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Income and wealth distribution in a simple model of growth★

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Summary. This paper studies a deterministic one-sector growth model with a constant returns to scale production function and endogenous labor supply. It is shown that the distribution of capital among the agents has an effect on the level of per-capita output. There exists a continuum of stationary equilibria with different levels of per-capita output. If the elasticity of intertemporal substitution is large, a higher output level can be achieved when income inequality is great, that is, when the income distribution is strongly dispersed. If the elasticity of intertemporal substitution is low, the reverse relation holds. The paper shows that countries with identical production technologies and identical preferences may have different GDP levels because wealth is distributed differently among their inhabitants.

Keywords and Phrases: Growth, Inequality, One-sector model, Elasticity of intertemporal substitution.

JEL Classification Numbers: O41, D31.

1 Introduction

This paper addresses two questions:

(i) Why do economies with similar demographic and technological characteristics often experience quite different economic developments?

(ii) What is the relation between the distribution of income or wealth and the level (or the growth rate) of gross domestic product in a closed economy?

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To this end we study a deterministic one-sector growth model with a constant returns to scale production function and endogenous labor supply. In this framework it is shown that the distribution of capital among the agents has an effect on the level of per-capita output. There exists a continuum of stationary equilibria with different levels of per-capita output. The paper shows that countries with identical production technologies and identical preferences may have different per-capita GDP levels if wealth is distributed differently among their inhabitants.

Since both of the questions mentioned above have been dealt with before we start by giving a brief survey of the literature in order to demonstrate how our approach and our results differ from previous work in this area. The first question was one reason for the development of the so-called new growth theories. According to traditional neoclassical growth theories, the growth rates of similar economies should asymptotically converge to each other (conditional convergence), whereas new growth theories include mechanisms which may work against convergence.[1] The most prominent mechanisms rely on differences in human capital development or knowledge formation (see, e.g., Romer [25] or Lucas [20]). Another branch of this literature is concerned with growth models in which the aggregate production technology exhibits increasing returns to scale. To accomodate this feature to a general equilibrium framework these models assume the presence of external effects or some form of imperfect competition. Here we refer in particular to the two symposium volumes of the *Journal of Economic Theory* (see Benahbib and Rustichini [10] and Benhabib [7]) and to the survey by Benhabib and Gali [9]. All these models usually describe economies that are populated by a large number of homogeneous, infinitely-lived households. Since it is commonly assumed that all households own identical initial stocks of wealth, they all behave identically throughout the planning period and, consequently, distributional considerations do not play any role. In the present paper we study a model that is very similar to many of those mentioned above, except that we allow for heterogeneous capital endowments of the (otherwise identical) households. The key features of our model are the endogenous labor supply, a constant returns to scale production function, a single capital stock (only physical capital exists but no human capital), no externalities, and perfect competition in all markets. It turns out that in this framework the distribution of wealth matters for the dynamic evolution and the long-run level of per-capita output. Depending on the parameter values, our model can predict both a positive and a negative relation between income dispersion and per-capita output. Formally, the model has a continuum of stationary equilibria differing in terms of their per-capita GDP levels and in terms of the income and wealth distribution. If the elasticity of intertemporal substitution is large, a higher output level can be achieved when income inequality is great, that is, when the income distribution is strongly dispersed. If the elasticity of intertemporal substitution is low, the reverse relation holds.

[1] For an excellent exposition of traditional and new growth theories as well as a discussion of the convergence hypothesis we refer to Barro and Sala-i-Martin [5].

Now let us consider the second question mentioned above. There are a number of models of capital accumulation and growth that can explain non-degenerate income distributions. Many of these models assume heterogeneity of households. Becker [6] and Lucas and Stokey [21], for example, derive a non-degenerate wealth and income distribution from the assumption that agents differ in terms of their time preferences. Other models assume that (otherwise identical) households face idiosyncratic risks (for example concerning their labor endowment); an example for this approach is Aiyagari [3]. Yet another explanantion is based on the presence of credit market imperfections, see, e.g., Aghion and Bolton [1], Galor and Zeira [17], or Piketty [23]. A model that is quite similar to the one studied here was presented in Chatterjee [13]. In that study, however, labor supply is exogenous and the development as well as the long-run level of per-capita output is independent of the distribution of wealth. This follows from the particular form of households' preferences that is assumed. Contrary to our approach, Chatterjee's model does not identify the income or wealth distribution as an important determinant of the long-run per-capita GDP level. Concerning the question whether income inequality is growth enhancing or growth reducing, theoretical models yield ambiguous results. Incentive consideration seem to point to a positive influence of inequality on growth, but capital market imperfections may mitigate or even reverse this influence. Persson and Tabellini [22] emphasize that strong inequality is harmful for growth (in democratic societies) because it triggers political decisions that promote redistribution at the expense of growth. As for the influence of development on income dispersion, the most prominent hypothesis stems from Kuznets [19]. It says that during the initial phases of development, income dispersion should increase, but eventually income inequality will be reduced again. For an enlightening discussion of these matters we refer to Aghion and Howitt [2, Chapter 9]. It has to be emphasized that our model does not have any of the features mentioned above. Households are homogeneous (except for their initial endowments), all markets are perfectly competitive, and no idiosyncratic risks are present. The only mechanism that creates the existence of a continuum of heterogeneous stationary equilibria is the interplay between the households' saving decision and their optimal labor supply. The endogenous determination of the labor supply is in fact crucial for our results, as it allows that the same capital-labor ratio and, hence, the same interest rate can occur at different levels of the aggregate capital stock.[2]

The empirical investigation of the relation between income dispersion and GDP has also a long tradition. The Kuznets hypothesis mentioned above as well as related questions have been tested with varying results (see, e.g., Bourguignon and Morrisson [11] and Fields [16, Chapter 4]). Strong empirical evidence on a negative relation between income inequality and growth is reported in Persson and Tabellini [22], who use both historical panel data and cross sectional data. An

[2] Other implications of an endogenous labor supply were derived by, e.g., de Hek [14] and Eriksson [15]. These papers, however, use a representative agent framework such that distributional influences are assumed away.

analogous negative relation between income inequality and the level of per-capita GDP was found by Kravis [18].

The paper is organized as follows. Section 2 introduces the model and states and discusses the equilibrium conditions. Section 3 proves the existence of a continuum of stationary equilibria and Section 4 states results on the stationary income distributions. These two sections contain the main results of the paper. Section 5 presents a few results on the transitional dynamics and Section 6 concludes.

