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How many voters are needed for paradoxes?*

James S. Weber

Department of Information and Decision Sciences, University of Illinois at Chicago, Chicago, IL 60607-7124, USA (e-mail: jsweber@uic.edu)

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Summary. This paper presents a general procedure for finding profiles with the minimum number of voters required for many important paradoxes. Borda's and Condorcet's classic examples are revisited as well as generalizations. Using Saari's procedure line, we obtain an upper bound for the minimum number of voters needed for a profile for which the Condorcet winner is not strictly top ranked for all w_s^3 weighted positional procedures. Also we give a simple upper bound on the minimum number of voters needed for a *set* of prescribed voting outcomes. In contrast to situations wherein small numbers of voters are needed, we consider paradoxes requiring arbitrarily large numbers of voters as well as large numbers of alternatives. Finally we indicate connections with statistical rank based tests.

Keywords and Phrases: Voting paradox, Minimum number of voters, Condorcet pairwise procedure, Borda Count, Plurality, w_s^3 procedure, Procedure line, Committees, Kruskal-Wallis Test.

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1 Introduction

"*How many voters are needed for paradoxes?*" is an important question. The relevance of voting paradoxes depends on their occurrence for real world numbers of voters! We need to understand and possibly eliminate paradoxes which occur for numbers of voters actually encountered in group decision making. Are conflicting

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outcomes a problem for groups regardless of size? If not, what is the threshold committee size? Do well known paradoxes pose a problem for corporate, political and academic committees and subcommittees? Do we need to worry only about village elections, but not smaller committee decisions? Although both Borda and Condorcet [1, 3, 4, 7, 12, 13, 16] were troubled by conflicting outcomes of different voting procedures, possibly neither considered the minimum number of voters for which problems occur. Consequently, managing at least some paradoxes by committee size does not appear to have been studied (although Taylor [23] comments on committee size.)

There also is a scientific interest in necessary numbers of voters. Significant recent work extends, explains and simplifies the existence of conflicting outcomes and cycles. Techniques used include, among others, the geometry of voting, profile decompositions and identification of cyclic components. [14, 15, 16, 17, 18, 19, 23, 26]. There are important simulated and theoretical relative frequencies of agreement of different rules [5, 6, 8] as well as asymptotic relative frequencies of election outcomes [2, 20, 22]. These recent advances, however, do not tell us how many voters are needed for paradoxes or, arguably, *real world* probabilities. Many of these discussions could appear to identify simulated and asymptotic frequencies with real world probabilities or frequencies of agreement of certain rules with the Condorcet procedure.

However these relative frequencies depend on the choice of distribution of profiles. This choice often is driven by a analytical tractability rather than reality. This is not meant as a criticism, but simply an observation about these well known studies. Certainly tractability has guided our own efforts. But notice that minimum voter requirements are robust in terms of profile frequency distributions: if a committee has fewer than the minimum number of voters required for a paradox, then the probability or relative frequency of that paradox is zero, for all distributions of profiles. In sum, although there are many examples with small numbers of voters, we need an organized knowledge of the minimum number of voters necessary and generally applicable methods. This study begins to provide this.

We begin our effort using integer programming (“IP”, [25]) to probe existence of individual paradoxes as well as the minimum number of voters needed. (Readers unfamiliar with IP will understand the problem statement and solution.) Then profiles for paradoxes may be used to give upper bounds for the number of voters needed for a set or class of possible voting outcomes.

We focus on three alternatives, the six voter types with strict preferences (shown in Table 1), and voting procedures expressed as linear inequalities. That is, many voting procedures are based upon rankings of tallies which are linear functions of the numbers of each voter type. Rankings without ties are equivalent to strict inequalities among the linear tally functions. Finding minimum integer numbers of voters for which linear inequalities are satisfied is a problem for which IP is usually applicable.

Consider a group ranking based on votes on three alternatives: A, B, C. Procedures of interest are Condorcet pairwise runoffs, plurality, Borda Count, antiplurality, and weighted scoring procedures[1] ($w_s^3 \equiv (1, s, 0)$ weights for 1st, 2nd, 3rd, $0 \leq s \leq 1$). Voter types are defined to have the strict preferences shown in Table 1, below, as in [15],[2] among others.

Table 1

Voter type	Preference ordering
1	A > B > C
2	A > C > B
3	C > A > B
4	C > B > A
5	B > C > A
6	B > A > C

Election outcome rankings are determined by comparing tallies for the candidates. The tallies may be taken all at once, as in plurality, Borda Count, antiplurality and weighted scoring procedures. Or they may be done in subset runoffs as in the Condorcet pairwise procedure.

