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Majority cycles in a multi-dimensional setting*

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Summary. We consider a set of alternatives (electoral platforms, bills, etc. ...) defined as a Cartesian product of k finite discrete sets. We assume that the preferences of the individuals (voters) are marginally single-peaked and separable. The main result of this paper states that the pairwise majority relation satisfies these two properties but that it might exhibit several cycles. This result is important when related to classical problems of multi-dimensional decisions such as logrolling and vote trading. We relate our result with a continuous version of it (McKelvey, 1976).

Keywords and Phrases: Majority cycles, Multi-dimensionnal vote, Logrolling and vote trading, McGarvey's theorem.

JEL Classification Numbers: C7, D7.

1 Introduction

It has been well known since Condorcet's (1785) pioneering work that the aggregation of individual transitive preferences through majority voting might lead to intransitivities (cycles). This celebrated "*paradox of voting*" was extended to a larger class of voting mechanisms by Arrow (1951).

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A steady debate took place in the 70's to check whether or not Arrow's result was related to another electoral paradox: the logrolling or vote trading. Let us give an example.

We consider a set of 5 voters labeled from 1 to 5, and a set of alternatives of the form (a,b) where $a \in \{a_1,a_2\}$ and $b \in \{b_1,b_2\}$. This multi-dimensionnal setting can be used in many contexts where alternatives can be measured or judged from several points of view. We say that a and b are two *components* and a_1 and a_2 are two possible *values* of the component a. The preferences of the voters are given in table 1. For instance, the preference of individual 1 is the ranking $(a_1,b_2) \succ (a_2,b_2) \succ (a_1,b_1) \succ (a_2,b_1)$. We shall always assume that individuals have strict preferences, *i.e.* indifference is forbidden.

Table 1. Five individuals and multi-dimensionnal preferences

1	2	3	4	5
(a_1,b_2)	(a_1,b_2)	(a_2,b_1)	(a_1,b_1)	(a_1,b_1)
(a_2,b_2)	(a_2,b_2)	(a_2,b_2)	(a_1,b_2)	(a_2,b_1)
(a_1,b_1)	(a_1,b_1)	(a_1,b_1)	(a_2,b_1)	(a_1,b_2)
(a_2,b_1)	(a_2,b_1)	(a_1,b_2)	(a_2,b_2)	(a_2,b_2)

The (pairwise) majority relation is established as follows: consider two distinct alternatives x,y and compute the numbers $n(x,y)$ of voters that prefer x to y.[1] The alternative x defeats y through the majority if and only if $n(x,y) > n(y,x)$.

The majority relation, in our example, is (a_1,b_1) defeats (a_1,b_2) and (a_2,b_1), (a_1,b_2) defeats (a_2,b_2) and (a_2,b_1), (a_2,b_2) defeats (a_1,b_1), and, finally, (a_2,b_1) defeats (a_2,b_2).

The majority voting over the four alternatives —the "*global vote*" procedure— does not achieve a unique decision because the majority relation contains a cycle encompassing the whole set of alternatives : (a_1,b_1) defeats (a_1,b_2), (a_1,b_2) defeats (a_2,b_1), (a_2,b_1) defeats (a_2,b_2) and (a_2,b_2) defeats (a_1,b_1). In words, if any one of the four alternatives is proposed to be the "*best*", there exists (at least) one other alternative which is preferred by a majority of voters to the former.

If the individuals vote for the best value of a and, separately[2], for the best value of b —the "*component vote*" procedure— the preferences are such that it is possible to determine the vote of each individuals. For instance, individual 1 will prefer a_1 to a_2 and b_2 to b_1. Indeed, she prefers (a_1,b) to (a_2,b) for any $b \in \{b_1,b_2\}$ and prefers (a,b_2) to (a,b_1) for any $a \in \{a_1,a_2\}$. In such a case, we say that the preferences are "*separable*". Under the component vote procedure, the alternative (a_1,b_1) is selected because a_1 is preferred by a majority to a_2 and b_1 is preferred by a majority to b_2.

[1] Because there is no indifference and that the number of voters is odd, $n(\bullet,\bullet)$ is unambiguously defined and $n(x,y) \neq n(y,x)$.

[2] We assume that there is a complete simultaneity of the procedure, *i.e.* nobody knows anything about the results of the two votes until each voter has voted in the two elections.

Although the component vote achieves a unique decision, individuals 1, 2 and 3 would have the opportunity to achieve the election of (a_2, b_2) if they had a possibility to negotiate before the vote takes place.[3]

Miller (1977) shows that if the global vote does not achieve a unique decision then logrolling opportunities exist in the component vote.

The purpose of this paper is to establish, under severe preference restrictions, the occurrence of these paradoxes by showing the existence of cycles.

The preference restrictions considered in this paper are the separability and the marginal single-peakedness. Separability roughly means that, when facing the comparison of two alternatives, an individual takes her decision on the exclusive basis of their differences —the common part, when it exists, is excluded from the evaluation. This assumption, just as in our example, allows to determine the choices of an individual in the global vote and in the component vote procedures.

Marginal single-peakedness states that any restriction of a preference to some alternatives having all but one component with the same value is single-peaked. Roughly speaking, single-peaked preferences can be represented with a single-peaked utility function. Black (1948) explains that paradoxes were impossible if the individual preferences were single-peaked. In other words, single-peakedness implies transitivity. Moulin (1988) shows that, in this case, the majority relation was itself single-peaked.

