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A comparison of Dodgson's method and the Borda count

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Summary. In an election without a Condorcet winner, Dodgson's Method is designed to find the candidate that is "closest" to being a Condorcet winner. In this paper, we show that the winner from Dodgson's Method can occur at any position in the ranking obtained from the Borda Count, the plurality method, or any other positional voting procedure. In addition, we demonstrate that Dodgson's Method does not satisfy the Independence of Irrelevant Alternatives axiom.

Keywords and Phrases: Voting theory, Dogson's method, Borda count.

JEL Classification Numbers: D71.

1 Introduction

The Condorcet criterion appears to be the most natural and reasonable criterion to apply to an election: if there is a single candidate that beats every other candidate in head-to-head elections then that candidate should be declared the winner. There is, however, the well-known problem that there are elections where a Condorcet winner may not exist. In 1874, Charles Dodgson (aka Lewis Carroll) proposed a voting method to extend the Condorcet criterion to elections without a Condorcet winner [1]. In essence, Dodgson's Method (DM) finds the candidate that is closest to being the Condorcet winner.

At first glance, DM appears quite similar to Kemeny's Rule. The major distinction is that Kemeny's Rule finds the closest complete transitive ranking of candidates whereas Dodgson's Method picks a single winner and allows there to be a cycle among the other candidates. This gives us enough leeway to show that with four or more candidates there is no connection between the DM winner and Kemeny's Rule. That is, the DM winner can occur at any position in the Kemeny ranking [7].

Given that Dodgson's Method is based on the Condorcet criterion, we should not be surprised that there can also be conflict between Dodgson's Method and the Borda Count. It is known that the Condorcet winner can never be ranked last in the Borda Count [4]. We will show that there is not even this level of consistency with Dodgson's Method: with four or more candidates, the DM winner can appear at any position in the Borda Count.

Further, the examples we generate also show that the DM winner can appear at any location in the ranking of any positional voting method, including plurality (where each voter gives 1 point to their top ranked candidate and 0 to all others), anti-plurality (where each voter gives 1 point to every candidate except their last ranked candidate who receives 0 points), and other variations on the Borda Count. These results highlight the need for extreme care when extending any criterion, no matter how reasonable it appears, to cases where the rule does not initially apply.

In Section 2, we give a simple example with four candidates to illustrate DM and to show how DM can differ from the Borda Count. In Section 3, we use Saari's decompositions of a voting profile [4] to show how to create four candidate profiles that generate conflict between DM and all positional procedures, and we explain why the example in Section 2 behaves as it does. Section 4 gives some geometric insight into why DM differs from the Borda Count. Section 5 gives an example to illustrate that DM does not satisfy Independence of Irrelevant Alternatives, and Section 6 contains the proof of our main result for more than four candidates.

2 Dodgson's Method

To illustrate Dodgson's Method, consider the voting profile in Table 1 among four candidates A,B,C, D with 30 voters where $A \succ B$ means that A is preferred to B. The head-to-head results are given in Table 2.

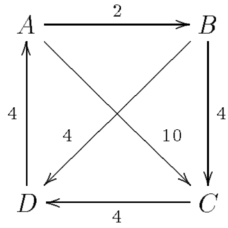
Table 1. An election with 30 voters

Number	Ranking
10	$A \succ B \succ C \succ D$ (1)
7	$C \succ D \succ B \succ A$ (2)
3	$A \succ D \succ C \succ B$ (3)
3	$D \succ C \succ A \succ B$ (4)
7	$B \succ D \succ A \succ C$ (5)

Notice there is no Condorcet winner since the first four head-to-head elections determine a cycle where every candidate loses at least one election. The intuition behind DM is that B is the closest to being the Condorcet winner since it loses a single election (to A) by two votes while every other candidate loses at least one election by four or more votes. Thus, if two voters with preference (1) change

Table 2. Head-to-head results from the example in Table 1

	Tally	Margin
$A \succ B$	16,14	2
$B \succ C$	17,13	4
$C \succ D$	17,13	4
$D \succ A$	17,13	4
$A \succ C$	20,10	10
$B \succ D$	17,13	4



to $B \succ A \succ C \succ D$, then B will become the Condorcet winner, but any other candidate will require more than two voters to change their rankings.

2.1 Precise statement of Dodgson's method

In our example, all four candidates are contained in the cycle. However, this may not always be the case as there may be a majority cycle where each candidate in the cycle is preferred to every candidate not in the cycle. In this situation, Dodgson restricts his attention to the majority cycle. We can state Dodgson's Method as follows:

1. If there is a Condorcet winner, then that is the Dodgson winner.
2. If not, there will be a majority cycle. For each candidate in the majority cycle, determine the number of adjacent switches in the voters' preferences that are necessary to make the candidate the Condorcet winner. The candidate in the majority cycle with the fewest required switches is the Dodgson winner.

Applying this to our example, Table 3 shows that B is the DM winner.

