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A simple proof of the necessity of the transversality condition*

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Summary. This note provides a simple proof of the necessity of the transversality condition for the differentiable reduced-form model. The proof uses only an elementary perturbation argument without relying on dynamic programming. The proof makes it clear that, contrary to common belief, the necessity of the transversality condition can be shown in a straightforward way.

Keywords and Phrases: Transversality condition, Reduced-form model, Dynamic optimization.

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1 Introduction

"The necessity of the transversality condition is a difficult issue," note Stokey and Lucas (1989, p. 102) after proving the sufficiency of the transversality condition. As a matter of fact, necessity of the transversality condition has long been widely perceived as a difficult issue, perhaps because the classical proofs of the necessity of the transversality condition are not easily understandable to nontechnical readers. What makes those proofs difficult, however, is not the difficulties in proving the transversality condition itself but the technical arguments required for proving the existence of support prices (Peleg, 1970; Peleg and Ryder, 1972; Weitzman, 1973; Araujo and Scheinkman, 1983) or for proving the envelope condition (Benveniste and Scheinkman, 1982). Though such arguments may be necessary when one wishes to establish a characterization theorem for a general maximization problem, they can in fact be entirely bypassed when one wishes

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only to prove the necessity of the transversality condition for the differentiable reduced-form model.

The purpose of this note is to offer a simple proof of the necessity of the transversality condition. The result proved in this note is a more or less well-known variant of Weitzman's (1973) theorem. The assumptions we use that are not assumed by Weitzman are the differentiability of the return functions and the interiority of a given optimal path. These assumptions allow us to work directly with derivatives, making it unnecessary to construct support prices. Another feature of our approach is that we do not use dynamic programming. Without relying on dynamic programming, we directly prove the necessity of the transversality condition using only an elementary perturbation argument.

While similar arguments are used in Kamihigashi (2000a, b, c), these papers do not provide a direct proof of the necessity of the transversality condition for the reduced-form model. Kamihigashi (2000a) focuses on Ekeland and Scheinkman's (1986) result. Instead of simplifying the proofs of well-known results, the other two papers generalize well-known results as well as establish new results. We believe that the direct proof offered in this note will benefit the profession by demystifying the necessity of the transversality condition.

The next section presents the model and states the result. Section 3 presents the proof. Section 4 discusses how the proof differs from those of Benveniste and Scheinkman (1983) and Ekeland and Scheinkman (1986), how the result can be generalized, and why the proof does not apply to the undiscounted stationary case. Section 5 concludes the note.

2 The transversality condition

Consider the following maximization problem.

$$\begin{cases} \max\limits_{\{x_t\}_{t=0}^{\infty}} & \sum\limits_{t=0}^{\infty} v_t(x_t, x_{t+1}) \\ \text{s.t.} & x_0 = \overline{x}_0, \quad \forall t \in \mathbb{Z}_+, (x_t, x_{t+1}) \in X_t. \end{cases} \tag{1}$$

Since the assumptions and definitions used here are standard, they are stated without comment.

Assumption 2.1. $\exists n \in \mathbb{N}, \overline{x}_0 \in \mathbb{R}^n_+$ *and* $\forall t \in \mathbb{Z}_+, X_t \subset \mathbb{R}^n_+ \times \mathbb{R}^n_+$.

Assumption 2.2. $\forall t \in \mathbb{Z}_+, X_t$ *is convex and* $(0,0) \in X_t$.

Assumption 2.3. $\forall t \in \mathbb{Z}_+, v_t : X_t \to \mathbb{R}$ *is* C^1 *on* $\overset{\circ}{X}_t$ *and concave.*

For $t \in \mathbb{Z}_+$ and $(y,z) \in \overset{\circ}{X}_t$, let $v_{t,2}(y,z)$ denote the partial derivative of v_t with respect to z; define $v_{t,1}(y,z)$ similarly.

Assumption 2.4. $\forall t \in \mathbb{Z}_+, \forall (y,z) \in \overset{\circ}{X}_t, v_{t,2}(y,z) \leq 0.$[1]

[1] Due to the Euler equation (5), Theorem 2.1 below holds even if this inequality is replaced by $v_{t,1}(y,z) \geq 0$.