When the individual preferences are separable and marginally single-peaked, the majority relation is separable, marginally single-peaked and marginally transitive.

Marginal transitivity means that any restriction of a preference to some alternatives having all but one component with the same value is transitive. It is important to remark that marginal transitivity does not imply transitivity. Any set of three or more alternatives having at least two components with different values can be involved in a cycle.

Our main result states that any complete, separable, marginally single-peaked and marginally transitive can be the majority relation of a set of separable and marginally single-peaked individual preferences.

This paper is organized as follows: in Section 2, we provide the definitions and notations necessary to our result. The main result is stated and proved in Section 3. As a conclusion, we shall discuss, in Section 4, the consequences of the result.

2 Definitions and notations

We consider alternatives defined as lists of k different components. To that respect, we denote $K \equiv \{1, \ldots, k\}$ and we define k sets $B_1, ..., B_i, ..., B_k$ containing the different possible values for the i^{th} component. We will assume that each component can take n_i different values ($\#B_i = n_i$). The set of alternatives X is

[3] These three individuals would be better off if 1 and 2 vote for a_2 (instead of a_1) and individual 3 votes for b_2 (instead of b_1).

defined as the Cartesian product of the sets B_i, $X = \prod_{i=1}^{k} B_i$. Each alternative in X can be considered as a point in a space and each of its components as a coordinate. For any $\alpha \subset K$, we denote x as $(x_\alpha, x_{K\setminus\alpha})$ where $x_\alpha = \{x_i : i \in \alpha\}$ and $x_{K\setminus\alpha} = \{x_i : i \notin \alpha\}$.[4]

Let $Ord(X)$ be the set of all linear orderings defined over a set X. Given N a set of v voters, we shall suppose that each voter is endowed with a preference given by a linear ordering $P_j \in Ord(X)$. A preference profile π is a v-tuple of individual preferences. The set of all possible profiles is denoted $Ord^N(X)$. For any profile $\pi \in Ord^N(X)$, the pairwise majority relation $M(\pi)$ is defined by $xM(\pi)y \iff \#\{j \in N : xP_jy\} \geq \#\{j \in N : yP_jx\}$. If v is odd then $M(\pi)$ is a complete and asymmetric binary relation, *i.e.* a tournament.

We shall now introduce some assumptions used in this article. The following definition of separability can be found in many papers.

Definition 1 *A binary relation R is separable over X if $\forall \alpha \subset K, \forall x_{K\setminus\alpha}, y_{K\setminus\alpha} \in \prod_{i\in K\setminus\alpha} B_i$ and $\forall z_\alpha, u_\alpha \in \prod_{i\in\alpha} B_i$, $(x_{K\setminus\alpha}, z_\alpha)R(y_{K\setminus\alpha}, z_\alpha) \iff (x_{K\setminus\alpha}, u_\alpha)$ $R(y_{K\setminus\alpha}, u_\alpha)$.*

Separability means that the choice reduces to the choice between the differences between the two alternatives.

The realism of separability clearly depends on the very nature of the set of alternatives and, in turn, on the nature of the components. Two simple examples : 1) if the alternatives imply some expenses, the effect of a budget constraint is a limit to the application of separability; 2) if the components describe complementary (or incompatible) issues, separability will hardly be satisfied. It turns out that the components must be independent to justify separability.[5]

Marginal single-peakedness, that we introduce now, is an extension of Black's (1948) single-peakedness.[6] It would be technically hazardous and quite unrealistic to assume that there exists a multi-dimensional underlying ordering or even a one-dimensional ordering of all the alternatives. Instead, the usual way to generate multi-dimensional single-peakedness is to define k underlying orderings.

Definition 2 *A linear ordering R is marginally single-peaked with respect to the collection S if and only if for any $i \in K$, any $u \in \prod_{j\in K\setminus\{i\}} B_j$, and any distinct $a, b, c \in B_i$, $aS_ibS_ic \Rightarrow \neg((a, u)R(b, u)$ and $(c, u)R(b, u))$*

[4] Although this *implicit transcription* of the alternatives is such that $(x_\alpha, x_{K\setminus\alpha}) = (x_{K\setminus\alpha}, x_\alpha)$ whereas the *explicit transcription* is such that $(0, 1, 0) \neq (1, 0, 0)$, there shall be no ambiguity due to this abuse of notation.

[5] This criticism of separability appears in Benoît and Kornhauser (1999) in the particular setting of assembly preferences. In this framework, a set of candidates is running for election to an assembly of m members. Each candidates announces a position and to each possible assembly is assigned an outcome according to the position of its members. The set of alternatives is different from ours but separability is the same. On the basis of the outcomes, the authors show that the preferences of the individuals may fail to be separable.

[6] Black's single-peakedness rests on the existence of an "*underlying ordering*" of the alternatives (*e.g.* high/low or left/right). Once an individual has determined her preferred alternative, her "*peak*", her preference between two alternatives lying on the same side of the underlying ordering with respect to the peak, will be based on the proximity to the peak. This proximity IS NOT involved when comparing a right/high alternative to a left/low alternative.