An integer profile (p_1, p_2, p_3, p_4, p_5, p_6) is a list of integer counts of each voter type.[3] As examples of tallies based on a profile, Table 2 presents tallies for the plurality and Borda Count procedures in terms of (p_i).

Table 2

Candidate	Plurality tally	Borda Count tally
A	$p_1 + p_2$	$2p_1 + 2p_2 + p_3 + p_6$
B	$p_5 + p_6$	$p_1 + p_4 + 2p_5 + 2p_6$
C	$p_3 + p_4$	$p_2 + 2p_3 + 2p_4 + p_5$

Now consider finding the minimum voter profile for which the plurality outcome is A > B > C and the Borda Count outcome is reversed, C > B > A. That such a profile exists is well known from the profile (0, 5, 0, 3, 4, 0) due to Borda [16, p. 352] showing that the plurality procedure can give a reversed Borda count ranking of three candidates. Obviously 12 voters are needed for this example.

[1] Recall that Borda count, plurality and antiplurality procedures are special cases of $\boldsymbol{w}_s^3$ procedures, for s = 1/2, 0, and 1 respectively. (Do not confuse $\boldsymbol{w}_s^3$ with $\boldsymbol{w}_s$, [15] Saari, 1995, p. 105, (4.1.9)).

[2] These are the voter types found in [15]. There are theoretical reasons why the voter types we use here are preferred to what might at first appear to be another more sensible way of keeping track of different individual rankings, namely, the lexicographic ordering: A > B > C, A > C > B, B > A > C, B > C > A, C > A > B, C > B > A.

[3] Such a list is called an *integer profile* or a *profile*. A *normalized profile* shows for each voter type its fraction of the total number of voters. Here, we focus on profiles, not normalized profiles.

Also Theorem 4.2.1,[4] p. 111, of Saari [15], assures the existence of such a profile. But neither Borda's example nor Theorem 4.2.1 indicate the profile with the fewest number of voters which leads to the reversed plurality and Borda Count outcomes, $A > B > C$ & $C > B > A$. Below is an IP minimization problem which yields such a profile, (p_i).

$$\begin{aligned}
&\text{Minimize} \quad p_1 + p_2 + p_3 + p_4 + p_5 + p_6 && \text{(Number of Voters)}\\
&\text{such that}\\
&p_1 + p_2 - p_5 - p_6 \geq 1 && \text{(P.ab)}\\
&p_5 + p_6 - p_3 - p_4 \geq 1 && \text{(P.bc)}\\
&-p_1 + p_2 + 2p_3 + p_4 - p_5 - 2p_6 \geq 1 && \text{(BC.cb)}\\
&-p_1 - 2p_2 - p_3 + p_4 + 2p_5 + p_6 \geq 1 && \text{(BC.ba)}\\
&p_1, p_2, p_3, p_4, p_5, p_6 \geq 0 && \text{(NonNegative)}
\end{aligned}$$

Constraints (P.ab) and (P.bc) are equivalent to the plurality outcomes of $A > B$ and $B > C$. Constraints (BC.cb) and (BC.ba) are equivalent (after simplification) to the Borda Count outcomes of $C > B$ and $B > A$. An IP solution[5] obtained using LINDO [21] is $(p_i) = (1, 3, 0, 2, 3, 0)$ and this proves Theorem 1, below.

Theorem 1. *The minimum number of voters required for a plurality outcome of $A > B > C$ and a Borda Count outcome of $C > B > A$ is 9. (A profile of 9 voters having these outcomes is (1, 3, 0, 2, 3, 0).)*

2 An example of Borda and variations

McLean, Hewitt, Urken, Saari and others [3, 4, 12, 13, 14, 15, 16, 19] discuss historically important examples of Condorcet and Borda which highlight the disagreements among the Condorcet pairwise, the Borda Count and plurality election procedures. Borda used the profile (0, 5, 0, 3, 4, 0) to show that the pairwise and Borda Count outcomes do not always agree with plurality outcomes. Tables 3 and 4 below (obtained with IP) show profiles of the fewest voters leading to varied combinations of outcomes. The outcomes of historical interest (in bold face) are those of the plurality, Borda Count and Condorcet pairwise procedures. However we can easily consider other outcomes. (Borda's example is *line 1* and Theorem 1 is given by *line 6*.) Theorem 1, above, and Theorem 2, below, are special cases of the results in Table 3.