We say that a path $\{x_t\}_{t=0}^{\infty}$ is *feasible* if $x_0 = \overline{x}_0$ and $\forall t \in \mathbb{Z}_+, (x_t, x_{t+1}) \in X_t$.

Assumption 2.5. *For any feasible path* $\{x_t\}$,

$$\sum_{t=0}^{\infty} v_t(x_t, x_{t+1}) \equiv \lim_{T\uparrow\infty} \sum_{t=0}^{T} v_t(x_t, x_{t+1}) \tag{2}$$

exists in $(-\infty, \infty)$.[2]

We say that a feasible path $\{x_t^*\}$ is *optimal* if for any feasible path $\{x_t\}$,

$$\sum_{t=0}^{\infty} v_t(x_t, x_{t+1}) \leq \sum_{t=0}^{\infty} v_t(x_t^*, x_{t+1}^*). \tag{3}$$

We say that a feasible path $\{x_t\}$ is *interior* if $\forall t \in \mathbb{Z}_+, (x_t, x_{t+1}) \in \overset{\circ}{X}_t$. The following result is proved in Section 2.1.

Theorem 2.1. *Under Assumptions 2.1–2.5, for any interior optimal path* $\{x_t^*\}$,

$$\lim_{T\uparrow\infty} [-v_{T,2}(x_T^*, x_{T+1}^*)x_{T+1}^*] = 0. \tag{4}$$

Theorem 2.1 is a variant of Weitzman (1973, Theorem) and a discrete-time version of Benveniste and Scheinkman (1982, Theorem 3.A). Since an interior optimal path $\{x_t^*\}$ satisfies the Euler equation

$$v_{t,2}(x_t^*, x_{t+1}^*) + v_{t+1,1}(x_{t+1}^*, x_{t+2}^*) = 0 \tag{5}$$

for $t \in \mathbb{Z}_+$, condition (4) can equivalently be expressed as

$$\lim_{T\uparrow\infty} v_{T,1}(x_T^*, x_{T+1}^*)x_T^* = 0. \tag{6}$$

Condition (4), or the above equivalent form, is the most commonly used transversality condition.

As the proof below shows, however, condition (4) is a necessary condition regardless of validity of the Euler equation. In addition, condition (4) better corresponds to the continuous-time version of the transversality condition.

3 Proof of Theorem 2.1

We prepare the following elementary lemma.

Lemma 3.1. *Let* $f : [0, 1] \to \mathbb{R} \cup \{-\infty\}$ *be a concave function with* $f(1) > -\infty$. *Then*

$$\forall\gamma \in [0, 1), \forall\lambda \in [\gamma, 1), \qquad \frac{f(1) - f(\lambda)}{1 - \lambda} \leq \frac{f(1) - f(\gamma)}{1 - \gamma}. \tag{7}$$

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[2] Throughout this note, $\sum_{t=0}^{\infty} \equiv \lim_{T\uparrow\infty} \sum_{t=0}^{T}$.

Proof. Let $\lambda \in [\gamma, 1)$ and $\mu = (1-\lambda)/(1-\gamma)$. By concavity, $f(\lambda) \geq \mu f(\gamma) + (1-\mu)f(1) = -\mu(f(1) - f(\gamma)) + f(1)$. Thus $f(1) - f(\lambda) \leq \mu(f(1) - f(\gamma))$; the inequality in (7) follows. □

Now to prove Theorem 2.1, let $\{x_t^*\}$ be an interior optimal path. Let $T \in \mathbb{Z}_+$. By interiority and Assumption 2.2, for $\lambda \in [0, 1)$ sufficiently close to one, the path

$$\{x_0^*, x_1^*, \cdots, x_T^*, \lambda x_{T+1}^*, \lambda x_{T+2}^*, \cdots\} \tag{8}$$

is feasible. Let $\lambda \in [0, 1)$ be so close to one that the above path is feasible. By optimality,

$$v_T(x_T^*, \lambda x_{T+1}^*) - v_T(x_T^*, x_{T+1}^*) + \sum_{t=T+1}^{\infty} [v_t(\lambda x_t^*, \lambda x_{t+1}^*) - v_t(x_t^*, x_{t+1}^*)] \leq 0. \tag{9}$$