A binary relation Q is marginally transitive if and only if for any $i \in K$*, any* $u \in \prod_{j \in K \setminus \{i\}} B_j$*, and any distinct* $a, b, c \in B_i$*,* $(a,u)Q(b,u)$ *and* $(b,u)Q(c,u) \Rightarrow (a,u)Q(c,u)$

We shall call *line of the set* X, a maximal (with respect to inclusion) subset of alternatives that have *exactly* $k-1$ components in common (*i. e.* these components take the same value). Marginal single-peakedness is satisfied if and only if single-peakedness is satisfied within each line.

When a relation is marginally transitive, transitivity holds only within lines. Cycles can occur between three or more alternatives that do no belong to the same line.

The single-peakedness assumption used in this paper is *marginal* by analogy with statistics where marginal series coexist with conditional series. What would be a conditional single-peakedness? Instead of one ordering per dimension, one ought to associate one underlying ordering per line.[7]

It turns out, as a straight consequence of Black's theorem, that marginally single-peaked preferences always induce a marginally transitive majority relation, and then does not preclude cycles. Moreover, the majority relation satisfies the marginal single-peakedness with respect to the same underlying orderings as the preferences from which the majority was computed (Moulin 1988).

For any $x \neq y$, there exists a non empty *separability set* $S_{\{x,y\}}$ containing unordered pairs $\{u,w\}$ such that the relation between u and w follows from the relation between x and y as a direct consequence of separability.

Definition 3 *Let* $x, y \in X$ *be two distinct alternatives. Let* $\alpha = \{i \in K : x_i = y_i\}$*. The separability set of* $\{x,y\}$ *is defined by*

$$S_{\{x,y\}} = \left\{ \{u,w\} : \begin{array}{ccc} u = (x_{K\setminus\alpha}, u_\alpha) & \textit{and} & w = (y_{K\setminus\alpha}, u_\alpha) \\ & \textit{or} & \\ u = (y_{K\setminus\alpha}, u_\alpha) & \textit{and} & w = (x_{K\setminus\alpha}, u_\alpha) \end{array} \right\}$$

It was shown in Vidu (1998) that the separability sets form a partition of the set of pairs of alternatives. The cardinality of a separability set $S_{\{x,y\}}$ is a function of the number of common components between x and y. In particular, when x and y have no common component, it turns out that $S_{\{x,y\}} = \{\{x,y\}\}$. In such a case, we shall say that x is an *opposite* of y, and we shall denote opp(y) the set of the opposites of y.

The *lexicographic ordering* of the alternatives is a generalization of the alphabetical ordering, and we shall handle it in this paper.

Proposition 4 *For any pair* $x \neq y$*, define the function* $I(x,y) = \min\{i \in K : x_i \neq y_i\}$*. The lexicographic ordering of X with respect to S, denoted* $\mathscr{L}(S)$*, is defined by* $x\mathscr{L}(S)y \iff x_{i'}S_{i'}y_{i'}$ *with* $i' = I(x,y)$*. The lexicographic ordering is complete, asymmetric, separable and marginally single-peaked with respect to* S.

7 A line can be viewed as a conditional series of alternatives.

Proof Consider two distinct alternatives x and y. If $x \in \text{opp}(y)$ then the separability is trivially satisfied. Let $x = (x_\alpha, x_{K\setminus\alpha})$ and $y = (x_\alpha, y_{K\setminus\alpha})$. For any $x' = (x'_\alpha, x_{K\setminus\alpha})$ and $y' = (x'_\alpha, y_{K\setminus\alpha})$, we have $\{x', y'\} \in S_{\{x,y\}}$ and $I(x,y) = I(x',y')$. By definition, we know that $I(x,y) \notin \alpha$, so that $x'_{I(x,y)} = x_{I(x,y)}$ and $y'_{I(x,y)} = y_{I(x,y)}$. Hence $x\mathscr{L}(S)y \iff x'\mathscr{L}(S)y'$, this proves the separability of $\mathscr{L}(S)$.

Consider three alternatives x, y, z over which marginal single-peakedness is effective, *i.e.* three alternatives with exactly $k-1$ common components. Let $i = I(x,y) = I(y,z) = I(x,z)$, ($x$, y and z only differ in the value of the i^{th} component). The lexicographic ordering of x, y, z is identical to the relation S_i over the elements x_i, y_i and z_i. This ordering, over these three, is then trivially single-peaked with respect to the underlying ordering S_i. Since this argument applies to any triple and any underlying ordering, the lexicographic ordering of X is marginally single-peaked with respect to the collection S. □

Proposition 5 *Let $X = \prod_{i=1}^{k} B_i$ be a set of alternatives, N be a set of v (odd) voters, π be a profile of v marginally single-peaked (with respect to a collection S) and separable preferences. The majority relation $M(\pi)$ is marginally transitive, marginally single-peaked (with respect to S), separable, complete and asymmetric.*

The proof of this proposition rests on well-known properties of majority rule and is therefore omitted.

The question answered in this paper is to know exactly to what extent the converse of this proposition holds. Does there always exist a set of marginally single-peaked and separable individual preferences over which the majority relation coincides with a given marginally transitive, marginally single-peaked, separable, complete and asymmetric binary relation?

We answer this question through a constructive method that we call the "*McGarvey's principle*". This principle was introduced in McGarvey (1953), and used in Deb (1976), Stearns (1958),[8] Holland and Le Breton (1996) and Vidu (1998, 1999). This method goes in two steps. The first one is concerned with the partition of $\mathbb{X}_2$ (defined as the collection of the subsets of cardinality 2 of X) and the second is concerned with the construction of *neutral preferences* around each part of $\mathbb{X}_2$.