2 Model formulation and equilibrium conditions

We consider a continuous-time model of a one-sector economy in which at each time $t \in [0, \infty)$ output $Y(t)$ is produced from capital $K(t)$ and labor $L(t)$ by the Cobb-Douglas technology

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}. \tag{1}$$

Output is the numeraire good and we denote by $r(t)$ and $w(t)$ the rental rate of capital and the wage rate at time t. At every instant t, the representative firm maximizes profits

$$\Pi(t) = Y(t) - r(t)K(t) - w(t)L(t)$$

taking factor prices as given. There are no production externalities, no technological progress, and $\alpha \in (0, 1)$ is constant.

There exists a continuum of measure 1 (identified with the unit interval $I = [0, 1]$) of households.[3] At time t, household $i \in I$ consumes at the rate $c_i(t) \geq 0$ and supplies labor at the rate $\ell_i(t) \in [0, 1]$ to the firms. All households have identical preferences described by the utility functional

$$J_i = \int_0^{+\infty} e^{-\rho t} U(c_i(t), \ell_i(t)) \, \mathrm{d}t,$$

where $\rho > 0$ denotes the time preference rate. We assume that the instantaneous utility function has the form

$$U(c, \ell) = \begin{cases} \dfrac{c^{1-1/\theta} - 1}{1 - 1/\theta} + \beta \ln(1 - \ell) & \text{if } \theta \in (0, 1) \cup (1, +\infty), \\ \ln c + \beta \ln(1 - \ell) & \text{if } \theta = 1. \end{cases}$$

The parameter $\theta > 0$ is the elasticity of intertemporal substitution, and $\beta > 0$ measures how much weight the households attach to the disutility of working.[4] Denoting by $\delta > 0$ the constant depreciation rate of capital, the interest rate is given by $r(t) - \delta$ and household i's flow of income at time t is

[3] When we use measure theoretic concepts we consider I as endowed with the Borel σ-algebra and Lebesgue measure.

[4] The case $\beta = 0$ would describe a situation where households inelastically supply 1 unit of labor per unit of time.

$$y_i(t) = [r(t) - \delta]k_i(t) + w(t)\ell_i(t), \tag{2}$$

where $k_i(t)$ is the wealth of household i at time t. A negative value of $k_i(t)$ would mean that household i is indebted at time t. We denote the household's initial wealth at time 0 by k_{i0}. With these notations, the intertemporal budget constraint of household i can be written as

$$\dot{k}_i(t) = [r(t) - \delta]k_i(t) + w(t)\ell_i(t) - c_i(t)\,,\ k_i(0) = k_{i0}, \tag{3}$$

$$\lim_{t\to+\infty} e^{-\int_0^t r(s)-\delta\,\mathrm{d}s} k_i(t) = 0. \tag{4}$$

Equation (3) is the wealth accumulation equation and (4) is a no-Ponzi-game condition. Note that we assume that households are identical in terms of their preferences, but that they may have different initial endowments k_{i0}. The function $i \mapsto k_{i0}$ can be any measurable function such that aggregate initial wealth $K_0 = \int_0^1 k_{i0}\,\mathrm{d}i$ is positive.

The factor markets are in equilibrium if

$$K(t) = \int_0^1 k_i(t)\,\mathrm{d}i\ ,\ L(t) = \int_0^1 \ell_i(t)\,\mathrm{d}i \tag{5}$$

holds for all t. The output market is in equilibrium if

$$Y(t) = \dot{K}(t) + \delta K(t) + C(t) \tag{6}$$

for all t, where $C(t) = \int_0^1 c_i(t)\,\mathrm{d}i$ denotes aggregate consumption.

Definition 1. A family of functions

$$E = (Y(\cdot), K(\cdot), L(\cdot), C(\cdot), r(\cdot), w(\cdot), \{k_i(\cdot), c_i(\cdot), \ell_i(\cdot) \,|\, i \in I\})$$

is called an *equilibrium* if
(i) the functions $i \mapsto k_i(t)$, $i \mapsto c_i(t)$, and $i \mapsto \ell_i(t)$ are measurable for all $t \in [0, +\infty)$,
(ii) for all $t \in [0, \infty)$, the pair $(K(t), L(t))$ maximizes profit $\Pi(t)$ subject to the technological constraint (1) and non-negativity constraints on the inputs,
(iii) for all $i \in I$, $(k_i(\cdot), c_i(\cdot), \ell_i(\cdot))$ is an optimal solution to the problem of maximizing J_i subject to (3) - (4),
(iv) the market clearing conditions (5) and (6) hold for all $t \in [0, +\infty)$.
An equilibrium is called *stationary* if it consists of constant functions, and it is called *homogeneous* if the conditions $k_i(t) = K(t)$, $c_i(t) = C(t)$, and $\ell_i(t) = L(t)$ hold for all $t \in [0, +\infty)$ and all $i \in I$.

Because the population in the economy has the constant measure 1, we can interpret the variables $Y(t)$, $K(t)$, $L(t)$, and $C(t)$ either as aggregate variables or as per-capita variables. Note, however, that they need not coincide with the corresponding individual variables $y_i(t)$, $k_i(t)$, $\ell_i(t)$, and $c_i(t)$ for any particular agent i.[5]

In a homogeneous equilibrium, wealth and income are uniformly distributed among the agents. This is the case typically considered in growth theory and real-business-cycle theory. In this paper we are particularly interested in equilibria displaying non-homogeneous wealth and income distributions. For some questions addressed in this paper it will not be useful to distinguish between equilibria which differ from each other only by a relabelling of households, or between equilibria which are identical except for the behavior of households in a set of measure 0. We formalize these ideas in the following definition.

Definition 2. Two equilibria E and $\tilde{E}$ are called *equivalent* if there exists a bijective and measure-preserving mapping $p : I \mapsto I$ such that $(\tilde{k}_i(t), \tilde{c}_i(t), \tilde{\ell}_i(t)) = (k_{p(i)}(t), c_{p(i)}(t), \ell_{p(i)}(t))$ for all $t \in [0, +\infty)$ and for almost all $i \in I$.

It is straightforward to see that the aggregate variables $Y(t)$, $K(t)$, $L(t)$, and $C(t)$ as well as the factor prices $r(t)$ and $w(t)$ in two equivalent equilibria must coincide.