Table 3. Switches required for the example in Table 1

Candidate	Election Lost	Margin	Switches	Ranking	Total
A	$D \succ A$	4	3	(5)	3
B	$A \succ B$	2	2	(1)	2
C	$B \succ C$	4	3	(1)	
	$A \succ C$	10	6	(5)	9
D	$C \succ D$	4	3	(2)	
	$B \succ D$	4	3	(5)	6

2.2 Comparison with Borda Count

Using the standard weights for the Borda Count with four candidates (3, 2, 1, and 0 points for first, second, third, and fourth place, respectively), Table 4 shows that A is the Borda Count winner and that the DM winner, B, places second.

Table 4. Borda count tallies for the example in Table 1

Candidate	First	Second	Third	Fourth	Borda Total
A	13	0	10	7	49
B	7	10	7	6	48
C	7	3	13	7	40
D	3	17	0	10	43

This illustrates our main result.

Theorem 1. *If there are four or more candidates, then there is no connection between the Dodgson winner and the Borda Count. That is, the Dodgson winner may occur at any position in the Borda Count ranking.*

Further, there is no connection between the Dodgson winner and any positional voting method, including plurality, anti-plurality, and variations on the Borda Count.

Notice that our example is not sufficient to illustrate the general case of the Theorem since B is the anti-plurality winner with a tally of 24 where A, C, and D have tallies of 23, 23, and 20, respectively.

We also note that with three candidates, DM is identical to Kemeny's rule and Saari and Merlin's analysis show that the Kemeny winner cannot be ranked last in the Borda Count [6], but can appear in either first or second place.

Intuitively, we should not be surprised that there can be conflict between the DM winner and the Borda Count. It is well known that the Borda Count for a candidate A depends only on the margins of all pairwise elections for A, including both wins and losses. However, DM depends on only the pairwise losses of A and not the pairwise wins of A. While A's margin of victory affects the DM calculations for the losing candidate (and thus helps A compared to the loser), this margin does not help A in comparison with other candidates. That is, if $A \succ B$ by a large margin, then this helps A compared to B in the DM calculations, but it does not aid A when comparing to other candidates. In contrast, this large margin of victory will help A in the Borda Count compared to all other candidates.

3 Decomposition of profiles

In order to understand how Dodgson's Method can differ from the Borda Count, we need to introduce a decomposition of profiles defined by Saari [4] into fundamental components. Our goal is to create a profile where A is the Dodgson winner but all positional procedures give a ranking of $B \succ A \succ C \succ D$. To do this, we will create a profile that consists entirely of

– a large *Condorcet component* that has no impact on any positional method but determines the DM winner to be A

- a small *Basic component* where all position methods give a ranking of $B \succ A \succ C \succ D$ and whose impact on the DM winner is overshadowed by the Condorcet component
- A *Kernel component* that has no impact on any method but guarantees we have a non-negative number of voters (see below for the justification of this component)

By modifying the Basic component, we will be able to change the ranking given by the positional methods without affecting the DM winner. We note that our example from Section 2 contains an additional component that generates conflict among the positional procedures (Recall that the Borda Count and antiplurality outcomes differed).

Definition 2. *The Kernel profile K contains one voter for each of the $n!$ rankings of the n candidates.*

Definition 3. *In an n candidate election, the Basic profile for the candidate A_i, denoted B_{A_i}, assigns one voter for each ranking where A_i is top-ranked and -1 voters for each ranking where A_i is bottom-ranked.*

For example, with four candidates B_A is the profile

1	$A \succ B \succ C \succ D$	-1	$D \succ C \succ B \succ A$
1	$A \succ B \succ D \succ C$	-1	$C \succ D \succ B \succ A$
1	$A \succ C \succ B \succ D$	-1	$D \succ B \succ C \succ A$
1	$A \succ C \succ D \succ B$	-1	$B \succ D \succ C \succ A$
1	$A \succ D \succ B \succ C$	-1	$C \succ B \succ D \succ A$
1	$A \succ D \succ C \succ B$	-1	$B \succ C \succ D \succ A$

The Basic profiles contain negative voters, but this does not cause a problem when computing the election outcomes. The Kernel profile K gives a complete tie for all positional procedures and all pairwise votes. By adding a sufficiently large multiple of K to a profile with negative entries, we obtain a profile with non-negative voters and exactly the same election outcomes. Further, when constructing examples, the Kernel allows us to avoid a problem in Dodgson's Method of adjacency switches (See [7] for details). Notice that the pairwise margins in the n candidate Basic profile B_{A_i} are

$$2(n-1)! \text{ for} A_i \succ A_j, i \neq j, \qquad 0 \text{ for all others}$$

The following definition for a Condorcet profile is only for a four candidate election, but it can clearly be generalized to n candidates.