Definition 6 *Let X be a set of alternatives, T be a binary relation on X, N' be a set of v' voters, $\pi = (P_1, ..., P_{v'})$ be a profile of preferences and γ be a subset of $\mathbb{X}_2$. The profile π is neutral around γ if :*

$$\begin{cases} xM(\pi)y \text{ and } \neg yM(\pi)x & \text{if } \{x,y\} \in \gamma \text{ and } xTy \text{ and } \neg yTx \\ xM(\pi)y \text{ and } yM(\pi)x & \text{if } \{x,y\} \notin \gamma \end{cases}$$

Neutrality around sets of alternatives was explicitly introduced by Holland and Le Breton (1996). The purpose of neutral preferences around a given subset

[8] Stearns gives an upper and a lower bound for the minimal number of individuals required to generate any tournament. Only the upper bound is obtained following McGarvey's principle.

γ of $\mathbb{X}_2$ is that the majority relation computed on these preferences leads to an indifference for any pair which does not belong to γ and leads to T (arbitrarily fixed) for any pair which belongs to γ. By considering each part of $\mathbb{X}_2$, we shall obtain a profile that will coincide to T.

3 Main result

The purpose of this section is to prove the following theorem.

Theorem 7 *Let $X = \prod_{i=1}^{k} B_i$ be a set of alternatives and $S \equiv S_1, \ldots, S_k$ be a collection of underlying orderings.*

For any binary relation T that is complete, asymmetric, separable, marginally transitive and marginally single-peaked with respect to S, there exists a profile π of marginally single-peaked and separable preferences such that $T = M(\pi)$.

We need to partition $\mathbb{X}_2$ with respect to the pairs of alternatives over which pairwise comparisons are related to each other by the separability assumption, *i.e.* the separability sets.

Before proving the theorem, we need a series of three lemmas. They are all stated in the same context and involve a complete, separable, marginally transitive and marginally single-peaked (w.r.t S) binary relation T.

Because the constructions are a little tricky and in order to make it easier to follow, we shall suppose, without loss of generality, that the sets B_i contain integers from 1 to n_i and that the underlying orderings are the natural ordering of these numbers[9], *i.e.* $a < b \iff aS_i b$ for all $i \in K$.

The first lemma shows the existence of neutral preferences around the lines of X.

Lemma 8 *Let $\gamma_i = \{\{x, y\} \in \mathbb{X}_2 : x_i \neq y_i$ and $x_{K\setminus\{i\}} = y_{K\setminus\{i\}}\}$.*

For any $i \in K$, there exist two individual preferences P_i and $\widetilde{P}_i$ marginally single-peaked with respect to the collection S, separable and neutral around γ_i.

Proof For any $i \in K$, we define the binary relations P_i and $\widetilde{P}_i$ as follows :

$$xP_i y \iff \begin{cases} xTy & \text{if } \{x, y\} \in \gamma_i \\ (x_i, x_{K\setminus\{i\}})\mathscr{L}(S)(x_i, y_{K\setminus\{i\}}) & \text{otherwise} \end{cases}$$
$$x\widetilde{P}_i y \iff \begin{cases} xTy & \text{if } \{x, y\} \notin \gamma_i \\ (x_i, y_{K\setminus\{i\}})\mathscr{L}(S)(x_i, x_{K\setminus\{i\}}) & \text{otherwise} \end{cases}$$

Clearly, P_i and $\widetilde{P}_i$ are neutral around γ_i.

We show that P_i is transitive. We partition γ_i into lines. Lines are defined as $L^i_{x_{K\setminus\{i\}}} = \{u \in X : u_{K\setminus\{i\}} = x_{K\setminus\{i\}}\}$. The lines are represented in Figure 1 by dotted sets. Clearly, for any $u \neq x$, $u \in L^i_{x_{K\setminus\{i\}}}$ iff $\{u, x\} \in \gamma_i$. If two alternatives

[9] Most of the constructions used in the remainder are based upon lexicographic ordering so that the rewriting of B_i are merely some one-to-one correspondences with respect to which the lexicographic ordering is *consistent*.

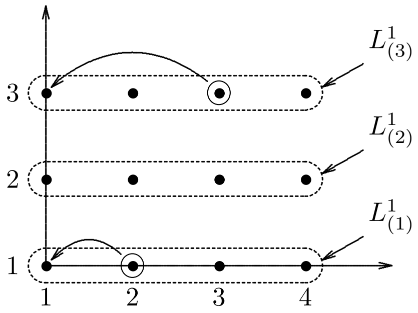

Figure 1. Construction of P_1

belong to the same line, then the first condition applies and we assumed that the binary relation T is marginally transitive. If two alternatives do not belong to the same line then the second condition applies. This relation consists in ordering the lines lexicographically with respect to their index $x_{K\setminus\{i\}}$, which is equivalent to making a projection of the alternatives as described by curved arrows in Figure 1. The orderings P_i and $\widetilde{P}_i$ are then transitive.

The marginal single-peakedness of P_i and $\widetilde{P}_i$ is easily established because single-peakedness is effective (w.r.t. S_i) and satisfied, by assumption, within the lines $L^i_\bullet$. The remaining underlying orderings are effective with respect to the second condition, that it to say, with the lexicographic ordering which is marginally single-peaked.

