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Exposita Notes

A simple random assignment problem with a unique solution★

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Summary. All agents have the same ordinal ranking over all objects, receiving no object (opting out) may be preferable to some objects, agents differ on which objects are worse than opting out, and the latter information is private. The Probabilistic Serial assignment, improves upon (in the Pareto sense) the Random Priority assignment, that randomly orders the agents and offers them successively the most valuable remaining object. We characterize Probabilistic Serial by efficiency in an ordinal sense, and envy-freeness. We characterize it also by ordinal efficiency, strategyproofness and equal treatment of equals.

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1 Introduction

An assignment problem consists of a finite set of indivisible objects, and a finite set of agents that can each consume at most one object. Randomization is the simplest and most common device to restore (ex ante) fairness in spite of the indivisibility of the objects ([7], [12], [1]).

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We consider the special case of random assignment where the objects are ranked in the same way (from best object to worst) by all agents, where opting out (receiving no object) is feasible and where the agents differ about which objects are desirable (i.e., preferred to opting out).

The main example is a scheduling problem where the server processes one agent per unit of time and each agent has a deadline, beyond which the service is worthless ([11], [5]). In another example, agents timeshare a set of machines of decreasing quality (with one machine usable by no more than one agent at a time) and each agent can only use machines with a certain minimal quality (think of PCs with decreasing memory size). In this second example, randomization is replaced by timesharing, and expected utility over lotteries by utility additive with respect to the time spent with a given machine.

Fix an arbitrary ordering of the agents and, following this ordering, let the agents successively either choose their preferred unassigned good or opt out. This *Priority* mechanism ([1] and [9] speak of "serial dictatorship") is efficient and incentive compatible, namely strategyproof. The *Random Priority* mechanism first selects at random and without bias an ordering of the agents, then implements the corresponding Priority mechanism. This mechanism is equitable (in the strong sense of No Envy, Lemma 4.3) and strategyproof. Yet it is Pareto inferior to another equitable and incentive compatible mechanism, the *Probabilistic Serial* one, introduced in [4]. The idea behind PS is that all agents for whom a certain object is desirable can claim a fair "share" of this object: if the object is valuable for m agents, each can claim the probability up to $1/m$ of receiving this object. Thus an agent gets a fair share of the best object, a fair share of the next best object, and so on until either she has accumulated a probability of one or no more object is desirable to her. This defines a random assignment, that differs in an interesting way from the RP assignment.

We compare PS and RP in an example. Assume four objects and four agents, and that agent i's deadline is i, for $i = 1, 2, 3, 4$: agent 1 finds the best object more desirable than opting out, but prefers the latter to the second best object (or any lower object); agent 2 finds only the first two objects desirable, and so on. One computes easily the RP assignment:

agents\objects	a	b	c	d
1	$1/4$	0	0	0
2	$1/4$	$1/3$	0	0
3	$1/4$	$1/3$	$3/8$	0
4	$1/4$	$1/3$	$3/8$	$1/24$

RP assignment (1)

For instance, agent 4 gets the worst object if and only if the priority ordering is $\{1, 2, 3, 4\}$, an event with probability $1/24$. And so on.

The PS allocation is computed from left to right by splitting equally the successive goods (starting from the best good) among all agents who desire it, until either there are no goods left or the agents still interested have already received a total probability of one. See Definition 4.1. In the example, the latter happens with the third good:

	a	b	c	d
1	$1/4$	0	0	0
2	$1/4$	$1/3$	0	0
3	$1/4$	$1/3$	$5/12$	0
4	$1/4$	$1/3$	$5/12$	0

PS assignement (2)

It is easy to see that the PS allocation is *Pareto superior* to the RP one in the stochastic dominance sense. Agents 1 and 2 are obviously indifferent to the switch but agent 3 receives the desirable good c with strictly higher probability; similarly agent 4's probability of receiving good d is transferred to that of receiving good c.

Crès and Moulin [4] show that the situation of this example is fully general: for any profile of ordinal deadlines, the PS assignment is guaranteed to be either identical or Pareto superior to the RP assignment, no matter what the cardinal preference profile is.

Note that, in order to compute the PS or the RP random assignments, we only need to know the profile of *deadlines*, i.e., for each agent her ordinal ranking of "opting out" (receiving the null object) among "real" objects; no further information about the cardinal intensity of VNM utilities over objects and opting out is needed.

We introduce the new concept of *ordinal efficiency* which relies only on the profile of (ordinal) deadlines. It views an assignment as inefficient if there is another feasible assignment that is Pareto superior for *all* cardinal utility profiles compatible with the given profile of (ordinal) deadlines (see Definition 3.2).

