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# Various New Expressions for Subresultants and Their Applications

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**Abstract.** This article is devoted to presenting new expressions for Subresultant Polynomials, written in terms of some minors of matrices different from the Sylvester matrix. Moreover, via these expressions, we provide new proofs for formulas which associate the Subresultant polynomials and the roots of the two polynomials. By one hand, we present a new proof for the formula introduced by *J. J. Sylvester* in 1839, formula written in terms of a single sum over the roots. By other hand, we introduce a new expression in terms of the roots by considering the Newton basis.

**Keywords:** Subresultant polynomials and roots, Matrix theory, Bezout matrix

## Introduction

One of the main tools in computer algebra to deal with polynomials in one variable is Subresultant polynomials. For example, they provide fraction free algorithms for computing the greatest common divisor of two polynomials, with a good behaviour under specialization. Their multiple properties over integral domains can be found in [8], [20], [21], [26], [27]. See [11] for extensions of the main results over integral domains to arbitrary commutative rings. They are also used in algorithms performing quantifier elimination or cylindrical algebraic decomposition (see [15] and [17]). In [13], an interesting historical discussion about Subresultant polynomials and polynomial remainder sequences is found.

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Given two polynomials, Subresultant polynomials are usually defined through their Sylvester matrix. Here, we present various alternative expressions which describe them in terms of some minors of matrices different from the Sylvester matrix. Such matrices are:

- the Bezout matrix,
- the Hybrid Bezout matrix,
- the Non–homogeneous Bezout matrix,
- the Barnett matrix, and
- the Hankel matrix.

Furthermore, these expressions allow us to provide new proofs for other expressions for Subresultant polynomials in terms of the roots of the two considered polynomials. By one hand, we prove the formula introduced by *J. J. Sylvester* in 1839, formula written in terms of a single sum over the roots. By other hand, by considering the Newton basis we obtain another expression in terms of the roots, which is very similar to the expression presented by H. Hong in [22].

All these expressions obviously provide new algorithms for computing Subresultant polynomials and new geometrical properties. However, our purpose is not to improve the sequential complexity of the best known algorithms, which are described by *the Subresultant Theorem* and its improved versions (see [3], [26], [27] for more details).

The paper is organized as follows. In the first section, some definitions and preliminaries are given. The second section introduces new expressions for Subresultant polynomials in terms of minors of the Bezout matrix, the Hybrid Bezout matrix, the Non–homogeneous Bezout matrix and the Barnett matrix. The third section introduces Subresultant polynomials written in terms of minors the Hankel matrix and the Horner basis. The proofs for results presented in the second and third sections are given in the fourth section. The fifth section presents as applications other expressions for Subresultant polynomials in terms of the roots of the given polynomials. We conclude with a remark on complexity in the sixth section.

## 1 Some Definitions and Preliminaries

Through the paper, $\mathrm{D}$ denotes an integral domain, $\mathrm{F}$ the fraction field of $\mathrm{D}$ and $J_n$ the backward identity matrix of order $n$:

$$J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

### *1.1 Sylvester Matrix and Subresultant Polynomials*

Let $P(x)$ and $Q(x)$ be two polynomials in $\mathrm{D}[x]$ of positive degrees,

$$P(x) = p_0x^n + p_1x^{n-1} + \ldots + p_n, \quad Q(x) = q_0x^m + q_1x^{m-1} + \ldots + q_m. \tag{1}$$

Next we introduce the well known definition of Sylvester matrix.

**Definition 1.1** *For $i \in \{0, \ldots, \inf(n, m) - 1\}$, the Sylvester matrix of index $i$ associated to $P(x)$, $n$, $Q(x)$ and $m$, denoted by $\mathbf{Sylv}_i(P, n, Q, m)$, is the $(n + m - 2i) \times (n + m - i)$ matrix:*

$$\mathbf{Sylv}_i(P, n, Q, m) = \overbrace{\begin{pmatrix} p_0 & \ldots & p_n & & \\ & \ddots & & \ddots & \\ & & p_0 & \ldots & p_n \\ q_0 & \ldots & q_m & & \\ & \ddots & & \ddots & \\ & & q_0 & \ldots & q_m \end{pmatrix}}^{n+m-i} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix}\right\} \begin{matrix} m-i \\ \\ \\ n-i \\ \\ \\ \end{matrix}$$

*The Sylvester matrix of index 0 associated to $P(x)$, $n$, $Q(x)$ and $m$ is denoted by* $\mathbf{Sylv}(P, n, Q, m)$.

If $\deg(P) = n$ and $\deg(Q) = m$ then the Sylvester matrix of index 0 is simply called the Sylvester matrix of $P(x)$ and $Q(x)$, denoted by $\mathbf{Sylv}(P, Q)$, and the Sylvester matrix of index $i \neq 0$ is denoted by $\mathbf{Sylv}_i(P, Q)$.

**Definition 1.2** *The determinant of* $\mathbf{Sylv}(P, Q)$ *is known as the resultant of $P(x)$ and $Q(x)$, denoted by* $\mathrm{res}(P, Q)$.

The concept of determinant polynomial associated to a matrix provides one of the usual ways to define Subresultant polynomials.

**Definition 1.3** *Let $\Delta$ be a $m \times n$ matrix with $m \leq n$. The determinant polynomial of $\Delta$,* $\mathbf{detpol}(\Delta)$*, is defined as:*

$$\mathbf{detpol}(\Delta) = \sum_{k=0}^{n-m} \det(\Delta_k) x^{n-m-k}$$

*where $\Delta_k$ is the square submatrix of $\Delta$ consisting of the first $m - 1$ columns and the $(k + m)$–th column.*

In these conditions, the Subresultant polynomial of index $i$ is defined as:

$$\mathbf{Sres}_i(P, Q) = \mathbf{detpol}(\mathbf{Sylv}_i(P, Q)).$$

One of the main characteristics of the Sylvester matrix and Subresultant polynomials is that they provide an algorithm for computing the greatest common divisor of two univariate polynomials. It is well known that Sylvester matrix verifies the following:

$$\deg(\gcd(P, Q)) = i \Longleftrightarrow \operatorname{rang}(\mathbf{Sylv}(P, Q)) = n + m - i$$

and in this case,

$$\gcd(P, Q) = \mathbf{Sres}_i(P, Q).$$

(See [8], [26] or [27] for more details).

### *1.2 Bezout Matrix*

Although the resultant of two univariate polynomials is known as the determinant of their Sylvester matrix, the original definition is given by the determinant of Bezout matrix, introduced by Bézout in 1748. The entries of Bezout matrix are bilinear functions of coefficients of the given polynomials and the most general definition of Bezout Matrix is the following.

Hereafter, we suppose that $n = \deg(P) \geq m = \deg(Q)$.

**Definition 1.4** *The Bezout Matrix associated to $P(x)$ and $Q(x)$ is the symmetric matrix:*

$$Bez(P, Q) = \begin{pmatrix} c_{0,0} & \cdots & c_{0,n-1} \\ \vdots & & \vdots \\ c_{n-1,0} & \cdots & c_{n-1,n-1} \end{pmatrix}$$

*where the $c_{i,j}$ are defined by the Cayley expression:*

$$\frac{P(x)Q(y) - P(y)Q(x)}{x - y} = \sum_{i,j=0}^{n-1} c_{i,j} x^i y^j.$$

The Bezout matrix is highly related to the Sylvester matrix and the greatest common divisor of polynomials. Similarly to Sylvester matrix, the Bezout matrix verifies the following:

$$\deg(\gcd(P, Q)) = n - \operatorname{rank}(\operatorname{Bez}(P, Q)). \tag{2}$$

In the literature, there are other matrices which verify Property 2 and are also associated to two polynomials. Next we are to present some of such matrices.

### *1.3 Hybrid Bezout Matrix*

The polynomials $P(x)$ and $x^{n-m}Q(x)$ have the same degree. If the first $i$ terms of each one are separated, we obtain two equations:

$$p_0x^n + \ldots + p_{i-1}x^{n-i+1} = -p_ix^{n-i} - \ldots - p_n$$

$$q_0x^n + \ldots + q_{i-1}x^{n-i+1} = -q_ix^{n-i} - \ldots - q_mx^{n-m}$$

Cancelling common factor of $x^{n-i+1}$ between the numerator and denominator yields

$$\frac{p_0x^{i-1} + \ldots + p_{i-1}}{q_0x^{i-1} + \ldots + q_{i-1}} = \frac{p_ix^{n-i} + \ldots + p_n}{q_ix^{n-i} + \ldots + q_mx^{n-m}}$$

Cross multiplying provides a polynomial which vanishes when $x$ is a zero of the $\gcd(P, Q)$:

$$k_i = (p_0x^{i-1} + \ldots + p_{i-1})(q_ix^{n-i} + \ldots + q_mx^{n-m})$$

$$-(p_ix^{n-i} + \ldots + p_n)(q_0x^{i-1} + \ldots + q_{i-1}) = \sum_{j=1}^{n} f_jx^{n-j}$$

and the coefficient of $x^{n-j}$, with $j \in \{1, \ldots, n\}$, in $k_i$ is

$$D_{0,j+i-1} + D_{1,j+i-2} + \ldots + D_{i-1,j} = \sum_{s=0}^{i-1} D_{s,i+j-1-s}$$

where

$$D_{r,t} = p_rq_t - p_tq_r$$

(if $r$ is out of range, $p_r = 0$ or $q_r = 0$).

Some authors define the next matrix as Bezout Matrix, for example, see [32] and [18] where the definition can be found with different order in rows. The Computer Algebra System `Maple` also defines the next matrix as Bezout Matrix.

**Definition 1.5** *The hybrid Bezout Matrix associated to $P(x)$ and $Q(x)$, denoted as $Hb(P, Q)$, is an $n$-square matrix whose entry $(i, j)$, with $1 \leq i \leq m$ and $1 \leq j \leq n$, is the coefficient of $x^{n-j}$ of the polynomial $k_{m-i+1}$, and the entry $(i, j)$, with $m + 1 \leq i \leq n$ and $1 \leq j \leq n$, is the coefficient of $x^{n-j}$ of the polynomial $x^{n-i}Q$.*

### *1.4 Non–homogeneous Bezout Matrix*

The Non–homogeneous Bezout Matrix of two polynomials is an $n$–square matrix whose first $m$ rows are the first $m$ rows of Bezout Matrix, and the last $n-m$ are the coefficients of polynomial $Q(x)$,

$$\mathrm{Nh}(P,Q)=\begin{pmatrix} c_{0,0} & \cdots & c_{0,m} & \cdots & c_{0,n-1} \\ \vdots & & \vdots & & \vdots \\ c_{m-1,0} & \cdots & c_{m-1,m} & \cdots & c_{m-1,n-1} \\ q_m & \cdots & q_0 & & \\ & \ddots & & \ddots & \\ & & q_m & \cdots & q_0 \end{pmatrix}.$$

This matrix can be found in [10]. For three bivariate polynomials of bidegree $(m,n)$, Dixon described three homogeneous determinants for the resultant; the similar to univariate resultants for two of such representations is the Sylvester and Bezout determinants. He also introduced a fourth determinant, a hybrid of the Sylvester and Bezout constructions. For univariate polynomials and in the case of that the degrees are different, the analogous to such a determinant is the Non–homogeneous determinant. This matrix is also found in [14].

