvee$ and $\bigwedge$). Following [14], for any $y \in L$ we write

$$(y] = \{x \in L \mid x \leq y\} \quad \text{and} \quad [y) = \{x \in L \mid x \geq y\}. \tag{5}$$

Note that $(y]$ will itself be a complete lattice for the ordering by $\leq$, with the same least element $\mathbf{0}$, the same supremum operation, and the same *non-empty* infimum operation as L, but its greatest element (or empty infimum $\bigwedge \emptyset$) will now be y instead of $\mathbf{1}$. For $M \subseteq L$, let us write $M(y]$ and $M[y)$ for $M \cap (y]$ and $M \cap [y)$, that is:

$$M(y] = \{x \in M \mid x \leq y\} \quad \text{and} \quad M[y) = \{x \in M \mid x \geq y\}. \tag{6}$$

A *lower set* is a subset S of L such that for $s \in S$ and $r \in L$ with $r \leq s$, we have $r \in S$; in other words, for any $s \in S$, $(s] \subseteq S$. For example, for any $y \in L$, $(y]$ is a lower set. A *sup-generating family* of L [17] is a subset G of L such that every element of L is the supremum of some elements of G; in fact for any $x \in L$ we have $x = \bigvee G(x]$.

The lattice L is *distributive* [5] if for every $a, b, c \in L$ we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$; equivalently, for all $a, b, c \in L$ we have $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. A stronger property is the *infinite supremum distributivity* [15]:

$$\forall a \in L, \quad \forall b_i \in L \ (i \in I), \qquad a \wedge \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \wedge b_i), \tag{7}$$

where the family I is arbitrary, and we say then that L is *infinitely supremum distributive*. There is also the dual property called *infinite infimum distributivity* [15]:

$$\forall a \in L, \quad \forall b_i \in L \ (i \in I), \qquad a \vee \left(\bigwedge_{i \in I} b_i \right) = \bigwedge_{i \in I} (a \vee b_i), \tag{8}$$

and we say then that L is *infinitely infimum distributive*. A *complete chain* is a totally ordered complete lattice; it is then both infinitely supremum and infimum distributive. Note that the lattices $\mathscr{P}(E)$ of binary images (sets) and T^E of grey-level images (numerical functions) are both infinitely supremum and infimum distributive; more generally, the same holds for the lattice C^E of functions $E \to C$, where C is a complete chain.

An *operator* on L is a map $L \to L$; operators are generally written by Greek lower-case letters $\alpha, \ldots, \omega$. The set L^L of operators is ordered componentwise:

$$\psi \leq \xi \quad \Longleftrightarrow \quad \forall x \in L, \qquad \psi(x) \leq \xi(x),$$

and this turns it into a complete lattice with componentwise supremum and infimum operations:

$$\bigvee_{i\in I} \psi_i : L \to L : x \mapsto \sup_{i\in I} \psi_i(x), \quad \bigwedge_{i\in I} \psi_i : L \to L : x \mapsto \inf_{i\in I} \psi_i(x).$$

Write **id** for the identity operator $x \to x$. Let ψ be an operator on L. The *domain of invariance* of ψ is the set $\mathrm{Inv}(\psi) = \{x \in L \mid \psi(x) = x\}$. Given a natural integer n, we define ψ^n recursively by $\psi^0 = \mathbf{id}$, $\psi^1 = \psi$, and $\psi^{n+1} = \psi\psi^n$ for $n \in \mathbf{N}$. An operator ψ is *increasing* if $x \leq y$ implies $\psi(x) \leq \psi(y)$, *extensive* [31] if $\psi \geq \mathbf{id}$ (i.e., $\psi(x) \geq x$ for all x), *anti-extensive* if $\psi \leq \mathbf{id}$ (i.e., $\psi(x) \leq x$ for all x), and *idempotent* if $\psi^2 = \psi$. A *closing* is an increasing, extensive and idempotent operator, while an *opening* is an increasing, anti-extensive and idempotent operator [31,32].

A *dilation* is an operator δ that commutes with the supremum operation: $\delta\left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I} \delta(x_i)$; dually, an *erosion* is an operator ε that commutes with the infimum operation: $\varepsilon\left(\bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I} \varepsilon(x_i)$ [32]. An *adjunction* is an ordered pair (ε, δ) such that for any $x, y \in L$, we have $\delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y)$; we say then that ε is the *upper adjoint* of δ and δ is the *lower adjoint* of ε [12,15,17]. In an adjunction (ε, δ), ε is an erosion and δ is a dilation; conversely, every dilation has a unique upper adjoint, and every erosion has a unique lower adjoint.

As an infimum of closings is a closing [15], for any increasing operator ψ, there is a least closing $\geq \psi$, which is characterized below:

Proposition 1 *Let L be a complete lattice.*

1. *For an increasing operator ψ on L, the least closing $\geq \psi$ is the unique closing φ on L such that* $\mathrm{Inv}(\varphi) = \mathrm{Inv}(\mathbf{id} \vee \psi)$, *in other words* [15,26,32]

$$\forall y \in L, \qquad \varphi(y) = y \iff (\mathbf{id} \vee \psi)(y) = y \iff \psi(y) \leq y,$$

   and for $x \in L$, $\varphi(x)$ is the least $y \in \mathrm{Inv}(\varphi)$ such that $y \geq x$.
2. *For a dilation δ on L, the least closing $\geq \delta$ is* [27]

$$\delta^\infty = \mathbf{id} \vee \bigvee_{i=1}^{\infty} \delta^i = \bigvee_{j=1}^{\infty} (\mathbf{id} \vee \delta)^j, \tag{9}$$

   and δ^∞ is also a dilation. When δ is extensive, $\delta = \mathbf{id} \vee \delta$ and (9) *reduces to $\delta^\infty = \bigvee_{i=1}^{\infty} \delta^i$.*

Given E a space of points and V a complete lattice of values, for $B \subseteq E$ and and $v \in V$, the *cylinder of base B and level v* is the function $C_{B,v}$ given by

$$\forall p \in E, \qquad C_{B,v}(p) = \begin{cases} v & \text{if } p \in B, \\ \bot & \text{if } p \notin B. \end{cases} \tag{10}$$

In particular the *impulse* $i_{h,v}$ is the cylinder $C_{\{h\},v}$, thus

$$\forall p \in E, \qquad i_{h,v}(p) = \begin{cases} v & \text{if } p = h, \\ \bot & \text{if } p \neq h. \end{cases} \tag{11}$$

For a function $F : E \to V$, the *support* of F is the subset $\mathrm{supp}(F)$ of E consisting of all points of E having value strictly above the least element $\bot$ of V:

$$\mathrm{supp}(F) = \{p \in E \mid F(p) > \bot\}. \tag{12}$$

After these preliminaries, let us now introduce the theory of geodesy in a complete lattice [27].

Given some $m \in L$ called a *mask* and a dilation δ on L, the *geodesic restriction to* m of δ is the operator $\delta_m : (m] \to (m]$ defined by

$$\forall x \in (m], \quad \delta_m(x) = \delta(x) \wedge m.$$

Some authors [32] call δ_m a *conditional dilation*.

In order for δ_m to be a dilation on $(m]$ (i.e., to commute with the supremum operation), the complete lattice L must be infinitely supremum distributive. Assuming this condition, we can iterate the geodesic dilation δ_m until convergence, in other words we construct

$$\delta_m^\infty = \bigvee_{i=0}^{\infty} \delta_m^i = \bigvee_{j=1}^{\infty} (\mathbf{id}_{(m]} \vee \delta_m)^j,$$

where $\mathbf{id}_{(m]}$ is the identity operator on $(m]$. Then δ_m^∞ is the least closing on $(m]$ which is $\geq \delta_m$, cf. (9).

An element of $(m]$ is usually called a *marker*. For $x \in (m]$, $\delta_m^\infty(x)$ is called the *geodesic reconstruction by dilation* from marker x under the mask m.

There is a dual theory of geodesic erosion and geodesic reconstruction by erosion. Given a mask $m \in L$ and an erosion $\varepsilon : L \to L$, we define the geodesic erosion $\varepsilon_m : [m) \to [m)$ by $\varepsilon_m(x) = \varepsilon(x) \vee m$ for all $x \in [m)$. It is an erosion on $[m)$ (it commutes with the infimum operation) if L is infinitely infimum distributive. Then we can iterate ε_m until convergence, constructing

$$\varepsilon_m^\infty = \bigwedge_{i=0}^{\infty} \varepsilon_m^i = \bigwedge_{j=1}^{\infty} (\mathbf{id}_{[m)} \wedge \varepsilon_m)^j,$$

where $\mathbf{id}_{[m)}$ is the identity operator on $[m)$. When ε_m is anti-extensive, $\varepsilon_m^\infty = \bigwedge_{i=1}^{\infty} \varepsilon_m^i$. Then ε_m^∞ is the greatest opening on $[m)$ which is $\leq \varepsilon_m$.

Calling a *marker* an element of $[m)$, for $x \in [m)$, $\varepsilon_m^\infty(x)$ is called the *geodesic reconstruction by erosion* from marker x above the mask m.

In [27] the notion of geodesic reconstruction received an elegant characterization. Call a *geodesic reconstruction system* a map $\rho : \{(s,r) \in L^2 \mid s \geq r\} \to L$ such that:

– For a fixed $s \in L$, $\rho(s, \cdot) : (s] \to (s] : r \mapsto \rho(s, r)$ is a closing on $(s]$.
– For a fixed $r \in L$, $\rho(\cdot, r) : [r) \to [r) : s \mapsto \rho(s, r)$ is an opening on $[r)$.

Then [27]:

– When L is infinitely supremum distributive, the geodesic reconstruction by dilation $\rho_\delta : (s, r) \mapsto \rho_\delta(s, r) = \delta_s^\infty(r)$ (for $s \geq r$) is a geodesic reconstruction system.
– Dually, when L is infinitely infimum distributive, the geodesic reconstruction by erosion with the order of the arguments exchanged, that is the map $(s, r) \mapsto \rho_\varepsilon(r, s) = \varepsilon_r^\infty(s)$ (for $s \geq r$), is a geodesic reconstruction system on L. Note the exchange of roles of marker and mask, due to duality: s and r are, respectively, mask and marker for the geodesic reconstruction system, but marker and mask for the iterated geodesic erosion.

Now we return to the lattice $\mathscr{P}(E)$ of subsets of a space E, provided with an adjacency relation $\sim$ (for example, $E = \mathbf{Z}^2$ with the four- or eight-adjacency), and consider δ_N, the neighbourhood dilation on $\mathscr{P}(E)$ defined in (2,3), which adds to a set all its neighbouring points. Then the geodesic reconstruction by dilation will fill every connected component of the mask intersecting the marker, that is, for $R \subseteq S$, $\rho_{\delta_N}(S, R)$ is the union of all connected components C of S such that $C \cap R \neq \emptyset$.

In order to generalize this idea to an arbitrary complete lattice, the notion of *connection* was introduced in [33]; we present the version with "canonical markers" studied in [27].

Definition 1 Let L be a complete lattice with least element $\mathbf{0}$ and having a sup-generating family G. A *connection* on L is a class $C \in \mathscr{P}(L)$ satisfying the following three conditions:

1. $\mathbf{0} \in C$;
2. $G \subseteq C$;
3. given $X \subseteq C$ such that $\bigwedge X \neq \mathbf{0}$, we have $\bigvee X \in C$.

A *system of connection openings* on L is a map $G \to L^L$ associating to every $g \in G$ an opening γ_g, such that for every $g, h \in G$ and $x \in L$ we have:

4. $\gamma_g(g) = g$;
5. $\gamma_g(x) \wedge \gamma_h(x) \neq \mathbf{0} \implies \gamma_g(x) = \gamma_h(x)$;
6. $g \not\leq x \implies \gamma_g(x) = \mathbf{0}$.

Here C means the family of connected objects, while for $g \in G$, γ_g is the operator associating to $x \in L$ the connected component of x containing g. There is a one-to-one correspondence between connections on L and systems of connection openings on L:

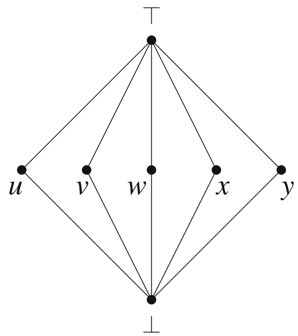
– for $g \in G$ and $x \in L$ with $g \leq x$, $\gamma_g(x)$ is the greatest $c \in C$ such that $g \leq c \leq x$;
– C is the set of $\gamma_g(x)$ for all $g \in G$ and $x \in L$.

For $L = \mathscr{P}(E)$, G is the set of all singletons, and for $p \in E$ and $X \in \mathscr{P}(E)$ we write $\gamma_p(X)$ instead of $\gamma_{\{p\}}(X)$. Connections on $\mathscr{P}(E)$ were already introduced in [32].