Definition 4. *Given the ranking $r = A \succ B \succ C \succ D$, define the Condorcet profile C_r to have one vote for each ranking consistent with the cycle $A \succ B \succ C \succ D \succ A$ and -1 vote for each ranking consistent with the reverse cycle $D \succ C \succ B \succ A \succ D$, as shown in Table 5.*

Table 5. The Condorcet profile C_r and associated cycle

#	Ranking	#	Ranking
1	$A \succ B \succ C \succ D$	−1	$D \succ C \succ B \succ A$
1	$B \succ C \succ D \succ A$	−1	$C \succ B \succ A \succ D$
1	$C \succ D \succ A \succ B$	−1	$B \succ A \succ D \succ C$
1	$D \succ A \succ B \succ C$	−1	$A \succ D \succ C \succ B$

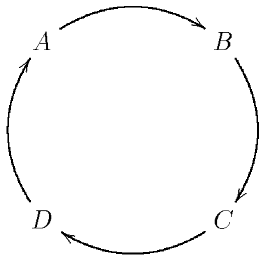
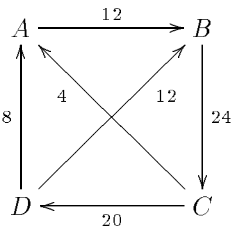


Table 6. The Condorcet component $\mathscr{C} = 4C_{r_1} + 2C_{r_2} + C_{r_3}$

#	Ranking	#	Ranking
4	$A \succ B \succ C \succ D$	−1	$C \succ A \succ B \succ D$
−1	$A \succ B \succ D \succ C$	2	$C \succ A \succ D \succ B$
2	$A \succ D \succ B \succ C$	4	$C \succ D \succ A \succ B$
−4	$A \succ D \succ C \succ B$	1	$C \succ D \succ B \succ A$
1	$A \succ C \succ D \succ B$	−2	$C \succ B \succ D \succ A$
−2	$A \succ C \succ B \succ D$	−4	$C \succ B \succ A \succ D$
1	$B \succ A \succ C \succ D$	4	$D \succ A \succ B \succ C$
−4	$B \succ A \succ D \succ C$	−2	$D \succ A \succ C \succ B$
−2	$B \succ D \succ A \succ C$	−1	$D \succ C \succ A \succ B$
−1	$B \succ D \succ C \succ A$	−4	$D \succ C \succ B \succ A$
4	$B \succ C \succ D \succ A$	2	$D \succ B \succ C \succ A$
2	$B \succ C \succ A \succ D$	1	$D \succ B \succ A \succ C$



Saari [5] proves that the Condorcet and Basic profiles have the fundamental properties we desire: All positional methods agree on linear combinations of the Basic profiles, and all positional methods give a complete tie on linear combinations of the Condorcet profiles. In particular, B_A gives a ranking of $A \succ B \sim C \sim D$ for all positional methods, where $B \sim C$ means that B and C are tied. We now have the tools to generate a profile where A is the DM winner, but every positional method gives a ranking of $B \succ A \succ C \succ D$.

3.1 The Condorcet component

Let $r_1 = A \succ B \succ C \succ D$, $r_2 = B \succ C \succ A \succ D$, and $r_3 = B \succ A \succ C \succ D$. Consider the Condorcet component $\mathscr{C} = 4C_{r_1} + 2C_{r_2} + C_{r_3}$ given in Table 6.

Notice that each candidate is in the majority cycle and that A is the DM winner requiring 8 switches whereas B, C, and D require 14, 13, and 11 switches, respectively. A straightforward calculation shows that every candidate has a Borda Count tally of 0 giving a complete tie, as we expect.

To understand why $\mathscr{C}$ behaves as it does with respect to Dodgson's Method, consider the pairwise outcomes of the Condorcet profiles $4C_{r_1}$, $2C_{r_2}$, and C_{r_3} shown in Figure 1, where a dashed line indicates a tie. The idea is to begin with the profile $4C_{r_1}$ that includes each candidate in the majority cycle and add profiles that will maintain the majority cycle while decreasing A's total margin of loss and increasing the total margin of loss of every other candidate. We add $2C_{r_2}$ to increase B and C's total margins of loss while not affecting either A or D's total margins. We add the profile C_{r_3} to increase D's total margin, decrease A's and not affect either B or C's margin. In particular, notice that the combined profile $2C_{r_2} + C_{r_3}$ will not disrupt the majority cycle. The case with more than four candidates is similar and is given in Section 6.

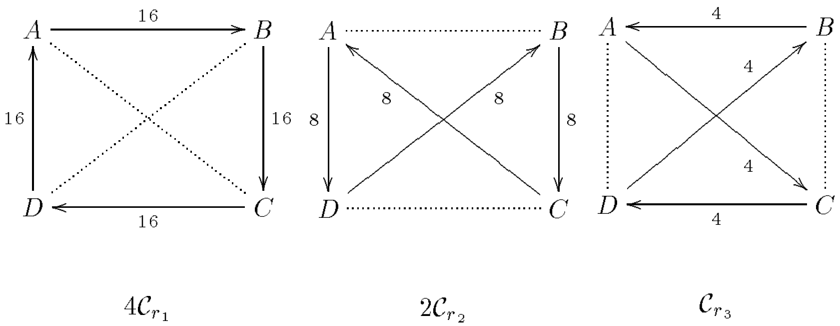


Figure 1. Pairwise outcomes for $4C_{r_1}$, $2C_{r_2}$, and C_{r_3}

After determining the Basic component $\mathscr{B}$, we will scale $\mathscr{C}$ sufficiently large to guarantee that the Basic component will not impact the DM winner.