### *1.5 Barnett Matrix*

Let $\Delta_P$ be the companion matrix of $P(x)$ given by

$$\Delta_P=\begin{pmatrix} 0 & 0 & \dots & 0 & -p_n \\ p_0 & 0 & \dots & 0 & -p_{n-1} \\ 0 & p_0 & \dots & 0 & -p_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & p_0 & -p_1 \end{pmatrix}. \tag{3}$$

The Barnett matrix associated to $P(x)$ and $Q(x)$ is the matrix $\widetilde{Q}(\Delta_P)$ with $\widetilde{Q}(x)=p_0^m Q(\frac{x}{p_0})$. Barnett originally used these matrices to obtain the greatest common divisor not only for two polynomials but for several ones (for more details, see [2] or [9]).

*Example 1.1* Consider the following polynomials in $\mathrm{Z}[x]$,

$$P(x)=6x^5-9x^4-3x^3-5x^2-4x-7,$$

$$Q(x)=x^4+7x^3+9x^2+3x-6.$$

Thus, the matrices introduced above are:

- The Sylvester matrix associated to $P(x)$ and $Q(x)$:

$$\mathbf{Sylv}(P, Q) = \begin{pmatrix} 6 & -9 & -3 & -5 & -4 & -7 & 0 & 0 & 0 \\ 0 & 6 & -9 & -3 & -5 & -4 & -7 & 0 & 0 \\ 0 & 0 & 6 & -9 & -3 & -5 & -4 & -7 & 0 \\ 0 & 0 & 0 & 6 & -9 & -3 & -5 & -4 & -7 \\ 1 & 7 & 9 & 3 & -6 & 0 & 0 & 0 & 0 \\ 0 & 1 & 7 & 9 & 3 & -6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 7 & 9 & 3 & -6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 7 & 9 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 7 & 9 & 3 & -6 \end{pmatrix}$$

- The Bezout matrix associated to $P(x)$ and $Q(x)$:

$$\mathrm{Bez}(P, Q) = \begin{pmatrix} 45 & 93 & 67 & 61 & -36 \\ 93 & 88 & 80 & -59 & 18 \\ 67 & 80 & -51 & -58 & 54 \\ 61 & -59 & -58 & -6 & 42 \\ -36 & 18 & 54 & 42 & 6 \end{pmatrix}$$

- The Hybrid Bezout Matrix associated to $P(x)$ and $Q(x)$:

$$\mathrm{Hb}(P, Q) = \begin{pmatrix} -32 & 89 & 103 & 105 & 21 \\ 23 & -24 & 125 & 103 & 63 \\ 57 & -37 & -24 & 89 & 49 \\ 51 & 57 & 23 & -32 & 7 \\ 1 & 7 & 9 & 3 & -6 \end{pmatrix}$$

- Non–homogeneous Bezout Matrix

$$\mathrm{Nh}(P, Q) = \begin{pmatrix} 45 & 93 & 67 & 61 & -36 \\ 93 & 88 & 80 & -59 & 18 \\ 67 & 80 & -51 & -58 & 54 \\ 61 & -59 & -58 & -6 & 42 \\ -6 & 3 & 9 & 7 & 1 \end{pmatrix}$$

- The Barnett matrix associated to $P(x)$ and $Q(x)$:

$$\widetilde{Q}(\Delta_P) = \begin{pmatrix} -7776 & 1512 & 12852 & 33642 & 62685 \\ 3888 & -6912 & 8856 & 32076 & 69462 \\ 11664 & 4968 & 2268 & 32886 & 76851 \\ 9072 & 12312 & 10476 & 16686 & 59751 \\ 1296 & 11016 & 28836 & 53730 & 97281 \end{pmatrix}$$

## 2 Subresultant Polynomials and Matrix Computation

In this section, we show how to express Subresultant polynomials in terms of minors of Bezout matrix. We also introduce relations of proportionality between the Bezout matrix and the other matrices presented in Section 1, which yield new expressions of Subresultant polynomials in terms of minors of such matrices.

It is well known that the sequence of principal subresultants can be obtained from principal minors of the Bezout matrix. In [31], a method to express Subresultant polynomials in terms of minors of the Hybrid Bezout Matrix is introduced. Here, we are going to obtain and to prove an expression of Subresultant polynomials in terms of minors of Bezout matrix.

For $k \in \{n-m+1, \ldots, n\}$ and $t \in \{0, \ldots, n-k\}$, let $\mathbf{B}^r_{k,t}$ denote the determinant of the submatrix

$$\left.\begin{pmatrix} c_{n-k,n-k-t} & \underbrace{c_{n-k,n-k-1} \cdots c_{n-k,n-1}}_{\text{the last } (k-1) \text{ columns}} \\ \vdots & \vdots \\ c_{n-1,n-k-t} & c_{n-1,n-k-1} \cdots c_{n-1,n-1} \end{pmatrix}\right\} \text{the last } k \text{ rows,}$$

the $(n-k-t+1)$-th column

extracted from $\mathrm{Bez}(P, Q)$. Thus, $\mathbf{B}^r_{k,0}$ denotes the principal minor of order $k$ but starting from the lower right hand corner of $\mathrm{Bez}(P, Q)$. Note that $\mathrm{Bez}(P, Q)$ is symmetric, and so the roles of rows and columns can be reversed in the definition of $\mathbf{B}^r_{k,t}$.

The following proposition describes the matricial relation between Sylvester and Bezout matrices.

**Proposition 2.1** *Let $P(x)$, $Q(x) \in \mathrm{D}[x]$, $n = \deg(P) \geq m = \deg(Q)$, denoted as*

$$P = p_0 x^n + p_1 x^{n-1} + \ldots + p_n, \quad Q = q_0 x^n + q_1 x^{n-1} + \ldots + + q_n,$$

*with $q_i = 0$, $0 \leq i \leq (n-m-1)$. Let the Sylvester matrix associated to $(P, n, Q, n)$ be partitioned as:*

$$\mathbf{Sylv}(P, n, Q, n) = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

*where*

$$T_1 = \begin{pmatrix} p_0 & \cdots & p_{n-1} \\ & \ddots & \vdots \\ O & & p_0 \end{pmatrix},\ T_2 = \begin{pmatrix} p_n & & \\ \vdots & \ddots & \\ p_1 & \cdots & p_n \end{pmatrix},$$

$$T_3 = \begin{pmatrix} q_0 & \cdots & q_{n-1} \\ & \ddots & \vdots \\ O & & q_0 \end{pmatrix},\ T_4 = \begin{pmatrix} q_n & & \\ \vdots & \ddots & \\ q_1 & \cdots & q_n \end{pmatrix}.$$

*Then*

$$\begin{pmatrix} \mathrm{I}_n & 0_n \\ -T_3 & T_1 \end{pmatrix} \mathbf{Sylv}(P, n, Q, n) = \begin{pmatrix} T_1 & T_2 \\ 0 & \mathrm{Bez}(P, Q) J_n \end{pmatrix}.$$

*Proof.* For a proof, see [2], [28] or the **Gohberg-Semencul** Formula in [24]. □

The next theorem shows how to express Subresultant polynomials in terms of minors of Bezout matrix.

**Theorem 2.1** *Let* $P(x),\ Q(x) \in \mathrm{D}[x]$, *with* $n = \deg(P) \geq m = \deg(Q)$ *and* $\mathrm{lcoef}(P) = p_0$. *Then the Subresultant polynomials of* $P(x)$ *and* $Q(x)$ *can be expressed in terms of minors of the matrix* $\mathrm{Bez}(P, Q)$ *as follows:*

$$(-1)^{k(k-1)/2} p_0^{n-m} \mathbf{Sres}_{n-k}(P, Q) = \mathbf{B}^r_{k,0} x^{n-k} + \mathbf{B}^r_{k,1} x^{n-k-1} + \ldots + \mathbf{B}^r_{k,n-k}.$$

*Proof.* The proof of this result is found in Section 4. □

*Example 2.1* Consider the polynomials given in Example 1.1. By Theorem 2.1, the sequence of Subresultant polynomials are given by:

- $\mathbf{k = 5}$ :
  $6\,\mathrm{Res}(P, Q) = \det(\mathrm{Bez}(P, Q)) = 7212464292.$
- $\mathbf{k = 4}$ :
  $6\,\mathbf{Sres}_1(P, Q) = 6(6181921x - 3813345) = \mathbf{B}^r_{4,0} x + \mathbf{B}^r_{4,1}$ where

$$\mathbf{B}^r_{4,0} = \begin{vmatrix} 88 & 80 & -59 & 18 \\ 80 & -51 & -58 & 54 \\ -59 & -58 & -6 & 42 \\ 18 & 54 & 42 & 6 \end{vmatrix} \text{ and } \mathbf{B}^r_{4,1} = \begin{vmatrix} 93 & 80 & -59 & 18 \\ 67 & -51 & -58 & 54 \\ 61 & -58 & -6 & 42 \\ -36 & 54 & 42 & 6 \end{vmatrix}.$$

- $\mathbf{k = 3}$ :
$-6\,\mathbf{Sres}_2(P, Q) = -6\,(28996x^2 + 56060x - 19168) = \mathbf{B}^r_{3,0}x^2 + \mathbf{B}^r_{3,1}x + \mathbf{B}^r_{3,2}$ where

$$\mathbf{B}^r_{3,0} = \begin{vmatrix} -51 & -58 & 54 \\ -58 & -6 & 42 \\ 54 & 42 & 6 \end{vmatrix}, \quad \mathbf{B}^r_{3,1} = \begin{vmatrix} 80 & -58 & 54 \\ -59 & -6 & 42 \\ 18 & 42 & 6 \end{vmatrix}, \text{ and}$$

$$\mathbf{B}^r_{3,2} = \begin{vmatrix} 67 & -58 & 54 \\ 61 & -6 & 42 \\ -36 & 42 & 6 \end{vmatrix}.$$

- $\mathbf{k = 2}$ :
$-6\,\mathbf{Sres}_3(P, Q) = -6\,(300x^3 + 436x^2 + 185x - 313) = \mathbf{B}^r_{2,0}x^3 + \mathbf{B}^r_{2,1}x^2 + \mathbf{B}^r_{2,2}x + \mathbf{B}^r_{2,3}$ where

$$\mathbf{B}^r_{2,0} = \begin{vmatrix} -6 & 42 \\ 42 & 6 \end{vmatrix}, \quad \mathbf{B}^r_{2,1} = \begin{vmatrix} -58 & 42 \\ 54 & 6 \end{vmatrix}, \text{ and}$$

$$\mathbf{B}^r_{2,2} = \begin{vmatrix} -59 & 42 \\ 18 & 6 \end{vmatrix}, \quad \mathbf{B}^r_{2,3} = \begin{vmatrix} 61 & 42 \\ -36 & 6 \end{vmatrix}.$$

Now, our next goal is to generalize the result of Theorem 2.1 to the other matrices introduced in Section 1.