Dividing through by $(1-\lambda)$ yields

$$\frac{v_T(x_T^*, \lambda x_{T+1}^*) - v_T(x_T^*, x_{T+1}^*)}{1-\lambda} \leq \sum_{t=T+1}^{\infty} \frac{v_t(x_t^*, x_{t+1}^*) - v_t(\lambda x_t^*, \lambda x_{t+1}^*)}{1-\lambda} \tag{10}$$

$$\leq \sum_{t=T+1}^{\infty} [v_t(x_t^*, x_{t+1}^*) - v_t(0, 0)], \tag{11}$$

where the last inequality holds by Assumption 2.3 and Lemma 3.1 with $\gamma = 0$. Applying $\lim_{\lambda \uparrow 1}$ to the left-hand side of (10) yields

$$0 \leq -v_{T,2}(x_T^*, x_{T+1}^*)x_{T+1}^* \leq \sum_{t=T+1}^{\infty} [v_t(x_t^*, x_{t+1}^*) - v_t(0, 0)], \tag{12}$$

where the first inequality holds by Assumption 2.4. Applying $\lim_{T \uparrow \infty}$ to (12) yields

$$0 \leq \lim_{T \uparrow \infty} [-v_{T,2}(x_T^*, x_{T+1}^*)x_{T+1}^*] \leq \lim_{T \uparrow \infty} \sum_{t=T+1}^{\infty} [v_t(x_t^*, x_{t+1}^*) - v_t(0, 0)] = 0, \tag{13}$$

where the equality holds by Assumption 2.5. Condition (4) now follows.

4 Discussions

4.1 Comparison with other methods

The crucial step in the above proof is the inequality in (11). Very roughly speaking, in Ekeland and Scheinkman's (1986) proof, $\lim_{\lambda \uparrow 1}$ is directly applied to both sides of (10). Again very roughly speaking, in Benveniste and Scheinkman's (1982) proof, $\lim_{\lambda \uparrow 1}$ is applied to the following inequality.

$$\frac{v_T(x_T^*, \lambda x_{T+1}^*) - v_T(x_T^*, x_{T+1}^*)}{1-\lambda} \leq \frac{V_{T+1}(x_{T+1}^*) - V_{T+1}(\lambda x_{T+1}^*)}{1-\lambda}, \tag{14}$$

where V_{T+1} is the value function for the maximization problem starting from period $T+1$.[3]

Though both methods work, technical arguments are needed to apply $\lim_{\lambda\uparrow 1}$ to the right-hand sides of (10) and (14), which depend on λ. In addition, both methods eventually use inequalities like (11).[4] In our proof, by contrast, the inequality in (11) is exploited at the earliest possible opportunity. This trivializes the process of applying $\lim_{\lambda\uparrow 1}$ since the right-hand side of (11) does not involve λ.

4.2 Generalization

All the assumptions, Assumptions 2.1–2.5, can considerably be weakened. In fact, for the basic argument of our proof to go through, only the following three assumptions are needed.[5]

Assumption 4.1. $\forall T \in \mathbb{Z}_+$, *for all* $\lambda < 1$ *sufficiently close to one, the path specified by (8) is feasible.*

Assumption 4.2. $\forall T \in \mathbb{Z}_+$, *the left-hand side of (10) has a limit as* $\lambda \uparrow 1$.

Assumption 4.3. *There exists a sequence* $\{B_T\}_{T=0}^{\infty}$ *with* $\lim_{T\uparrow\infty} B_T = 0$ *such that* $\forall T \in \mathbb{Z}_+$, *for all* $\lambda < 1$ *sufficiently close to one, the right-hand side of (10) is bounded above by* B_T.

Assumptions 4.1 and 4.2 are satisfied in standard models. For example, these assumptions are implied by Assumptions 2.2 and 2.3.