3.2 The basic component

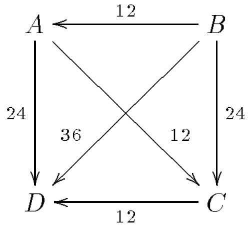
As previously noted, all positional methods give the ranking $A \succ B \sim C \sim D$ on the Basic profile B_A. Then the Basic profile $\mathscr{B} = 3B_B + 2B_A + B_C$ given in Table 7 will give a ranking of $B \succ A \succ C \succ D$ for all position methods.

For example, the Borda Count tallies are $A : 12, B : 36, C : -12, D : -36$. Notice that if we simply add the Basic component $\mathscr{B}$ to the Condorcet component $\mathscr{C}$, then we will reverse some of the pairwise outcomes and, therefore, affect the DM winner. For example, the $D \succ A$ outcome in $\mathscr{C}$ becomes $A \succ D$ in $\mathscr{C} + \mathscr{B}$. Thus, we need to scale the Condorcet component large enough to overcome the effect of the Basic component.

Consider the profile $5\mathscr{C} + \mathscr{B}$ given in Table 8. Clearly A is the DM winner for this profile. Thus, the profile $5\mathscr{C} + \mathscr{B}$ will have a DM winner of A and a positional outcome of $B \succ A \succ C \succ D$. The final piece is to add the component $22K$ to give a non-negative number of voters for each ranking so that our desired profile is $5\mathscr{C} + \mathscr{B} + 22K$.

Table 7. The basic component $\mathscr{B} = 3B_B + 2B_A + B_C$

#	Ranking	#	Ranking
2	$A \succ B \succ C \succ D$	1	$C \succ A \succ B \succ D$
1	$A \succ B \succ D \succ C$	−2	$C \succ A \succ D \succ B$
1	$A \succ D \succ B \succ C$	−2	$C \succ D \succ A \succ B$
−1	$A \succ D \succ C \succ B$	−1	$C \succ D \succ B \succ A$
−1	$A \succ C \succ D \succ B$	−1	$C \succ B \succ D \succ A$
2	$A \succ C \succ B \succ D$	1	$C \succ B \succ A \succ D$
3	$B \succ A \succ C \succ D$	−1	$D \succ A \succ B \succ C$
2	$B \succ A \succ D \succ C$	−3	$D \succ A \succ C \succ B$
2	$B \succ D \succ A \succ C$	−3	$D \succ C \succ A \succ B$
1	$B \succ D \succ C \succ A$	−2	$D \succ C \succ B \succ A$
1	$B \succ C \succ D \succ A$	−2	$D \succ B \succ C \succ A$
3	$B \succ C \succ A \succ D$	−1	$D \succ B \succ A \succ C$



Also note that we can easily modify $\mathscr{B}$ to obtain any positional ranking without affecting the DM winner. For example, $3B_D + 2B_C + B_B$ will give a positional ranking of $D \succ C \succ B \succ A$, and we then add a sufficiently large multiple of $\mathscr{C}$ to guarantee that we do not reverse the pairwise outcomes and inadvertently change the DM winner.

4 Geometric comparison of the methods

We can gain additional insight into the conflict between Dodgson's Method and the Borda Count by understanding the geometric behavior of these methods for elections with n candidates.

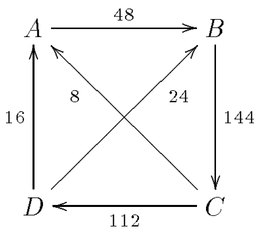
4.1 Geometric framework

We will use the geometric model developed by Saari [4, 5]. Each voter profile specifies the number of voters who prefer each of the $n!$ rankings of the candidates. With n candidates $A_1, A_2, \ldots, A_n$, the profile defines a point in $\mathbb{R}^{n!}$ space. For each of the $\binom{n}{2} = \frac{n(n-1)}{2}$ pairwise elections, pick an ordering of the pairwise elections $A_i \succ A_j$, and let a_{ij} denote the margin by which A_i is preferred to A_j in the pairwise vote (if A_j is preferred to A_i, then a_{ij} will be negative). Therefore, the pairwise votes define a point in $\mathbb{R}^{\binom{n}{2}}$ where the sign of any component indicates which candidate won the corresponding election (a zero value indicates a tie).

For example, the profile from Table 1 defines a point in the profile space R^{24} (where 19 of the components are zero), and the corresponding point in the pairwise space $\mathbb{R}^6$ is $(2, 10, -4, 4, 4, 4)$ with the pairwise elections ordered $(A \succ B, A \succ C, A \succ D, B \succ C, B \succ D, C \succ D)$.