**Proposition 2.2** *1. The Hybrid Bezout matrix factors as*

$$Hb(P, Q) = S \cdot Bez(P, Q)J_n, \tag{4}$$

*where*

$$S = \begin{pmatrix} 1 & & p_m & \cdots & p_{n-1} \\ & \ddots & \vdots & & \vdots \\ & & 1 & p_1 & \cdots & p_{n-m} \\ & & & p_0 & \cdots & p_{n-m-1} \\ & & & & \ddots & \vdots \\ & & & & & p_0 \end{pmatrix}^{-1}.$$

*2. The Non–homogeneous Bezout Matrix factors as*

$$Bez(P, Q) = \begin{pmatrix} \mathrm{I}_m & 0_{m,n-m} \\ 0_{n-m,m} & A_{n-m,n-m} \end{pmatrix} Nh(P, Q),$$

*where*

$$A_{n-m,n-m} = \begin{pmatrix} p_{n-m-1} & \cdots & p_0 \\ \vdots & \iddots & \\ p_0 & & \end{pmatrix}.$$

*3. The Barnett's Factorization claims the following:*

$$p_0^m \mathrm{Bez}(P, Q) = \widetilde{Q}(\Delta_P)\mathrm{Bez}(P, 1) = \mathrm{Bez}(P, 1)\widetilde{Q}(\Delta_P^t).$$

*Proof.* For a proof, 1.: see [9]; 2.: see [14] and 3.: see [2] or [4]. □

We must introduce here some notation in order to distinguish better the different minors which we are going to consider. Given an $n$–square matrix $A = (a_{i,j})$, for $j \in \{n, \ldots, 1\}$ and $t \in \{0, \ldots, n-j\}$:

- $\mathbf{A}^l_{j,t}$ will denote the determinant of the $j$–square submatrix

$$\underbrace{\begin{pmatrix} a_{n-j+1-t,1} & a_{n-j+1-t,2} & \cdots & a_{n-j+1-t,j} \\ a_{n-j+2,1} & a_{n-j+2,2} & \cdots & a_{n-j+2,j} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,j} \end{pmatrix}}_{\text{the first } j \text{ columns}} \begin{matrix} \text{--- > the } (n-j+1-t)\text{-th row} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{the last } (j-1) \text{ rows} \end{matrix}$$

  of $A$. Thus, $\mathbf{A}^l_{j,0}$ denotes the principal minor of order $j$ starting from the lower left hand corner of $A$.
- $\mathbf{A}^r_{j,t}$ will denote the determinant of the $j$–square submatrix

$$\underbrace{\begin{pmatrix} a_{n-j+1-t,n-j+1} & a_{n-j+1-t,n-j+2} & \cdots & a_{n-j+1-t,n} \\ a_{n-j+2,n-j+1} & a_{n-j+2,n-j+2} & \cdots & a_{n-j+2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,n-j+1} & a_{n,n-j+2} & \cdots & a_{n,n} \end{pmatrix}}_{\text{the last } j \text{ columns}} \begin{matrix} \text{--- > the } (n-j+1-t)\text{-th row} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{the last } (j-1) \text{ rows} \end{matrix}$$

  of $A$. Thus, $\mathbf{A}^r_{j,0}$ denotes the principal minor of order $j$ but starting from the lower right hand corner of $A$.

Once introduced the required notation, we can already present and prove the following result.

**Theorem 2.2** *The Subresultant polynomials can be computed as follows:*

1. $p_0^{m(n-k)+(k-m)} \cdot \mathbf{Sres}_k(P, Q) = \widetilde{\mathbf{Q}}^l_{n-k,0}x^k + \widetilde{\mathbf{Q}}^l_{n-k,1}x^{k-1} + \ldots + \widetilde{\mathbf{Q}}^l_{n-k,k}$
2. $\mathbf{Sres}_k(P, Q) = \mathbf{Hb}^l_{n-k,0}x^k + \mathbf{Hb}^l_{n-k,1}x^{k-1} + \ldots + \mathbf{Hb}^l_{n-k,k}$
3. $(-1)^{(n-k)(n-k-1)/2}\mathbf{Sres}_k(P, Q) = (-1)^{(n-m)(n-m-1)/2}(\mathbf{Nh}^r_{n-k,0}x^k + \mathbf{Nh}^r_{n-k,1}x^{k-1} + \ldots + \mathbf{Nh}^r_{n-k,k})$

*Proof.* The proof of this result is found in Section 4. □

As we said before, the statement (2) of the previous theorem is also proved in [31] in a different way.

*Example 2.2* Consider the polynomials given in Example 1.1.

By Theorem 2.2, the Subresultant polynomial $\mathbf{Sres}_2(P, Q) = 28996x^2 + 56060x - 19168$ is given:

- By the Barnett matrix, $6^{10}\,\mathbf{Sres}_2(P, Q) = \widetilde{\mathbf{Q}}^l_{3,0}x^2 + \widetilde{\mathbf{Q}}^l_{3,1}x + \widetilde{\mathbf{Q}}^l_{3,2}$ , where

$$\widetilde{\mathbf{Q}}^l_{3,0} = \begin{vmatrix} 11664 & 4968 & 2268 \\ 9072 & 12312 & 10476 \\ 1296 & 11016 & 28836 \end{vmatrix},\ \widetilde{\mathbf{Q}}^l_{3,1} = \begin{vmatrix} 3888 & -6912 & 8856 \\ 9072 & 12312 & 10476 \\ 1296 & 11016 & 28836 \end{vmatrix} \text{ and}$$

$$\widetilde{\mathbf{Q}}^l_{3,2} = \begin{vmatrix} -7776 & 1512 & 12852 \\ 9072 & 12312 & 10476 \\ 1296 & 11016 & 28836 \end{vmatrix}.$$

- By The Hybrid Bezout Matrix, $\mathbf{Sres}_2(P, Q) = \mathbf{Hb}^l_{3,0}x^2 + \mathbf{Hb}^l_{3,1}x + \mathbf{Hb}^l_{3,2}$, where

$$\mathbf{Hb}^l_{3,0} = \begin{vmatrix} 57 & -37 & -24 \\ 51 & 57 & 23 \\ 1 & 7 & 9 \end{vmatrix},\ \mathbf{Hb}^l_{3,1} = \begin{vmatrix} 23 & -24 & 125 \\ 51 & 57 & 23 \\ 1 & 7 & 9 \end{vmatrix} \text{ and}$$

$$\mathbf{Hb}^l_{3,2} = \begin{vmatrix} -32 & 89 & 103 \\ 51 & 57 & 23 \\ 1 & 7 & 9 \end{vmatrix}.$$

- By the Non–homogeneous Bezout Matrix, $\mathbf{Sres}_2(P, Q) = -(\mathbf{Nh}^r_{3,0}x^2 + \mathbf{Nh}^r_{3,1}x + \mathbf{Nh}^r_{3,2})$, where

$$\mathbf{Nh}^r_{3,0} = \begin{vmatrix} -51 & -58 & 54 \\ -58 & -6 & 42 \\ 9 & 7 & 1 \end{vmatrix},\ \mathbf{Nh}^r_{3,1} = \begin{vmatrix} 80 & -59 & 18 \\ -58 & -6 & 42 \\ 9 & 7 & 1 \end{vmatrix} \text{ and}$$

$$\mathbf{Nh}^r_{3,2} = \begin{vmatrix} 67 & 61 & -36 \\ -58 & -6 & 42 \\ 9 & 7 & 1 \end{vmatrix}$$

## 3 The Hankel Matrix and Horner Polynomials

In the literature, the Hankel Matrix is another well known matrix which is highly related to the greatest common divisor of two univariate polynomials. Moreover, Kronecker already investigated this matrix obtaining the first occurrence of Subresultant polynomials (see [13]).

Hereafter, we assume for simplicity that $p_0 = 1$. Let $R(x)$ be the power series expansion of the function $Q(x)/P(x)$ with $m < n$

$$R(x) = \frac{Q(x)}{P(x)} = \sum_{i=1}^{\infty} h_i x^{-i}.$$

This power series defines the $n \times n$ Hankel matrix, $\mathrm{H}(P, Q)$, whose $(i, j)$ entry is $h_{i+j-1}$ $(i, j \in \{1, \ldots, n\})$

$$\mathrm{H}(P, Q) = \begin{pmatrix} h_1 & h_2 & \cdots & h_n \\ h_2 & h_3 & \cdots & h_{n+1} \\ \vdots & \vdots & & \vdots \\ h_n & h_{n+1} & \cdots & h_{2n-1} \end{pmatrix}.$$

The Hankel matrix can be factored as follows:

$$\mathrm{H}(P, Q) = \mathrm{Bez}^{-1}(P, 1)\mathrm{Bez}(P, Q)\ \mathrm{Bez}^{-1}(P, 1) = \mathrm{Bez}^{-1}(P, 1)Q(\Delta_P).$$

*Example 3.1* Consider the following polynomials in $\mathrm{Z}[x]$,

$$P(x) = x^5 - 9x^4 - 3x^3 - 5x^2 - 4x - 7,$$

$$Q(x) = x^4 + 7x^3 + 9x^2 + 3x - 6,$$

the Hankel matrix associated to $P(x)$ and $Q(x)$ is:

$$\mathrm{H}(P, Q) = \begin{pmatrix} 1 & 16 & 156 & 1460 & 13686 \\ 16 & 156 & 1460 & 13686 & 128405 \\ 156 & 1460 & 13686 & 128405 & 1204739 \\ 1460 & 13686 & 128405 & 1204739 & 11303228 \\ 13686 & 128405 & 1204739 & 11303228 & 106050258 \end{pmatrix}$$

If $\mathrm{F}_n[x]$ is the F-vector space of polynomials in $\mathrm{F}[x]$ with degree smaller than $n$, then the usual basis of $\mathrm{F}_n[x]$ is the Standard Basis given by:

$$\mathcal{B}_{\mathrm{St}} = \{1, x, \ldots, x^{n-1}\}.$$

There is another basis in $\mathrm{F}_n[x]$ called the Horner Basis and denoted by $\mathcal{B}_{\mathrm{Ho}}$, which is defined from Horner polynomials associated to $P(x)$.

**Definition 3.1** *Horner polynomials associated to $P(x)$, denoted by $\alpha_1, \ldots, \alpha_n$, are defined by recursion in the following way:*

$$\alpha_n(x) = 1, \qquad \alpha_{n-k}(x) = x\alpha_{n-k+1}(x) + p_k, \qquad k = 1, \ldots, n-1$$

Our next result shows that the minors of this matrix are also valid for obtaining Subresultant polynomials, written not in the Standard Basis but in the Horner Basis. For $k \in \{0, \ldots, m-1\}$ and $t \in \{0, \ldots, k\}$, let $\mathbf{H}_{n-k,t}$ denote the determinant of the $(n-k)$–square submatrix

$$\underbrace{\left.\begin{pmatrix} h_1 & \cdots & h_{n-k} \\ h_2 & \cdots & h_{n-k+1} \\ \vdots & & \vdots \\ h_{n-k-1} & \cdots & h_{2(n-k-1)} \\ h_{n-k+t} & \cdots & h_{2(n-k)-1+t} \end{pmatrix}\right\}}_{\text{the first } (n-k) \text{ columns}} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \\ a \end{matrix}}\right\} \text{the first } (n-k-1) \text{ rows} \\ --> \text{the } (n-k+t)\text{-th row} \end{matrix}$$

of $\mathrm{H}(P, Q)$. Thus, $\mathbf{H}_{n-k,0}$ denotes the principal minor of order $(n-k)$ of the matrix $\mathrm{H}(P, Q)$.