In our axiomatic analysis, we restrict attention to those mechanisms that only elicit the ordinal component of the agents' preferences, namely the deadline. On the other hand we assume that agents compare random allocations by their expected utility (they have standard Von Neumann-Morgenstern utilities over lotteries). These two assumptions are commonplace in the literature on probabilistic voting (e.g., [6], [2]). They mean that the mechanism is informationally less sophisticated than the agents themselves.

In particular, the mechanism cannot achieve full ex ante efficiency (i.e., efficiency with respect to the VNM utility functions): ordinal efficiency is the most demanding efficiency test relying only on the profile of deadlines. On the other hand, the properties No Envy and strategyproofness rely on individual comparisons between lotteries (e.g., an agent compares her own allocation to that of another agent), performed with (VNM) expected cardinal utilities.

The paper is organized as follows. Section 2 defines the model and Section 3 discusses the central concept of ordinal efficiency. The Probabilistic Serial *assignment* is introduced in Section 4 and characterized by No Envy and ordinal efficiency (Theorem 4.1). The Probabilistic Serial *mechanism* is then characterized in Section 5 by the combination of Ordinal Efficiency, Equal Treatment of Equals and strategyproofness (Theorem 5.1).

A few comments about related literature. In the short literature on random assignment, Zhou [12] offers an impossibility result that can be applied to our model: the three requirements of equal treatment of equals, strategyproofness and the usual ex ante efficiency w.r.t. VNM cardinal utilities are incompatible. We go around this impossibility by restricting the information elicited by the mechanisms, and weakening accordingly the efficiency requirement.

Abdulkadiroglu and Sönmez [1] offer an alternative definition of the RP assignment based on the top-trading cycles of Shapley and Scarf [10].

In the companion paper [3], we apply the concept of ordinal efficiency to random assignment with arbitrary preferences and define the Probabilistic Serial assignment, a central point within the set of ordinally efficient assignments: this definition generalizes that of the current paper. The No Envy property of the PS assignment is maintained, however the PS *mechanism* is no longer strategyproof.

2 The model

Given are the set $N = \{1, 2, \ldots, n\}$ of agents, the set $A = \{1, 2, \ldots, m\}$ of objects and a null object denoted $\varnothing$.

A *deterministic assignment* is a mapping Π from N into $A \cup \{\varnothing\}$ that is one-to-one on A (no two agents can have the same "real" object; each agent gets either a real object or the null object). In matrix representation, Π is a $(n \times m)$-matrix filled with 0 and 1, and with at most one 1 per column and per row.

A *random assignment is a mapping* from N into

$$\Delta(A) = \left\{ P \in \mathbb{R}^m \,\middle|\, \sum_{a=1}^{m} p_a \leq 1,\ p_a \geq 0 \text{ for all } a \in A \right\}$$

associating to each agent $i \in N$ an allocation $P_i = (p_{ia})_{a \in A}$ in $\Delta(A)$. Of course, p_{ia} represents the probability that an agent i receives object a, and $1 - \sum_A p_{ia}$ the probability that she receives the null object. In matrix notation, P is a $(n \times m)$-matrix $P = [p_{ia}]_{i \in N, a \in A}$. Throughout the paper, we identify a random assignment P with its matrix representation and the allocation P_i with the i-th row of P.

A random assignment is feasible if and only if it is generated by a probability distribution over deterministic assignments. Thus a random assignment matrix P is feasible if and only if it is a convex combination of deterministic assignment matrices Π.

Lemma 2.1. *A random assignment matrix $P = [p_{ia}]$ is feasible if and only if it is* **substochastic**, *namely if we have*

$$p_{ia} \geq 0 \text{ for all } i, a; \sum_{a \in A} p_{ia} \leq 1 \text{ for all } i; \sum_{i \in N} p_{ia} \leq 1 \text{ for all } a. \tag{3}$$

We denote by $\mathscr{P}$ the set of substochastic matrices.

The lemma is an easy variant of the Birkhoff-Von Neumann theorem [8]. Every individual preference agrees with the given ordering of A: for all k, $k = 1, 2, \ldots, (m-1)$, every agent strictly prefers object k to object $(k+1)$. We assume that agent i's preferences over $\Delta(A)$ are represented by a Von Neumann Morgenstern utility function u_i defined on A. Her utility for an allocation $P_i \in \Delta(A)$ is:

$$u_i \cdot P_i = \sum_{k=1}^{m} u_i(k) \cdot p_{ik} \,.$$

Note that we have normalized u_i so that $u_i(\varnothing) = 0$.

Finally we assume that no agent is indifferent between the null house $\varnothing$ and a real house. Thus our domain U of utility functions is

$$U = \{u_i \in \mathbb{R}^m \,|u_i(k) > u_i(k+1) \text{ for } 1 \leq k \leq m-1 \text{ and } u_i(k) \neq 0 \text{ for all } k\}\,.$$

We say that a utility function u_i in U is of type k if k is the worst real object preferred to the null object; we say that u_i is of type 0 if $u_i(1) < 0$. We denote by U_k the subset of utility functions of type k.