Given a connection C on L, we define [7,27] the reconstruction operator ρ_C by setting for $s, r \in L$ with $s \geq r$:

$$\rho_C(s, r) = \bigvee \{\gamma_g(s) \mid g \in G(r]\}. \tag{13}$$

Fig. 2 Hasse diagram of the lattice U of labels with five proper labels u, v, w, x, y, and the two dummy labels $\bot$ and $\top$



In the case of sets, this gives what we had above: the union of all connected components of the mask intersecting the marker.

When L is infinitely supremum distributive, ρ_C is a geodesic reconstruction system on L [7,27]. In fact (see [27, Proofs of Theorem 3.2], [7, Theorem 5.2]), the infinite supremum distributivity property is necessary only for showing the idempotence of the map $\rho_C(s, \cdot) : (s] \to (s] : r \mapsto \rho_C(s, r)$ for $s \in L$.

Let us now show that this theory is inadequate for non-distributive lattices. We will use the example of *label images* [24]. The lattice U of labels is made of a set U_* of symbols called *proper labels*, which are not comparable to each other in terms of order (for $u, u' \in U_*$ with $u \neq u'$, neither $u < u'$ nor $u > u'$ holds), to which one adds, as least and greatest elements, two *dummy labels* $\bot$ and $\top$ (meaning, respectively, "no label" and "conflicting labels"); thus $U = U_* \cup \{\bot, \top\}$. Then the partial order $\leq$ on U is given by: $\bot \leq \top$, and for all $u \in U_*$, $\bot \leq u$, $u \leq u$ and $u \leq \top$; this turns U into a complete lattice. We illustrate in Fig. 2 the *Hasse diagram* [5,15] of this lattice for $|U_*| = 5$.

For $|U_*| \geq 3$, the lattice U is not distributive: given three distinct $a, b, c \in U_*$, we have $a \vee b = \top$ and $a \wedge c = b \wedge c = \bot$, so that

$$(a \vee b) \wedge c = c > \bot = (a \wedge c) \vee (b \wedge c).$$

We call a *label image* a map $E \to U$. Thus for $|U_*| \geq 3$, the lattice U^E of label images is not distributive, and then the geodesic restriction of a dilation will usually not be a dilation. We illustrate this in Fig. 3. Here we take $E = \mathbf{Z}^2$ and the neighbourhood dilation δ on $\mathscr{P}(E)$ corresponding to four-adjacency; in other words δ adds to a set its horizontal and vertical neighbours (2,3). Its *flat extension* on label images [22–24] is the dilation δ^U on U^E such that for any $F \in U^E$ and $p \in E$, $\delta^U(F)(p)$ is the supremum of $F(p)$ and of the $F(q)$ for q in the four-neighbourhood of p. Then, on the pixel located between the markers R_a and R_b, $\delta_S^U(R_a)$ and $\delta_S^U(R_b)$ will give the values $a \wedge c = \bot$ and $b \wedge c = \bot$, while $\delta_S^U(R_a \vee R_b)$ will give the value $(a \vee b) \wedge c = c$. Hence $\delta_S^U(R_a \vee R_b) > \delta_S^U(R_a) \vee \delta_S^U(R_b)$.

Let us now consider the reconstruction by connected components according to (13). Given a connection $\mathscr{C}$ on $\mathscr{P}(E)$, it gives rise to a connection $\mathscr{C}^U$ on U^E [24], made of all *cylinders of connected base and proper level*, that is, cf. (10), the label

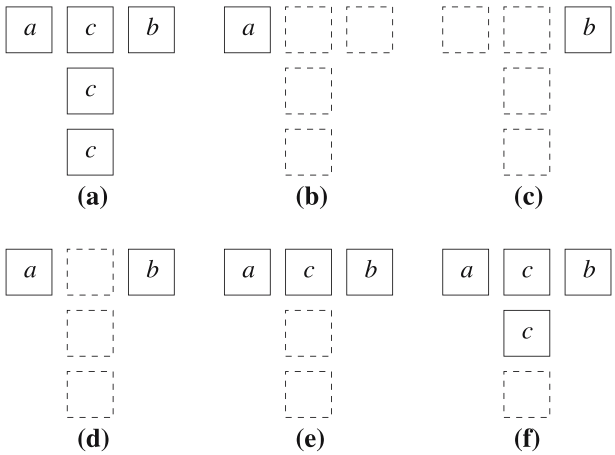


Fig. 3 $E = \mathbf{Z}^2$ and $|U_*| \geq 3$; a pixel is shown as a *square*, with its label value inside it. Let δ be the neighbourhood dilation corresponding to the four-adjacency, and let δ^U be its flat extension on label images. **a** The mask image $S \in U^E$, where a, b, c are pairwise distinct proper labels, and with value $\perp$ outside the support shown here; consider the geodesic dilation $\delta_S^U : (S] \to (S] : G \mapsto \delta^U(G) \wedge S$. **b** The marker image R_a verifies $\delta_S^U(R_a) = R_a$. **c** The marker image R_b verifies $\delta_S^U(R_b) = R_b$. **d** $R_a \vee R_b$. **e** $\delta_S^U(R_a \vee R_b)$, which is greater than $\delta_S^U(R_a) \vee \delta_S^U(R_b) = R_a \vee R_b$. **f** $\left[\delta_S^U\right]^2 (R_a \vee R_b)$; finally $\left[\delta_S^U\right]^3 (R_a \vee R_b) = S$, which is invariant under further applications of δ_S^U

images $C_{B,u}$ for $B \in \mathscr{C}$ and $u \in U_*$. The example in Fig. 4 shows that $\rho_{\mathscr{C}^U}$ is not a geodesic reconstruction system, because for the mask image S (in b), $\rho_{\mathscr{C}^U}(S, \cdot)$ is not idempotent, as for the marker image $R \leq S$ (in c), $\rho_{\mathscr{C}^U}(S, R)$ (in f) is smaller than $\rho_{\mathscr{C}^U}\big(S, \rho_{\mathscr{C}^U}(S, R)\big) = S$. Thus for a fixed mask, the reconstruction from a marker can be further reconstructed!

Is there a solution? In [25], we will study in detail geodesic operations on label images. We can simply say that in Fig. 3 we took a bad definition of the geodesic dilation, and that in Fig. 4 the mask is inappropriate, because of the value $\top$ appearing in it.

The example of label images is the driving impetus behind our study. In Sect. 4, we will define a variant of geodesic dilation, called the *impulsive geodesic dilation*, that will work well in the example of Fig. 3. Now the lattice of labels has some similarity with the lattice of numerical values with the reference ordering [16,18,19], and indeed the impulsive geodesic dilation works correctly for images with values in this lattice. Hence in Sect. 5.1, we will explain that labels and reference ordering are both members of a class of non-distributive lattices that we call *bundle lattices*, to which the impulsive geodesic dilation is well suited, as long as the value $\top$ does not appear in the mask.

In Sect. 5.2 we will see that for partitions, the definition of geodesic dilation must also be modified, but in a different way.

These examples show that we need a general theory of geodesic operations, explaining what goes wrong and what works correctly. This is the topic of the next section.

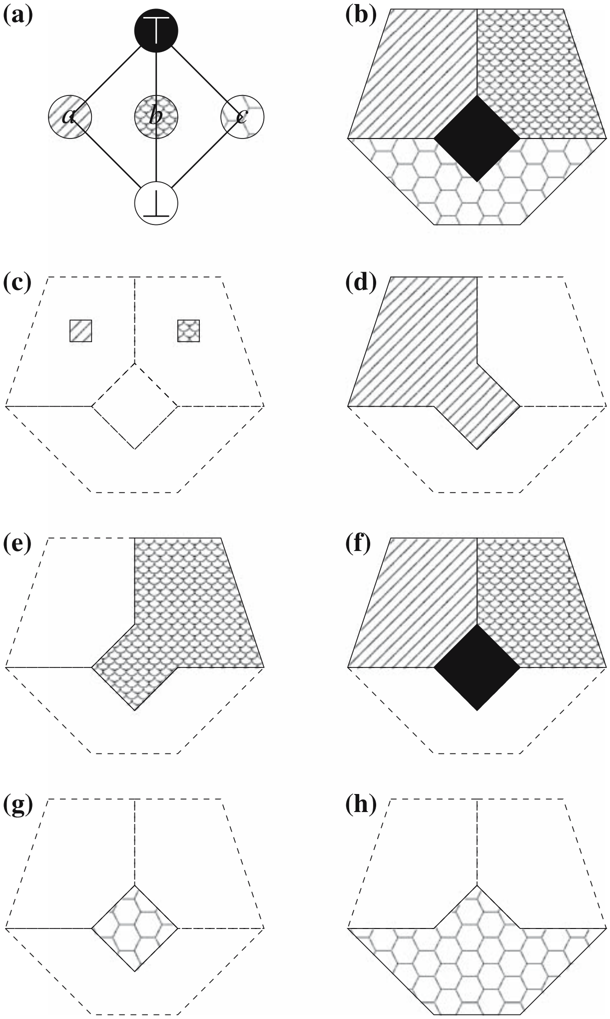


Fig. 4 **a** We show the lattice $U = \{\bot, a, b, c, \top\}$ with a hatching associated to each proper label, plus *black* for $\top$ and *white* for $\bot$. **b** The label image S is the mask (it has value $\bot$ outside the support shown here). **c** The marker R, with two labels a and b. **d** The connected component associated to the subset of R with label a. **e** The connected component associated to the subset of R with label b. **f** The join of these two connected components is the reconstruction $\rho_{\mathscr{C}U}(S, R)$. **g** It majorates a marker with label c. **h** This marker allows the reconstruction of a third connected component. The final reconstruction is the initial mask image: $\rho_{\mathscr{C}U}\left(S, \rho_{\mathscr{C}U}(S, R)\right) = S$

3 Geodesic map and reconstruction systems

As we saw in the preceding section, in a non-distributive complete lattice L, the general definitions of geodesic dilation and reconstruction must be modified in two ways:

– Masks and markers are not always arbitrary elements of L. We will thus assume two subsets $R, S \subseteq L$ of markers and masks, respectively.

– For a mask s and a dilation δ, the geodesic restriction of δ to markers $r \leq s$ will not necessarily be the conditional dilation $r \mapsto \delta(r) \wedge s$.

Recall the notation $(y]$ and $[y)$ from (5), as well as the abbreviations $M(y]$ and $M[y)$ for $M \cap (y]$ and $M \cap [y)$, cf. (6). Let us write RS for $R \cap S$ (the set of markers that are at the same time masks); we will in particular consider sets $RS(y] = R \cap S \cap (y]$. Our considerations lead to the following first definition:

Definition 2 Let L be a complete lattice, let $R, S \subseteq L$ be sets of markers and masks, respectively. A *geodesic map system on* (S, R), written $(\psi_s, s \in S)$, associates to each $s \in S$ a map $\psi_s : R(s] \to RS(s]$ such that for any $r \in R$ and $s, s' \in S$ we have:

1. if $r \leq s' \leq s$, then $\psi_{s'}(r) \leq \psi_s(r)$;
2. if $r \leq s' \leq s$ and $\psi_s(r) \leq s'$, then $\psi_{s'}(r) = \psi_s(r)$.

When $S = R$, we will call the geodesic map system "on S" instead of "on (S, R)".