Table 8. The profile $5\mathscr{C}+\mathscr{B}$ and pairwise tallies

#	Ranking	#	Ranking
22	$A \succ B \succ C \succ D$	-4	$C \succ A \succ B \succ D$
-4	$A \succ B \succ D \succ C$	8	$C \succ A \succ D \succ B$
11	$A \succ D \succ B \succ C$	18	$C \succ D \succ A \succ B$
-21	$A \succ D \succ C \succ B$	4	$C \succ D \succ B \succ A$
4	$A \succ C \succ D \succ B$	-11	$C \succ B \succ D \succ A$
-8	$A \succ C \succ B \succ D$	-19	$C \succ B \succ A \succ D$
8	$B \succ A \succ C \succ D$	19	$D \succ A \succ B \succ C$
-18	$B \succ A \succ D \succ C$	-13	$D \succ A \succ C \succ B$
-8	$B \succ D \succ A \succ C$	-8	$D \succ C \succ A \succ B$
-4	$B \succ D \succ C \succ A$	-22	$D \succ C \succ B \succ A$
21	$B \succ C \succ D \succ A$	8	$D \succ B \succ C \succ A$
13	$B \succ C \succ A \succ D$	4	$D \succ B \succ A \succ C$



4.2 Geometry of Dodgson's method and the Borda count

Note that each orthant in $\mathbb{R}^{\binom{n}{2}}$ determines a ranking, possibly with cycles. In comparing how voting methods based on the pairwise votes treat cycles, the real issue is understanding how each method moves from an orthant representing a cycle (or cycles) to an orthant representing a transitive ranking (in the case of the positional methods) or a Condorcet winner (in the case of Dodgson's method). DM is closely related to finding the orthant representing the ranking with a Condorcet winner that has the closest ℓ_1 distance to the profile's point in pairwise space. The ℓ_1 metric, also called the *taxicab* or *Manhattan* metric, determines the distance between two points by summing the absolute value of the differences between the coordinates. For example, the ℓ_1 distance between $(2,-1,3)$ and $(1,4,-2)$ is $|2-1|+|-1-4|+|3-(-2)|=11$. Intuitively, we can think of this metric as the shortest driving distance between the points where we are allowed to travel only east-west and/or north-south. Notice that the shortest ℓ_1 distance to an orthant with A_i as the Condorcet winner is found by summing the margins of loss of A_i in the pairwise elections. Thus, the ℓ_1 winner is the candidate with the smallest total margin of loss in the pairwise outcomes.

Although the ℓ_1 winner and the DM winner may differ for a profile P, if all candidates are included in the majority cycle, then for sufficiently large scalars c, the ℓ_1 winner and the DM winner agree on cP [7]. In contrast, the Borda Count behaves as a projection onto the *transitivity plane* spanned by the images of the Basic vectors in pairwise space. A key factor is that the images of the Condorcet profiles are orthogonal to the transitivity plane in pairwise space and therefore have no impact on the Borda Count.

Thus, our Condorcet component $\mathscr{C}$ is a large vector in an orthant defining a cycle that is close in ℓ_1 distance to an orthant where A is the Condorcet winner. Further, the orthogonal projection of $\mathscr{C}$ onto the transitivity plane lands at the origin since the Borda Count gives a complete tie on $\mathscr{C}$. A small tweak to $\mathscr{C}$ by adding the Basic component $\mathscr{B}$ does not change the closest orthant with A as the Condorcet winner, but does change the orthogonal projection onto the transitivity plane to an orthant with A in the desired position.

5 Dodgson's method and the independence of irrelevant alternatives

In this section we will show that Dodgson's Method does not satisfy the Independence of Irrelevant Alternatives axiom.

Definition 5. *Let P_1 and P_2 be two profiles for candidates $A_1, \ldots, A_n$ with the same set of voters. Suppose that the exact same set of voters prefer A_i to A_j in both profiles P_1 and P_2. Then a social welfare function satisfies the Independence of Irrelevant Alternatives (IIA) property if the outcome of the function on the two profiles with respect to A_i and A_j is the same. That is, either $A_i \succ A_j$ for both profiles, $A_j \succ A_i$ for both profiles, or $A_i \sim A_j$ for both profiles.*

Consider the profile P_1 given in Table 9 whose head-to-head results are given in Table 10. To determine the DM winner, we restrict our attention to the majority cycle consisting of candidates A, B, and C. Then A is the Dogdson winner, requiring 5 voters with preference (6) to switch their preference to $A \succ C$, whereas both B and C will require at least 6 switches.

Table 9. The profile P_1

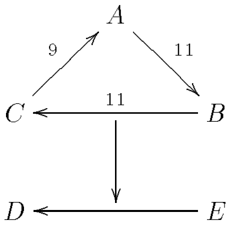
#	Ranking
6	$A \succ B \succ C \succ E \succ D$ (1)
5	$A \succ C \succ E \succ B \succ D$ (2)
5	$A \succ B \succ E \succ D \succ C$ (3)
5	$B \succ C \succ A \succ E \succ D$ (4)
5	$B \succ C \succ E \succ A \succ D$ (5)
10	$D \succ E \succ C \succ A \succ B$ (6)
5	$D \succ E \succ B \succ C \succ A$ (7)

Now consider the profile P_2 given in Table 11 that is obtained from P_1 by having one voter with preference (5) switch the location of candidates C and D. Notice that in these profiles, no voter has changed their preference with respect to the candidates A and E. As demonstrated in Table 12, the $C \succ D$ result has changed, and thus all candidates are included in the majority cycle. As a consequence, we see that E is the DM winner, requiring only one voter with ranking (4) to switch their $A \succ E$ preference and one voter with ranking (3) to switch their $B \succ E$ preference. Since every other candidate loses an election by a margin of at least 7, we know that E is the DM winner.