[4] **Theorem 4.2.1** ([15], p. 111). "Let the voting vectors $\boldsymbol{w}_{s1} \neq \boldsymbol{w}_{s2}$ be given. Let β_1, β_2 be any two rankings of three candidates – the rankings may be the same or they may differ. There exists a profile so that when $\boldsymbol{w}_{sj}$ is used to tally the ballots of the voters, the outcome is β_j; $j = 1, 2$."

[5] The IP formulations used for this research do not report all occurrences or the number of minima due to profile symmetries. In contrast to the voting theory applications presented here, many real world applications have unique extreme points. And when they are not unique, the multiplicity often results from conditions other than profile symmetries present with unrestricted voter profile domains.

Table 3. Borda's example and variations (1770) (0, 5, 0, 3, 4, 0). Examples 2 through 16 are minimum voter profiles; 0 & 1 are not

	# vtrs	BC	Pl	Con	APl	Profile	Profile decomposition 1/6 (a_B	b_B	a_R	b_R	γ	k)
0.	54	C > B > A	A > B > C	...	...	5,18,2,9,20,0	-12	-6	36	24	0	54
1.	12	C > B > A	A > B > C	C > B > A	C > B > A	050340	-5	-4	9	6	-4	12
2.	1	C > B > A	...	...	...	000100	-2	-1	0	-1	-1	1
3.	3	...	A > B > C	...	...	200010	3	3	1	-1	3	3
4.	1	...	...	C > B > A	...	000100	-2	-1	0	-1	-1	1
5.	3	...	...	...	C > B > A	010200	-3	-3	1	-1	-3	3
6.	*9*	*C > B > A*	*A > B > C*	...	...	*130230*	*-2*	*-1*	*6*	*3*	*-1*	*9*
7.	1	C > B > A	...	C > B > A	...	000100	-2	-1	0	-1	-1	1
8.	3	C > B > A	...	...	C > B > A	001200	-5	-4	-1	-2	-1	3
9.	9	...	A > B > C	C > B > A	...	130230	-2	-1	6	3	-1	9
10.	6	...	A > B > C	...	C > B > A	120120	0	0	4	2	0	6
11.	3	...	...	C > B > A	C > B > A	001200	-5	-4	-1	-2	-1	3
12.	9	C > B > A	A > B > C	C > B > A	...	130230	-2	-1	6	3	-1	9
13.	9	C > B > A	A > B > C	...	C > B > A	130230	-2	-1	6	3	-1	9
14.	3	C > B > A	...	C > B > A	C > B > A	001200	-5	-4	-1	-2	-1	3
15.	9	...	A > B > C	C > B > A	C > B > A	040230	-3	-3	7	5	-3	9
16.	9	C > B > A	A > B > C	C > B > A	C > B > A	130230	-2	-1	6	3	-1	9

The profiles 1–16 are shown as 6 digits since all profiles have are 9 or fewer of each voter type. Profiles # 0 & # 1 obviously are not minimum voter profiles. The 16 sets of constraints lead to seven distinct minimum voter profiles. Also shown are the profile decompositions studied by Saari [16, 17, 18, 19], written as 1/6($6a_B$ $6b_B$ $6a_R$ $6b_R$ 6γ 6k).)

Theorem 2. *The minimum number of voters with strict preferences which lead to the same strict Borda Count, plurality, Condorcet and antiplurality outcomes as Borda's (0, 5, 0, 3, 4, 0) example is 9. A profile of 9 voters having these outcomes is (1, 3, 0, 2, 3, 0).*

When we enlarge our view to include outcomes of voting procedures other than the ones for which the example was created, we find profiles with small numbers of voters have additional interesting outcomes. In particular, the profile with the restrictions that the plurality and Borda Count outcomes are, respectively, A > B > C and C > B > A, respectively, has the same Condorcet and antiplurality outcomes as Borda's original example with 12 voters. Here, imposing the Condorcet and antiplurality outcomes does not change the minimum voter profile. A consideration of the procedure line suggests that we should expect this when we require an antiplurality outcome which is the same as the BC outcome [15].

3 An example of Condorcet and variations

Condorcet argued against the Borda Count procedure with the profile (30, 1, 10, 1, 10, 29) of 81 voters. With this profile, no positional procedure will elect the Condorcet winner [12, p. 137, 16, p. 352, 19, p. 117]. Table 4 gives profiles leading to various procedure outcome combinations for Condorcet's (30, 1, 10, 1, 10, 29) profile. (*Line 7* gives the minimum voter profile with the same Borda Count and Condorcet outcomes.) Theorem 3 below is a special case among the results shown in Table 4.