This definition implicitly hints that for all $s \in S$, $RS(s] \neq \emptyset$ (although we can without problem consider a map $\psi_s : \emptyset \to \emptyset$). Let us notice that the usual definition [27] of the geodesic restriction of an operator, is indeed a geodesic map system:

Proposition 2 *Let L be a complete lattice and let ψ be an operator on L. For every marker $s \in L$, define $\psi_s : (s] \to (s] : r \mapsto \psi(r) \wedge s$. Then $(\psi_s, s \in L)$ is a geodesic map system on L.*

Proof If $r \leq s' \leq s$, then

$$\psi_{s'}(r) = \psi(r) \wedge s' \leq \psi(r) \wedge s = \psi_s(r).$$

If $r \leq s' \leq s$ and $\psi_s(r) \leq s'$, then

$$\psi_s(r) = \psi_s(r) \wedge s' = (\psi(r) \wedge s) \wedge s' = \psi(r) \wedge (s \wedge s') = \psi(r) \wedge s' = \psi_{s'}(r). \quad \square$$

We give now some rules for obtaining new geodesic map systems from existing ones:

Proposition 3 *Let L be a complete lattice, let $R, S \subseteq L$ be sets of markers and masks, respectively.*

1. *Let $RS(s]$ be closed under the supremum operation for all $s \in S$. For every family $(\psi_{i,s}, s \in S)$ $(i \in I)$ of geodesic map systems on (S, R), $\left(\bigvee_{i \in I} \psi_{i,s}, s \in S\right)$ will be a geodesic map system on (S, R).*
2. *If $R \subseteq S$, then $(\mathbf{id}_s, s \in S)$, defined by $\mathbf{id}_s : R(s] \to R(s] : r \mapsto r$, is a geodesic map system on (S, R).*
3. *Given two geodesic map systems $(\psi_s, s \in S)$ and $(\xi_s, s \in S)$ on (S, R), such that for every $s \in S$, ξ_s is increasing and extensive, $(\xi_s \psi_s, s \in S)$ will be a geodesic map system on (S, R).*

Proof 1. For $s \in S$ and $r \in R(s]$, $\psi_{i,s}(r) \in RS(s]$ for all $i \in I$; as $RS(s]$ is closed under the supremum operation, we will have $\bigvee_{i \in I} \psi_{i,s}(r) \in RS(s]$. For $r \leq s' \leq s$ ($r \in R$, $s, s' \in S$), for all $i \in I$ we have $\psi_{i,s'}(r) \leq \psi_{i,s}(r)$ (by item 1 of Definition 2), hence $\bigvee_{i \in I} \psi_{i,s'}(r) \leq \bigvee_{i \in I} \psi_{i,s}(r)$. For $r \leq s' \leq s$ with $\bigvee_{i \in I} \psi_{i,s}(r) \leq s'$, each $i \in I$ verifies $\psi_{i,s}(r) \leq s'$, hence $\psi_{i,s'}(r) = \psi_{i,s}(r)$ (by item 2 of Definition 2), so $\bigvee_{i \in I} \psi_{i,s'}(r) = \bigvee_{i \in I} \psi_{i,s}(r)$. For I empty, all this remains true: $\psi_s : r \mapsto \mathbf{0}$ (for all $s \in S$ and $r \in R(s]$) gives a geodesic map system (since $\mathbf{0} = \bigvee \emptyset \in RS(s]$).

2. As $R \subseteq S$, $\mathbf{id}_s$ is $R(s] \to RS(s]$, and for $r \leq s' \leq s$, we have $\mathbf{id}_{s'}(r) = \mathbf{id}_s(r) = r$.
3. Clearly $\xi_s \psi_s$ is $R(s] \to RS(s]$ for all $s \in S$. Let $r \leq s' \leq s$ ($r \in R$, $s, s' \in S$). Applying item 1 of Definition 2 for ψ_s, we have $\psi_{s'}(r) \leq \psi_s(r)$, and as ξ_s is increasing, $\xi_s(\psi_{s'}(r)) \leq \xi_s(\psi_s(r))$; now $\psi_{s'}(r) \leq s' \leq s$, so applying item 1 of Definition 2 for ξ_s and with $\psi_{s'}(r)$ instead of r, we get $\xi_{s'}(\psi_{s'}(r)) \leq \xi_s(\psi_{s'}(r))$; hence $\xi_{s'}(\psi_{s'}(r)) \leq \xi_s(\psi_{s'}(r)) \leq \xi_s(\psi_s(r))$. If $\xi_s(\psi_s(r)) \leq s'$, as ξ_s is extensive, we get $\psi_s(r) \leq s'$, and item 2 of Definition 2 for ψ_s gives $\psi_{s'}(r) = \psi_s(r)$, so $\xi_s(\psi_{s'}(r)) = \xi_s(\psi_s(r))$; now $\psi_{s'}(r) \leq s' \leq s$ and $\xi_s(\psi_{s'}(r)) = \xi_s(\psi_s(r)) \leq s'$, so applying item 2 of Definition 2 for ξ_s and with $\psi_{s'}(r)$ instead of r, we get $\xi_{s'}(\psi_{s'}(r)) = \xi_s(\psi_{s'}(r))$; hence $\xi_{s'}(\psi_{s'}(r)) = \xi_s(\psi_{s'}(r)) = \xi_s(\psi_s(r))$. □

Let us now slightly modify the definition given in [27] of a geodesic reconstruction system, by specifying the sets of markers and masks:

Definition 3 Let L be a complete lattice, let $R, S \subseteq L$ be sets of *markers* and *masks*, respectively. A *geodesic reconstruction system on* (S, R) is a map $\rho : \{(s, r) \in S \times R \mid s \geq r\} \to RS$ such that:

1. for every $s \in S$, $\rho(s, \cdot) : r \mapsto \rho(s, r)$ is a closing on $R(s]$;
2. for every $r \in R$, $\rho(\cdot, r) : s \mapsto \rho(s, r)$ is an opening on $S[r)$.

When $S = R$, we will call the geodesic reconstruction system "on S" instead of "on (S, R)".

Concretely, the two conditions 1 and 2 mean that for $s \in S$ and $r \in R$ with $s \geq r$,

- $\rho(s, r) \in RS$ and $s \geq \rho(s, r) \geq r$ (in particular, $R(s] \cap S[r) \neq \emptyset$),
- $\rho(s, r)$ is increasing in both s and r,
- $\rho(s, \rho(s, r)) = \rho(\rho(s, r), r) = \rho(s, r)$.

Note that masks and markers play dual roles in a geodesic reconstruction system, that is, the definition remains the same if we invert the order $\leq$ into $\geq$, and at the same time exchange the roles of S and R, writing thus $\rho(r, s)$ instead of $\rho(s, r)$.

We will now give the relation between geodesic reconstruction systems and geodesic map systems:

Theorem 1 *Let L be a complete lattice, let $R, S \subseteq L$ be sets of markers and masks, respectively.*

1. *Given for each $s \in S$ a map $\psi_s : R(s] \to (s]$, the map $\rho : \{(s, r) \in S \times R \mid s \geq r\} \to L$ defined by $\rho(s, r) = \psi_s(r)$, is a geodesic reconstruction system on (S, R) iff $(\psi_s, s \in S)$ is a geodesic map system on (S, R), and for every $s \in S$, ψ_s is a closing on $R(s]$.*

2. *Let $R \subseteq S$ and $R(s]$ be closed under the supremum operation for all $s \in S$. Let $(\psi_s, s \in S)$ be a geodesic map system on (S, R), such that for every $s \in S$, ψ_s is increasing; for every $s \in S$, let φ_s be the least closing on the complete lattice $R(s]$, such that $\psi_s \leq \varphi_s$. Then $(\varphi_s, s \in S)$ is a geodesic map system on (S, R), and the map $\rho : \{(s, r) \in S \times R \mid s \geq r\} \to L$ defined by $\rho(s, r) = \varphi_s(r)$, is a geodesic reconstruction system on (S, R).*

Proof 1. Clearly ρ is defined on the set $\{(s, r) \in S \times R \mid s \geq r\}$, with $\rho(s, r) \in (s]$. In order to be a geodesic reconstruction system on (S, R), the following conditions must be verified for $(s, r) \in S \times R$ with $s \geq r$:

– $\rho(s, r) \in RS$: that is, $\psi_s(r) \in RS$, so ψ_s must be $R(s] \to RS(s]$.
– $s \geq \rho(s, r) \geq r$: we already have $\psi_s(r) \in (s]$, that is $s \geq \psi_s(r)$, so it is required that $\psi_s(r) \geq r$, in other words ψ_s must be extensive.
– For $s' \in S$ with $s \geq s' \geq r$, $\rho(s, r) \geq \rho(s', r)$: that is, $\psi_s(r) \geq \psi_{s'}(r)$, in other words item 1 of Definition 2.
– For $r' \in R$ with $s \geq r' \geq r$, $\rho(s, r') \geq \rho(s, r)$: that is, $\psi_s(r') \geq \psi_s(r)$, in other words ψ_s must be increasing.
– $\rho\left(s, \rho(s, r)\right) = \rho(s, r)$: that is, $\psi_s(\psi_s(r)) = \psi_s(r)$, in other words ψ_s must be idempotent.
– $\rho\left(\rho(s, r), r\right) = \rho(s, r)$: that is, $\psi_{\psi_s(r)}(r) = \psi_s(r)$; but by item 1 of Definition 2 (obtained in the third item above), for $s' \in S$ with $s \geq s' \geq r$ and $s' \geq \psi_s(r)$, we have then $\psi_s(r) \geq \psi_{s'}(r) \geq \psi_{\psi_s(r)}(r) = \psi_s(r)$, thus $\psi_s(r) = \psi_{s'}(r)$, in other words item 2 of Definition 2.

Collecting these requirements, they mean that $(\psi_s, s \in S)$ is a geodesic map system on (S, R), and for every $s \in S$, ψ_s is a closing on $R(s]$.

2. We have $R \subseteq S$, so $R = RS$, and for every $s \in S$, $R(s] = RS(s]$ is closed under the supremum operation. Applying items 1 and 2 of Proposition 3, $(\mathbf{id}_s \vee \psi_s, s \in S)$ is a geodesic map system on (S, R), where for all $s \in S$, $\mathbf{id}_s \vee \psi_s$ is increasing and extensive, and $\mathbf{id}_s \vee \psi_s \leq \varphi_s$. Consider the family $(\psi_{i,s}, s \in S)$ $(i \in I)$ of all geodesic map systems on (S, R) such that for every $i \in I$ and $s \in S$, $\psi_{i,s}$ is increasing and $\mathbf{id}_s \vee \psi_s \leq \psi_{i,s} \leq \varphi_s$ (so $\psi_{i,s}$ is extensive); this family contains $(\mathbf{id}_s \vee \psi_s, s \in S)$. Setting $\eta_s = \bigvee_{i \in I} \psi_{i,s}$, by item 1 of Proposition 3, $(\eta_s, s \in S)$ is a geodesic map system on (S, R), and clearly η_s is increasing and extensive and $\mathbf{id}_s \vee \psi_s \leq \eta_s \leq \varphi_s$; by item 3 of Proposition 3, $(\eta_s \eta_s, s \in S)$ is a geodesic map system on (S, R), with $\mathbf{id}_s \vee \psi_s \leq \eta_s \leq \eta_s \eta_s \leq \varphi_s \varphi_s = \varphi_s$, so it must be one $(\psi_{i,s}, s \in S)$, hence $\eta_s \eta_s = \psi_{i,s} \leq \bigvee_{i \in I} \psi_{i,s} = \eta_s$. Thus η_s is increasing, extensive and is idempotent, that is, a closing. Now $\mathbf{id}_s \vee \psi_s \leq \eta_s = \bigvee_{i \in I} \psi_{i,s} \leq \varphi_s$, where φ_s is the least closing on $R(s]$ such that $\psi_s \leq \varphi_s$, we deduce that $\eta_s = \varphi_s$. Applying the present item 1, ρ is a geodesic reconstruction system on (S, R). □

Let us now consider what happens when one reduces the sets of markers and masks. The following is straightforward:

Proposition 4 *Let L be a complete lattice, let $R, R', S, S' \subseteq L$ with $R' \subseteq R$ and $S' \subseteq S$.*

1. *Given a geodesic map system* $(\psi_s, s \in S)$ *on* (S, R)*, restricting* ψ_s *to* $R'(s]$ *for each* $s \in S'$*, the* $(\psi_s, s \in S')$ *will form a geodesic map system on* (S', R') *iff for any* $s \in S'$ *and* $r' \in R'(s]$*,* $\psi_s(r) \in R'S'$.
2. *A geodesic reconstruction system* ρ *on* (S, R)*, restricted to* $\{(s,r) \in S' \times R' \mid s \geq r\}$*, will give a geodesic reconstruction system on* (S', R') *iff for any* $s \in S'$ *and* $r' \in R'(s]$*,* $\rho(s,r) \in R'S'$.

In view of item 2 of Proposition 1, given a geodesic map system $(\delta_s, s \in S)$ such that δ_s is a dilation for each $s \in S$, we will have $\varphi_s = \delta_s^\infty$, according to (9). We will thus obtain a geodesic reconstruction by dilation $(s,r) \mapsto \delta_s^\infty(r)$. This justifies the interest of the construction of such geodesic map systems $(\delta_s, s \in S)$ made of dilations.

4 Generated geodesic dilation

We will consider a general method that derives from a dilation δ a geodesic map system. It applies the conditional dilation to the sup-generators of the marker, we call it the *generated geodesic dilation*. In the case of images $E \to V$ (E a space, V a complete lattice of values), the sup-generators will be the impulses (11), and we have then the *impulsive geodesic dilation*; we obtain interesting results when the dilation is *flat*. In Sect. 5.1 we will consider impulsive geodesic dilations when V is a *bundle lattice*, and for a restricted family of masks, they will indeed be dilations.

We assume a complete lattice L with sup-generating family G, that is, for every $x \in L$ we have $x = \bigvee G(x]$. Given a dilation δ on L, for any $s \in L$ we have the conditional dilation $\delta_s : (s] \to (s] : r \mapsto \delta(r) \wedge s$; when L is infinitely supremum distributive, δ_s is indeed a dilation on $(s]$. We define a variant of this operator, which applies δ_s to every sup-generator of the marker:

Definition 4 Let L be a complete lattice with sup-generating family G, and let δ be a dilation on L. For any $s \in L$, define $\delta_{G,s} : (s] \to (s]$ by

$$\forall r \in (s], \quad \delta_{G,s}(r) = \bigvee \{\delta(g) \wedge s \mid g \in G(r]\}. \tag{14}$$

$\delta_{G,s}$ is called the *generated geodesic dilation* with mask s derived from δ.