Thus, we have two profiles P_1 and P_2 where all voters have the same preference with respect to candidates A and E, but in one profile, A is the DM winner,

Table 10. Head-to-head results from Table 9

	Margin
$A \succ B$	11
$B \succ C$	11
$C \succ A$	9
$A \succ D$	11
$A \succ E$	1
$B \succ D$	11
$B \succ E$	1
$C \succ D$	1
$C \succ E$	1
$E \succ D$	1



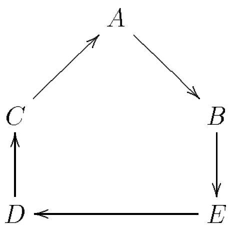
and in the other, E is the DM winner. Thus, DM does not satisfy Independence of Irrelevant Alternatives.

Table 11. The profile P_2

#	Ranking
6	$A \succ B \succ C \succ E \succ D$ (1)
5	$A \succ C \succ E \succ B \succ D$ (2)
5	$A \succ B \succ E \succ D \succ C$ (3)
5	$B \succ C \succ A \succ E \succ D$ (4)
4	$B \succ C \succ E \succ A \succ D$ (5a)
1	$B \succ D \succ E \succ A \succ C$ (5b)
10	$D \succ E \succ C \succ A \succ B$ (6)
5	$D \succ E \succ B \succ C \succ A$ (7)

Table 12. Head-to-head results from Table 11

	Margin
$A \succ B$	11
$B \succ C$	11
$C \succ A$	7
$A \succ D$	13
$A \succ E$	1
$B \succ D$	11
$B \succ E$	1
$D \succ C$	1
$E \succ C$	1
$E \succ D$	13



6 Proof of Theorem 1

Suppose that our candidates are $A_1, A_2, \ldots, A_n$. We will construct profiles where A_1 is the Dodgson winner, but A_1 can appear at any position in the ranking given by any positional method.

The construction will mimic that of our example from Section 3. We will construct a Condorcet component $\mathscr{C}$ from three Condorcet profiles that gives an ℓ_1 winner of A_1 and scale $\mathscr{C}$ sufficiently so that we can add a Basic component $\mathscr{B}$ to create conflict with any positional method without affecting the ℓ_1 winner. Then we scale this profile sufficiently large to guarantee that A_1 is the DM winner as well as the ℓ_1 winner. To construct the Condorcet component $\mathscr{C}$, it is easiest to deal with the cases of n odd and n even separately.

6.1 The Condorcet component for n odd

Consider the Condorcet profiles C_{r_1}, C_{r_2}, and C_{r_3} defined by the rankings

$$\begin{aligned} r_1 &= A_1 \succ A_2 \succ A_3 \succ \cdots \succ A_{q-1} \succ A_q \succ A_{q+1} \succ \cdots \succ A_n \\ r_2 &= A_q \succ A_2 \succ A_3 \succ \cdots \succ A_{q-1} \succ A_1 \succ A_{q+1} \succ \cdots \succ A_n \\ r_3 &= A_2 \succ A_q \succ A_3 \succ \cdots \succ A_{q-1} \succ A_1 \succ A_{q+1} \succ \cdots \succ A_n \end{aligned}$$

where $q = \frac{n+1}{2}$. The cycles associated with C_{r_1}, C_{r_2}, and C_{r_3} are shown in Figure 2. In Theorem 8 of [5], Saari shows that if A_i is ranked s candidates above A_j in a cycle, then the pairwise tallies for the corresponding Condorcet profile are

$$A_i : n - 2s \qquad\qquad A_j : 2s - n$$

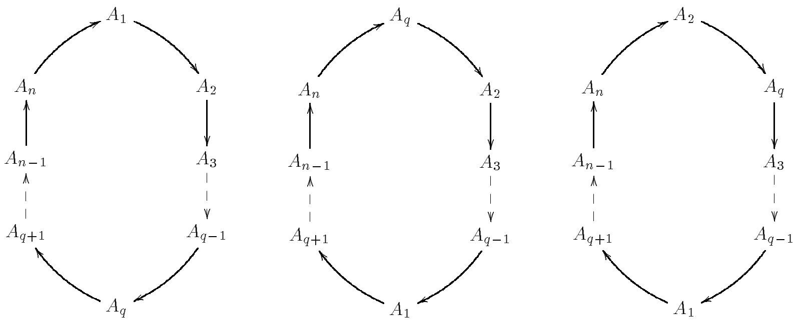
Thus, the largest margin of victory is $2n - 4$ (when $s = 1$) and the smallest is 2 (when $s = q$). For example, the pairwise margins involving A_1 are

$$2n - 4s \text{ for } A_1 \succ A_i, 1 < i \le q, \qquad 4s - 2n \text{ for } A_j \succ A_1, q < j \le n$$

In other words, candidate A_i beats A_j in the pairwise election when A_j is within q candidates of A_i moving clockwise around the cycle from A_i, and otherwise, A_j beats A_i. Notice that the margin decreases as the distance between A_i and A_j increases.