**Theorem 3.1** *Let $P(x),\ Q(x) \in \mathrm{D}[x]$, with $n = \deg(P) \geq m = \deg(Q)$ and $p_0 = 1$. Then:*

$$\mathbf{Sres}_k(P, Q) = (-1)^{(n-k)(n-k-1)/2} \times \left(\mathbf{H}_{n-k,0} \cdot \alpha_{n-k} + \mathbf{H}_{n-k,1} \cdot \alpha_{n-k+1} + \ldots + \mathbf{H}_{n-k,k} \cdot \alpha_n\right).$$

*Proof.* The proof of this result is found in Section 4. □

*Remark 1* ($\mathbf{p_0 \neq 1}$) If $P(x)$ is not monic, then the Hankel matrix factorizes as follows:

$$\mathrm{H}(P, Q) = \mathrm{Bez}^{-1}(P, 1)\mathrm{Bez}(P, Q)\ \mathrm{Bez}^{-1}(P, 1) = \mathrm{Bez}^{-1}(P, 1)\, Q(\Delta_{P/p_0}). \tag{5}$$

Since $p_0^m\ Q(\Delta_{P/p_0}) = \widetilde{Q}(\Delta_P)$, by Theorem 2.2 we obtain the following expression for Subresultant polynomials in terms of minors of $Q(\Delta_{P/p_0})$:

$$\mathbf{Sres}_k(P, Q) = p_0^{m-k}\ \left(\mathbf{Q}^l_{n-k,0} x^k + \mathbf{Q}^l_{n-k,1} x^{k-1} + \ldots + \mathbf{Q}^l_{n-k,k}\right). \tag{6}$$

Thus, when $p_0 \neq 1$, following the same reasoning of the proof for Theorem 3.1 but considering Equations (5) and (6), we easily obtain:

$$\mathbf{Sres}_k(P, Q) = p_0^{n+m-2k-1}\ (-1)^{(n-k)(n-k-1)/2} \times \left(\mathbf{H}_{n-k,0} \cdot \alpha_{n-k} + \mathbf{H}_{n-k,1} \cdot \alpha_{n-k+1} + \ldots + \mathbf{H}_{n-k,k} \cdot \alpha_n\right).$$

## 4 Proofs

*Proof of Theorem 2.1.* Throughout the proof, we will use the following lemma.

**Lemma 4.1** *Let $A$, $B$, $C$ and $D$ be square matrices such that $A$ is non–singular and $A$ and $C$ commute. Then*

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

Now, we start with the proof:

Since:

$$\begin{aligned}
&\mathrm{Bez}(P, Q) \\
&= (T_1 \cdot T_4 - T_3 \cdot T_2) \cdot J_n \\
&= \left[ \begin{pmatrix} p_0 & \cdots & p_{n-1} \\ & \ddots & \vdots \\ & & p_0 \end{pmatrix} \begin{pmatrix} q_n & & \\ \vdots & \ddots & \\ q_1 & \cdots & q_n \end{pmatrix} - \begin{pmatrix} q_0 & \cdots & q_{n-1} \\ & \ddots & \vdots \\ & & q_0 \end{pmatrix} \begin{pmatrix} p_n & & \\ \vdots & \ddots & \\ p_1 & \cdots & p_n \end{pmatrix} \right] \cdot J_n,
\end{aligned}$$

the $(k-1)$ last columns of $\mathrm{Bez}(P, Q)$ are the first $(k-1)$ columns of $T_1 \cdot T_4 - T_3 \cdot T_2$, and the $(n-k-t+1)$–th column of $\mathrm{Bez}(P, Q)$ is the $(k+t)$–th column of $T_1 \cdot T_4 - T_3 \cdot T_2$. Hence:

$$\mathbf{B}^r_{k,t} = (-1)^{k(k-1)/2} \left| \begin{pmatrix} p_0 & \cdots & p_{k-1} \\ & \ddots & \vdots \\ & & p_0 \end{pmatrix} \cdot \begin{pmatrix} q_k & \cdots & q_n & & & \\ \vdots & & & \ddots & & \\ q_{n-k+2} & & & & q_n & \\ \vdots & & & & \vdots & \\ q_{n-k+1-t} & & & & q_{n-t-1} & q_n \\ \vdots & & & & \vdots & \vdots \\ q_1 & \cdots & q_{n-k+1} & \cdots & q_{k-1} & q_{k+t} \end{pmatrix} \right.$$

$$\left. - \begin{pmatrix} q_0 & \cdots & q_{k-1} \\ & \ddots & \vdots \\ & & q_0 \end{pmatrix} \cdot \begin{pmatrix} p_k & \cdots & p_n & & & \\ \vdots & & & \ddots & & \\ p_{n-k+2} & & & & p_n & \\ \vdots & & & & \vdots & \\ p_{n-k+1-t} & & & & p_{n-t-1} & p_n \\ \vdots & & & & \vdots & \vdots \\ p_1 & \cdots & p_{n-k+1} & \cdots & p_{k-1} & p_{k+t} \end{pmatrix} \right| .$$

Furthermore, since $p_0 \neq 0$ and the matrices

$$\begin{pmatrix} p_0 & \cdots & p_{k-1} \\ & \ddots & \vdots \\ & & p_0 \end{pmatrix} \quad , \quad \begin{pmatrix} q_0 & \cdots & q_{k-1} \\ & \ddots & \vdots \\ & & q_0 \end{pmatrix}$$

commute, by Lemma 4.1, it follows that

$$(-1)^{\frac{k(k-1)}{2}}\mathbf{B}^r_{k,t}$$

is equal to:

$$\begin{vmatrix}
p_0 & \cdots & \cdots & p_{k-1} & p_k & \cdots & p_n & & \\
 & \ddots & & \vdots & \vdots & & & \ddots & \\
 & & \ddots & p_{n-k+1} & p_{n-k+2} & & & p_n & \\
 & & & \vdots & \vdots & & & \vdots & \\
 & & & p_{n-k-t} & p_{n-k+1-t} & & & p_{n-t-1} & p_n \\
 & & & \vdots & \vdots & & & \vdots & \vdots \\
 & & & p_0 & p_1 & \cdots & p_{n-k+1} \cdots & p_{k-1} & p_{k+t} \\
q_0 & \cdots & \cdots & q_{k-1} & q_k & \cdots & q_n & & \\
 & \ddots & & \vdots & \vdots & & & \ddots & \\
 & & \ddots & q_{n-k+1} & q_{n-k+2} & & & q_n & \\
 & & & \vdots & \vdots & & & \vdots & \\
 & & & q_{n-k-t} & q_{n-k+1-t} & & & q_{n-t-1} & q_n \\
 & & & \vdots & \vdots & & & \vdots & \vdots \\
 & & & q_0 & q_1 & \cdots & q_{n-k+1} \cdots & q_{k-1} & q_{k+t}
\end{vmatrix}$$

that is the coefficient in $x^{n-k-t}$ of the polynomial $\mathbf{Sres}_{n-k}(P, n, Q, n)$. Consequently:

$$(-1)^{k(k-1)/2}\mathbf{Sres}_{n-k}(P, n, Q, n) = \mathbf{B}^r_{k,0}x^{n-k} + \mathbf{B}^r_{k,1}x^{n-k-1} + \ldots + \mathbf{B}^r_{k,n-k}.$$

If $n > m$, by applying the property

$$\mathbf{Sres}_{n-k}(P, n, Q, n) = p_0^{n-m}\mathbf{Sres}_{n-k}(P, Q),$$

the result is obtained. □

*Proof of Theorem 2.2.* (1) First, let us denote $\widetilde{Q}(\Delta_P) = (q_{i,j})$ and suppose that $p_0 = 1$. Let $t \in \{0, \ldots, k\}$. By Barnett's factorization we have that

$$\mathrm{Bez}(P, Q) = Q(\Delta_P)\mathrm{Bez}(P, 1),$$

and so

$$\mathbf{B}^r_{n-k,t} = \det\left(\begin{pmatrix} q_{k+1-t,1} & q_{k+1-t,2} & \cdots & q_{k+1-t,n} \\ q_{k+2,1} & q_{k+2,2} & \cdots & q_{k+2,n} \\ \vdots & \vdots & & \vdots \\ q_{n,1} & q_{n,2} & \cdots & q_{n,n} \end{pmatrix} \cdot \begin{pmatrix} p_{n-k-1} & \cdots & p_0 \\ \vdots & \iddots & \\ p_0 & & \\ 0 & & \\ \vdots & & \\ 0 & & \end{pmatrix}\right),$$

and simplifying

$$\mathbf{B}^r_{n-k,t} = \det\left(\begin{pmatrix} q_{k+1-t,1} & q_{k+1-t,2} & \cdots & q_{k+1-t,n-k} \\ q_{k+2,1} & q_{k+2,2} & \cdots & q_{k+2,n-k} \\ \vdots & \vdots & & \vdots \\ q_{n,1} & q_{n,2} & \cdots & q_{n,n-k} \end{pmatrix} \cdot \begin{pmatrix} p_{n-k-1} & \cdots & p_0 \\ \vdots & \iddots & \\ p_0 & & \end{pmatrix}\right)$$

Hence, with $p_0 = 1$,

$$\mathbf{B}^r_{n-k,t} = \mathbf{Q}^l_{n-k,t}(-1)^{(n-k)(n-k-1)/2}$$

and by applying Theorem 2.1 it follows that

$$\text{coef } x^{k-t} \text{in } \mathbf{Sres}_k(P, Q) = \mathbf{Q}^l_{n-k,t}.$$

If $p_0 \neq 1$ then the factorization

$$p_0^m \text{Bez}(P, Q) = \widetilde{Q}(\Delta_P)\text{Bez}(P, 1)$$

and the property

$$\det(aM) = a^n \det(M)$$

provide that

$$p_0^{m(n-k)+(k-m)} \cdot \text{coef } x^{k-t} \text{in } \mathbf{Sres}_k(P, Q) = \widetilde{\mathbf{Q}}^l_{n-k,t},$$

which completes the proof of the statement (1).