Theorem 3. *The minimum number of voters with strict preferences which lead to the same strict Borda Count and Condorcet outcomes as Condorcet's (30, 1, 10, 1, 10, 29) profile is five.* (That is, five is the smallest number of voters for three candidates for which the Condorcet winner is ranked second by the Borda Count. Such a profile is (3, 0, 0, 0, 2, 0).) (Note that the Condorcet outcome A $>$ B $>$ C implies that the Borda Count outcome can only be B $>$ A $>$ C or A $>$ B $>$ C, since A cannot be Borda Count bottom ranked & C cannot be Borda Count top ranked.)

We see from its profile decomposition [16, 17, 18, 19] that with the fewest number of voters for which these outcomes happen, there is a cyclic component of 5/6 which shifts the Borda Count winner, B, to Condorcet pairwise second place. This component shows the effect of cyclic intransitivity of Condorcet pairwise rankings with the smallest possible number of voters.

The 15 sets of constraints lead to 6 distinct minimum voter profiles.

4 Number of voters needed for possible plurality, Borda Count and Condorcet outcomes

In the previous sections, Tables 3 and 4 present minimum voter profiles leading to historically significant outcomes for Borda Count and plurality procedures.

Notice that Theorem 4.2.1[4] ([15], p. 111) involves only positional procedures. Our knowledge of all of the outcome combinations for Condorcet and Borda Count has been less complete, or not so easily summarized by a single general fact such as Theorem 4.2.1 (also see [5, 6, 20, 22]). Hence we look at all the outcome combinations for Borda Count and Condorcet procedures. Table 5 shows the results of using IP to find the fewest number of voters for various possibilities. This systematically elaborates upon the well known fact that a Condorcet winner cannot be bottom ranked via Borda Count.

In contrast to profile Tables 3 and 4, all of the profiles in Table 5 necessarily are different.

Table 5 proves Theorem 4a while Table 5 and Lemma 1a lead to or motivate Theorems 4b, 4c, 4d, and 4e.

Theorem 4a. *Each of the possible strict plurality, Borda count and Condorcet (both noncyclic and cyclic) outcomes may occur for some voter profile of at least three and at most 21 voters.*

Table 4. Condorcet's example and variations (30, 1, 10, 1, 10, 29). Examples 2 through 16 are minimum voter profiles; 1 is not

	# vtrs	BC	Pl	Con	APl	P & 1/6 (a_B b_B a_R b_R γ k)						
1.	81	B > A > C	B > A > C	A > B > C	A ∼ B > C	30, 1, 10, 1, 10, 29	68	76	-28	-20	19	81
2.	1	B > A > C	...	...	...	000001	1	2	-1	0	-1	1
3.	3	...	B > A > C	...	...	100020	0	3	2	1	3	3
4.	1	...	...	A > B > C	...	100000	2	1	0	-1	1	1
5.	1	...	...	...	A ∼ B > C	100000	2	1	0	-1	1	1
6.	3	B > A > C	B > A > C	...	...	100002	4	5	-2	-1	-1	3
7.	*5*	*B > A > C*	...	*A > B > C*	...	*300020*	*4*	*5*	*2*	*-1*	*5*	*5*
8.	1	B > A > C	...	...	A ∼ B > C	000001	1	2	-1	0	-1	1
9.	9	...	B > A > C	A > B > C	...	302022	4	5	-2	-1	5	9
10.	3	...	B > A > C	...	A ∼ B > C	00002	4	5	-2	-1	-1	3
11.	1	...	...	A > B > C	A ∼ B > C	100000	2	1	0	-1	1	1
12.	9	B > A > C	B > A > C	A > B > C	...	302022	4	5	-2	-1	5	9
13.	3	B > A > C	B > A > C	...	A ∼ B > C	100002	4	5	-2	-1	-1	3
14.	9	B > A > C	...	A > B > C	A ∼ B > C	302022	4	5	-2	-1	5	9
15.	9	...	B > A > C	A > B > C	A ∼ B > C	302022	4	5	-2	-1	5	9
16.	9	B > A > C	B > A > C	A > B > C	A ∼ B > C	302022	4	5	-2	-1	5	9

Table 5. Minimum voter profiles for all possible Borda Count and Condorcet outcomes without ties, when plurality outcome is A > B > C. Columns distinguish Borda count rankings; Rows distinguish Condorcet rankings