A profile of utility functions is an element u of U^N. However, for the definition of ordinal efficiency as well as of the PS and RP assignments, we only use the profile of types t, $t \in \{0, 1, \ldots, m\}^N$, where $t_i = k \Leftrightarrow u_i \in U_k$.

Remark 2.1. The assumption ruling out indifference between two objects is crucial. If indifferences between objects are permitted, the PS assignment is no longer ordinally efficient and none of our results survive. On the other hand, it is easy to adapt our results to allow for indifferences between the null house and a real object.

3 Ordinal efficiency

Given a profile of utility functions u, $u \in U^N$, a substochastic matrix P in $\mathscr{P}$ is *efficient* (Pareto optimal) *at* u if there is no matrix Q in $\mathscr{P}$, such that

$$u_i \cdot Q_i \geq u_i \cdot P_i \text{ for all } i, \text{ with at least one strict inequality}\,. \tag{4}$$

Ordinal efficiency relies on the notion of stochastic dominance.

Definition 3.1. *Fix an agent i of type t, and two allocations P_i, Q_i in $\Delta(A)$. We say that P_i i-stochastically dominates Q_i, and we write $P_i \succ_i Q_i$, if the two following equivalent conditions are satisfied:*

i) the following system of inequalities holds, with at least one strict inequality

$$\begin{aligned} \sum_{a=1}^{k} p_{ia} &\geq \sum_{a=1}^{k} q_{ia} \text{ for all } k = 1, \ldots, t, \\ \sum_{a=t+1}^{k} p_{ia} &\leq \sum_{a=t+1}^{k} q_{ia} \text{ for all } k = t+1, \ldots, m, \end{aligned} \tag{5}$$

ii)

$$\textit{for all } u_i \in U_t \quad : \quad u_i \cdot P_i > u_i \cdot Q_i \,. \tag{6}$$

The equivalence of properties *i)* and *ii)* is straightforward. We write $P_i \succsim_i Q_i$ if $P_i \succ Q_i$ or $P_i = Q_i$; this property is equivalent to $\{u_i.P_i \geq u_i.Q_i$ for all u_i in $U_t\}$.

Definition 3.2. Given a profile of types t, $t \in \{0, 1, \ldots, m\}^N$, and a substochastic matrix P in $\mathscr{P}$, we say that P is *ordinally efficient at* t if there is no other substochastic matrix Q in $\mathscr{P}$ such that

$$\text{for all } \; i \in N \quad : \quad Q_i \succsim_i P_i \;\text{ and }\; Q \neq P\,.$$

Given a utility profile u, $u \in U^N$, with associated profile of types t, if P in $\mathscr{P}$ is efficient at u then it is ordinally efficient at t. In the next two Lemmas, we characterize the ordinally efficient feasible assignments first in the case of deterministic, then of random assignments.

Fix a profile of types t, and an ordering σ of N, namely a one to one mapping from $\{1, 2, \ldots, n\}$ into N: $\sigma(1)$ is the agent with the highest priority, $\sigma(2)$ is the agent with the second highest priority and so on. Construct a deterministic assignment $\Pi(t; \sigma)$ as follows: according to the priority ordering σ, the agents are successively "offered" the best unassigned object; an agent receives the object offered to her if it is a desirable one, otherwise she gets the null object (and the best current object is offered to the next agent in the priority order).

Lemma 3.1. *Given a profile of types t, the deterministic assignment Π is efficient if and only if it is the priority assignment $\Pi(t; \sigma)$ for some ordering σ of N.*

We omit the straightforward proof.

We turn to random assignments. Recall that a feasible assignment matrix P is a convex combination of deterministic assignment matrices Π:

$$P = \sum_\alpha \lambda_\alpha \cdot \Pi_\alpha \quad \lambda_\alpha \geq 0, \; \sum_\alpha \lambda_\alpha = 1\,,$$

where the sum runs over all deterministic assignments. A *necessary* condition for ordinal efficiency of P at a given profile of types t, is that each matrix Π_α receiving a positive weight in the above sum is ordinally efficient at t: for if it is not, we simply replace in the above sum Π_α by a t-stochastically superior matrix Q and the resulting convex combination t-stochastically dominates P. In view of Lemma 3.1, this means that a matrix P ordinally efficient at t must take the form

$$P = \sum_\sigma \lambda_\sigma \Pi(t; \sigma)\,. \tag{7}$$

That the above condition is, however, not sufficient for ordinal efficiency is established by the numerical example with 4 agents and $t = (1, 2, 3, 4)$ discussed in the introduction (see (1), (2)). Indeed the *Random Priority* assignment is *defined*, at any profile of types, as the uniform average of all priority allocations.