Profit maximization under perfect competition implies that capital and labor earn their marginal products, that is

$$r(t) = \alpha[K(t)/L(t)]^{-(1-\alpha)}, \; w(t) = (1-\alpha)[K(t)/L(t)]^{\alpha}. \tag{7}$$

Conditions (1), (3), (5), and (7) imply that (6) is satisfied (Walras' law). Thus, we may disregard condition (6). The first order optimality conditions for the optimization problem of household i are

$$c_i(t) = [w(t)/\beta]^{\theta}[1 - \ell_i(t)]^{\theta} \tag{8}$$

and

$$\dot{c}_i(t)/c_i(t) = \theta[r(t) - \delta - \rho]. \tag{9}$$

Condition (8) shows that $c_i(t)$ is a strictly concave function of $\ell_i(t)$ if $\theta \in (0,1)$, whereas it is a strictly convex function of $\ell_i(t)$ if $\theta \in (1,+\infty)$. In the borderline case of logarithmic consumption utility ($\theta = 1$), $c_i(t)$ is a linear function of $\ell_i(t)$. These curvature properties as well as their dependence on the parameter θ will turn out to be essential for most of the results in this paper. In the following lemma we summarize the equilibrium conditions and draw a simple conclusion.

Lemma 1. *(i) A family of functions* $(Y(\cdot), K(\cdot), L(\cdot), C(\cdot), r(\cdot), w(\cdot), \{k_i(\cdot), c_i(\cdot), \ell_i(\cdot) \,|\, i \in I\})$ *is an equilibrium if and only if the conditions (1), (3) - (5), and (7) - (9) hold for all* $t \in [0, +\infty)$ *and all* $i \in I$.
(ii) In every equilibrium and for every household $i \in I$ *there exist constants* $\mu_i > 0$ *and* $\nu_i > 0$ *such that* $c_i(t)/c_0(t) = \mu_i$ *and* $[1 - \ell_i(t)]/[1 - \ell_0(t)] = \nu_i$ *for all* $t \in [0, +\infty)$.

[5] Note also that $Y(t)$ is gross domestic product whereas $y_i(t)$ denotes individual income net of depreciation. Thus, one has $\int_0^1 y_i(t)\,\mathrm{d}i = Y(t) - \delta K(t)$. The other three aggregate variables are simply the integrals over I of the corresponding individual variables.

Proof. Statement (i) is well known. From condition (9) follows $\dot{c}_i(t)/c_i(t) - \dot{c}_0(t)/c_0(t) = 0$. It is straightforward to see that this implies that $c_i(t)/c_0(t)$ is constant with respect to time. Together with (8) this shows that $[1 - \ell_i(t)]/[1 - \ell_0(t)]$ is constant, too. □

For every measurable function $\nu : I \mapsto (0, +\infty)$ let us define[6]

$$B(\nu) = \frac{\int_0^1 \nu_i^\theta \,\mathrm{d}i}{\left(\int_0^1 \nu_i \,\mathrm{d}i\right)^\theta}.$$

It follows from Jensen's inequality that $B(\nu) \le 1$ if $\theta \in (0,1)$, $B(\nu) = 1$ if $\theta = 1$, and $B(\nu) \ge 1$ if $\theta \in (1, +\infty)$. The inequalities hold strictly if and only if there does not exist any constant function which coincides with ν almost everywhere.

The next lemma presents a set of equilibrium conditions which is different from the one stated in Lemma 1(i). The advantage of these alternative conditions is that the equilibrium dynamics of the two variables $K(t)$ and $L(t)$ are isolated, and that all other variables are expressed in terms of $K(t)$ and $L(t)$.

Lemma 2. *A family of functions* $(Y(\cdot), K(\cdot), L(\cdot), C(\cdot), r(\cdot), w(\cdot), \{k_i(\cdot), c_i(\cdot), \ell_i(\cdot) \,|\, i \in I\})$ *is an equilibrium if and only if there exists a measurable function* $\nu : I \mapsto (0, +\infty)$ *such that the differential equations*

$$\begin{aligned} \dot{K}(t) &= K(t)^\alpha L(t)^{1-\alpha} - \delta K(t) \\ &\quad - \left(\frac{1-\alpha}{\beta}\right)^\theta B(\nu) \frac{K(t)^{\alpha\theta}[1 - L(t)]^\theta}{L(t)^{\alpha\theta}}, \; K(0) = K_0, \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{L}(t) &= \frac{L(t)[1 - L(t)]}{L(t) + \alpha[1 - L(t)]} \left\{ (1-\alpha)\delta + \rho - \alpha \left(\frac{1-\alpha}{\beta}\right)^\theta \right. \\ &\quad \left. \times B(\nu) \frac{K(t)^{\alpha\theta-1}[1 - L(t)]^\theta}{L(t)^{\alpha\theta}} \right\}, \end{aligned} \tag{11}$$

$$\begin{aligned} \dot{k}_i(t) &= \left[\alpha \frac{K(t)^{\alpha-1}}{L(t)^{\alpha-1}} - \delta\right] k_i(t) + (1-\alpha) \left\{ 1 - \frac{\nu_i[1 - L(t)]}{\int_0^1 \nu_j \,\mathrm{d}j} \right\} \frac{K(t)^\alpha}{L(t)^\alpha} \\ &\quad - \left(\frac{1-\alpha}{\beta}\right)^\theta \frac{\nu_i^\theta K(t)^{\alpha\theta}[1 - L(t)]^\theta}{\left(\int_0^1 \nu_j \,\mathrm{d}j\right)^\theta L(t)^{\alpha\theta}}, \; k_i(0) = k_{i0} \end{aligned} \tag{12}$$

hold for all $i \in I$ *and conditions (1), (4), and (7) as well as*

$$C(t) = \left(\frac{1-\alpha}{\beta}\right)^\theta B(\nu) \frac{K(t)^{\alpha\theta}[1 - L(t)]^\theta}{L(t)^{\alpha\theta}}, \tag{13}$$

$$c_i(t) = \left(\frac{1-\alpha}{\beta}\right)^\theta \frac{\nu_i^\theta K(t)^{\alpha\theta}[1 - L(t)]^\theta}{\left(\int_0^1 \nu_j \,\mathrm{d}j\right)^\theta L(t)^{\alpha\theta}}, \tag{14}$$

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[6] The value of ν at $i \in I$ will be denoted by ν_i.

$$\ell_i(t) = 1 - \frac{\nu_i}{\int_0^1 \nu_j \, \mathrm{d}j}[1 - L(t)], \quad (15)$$

$$1 > L(t) \geq 1 - \frac{\int_0^1 \nu_j \, \mathrm{d}j}{\nu_i} \quad (16)$$

hold for all $i \in I$ *and all* $t \in [0, +\infty)$.