(2) Equality (4) implies that

$$\text{Bez}(P, Q) = \begin{pmatrix} 1 & & p_m & \cdots & p_{n-1} \\ & \ddots & \vdots & & \vdots \\ & & 1 & p_1 & \cdots & p_{n-m} \\ & & & p_0 & \cdots & p_{n-m-1} \\ & & & & \ddots & \vdots \\ & & & & & p_0 \end{pmatrix} \cdot \text{Hb}(P, Q) \cdot J_n,$$

that means that if $H_{*,j}$ denotes the $j$–th column of $\text{Hb}(P, Q)$, then

$$\text{Bez}(P, Q) = \begin{pmatrix} 1 & & p_m & \cdots & p_{n-1} \\ & \ddots & \vdots & & \vdots \\ & & 1 & p_1 & \cdots & p_{n-m} \\ & & & p_0 & \cdots & p_{n-m-1} \\ & & & & \ddots & \vdots \\ & & & & & p_0 \end{pmatrix} \cdot (H_{*,n}, \ldots, H_{*,1}).$$

Therefore, if $t \in \{0, \ldots, k\}$ and $\text{Hb}(P, Q) = (Hb_{i,j})$ then:

$$\mathbf{B}^r_{n-k,t} = \det\left(\left(\begin{pmatrix} 0\dots0\,1\,0\dots\,0 & p_{m-k+t} & \cdots & p_{n-k-1+t} \\ \quad 1 & p_{m-k-1} & \cdots & p_{n-k-2} \\ \quad\ddots & \vdots & & \vdots \\ \quad 1 & p_1 & \cdots & p_{n-m} \\ & p_0 & \cdots & p_{n-m-1} \\ & & \ddots & \vdots \\ & & & p_0 \end{pmatrix} \cdot (H_{*,n-k}, \dots, H_{*,1})\right),$$

and simplifying:

$$\mathbf{B}^r_{n-k,t} = \det\left(\left(\begin{pmatrix} 1 & p_{m-k+t} & \cdots & p_{n-k-1+t} \\ \quad 1 & p_{m-k-1} & \cdots & p_{n-k-2} \\ \quad\ddots & \vdots & & \vdots \\ \quad 1 & p_1 & \cdots & p_{n-m} \\ & p_0 & \cdots & p_{n-m-1} \\ & & \ddots & \vdots \\ & & & p_0 \end{pmatrix} \cdot \begin{pmatrix} Hb_{k+1-t,n-k} & \cdots & Hb_{k+1-t,1} \\ Hb_{k+2,n-k} & \cdots & Hb_{k+2,1} \\ \vdots & & \vdots \\ Hb_{n,n-k} & \cdots & Hb_{n,1} \end{pmatrix}\right)$$

Hence,

$$\mathbf{B}^r_{n-k,t} = p_0^{n-m}(-1)^{(n-k)(n-k-1)/2}\mathbf{Hb}^l_{n-k,t}$$

and by applying Theorem 2.1 it follows that

$$\operatorname{coef} x^{k-t} \text{in } \mathbf{Sres}_k(P, Q) = \mathbf{Hb}^l_{n-k,t}.$$

(3) Since

$$\operatorname{Bez}(P, Q) = \begin{pmatrix} \mathrm{I}_m & 0_{m,n-m} \\ 0_{n-m,m} & A_{n-m,n-m} \end{pmatrix} \operatorname{Nhbez}(P, Q),$$

if $t \in \{0, \dots, k-1\}$, we have that:

$$\mathbf{B}^r_{n-k,t} = \det\left(\begin{pmatrix} \mathrm{I}_{m-k} & 0_{m-k,n-m} \\ 0_{n-m,m-k} & A_{n-m,n-m} \end{pmatrix}\right) \cdot \mathbf{Nh}^r_{n-k,t}.$$

Hence

$$\mathbf{B}^r_{n-k,t} = (-1)^{(n-m)(n-m-1)/2} p_0^{n-m}\mathbf{Nh}^r_{n-k,t}$$

and by Theorem 2.1,

$$(-1)^{(n-k)(n-k-1)/2} \cdot \operatorname{coef} x^{k-t} \text{in } \mathbf{Sres}_k(P, Q) = (-1)^{(n-m)(n-m-1)/2}\mathbf{Nh}^r_{n-k,t},$$

which completes the proof. □

*Proof of Theorem 3.1.* Throughout the proof, we will use the Binet–Cauchy Theorem.

**Theorem 4.1** (Binet–Cauchy) *If $X$ and $Y$ are matrices of $p$ columns and $n$ rows each, $p \leq n$, then the determinant*

$$\det(Y^t X)$$

*is equal to the sum of the $\binom{n}{p}$ products of pairs of p–order determinants that can be formed by selection $p$ rows from $Y$ and the same $p$ rows of $X$.*

Now, we start with the proof:

Theorem 2.2 states that

$$\mathbf{Sres}_k(P, Q) = \mathbf{Q}^l_{n-k,0} x^k + \mathbf{Q}^l_{n-k,1} x^{k-1} + \ldots + \mathbf{Q}^l_{n-k,k},$$

and so $\{\mathbf{Q}^l_{n-k,k}, \ldots, \mathbf{Q}_{n-k,0}\}$ are the coordinates of $\mathbf{Sres}_k(P, Q)$ with respect the Standard Basis.

Thus, given the basis change matrix of $\mathcal{B}_{\mathrm{Ho}}$ to $\mathcal{B}_{\mathrm{St}}$ by the triangular matrix $\mathrm{Bez}(P, 1)$,

$$\mathrm{Bez}(P, 1) = \begin{pmatrix} p_{n-1} & p_{n-2} & \cdots & p_1 & p_0 \\ p_{n-2} & p_{n-3} & \ldots & p_0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ p_1 & p_0 & \ldots & 0 & 0 \\ p_0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

we have only to see that

$$\begin{pmatrix} \mathbf{Q}^l_{n-k,k} \\ \vdots \\ \mathbf{Q}^l_{n-k,0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathrm{Bez}(P, 1) \cdot (-1)^{(n-k)(n-k-1)/2} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{H}_{n-k,0} \\ \mathbf{H}_{n-k,1} \\ \vdots \\ \mathbf{H}_{n-k,k} \end{pmatrix}$$

by proving the next equality for $t \in \{0, \ldots, k\}$:

$$\mathbf{Q}^l_{n-k,t} = (p_{n-k+t-1}, \ldots, p_1, p_0, 0, \ldots, 0) \cdot (-1)^{(n-k)(n-k-1)/2} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{H}_{n-k,0} \\ \mathbf{H}_{n-k,1} \\ \vdots \\ \mathbf{H}_{n-k,k} \end{pmatrix}$$

$$= (-1)^{(n-k)(n-k-1)/2} \cdot \left(p_0 \mathbf{H}_{n-k,t} + p_1 \mathbf{H}_{n-k,t-1} + \ldots + p_t \mathbf{H}_{n-k,0}\right)$$

Since

$$Q(\Delta_P) = \mathrm{Bez}(P, 1) \cdot \mathrm{H}(P, Q),$$

then,

$$(\underbrace{Q_1}_{n-k}\ \underbrace{Q_2}_{k}) = \begin{pmatrix} p_{n-1} & \cdots & p_0 \\ \vdots & \iddots & \\ p_0 & & \end{pmatrix} (\underbrace{H_1}_{n-k}\ \underbrace{H_2}_{k}),$$

and by taking the first $(n-k)$ columns of both sides, we have:

$$(Q_1) = \begin{pmatrix} p_{n-1} & \cdots & p_0 \\ \vdots & \iddots & \\ p_0 & & \end{pmatrix} (H_1).$$

Hence,

$$\begin{aligned} \mathbf{Q}^l_{n-k,t} &= \left| \begin{pmatrix} p_{n-k-1+t} & \cdots & p_{t+1} & \cdots & p_0 & \cdots \\ p_{n-k-2} & \cdots & p_0 & & & \\ \vdots & \iddots & & & & \\ p_0 & & & & & \end{pmatrix} \cdot (H_1) \right| \\ &\overset{\text{Binet-Cauchy}}{=} \sum_{0\le i\le t} \left| \begin{pmatrix} p_{n-k-1+t} & \cdots & p_{t+1} & p_{t-i} \\ p_{n-k-2} & \cdots & p_0 & \\ \vdots & \iddots & & \\ p_0 & & & \end{pmatrix} \right| \cdot \mathbf{H}_{n-k,i} \\ &\overset{p_0=1}{=} \sum_{0\le i\le t} (-1)^{(n-k-1)(n-k-2)/2} (-1)^{n-k+1} \cdot p_{t-i} \cdot \mathbf{H}_{n-k,i} \\ &= (-1)^{(n-k)(n-k-1)/2} \sum_{0\le i\le t} p_{t-i} \cdot \mathbf{H}_{n-k,i} \end{aligned}$$

which completes the proof. □

## 5 Applications: Subresultants and Roots of $P(x)$

### *5.1 Newton and Standard Bases – Hong's Formula*

Suppose

$$P(x) = \prod_{i=1}^{n} (x - \lambda_i).$$

In this section we consider the Newton Basis of $\mathrm{F}_n[x]$, given by

$$\mathcal{B}_{\mathrm{Nw}} = \{(x-\lambda_n)\cdot \ldots \cdot (x-\lambda_2), \ldots, (x-\lambda_n), 1\},$$

and our first goal is to describe Subresultant polynomials in terms of this basis.

It's well known that the basis change matrix of $\mathcal{B}_{\mathrm{Nw}}$ to $\mathcal{B}_{\mathrm{St}}$ and its inverse are given by elementary symmetric functions and complete symmetric functions on roots of $P(x)$ respectively.

The companion matrix of $P(x)$ given by Equation (3) represents the endomorphism of $\mathrm{F}_n[x]$ defined by the multiplication by $p_0\,x$ with respect to the Standard Basis. Note that in this case $p_0 = 1$ because $P(x)$ is monic. If we consider the Newton basis, we obtain the following companion matrix

$$\Lambda_P = \begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_{n-1} & 1 \\ & & & & \lambda_n \end{pmatrix},$$

such that

$$\Lambda_P = \begin{pmatrix} & & & 1 \\ & & \iddots & \vdots \\ & 1 & \dots & c_{n-2}(\lambda_n, \lambda_{n-1}) \\ 1 & c_1(\lambda_n) & \dots & c_{n-1}(\lambda_n) \end{pmatrix}$$

$$\times \Delta_P \cdot \begin{pmatrix} (-1)^{n-1} e_{n-1}(\lambda_n, \dots, \lambda_2) & (-1)^{n-2} e_{n-2}(\lambda_n, \dots, \lambda_3) & \dots & 1 \\ \vdots & \vdots & \iddots & \\ -e_1(\lambda_n, \dots, \lambda_2) & 1 & & \\ 1 & & & \end{pmatrix}$$

where

- $e_i(\lambda_n, \dots, \lambda_j)$ denotes the $i$–th elementary symmetric function on $\{\lambda_n, \dots, \lambda_j\}$,
- $c_i(\lambda_n, \dots, \lambda_j)$ denotes the $i$–th complete symmetric function on $\{\lambda_n, \dots, \lambda_j\}$,
- $\Delta_P$ denotes the usual companion matrix of $P(x)$.

Moreover, if $Q(x) = \prod_{j=1}^{m} (x - \beta_j)$, then

$$Q(\Lambda_P) = \prod_{j=1}^{m} (\Lambda_P - \beta_j \mathrm{I}).$$

The next proposition describes Subresultant polynomials in terms of minors of the matrix $Q(\Lambda_P)$ and the Newton Basis.

We must first introduce some notation. For $k \in \{0, \dots, m-1\}$ and $t \in \{0, \dots, k\}$, then $\mathbf{Nw}^{r,u}_{n-k,t}$ will denote the determinant of the $(n-k)$–square submatrix of $Q(\Lambda_P)$ consisting of the last $(n-k)$ columns, the first $(n-k-1)$ rows and the $(n-k-t)$–th row. Thus, $\mathbf{Nw}^{r,u}_{n-k,0}$ denotes the principal minor of order $(n-k)$ but starting from the upper right hand corner of the matrix $Q(\Lambda_P)$.

**Proposition 5.1**

$$\mathbf{Sres}_k(P, Q) = \mathbf{Nw}^{r,u}_{n-k,0} \cdot (x - \lambda_n) \dots (x - \lambda_{n-k+1}) + \mathbf{Nw}^{r,u}_{n-k,1} \cdot (x - \lambda_n) \dots (x - \lambda_{n-k+2}) + \dots + \mathbf{Nw}^{r,u}_{n-k,k}$$

*Proof.* Theorem 2.2 states that

$$\mathbf{Sres}_k(P, Q) = \mathbf{Q}^l_{n-k,0} x^k + \mathbf{Q}^l_{n-k,1} x^{k-1} + \dots + \mathbf{Q}^l_{n-k,k},$$

and so $\{\mathbf{Q}^l_{n-k,k}, \dots, \mathbf{Q}_{n-k,0}\}$ are the coordinates of $\mathbf{Sres}_k(P, Q)$ with respect the Standard Basis.