The key to the necessity of the transversality condition is Assumption 4.3.[6] This assumption is substantially weaker than Assumption 2.5 and is useful particularly when $v_t(0,0) = -\infty$, which is the case in many parametric models. In such cases, the above proof, which does not work in its current form, can easily be modified as follows. Assume

$$\sum_{t=1}^{\infty} [v_t(x_t^*, x_{t+1}^*) - v_t(\gamma x_t^*, \gamma x_{t+1}^*)] \tag{15}$$

exists in $(-\infty, \infty)$ for some $\gamma \in [0,1)$. By Lemma 3.1, for $\lambda \in [\gamma, 1)$,

[3] To be more specific, the corresponding Bellman equation is $V_T(x) = \max_y \{v_T(x,y) + V_{T+1}(y) \mid (x,y) \in X_T\}$. By optimality, $v_T(x_T^*, \lambda x_{T+1}^*) + V_{T+1}(\lambda x_{T+1}^*) \leq v_T(x_T^*, x_{T+1}^*) + V_{T+1}(x_{T+1}^*)$. This implies (14).

[4] Benveniste and Scheinkman use essentially the same inequality as (11). Ekeland and Scheinkman use a more general inequality somewhat similar to (16) below.

[5] The result that can be shown under Assumptions 4.1–4.3 is not identical to Theorem 2.1. See Kamihigashi (2000c) for general results established under minimal assumptions.

[6] This assumption can be avoided in some cases, however. See Kamihigashi (2000b, c).

$$\sum_{t=T+1}^{\infty} \frac{v_t(x_t^*, x_{t+1}^*) - v_t(\lambda x_t^*, \lambda x_{t+1}^*)}{1-\lambda} \leq \sum_{t=T+1}^{\infty} \frac{v_t(x_t^*, x_{t+1}^*) - v_t(\gamma x_t^*, \gamma x_{t+1}^*)}{1-\gamma}. \quad (16)$$

Use this inequality in place of (11). The rest of the proof then goes through.

The above argument shows that the transversality condition is necessary as long as expression (15) is finite for some $\gamma \in [0, 1)$.[7] This result is useful for models with unbounded return functions since it does not require the objective function to be finite, or even well-defined, for all feasible paths.

4.3 The undiscounted stationary case

As is well-known, the transversality condition fails to be necessary for optimality in the undiscounted Ramsey model or, more generally, in undiscounted stationary models in which the Turnpike Theorem (cf. Gale, 1967, Theorem 8; Becker and Boyd, 1997, p. 188) holds.

To be more specific, assume Assumptions 2.1–2.4, and suppose v_t and X_t do not depend on t; i.e., there exist v and X such that $\forall t \in \mathbb{Z}_+, v_t = v$ and $X_t = X$. Suppose v is strictly concave. Let $\hat{x} = \text{argmax}_x \{v(x, x) \,|\, (x, x) \in X\}$. Suppose $\hat{x} \gg 0$ and $v_2(\hat{x}, \hat{x}) < 0$. Then under additional assumptions, an optimal path $\{x_t^*\}$ (optimal in the sense of Gale, p. 3) converges to $\hat{x}$ by the Turnpike Theorem. Consequently,

$$\lim_{T \uparrow \infty} [-v_2(x_T^*, x_{T+1}^*) x_{T+1}^*] = -v_2(\hat{x}, \hat{x})\hat{x} > 0, \quad (17)$$

i.e., the transversality condition fails. The proof in Section 3 breaks down at the very last step in this case since the equality in (13) no longer holds.

As is clear from Sections 3 and 4.2, optimality implies the transversality condition *if* perturbing the optimal path proportionally downward results in a finite loss. This "finite loss" condition—or, more precisely, Assumption 4.3—is violated here since the right-hand side of (10) becomes ∞ for any $\lambda < 1$.[8] See Michel (1990) and Becker and Boyd (1997) for related results and discussions.

5 Conclusion

This note proved the necessity of the transversality condition for the differentiable reduced-form model using only an elementary perturbation argument. The proof is short and simple because it bypasses the technical arguments required for constructing support prices or for showing the envelope condition. We hope, and believe, that the direct proof offered in this note will help the profession better understand the transversality condition.

[7] Results of this nature are established in Kamihigashi (2000b, c).

[8] This is because $v(\hat{x}, \hat{x}) - v(\lambda\hat{x}, \lambda\hat{x}) > 0$ by the definition of $\hat{x}$ and the strict concavity of v.

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