Proof. We first proof necessity. Condition (15) follows from Lemma 1(ii) and (5). Condition (16) follows from (15) and $0 \leq \ell_i(t) < 1$. Condition (14) follows from (7), (8), and (15). Integrating (14) over $i \in I$ yields (13). Substituting (7), (14), and (15) into (3) yields (12). Integrating (12) over $i \in I$ and using (5) yields (10). To prove (11) it will be convenient to use the variables $x_i(t) = 1 - \ell_i(t)$ and $X(t) = 1 - L(t)$. Note that $X(t) = \int_0^1 x_i(t) \, \mathrm{d}i$ and $\dot{X}(t) = -\dot{L}(t)$. From (15), (8), and (7) we obtain

$$\frac{\dot{X}(t)}{X(t)} = \frac{\dot{x}_i(t)}{x_i(t)} = \frac{\dot{c}_i(t)}{\theta c_i(t)} - \frac{\dot{w}(t)}{w(t)} = \frac{\dot{c}_i(t)}{\theta c_i(t)} - \alpha \frac{\dot{K}(t)}{K(t)} - \alpha \frac{\dot{X}(t)}{L(t)}.$$

Solving this equation for $\dot{X}(t)$ and using (9), (6), and (7) one gets

$$\begin{aligned} \dot{X}(t) &= \frac{L(t)[1 - L(t)]}{\alpha + (1 - \alpha)L(t)} \left[r(t) - \delta - \rho - \alpha \frac{\dot{K}(t)}{K(t)} \right] \\ &= \frac{-L(t)[1 - L(t)]}{\alpha + (1 - \alpha)L(t)} \left[(1 - \alpha)\delta + \rho - \alpha \frac{C(t)}{K(t)} \right]. \end{aligned}$$

Substituting $C(t)$ from (13) into this equation one obtains (11). Thus, we have proved the necessity of the conditions stated in the lemma. Since all steps are reversible, sufficiency follows as well. □

3 Multiplicity of stationary equilibria

In the following two sections we deal with stationary (i.e. time-independent) equilibria. We shall therefore omit the time argument t whenever this is possible. For notational simplicity we define the constant ξ by

$$\xi = \left(\frac{\beta}{1 - \alpha} \right)^{\theta} \left(\frac{\alpha}{\delta + \rho} \right)^{\alpha(1-\theta)/(1-\alpha)} \frac{(1 - \alpha)\delta + \rho}{\delta + \rho}.$$

The following lemma summarizes some properties that hold in every stationary equilibrium (whether it is homogeneous or not).

Lemma 3. *(i) In every stationary equilibrium it holds that*

$$\begin{aligned} r &= \delta + \rho, \; w = (1 - \alpha) \left(\frac{\alpha}{\delta + \rho} \right)^{\alpha/(1-\alpha)}, \\ \frac{K}{L} &= \left(\frac{\alpha}{\delta + \rho} \right)^{1/(1-\alpha)}, \; \frac{Y}{L} = \left(\frac{\alpha}{\delta + \rho} \right)^{\alpha/(1-\alpha)}. \end{aligned}$$

(ii) In every stationary equilibrium and for all $i \in I$ it holds that

$$k_i = \frac{1}{\rho}\left[\left(\frac{1-\alpha}{\beta}\right)^{\theta}\left(\frac{\alpha}{\delta+\rho}\right)^{\alpha\theta/(1-\alpha)}(1-\ell_i)^{\theta} - (1-\alpha)\left(\frac{\alpha}{\delta+\rho}\right)^{\alpha/(1-\alpha)}\ell_i\right] \tag{17}$$

and

$$y_i = c_i = \left(\frac{1-\alpha}{\beta}\right)^{\theta}\left(\frac{\alpha}{\delta+\rho}\right)^{\alpha\theta/(1-\alpha)}(1-\ell_i)^{\theta}. \tag{18}$$

(iii) In every stationary equilibrium it holds that

$$B(\nu) = \xi L/(1-L)^{\theta} \tag{19}$$

where $\nu : I \mapsto (0, +\infty)$ is the function mentioned in Lemma 2.

Proof. The results stated in (i) follow easily from the stationarity assumption $\dot{c}_i(t) = 0$ and conditions (1), (7), and (9). The results stated in (ii) follow from those stated in (i), the stationarity assumption $\dot{k}_i(t) = 0$, and conditions (2), (3), and (8). The condition stated in (iii) follows from the results in (i) and from (11) upon setting $\dot{L}(t) = 0$. □

The first part of Lemma 3 shows that the factor prices, the capital-labor ratio, and the output-labor ratio are the same in all stationary equilibria. The second part of the lemma expresses wealth, income, and consumption of household i in a stationary equilibrium in terms of the household's labor supply. These relations are also independent of which particular stationary equilibrium is considered. It is obvious from (17) and (18) that wealth, income, and consumption are negatively related to the labor supply. Households who provide much labor hold less capital and have a lower income than households who provide only little labor. The third part of Lemma 3 is merely of technical nature and will be used later.

We denote by L^* the unique value in the interval $[0, 1]$ which satisfies the equation[7]

$$(1-L^*)^{\theta} = \xi L^* \tag{20}$$

and we define

$$\bar{L} = 1/(1+\xi). \tag{21}$$

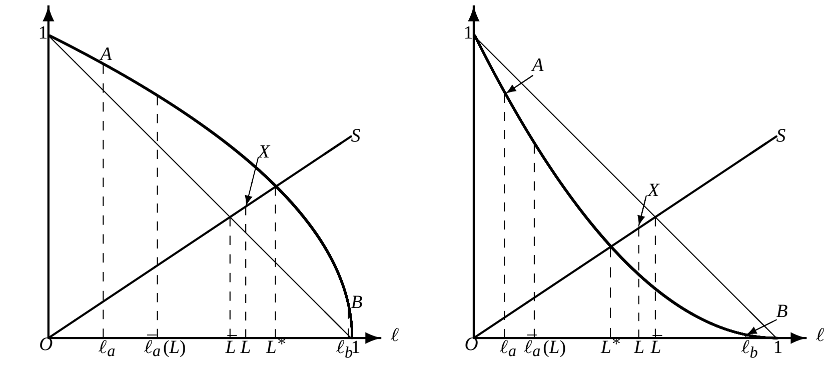
Note that $\bar{L} < L^*$ if $\theta \in (0, 1)$, $\bar{L} > L^*$ if $\theta \in (1, +\infty)$, and $\bar{L} = L^*$ if $\theta = 1$.[8] In the following we shall restrict ourselves to the case $\theta \neq 1$. To simplify the notation we denote the open interval of all values between L^* and $\bar{L}$ by $\langle L^*, \bar{L}\rangle$, that is,

$$\langle L^*, \bar{L}\rangle = \begin{cases} (\bar{L}, L^*) & \text{if } \theta \in (0,1), \\ (L^*, \bar{L}) & \text{if } \theta \in (1, +\infty). \end{cases}$$

Before we can prove the main result of this section we establish the following auxiliary result.