Proposition 5 *Given a complete lattice L with sup-generating family G, and a dilation* δ *on L:*

1. *For any lower set S in L,* $(\delta_{G,s}, s \in S)$ *is a geodesic map system on S.*
2. *For any* $s \in L$*:* $\delta_{G,s}$ *is increasing,* $\delta_{G,s} \leq \delta_s$ *and for every dilation* δ'_s *on* $(s]$ *such that* $\delta'_s \leq \delta_s$*, we have* $\delta'_s \leq \delta_{G,s}$*; in particular if* δ_s *is a dilation on* $(s]$*, then* $\delta_{G,s} = \delta_s$.
3. *If L is infinitely supremum distributive, then for every* $s \in L$*,* δ_s *is a dilation on* $(s]$ *and* $\delta_{G,s} = \delta_s$.

Proof 1. As S is a lower set, for $s \in S$, $(s] = S(s]$, so $\delta_{G,s}$ is $S(s] \to S(s]$. Let $r \leq s' \leq s \in S$. For all $g \in G(r]$ we have $\delta(g) \wedge s' \leq \delta(g) \wedge s$, hence

$\delta_{G,s'}(r) \leq \delta_{G,s}(r)$. If $\delta_{G,s}(r) \leq s'$, then for all $g \in G(r]$ we have $\delta(g) \wedge s \leq s'$, so $\delta(g) \wedge s = (\delta(g) \wedge s) \wedge s' = \delta(g) \wedge (s \wedge s') = \delta(g) \wedge s'$, hence $\delta_{G,s}(r) = \delta_{G,s'}(r)$. Therefore $(\delta_{G,s}, s \in S)$ satisfies the two conditions of Definition 2.

2. Let $r \leq r' \leq s$; for every $g \in G$, $g \leq r$ implies $g \leq r'$, that is $G(r] \subseteq G(r']$, so $\delta_{G,s}(r) \leq \delta_{G,s}(r')$. Hence $\delta_{G,s}$ is increasing. Given $r \leq s$, for every $g \in G(r]$, we have $\delta(g) \wedge s = \delta_s(g) \leq \delta_s(r)$, hence $\delta_{G,s}(r) = \bigvee\{\delta(g) \wedge s \mid g \in G(r]\} \leq \delta_s(r)$. Therefore $\delta_{G,s} \leq \delta_s$. Consider a dilation δ'_s on $(s]$ such that $\delta'_s \leq \delta_s$; for $r \leq s$ we have $r = \bigvee G(r]$, so

$$\delta'_s(r) = \delta'_s\left(\bigvee G(r]\right) = \bigvee_{g \in G(r]} \delta'_s(g) \leq \bigvee_{g \in G(r]} \delta_s(g) = \delta_{G,s}(r).$$

Therefore $\delta'_s \leq \delta_{G,s}$. If δ_s is a dilation on $(s]$, as $\delta_s \leq \delta_s$, we have $\delta_s \leq \delta_{G,s}$, but $\delta_{G,s} \leq \delta_s$ also, hence the equality $\delta_{G,s} = \delta_s$.

3. As δ is a dilation, for $r_i \in (s]$ $(i \in I)$, we have $\delta\left(\bigvee_{i \in I} r_i\right) = \bigvee_{i \in I} \delta(r_i)$. If L is infinitely supremum distributive, then

$$\delta_s\left(\bigvee_{i \in I} r_i\right) = \delta\left(\bigvee_{i \in I} r_i\right) \wedge s = \left[\bigvee_{i \in I} \delta(r_i)\right] \wedge s = \bigvee_{i \in I} [\delta(r_i) \wedge s] = \bigvee_{i \in I} \delta_s(r_i).$$

Thus δ_s is a dilation on $(s]$. By item 2, $\delta_{G,s} = \delta_s$. □

Concerning item 1, we note that since $R = S$ and S is a lower set, for every $s \in S$, $R(s] = S(s] = (s]$, which is closed under the supremum operation; hence the requirements of Proposition 3 and Theorem 1 are satisfied.

In relation to item 2, one can wonder whether we could not associate to each mask $s \in S$ the greatest dilation δ'_s on $(s]$ such that $\delta'_s \leq \delta_s$ (this δ'_s exists, since the set of dilations is closed under the supremum operation). However, for a non-distributive lattice L, this does in general *not* provide a geodesic map system, as shows the following counter-example:

Example 1 Let U be the lattice of labels (with $|U_*| \geq 3$), and let $L = U^4$, the set of four-tuples $\left(u|v|w|x\right)$ with coordinates in U, ordered componentwise; L can be viewed as the set of label images on four points. Take the dilation

$$\delta : \left(u|v|w|x\right) \mapsto \left(u|u \vee v \vee w|w|v \vee x\right).$$

For each $s \in L$, let δ'_s be the greatest dilation on $(s]$ such that $\delta'_s \leq \delta_s$. Let a, b, c be three distinct elements of U_*. Consider the following three masks and associated conditional dilations:

$$s_0 = \left(a|a|\bot|\bot\right), \quad \delta_{s_0} : \left(u|v|\bot|\bot\right) \mapsto \left(u|u \vee v|\bot|\bot\right);$$
$$s_1 = \left(\bot|b|b|\bot\right), \quad \delta_{s_1} : \left(\bot|v|w|\bot\right) \mapsto \left(\bot|v \vee w|w|\bot\right);$$
$$s_2 = \left(\bot|c|\bot|c\right), \quad \delta_{s_2} : \left(\bot|v|\bot|x\right) \mapsto \left(\bot|v|\bot|v \vee x\right).$$

Then each δ_{s_i} is a dilation on $(s_i]$, so $\delta'_{s_i} = \delta_{s_i}$ $(i = 0, 1, 2)$. Now let $s_3 = \left(a\middle|\top\middle|b\middle|c\right)$. Suppose that the $(\delta'_s, s \in U^4)$ constitute a geodesic map system. For $i = 0, 1, 2$, as $s_i \leq s_3$, by item 1 of Definition 2, for each $r \in (s_i]$ we have $\delta_{s_i}(r) = \delta'_{s_i}(r) \leq \delta'_{s_3}(r) \leq \delta_{s_3}(r)$, so

$$\forall r \in (s_i], \quad \left[\delta_{s_i}(r) = \delta_{s_3}(r)\right] \implies \left[\delta'_{s_3}(r) = \delta_{s_i}(r)\right].$$

We can check that for $i = 0, 1$, $\delta_{s_i}(s_i) = \delta_{s_3}(s_i) = s_i$, so $\delta'_{s_3}(s_i) = s_i$, while for $r = \left(\bot\middle|c\middle|\bot\middle|\bot\right)$, $\delta_{s_2}(r) = \delta_{s_3}(r) = s_2$, so $\delta'_{s_3}(r) = s_2$. But then

$$r \leq s_0 \vee s_1 \quad \text{with} \quad \delta'_{s_3}(r) = s_2 \not\leq s_0 \vee s_1 = \delta'_{s_3}(s_0) \vee \delta'_{s_3}(s_1),$$

which contradicts the fact that δ'_{s_3} is a dilation on $(s_3]$.

The following will be useful in Sect. 5.1 (and in [25]):

Proposition 6 *Consider a complete lattice L with sup-generating family G, and a dilation δ on L. Let $s \in L$ such that $(s]$ is infinitely supremum distributive. Then for any $s' \in (s]$ and $r \in (s']$, $\delta_{G,s'}(r) = \delta_{G,s}(r) \wedge s'$.*

Proof For $g \in G(r]$ we have $(\delta(g) \wedge s) \wedge s' = \delta(g) \wedge (s \wedge s') = \delta(g) \wedge s'$. Applying infinite supremum distributivity (7) and (14),

$$\begin{aligned} \delta_{G,s}(r) \wedge s' &= \left[\bigvee_{g \in G(r]} (\delta(g) \wedge s)\right] \wedge s' = \bigvee_{g \in G(r]} \left[(\delta(g) \wedge s) \wedge s'\right] \\ &= \bigvee_{g \in G(r]} \left(\delta(g) \wedge s'\right) = \delta_{G,s'}(r). \end{aligned}$$

□

4.1 Impulsive geodesic dilation

Let us now consider the case when $L = V^E$, the set of functions $E \to V$, where E is a space of points and V a complete lattice of values. The least and greatest elements of V are written $\bot$ and $\top$, respectively, the other elements being written v, w, etc.; functions $E \to V$ are written F, G, H, etc. Recall the above definition of a *cylinder* (10) and of an *impulse* (11). For $F : E \to V$ and $v \in V$, the *threshold set* $X_v(F)$ [15,22,23] is defined by

$$X_v(F) = \{p \in E \mid F(p) \geq v\}. \tag{15}$$

Let W be a sup-generating family of V; in other words, for every $v \in V$ we have $v = \bigvee W(v]$. For example for $V = \overline{\mathbf{R}}$ we can choose $W = \mathbf{R}$, while for $V = U$ (the lattice of labels) we can take $W = U_*$ (the set of proper labels). Let IW be the set of impulses $i_{h,v}$ for $h \in E$ and $v \in W$. Then IW is a sup-generating family of V^E: for any $F \in V^E$, $F = \bigvee IW(F]$, where

$$
\begin{aligned}
IW(F] &= \left\{ i_{h,v} \mid h \in E, v \in W, v \leq F(h) \right\} \\
&= \left\{ i_{h,v} \mid h \in E, v \in W\,(F(h)] \right\} \\
&= \left\{ i_{h,v} \mid v \in W, h \in X_v(F) \right\}.
\end{aligned} \tag{16}
$$

Definition 5 Consider a mask function $F \in V^E$ and a dilation Δ on V^E. The *impulsive geodesic dilation* with mask F derived from Δ is $\Delta_{IW,F}$, the generated geodesic dilation on $(F]$ for the sup-generating family IW:

$$\forall H \in (F], \quad \Delta_{IW,F}(H) = \bigvee \left\{ \Delta(i_{h,v}) \wedge F \mid i_{h,v} \in IW(H] \right\}. \tag{17}$$

Of particular interest is the case when the dilation is *flat*. Every increasing operator $\psi : \mathscr{P}(E)^n \to \mathscr{P}(E)$ has a flat extension $\psi^V : (V^E)^n \to V^E$ defined by [23]:

$$\psi^V(F_1, \ldots, F_n) = \bigvee_{v \in V} C_{\psi(X_v(F_1), \ldots, X_v(F_n)), v}, \tag{18}$$

so that

$$\forall p \in E, \quad \psi^V(F_1, \ldots, F_n)(p) = \bigvee \{ v \in V \mid p \in \psi\left(X_v(F_1), \ldots, X_v(F_n)\right) \}. \tag{19}$$

Note that in these two equations, we can restrict the values v to any sup-generating family W of V, in other words we can replace "$v \in V$" by "$v \in W$", this does not change the result [22,23]. For example, we can take $W = V \setminus \{\bot\}$. Usually one restricts oneself to $n = 1$, that is the flat extension of an increasing operator $\psi : \mathscr{P}(E) \to \mathscr{P}(E)$, which gives then a *flat operator* $\psi^V : V^E \to V^E$ [15,22,23]:

$$\psi^V(F) = \bigvee_{v \in V} C_{\psi(X_v(F)), v} = \bigvee_{v \in V \setminus \{\bot\}} C_{\psi(X_v(F)), v}. \tag{20}$$

Given a dilation δ on $\mathscr{P}(E)$, its flat extension δ^V is a dilation on V^E (i.e., it commutes with the supremum operation) [22,23], and it is given by setting for $F \in V^E$ and $p \in E$:

$$\delta^V(F)(p) = \bigvee_{q \in \check{\delta}(p)} F(q), \tag{21}$$

where $\check{\delta}$ is the transpose of δ, cf. (1). It acts on cylinders by dilating the basis [23]:

$$\forall B \subseteq E, \ \forall v \in V, \quad \delta^V(C_{B,v}) = C_{\delta(B), v}. \tag{22}$$

In particular, for an impulse we have:

$$\forall (h, v) \in E \times W, \quad \delta^V(i_{h,v}) = C_{\delta(h), v}. \tag{23}$$

Proposition 7 *Let δ be a dilation on $\mathscr{P}(E)$ and $F \in V^E$; consider the flat dilation δ^V on V^E and the impulsive geodesic dilation $\delta^V_{IW,F}$ with mask F. Then for any $H \in (F]$:*

1. *For every $p \in E$, we have*

$$\delta^V_{IW,F}(H)(p) = \bigvee_{q \in \check{\delta}(p)} (H(q) \wedge F(p))\,. \tag{24}$$

2. *$\delta^V_{IW,F}(H)$ results from the application to (F, H) of the flat extension of the set operator $\mathscr{P}(E)^2 \to \mathscr{P}(E) : (S, R) \mapsto \delta(R) \cap S$. In other words,*

$$\delta^V_{IW,F}(H) = \bigvee_{v \in V} C_{\delta(X_v(H)) \cap X_v(F), v}. \tag{25}$$

In particular, $\delta^V_{IW,F}(H)$ does not depend on the choice of the sup-generating family W.