Thus, given the basis change matrix of $\mathcal{B}_{\mathrm{Nw}}$ to $\mathcal{B}_{\mathrm{St}}$ by the triangular matrix $B$,

$$B = \begin{pmatrix} (-1)^{n-1} e_{n-1}(\lambda_n, \dots, \lambda_2) & (-1)^{n-2} e_{n-2}(\lambda_n, \dots, \lambda_3) & \dots & 1 \\ \vdots & \vdots & \iddots & \\ -e_1(\lambda_n, \dots, \lambda_2) & 1 & & \\ 1 & & & \end{pmatrix},$$

we have only to see that

$$\begin{pmatrix} \mathbf{Q}^l_{n-k,k} \\ \vdots \\ \mathbf{Q}^l_{n-k,0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = B \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{Nw}^{r,u}_{n-k,0} \\ \mathbf{Nw}^{r,u}_{n-k,1} \\ \vdots \\ \mathbf{Nw}^{r,u}_{n-k,k} \end{pmatrix}.$$

by proving the next equality for $t \in \{0, \dots, k\}$:

$$\begin{aligned} \mathbf{Q}^l_{n-k,t} &= ((-1)^{n-k+t-1} e_{n-k+t-1}(\lambda_n, \dots, \lambda_2), \dots, -e_1(\lambda_n, \dots, \lambda_{n-k+t}), 1, 0, \dots, 0) \\ &\quad \times \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{Nw}^{r,u}_{n-k,0} \\ \mathbf{Nw}^{r,u}_{n-k,1} \\ \vdots \\ \mathbf{Nw}^{r,u}_{n-k,k} \end{pmatrix} \\ &= \mathbf{Nw}^{r,u}_{n-k,t} + \dots + (-1)^t e_t(\lambda_n, \dots, \lambda_{n-k+1}) \mathbf{Nw}^{r,u}_{n-k,0} \end{aligned}$$

Since

$$Q(\Delta_P) = B \cdot Q(\Lambda_P) \cdot B^{-1}$$

$$= \begin{pmatrix} (-1)^{n-1}e_{n-1} & (-1)^{n-2}e_{n-2} & \dots & 1 \\ \vdots & \vdots & \iddots & \\ -e_1 & 1 & & \\ 1 & & & \end{pmatrix} \cdot Q(\Lambda_P) \cdot \begin{pmatrix} & & & 1 \\ & & \iddots & \vdots \\ & & 1 & \dots\ c_{n-2} \\ 1 & c_1 & \dots & c_{n-1} \end{pmatrix}$$

then,

$$(\underbrace{Q_1}_{n-k}\ \underbrace{Q_2}_{k}) = \begin{pmatrix} (-1)^{n-1}e_{n-1} & (-1)^{n-2}e_{n-2} & \dots & 1 \\ \vdots & \vdots & \iddots & \\ -e_1 & 1 & & \\ 1 & & & \end{pmatrix} \cdot Q(\Lambda_P) \cdot (\underbrace{B_1^{-1}}_{n-k}\ \underbrace{B_2^{-1}}_{k})$$

and by taking the first $(n-k)$ columns of both sides, we have:

$$(\,Q_1\,) = \begin{pmatrix} (-1)^{n-1}e_{n-1} & (-1)^{n-2}e_{n-2} & \dots & 1 \\ \vdots & \vdots & \iddots & \\ -e_1 & 1 & & \\ 1 & & & \end{pmatrix} \cdot Q(\Lambda_P) \cdot \begin{pmatrix} & & & 0 \\ & & & \vdots \\ & & & 1 \\ & & \iddots & \vdots \\ 1 & c_1 & \dots & c_{n-k-1} \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{n-1}e_{n-1} & (-1)^{n-2}e_{n-2} & \dots & 1 \\ \vdots & \vdots & \iddots & \\ -e_1 & 1 & & \\ 1 & & & \end{pmatrix} \cdot \underbrace{(\,Q_2(\Lambda_P)\,)}_{\text{last}\,n-k\,\text{col.}} \cdot \begin{pmatrix} & & 1 \\ & \iddots & \vdots \\ 1 & c_1 & \dots\ c_{n-k-1} \end{pmatrix}$$

Hence,

$$\mathbf{Q}^{l}_{n-k,t} = \left| \begin{pmatrix} (-1)^{n-k-1+t}e_{n-k-1+t} & \cdots & (-1)^{t+1}e_{t+1} & \cdots & 1 & \cdots & 0 \\ (-1)^{n-k-2}e_{n-k-2} & \cdots & 1 & & & & \\ \vdots & \iddots & & & & & \\ 1 & & & & & & \end{pmatrix} (Q_2(\Lambda_P)) \right| \cdot \left| \begin{matrix} & & 1 \\ & \iddots & \vdots \\ 1 & c_1 \dots & c_{n-k-1} \end{matrix} \right|$$

$$\overset{\text{Binet–Cauchy}}{=} \sum_{0 \le i < t} \left| \begin{pmatrix} (-1)^{n-k-1+t}e_{n-k-1+t} & \cdots & (-1)^{t+1}e_{t+1} & (-1)^{t-i}e_{t-i} \\ (-1)^{n-k-2}e_{n-k-2} & \cdots & 1 & \\ \vdots & \iddots & & \\ 1 & & & \end{pmatrix} \right| \cdot \mathbf{Nw}^{r,u}_{n-k,i}$$

$$= \sum_{0 \le i \le t} (-1)^{t-i}e_{t-i} \cdot \mathbf{Nw}^{r,u}_{n-k,i}$$

which completes the proof. □

Finally, we compare the last expression for Subresultant polynomials with the formula introduced by H. Hong in [22]. They both provide expressions for Subresultant polynomials in terms of the roots.

Given the matrix $Q(\Lambda_P)$ and $k \in \{0, \ldots, m-1\}$ and $t \in \{0, \ldots, k\}$:

- $\mathbf{Nw}^{r,u}_{n-k,t}$ denotes the determinant of the $(n-k)$–square submatrix of $Q(\Lambda_P)$ formed by taking the last $(n-k)$ columns, the first $(n-k-1)$ rows and the $(n-k-t)$–th row.
- $\mathbf{Nw}_{n-k,t}$ denotes the determinant of the $(n-k)$–square submatrix of $Q(\Lambda_P)$ formed by taking the $(n-k)$ first rows, the last $(n-k-1)$ columns and the $(n-k-t)$–th column.

Then Proposition 5.1 claims that:

$$\begin{aligned}\mathbf{Sres}_k(P, Q) = \mathbf{Nw}^{r,u}_{n-k,0} \cdot (x-\lambda_n)\ldots(x-\lambda_{n-k+1}) \\ +\mathbf{Nw}^{r,u}_{n-k,1} \cdot (x-\lambda_n)\ldots(x-\lambda_{n-k+2}) + \ldots + \mathbf{Nw}^{r,u}_{n-k,k},\end{aligned}$$

and the expression introduced by H. Hong in [22] is the following:

$$\begin{aligned}\mathbf{Sres}_k(P, Q) = \mathbf{Nw}_{n-k,0} \cdot (x-\lambda_1)\ldots(x-\lambda_k) \\ +\mathbf{Nw}_{n-k,1} \cdot (x-\lambda_1)\ldots(x-\lambda_{k-1}) + \ldots + \mathbf{Nw}_{n-k,k}.\end{aligned}$$

Observe that there is a slight difference between both expressions.

*Remark 2* ($\mathbf{p_0} \neq \mathbf{1}$) If $P(x)$ is not monic, then $\Lambda_P$ factorizes as follows:

$$\begin{aligned}\Lambda_P = &\begin{pmatrix} & & & 1 \\ & & \iddots & \vdots \\ & 1 & \ldots & c_{n-2}(\lambda_n, \lambda_{n-1}) \\ 1 & c_1(\lambda_n) & \ldots & c_{n-1}(\lambda_n) \end{pmatrix} \cdot \Delta_{P/p_0} \\ &\times \begin{pmatrix} (-1)^{n-1}e_{n-1}(\lambda_n, \ldots, \lambda_2) & (-1)^{n-2}e_{n-2}(\lambda_n, \ldots, \lambda_3) & \ldots & 1 \\ \vdots & \vdots & \iddots & \\ -e_1(\lambda_n, \ldots, \lambda_2) & 1 & & \\ 1 & & & \end{pmatrix} \quad (7)\end{aligned}$$

Thus, following the same reasoning of the proof for Proposition 5.1 but considering Expression (6) and Factorization (7), we easily obtain:

$$\begin{aligned}\mathbf{Sres}_k(P, Q) = p_0^{m-k} \left(\mathbf{Nw}^{r,u}_{n-k,0} \cdot (x-\lambda_n)\ldots(x-\lambda_{n-k+1})\right. \\ \left.+\mathbf{Nw}^{r,u}_{n-k,1} \cdot (x-\lambda_n)\ldots(x-\lambda_{n-k+2}) + \ldots + \mathbf{Nw}^{r,u}_{n-k,k}\right).\end{aligned}$$

### *5.2 Interpolation and Standard Bases – Single Sylvester Sum*

Suppose that $P(x)$ is monic and squarefree. Let $\lambda_1, \ldots, \lambda_n$ be the different roots of $P(x)$. J. J. Sylvester introduced in [29] the following single sum for Subresultant polynomials:

$$\mathbf{Sres}_k(P, Q) = \sum_{\substack{I \uplus J = N \\ |J| = k}} \frac{\operatorname{res}(P_I, Q)}{\operatorname{res}(P_I, P_J)} P_J(x), \tag{8}$$

where

$$k \in \{0, \ldots, m-1\}, \quad N = \{1, \ldots, n\}, \quad P_I(x) = \prod_{i \in I} (x - \lambda_i).$$

Proofs of this formula can be found in [6] or [21]. In [25], they prove a more general expression for Subresultant polynomials, the double Sylvester Sum, introduced by Sylvester in [30]. Here we present a new proof for the single Sylvester sum, which turns out to be a special case of the double Sylvester sum.

In our proof, we use the relation between the Standard basis and the Interpolation basis (defined by Equation (9) below), and the expression of the Subresultant polynomials in terms of minors of the matrix $\widetilde{Q}(\Delta_P)$. Note that $\widetilde{Q}(\Delta_P)$ is equal to $Q(\Delta_P)$ because $P(x)$ is monic.