[7] The left-hand side of (20) is strictly decreasing and the right-hand side is strictly increasing. This proves uniqueness of L^*. Existence follows by a standard continuity argument.

[8] The definitions of L^* and $\bar{L}$ are explained in more detail in the proof of Lemma 4 and illustrated in Figure 1. The properties mentioned in this paragraph are obvious from that figure.



a **b**

Figure 1. a Proof of Lemma 4 for $\theta < 1$. **b** Proof of Lemma 4 for $\theta > 1$

Lemma 4. *Assume $\theta \neq 1$ and $L \in \langle L^*, \bar{L}\rangle$. There exists a continuum of triples $(\ell_a, \ell_b, \lambda)$ such that $0 \leq \ell_a < L < \ell_b < 1$, $\lambda = (L - \ell_a)/(\ell_b - \ell_a) \in (0, 1)$, $(1 - \lambda)\ell_a + \lambda\ell_b = L$, and $(1 - \lambda)(1 - \ell_a)^\theta + \lambda(1 - \ell_b)^\theta = \xi L$.*

Proof. The proof is illustrated in Figure 1. Part (a) of the figure illustrates the case $\theta \in (0, 1)$ whereas part (b) illustrates the case $\theta \in (1, +\infty)$. The downward sloping parabolic curve in the figure is the graph of the function $\ell \mapsto (1-\ell)^\theta$, the upward sloping straight line OS is the graph of $\ell \mapsto \xi\ell$. The unique intersection of the two graphs determines the value $\ell = L^*$ (see Equation (20)). The straight line connecting the points $(0, 1)$ and $(1, 0)$ is the graph of the function $\ell \mapsto 1 - \ell$. Its unique intersection with OS occurs at the value $\ell = \bar{L}$ (see Equation (21)). Now fix any $L \in \langle L^*, \bar{L}\rangle$ and choose a sufficiently small non-negative value ℓ_a. Find the point on OS with $\ell = L$ (point X) and the point on the parabolic curve with $\ell = \ell_a$ (point A). The intersection of the parabola and the straight line through A and X to the right of X is called B. It is straightforward to verify that point B exists and has a ℓ-coordinate smaller than 1 if $\ell_a < \bar{\ell}_a(L) := 1 - [(1-L)/(\xi L)]^{1/(1-\theta)}$. Point B's uniqueness is ensured by the parabolic shape of the curve. We denote the ℓ-coordinate of B by ℓ_b. Finally define $\lambda = (L - \ell_a)/(\ell_b - \ell_a)$. This construction ensures that all conditions stated in the lemma are satisfied. Because $\bar{\ell}_a(L) > 0$ whenever $L \in \langle L^*, \bar{L}\rangle$, there exists a continuum of triples $(\ell_a, \ell_b, \lambda)$ with the desired properties. □

We are now ready to characterize the set of stationary equilibria.

Theorem 1. *Assume $\theta \neq 1$.*
(i) There exists a unique homogeneous stationary equilibrium. In this equilibrium, per-capita (and aggregate) labor supply is given by L^*.[9]
(ii) Every stationary equilibrium in which the per-capita labor supply is equal to L^* is equivalent (in the sense of Definition 2) to the homogeneous stationary equilibrium.

[9] Per-capita capital and per-capita output can be computed from the results stated in Lemma 3(i).

(iii) For every $L \in \langle L^*, \overline{L}\rangle$ there exists a continuum of non-equivalent stationary equilibria in which per-capita labor supply is equal to L.[9]
(iv) There does not exist a stationary equilibrium in which per-capita labor supply satisfies $L \notin \langle L^*, \overline{L}\rangle \cup \{L^*\}$.

Proof. (i) Homogeneity of an equilibrium implies that $\ell_i(t) = \ell_j(t)$ for all $i, j \in I$ and, because of (15), $\nu_i = \nu_j$ for all $i, j \in I$. Thus, ν must be constant which implies $B(\nu) = 1$. Stationarity of an equilibrium implies $\dot{L}(t) = 0$ and $K(t)/L(t) = [\alpha/(\delta + \rho)]^{1/(1-\alpha)}$; see Lemma 3(i). Together with (11) these conditions imply that $L(t)$ must satisfy equation (20) so that $L(t) = L^*$ for all t. The values of the other variables can easily be computed from Lemma 3 and they satisfy all the conditions in Lemma 2. This proves statement (i).

(ii) Stationarity implies $\dot{L}(t) = 0$ and $K(t)/L(t) = [\alpha/(\delta + \rho)]^{1/(1-\alpha)}$. Substituting these two properties as well as $L(t) = L^*$ into (11) it follows that $B(\nu) = 1$. Because $\theta \neq 1$ this implies that ν coincides with a constant function almost everywhere. It is easy to see that this can be the case only if the equilibrium is equivalent to the stationary homogeneous equilibrium.

(iii) Define the function $\nu : I \mapsto (0, +\infty)$ by

$$\nu_i = \begin{cases} 1 & \text{if } i \in [0, 1-\lambda), \\ (1-\ell_b)/(1-\ell_a) & \text{if } i \in [1-\lambda, 1], \end{cases}$$

where $(\ell_a, \ell_b, \lambda)$ is specified in Lemma 4. Because of the properties stated in Lemma 4 we have

$$B(\nu) = \frac{(1-\lambda)(1-\ell_a)^\theta + \lambda(1-\ell_b)^\theta}{[(1-\lambda)(1-\ell_a) + \lambda(1-\ell_b)]^\theta} = \frac{\xi L}{(1-L)^\theta}.$$

Using this value for $B(\nu)$ one can easily verify that $L(t) = L$ and $K(t) = L[\alpha/(\delta + \rho)]^{1/(1-\alpha)}$ are constant solutions of (10) - (11). It is also straightforward to verify that condition (16) holds. The values of the remaining variables can be computed from (15) and Lemma 3 and they satisfy all equilibrium conditions stated in Lemma 2.