We show the separability of P_i. Consider $\{u,w\} \in S_{\{x,y\}}$ where $x = (x_\alpha, x_{K\setminus\alpha})$, $y = (x_\alpha, y_{K\setminus\alpha})$, $u = (u_\alpha, x_{K\setminus\alpha})$ and $w = (u_\alpha, y_{K\setminus\alpha})$. If $\{x,y\} \in \gamma_i$, then, by the separability of T, we have $uTw \Longleftrightarrow xTy$ and by the first condition, we get $uP_iw \Longleftrightarrow xP_iy$.

If $i \in \alpha$, $(u_\alpha, x_{K\setminus\alpha})P_i(u_\alpha, y_{K\setminus\alpha}) \Longleftrightarrow (x_i, u_{\alpha\setminus\{i\}}, x_{K\setminus\alpha})\mathscr{L}(S)(x_i, u_{\alpha\setminus\{i\}}, y_{K\setminus\alpha}) \Longleftrightarrow (x_i, x_{\alpha\setminus\{i\}}, x_{K\setminus\alpha})\ \mathscr{L}(S)\ (x_i, x_{\alpha\setminus\{i\}}, y_{K\setminus\alpha}) \Longleftrightarrow xP_iy$.

If $i \notin \alpha$, $(u_\alpha, x_{K\setminus\alpha})P_i(u_\alpha, y_{K\setminus\alpha}) \Longleftrightarrow (x_i, u_\alpha, x_{K\setminus\alpha\cup\{i\}})\mathscr{L}(S)(x_i, u_\alpha, y_{K\setminus\alpha\cup\{i\}}) \Longleftrightarrow (x_i, x_\alpha, x_{K\setminus\alpha\cup\{i\}})\mathscr{L}(S)(x_i, x_\alpha, y_{K\setminus\alpha\cup\{i\}}) \Longleftrightarrow xP_iy$. The separability is then satisfied.

The separability of $\widetilde{P}_i$ is established by the same way. □

The next lemma establishes the existence of neutral preferences around the separability sets of opposite alternatives.

Lemma 9 *For any $x \in X$ and $y \in opp(x)$, there exist two orderings P and $\widetilde{P}$ that are separable, marginally single-peaked (with respect to S) and neutral around $S_{\{x,y\}} = \{\{x,y\}\}$.*

Proof Let x and y be two opposite alternatives. Without loss of generality, assume that x and y are such that $x_i < y_i$ for any $i \in K$.[10]

[10] Indeed, if it is not the case, we can turn X upside down. Starting from the collection S, we define S' such that $S'_i = \pm S_i$ and $x_i S'_i y_i$. One can then go on by replacing S with S' in the remainder of the proof.

We partition X in 3^k subsets[11] by defining for any $a \in \{-1,0,1\}^k$ the subset X_a as follows.

$$X_a = \left\{ z \in X : \begin{array}{ll} z_i < x_i & \text{if } a_i = -1 \\ x_i \le z_i \le y_i & \text{if } a_i = 0 \\ y_i < z_i & \text{if } a_i = 1 \end{array} \right\} \tag{1}$$

It is easy to check that x and y both belong to X_{0_K}, where 0_α means that all coordinates in α are zeros.

For any $b \in \{0,1\}^k$, we define $X_a^b \subset X_a$ as follows :

$$X_a^b = \left\{ z \in X_a : \begin{array}{ll} z_i = y_i & \text{if } b_i = 1 \text{ and } i \neq 1 \\ z_1 = x_1 & \text{if } b_1 = 1 \end{array} \right\} \tag{2}$$

At this point, it is crucial to determine the effect of these two partitions on the separability sets, namely where (in which subset X_a^b) are located the alternatives l and m whose comparison is related to that of u and w by separability?

Let us consider two alternatives $u = (u_\alpha, u_{K\setminus\alpha}) \in X_a^b$ and $w = (u_\alpha, w_{K\setminus\alpha}) \in X_c^d$, and two other alternatives such that $l = (l_\alpha, u_{K\setminus\alpha}) \in X_e^f$ and $m = (l_\alpha, w_{K\setminus\alpha}) \in X_g^h$. It is clear that $(l,m) \in S_{\{u,w\}}$. From the definition of the first partition of X, we have $a = (a_\alpha, a_{K\setminus\alpha})$ and $c = (a_\alpha, c_{K\setminus\alpha})$. Moreover, $e_{K\setminus\alpha} = a_{K\setminus\alpha}$ and $g_{K\setminus\alpha} = c_{K\setminus\alpha}$. It follows that $\{e,g\} \in S_{\{a,c\}}$. From the definition of the second partition, we have $f_{K\setminus\alpha} = b_{K\setminus\alpha}$, $h_{K\setminus\alpha} = d_{K\setminus\alpha}$ and $f_\alpha = h_\alpha$. It follows that $\{f,h\} \in S_{\{b,d\}}$.

This means that the subsets X_a^b are labeled in a *consistent* way with respect to separability so that we have $\{e,g\} \in S_{\{a,c\}}$ and $\{f,h\} \in S_{\{b,d\}}$.