$$RP(t) = \frac{1}{n!}\sum_{\sigma} \Pi(t;\sigma) \text{ where the sum runs over all orderings of } N\,. \tag{8}$$

Lemma 3.2. *Given a profile of types t, $t \in \{0,1,\ldots,m\}^N$, and a substochastic matrix P in $\mathscr{P}$, the matrix P is ordinally efficient at t* **if and only if** *it satisfies the four following properties:*

a) for all k, all i : $t_i < k \Rightarrow p_{ik} = 0$
Let $k^* = \max\{k|\sum_{i\in N} p_{ik} > 0\}$ be the worst object assigned with some positive probability;
b) for all $k < k^*$: $\sum_{i\in N} p_{ik} = 1$
c) for all i : $\{t_i > k^*\} \Rightarrow \sum_{k=1}^{k^*} p_{ik} = 1$
d) if $\sum_{i\in N} p_{ik^*} < 1$ then for all i : $\{t_i \geq k^*\} \Rightarrow \{\sum_{k=1}^{k^*} p_{ik} = 1\}$.

Proof. **Only if**: if any one of properties a, b, c or d is violated, it is easy to construct a matrix Q stochastically dominating P. For instance if P fails b, i.e., there are two columns k, k' such that

$$k < k',\ \sum_j p_{ik} < 1,\ \sum_j p_{ik'} > 0\,,$$

then choose an agent i such that $p_{ik} < 1$, $p_{ik'} > 0$. A matrix Q which differs from P only by $q_{ik} = p_{ik} + \varepsilon$, $q_{ik'} = p_{ik'} - \varepsilon$ is feasible for ε small enough (Lemma 2.1) and stochastically dominates P.

If: Fix a matrix P in $\mathscr{P}$ meeting a, b, c, d, and assume there is another matrix Q in $\mathscr{P}$ such that $Q_i \succsim_i P_i$ for all i. From property a for P and the definition of stochastic dominance it follows that:

$$\text{for all } i,k \quad : \quad \{t_i < k\} \Rightarrow p_{ik} = q_{ik} = 0\,. \tag{9}$$

Let k^* be the last nonzero column of P. Stochastic dominance implies $q_{i1} \geq p_{i1}$ if $t_i \geq 1$, thus by (9) we have $Q^1 \geq P^1$. As $\sum_i p_{i1} = 1$ (property b for P), we get $Q^1 = P^1$. Next, stochastic dominance yields $q_{i1} + q_{i2} \geq p_{i1} + p_{i2}$ if $t_i \geq 2$, and so (9) implies $q_{i2} = p_{i2} = 0$ if $t_i < 2$. Therefore $Q^1 + Q^2 = P^1 + P^2$. The obvious induction argument gives $Q^k = P^k$ for $k = 1,\ldots,k^* - 1$.

Next consider columns $k^*, k^*+1, \ldots, m$. Assume first $\sum_i p_{ik^*} < 1$. By property d we have $\sum_1^{k^*} p_{ik} = 1$ for all i such that $t_i \geq k^*$. In view of $p_{ik} = q_{ik}$ for $k = 1,\ldots,k^* - 1$, and of $Q_i \succsim_i P_i$ this implies $Q_i = P_i$. Turning to an agent i such that $t_i < k^*$, we have $p_{ik} = q_{ik} = 0$ (by (9)) for all $k \geq k^*$ and the equality $Q_i = P_i$ holds as well.

We are left with the case $\sum_i p_{ik^*} = 1$. If i is such that $t_i < k^*$, then as above $p_{ik} = q_{ik} = 0$ for all $k \geq k^*$. If i is such that $t_i = k^*$ then $q_{ik} = p_{ik} = 0$ for $k \geq k^* + 1$ and $Q_i \succsim_i P_i$ implies $q_{ik^*} \geq p_{ik^*}$ (recall $q_{ik} = p_{ik}$ for $k = 1,\ldots,k^* - 1$). If i is such that $t_i > k^*$, then $q_{ik} = p_{ik} = 0$ for $k \geq k^* + 1$, and property c gives $\sum_1^{k^*} p_{ik} = 1$ so that $Q_i \succsim_i P_i$ implies $q_{ik^*} = p_{ik^*}$. Finally, all inequalities $q_{ik^*} \geq p_{ik^*}$ must be equalities, because $\sum_i q_{ik^*} \leq 1$, and we have shown $Q = P$ as desired. □

4 The Probabilistic Serial assignment: first characterization result

We use the following notation: given a profile of types t and an object k, $1 \leq k \leq m$, the set N_k is made of all the agents of type at least k, $N_k = \{i \in N | t_i \geq k\}$ and its cardinality is n_k, $0 \leq n_k \leq n$.