(iv) Assume there exists an equilibrium with per-capita labor supply L. Integrating (17) over $i \in I$ and using (5) one obtains

$$\begin{aligned} K &= \frac{1}{\rho}\left[\left(\frac{1-\alpha}{\beta}\right)^\theta \left(\frac{\alpha}{\delta+\rho}\right)^{\alpha\theta/(1-\alpha)} \right. \\ &\quad \left. \times \int_0^1 (1-\ell_i)^\theta \,\mathrm{d}i - (1-\alpha)\left(\frac{\alpha}{\delta+\rho}\right)^{\alpha/(1-\alpha)} L\right]. \end{aligned}$$

Substituting this expression for K into $K/L = [\alpha/(\delta + \rho)]^{1/(1-\alpha)}$ from Lemma 3(i) yields after simplifications $\int_0^1 (1-\ell_i)^\theta \,\mathrm{d}i = \xi L$. This condition and (5) imply that the point $(L, \xi L)$ must be contained in the convex hull of the set $\Gamma = \{(\ell, (1-\ell)^\theta) \,|\, 0 \leq \ell < 1\}$. Note that Γ is the graph of the parabolic curve in Figure 1. Thus, $(L, \xi L)$ must be lying in the intersection of the line OS and the convex hull of the parabola. This implies obviously that $L \in \langle L^*, \bar{L}\rangle \cup \{L^*\}$ and the proof is complete. □

A consequence of Theorem 1 is that the model does not predict a unique value for the long-run per-capita output if we allow for non-homogeneous wealth distributions. Note that by Lemma 3(i) $Y = L[\alpha/(\delta + \rho)]^{\alpha/(1-\alpha)}$ such that the range of per-capita output levels in stationary equilibria is the interval between $L^*[\alpha/(\delta+\rho)]^{\alpha/(1-\alpha)}$ and $\bar{L}[\alpha/(\delta+\rho)]^{\alpha/(1-\alpha)}$. Therefore, differences between per-capita output levels in different countries can (at least partly) be explained by different wealth distributions. It is important to emphasize that we have assumed that there are neither production externalities, nor increasing returns to scale on the aggregate level, nor imperfect competition, nor any market distortions. Theorem 1 therefore shows that none of these features (often assumed in the new growth theories mentioned in Section 1) is necessary to explain different GDP levels, if households have different initial wealth stocks and a labor/leisure choice.

Example 1. To get an idea of how much variance in per-capita output levels can be explained by the wealth distribution, we consider a simple numerical example. Let us choose $\alpha = 1/3$, $\delta = 3/100$, $\rho = 2/100$, and $\underline{\beta}$ such that $L^* = 1/3$.[10] Because of (20) this yields $\xi = 3(2/3)^\theta$ and, hence, $\bar{L} = 1/[1 + 3(2/3)^\theta]$. For example, if $\theta = 1/2$, then the range of per-capita output levels in stationary equilibria for these parameters is $Y \in (0.748513, 0.860663]$. If $\theta = 2$, the range of per-capita output levels is $Y \in [0.860663, 1.10657)$. In both cases the variance in per-capita output caused by different wealth distributions is substantial. For values of θ closer to 1, the variance becomes smaller and vanishes completely if $\theta = 1$ (because $\bar{L} = L^*$ in this case).

At first sight, the existence of non-homogeneous stationary equilibria may seem to be counterintuitive. After all, all households face exactly the same dynamic optimization problem, the same interest rate, and the same wage rate. They differ only with respect to their initial wealth. How can it be that they find it optimal to hold different wealth levels also in the long run? To understand this seemingly paradoxical result, consider first the case of homogeneous equilibria. From the familiar phase diagram in capital-consumption space we know that optimally behaving households increase consumption and wealth (capital stock) if the interest rate $r(t) - \delta$ exceeds the time preference rate ρ, and they decrease consumption and wealth in the case $r(t) - \delta < \rho$ (c.f. the Euler equation (9)). In the borderline case where the interest rate coincides with the time preference rate, both consumption and wealth should optimally be kept constant. Note, however, that this borderline case is exactly the situation that prevails in any stationary equilibrium, also in a non-homogeneous one. Theorem 1(iii) demonstrates that there may be situations (in fact, infinitely many of them) in which different households have different wealth levels and find it optimal to keep their wealth at those levels because they face an interest rate equal to their time preference

[10] These parameter values are typically used in calibrations of real-business-cycle models.

rate. So far, this intuitive explanation does not involve the endogenous labor supply, and it is indeed true that there exists a continuum of stationary equilibria in the model with fixed (exogenous) labor supply, too. However, if the labor supply is exogenous, all these stationary equilibria give rise to the same per-capita output, whereas this is not true in the situation considered in this paper. To see this, note that the output-labor ratio is the same in all stationary equilibria. If the labor supply is fixed, the aggregate output level in a stationary equilibrium is therefore uniquely determined. Theorem 1 shows that this result breaks down if the labor supply is endogenous. In that case the same output-labor ratio can occur for different output levels.

4 Stationary income distributions

It follows from Theorem 1 that the homogeneous stationary equilibrium is the stationary equilibrium with the lowest per-capita output if $\theta \in (1, +\infty)$, and that it is the stationary equilibrium with the highest per-capita output if $\theta \in (0, 1)$. This seems to indicate that the model predicts a positive relation between income dispersion (inequality) and per-capita output if the elasticity of intertemporal substitution is large, and a negative relation between income dispersion and output if θ is small. In the remainder of this section we explore this question in more detail.

There are various ways to measure income inequality or the dispersion of the income distribution. By income distribution we mean the distribution of the function $y : I \mapsto (0, +\infty)$ with values y_i given by (18).[11] The most widely used measure of income inequality is the Gini coefficient. The present model does not predict an unambiguous relation between the Gini coefficient G and per-capita output. Since in all stationary equilibria, per-capita output is proportional to per-capita labor supply (with the factor of proportionality not depending on the particular equilibrium), such a relation could be written in the form $L \mapsto G(L)$. One can show that G is multivalued, that is to a given level of per capita labor supply (or, per capita output) there exist many different stationary equilibria with different income Gini coefficients. It is possible, however, to determine the limits of the Gini coefficients if L approaches its extreme values, i.e., the boundaries of the interval $\langle L^*, \bar{L} \rangle$. This is done in the following theorem.