Proof 1. Let $x, y \in V$. We remark that: (a) for $v \in W(x]$, we have $x \geq v$ and so $x \wedge y \geq v \wedge y$; (b) $W(x] \supseteq W(x \wedge y]$; (c) for $v \in W(x \wedge y]$ we have $v \leq y$ and so $v \wedge y = v$. It follows that

$$x \wedge y \geq \bigvee_{v \in W(x]} [v \wedge y] \geq \bigvee_{v \in W(x \wedge y]} [v \wedge y] = \bigvee_{v \in W(x \wedge y]} v = x \wedge y,$$

hence $\bigvee_{v \in W(x]} [v \wedge y] = x \wedge y$. For $x = H(h)$ ($h \in E$) and $y = F(p)$, this gives $\bigvee_{v \in W(H(h)]} [v \wedge F(p)] = H(h) \wedge F(p)$.

Let $h \in \check{\delta}(p)$. Then $p \in \delta(h)$ and for every v we have $C_{\delta(h),v}(p) = v$, hence

$$\bigvee_{v \in W(H(h)]} \left[C_{\delta(h),v}(p) \wedge F(p)\right] = \bigvee_{v \in W(H(h)]} [v \wedge F(p)] = H(h) \wedge F(p).$$

Now for $h \notin \check{\delta}(p)$, $p \notin \delta(h)$ so for every v we have $C_{\delta(h),v}(p) = \perp$, hence

$$\bigvee_{v \in W(H(h)]} \left[C_{\delta(h),v}(p) \wedge F(p)\right] = \perp.$$

Combining (16,17,23), we get

$$\delta^V_{IW,F}(H) = \bigvee \left\{C_{\delta(h),v} \wedge F \mid h \in E, v \in W\,(H(h)]\right\};$$

therefore

$$\delta^V_{IW,F}(H)(p) = \bigvee_{h \in E} \left(\bigvee_{v \in W(H(h)]} \left[C_{\delta(h),v}(p) \wedge F(p)\right] \right)$$

$$= \bigvee_{h \in \check{\delta}(p)} \left(\bigvee_{v \in W(H(h))} \left[C_{\delta(h),v}(p) \wedge F(p) \right] \right)$$
$$\vee \bigvee_{h \notin \check{\delta}(p)} \left(\bigvee_{v \in W(H(h))} \left[C_{\delta(h),v}(p) \wedge F(p) \right] \right)$$
$$= \bigvee_{h \in \check{\delta}(p)} (H(h) \wedge F(p)) .$$

This gives (24). Clearly this expression does not depend on the choice of W.

2. Let $G = \bigvee_{v \in V} C_{\delta(X_v(H)) \cap X_v(F), v}$. We must show that $G = \delta^V_{IW,F}(H)$, in other words for every $p \in E$, $G(p) = \delta^V_{IW,F}(H)(p)$. Now by (19),

$$\begin{aligned} G(p) &= \bigvee \{v \in V \mid p \in \delta(X_v(H)) \cap X_v(F)\} \\ &= \bigvee \left\{ v \in V \mid \check{\delta}(p) \cap X_v(H) \neq \emptyset, p \in X_v(F) \right\} \\ &= \bigvee \left\{ v \in V \mid \exists q \in \check{\delta}(p), H(q) \geq v, F(p) \geq v \right\} \\ &= \bigvee \left\{ v \in V \mid \exists q \in \check{\delta}(p), H(q) \wedge F(p) \geq v \right\} \\ &= \bigvee \left(\bigcup_{q \in \check{\delta}(p)} \{v \in V \mid H(q) \wedge F(p) \geq v\} \right) \\ &= \bigvee_{q \in \check{\delta}(p)} \left(\bigvee \{v \in V \mid H(q) \wedge F(p) \geq v\} \right) \\ &= \bigvee_{q \in \check{\delta}(p)} (H(q) \wedge F(p)). \end{aligned}$$

By item 1, this is equal to $\delta^V_{IW,F}(H)(p)$. □

Since the operator $\delta^V_{IW,F}$ *does not depend on the choice of the sup-generating family* W*, we can write it* $\delta^V_{I,F}$. By (21) we have

$$\left(\delta^V(H) \wedge F \right)(p) = \delta^V(H)(p) \wedge F(p) = \left(\bigvee_{q \in \check{\delta}(p)} H(q) \right) \wedge F(p).$$

Comparing with (24), we see again that $\delta^V_{I,F}(H)(p) \leq \left(\delta^V(H) \wedge F \right)(p)$, with the equality holding when V is infinitely supremum distributive, cf. Proposition 5.

Geodesic dilation for numerical functions $E \to T$ has been considered in [6] as an operator acting on markers for a fixed mask, in other words as $(\delta^T)_F$ for a fixed F, and as such this operator is *semi-flat* in the sense of [15] (see [23] for details). We see however here that for an arbitrary lattice V, if we consider both mask and markers as variables, the flat extension of geodesic dilation for sets is the map $(F, H) \mapsto \delta^V_{I,F}(H)$.

Note also that when $E = \mathbf{R}^n$ or $\mathbf{Z}^n$ and $\delta = \delta_A$, the dilation $X \mapsto X \oplus A$ by a structuring element A, (24) becomes

$$\delta^V_{I,F}(H)(p) = \bigvee_{a \in A} (H(p-a) \wedge F(p)) \,.$$

5 Examples

We will consider here two types of non-distributive lattices considered in morphological image processing, and discuss briefly (without proofs) the design of geodesic map systems from dilations for each. They are:

1. The lattice of images $E \to V$, where V is a *bundle lattice*; particular cases are the lattice of label images [24] and the one of images with reference ordering [16,18,19]. Here the impulsive geodesic dilation will be a dilation, provided that no point has value $\top$ in the mask image.
2. The lattice of partitions of a space E; we give here a specific form of geodesic dilation, the *partitioned geodesic dilation*, which will indeed be a dilation, and will provide a geodesic map system.

5.1 Bundle lattices

In Sect. 2 we described the lattice U of labels (whose Hasse diagram is shown in Fig. 2) and discussed problems of geodesy in the lattice U^E of label images $E \to U$.

Another lattice of image values has been considered in mathematical morphology, and it appears that it has some similarity with the lattice U of labels. Let $T = \mathbf{R}$ or $\mathbf{Z}$; Keshet (Kresch) [18,19] defined on T the *reference order*: choose a fixed $r \in T$, and define the relation $\leq_r$ on T by $a \leq_r b$ if either $r \leq a \leq b$ or $r \geq a \geq b$ (numerically). This is a partial order relation which turns T into a *complete inf-semilattice* [16], in other words every non-void subset of T has an infimum, but not necessarily a supremum. Note that the map $T \to T : x \mapsto x - r$ transforms the order $\leq_r$ into $\leq_0$; the latter order is called the *difference order* in [18,19]. Although Keshet restricted himself to the complete inf-semilattice $(T, \leq_r)$, adding to T as greatest element the unsigned infinity ∞, $D = T \cup \{\infty\}$ becomes a complete lattice. We will consider D with the difference order $\leq_0$. The construction of the lattice $(D, \leq_0)$ in the discrete case $T = \mathbf{Z}$ is illustrated in Fig. 5.

Both U and D are particular cases of a class of lattices built from a chain and a set:

Definition 6 Let B be a set with $|B| \geq 2$ and let C be a complete chain with $|C| \geq 3$, whose least and greatest elements are $\bot$ and $\top$. Let $C' = C \setminus \{\bot, \top\}$. The *bundle product* of B by C is the set

$$B \mathbin{\lhd\!\rhd} C = (B \times C') \cup \{\bot, \top\}$$

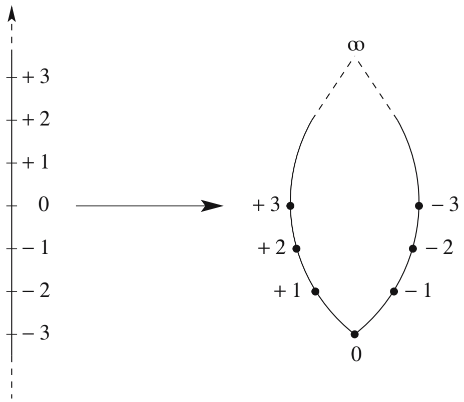


Fig. 5 *Left* **Z** with the numerical ordering. *Right* the difference order is like a "folding" of **Z**, to which ∞ is added as greatest element

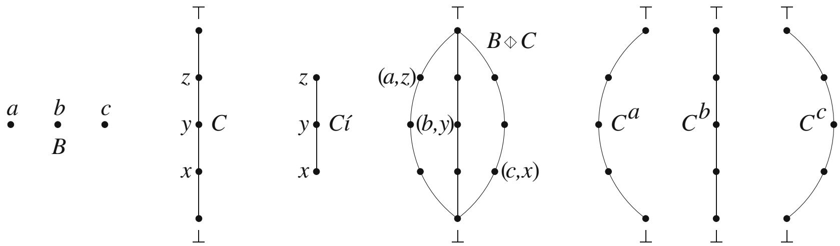


Fig. 6 Hasse diagrams of a set $B = \{a, b, c\}$, a chain C, the reduced chain $C' = C \setminus \{\bot, \top\}$, the bundle product $B \mathrel{\triangleleft\!\triangleright} C = (B \times C') \cup \{\bot, \top\}$, and the extended branches $C^p = (\{p\} \times C') \cup \{\bot, \top\}$ for $p \in B$

ordered as follows:

$$\begin{aligned} \forall b \in B,\ \forall c_1, c_2 \in C', \quad & (b, c_1) \leq (b, c_2) \iff c_1 \leq c_2, \\ \forall v \in B \mathrel{\triangleleft\!\triangleright} C, \quad & \bot \leq v \quad \text{and} \quad v \leq \top. \end{aligned}$$

For $b \in B$, the *b-branch* of $B \mathrel{\triangleleft\!\triangleright} C$ is the set $b \times C'$, and the *extended b-branch* of $B \mathrel{\triangleleft\!\triangleright} C$ is the set

$$C^b = (b \times C') \cup \{\bot, \top\},$$

where we write $b \times C'$ for $\{b\} \times C' = \{(b, c) \mid c \in C'\}$.

We illustrate this notion in Fig. 6. Clearly, for any set B and complete chain C, $B \mathrel{\triangleleft\!\triangleright} C$ is a complete lattice having $B \times C'$ as sup-generating family; we call a *bundle lattice* any complete lattice isomorphic to $B \mathrel{\triangleleft\!\triangleright} C$. Now the two lattices discussed above are indeed examples of bundle lattices:

- Taking the chain $\{\bot, m, \top\}$ (where $\bot < m < \top$), the lattice U of labels is isomorphic to $U_* \mathrel{\triangleleft\!\triangleright} \{\bot, m, \top\}$; here each proper label u will correspond to (u, m).
- The difference lattice $D = T \cup \{\infty\}$ ordered by $\leq_0$ is isomorphic to $\{+, -\} \mathrel{\triangleleft\!\triangleright} [0, \infty]$: for $0 < x < \infty$, $(+, x)$ and $(-, x)$ correspond to $+x$ and $-x$, respectively. Note

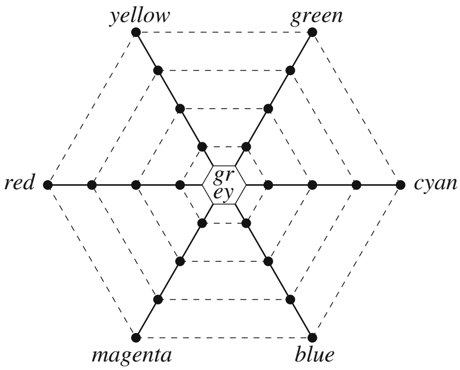


Fig. 7 We take the set H of six primary and secondary hues: *red*, *yellow*, *green*, *cyan*, *blue*, *magenta*; saturations range from 0 (*grey*) to 4 (fully saturated), they form the set $S = \{0, \ldots, 4\}$. In the bundle lattice $H \lhd\!\rhd (S \cup \{\top\})$, 0 (*grey*) is the least element, $\top$ (hue conflict, not shown in the figure) is the greatest element, and between them are the pairs $(h, s) \in H \times \{1, \ldots, 4\}$; for a fixed hue h, (h, s) grows with s. The *dashed hexagonal rings* correspond to the various values of saturation, growing from centre to periphery

that for $r \in T$, the map $x \mapsto x - r$ (fixing ∞) is an isomorphism between the reference lattice $(D, \leq_r)$ and the difference lattice $(D, \leq_0)$.