Definition 4.1. Given a profile of types t, $t \in \{0, 1, \dots, m\}^N$, the *Probabilistic Serial* assignment is the random assignment denoted $PS(t) = [p_{ik}]$ and defined recursively by system (10) or explicitly by the equivalent system (11):

$$p_{ik} = 0 \text{ if } t_i < k;\ p_{ik} = \min\left\{\frac{1}{n_k}, 1 - \sum_{k'=1}^{k-1} p_{ik'}\right\} \text{ if } k \leq t_i \tag{10}$$

set

$$k^* = 1 + \max\left\{k | 1 \leq k \leq m \text{ and } \sum_{k'=1}^{k} \frac{1}{n_{k'}} < 1\right\}$$

$$\begin{aligned} p_{ik} &= 0 \text{ if } t_i < k \\ p_{ik} &= \frac{1}{n_k} \text{ if } k \leq k^* - 1 \text{ and } t_i \geq k \\ p_{ik^*} &= 1 - \sum_{k'=1}^{k^*-1} \frac{1}{n_{k'}} \text{ if } t_i \geq k^* \\ p_{ik} &= 0 \text{ if } k \geq k^* + 1 \text{ and } t_i \geq k \end{aligned} \tag{11}$$

(note that in (11) we set $k^* = 1$ if $n_1 = 1$ or $n_1 = 0$. In the latter case, $PS(t)$ is the null matrix; in the former case, it gives the best object to the single interested agent).

For instance, if $N = \{1, 2, 3, 4\}$ and $t = (1, 2, 3, 4)$, the matrix $PS(t)$ is given by (2) and differs from $RP(t)$ given by (1). On the other hand, if $t' = (1, 1, 3, 3)$, the two matrices coincide:

$$PS(t') = RP(t') = \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \end{bmatrix} \tag{12}$$

Lemma 4.1. *For all $t \in \{0, 1, \dots, m\}^N$, the Probabilistic Serial assignment is ordinally efficient.*

Proof. System (11) implies easily that the matrix $PS(t)$ is substochastic.

To prove ordinal efficiency, we check that $PS(t)$ meets the properties a, b, c, d in Lemma 3.2. This is straightforward, in view of the fact that k^* defined in (11) is precisely the last column of $PS(t)$ with positive weight as in Lemma 3.2. In particular, note that $\sum_1^{k^*} p_{ik} = 1$ holds for any agent i such that $t_i \geq k^*$, implying c and d. □

Definition 4.2. Given a profile u in U^N, and a feasible random assignment matrix P in $\mathscr{P}$ we say that P meets *No Envy* (or is nonenvious) if we have

$$\text{for all } \; i,j \; \text{ in } \; N \quad : \quad u_i \cdot P_i \geq u_i \cdot P_j \; .$$

Given a profile of types t in $\{0,1,\ldots,m\}^N$, we say that P meets No Envy at t if we have

$$\text{for all } \; i,j \; \text{ in } \; N \quad : \quad P_i \succsim P_j \; ,$$

or equivalently, P meets No Envy at all profiles u compatible with t.

Lemma 4.3. *Given a profile of types t, the matrices $PS(t)$ and $RP(t)$ are nonenvious at t.*

Proof. Compare the two allocations P_i and P_j in $PS(t)$, when $t_i \leq t_j$. We have $p_{ik} = p_{jk}$ for $k = 1,\ldots,t_i$ and $p_{ik} = 0$, $u_i(k) \leq 0$ for $k \geq t_i + 1$; therefore agent i does not envy j. As $u_j(k) > 0$ for $k = t_i + 1,\ldots,t_j$, agent j does not envy i either.

To prove that $RP(t)$ is nonenvious, fix two agents i,j and an arbitrary ordering σ where i precedes j; let $\tilde{\sigma}$ be obtained from σ by exchanging i and j. The following is easy to check

$$u_i \cdot \frac{(\Pi_i(t;\sigma) + \Pi_i(t;\tilde{\sigma}))}{2} \geq u_j \cdot \frac{(\Pi_j(t,\sigma) + \Pi_j(t,\tilde{\sigma}))}{2} \; .$$

The conclusion follows by summing up over all orderings where i precedes j. □

Theorem 4.1. *Given a profile of types t, the random assignment $PS(t)$ is the only feasible assignment satisfying Ordinal Efficiency and No Envy at t.*

Proof. We fix the profile t and a substochastic matrix $P = [p_{ik}]$, ordinally efficient and nonenvious at t. No Envy implies the following facts:

$$\text{for all } \; i,j, \;\; \text{all } \; k \; : \; t_i, t_j \geq k \Rightarrow p_{ik} = p_{jk} \tag{13}$$

$$\text{for all } \; i,j, \qquad\qquad t_i = t_j \Rightarrow P_i = P_j \tag{14}$$

Indeed let $\bar{k}$ be the first column where (13) fails, say $p_{i\bar{k}} < p_{j\bar{k}}$. We can find a utility vector u_i in U such that $u_i(k) \approx 1$ for $k = 1,\ldots,\bar{k}, u_i(k)$ is a small positive number for $k = \bar{k}+1,\ldots,t_i$, and $u_i(k) < 0$ for $k > t_i$, and such that $u_i \cdot P_j \geq u_i \cdot P_i$, violating No Envy. Property (14) follows from (13) because ordinal efficiency implies $p_{ik} = 0$ for $k > t_i$.