Let us define $\mathscr{N}$ over $\{0,1\}^k$ as the lexicographic ordering $\mathscr{L}((0 \succ 1)^k)$

For any X_a ($a \neq 0_K$), we construct the relation $\mathscr{P}_a$ as follows:

$$u \mathscr{P}_a w \iff \left\{ \begin{array}{l} u \in X_a^b \text{ and } w \in X_a^c \text{ and } b \mathscr{N} c \\ u,w \in X_a^b \text{ and } u \mathscr{L}(-S_1, S_2, \ldots, S_k) w \end{array} \right. \tag{3}$$

We show that $\mathscr{P}_a$ is separable: if $u,w \in X_a^b$, then we have established that for any $\{l,m\} \in S_{\{u,w\}}$, we have $l,m \in X_a^c$ and, since the same lexicographic ordering is used in these two sets, separability is satisfied. By the same argument, $u \in X_a^b$ and $w \in X_a^c$ imply that for any $\{l,m\} \in S_{\{u,w\}}$, it must be that $l \in X_a^d$ and $m \in X_a^e$ with $\{d,e\} \in S_{\{b,c\}}$. By the separability of $\mathscr{N}$, we have $b \mathscr{N} c \iff d \mathscr{N} e$ and then $u \mathscr{P}_a w \iff d \mathscr{P}_a e$.

The ranking of the alternatives in X_{0_K} is achieved in two steps. First we define $\mathscr{P}'_{0_K}$ in the same way as $\mathscr{P}_a$.[12] Notice that $y \in X_{0_K}^{(0,1_{K\setminus\{1\}})}$ and $x \in X_{0_K}^{(1,0_{K\setminus\{1\}})}$

[11] At most 3^k since some of these subsets may be empty.

[12] We obtain:

$$u \mathscr{P}'_{0_K} w \iff \left\{ \begin{array}{l} u \in X_{0_K}^b \text{ and } w \in X_{0_K}^c \text{ and } b \mathscr{N} c \\ u,w \in X_{0_K}^b \text{ and } u \mathscr{L}(-S_1, S_2, \ldots, S_k) w \end{array} \right.$$

Consider $u(\neq y) \in X_{0_K}^{(0,1_{K\setminus\{1\}})}$. By (1) and (2), $u = (u_1, y_{K\setminus\{1\}})$. Since $u_1 < y_1$ implies $y\mathscr{L}(-S_1, S_2, \dots, S_k)u$, we get $y\mathscr{P}'_{0_K}u$.

Consider $w(\neq x) \in X_{0_K}^{(1,0_{K\setminus\{1\}})}$. By (1) and (2), $w = (x_1, w_{K\setminus\{1\}})$. Since $I(x,w) \neq 1$, we have $x_{I(x,w)} < w_{I(x,w)}$ which implies $x\mathscr{L}(-S_1, S_2, \dots, S_k)w$. We obtain $x\mathscr{P}'_{0_K}w$.

Now, we observe that any $u \in X_{0_K}^{(0,1_{K\setminus\{1\}})}$ is an opposite of x and, since $(0, 1_{K\setminus\{1\}})$ comes immediately before $(1, 0_{K\setminus\{1\}})$ in the relation $\mathscr{N}$, we can improve the ranking of x in $\mathscr{P}'_{0_K}$ so that x and y are consecutive, without any consequence on the separability of the relation. If xTy then we let x come just before y. If yTx then we let x come just after y. This modified relation $\mathscr{P}'_{0_K}$ defines the relation $\mathscr{P}_{0_K}$.

Let $\mathscr{M}$ be the lexicographic ordering of $\{-1,0,1\}^k$ with respect to $1 \succ 0 \succ -1$ for the first coordinate and $-1 \succ 0 \succ 1$ for the $k-1$ remaining ones.

Let us now construct the two preferences P and $\widetilde{P}$ as follows:

$$aPb \iff \begin{cases} a\mathscr{P}^c b & \text{if } a,b \in X^c \\ c.\mathscr{M}d & \text{if } a \in X_c \text{ and } b \in X_d \end{cases}$$
$$a\widetilde{P}b \iff \begin{cases} aPb & \text{if } \{a,b\} = \{x,y\} \\ bPa & \text{otherwise} \end{cases}$$

P and $\widetilde{P}$ are clearly neutral around $S_{\{x,y\}}$.

The transitivity of P is obvious because it is the lexicographic ordering of the $X_\bullet$, the subsets of $X_\bullet$ are ordered with respect to $\mathscr{N}$ and the elements of $X_\bullet^\bullet$ are lexicographically ordered. The transitivity of $\widetilde{P}$ is guaranteed because x is ranked immediately before (depending on the relation T, it may be after) y in P.

The separability of the relation P is almost established by the *consistency* of partitions (1) and (2). When two alternatives belong to the same X_b^a, their relative ranking follows from the lexicographic ordering $\mathscr{L}(-S_1, S_2, \dots, S_k)$.

The marginal single-peakedness of P is established in two steps. We consider two alternatives a and b such that $a = (a_i, a_{K\setminus\{i\}})$, $b = (b_i, a_{K\setminus\{i\}})$. At a first step, we show that $a_iS_ib_i \iff aPb$ for any $i \neq 1$. At the second step, we show that $a_1S_1b_1 \iff bPa$ when $i = 1$.