BordaCt **Condor.**	1. $A>B>C$	2. $A>C>B$	3. $C>A>B$	4. $C>B>A$	5. $B>C>A$	6. $B>A>C$
1. $A>B>C$	**3*** 110001	**11** 332030	NP	NP	NP	**5** 300020
2. $A>C>B$	**5** 030002	**3** 020010	**11** 060230	NP	NP	NP
3. $C>A>B$	NP	**13** 243040	**9** 041130	**21** 353370	NP	NP
4. $C>B>A$	NP	NP	**13** 060340	**9** 130230	**21** 350652	NP
5. $B>C>A$	NP	NP	NP	**21** 442470	**9** 220230	**9** 400230
6. $B>A>C$	**9** 220203	NP	NP	NP	**21** 440643	**9** 400221
** $A>B>C>A$	**9** 312030	**9** 222030	**15** 333150	**15** 332250	**15** 422250	**9** 311130
** $A>C>B>A$	**9** 130212	**9** 040212	**9** 040221	**15** 150432	**15** 240432	**15** 240423

"NP" abbreviates "not possible"

*- While there is no conflict when all three procedures have the same A > B > C ranking, 3 voters are needed for the plurality ranking to distinguish B & C; otherwise B & C would be tied at zero to zero, for just one voter with preference A > B > C.

** 1–6 are outcome types which are the same as the voter types of Table 1. The cycles $A > B > C > A$ and $A > C > B > A$ are not among the 13 non cyclic numbered outcome types considered in [15] (among others).

Proof. Table 5. □

Saari and Zwicker [16, 17, 18, 19, 26] give theoretical clarifications of the linear structure of cycles from which we observe that ranks of tallies of positional procedures are not changed by augmenting a profile by (1, 0, 1, 0, 1, 0) or (0, 1, 0, 1, 0, 1). Also the net margin of winning for Condorcet's pairwise procedure does not change by more than one.

From Saari ([15], p. 104 (4.1.7), p. 108 4.1.4# 1) we see how equivalence classes for voting vectors for positional procedures can be defined. Also it is clear that the ranking outcome of any positional procedure and Condorcet's pairwise procedure is not changed by augmenting any profile by multiples of the kernel profile (1, 1, 1, 1, 1, 1).[6] In this context, Lemma 1a leads to an association of numbers of voters with profiles.

Lemma 1a. Let p be a profile leading to *strict* rankings for Condorcet's pairwise procedure and *any* positional procedure rankings.

[6] Notice that positional procedures for three alternatives have a 2 dimensional kernel: (1,0,1,0,1,0) & (0,1,0,1,0,1) whereas the kernel for both positional and the Condorcet procedures is one dimensional (1,1,1,1,1,1).

Let $p* \equiv qp + k(1,1,1,1,1,1) + m(1,0,1,0,1,0) + n(0,1,0,1,0,1)$; wherein q, k, m, n are integers with $q \geq 1; k, m, n \geq 0; (q-1) \geq |m-n| \geq 0$.

Normalize k, m, n so that at most one of m, n is nonzero with

$$\begin{aligned} k^* &= k + \min\{m, n\} \\ m^* &= m - \min\{m, n\} \\ n^* &= n - \min\{m, n\} \end{aligned}$$

Write p^* as

$$p^* \equiv qp + k^*(1,1,1,1,1,1) + m^*(1,0,1,0,1,0)$$

or

$$p^* \equiv qp + k^*(1,1,1,1,1,1) + n^*(0,1,0,1,0,1)$$

Then the profiles p and p^* have the same ranking outcomes for Condorcet's pairwise procedure and all w_s^3 positional voting procedures.

Proof. For all w_s^3 positional procedures k^* $(1,1,1,1,1,1) + m^*(1,0,1,0,1,0) + n^*(0,1,0,1,0,1)$ increases all the tallies by a constant, $(2k^* + m^* + n^*)(1+s)$, but this does not change the rankings. (Although the percentage margins of victory become smaller.) For the Condorcet pairwise procedures, strict outcomes mean that all pairwise tallies differ by at least 1 for p and at least q for qp. Hence $(q-1)(1,0,1,0,1,0)$ or $(q-1)(0,1,0,1,0,1)$ can change the tallies by at most $(q-1)$, but not change rankings which have margins of q or more. □

(Also see [16, 17, 18, 19]).

Lemma 1b. Let p be a profile with *at least one tied* ranking for Condorcet's pairwise procedure and any rankings for any w_s^3 positional procedure.

Let $p^* \equiv qp + k^*$ $(1,1,1,1,1,1) + m^*$ $(1,0,1,0,1,0) + n^*$ $(0,1,0,1,0,1)$ as in Lemma 1a.

Then p^* has the same positional ranking and strict Condorcet pairwise ranking outcomes as p, but all tied rankings change when m^* or $n^* \geq 1$.