Let us suggest another possible example. In digital images, colour is represented by a 3D vector whose components are the red, green and blue intensities. Transforms of the 3D RGB space suitable for morphological processing are discussed in [13]. In colorimetry, one usually transforms this vector into another one with one component corresponding to luminance and the other two representing chrominance (for gamma-corrected signals, they are called luma and chroma) [21]. The luminance/luma corresponds to intensity in grey-level images, and it is numerically ordered. One can represent the chrominance/chroma in polar coordinates, where the angle corresponds to the hue (say: red, orange, yellow, etc.), and the radius corresponds to saturation (zero for grey, maximum for pure colours) [4]. It is possible to order the radius (saturation) numerically in an interval $[0, r_{max}]$, but one should note that for $r = 0$, the angle (hue) is not defined (grey has no hue). This suggests to represent chrominance/chroma, i.e., the set of possible hue and saturation pairs, as the bundle lattice $H \lhd\!\rhd ([0, r_{max}] \cup \{\top\})$, where H is the set of hues; here $\bot = 0$ corresponds to grey, (h, r) (for $0 < r \leq r_{max}$) to the colour with hue h and saturation r, and $\top$ is just a symbol indicating hue conflict (removing $\top$, the colours would form a complete inf-semilattice instead of a complete lattice). We illustrate this in Fig. 7 for six hues and four non-zero saturations. Note that for $r, r' > 0$, $(h, r) \vee (h, r') = (h, r \vee r')$ and $(h, r) \wedge (h, r') = (h, r \wedge r')$, while for $h' \neq h$ we have $(h, r) \vee (h', r') = \top$ and $(h, r) \wedge (h', r') = 0$; hence the set H of hues should be discrete, otherwise the supremum and infimum operations would be discontinuous.

Let us now describe flat morphology and geodesy on images with values in a bundle lattice, in particular the difference lattice $(D, \leq_0)$. No proofs are given, the topic will be further developed in a future paper. For the particular case of label images, flat morphological operators were studied in [24], while in [25] we will deal with the impulsive geodesic dilation and the resulting geodesic reconstruction.

One important property of label images (see Proposition 8 of [24]) is that a flat operator behaves by processing independently the image portion corresponding to each label, then joining the results. Something similar happens for images with values in a bundle lattice, that is functions $E \to B \mathbin{\triangleleft\!\triangleright} C$.

For $b \in B$ and $F : E \to B \mathbin{\triangleleft\!\triangleright} C$, we define $F^b : E \to C^b$, the *restriction of F to the extended b-branch*, by

$$\forall p \in E, \qquad F^b(p) = \begin{cases} F(p) & \text{if } F(p) \in C^b, \\ \bot & \text{if } F(p) \notin C^b. \end{cases}$$

Note that C^b is a complete chain isomorphic to C, see Fig. 6. Now for an increasing operator ψ on $\mathscr{P}(E)$ in order to compute $\psi^{B \triangleleft\!\triangleright C}(F)$, one does the following:

1. Decompose F into the $F^b : E \to C^b$, $b \in B$.
2. For each $b \in B$, compute $\psi^{C^b}(F^b)$; since C^b is isomorphic to C, this amounts to applying ψ^C to a function $E \to C$, and as C is a complete chain, this is computed as the usual flat operator for numerically ordered grey-level images [23].
3. Join the resulting $\psi^{C^b}(F^b)$, $b \in B$.

For example let $V = D$ with the difference ordering $\leq_0$, which is isomorphic to $\{+, -\} \mathbin{\triangleleft\!\triangleright} [0, \infty]$; then F^+ and F^- will be the positive and negative parts of F:

$$F^+(p) = \begin{cases} F(p) & \text{if } F(p) \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad F^-(p) = \begin{cases} F(p) & \text{if } F(p) \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

with the convention that both $\infty \geq 0$ and $\infty \leq 0$ (the unsigned infinity is both positive and negative). Then a flat operator will act by processing the positive and negative parts of a function, each one being rectified as a numerical function $E \to [0, \infty]$, then joining the two results. We illustrate in Fig. 8 this behaviour for a dilation.

In [25] we show that for label images, if the mask image F has all its values $< \top$ (that is, F is $E \to U \setminus \{\top\}$), then the impulsive geodesic dilation $\delta^U_{I,F}$ is a dilation in the algebraic sense, which acts like an ordinary geodesic dilation on the image portion corresponding to each proper label. The same happens for images with values in a bundle lattice.

Let $V = B \mathbin{\triangleleft\!\triangleright} C$ and consider a mask image $F : E \to V \setminus \{\top\}$ (i.e., $F(p) < \top$ for all $p \in E$). In order to apply the impulsive geodesic dilation $\delta^V_{I,F}$ to a marker H, one does the following:

1. Decompose F and H into the $F^b, H^b : E \to C^b$, $b \in B$ (we have $H^b \leq F^b$).
2. For each $b \in B$, compute the conditional dilation $\delta^{C^b}(H^b) \wedge F^b$ in the lattice of images $E \to C^b$ (as the usual flat conditional dilation for numerically ordered grey-level images).
3. Join the resulting $\delta^{C^b}(H^b) \wedge F^b$, $b \in B$.

In practice, one can reduce B to $B' = \{b \in B \mid \exists p \in E,\ F(p) \in b \times C'\}$, in other words remove from V the branches $b \times C'$ for $b \in B \setminus B'$. Then $B' \mathbin{\triangleleft\!\triangleright} C$ is either a chain (for $|B'| = 1$) or a bundle lattice (for $|B'| > 1$) and F is $E \to$

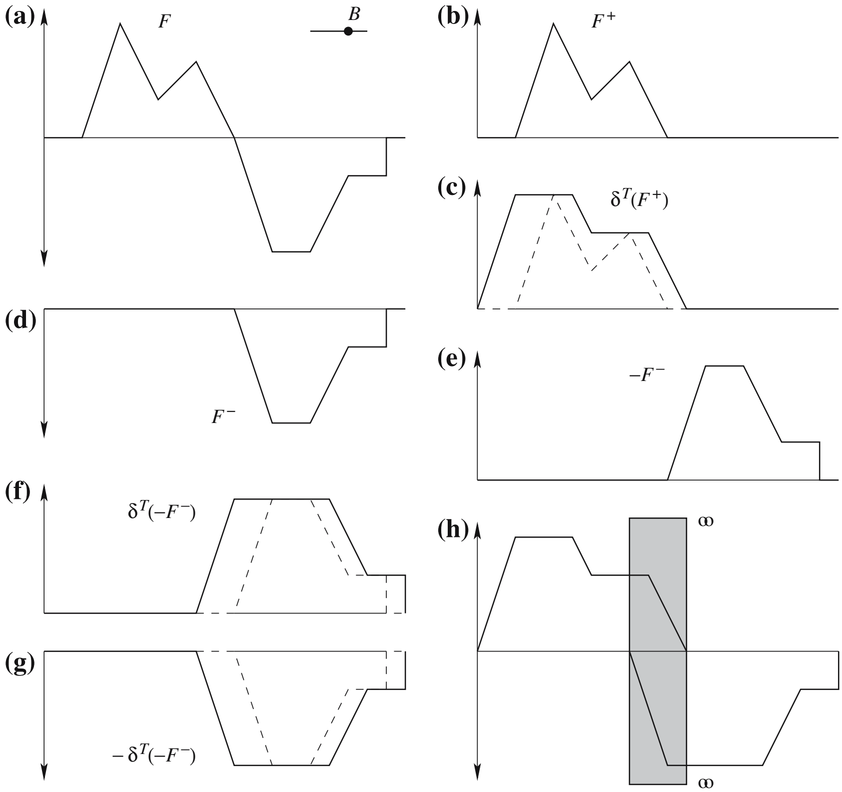


Fig. 8 We take $V = D = \mathbf{R} \cup \{\infty\}$ ordered by $\leq_0$. **a** A function $F : E \to \mathbf{R}$, and the structuring element used in the dilation; the *thick dot* indicates the position of the origin. **b** The positive part F^+ of F. **c** We apply to F^+ the flat dilation δ^T for numerically ordered functions. **d** The negative part F^- of F. **e** The rectified negative part $-F^-$. **f** We apply to $-F^-$ the flat dilation δ^T for numerically ordered functions. **g** The unrectified dilate $-\delta^T(-F^-)$. **h** The superposition of $\delta^T(F^+)$ and $-\delta^T(-F^-)$; $\delta^V(F)$ is the supremum (for the difference order $\leq_0$) of these two functions, the points p where $\delta^T(F^+)(p) > 0 > -\delta^T(-F^-)(p)$ (numerically) will get the value ∞ (the corresponding zone is shown in *grey*)

$B' \lhd\!\rhd C$. The support $\text{supp}(F)$ is partitioned into the sets $\text{supp}(F^b) = F^{-1}(b \times C')$ for $b \in B'$. In each $\text{supp}(F^b)$, $b \in B'$, $\text{supp}(H^b)$ will be geodesically dilated into $\delta(\text{supp}(H^b)) \wedge \text{supp}(F^b)$, and (24) gives here

$$\forall b \in B',\ \forall p \in \text{supp}(F^b), \quad \delta^V_{I,F}(H)(p) = \bigvee_{q \in \check{\delta}(p) \cap \text{supp}(F^b)} (H(q) \wedge F(p))$$
$$= \left(\bigvee_{q \in \check{\delta}(p) \cap \text{supp}(F^b)} H(q) \right) \wedge F(p).$$

We illustrate in Fig. 9 the impulsive geodesic dilation, and the resulting reconstruction, for functions with values in the bundle lattice $\{+, -\} \lhd\!\rhd [0, \infty]$, that is $\mathbf{R} \cup \{\infty\}$ with

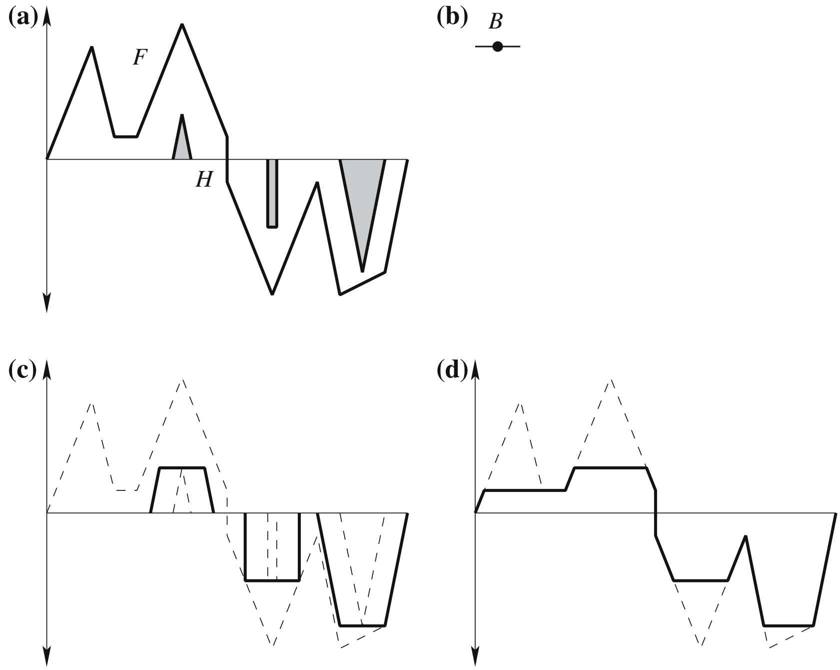


Fig. 9 We take $V = D = \mathbf{R} \cup \{\infty\}$ ordered by $\leq_0$. **a** The mask function F and the marker function H (with its hypograph in *grey*). **b** The structuring element used in the dilation; the *thick dot* indicates the position of the origin. **c** The impulsive geodesic dilation $\delta^V_{I,F}$ is applied to H, it acts separately on the positive and negative parts of F. **d** Iterating $\delta^V_{I,F}$ provides a geodesic reconstruction

the difference order $\leq_0$. One sees that this gives exactly what one expects: the marker is expanded "horizontally" within the mask.

By comparison, we show in Fig. 10 that the standard geodesic dilation $\delta^V(H) \wedge F$ produces an aberrant result in such a case.

Note that for a mask image $F : E \to V \setminus \{\top\}$, for every $x \in E$, $(F(x)]$ is a chain, so $(F]$ is infinitely supremum distributive. Thus Proposition 6 applies here: for $H \leq F' \leq F$, $\delta^V_{I,F'}(H) = \delta^V_{I,F} \wedge F'$.