Denote by k^* the last column with positive sum in P (Lemma 3.2) and recall that N_k is the set of agents of type at least k. By properties a, b in Lemma 3.2 and (13) we have

$$k < k^* \Rightarrow p_{ik} = \frac{1}{n_k} \;\; \text{for all} \;\; i \in N_k \; ; \quad p_{ik} = 0 \;\; \text{for} \;\; i \notin N_k \; ; \tag{15}$$

$$p_{ik^*} = p_{jk^*} \;\; \text{for all} \;\; i,j \in N_{k^*} \; ; \quad p_{ik^*} = 0 \;\; \text{for} \;\; i \notin N_{k^*} \; . \tag{16}$$

Let $\tilde{k}$ be the last non zero column in $Q = PS(t)$. Assume first $k^* < \tilde{k}$. The definition of PS plus (15) imply $Q^k = P^k$ for $k = 1, \ldots, k^* - 1$ and the definition of PS plus (16) give $Q^{k^*} \geq P^{k^*}$. Now the matrix Q has some positive weight beyond column k^* but P does not, which results in $Q \succ_t P$, contradiction.

Next assume $\tilde{k} < k^*$. As above, properties (15), (16) and the definition of PS give $Q^k = P^k$ for $k = 1, \ldots, \tilde{k} - 1$ and $Q^{\tilde{k}} \leq P^{\tilde{k}}$. Because there is at least one agent i with $t_i > \tilde{k}$ (the k^* column of P is not zero), property c in Lemma 3.2 gives

$$1 = q_{i\tilde{k}} + \sum_{k=1}^{\tilde{k}-1} q_{ik} = q_{i\tilde{k}} + \sum_{k=1}^{\tilde{k}-1} p_{ik} \leq p_{i\tilde{k}} + \sum_{k=1}^{\tilde{k}-1} p_{ik}\,, \tag{17}$$

hence $p_{i\tilde{k}} = q_{i\tilde{k}}$ and $Q^{\tilde{k}} = P^{\tilde{k}}$ (because both columns treat equally all agents in $N_{\tilde{k}}$). So we have $P \succ_t Q$ in contradiction of the ordinal efficiency of Q. We have shown $\tilde{k} = k^*$. Thus Q and P coincide except perhaps in column k^*. Moreover, by (16) and the analogous property for Q, we have $Q^{k^*} \geq P^{k^*}$ and/or $P^{k^*} \geq Q^{k^*}$. Ordinal efficiency of P and Q implies then $P = Q$. $\square$

5 The Probabilistic Serial mechanism: second characterization result

We discuss incentive compatibility in a special class of revelation mechanisms. These mechanisms only elicit the type (deadline) of each agent. On the other hand, the requirement of strategyproofness relies on the full-fledged (cardinal) utility functions.

Definition 5.1. Given N and A, a *random assignment mechanism* is a mapping P from $\{0, 1, \ldots, m\}^N$ into $\mathscr{P}$, associating to each profile of types t a feasible random assignment $P(t)$. We say that P is *strategyproof* if for any agent i in N, any two profiles t, t^* in $\{0, 1, \ldots, m\}^N$ such that $t_j = t_j^*$ for all j different from i, we have:

$$P_i(t) \succsim_i P_i(t^*) \text{ where } \succsim_i \text{ refers to type } t_i\,. \tag{18}$$

We say that P is *ordinally efficient* if $P(t)$ is ordinally efficient at t for all t. We say that P meets *Equal Treatment of Equals* if for all t and all i, j, $t_i = t_j$ implies $P_i(t) = P_j(t)$.

Equal Treatment of Equals is a minimal equity requirement implied by and much weaker than the requirement that $P(t)$ is nonenvious for all t .

For any fixed ordering σ of N, the priority mechanism $t \to \Pi(t; \sigma)$ is strategyproof. Strategyproofness is preserved by fixed convex combinations (because (19) is a system of inequalities linear in $P(t)$), therefore the Random Priority mechanism $t \to RP(t)$ is strategyproof; it is also equitable (even nonenvious: Lemma 4.3) but not ordinally efficient.