Proof. Similar to the proof of Lemma 1a, upon observing that Condorcet qp tallies will differ by zero or nonzero multiples of q so that Condorcet ties for p lead to a difference of m^* or n^* for p^*. □

Theorem 4b. *All of the possible* **strict** *plurality, Borda count and Condorcet (noncyclic and cyclic) outcomes may occur with 39 voters. (And obviously with* $q39 + 6k + 3m$ *voters, for integers* $q > 0$, $k \geq 0$, $(q-1) \geq m \geq 0$.)

Proof. We simply observe that profiles with 3, 5, 9, 11, 13, 15, and 21 voters may be enlarged to 39 voters without changing any of the strict outcomes. Each of the situations *i, ii, iii* below is an instance of Lemma 1a.

i) **3**, **13:** 39 is divisible by 3 and 13, so the profiles of 3 and 13 may be multiplied by 13 and 3 to obtain profiles of 39 voters having the same strict outcomes.

ii) **5**, **15:** 39 is 9 more than 30. Each strict outcome with 5 voters will have a margin of at least 1 so the margin will be at least 6 when a 5 voter profile is multiplied by 6 and at least 2 when a 15 voter profile is multiplied by 2. Adding 1 of each voter type will not change any outcome or margin of victory. Adding one each of A $>$ B $>$ C, B $>$ C $>$ A, C $>$ A $>$ B (or A $<$ B $<$ C, B $<$ C $<$ A, C $<$ A $<$ B) will not change any margin by more than 1, so there are profiles of 39 voters with the same outcomes as for profiles of 5 or 15 voters.

iii) **9**, **11, 21:** 39 is 12 more than 27, 6 more than 33 and 18 more than 21. Consequently there are profiles of 39 voters with the same outcomes as profiles of 9, 11 and 21 voters (by adding 2, 1 or 3 each of the six voter types with strict preferences to qp for q = $3, 3, 1$, respectively.) □

Theorem 4c. *Consider a set of K voter profiles,* $\boldsymbol{p}_j$, $j = 1, \ldots K$, *of 6 voter types with strict preferences among three candidates, A, B, C, with each profile having* n_j *voters. An upper bound for the minimum number of voters having ALL of the positional and Condorcet outcomes of the individual profiles is given by*

$$B_{\min} = Min \left\{ \bigcap_{j=1\ldots K} \left\{ qn_j + 6k + 3m \,|\, q = 1, \ldots \infty, k = 0, \ldots \infty, 0 \leq m \leq (q-1) \right\} \right\}$$

Proof. Replicating voters leads to multiples of tallies, so rankings remain the same. Adding kernel profiles adds a constant to all tallies, so rankings remain the same. Adding the profiles (1,0,1,0,1) or (0,1,0,1,0,1) only changes margins of the pairwise tallies by 1. A strict procedure outcome will have pairwise margins of 1 or more and a q replicated profile will have margins of q or more. Hence up to ($q - 1$) sets of voters of types 1, 3, 5 or sets of types 2, 4, 6 may be added without changing the pairwise rankings. □

(Note: $\mathbf{B}_{\min} < \infty$ since the r.h.s. is not an empty set. The product, $\prod_{j=1\ldots K} n_j$ occurs in every intersection set (for $n_k, q = \prod_{j=1\ldots K} n_{j/n_k}$)).

Obviously, we would like to upgrade Theorem 4c by any of the following conjectures.

Conjecture 4c.1. If each profile is a minimum voter profile, then $\mathbf{B}_{\min}$ is the minimum number of voters for which there are profiles which have every outcome of a set of profiles.

Conjecture 4c.2. If (or iff) each profile is a minimum voter profile and if $\{n_i\}$ satisfy certain conditions (e.g. possibly $\{n_i\}$ are relatively prime, or prime), then $\mathbf{B}_{\min}$ is the minimum number of voters for which there are profiles which have every outcome of a set of profiles.

Theorem 4d. *An upper bound for the minimum number of voters required for all of the positional and all strict pairwise outcomes of a given profile is*

$$\text{Min } \{p_1^* + p_2^* + p_3^* + p_4^* + p_5^* + p_6^* \mid \boldsymbol{p}^* \text{can be augmented to } p \text{ by the operations of Lemma 1a}\}$$

Proof. The set of p^* includes p which yields a valid upper bound, $p_1 + p_2 + p_3 + p_4 + p_5 + p_6$. All other values also are upper bounds and the smallest of these is an upper bound. □

The uninspiring proof of Theorem 4d hides possible utility. Consider its application to the example of Condorcet which we have just studied: (30,1,10,1,10,29) = 2(15,0,5,0,5,14) + (0,1,0,1,0,1). *Hence all of the positional and strict pairwise outcomes are the same for Condorcet's example and the profile of 39 voters: (15,0,5,0,5,14)*! We must be a little intrigued that this minimum is the same as the minimum obtained in Theorem 4b. That is, the same minimum number of voters arises i) by combining all the profiles of Table 5 and ii) by shrinking Condorcet's profile as small as possible! This motivates the following result.