Theorem 2. *Assume $\theta \neq 1$. Consider any sequence of stationary equilibria $(E^{(n)})_{n=0}^{+\infty}$ such that the corresponding sequence of labor supplies is $(L^{(n)})_{n=0}^{+\infty}$ and the corresponding sequence of Gini coefficients of the income distribution is $(G^{(n)})_{n=0}^{+\infty}$.*
(i) If $\lim_{n\to+\infty} L^{(n)} = L^*$ then $\lim_{n\to+\infty} G^{(n)} = 0$.
(ii) If $\lim_{n\to+\infty} L^{(n)} = \bar{L}$ then $\lim_{n\to+\infty} G^{(n)} = \bar{L}$.

Proof. (i) It can be seen from Figure 1 that, if $L^{(n)} \to L^*$, then the labor supply distribution in $E^{(n)}$ converges to the degenerate distribution in which all households

[11] Analogously, we shall talk about the labor supply distribution $\ell : I \mapsto [0, 1)$ or the wealth distribution $k : I \mapsto \mathbb{R}$.

provide L^* units of labor per period. Because of (18), the income distribution converges also to a degenerate distribution. Since the Gini coefficient depends continuously on the underlying distribution and is obviously equal to 0 if the distribution is degenerate, statement (i) follows.

(ii) Appealing again to Figure 1, convergence of $L^{(n)}$ to $\bar{L}$ implies that the labor supply distribution in $E^{(n)}$ converges to a distribution in which a fraction $1-\lambda$ of households provides no labor at all ($\ell_i = 0$) and the remaining fraction λ provides the maximal amount of labor ($\ell_i = 1$). Since for this limit distribution $\int_0^1 \ell_i \, \mathrm{d}i = L = \bar{L}$ must hold, it follows that $\lambda = \bar{L}$. In the corresponding income distribution $1-\lambda$ households earn $[(1-\alpha)/\beta]^\theta[\alpha/(\delta+\rho)]^{\alpha\theta/(1-\alpha)}$ and the remaining λ households earn 0 (see (18)). Using (21) it is straightforward to see that the Gini coefficient of this limit distribution is $\bar{L}$. □

If $L \neq L^*$ then the labor supply distribution in a stationary equilibrium with aggregate labor supply L cannot be degenerate and, therefore, neither can the income distribution. It follows that the Gini coefficient of the income distribution in such an equilibrium must be positive. This means that there is income inequality. Theorem 2 shows that, as the stationary labor supply converges to its extreme value $\bar{L}$, the Gini index of the income distribution in any corresponding stationary equilibrium must also converge to $\bar{L}$. If the stationary labor supply converges to its other extreme, namely L^*, the income distribution becomes more equal in the sense that its Gini coefficient converges to 0. A complete characterization of the set of all Gini coefficients that can occur in stationary equilibria with a given per capita labor supply L seems to be quite difficult. Figure 2 shows the graph of the multivalued mapping $L \mapsto G_2(L)$, where $G_2(L)$ is defined as the set of all income Gini coefficients in stationary equilibria with the following two properties: (i) the per capita labor supply is L and (ii) the income distribution is a Bernoulli distribution. Property (ii) means that there is a number $\lambda \in [0, 1]$ and two values y_a and y_b such that the measure of the set $\{i \in I \,|\, y_i = y_a\}$ is $1-\lambda$ and the measure of the set $\{i \in I \,|\, y_i = y_b\}$ is λ. In other words, all households earn either y_a or y_b whereby the fraction of households earning y_b is λ. Figure 2a shows the graph of G_2 over $L \in \langle L^*, \bar{L}\rangle$ if the parameters are as in Example 1 and $\theta = 1/2$, whereas Figure 2b shows the graph of G_2 for the same parameter values but $\theta = 2$. Figure 2(b) shows in particular that the Gini coefficient is not monotonically related to the stationary labor supply. There may exist stationary equilibria with $L \in \langle L^*, \bar{L}\rangle$ and a Gini coefficient strictly larger than $\bar{L}$. Since the stationary equilibria with a Bernoulli income distribution form a subset of all stationary equilibria, the graph of G_2 must be contained in the graph of G. It follows from Theorem 2 that the set of possible Gini coefficients of the income distribution in stationary equilibria contains the interval $[0, \bar{L}]$. Under the parameter specification of Example 1 this is the interval $[0, 0.289898]$ if $\theta = 1/2$ and the interval $[0, 0.428571]$ if $\theta = 2$.

Income distributions can also be ranked according to second order stochastic dominance, whereby it is useful to normalize the distributions such that they have the same expected value. Thus, if $y : I \mapsto (0, +\infty)$ is an income distribution, we denote by $\tilde{y} : I \mapsto (0, +\infty)$ its normalization defined by

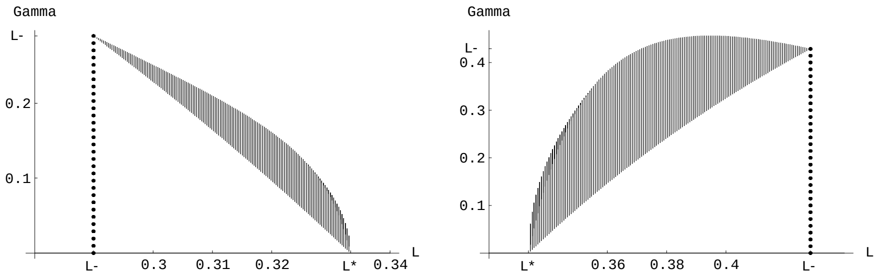


Figure 2. **a** The graph of Γ_2 when $\theta = 1/2$. **b** The graph of Γ_2 when $\theta = 2$

$$\tilde{y}_i = \frac{y_i}{\int_0^1 y_j \,\mathrm{d}j}.$$

The following result proves that, when the (normalized) income distribution in a stationary equilibrium $E^{(1)}$ dominates that of $E^{(2)}$ in the sense of second order stochastic dominance, then the per-capita output levels $Y^{(1)}$ and $Y^{(2)}$ must satisfy $Y^{(1)} \in \langle Y^*, Y^{(2)} \rangle$. Here, Y^* denotes the per-capita output in the homogeneous stationary equilibrium and $\langle Y^*, Y^{(2)} \rangle$ is the interval of values between Y^* and $Y^{(2)}$.