Theorem 5.1. *The Probabilistic Serial mechanism $t \to PS(t)$ is characterized, within the set of random assignment mechanisms, by the combination of Strategyproofness, Ordinal Efficiency and Equal Treatment of Equals.*

Proof. The PS mechanism is ordinally efficient and nonenvious (Lemmas 4.1, 4.3). We will need the following

Lemma 5.1. *An ordinally efficient mechanism P is strategyproof if and only if for any two profiles t, t^* as in Definition 5.1*

$$p_{ik}(t) = p_{ik}(t^*) \ \ \textit{for all} \ \ k \leq \min\{t_i, t_i^*\}$$

Proof. **If:** easily follows, as efficiency guarantees

$$p_{ik}(t) = 0 \ \forall \ k > t_i$$

Only if: by contradiction consider the smallest $\overline{k}$ such that $\overline{k} < \min\{t_i, t_i^*\}$ and $p_{i\overline{k}}(t) \not\geq p_{i\overline{k}}(t^*)$. We can find a utility vector u_i, such that $u_i(k) \approx 1$ for $k = 1, \ldots, \overline{k}$, $u_i(k) \approx 0$ for $k > \overline{k}$, and that $u_i \cdot P_i(t) \ngeq u_i \cdot P_i(t^*)$, providing for i a possibility to manipulate. □

Lemma 5.1 and Definition 4.1 imply that the PS mechanism is strategyproof.

Now we fix a mechanism P satisfying the three announced properties and prove the equality $P(t) = PS(t)$ by induction on $|t| = \sum_i t_i$. This equality is obvious for $|t| = 0, 1, 2$. Consider an arbitrary profile of types t that remains fixed throughout the rest of the proof; we simply write $P(t) = P = [p_{ik}]$ and $PS(t) = Q = [q_{ik}]$. Without loss of generality we label the agents in such a way that $t_1 \leq t_2 \leq \ldots \leq t_n$ and define the increasing subsequence $i_0, i_1, \ldots, i_r$ as follows:

$$\begin{aligned} t_1 &= \ldots = t_{i_0} = 0 < t_{i_0+1} = t_{i_0+2} = \ldots = t_{i_1} < t_{i_1+1} = t_{i_1+2} = \ldots \\ &= t_{i_2} < \ldots < t_{i_{r-1}+1} = t_{i_{r-1}+2} = \ldots = t_{i_r} \\ &\quad \text{where } \ i_r = n \ \text{ and, by convention, } \ i_0 = 0 \ \text{ if } \ N_1 = N \ \ \text{(i.e., if } \ t_1 > 0). \end{aligned}$$

Thus the range of types is $t_{i_1}, \ldots, t_{i_r}$ if $i_0 = 0$ and $0, t_{i_1}, \ldots, t_{i_r}$ if $i_0 \geq 1$. In the latter case, ordinal efficiency implies that the first i_0 rows of P and Q are null. We now show that their nonnull rows are equal as well. By Equal Treatment of Equals, we have

$$P_{i_{s-1}+1} = P_{i_{s-1}+2} = \ldots = P_{i_s} \ \ \text{for all} \ \ s = 1, \ldots, r \,. \tag{19}$$

For any s, $s = 1, \ldots, r$, let t^s be the following profile of types:

$$t^s_{i_s} = t_{i_s} - 1 \quad ; \quad t^s_j = t_j \ \ \text{for all} \ \ j \neq i_s \,.$$

The induction hypothesis implies $P(t^s) = PS(t^s) = [p^s_{ik}]$. From Definition 4.1

$$p^s_{(i_{s-1}+1)_k} = p^s_{(i_{s-1}+2)k} = \ldots = p^s_{i_s k} \ \ \text{for all} \ \ s = 1, \ldots, r \ \ \text{and all} \ \ k < t_{i_s} \,. \tag{20}$$

As t and t^s only differ in the i_s component, Lemma 5.1 gives

$$p_{i_s k} = p^s_{i_s k} \ \ \text{for all} \ \ s = 1, \ldots, r \ \ \text{and all} \ \ k < t_{i_s} \,. \tag{21}$$

Gathering (19)–(21), we have shown that the following submatrices coincide:

$$\begin{aligned} p_{ik} &= p^s_{ik} \text{ for all } s = 1, \ldots, r, \text{ all } i, \ i_{s-1} < i \leq i_s, \\ &\text{and all } k, \ 1 \leq k < t_{i_s} . \end{aligned} \tag{22}$$

For Probabilistic Serial, when the profile changes, the allocation of the columns $k = 1, \ldots, t_{i_s} - 1$ does not change from t^s to t (this is clear from Definition 4.1, see also the related property stated in Remark 4.1). Hence, from (22):

$$\begin{aligned} q_{ik} &= p^s_{ik} = p_{ik} \text{ for all } s = 1, \ldots, r, \text{ all } i, \ i_{s-1} < i \leq i_s, \\ &\text{and all } k, \ 1 \leq k < t_{i_s} . \end{aligned} \tag{23}$$