Theorem 4e. *Consider a set of* **K** *profiles,* $\boldsymbol{p}_j$, *as in Theorem 4c. Consider the set of* **K** *profiles,* $\boldsymbol{p}_j^*$, *from application of Theorem 4d. An improved upper bound may emerge from application of Theorem 4c to the profiles* $\boldsymbol{p}_j^*$ *obtained from Theorem 4d.*

Proof. Similar to 4d, the minimum of a union of sets of numbers (of voters) is less than or equal to the minimum of any of the individual sets. □

Use of the procedure line with profiles shown in Table 5 allows us to obtain theorem 5 below.

Theorem 5. *For three alternatives, 9 is an upper bound for the minimum number of voters for which the Condorcet winner is not strictly top ranked by any positional* $\boldsymbol{w}_s^3$ *procedures.*

Proof of Theorem 5 (see Saari [15], pp. 35, 179). From Table 5, the profile (220203) with nine voters has plurality outcome of $A > B > C$ (type 1) and Condorcet outcome of $B > A > C$ (type 6). Notice that the antiplurality outcome is $A \sim B > C$ (type 12). Hence in the representation triangle, the procedure line [15, pp. 35, 179] [19, p. 47] begins as a type 1 outcome and ends on the right boundary of the type 1 outcomes, namely a type 12 outcome. Hence no $\boldsymbol{w}_s^3$ procedure has a type 5, 11, or 6 outcome, and for this profile B is not top ranked by any $\boldsymbol{w}_s^3$ procedure. □

Theorem 5 needs to be discussed in the context of Fishburn's 1974 paper, [6]. In that study, Fishburn simulates the relative frequency with which certain weighted scoring rules select a Condorcet winner when one exists and refers to this simulated relative frequency as the *efficiency* of the procedure. Figure 1 op. cit. estimates the efficiency of plurality with 3 alternatives to be about 92%. It could appear that 8% of the simulated profiles which have a Condorcet winner did not lead to the same alternative as the **unique** plurality winner. However Fishburn includes in his measure of efficiency 1/2 of the number of profiles for which the

Condorcet winner ties for top ranking with the plurality ranking procedure. It is one half of the tied outcomes which account for this 8% inefficiency reported by Fishburn. However, there are no profiles of three voters for which the Condorcet winner is strictly second or lower ranked by the plurality procedure. To gain some confidence that the upper bound of 9 might be a minimum, consider Table 6. Table 6 presents minimum voter profiles for which the Condorcet procedure has a strict winner while plurality outcomes advance from completely tied to a strict ranking. Notice that our Theorem 5 deals with all positional procedures whereas the Fishburn study [6] considers only selected positional procedures without the insight gained from the procedure line. (Our Theorem 3 gives a profile of 5 voters for which the Condorcet winner is 2nd ranked by the Borda procedure. However for this profile, the Condorcet winner is plurality top ranked. Also note Table 4, profile #15.)

Table 6. More IP minimum voter profiles

Number of voters	Minimum voter profile	Condorcet outcome*	Plurality outcome
3	010110	$C > B$, $B > A$, $C > A$*	$A \sim B \sim C$
5	020120	$C > B$, $B > A$, $C > A$*	$A \sim B > C$
7	030220	$C > B$, $B > A$, $C > A$*	$A > B \sim C$
9	130230	$C > B$, $B > A$, $C > A$*	$A > B > C$

* These could be regarded as redundant, but we want to explicitly exclude cycles, since Table 5 does include cycles.

Note added in proof to Section 4

1. Plurality outcomes of $B > A \sim C$ and $B > C > A$ would have been preferred for lines three and four of Table 6.

2. We describe a way to improve Theorem 5 by finding the minimum number of voters for which the Condorcet winner is second or lower ranked or not strictly top ranked for any w_s^3 positional procedure. Create additional tables similar to Table 5. Instead of plurality, as in Table 5, fix the Condorcet outcome as $A > B > C$. Use rows and columns for different outcomes for two positional procedures: plurality and antiplurality. Let these outcomes have ranking types 1–6 or 1–13 (ranking types 1–6 are identical to voter types for 1–6. Usually voters are assumed not to have ranking types 7–13. See Saari [15] pp 35, 179). Such tables together with the procedure line should lead to a minimum voter profile for which the Condorcet winner is second or lower ranked for all w_s^3 positional procedure for 3 alternatives for the type 1–6 table. The type 1–13 table should lead to a minimum voter profile for which the Condorcet winner is not strictly top ranked for all w_s^3 procedures.