Our plan is to form a linear combination $\mathscr{C}$ of C_{r_1}, C_{r_2}, and C_{r_3} where the pairwise outcomes of $\mathscr{C}$ agree with those of C_{r_1} (so that every candidate is included in the majority cycle) but we use C_{r_2} and C_{r_3} to manipulate the pairwise margins so that A_1 has the smallest total margin of defeat, and thus is the ℓ_1 winner.

Consider the profile $\mathscr{C}' = nC_{r_1} + C_{r_2}$. Notice that the smallest pairwise election margin in nC_{r_1} is $2n$ and the largest in C_{r_2} is $2n-4$. Thus, we are guaranteed that the pairwise outcomes in $\mathscr{C}'$ agree with those of C_{r_1}, although the C_{r_2} component will affect the margins. In particular, by switching the locations of A_1 and A_q, both A_1 and A_q reduce their margin of loss for every pairwise election lost in



$\mathcal{C}_{r_1}$ $\mathcal{C}_{r_2}$ $\mathcal{C}_{r_3}$

Figure 2. The cycles for C_{r_1}, C_{r_2}, and C_{r_3}, n odd

C_{r_1}. For example, the $A_{q+1} \succ A_1$ margin is reduced by $2n - 4$, the $A_{q+2} \succ A_1$ margin is reduced by $2n - 8$, etc. Table 13 summarizes the effects of C_{r_2} on the pairwise results of C_{r_1}.

Table 13. Effects of C_{r_2} on the C_{r_1} pairwise margins of loss

	Margin Decreased	Margin Increased
A_1	All	None
$A_i, 1 < i < q$	$A_1 \succ A_i$	All others
A_q	All	None
$A_j, q < j \leq n$	$A_q \succ A_j$	All others

Note that A_1 and A_q have all $q - 1$ pairwise losses reduced by the maximum amount since they are ranked immediately ahead of the $q - 1$ candidates they lose to in C_{r_1}. Since every other candidate has at least one loss reinforced, the cumulative effect is to reduce the total margin of loss of A_1 and A_q more than any other candidate. Thus A_1 and A_q are tied as ℓ_1 winners in $\mathscr{C}'$. We now need to add a component to break this tie and make A_1 the ℓ_1 winner.

Now consider $\mathscr{C} = nC' + C_{r_3} = n^2 C_{r_1} + nC_{r_2} + C_{r_3}$. The same argument as above shows that the pairwise outcomes of $\mathscr{C}$ agree with those of $\mathscr{C}'$, and hence with those of C_{r_1}. Table 14 summarizes the effects of C_{r_3} on the pairwise results of C_{r_1}. As above, C_{r_3} reduces the pairwise losses of A_1 by the maximum amount, but every other candidate, including A_q, has at least one loss reinforced.

Thus, the cumulative effect of $\mathscr{C}$ on the pairwise losses of C_{r_1} is to reduce the losses of A_1 at each stage while every other candidate has at least one pairwise loss reinforced at one stage. Thus, A_1 is the ℓ_1 winner of $\mathscr{C}$.

Table 14. Effects of C_{r_3} on the C_{r_1} pairwise margins of loss

	Margin decreased	Margin increased
A_1	All	None
$A_i, 1 < i < q$	$A_1 \succ A_i$	All others
A_q	All others	$A_2 \succ A_q$
A_{q+1}	$A_2 \succ A_{q+1}$	All others
$A_j, q+1 < j \leq n$	$A_q \succ A_j$	All others

6.2 The Condorcet component for n even

When n is even, let $q = \frac{n}{2} + 1$ and define r_1, r_2, C_{r_1}, and C_{r_2} as in the odd case. However, we now define

$$r_3 = A_n \succ A_2 \succ \cdots \succ A_{q-1} \succ A_1 \succ A_{q+1} \succ \cdots \succ A_q$$

and C_{r_3} as the corresponding Condorcet profile. The cycles for these profiles are shown in Figure 3. Saari's results for the pairwise margins still hold, but now when A_i is ranked $s = q$ candidates ahead of A_j in a cycle, the pairwise outcome gives a complete tie. Although we will use the same construction as before of

$$\mathscr{C} = n^2 C_{r_1} + nC_{r_2} + C_{r_3},$$

the introduction of ties requires slightly more care in the argument since we introduce two pairwise outcomes in $\mathscr{C}$ ($A_1 \succ A_q$ and $A_n \succ A_{q-1}$) that are ties in C_{r_1}.