Recall that if $V_P$ denotes the Vandermonde matrix associated to $P(x)$, then its inverse is given by the Horner polynomials associated to $P(x)$ in the following way:

$$V_P = \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & & \lambda_2^{n-1} \\ \vdots & & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

$$V_P^{-1} = \begin{pmatrix} \alpha_1(\lambda_1) & \alpha_1(\lambda_2) & \cdots & \alpha_1(\lambda_n) \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1}(\lambda_1) & \alpha_{n-1}(\lambda_2) & \cdots & \alpha_{n-1}(\lambda_n) \\ \alpha_n(\lambda_1) & \alpha_n(\lambda_2) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \cdot \operatorname{diag}(c_1, \ldots, c_n)$$

where

$$c_i = \frac{1}{P'(\lambda_i)} = \frac{1}{\prod_{\substack{k=1 \\ k \neq i}}^{n} (\lambda_i - \lambda_k)}$$

and $\alpha_1, \ldots, \alpha_n$ are the Horner polynomials. It is well known that the (usual) companion matrix $\Delta_P$ can be factored as follows:

$$\Delta_P = V_P^{-1} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} V_P = V_P^{-1} \cdot D \cdot V_P,$$

such that on the one hand $D$ is the Jordan form of $\Delta_P$ (which is diagonalizable because $P(x)$ is squarefree), and on the other hand, if $\mathcal{B}_{\text{Int}}$ denotes the Interpolation Basis of $\mathrm{F}_n[x]$, given by

$$\mathcal{B}_{\text{Int}} = \left\{ \frac{\prod_{i=2}^{n}(x-\lambda_i)}{\prod_{i=2}^{n}(\lambda_1-\lambda_i)}, \ldots, \frac{\prod_{i=1}^{n-1}(x-\lambda_i)}{\prod_{i=1}^{n-1}(\lambda_n-\lambda_i)} \right\}, \tag{9}$$

$V_P$ is the basis change matrix of $B_{\text{St}}$ to $B_{\text{Int}}$.

Hence, evaluating $Q(x)$ in $\Delta_P$ yields the following:

$$\begin{aligned} Q(\Delta_P) &= Q(V_P^{-1}\cdot D\cdot V_P) = q_0(V_P^{-1}\cdot D\cdot V_P)^m \\ &\quad + q_1(V_P^{-1}\cdot D\cdot V_P)^{m-1} + \ldots + q_m \\ &= q_0 V_P^{-1} D^m V_P + q_1 V_P^{-1} D^{m-1} V_P + \ldots + q_m = V_P^{-1} Q(D) V_P \\ &= V_P^{-1} \begin{pmatrix} Q(\lambda_1) & & \\ & \ddots & \\ & & Q(\lambda_n) \end{pmatrix} V_P. \end{aligned} \tag{10}$$

Observe that Equation (10) enables us to relate minors of $Q(\Delta_P)$ to minors of $V_P$ and $V_P^{-1}$ and therefore, by Theorem 2.2, we are able to obtain Subresultant polynomials from the roots of $P(x)$ and the Horner polynomials.

In order to prove Formula (8) with our results, we first introduce a well known property of determinant computations.

**Theorem 5.1** (Minors of the inverse) *Let $A \in M_{n,n}(K)$. For index sets $\alpha \subseteq \{1,\ldots,n\}$ and $\beta \subseteq \{1,\ldots,n\}$, let $A(\alpha,\beta)$ be the submatrix that lies in the rows of $A$ indexed by $\alpha$ and the columns indexed by $\beta$ and let $A(\alpha',\beta')$ be the result of deleting the rows indicated by $\alpha$ and the columns indicated by $\beta$. Then given a square nonsingular matrix $A$, the minors of $A^{-1}$ are related to those of $A$ by the next formula:*

$$\det A^{-1}(\alpha',\beta') = (-1)^{(\sum_{i\in\alpha} i + \sum_{j\in\beta} j)} \frac{\det A(\beta,\alpha)}{\det A}.$$

**Theorem 5.2** (Single Sylvester Sum) *Assume that $P(x)$ is squarefree, then*

$$\mathbf{Sres}_k(P,Q) = \sum_{\substack{I \uplus J = N \\ |J|=k}} \frac{\operatorname{res}(P_I, Q)}{\operatorname{res}(P_I, P_J)} P_J(x)$$

*Proof.* We are going to prove that the coefficients in both polynomials are equal. Recall that given a polynomial $A = \sum_{i=0}^{n} a_i x^i$ then

$$a_i = \frac{A^{(i)}(0)}{i!}.$$

Thus, given the polynomial

$$\sum_{\substack{I \uplus J = N \\ |J|=k}} \frac{\operatorname{res}(P_I, Q)}{\operatorname{res}(P_I, P_J)} P_J(x) = \sum_{\substack{I \uplus J = N \\ |J|=k}} \frac{\operatorname{res}(P_I, Q)}{\operatorname{res}(P_I, P_J)} \prod_{\substack{j_i \in J \\ i=1}}^{k} (x - \lambda_{j_i}),$$

its coefficient of $x^r$ is given by the next expression:

$$\frac{\displaystyle\sum_{\substack{I \uplus J = N \\ |J|=k}} \left( \frac{\operatorname{res}(P_I, Q)}{\operatorname{res}(P_I, P_J)}\, r! \sum \prod_{j_1 \le h_1 < \ldots < h_{k-r} \le j_k} (-\lambda_{h_i}) \right)}{r!}$$

$$= \sum_{\substack{I \uplus J = N \\ |J|=k}} \left( \frac{\operatorname{res}(P_I, Q)}{\operatorname{res}(P_I, P_J)} \sum \prod_{\substack{j_1 \le h_1 < \ldots < h_{k-r} \le j_k \\ h_i \in J}} (-\lambda_{h_i}) \right).$$

In the other hand, the coefficient of $x^r$ in $\mathbf{Sres}_k(P, Q)$ is given by:

$$\begin{aligned}
\mathbf{Q}^l_{n-k,k-r} &= \left| \begin{pmatrix} r_{r+1} \\ V^{-1}_{k+2..n,1..n} \end{pmatrix} Q(D) \left( V_{1..n,1..n-k} \right) \right| \\
&= \left| \begin{pmatrix} \alpha_{r+1}(\lambda_1) & \cdots & \alpha_{r+1}(\lambda_n) \\ \alpha_{k+2}(\lambda_1) & \cdots & \alpha_{k+2}(\lambda_n) \\ \vdots & & \vdots \\ \alpha_n(\lambda_1) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \begin{pmatrix} \frac{Q(\lambda_1)}{P'(\lambda_1)} & & \\ & \ddots & \\ & & \frac{Q(\lambda_n)}{P'(\lambda_n)} \end{pmatrix} \right. \\
&\quad \left. \times \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-k-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-k-1} \end{pmatrix} \right| \\
&\overset{\text{Binet–Cauchy}}{=} \sum_{1 \le h_1 < \ldots < h_{n-k} \le n} \left| \begin{pmatrix} \alpha_{r+1}(\lambda_{h_1}) & \cdots & \alpha_{r+1}(\lambda_{h_{n-k}}) \\ \alpha_{k+2}(\lambda_{h_1}) & \cdots & \alpha_{k+2}(\lambda_{h_{n-k}}) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{h_1}) & \cdots & \alpha_n(\lambda_{h_{n-k}}) \end{pmatrix} \right. \\
&\quad \left. \times \begin{pmatrix} \frac{Q(\lambda_{h_1})}{P'(\lambda_{h_1})} & & \\ & \ddots & \\ & & \frac{Q(\lambda_{h_{n-k}})}{P'(\lambda_{h_{n-k}})} \end{pmatrix} \begin{pmatrix} 1 & \lambda_{h_1} & \cdots & \lambda_{h_1}^{n-k-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_{h_{n-k}} & \cdots & \lambda_{h_{n-k}}^{n-k-1} \end{pmatrix} \right|,
\end{aligned}$$

where $V_{k+2..n,1..n}^{-1}$ denotes the last $k+2$ rows of $V_P^{-1}$, $r_{r+1}$ the $(r+1)$-th row of $V_P^{-1}$, and $\left(V_{1..n,1..n-k}\right)$ the last $n-k$ columns of $V_P$.

Next we prove that if $J$ and $I$ are given by

$$J = \{j_1, \ldots, j_k\},$$

$$I = \{i_1, \ldots, i_{n-k}\}$$

then

$$\frac{\operatorname{res}(P_I, Q)}{\operatorname{res}(P_I, P_J)} \sum \prod_{\substack{j_1 \leq h_1 < \ldots < h_{k-r} \leq j_k \\ h_i \in J}} (-\lambda_{h_i}) = \left| \begin{pmatrix} \alpha_{r+1}(\lambda_{i_1}) & \cdots & \alpha_{r+1}(\lambda_{i_{n-k}}) \\ \alpha_{k+2}(\lambda_{i_1}) & \cdots & \alpha_{k+2}(\lambda_{i_{n-k}}) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{i_1}) & \cdots & \alpha_n(\lambda_{i_{n-k}}) \end{pmatrix} \right.$$

$$\left. \times \begin{pmatrix} \frac{Q(\lambda_{i_1})}{P'(\lambda_{i_1})} & & \\ & \ddots & \\ & & \frac{Q(\lambda_{i_{n-k}})}{P'(\lambda_{i_{n-k}})} \end{pmatrix} \begin{pmatrix} 1 & \lambda_{i_1} & \cdots & \lambda_{i_1}^{n-k-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_{i_{n-k}} & \cdots & \lambda_{i_{n-k}}^{n-k-1} \end{pmatrix} \right|.$$

In order to simplify the notation, suppose that

$$J = \{1, \ldots, k\}$$

$$I = \{k+1, \ldots, n\}.$$

and let $\mathbf{V}(\lambda_1, \ldots, \lambda_k)$ denote the determinant of the matrix $V(\lambda_1, \ldots, \lambda_k)$.

Then since

$$\left| \begin{pmatrix} \alpha_{r+1}(\lambda_{k+1}) & \cdots & \alpha_{r+1}(\lambda_n) \\ \alpha_{k+2}(\lambda_{k+1}) & \cdots & \alpha_{k+2}(\lambda_n) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{k+1}) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \begin{pmatrix} \frac{Q(\lambda_{k+1})}{P'(\lambda_{k+1})} & & \\ & \ddots & \\ & & \frac{Q(\lambda_n)}{P'(\lambda_n)} \end{pmatrix} \begin{pmatrix} 1 & \lambda_{k+1} & \cdots & \lambda_{k+1}^{n-k-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-k-1} \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} \alpha_{r+1}(\lambda_{k+1}) & \cdots & \alpha_{r+1}(\lambda_n) \\ \alpha_{k+2}(\lambda_{k+1}) & \cdots & \alpha_{k+2}(\lambda_n) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{k+1}) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \begin{pmatrix} \frac{1}{P'(\lambda_{k+1})} & & \\ & \ddots & \\ & & \frac{1}{P'(\lambda_n)} \end{pmatrix} \right| \prod_{i=k+1}^{n} Q(\lambda_i) \mathbf{V}(\lambda_{k+1}, \ldots, \lambda_n)$$

$$= \left| \begin{pmatrix} \alpha_{r+1}(\lambda_{k+1}) & \cdots & \alpha_{r+1}(\lambda_n) \\ \alpha_{k+2}(\lambda_{k+1}) & \cdots & \alpha_{k+2}(\lambda_n) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{k+1}) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \begin{pmatrix} \frac{1}{P'(\lambda_{k+1})} & & \\ & \ddots & \\ & & \frac{1}{P'(\lambda_n)} \end{pmatrix} \right| \operatorname{res}(P_I, Q) \mathbf{V}(\lambda_{k+1}, \ldots, \lambda_n),$$

it suffices now to prove the next equality:

$$\frac{\sum \prod_{\substack{1 \leq h_1 < \ldots < h_{k-r} \leq k \\ h_i \in J}} (-\lambda_{h_i})}{\operatorname{res}(P_I, P_J)} = \left| \begin{pmatrix} \alpha_{r+1}(\lambda_{k+1}) & \cdots & \alpha_{r+1}(\lambda_n) \\ \alpha_{k+2}(\lambda_{k+1}) & \cdots & \alpha_{k+2}(\lambda_n) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{k+1}) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \right.$$

$$\times \left( \begin{array}{ccc} \frac{1}{P'(\lambda_{k+1})} & & \\ & \ddots & \\ & & \frac{1}{P'(\lambda_n)} \end{array} \right) \Bigg| \mathbf{V}(\lambda_{k+1}, \ldots, \lambda_n) \qquad (11)$$