5 Paradoxes which require large numbers of voters

It could appear from our findings thus far that all paradoxes occur for relatively small numbers of voters. This certainly is not the case. Theorems 6 and 7 present two well known situations wherein arbitrarily large numbers of voters are necessary. For Theorem 6, as before, a positional procedure $\boldsymbol{w}_s^3$ has weights of $(1, s, 0)$ for first, second and third ranked alternatives, with $0 \leq s \leq 1$. For $s = 1/2$, $\boldsymbol{w}_s^3$ yields the rankings given by the Borda Count procedure for three alternatives [16, 17, 18, 19].

Theorem 6. *There is no upper bound on the minimum number of voters necessary for a Condorcet winner to be $\boldsymbol{w}_s^3$ bottom ranked for s ε [0, 1/2) or s ε (1/2, 1].*

Proof. As s $\to$ 1/2, the IP minimum grows arbitrarily large. For s = 1/2, the LP is not feasible. □

Theorem 7. *There is no upper bound on the minimum number of voters required for a super majority cycle among 3 candidates as the fraction of voters required for a super majority pairwise winner approaches 1.*

Proof. As the fraction of votes required for a super majority, ϕ, approaches 1, less than unanimous votes for arbitrarily large numbers of voters will not equal or exceed ϕ unless the number of voters, V, is arbitrarily large. That is, since Condorcet cycles cannot occur when any alternative wins unanimously against all others, we need:

$$(V - 1)/V \geq \phi \Leftrightarrow V \geq 1/(1 - \phi). \qquad \square$$

6 The number of candidates needed for paradoxes

Since the number of voter types depends on the number of candidates, IP may not be as readily applicable to the consideration of the relationship of the number of candidates to the existence of voting paradoxes. But we present the following quick result, since, interestingly, we obtain the same lower bound for the number of alternatives as for the number of voters in Theorem 7.

Theorem 8. *There is no upper bound on the minimum number, A, of alternatives required for a super majority cycle among V voters as the fraction of voters required for a super majority pairwise winner approaches 1.*

Proof. Let ϕ denote the fraction needed for a super majority pairwise winner. From [24][7] we have

$$A \geq V/(V - M) \Leftrightarrow A \geq V/(V - \phi V) = 1/(1 - \phi). \qquad \square$$

We have the same lower bound for both the number of voters and the number of candidates needed for the possibility of a cycle in terms of the fraction, ϕ, required for a super majority pairwise winner. This lower bound becomes large as ϕ approaches 1.

[7] Errata: [24]) incorrectly attributes the elementary condition, A $\geq$ V/(V $-$ 1) to [9]. This error was called to the author's attention in a note from Professor A. K. Sen. Also notice [13, pp. 88-89] that this condition, A $\geq$ V/(V $-$ 1), [24], is similar, although not identical to Borda's condition for the minimum size of a plurality required to insure that a plurality winner will win all pairwise votes.

Theorems 7 and 8 above have an intuitive consistency with the result obtained by Balasko and Cres [2] wherein it is shown that as the number of alternatives and universe of profiles becomes large, the probability of a super majority Condorcet Cycle becomes a rare event if the percent required for the supermajority exceeds 53% .

7 Conclusion

We have used IP to answer questions about both the existence of paradoxes and the necessary numbers of voters. We found the minimum numbers of voters needed for historically important examples due to Borda and Condorcet; various subsets of outcomes which can be associated with those examples; and the plurality, Borda Count and Condorcet outcomes which can occur. We have provided simple upper bounds on the minimum number of voters needed for sets of paradoxes when profiles have already been determined for each paradox. In other reports, we expect to study minimum voter profiles for paradoxes involving four and five alternatives.

A final afterthought is that Haunsperger has explained the connection of voting paradoxes to the Kruskal-Wallis and related tests [10, 11]. An important question in applied statistics is sample size. We expect that avoiding certain statistical paradoxes discussed by Haunsperger may be approached via sample size, using some of the tools and results presented here. Avoidance of paradoxes via small samples may need to be balanced against larger samples for reduction of estimation variance and decision error probabilities.

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