Remark 1 Let E be a digital space provided with an adjacency relation $\sim$. If one takes for δ the neighbourhood dilation (2,3) corresponding to $\sim$, one can iterate the application of $\delta^V_{I,F}$ to a marker $H \in (F]$, leading to a reconstruction $\rho(F, H) = \bigvee_{n \in \mathbf{N}} [\delta^V_{I,F}]^n(H)$. For $V = \mathbf{R} \cup \{\infty\}$ with the difference ordering $\leq_0$ and for $F : E \to \mathbf{R}$, this gives a result like in Fig. 9d, where $\rho(F, H)$ lies between 0 and the extrema of F.

In [20], one defines a *levelling* of a numerical function F to be any function G such that

$$\forall x, y \in E, \quad [x \sim y,\ G(x) > G(y)] \implies [F(x) \geq G(x),\ G(y) \geq F(y)]. \quad (26)$$

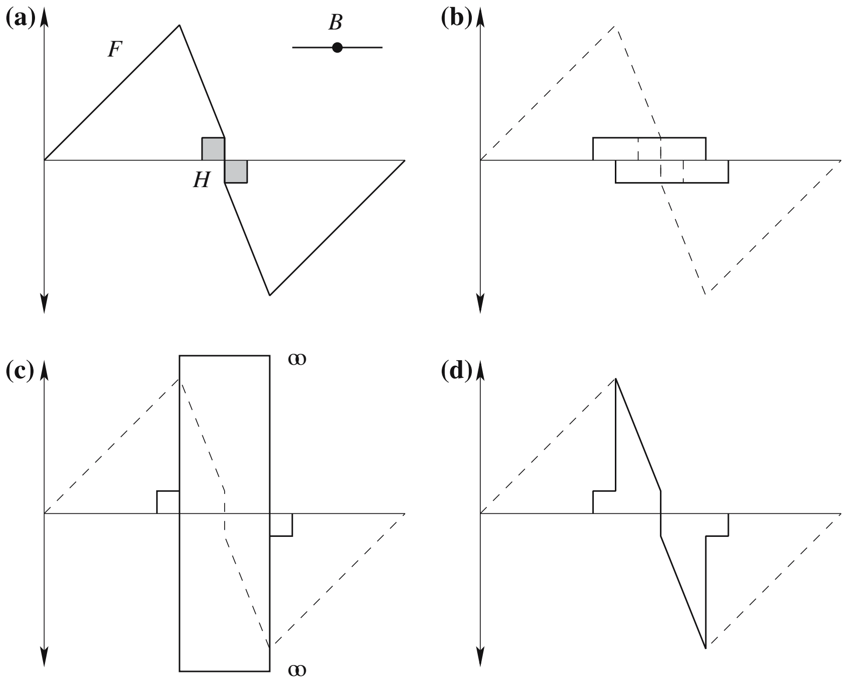


Fig. 10 We take $V = D = \mathbf{R} \cup \{\infty\}$ ordered by $\leq_0$. **a** The mask function F, the marker function H (with its hypograph in *grey*), and the structuring element used in the dilation; the *thick dot* indicates the position of the origin. **b** The negative and positive parts of H are dilated. **c** Where their supports overlap, the dilation $\delta^V(H)$ gets the value ∞. **d** The geodesic restriction $\delta^V(H) \wedge F$

Then $\rho(F, H)$, the reconstruction of mask F from marker H *in the difference order* $\leq_0$, will be a levelling of F *for the numerical order* $\leq$.

Indeed, let $G = \rho(F, H)$. In the positive part $\text{supp}(F^+)$, $\delta^V_{I,F}$ applies to H^+ the usual conditional dilation $\delta^T_{F^+}$, so by iteration the reconstruction G will give on $\text{supp}(F^+)$ the geodesic reconstruction by dilation from marker H^+ under the mask F^+, in the usual sense for numerical functions. This is a levelling, so (26) holds for $x, y \in \text{supp}(F^+)$. In the negative part $\text{supp}(F^-)$ the behaviour is, from a numerical point of view, the dual by inversion of what happens in $\text{supp}(F^+)$, so here G will be the geodesic reconstruction by erosion from marker H^- above the mask F^-. This is again a levelling, so (26) holds for $x, y \in \text{supp}(F^-)$. For $x \in \text{supp}(F^-)$ and $y \in \text{supp}(F^+)$, we have $G(y) \geq 0 \geq G(x)$, so $G(x) \not> G(y)$ and (26) holds. For $x \in \text{supp}(F^+)$ and $y \in \text{supp}(F^-)$, as $G \leq_0 F$, we have $F(x) \geq G(x) \geq 0$ and $F(y) \leq G(y) \leq 0$, so (26) holds.

On the other hand, $G = \rho(F, H)$ will in general not be a levelling of F *for the difference order* $\leq_0$. We can have $x \sim y$ with $F(x) >_0 G(x) >_0 0$ and $F(y) >_0 G(y) = 0$, so $F(x) >_0 G(x) >_0 G(y) <_0 F(y)$, contradicting (26).

In [25], we will study in detail the impulsive geodesic dilation, and the reconstruction arising from it, in the case of label images. When in the mask image S no point has value $\top$, the impulsive geodesic dilation will indeed be a dilation on $(S]$, that acts like a conditional dilation on each labelled zone. For the neighbourhood dilation, the iterated impulsive geodesic dilation will indeed reconstruct all connected components of the mask meeting the marker. Given an arbitrary connection $\mathscr{C}$ on $\mathscr{P}(E)$, and the associated connection $\mathscr{C}^U$ on U^E, the reconstruction operator $\rho_{\mathscr{C}^U}$ defined according to (13), will be a geodesic reconstruction system for masks and markers $E \to U \setminus \{\top\}$, but also for arbitrary masks $E \to U$ and markers restricted to cylinders $C_{B,u}$ with $u \in U_*$.

5.2 Partitions

A *partition* of a set E is a set π of subsets of E, $\pi \in \mathscr{P}(\mathscr{P}(E))$, that are all non-void ($\emptyset \notin \pi$) and mutually disjoint (for $X, Y \in \pi$, $X \neq Y \Rightarrow X \cap Y = \emptyset$), and which cover E ($\bigcup \pi = E$). Equivalently, for every $p \in E$, there is a unique $X \in \pi$ such that $p \in X$, and that set X is called the *class of p in π*; thus the elements of π will be called the *classes* of the partition.

If $E = \emptyset$, then the unique partition of E is the empty one (without class), otherwise a partition will always be non-void.

Let us write $cl_\pi(p)$ for the class of p in partition π, π_0 for the partition whose classes are the singletons, π_E for the partition with E as unique class, and $\Pi(E)$ for the set of all partitions of E. This set is ordered as follows [32,34]: given $\pi_1, \pi_2 \in \Pi(E)$, we say that π_1 is *finer* than π_2, or that π_2 is *coarser* than π_1, and write $\pi_1 \leq \pi_2$ (or $\pi_2 \geq \pi_1$), iff every class of π_1 is included in a class of π_2, in other words for every $p \in E$, $cl_{\pi_1}(p) \subseteq cl_{\pi_2}(p)$. Then $\Pi(E)$ with this fine-to-coarse ordering is a complete lattice, with least element π_0, greatest element π_E, and where the infimum of a family π_i $(i \in I)$ of partitions is given by $cl_{\bigwedge_{i \in I} \pi_i}(p) = \bigcap_{i \in I} cl_{\pi_i}(p)$ for all $p \in E$. The supremum $\bigvee_{i \in I} \pi_i$ is more complicated, it relies on a chaining of the overlapping classes from all π_i, see [37] for more details.

Partitions intervene in image segmentation [37]. Generally, the classes of a segmentation are constrained to be connected. However we do not make any such assumption, *we admit partitions with disconnected classes*. Sometimes one considers *labelled partitions*, where a label or some other sort of information is attached to each class; then the partition can be considered as a label image. Otherwise one can attach a smooth function to each class, leading thus to a *weighted partition* [34]. Let us stress that *we consider here only unlabelled partitions*, no information is attached to the classes.

Due to its complexity, morphology on partitions constitutes an almost virgin territory. Most authors have considered only the following operations: the infimum and supremum of partitions, the splitting of the classes of a partition into their connected components, and the splitting and merging of classes (in a non-deterministic order) according to some heuristic. Let us note however a first investigation of the deep links between partitions and connections [37]. In future papers, the author will give a general theory of morphology and geodesy on partitions, with its links to connections; in fact we consider the more general framework of *partial partitions* (where one removes the axiom that partition classes cover the set) and *partial connexions* (where one removes the axiom that singletons are connected, so the connected components do not always cover a set). However we will briefly describe here erosion, dilation, and geodesic dilation, and state some facts without proof.

In [35], Serra introduced an erosion on partitions. Let ε be an erosion on $\mathscr{P}(E)$ such that $\varepsilon(\emptyset) = \emptyset$ (for example, the erosion by a non-empty structuring element, when $E = \mathbf{R}^n$ or $\mathbf{Z}^n$). Then one derives from ε an erosion ε' on $\Pi(E)$; for a partition π, $\varepsilon'(\pi)$ is obtained as follows:

1. Erode by ε all classes of π, and keep the non-void eroded classes;
2. all points $p \in E$ which do not belong to an eroded class are constituted into singleton classes $\{p\}$.

In other words,

$$\varepsilon'(\pi) = \{\varepsilon(C) \mid C \in \pi,\ \varepsilon(C) \neq \emptyset\} \cup \left\{\{p\} \mid p \in E \setminus \left(\bigcup_{C \in \pi} \varepsilon(C)\right)\right\}.$$

Serra [35] expressed ε' in terms of the class of a point, by $cl_{\varepsilon'(\pi)}(p) = \varepsilon(cl_\pi(p))$ if $p \in \varepsilon(cl_\pi(p))$, and $\{p\}$ if $p \notin \varepsilon(cl_\pi(p))$, but this formulation *is valid only if ε is anti-extensive* [i.e., $\varepsilon(X) \subseteq X$ for all $X \in \mathscr{P}(E)$]. Indeed, if ε is not anti-extensive, we may have $p \notin \varepsilon(cl_\pi(p))$ but $p \in \varepsilon(cl_\pi(q))$ for some $cl_\pi(q) \neq cl_\pi(p)$, and in this case we have $cl_{\varepsilon'(\pi)}(p) = \varepsilon(cl_\pi(q))$.

Now let δ be the dilation on $\mathscr{P}(E)$ that is the lower adjoint of ε; the fact that $\varepsilon(\emptyset) = \emptyset$ is equivalent to $\forall X \in \mathscr{P}(E),\ X \neq \emptyset \;\Rightarrow\; \delta(X) \neq \emptyset$. Then we can derive from δ a dilation δ' on $\Pi(E)$; for a partition π, $\delta'(\pi)$ is obtained as follows:

1. remove all singleton classes in π;
2. dilate by δ the remaining classes;
3. recursively fuse all overlapping dilated classes, until only disjoint classes remain;
4. all points $p \in E$ which do not belong to a class are constituted into singleton classes $\{p\}$.

In other words, $\delta'(\pi)$ is the least partition ρ such that for every non-singleton class C of π, $\delta(C)$ is included in one class of ρ. Furthermore, given that (ε, δ) is an adjunction on $\mathscr{P}(E)$, (ε', δ') will be an adjunction on $\Pi(E)$.

Assume that $|E| \geq 2$. Then a sup-generating family of $\Pi(E)$ is given by the set Π^2 of all partitions having exactly one non-singleton class that is a pair, in other words, partitions of the form $\{P\} \cup \{\{p\} \mid p \in E \setminus P\}$ for a pair $P \subseteq E$.