Using now the following property of the resultant of two monic polynomials:

$$\operatorname{res}(P_I, P_J) = \prod_{j \in J} \left( \prod_{i \in I} (\lambda_j - \lambda_i) \right),$$

we obtain:

$$\frac{V(\lambda_1, \ldots, \lambda_k)}{V(\lambda_1, \ldots, \lambda_k)} \frac{\sum \prod_{\substack{j_1 \le h_1 < \ldots < h_{k-r} \le j_k \\ h_i \in J}} (-\lambda_{h_i})}{\prod_{j=1}^{k} \left( \prod_{i=k+1}^{n} (\lambda_j - \lambda_i) \right)}$$

$$= \left| \begin{pmatrix} \alpha_{r+1}(\lambda_{k+1}) & \cdots & \alpha_{r+1}(\lambda_n) \\ \alpha_{k+2}(\lambda_{k+1}) & \cdots & \alpha_{k+2}(\lambda_n) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{k+1}) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \begin{pmatrix} \frac{1}{P'(\lambda_{k+1})} & & \\ & \ddots & \\ & & \frac{1}{P'(\lambda_n)} \end{pmatrix} \right| \mathbf{V}(\lambda_{k+1}, \ldots, \lambda_n)$$

$$\Updownarrow$$

$$\frac{\mathbf{V}(\lambda_1, \ldots, \lambda_k)}{\mathbf{V}(\lambda_1, \ldots, \lambda_k)} \frac{\sum \prod_{\substack{j_1 \le h_1 < \ldots < h_{k-r} \le j_k \\ h_i \in J}} (-\lambda_{h_i})}{\prod_{j=1}^{k} \left( \prod_{i=k+1}^{n} (\lambda_j - \lambda_i) \right) \mathbf{V}(\lambda_{k+1}, \ldots, \lambda_n)}$$

$$= \left| \begin{pmatrix} \alpha_{r+1}(\lambda_{k+1}) & \cdots & \alpha_{r+1}(\lambda_n) \\ \alpha_{k+2}(\lambda_{k+1}) & \cdots & \alpha_{k+2}(\lambda_n) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{k+1}) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \begin{pmatrix} \frac{1}{P'(\lambda_{k+1})} & & \\ & \ddots & \\ & & \frac{1}{P'(\lambda_n)} \end{pmatrix} \right|$$

$$\Updownarrow$$

$$\mathbf{V}(\lambda_1, \ldots, \lambda_k) \frac{\sum \prod_{\substack{j_1 \le h_1 < \ldots < h_{k-r} \le j_k \\ h_i \in J}} (-\lambda_{h_i})}{\mathbf{V}(\lambda_1, \ldots, \lambda_n)}$$

$$= \left| \begin{pmatrix} \alpha_{r+1}(\lambda_{k+1}) & \cdots & \alpha_{r+1}(\lambda_n) \\ \alpha_{k+2}(\lambda_{k+1}) & \cdots & \alpha_{k+2}(\lambda_n) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{k+1}) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \begin{pmatrix} \frac{1}{P'(\lambda_{k+1})} & & \\ & \ddots & \\ & & \frac{1}{P'(\lambda_n)} \end{pmatrix} \right| \qquad (12)$$

and observe that the right side of (12) is a minor of the Vandermonde matrix associated to $\{\lambda_1, \ldots, \lambda_n\}$.

Moreover, since Theorem 5.1 provides that:

$$\left|\begin{pmatrix} \alpha_{r+1}(\lambda_{k+1}) & \cdots & \alpha_{r+1}(\lambda_n) \\ \alpha_{k+2}(\lambda_{k+1}) & \cdots & \alpha_{k+2}(\lambda_n) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{k+1}) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \begin{pmatrix} \frac{1}{P'(\lambda_{k+1})} & & \\ & \ddots & \\ & & \frac{1}{P'(\lambda_n)} \end{pmatrix}\right| = \frac{(-1)^{(k-r)} \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{r-1} & \lambda_1^{r+1} & \cdots & \lambda_1^k \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{r-1} & \lambda_k^{r+1} & \cdots & \lambda_k^k \end{vmatrix}}{\mathbf{V}(\lambda_1, \ldots, \lambda_n)}$$

and

$$\begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{r-1} & \lambda_1^{r+1} & \cdots & \lambda_1^k \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{r-1} & \lambda_k^{r+1} & \cdots & \lambda_k \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{r-1} & \lambda_1^{r} & \cdots & \lambda_1^{k-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{r-1} & \lambda_k^{r} & \cdots & \lambda_k^{k-1} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & & & 0 & & (-1)^{k-1}\prod_{i=1}^{k}\lambda_i \\ & \ddots & & & & \vdots \\ & & 1_{r,r} & 0 & (-1)^{k-r}\sum \prod_{\substack{1\le h_1<\ldots<h_{k-r+1}\le k \\ h_i\in J}} (\lambda_{h_i}) \\ & & 0 & 0 & (-1)^{k-(r+1)}\sum \prod_{\substack{1\le h_1<\ldots<h_{k-r}\le k \\ h_i\in J}} (\lambda_{h_i}) \\ & & 1 & 0 & (-1)^{k-(r+2)}\sum \prod_{\substack{1\le h_1<\ldots<h_{k-r-1}\le k \\ h_i\in J}} (\lambda_{h_i}) \\ & & & \ddots & \vdots \\ & & & & 1 \quad \sum_{i=1}^{k}\lambda_i \end{pmatrix},$$

we have that

$$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{r-1} & \lambda_1^{r+1} & \cdots & \lambda_1^k \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{r-1} & \lambda_k^{r+1} & \cdots & \lambda_k^k \end{vmatrix} = \mathbf{V}(\lambda_1, \ldots, \lambda_k)\sum \prod_{\substack{1\le h_1<\ldots<h_{k-r}\le k \\ h_i\in J}} (\lambda_{h_i})$$

Hence

$$\left|\begin{pmatrix} \alpha_{r+1}(\lambda_{k+1}) & \cdots & \alpha_{r+1}(\lambda_n) \\ \alpha_{k+2}(\lambda_{k+1}) & \cdots & \alpha_{k+2}(\lambda_n) \\ \vdots & & \vdots \\ \alpha_n(\lambda_{k+1}) & \cdots & \alpha_n(\lambda_n) \end{pmatrix} \begin{pmatrix} \frac{1}{P'(\lambda_{k+1})} & & \\ & \ddots & \\ & & \frac{1}{P'(\lambda_n)} \end{pmatrix}\right|$$

$$= \frac{(-1)^{(k-r)}\mathbf{V}(\lambda_1, \ldots, \lambda_k) \sum \prod_{\substack{1 \le h_1 < \ldots < h_{k-r} \le k \\ h_i \in J}} (\lambda_{h_i})}{\mathbf{V}(\lambda_1, \ldots, \lambda_n)},$$

which completes the proof of (12). □

### 5.2.1 When $P(x)$ is not squarefree

Observe that previous result only is valid when $P(x)$ is squarefree. When $P(x)$ is monic but not squarefree, if $\lambda_1, \ldots, \lambda_t$ denote the different roots of $P(x)$ with multiplicity $e_1, \ldots, e_t$ respectively, then the Jordan form of $\Delta_P$ is given by the following diagonal block matrix:

$$J_p = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_t \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{e_i, e_i}.$$

So, if $N$ denotes the confluent Vandermonde matrix associated to zeros of $P(x)$ then:

$$\Delta_P = N^{-1} \cdot J_p \cdot N$$

and thus

$$Q(\Delta_P) = N^{-1} \cdot Q(J_p) \cdot N.$$

The confluent Vandermonde matrix is made up of $t$ blocks in the form:

$$\begin{pmatrix} & & & \binom{e_i-1}{e_i-1} & \cdots & \binom{n-2}{e_i-1}\lambda_i^{n-e_i-1} & \binom{n-1}{e_i-1}\lambda_i^{n-e_i} \\ & & \iddots & \vdots & & \vdots & \vdots \\ & \binom{2}{2} & \binom{3}{2}\lambda_i & \cdots \binom{e_i-1}{2}\lambda_i^{e_i-3} & \cdots & \binom{n-2}{2}\lambda_i^{n-4} & \binom{n-1}{2}\lambda_i^{n-3} \\ 1 & 2\lambda_i & 3\lambda_i^2 & \cdots (e_i-1)\lambda_i^{e_i-2} & \cdots & (n-2)\lambda_i^{n-3} & (n-1)\lambda_i^{n-2} \\ 1 & \lambda_i & \lambda_i^2 & \lambda_i^3 \cdots & \lambda_i^{e_i-1} & \cdots & \lambda_i^{n-2} & \lambda_i^{n-1} \end{pmatrix}.$$

The matrix $N^{-1}$ can be computed by the recursive algorithm for inverting confluent Vandermonde matrices presented in [23]. Note that columns of $N^{-1}$ are the coordinates with respect to the Standard Basis of $\mathrm{F}_n[x]$,

$$\mathcal{B}_{\mathrm{St}} = \{1, x, \ldots, x^{n-1}\},$$

of polynomials denoted by

$$p_{1,1}(x), \dots, p_{e_1,1}(x), p_{1,2}(x), \dots, p_{e_2,2}(x), \dots, \dots, p_{e_t,t}(x)$$

and defined for every $i \in \{1, \dots, t\}$ by the following relations:

$$p_{e_i,i}(\lambda_i) = 1, \quad \frac{p_{e_i,i}^{(m)}(\lambda_i)}{m!} = 0, \quad (1 \leq m \leq e_i - 1)$$

$$p_{k,i}(\lambda_i) = 0, \ \frac{p_{k,i}^{(e_i-k)}(\lambda_i)}{(e_i-k)!} = 1, \ \frac{p_{k,i}^{(m)}(\lambda_i)}{m!} = 0, \quad (k < e_i, m \neq e_i - k)$$

$$p_{k,i}(\lambda_j) = 0, \quad \frac{p_{k,i}^{(m)}(\lambda_j)}{m!} = 0, \quad (k \leq e_i, \ j \neq i, 1 \leq m \leq e_j - 1).$$

Hence the rows of $N$ provide the coordinates with respect to the Standard Basis in the dual space of $\mathrm{F}_n[x]$, $\mathrm{F}_n^*[x]$, of the dual basis of the basis of $\mathrm{F}_n[x]$ given by the polynomials $p_{k,i}(x)$.

## 6 Remark on Complexity

In this paper, our purpose has been to present new expressions for Subresultant polynomials, which can be useful for example to obtain other formulas or to write them in terms of other bases.

These expressions also provide new algorithms for computing Subresultants, by determinant computation. However, when the coefficients of the given polynomials do not depend of parameters, they do not improve the sequential complexity of the best known algorithms, given by the Subresultant Theorem and its multiple variants (see [27] for more details). When the coefficients depend on parameters, a careful discussion about the use of Bezout matrices and classical algorithms is presented in [1].

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