- If $a, b \in X_l^m$, the lexicographic ordering $\mathscr{L}(-S_1, S_2, \dots, S_k)$ is used, with $I(a,b) = i$ so that $a_iS_ib_i \iff aPb$.
- If $a \in X_l^m$ and $b \in X_l^r$, then, by the *consistency* of the partitions, it must be that $r_{K\setminus\{i\}} = m_{K\setminus\{i\}}$.
  We must consider two cases :
  - Case 1 : If $m_i = 1$ and $r_i = 0$, then $r.\mathscr{N}m$ which, by (3), implies bPa. From (2), $m_i = 1$ implies $a_i = y_i$. Since $(x_i, a_{K\setminus\{i\}})$ necessarily belongs to X_l^r (just as b) and $x_iS_iy_i$, we have $b_iS_ia_i$.
  - Case 2 : If $m_i = 0$ and $r_i = 1$, then by the same arguments, we obtain aPb and $a_iS_ib_i$.

– If $a \in X_l$ and $b \in X_m$, then l and m are ranked according to $\mathscr{M}$ and we observe that $l_i < m_i \iff l\mathscr{M}m \iff a_i S_i b_i \iff aPb$.

When $i = 1$, then if $a, b \in X_l^m$, the lexicographic ordering $\mathscr{L}(-S_1, S_2, \ldots, S_k)$ in use implies $a_1 S_1 b_1 \iff bPa$.

If $a \in X_l^m$ and $b \in X_l^r$, then again we must consider two cases :

– Case 1 : If $m_1 = 1$ and $r_1 = 0$, then $r\mathscr{N}m$ which, by (3), implies bPa. From (2), $m_1 = 1$ implies $a_i = x_i$. Since $(y_1, a_{K\setminus\{1\}})$ necessarily belongs to X_l^r (just as b) and $x_1 S_1 y_1$, we have $a_i S_i b_i$.
– Case 2 : If $m_1 = 0$ and $r_1 = 1$, then by the same arguments, we obtain aPb and $b_1 S_1 a_1$.

If $a \in X_l$ and $b \in X_m$, then l and m are ranked according to $\mathscr{M}$ and we observe that $l_1 > m_1 \iff l\mathscr{M}m \iff b_1 S_1 a_1 \iff aPb$.

This proves the marginal single-peakedness of P. The marginal single-peakedness of $\widetilde{P}$ is established by showing that $a_i S_i b_i \iff b\widetilde{P}a$ for any $i \neq 1$ and that $a_1 S_1 b_1 \iff a\widetilde{P}b$ when $i = 1$. The conclusion is obtained through the same reasoning. □

The last lemma shows the existence of convenient neutral preferences around any remaining separability set $S_{\{x,y\}}$.

Lemma 10 *For any $x, y \in X$, there exist two orderings P and $\widetilde{P}$ that are separable, marginally single-peaked (with respect to S) and neutral around $S_{\{x,y\}}$*

Proof Let us consider two alternatives x, y such that $x \notin \text{opp}(y)$ and that they do not belong to the same line. We can then write $x = (x_\alpha, x_{k\setminus\alpha})$ and $y = (x_\alpha, y_{K\setminus\alpha})$ where $1 \leq \#\alpha < k - 1$. Let us define a partition of X such that for any $z_\alpha \in \prod_{i\in\alpha} B_i$, $F^{z_\alpha} = \{u \in X : u_\alpha = z_\alpha\}$.

Let $X' = \prod_{i\notin\alpha} B_i$ be a $(k-\#\alpha)$-dimensional set of alternatives. One can easily observe that $\#F^{z_\alpha} = \#X'$. Moreover, we can make a one-to-one correspondence between F^{z_α} and X', because for any $u \in F^{z_\alpha}$, we have $u_{K\setminus\alpha} \in X'$ and for any $w \in X'$ and any $z_\alpha \in \prod_{i\in\alpha} B_i$, we have $(z_\alpha, w) \in F^{z_\alpha}$. There are as many sets F^{z_α} as pairs in $S_{\{x,y\}}$.

Since the two alternatives $x_{K\setminus\alpha}$ and $y_{K\setminus\alpha}$ are opposite in X', it is possible, by Lemma 9, to construct two separable, marginally single-peaked preferences defined over X' that are neutral around $\{x_{K\setminus\alpha}, y_{K\setminus\alpha}\}$. Let P' and $\widetilde{P}'$ be these two preferences.

Finally, let us denote S' the sub-collection of S such that $S_i \in S' \iff i \in \alpha$.

We construct the following preferences:

$$uPw \iff \begin{cases} u_{K\setminus\alpha} P' w_{K\setminus\alpha} & \text{if } u_\alpha = w_\alpha \\ u_\alpha \mathscr{L}(S') w_\alpha & \text{otherwise} \end{cases}$$

$$u\widetilde{P}w \iff \begin{cases} u_{K\setminus\alpha} \widetilde{P}' w_{K\setminus\alpha} & \text{if } u_\alpha = w_\alpha \\ w_\alpha \mathscr{L}(S') u_\alpha & \text{otherwise} \end{cases}$$

These two binary relations P and $\widetilde{P}$ are clearly transitive since they are respectively the lexicographic ordering and its reverse of the sets $F^{\bullet}$. Within these sets, the relations P and $\widetilde{P'}$ respectively apply.

The same argument allows to consider that P and $\widetilde{P}$ are separable and marginally single-peaked.