Let us now consider geodesy. Assume that the dilation δ on $\mathscr{P}(E)$ is *extensive*, i.e., $\forall X \in \mathscr{P}(E),\ X \subseteq \delta(X)$. Given the derived dilation δ' on $\Pi(E)$ we can, for any mask partition σ, define the usual geodesic dilation

$$[\delta']_\sigma : (\sigma] \to (\sigma] : \rho \to \delta'(\rho) \wedge \sigma,$$

as well as the generated geodesic dilation

$$[\delta']_{\Pi^2,\sigma} : (\sigma] \to (\sigma] : \rho \to \bigvee \left\{\delta'(\pi) \wedge \sigma \mid \pi \in \Pi^2(\rho]\right\}.$$

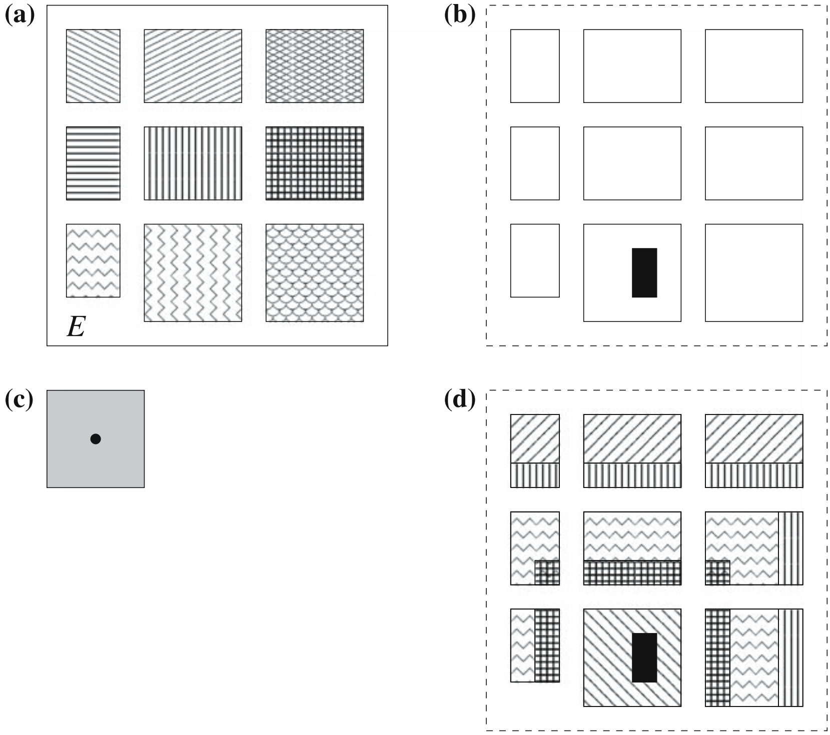


Fig. 11 In each partition, we identify every non-singleton class with a distinctive *grey*-level or *hatching*, while the *white zone* represents the union of all singleton classes. **a** In the space E (a *square*), the mask partition σ with nine non-singleton classes. **b** The marker partition ρ has one non-singleton class (in *black*). **c** δ is the dilation by the square structuring element shown here, that is centered about the origin (shown as a *black dot*). **d** Iterating $[\delta']_\sigma$ will progressively reconstruct all non-singleton classes of the mask σ; the marker ρ is shown in *black*, and the five successive layers (*hatched*) give the growth of the classes between $([\delta']_\sigma)^n(\rho)$ and $([\delta']_\sigma)^{n+1}(\rho)$ ($n = 0, \ldots, 4$); in particular, $([\delta']_\sigma)^5(\rho) = \sigma$

Both have the effect that the classes of the marker ρ within one class C of the mask σ will usually affect the classes of ρ within another class C' of σ, and iterating them will usually reconstruct the whole mask σ, see Fig. 11. Anyway, none of them is an algebraic dilation.

Therefore we propose another approach, that we call the *partitioned geodesic dilation*. Let σ be a mask partition. Let $\delta_{cl,\sigma}$ be the dilation that applies within each class C of σ the conditional dilation δ_C; in other words,

$$\forall X \in \mathscr{P}(E), \quad \delta_{cl,\sigma}(X) = \bigcup_{x \in X} [\delta(x) \cap cl_\sigma(x)] = \bigcup_{C \in \sigma} [\delta(X \cap C) \cap C].$$

We consider then its extension $[\delta_{cl,\sigma}]'$ to partitions (by the above construction), and the restriction of $[\delta_{cl,\sigma}]'$ to marker partitions $\rho \leq \sigma$ will be an operator $\delta^*_\sigma : (\sigma] \to (\sigma]$, the *partitioned geodesic dilation under the mask* σ.

Alternately, given a mask partition σ and a marker partition ρ (where $\rho \leq \sigma$), for every class $C \in \sigma$, $\rho_{|C} = \{D \in \rho \mid D \subseteq C\}$ is a partition of C, and the $\rho_{|C}$, $C \in \sigma$, constitute a partition of ρ (viewed as a set of classes). For each class $C \in \sigma$, consider the geodesic dilation $\delta_C : \mathscr{P}(C) \to \mathscr{P}(C) : X \mapsto X \cap C$ and the derived dilation $[\delta_C]'$ on $\Pi(C)$; then we apply $[\delta_C]'$ to $\rho_{|C}$ for each $C \in \sigma$, so our operator is

$$\delta_\sigma^* : (\sigma] \to (\sigma] : \rho \mapsto \bigvee_{C \in \sigma} [\delta_C]'(\rho_{|C}).$$

Thus $\delta_\sigma^*(\rho)$ is the least partition $\pi \in \Pi(E)$ such that for every $C \in \sigma$ and every non-singleton class $D \in \rho_{|C}$, $\delta(D) \cap C$ is included in one class of π. This construction is illustrated in Fig. 12.

Since δ is extensive, so is $\delta_{cl,\sigma}$, in particular it satisfies the condition $\forall X \in \mathscr{P}(E)$, $X \neq \emptyset \;\Rightarrow\; \delta_{cl,\sigma}(X) \neq \emptyset$. Thus $[\delta_{cl,\sigma}]'$ is a dilation on $\Pi(E)$, and its restriction δ_σ^* to $(\sigma]$ is a dilation on $(\sigma]$. We see now that $\big(\delta_\sigma^*, \sigma \in \Pi(E)\big)$ is a geodesic map system. For $\sigma' \leq \sigma$, we have $\delta_{cl,\sigma'} \leq \delta_{cl,\sigma}$, so $[\delta_{cl,\sigma'}]' \leq [\delta_{cl,\sigma}]'$, hence for $\rho \leq \sigma'$, $\delta_{\sigma'}^*(\rho) \leq \delta_\sigma^*(\rho)$. If $\delta_\sigma^*(\rho) \leq \sigma'$, for every $C \in \sigma$ and every non-singleton class $D \in \rho$ such that $D \subseteq C$, $\delta(D) \cap C$ is included in one class C' of σ', but as δ is extensive, $D \subseteq \delta(D) \cap C$, so C' is the class containing D; thus $D \subseteq C' \subseteq C$ and $\delta(D) \cap C \subseteq C'$, so $\delta(D) \cap C = \delta(D) \cap C'$; as $\delta_{\sigma'}^*(\rho)$ [respectively, $\delta_\sigma^*(\rho)$] is spanned by such $\delta(D) \cap C'$ [respectively, $\delta(D) \cap C$], we deduce then that $\delta_{\sigma'}^*(\rho) = \delta_\sigma^*(\rho)$.

If one takes for δ the neighbourhood dilation (2,3), the geodesic reconstruction derived from $\delta_\sigma^*(\rho)$ will reconstruct within each class of the mask all connected components that intersect at least one class of the marker. It could then be possible to define connections on $\Pi(E)$, or on the lattice of partitions with connected classes, cf. [34], associated to such reconstructions.

6 Conclusion

In this paper, we have seen that the standard theory of geodesic dilation and reconstruction [27], based on the geodesic restriction of dilations, is adapted to the case when the lattice of images is infinitely supremum distributive, but that it fails for several non-distributive lattices. We have proposed as an alternative to geodesic restriction the wider notion of a *geodesic map system*. We have seen in Theorem 1 that it allows us to build a *geodesic reconstruction system* by taking the closings generated by the geodesic maps.

The main problem is to derive from a dilation a geodesic map system with good properties. We have studied in detail one such construction, the *generated geodesic dilation*, which in the case of the lattice of functions $E \to V$ becomes the *impulsive geodesic dilation*. The generated (respectively, impulsive) geodesic dilation is not necessarily a dilation in the algebraic sense. Therefore special care should be devoted to conditions ensuring that it is indeed a dilation. This allows in particular to describe the geodesic reconstruction given by the closing generated by that dilation, cf. (9).

We have described one class of lattice of values, the *bundle lattices*. It has interesting features concerning flat operators and impulsive geodesic dilation: when the mask

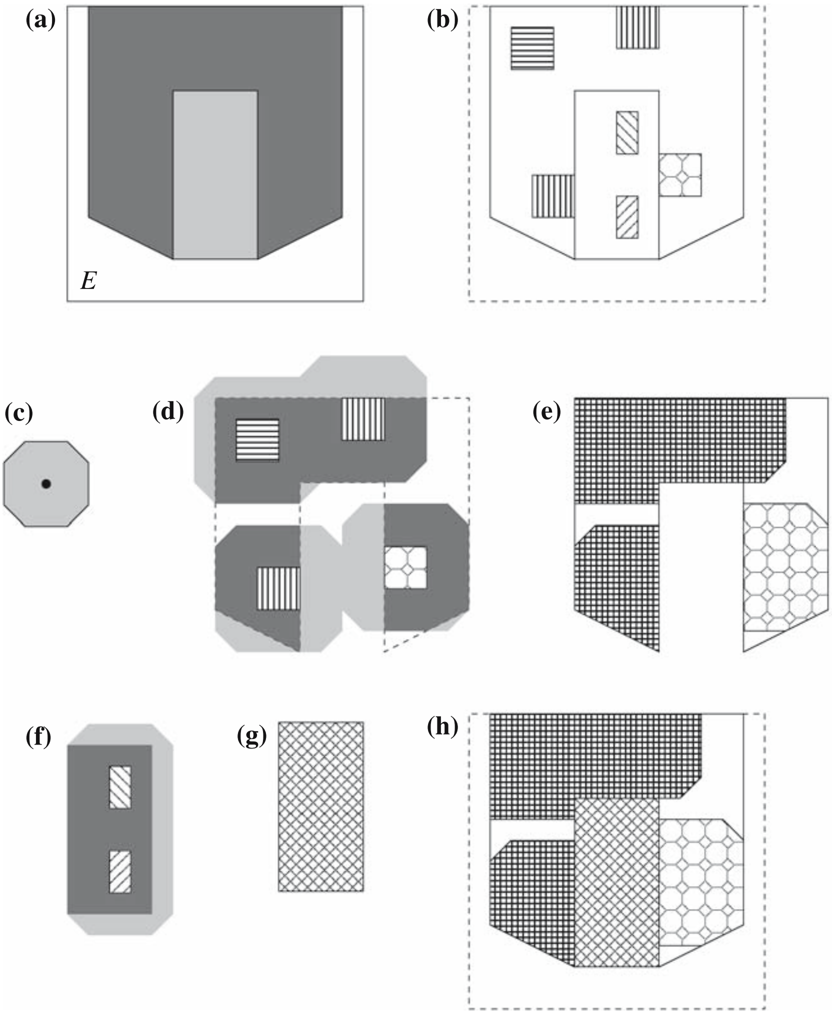


Fig. 12 In each partition, we identify every non-singleton class with a distinctive *grey*-level or *hatching*, while the *white zone* represents the union of all singleton classes. **a** In the space E (a *square*), the mask partition σ with two non-singleton classes (*dark* and *light grey*). **b** The marker partition ρ, with five non-singleton classes (one is disconnected). **c** δ is the dilation by the octogonal structuring element shown here, that is centered about the origin (shown as a *black dot*). **d** In the first non-singleton class C_1 of σ, the non-singleton classes of ρ are geodesically dilated in C_1 (the dilated classes in *light grey*, their geodesic restrictions in *dark grey*, and the original classes *hatched*). **e** Merging the overlapping geodesically dilated non-singleton classes of ρ, we obtain a dilation of the partition $\rho_{|C_1}$ of C_1. **f**, **g** Same as **d** and **e** for the second non-singleton class C_2 of σ. **h** Collecting the two partitions **f** and **g**, we obtain $\delta^*_\sigma(\rho)$, the partitioned geodesic dilation of ρ under σ

function has all its values $< \top$ (a standard requirement in practical situations), the impulsive geodesic dilation is a dilation.

Two examples of bundle lattices, as lattice V of function values, have been considered in the literature, both in relation to the practical problem of motion analysis in video sequences:

– Agnus [1–3] has used the so-called “object-oriented erosion” (an erosion on label images [24]) and “object-oriented reconstruction” (in fact, a geodesic reconstruction obained from an impulsive geodesic dilation, see [25]).
– Keshet [16, 18, 19] has introduced the *reference lattice*, and proposed some morphological operators for images with values in such a lattice. He did not study geodesy however.

Another possible bundle lattice consists in the polar representation of colour chrominance information. A further paper will analyse in depth morphology and geodesy on images with values in a bundle lattice.

A well-known example of non-distributive lattice is the lattice of partitions, which intervenes in particular in the study of image segmentation [34, 37]. We have briefly explained how to extend set erosions and dilations to partitions, and how to build a geodesic map system on partitions, made of dilations. The detailed study of morphological and geodesic operators on partitions will thus be the subject of separate papers.

A lesser known non-distributive lattice of images is the so-called *viscous lattice* [36]. It bears some resemblance to the lattice of convex sets [32]. It can easily be seen that in these two lattices, both the usual and the generated geodesic dilations are not dilations, and they give unsatisfactory results. Hence geodesy on such lattices is an open problem for future research.

It seems thus that each non-distributive lattice requires a specific method for the design of a geodesic map system made of dilations.

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