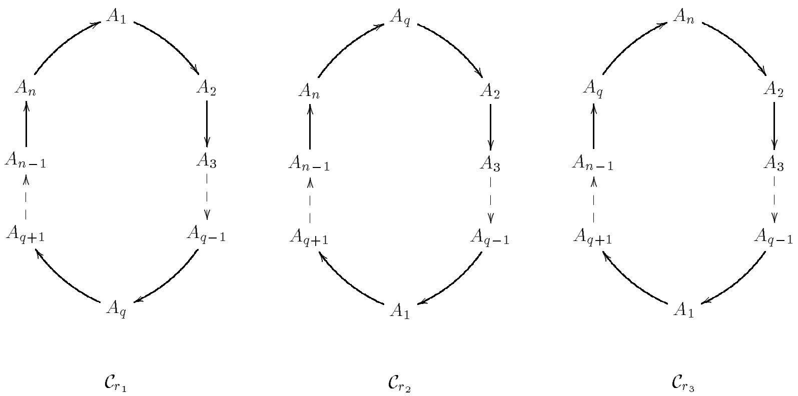


Figure 3. The cycles for C_{r_1}, C_{r_2}, and C_{r_3}, n even

As above, define $\mathscr{C}' = nC_{r_1} + C_{r_2}$ and note that C_{r_2} will not reverse any pairwise outcome from C_{r_1} nor will it break any of the pairwise ties existing in C_{r_1}. The effects of C_{r_2} on the pairwise losses is identical to that in Table 13, and thus A_1 and A_q are tied as ℓ_1 winners in $\mathscr{C}'$.

As before, let $\mathscr{C} = n\mathscr{C}' + C_{r_3}$. Then the C_{r_3} component will not reverse any pairwise preferences from C_{r_1}, although it does change the ties $A_1 \sim A_q$ and $A_n \sim A_{q-1}$ to $A_1 \succ A_q$ and $A_n \succ A_{q-1}$. Now consider the impact that C_{r_3} has on the pairwise losses of C_{r_1}.

First notice that A_1 reduces $q-3$ of its $q-2$ pairwise losses by the maximum possible amount since A_1 is ranked immediately ahead of $A_{q+1}, \ldots, A_{n-1}$ in C_{r_3}. Further, C_{r_3} has no impact A_1's other loss (to A_n) but does break the $A_1 \sim A_q$ tie in A_1's favor. Thus the only ways for another candidate to have its total margin of loss reduced as much as A_1's is to either decrease all $q-2$ of its pairwise losses or to decrease $q-3$ of its pairwise losses by the maximum amount and have no negative impact on its remaining loss or tie.

Clearly $A_2, \ldots, A_{q-1}$ and $A_{q+1}, \ldots, A_n$ have at least two pairwise losses increased and thus cannot have $q-3$ of their losses reduced. Although A_q does reduce $q-3$ of its pairwise losses (to $A_2, \ldots, A_{q-2}$), it is not by the maximum amount since A_q is not ranked immediately before $A_2, \ldots, A_{q-2}$ in C_{r_3}. Further, C_{r_3} changes the $A_1 \sim A_q$ tie to $A_1 \succ A_q$ so that A_q does not have its total margin of loss reduced as much as A_1. Thus, A_1 is the ℓ_1 winner in $\mathscr{C}$.

6.3 *The basic component*

We can easily place A_1 in the ith position using any positional voting procedure by taking an appropriate linear combination of Basic profiles B_{A_i}. For example, to place A_1 in the third position, we can form

$$\mathscr{B} = (n-1)B_{A_2} + (n-2)B_{A_3} + (n-3)B_{A_1} + (n-4)B_{A_4} + (n-5)B_{A_5} + \cdots + B_{A_{n-1}}$$

Then every positional method will give the outcome $A_2 \succ A_3 \succ A_1 \succ A_4 \succ \cdots \succ A_n$.

Recall that the pairwise margins for B_{A_i} are

$$2(n-1)! \text{ for} A_i \succ A_j, i \neq j, \qquad 0 \text{ for all others}$$

Thus, the largest margin in $\mathscr{B}$ is $2(n-1)!\,(n-1)$, which occurs for the first place candidate over the last. Since each candidate is involved in $n-1$ pairwise elections, an upper bound for the impact of $\mathscr{B}$ on the pairwise outcomes for any candidate is $2(n-1)!(n-1) \cdot (n-1) = 2(n-1)!(n-1)^2$. Notice that since $n > 4$, we have $2(n-1)!(n-1)^2 < 2(n+1)!$.

Therefore, if we scale $\mathscr{C}$ by $2(n+1)!$, we guarantee that A_1 is the ℓ_1 winner by a margin of at least $2(n+1)!$ in $2(n+1)!\mathscr{C}$, and the additional $\mathscr{B}$ component in $2(n+1)!\mathscr{C}+\mathscr{B}$ will not affect the ℓ_1 winner. Now we can scale $2(n+1)!\mathscr{C}+\mathscr{B}$ sufficiently large to ensure that A_1 is the DM winner as well as the ℓ_1 winner. By adding the appropriate multiple of K, we obtain a profile where A_1 is the DM winner but is located in the desired position using any positional method.

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