We prove the neutrality of P and $\widetilde{P}$ around $S_{\{x,y\}}$. Consider two alternatives $a \in F^{z_\alpha}$ and $b \in F^{z'_\alpha}$. It is clear that $\{a, b\} \notin S_{\{x,y\}}$. Since $a = (z_\alpha, a_{K\setminus\alpha})$ and $b = (z'_\alpha, b_{K\setminus\alpha})$, the lexicographic ordering and its converse are respectively used for P and $\widetilde{P}$. Hence there is an indifference between a and b through the majority rule. If a and b belong to the same F^{z_α}, the construction of Lemma 9 ensures that we obtain an indifference if and only if $\{a_{K\setminus\alpha}, b_{K\setminus\alpha}\} = \{x_{K\setminus\alpha}, y_{K\setminus\alpha}\}$. □

We can then prove our main result.

Proof (of Theorem 7) Let us consider the partition of $\mathbb{X}_2$ with respect to the various separability sets. Let $\gamma_i = \{\{x, y\} \in \mathbb{X}_2 : x_i \neq y_i \text{ and } x_j = y_j, \forall j \neq i\}$ be the set of pairs on which single-peaked with respect to S_i is effective.

By Lemma 8, for any $i \in K$, there exist two convenient preferences P_i and $\widetilde{P_i}$ that are neutral around γ_i.

We know (Vidu 1998) that there are $\frac{1}{2} \sum_{\beta \subset K} \prod_{i \in \beta} n_i(n_i - 1)$ separability sets and that this number can be written $\frac{1}{2} \prod_{i \in K} (n_i(n_i - 1) + 1) - 1$. Moreover, it is easy to determine that $\sum_{i \in K} \frac{n_i(n_i-1)}{2}$ separability sets are included in the k sets γ_i.

Let us label the remaining separability sets from J' to J where

$$J' = 1 + \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$
$$J = \frac{1}{2} \prod_{i=1}^{k} (n_i(n_i - 1) + 1) - 1$$

Consider any separability set $S_{\{x,y\}}$ and suppose that it is labeled $J' \leq p \leq J$.

Depending on whether x is an opposite of y or not, Lemma 9 or Lemma 10 (respectively) ensure that there exist two convenient preferences that are neutral around $S_{\{x,y\}}$. Let P_p and $\widetilde{P_p}$ be these two preferences.

The proof is completed by proposing the following profile:

$$\pi = (P_1, \widetilde{P_1}, \ldots, P_k, \widetilde{P_k}, P_{J'}, \widetilde{P_{J'}}, \ldots, P_J, \widetilde{P_J})$$

It is clear that, from this profile, the majority relation will coincide with the relation T, which is the desired result.[13] □

[13] This construction requires an even number of voters, but if the number of voter is required to be odd, then one can remove any single individual from this profile without any alteration of the majority outcome since for any pair of options, we have $\#\{j \in N : xP_j y\} - \#\{j \in N : yP_j x\} = \pm 2$.

4 Concluding remarks and open problems

The purpose of this paper was to show the extent of cycles in the majority relation when severe restrictions are made on the admissible preferences of individuals having to consider multi-dimensional alternatives. We first observed that when the preferences of an odd number of individuals are separable and marginally single-peaked, the majority relation is a separable, marginally single-peaked and marginally transitive tournament. Our main result shows that any separable, marginally single-peaked and marginally transitive tournament could be the majority relation obtained from a set of separable and marginally single-peaked individual preferences.

On the one hand, this result has a negative consequence : adding new preference restrictions prevents neither the Condorcet paradox, nor the logrolling paradox because the majority relation can contain cycles. It would be very interesting to find some natural conditions which could avoid these paradoxes. Should these conditions be imposed to the method of decision or to the individual preferences? The constructive approach of McGarvey's type theorems makes it tempting to consider restrictions from a normative point of view. On the other hand, having in mind Arrow's "*unrestricted domain*" axiom, by which the author intended to guarantee the individuals a minimal "*freedom*", this should be done with the utmost care.

On the other hand, there is a positive aspect to this result: McKelvey (1979) studied the properties of the majority relation in an infinite, continuous multi-dimensional space of alternatives. His conclusions can be stated as follows: Either there is a uniquely defined best alternative in the global vote, or there exists a cycle encompassing the whole space. In this most likely case, it becomes impossible to discard any alternative with the global vote. The positive consequence of our result is that cycles, when they occur, may not encompass the whole set of alternatives and then allow to discard some very bad alternatives.

The contrast between the continuous case of McKelvey and our setting, though not a surprise, is still intuitively interesting. The richness of a continuous and infinite space makes it "*easy*" (or at least likely) to find a majority sequence starting from any x to any y while the majority relation goes directly from y to x. In a finite setting, this "*likelihood*" must undoubtedly be lower but surely increases with the number of alternatives. The situation is that McKelvey's result is never approached even when considering a set of finitely many alternatives. This is an example of a situation where the modelization of reality by assuming infinity and continuity leads to diametrically different conclusions.

The results contained in the present paper, as well as those in Holland and Le Breton (1996) and Vidu (1998, 1999), are to be considered as extensions of McGarvey's theorem (1953) and may allow the extension of the domain of tournament theory.[14]

[14] McGarvey's theorem states that any binary relation, including cyclic ones, can be obtained via the pairwise majority aggregation of individual preferences. It is the cornerstone of the development of the theory of tournaments from a voting point of view. See Laslier (1997) for a comprehensive exposition of the topic.

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