Moreover, ordinal efficiency implies $q_{ik} = p_{ik} = 0$ for all s, all i, $i_{s-1} < i \leq i_s$ and all k, $t_{i_s} < k$. Therefore we are left to check that for all $s = 1, \ldots, r$ the following subcolumns of P and Q coincide:

$$q_{ik}, \ p_{ik} \quad \text{for } i_{s-1} < i \leq i_s, \ \ k = t_{i_s} .$$

By Equal Treatment of Equals, each one of these $2r$ subcolumns is made of identical elements. Moreover the complete columns $Q^{t_{i_s}}$ and $P^{t_{i_s}}$ have the same entry in any row i such that $i \leq i_{s-1}$ (by (23)), or such that $i > i_s$ (where they are both zero). Thus $Q^{t_{i_s}} \geq P^{t_{i_s}}$ and/or $P^{t_{i_s}} \geq Q^{t_{i_s}}$.

Let k^* (resp.$\tilde{k}$) be the last column with positive sum in P (resp. Q). By Lemma 3.2, $\sum_i p_{ik} = 1$ (resp. $\sum_i q_{ik} = 1$) for $k = 1, \ldots, k^* - 1$ (resp. $\tilde{k} - 1$). Since P, Q are both ordinally efficient, neither of them can be stochastically dominated by the other. Together with all the above, this implies that $k^* = \tilde{k}$, and, given this equality, that $\sum_i p_{ik^*} = \sum_i q_{ik^*}$, which finally gives $Q^{t_{i_s}} = P^{t_{i_s}}$ for all s, and $P = Q$.

Remark 5.1. Crès and Moulin [4] show that both mechanisms *PS* and *RP* meet the stronger incentive compatibility requirement of *groupstrategyproofness* ruling out profitable misreports by any coalition of agents.

Remark 5.2. As mentioned in the introduction, the proof of the main result in [12] implies the following impossibility result in our problem: among mechanisms eliciting the full fledged cardinal utility functions, the requirements of Strategyproofness, Efficiency w.r.t. cardinal utilities and Equal Treatment of Equals are incompatible.

Remark 5.3. Theorems 4.1 and 5.1 are tight results.

Drop Equal Treatment of Equals in Theorem 5.1 or No Envy in Theorem 4.1: then all priority mechanisms $\Pi(\cdot, \sigma)$ meet the other assumptions. Drop Ordinal Efficiency in either theorem, then any constant and egalitarian mechanism ($P(t) = P$ all t, $P_i = P_j$ all i, j) meets the other assumptions. Drop Strategyproofness in Theorem 5.1. Then construct an equitable mechanism where priority is given to the more impatient agents (those with a shorter deadline) as follows. Given t, let $S(t)$ be the set of all orderings σ of N such that $t_i < t_j \Rightarrow \sigma(i) < \sigma(j)$. Define

$$P(t) = \frac{1}{\#\{S(t)\}} \sum_{\sigma \in S(t)} \Pi(t, \sigma) .$$

We let the reader check, with the help of Lemma 3.2, that this mechanism is ordinally efficient. Equal Treatment of Equals is obvious.

References

1. Abdulkadiroglu, A., Sönmez, T.: Random serial dictatorship and the core from random endowments in house allocation problems. Econometrica **66**, 689–701 (1998)
2. Barbera, S.: Majority and positional voting in a probabilistic framework. Review of Economic Studies **46**, 389–397 (1979)
3. Bogomolnaia, A., Moulin, H.: A new solution to the random assignment problem. Journal of Economic Theory (forthcoming)
4. Crès, H., Moulin, H.: Scheduling with opting out: Improving upon random priority. Operations Research (forthcoming)
5. Friedman, E.: An incentive compatible scheduling algorithm. Mimeo, Duke University, Durham, NC (1994)
6. Gibbard, A.: Straightforwardness of game forms with lotteries as outcomes. Econometrica **46**, 595–602 (1978)
7. Hylland, A., Zeckhauser, R.: The efficient allocation of individuals to positions. Journal of Political Economy **91**, 293–313 (1979)
8. Von Neumann, J.: A certain zero-sum two person game equivalent to the optimal assignment problem. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the theory of games, vol. II. Annals of Mathematics Studies 28. Princeton, NJ.: Princeton University Press 1953
9. Pápai, S.: Strategyproof assignment by hierarchical exchange. Econometrica **68**, 1403–1434 (2000)
10. Shapley, L., Scarf, H.: On cores and indivisibility. Journal of Mathematical Economics **1**, 23–37 (1974)
11. Svensson, L.-G.: Queue allocation of indivisible goods. Social Choice and Welfare **11**, 323–330 (1994)
12. Zhou, L.: On a conjecture by Gale about one-sided matching problems. Journal of Economic Theory **52**, 123–135 (1990)