Theorem 3. *Consider two stationary equilibria $E^{(1)}$ and $E^{(2)}$ and let $\tilde{y}^{(1)}$ and $\tilde{y}^{(2)}$ be the corresponding normalized income distributions. Analogously, let $L^{(1)}$ and $L^{(2)}$ be the per-capita labor supplies in the two equilibria. Assume that $\tilde{y}^{(1)}$ dominates $\tilde{y}^{(2)}$ in the sense of second order stochastic dominance. If $\theta \in (0,1)$ then $L^{(2)} \le L^{(1)} \le L^*$, if $\theta \in (1,+\infty)$ then $L^* \le L^{(1)} \le L^{(2)}$. These inequalities hold strictly if the stochastic dominance relation is a strict one.*

Proof. We first show that in every stationary equilibrium

$$\int_0^1 \tilde{y}_i^{1/\theta} \,\mathrm{d}i = \frac{1-L}{(\xi L)^{1/\theta}}. \tag{22}$$

To see this, note that (15) and (18) imply $y_i = D\nu_i^\theta$ where the constant D is independent of i. This implies

$$\int_0^1 \tilde{y}_i^{1/\theta} \,\mathrm{d}i = \frac{\int_0^1 y_i^{1/\theta} \,\mathrm{d}i}{\left(\int_0^1 y_i \,\mathrm{d}i\right)^{1/\theta}} = \frac{\int_0^1 \nu_i \,\mathrm{d}i}{\left(\int_0^1 \nu_i^\theta \,\mathrm{d}i\right)^{1/\theta}} = B(\nu)^{-1/\theta}.$$

Combining this equation with (19) yields (22). If $\tilde{y}^{(1)}$ dominates $\tilde{y}^{(2)}$ in the sense of second order stochastic dominance, then the expected value of $W(\tilde{y}^{(1)})$ must be smaller than or equal to the expected value of $W(\tilde{y}^{(2)})$ for every convex function W. Now assume $\theta \in (0,1)$. In this case the function $W(y) = y^{1/\theta}$ is strictly convex and it follows therefore from (22) that $(1-L^{(1)})/(\xi L^{(1)})^{1/\theta} \le (1-L^{(2)})/(\xi L^{(2)})^{1/\theta}$. Since the function $L \mapsto (1-L)/(\xi L)^{1/\theta}$ is strictly decreasing we have proved the theorem in the case $\theta \in (0,1)$. The case $\theta \in (1,+\infty)$ can be proved in an analogous way. □

5 Dynamics

So far we have focussed on stationary equilibria. In this section we derive a few results concerning the transitional dynamics of non-stationary equilibria. We begin by analysing the dynamics of the aggregate variables $K(t)$ and $L(t)$, which are described by the two differential equations (10) and (11).

Lemma 5. *For any given measurable function* $\nu : I \mapsto (0, +\infty)$ *the system (10) - (11) has a unique fixed point* (K_ν, L_ν) *such that* $K_\nu > 0$ *and* $L_\nu > 0$*. This fixed point is a saddle point.*

Proof. From Lemma 3(i) we know that, if (K_ν, L_ν) is a fixed point, then $K_\nu/L_\nu = [\alpha/(\delta+\rho)]^{1/(1-\alpha)}$ must hold independently of ν. Substituting this into (11) it is easily found that there exist unique steady state values $K_\nu > 0$ and $L_\nu > 0$. To prove the second statement of the lemma, let us denote by

$$J(\nu) = \begin{pmatrix} J_{KK}(\nu) & J_{KL}(\nu) \\ J_{LK}(\nu) & J_{LL}(\nu) \end{pmatrix} = \begin{pmatrix} \mathrm{d}\dot{K}(t)/\mathrm{d}K(t) & \mathrm{d}\dot{K}(t)/\mathrm{d}L(t) \\ \mathrm{d}\dot{L}(t)/\mathrm{d}K(t) & \mathrm{d}\dot{L}(t)/\mathrm{d}L(t) \end{pmatrix}\Bigg|_{K(t)=K_\nu, L(t)=L_\nu}$$

the Jacobian matrix of system (10) - (11) evaluated at $K(t) = K_\nu$ and $L(t) = L_\nu$. We have

$$\begin{aligned} J_{KK}(\nu) &= \rho(1-\theta) - (1-\alpha)\delta\theta, \\ J_{KL}(\nu) &= \left(\frac{\alpha}{\delta+\rho}\right)^{\alpha/(1-\alpha)} \left\{1 - \alpha + \theta\frac{[(1-\alpha)\delta+\rho][L_\nu + \alpha(1-L_\nu)]}{(\delta+\rho)(1-L_\nu)}\right\}, \\ J_{LK}(\nu) &= (1-\alpha\theta)[(1-\alpha)\delta+\rho]\left(\frac{\alpha}{\delta+\rho}\right)^{-1/(1-\alpha)} \frac{1-L_\nu}{L_\nu+\alpha(1-L_\nu)}, \\ J_{LL}(\nu) &= \theta[(1-\alpha)\delta+\rho]. \end{aligned}$$

The trace of $J(\nu)$ is ρ and the determinant is

$$\mathrm{Det}\, J(\nu) = -\frac{(1-\alpha)(\delta+\rho)[(1-\alpha)\delta+\rho][1-L_\nu(1-\theta)]}{\alpha[L_\nu+\alpha(1-L_\nu)]}.$$

Since $\theta > 0$ and $L_\nu < 1$ it follows immediately that $\mathrm{Det}\, J(\nu) < 0$. Obviously, this implies that the Jacobian $J(\nu)$ has one negative and one positive eigenvalue such that (K_ν, L_ν) is a saddlepoint. □

Lemma 5 shows that for every given function ν and every initial stock of aggregate wealth $K_0 = K(0)$ there exists a unique initial value $L_0 = L(0)$ such that the trajectory of system (10) - (11) which starts in (K_0, L_0) converges to (K_ν, L_ν).[12] Note, however, that the function ν is also endogenously determined by the model. The following lemma is a mirror image of Lemma 5 in the sense that it takes a path $(K(\cdot), L(\cdot))$ as given and shows that there exists a unique

[12] Actually, Lemma 5 implies this property only locally, that is for initial stocks K_0 sufficiently close to K_ν. One could prove the global saddlepoint stability of the system by carrying out a phase diagram analysis.

function ν such that the equilibrium conditions (12) and (4) hold. This result requires that the initial capital stocks k_{i0} are sufficiently large for all $i \in I$.[13]

Lemma 6. *Assume that the initial wealth k_{i0} is sufficiently large for all $i \in I$. Let $(K(\cdot), L(\cdot))$ be any given pair of positive functions such that $L(t) < 1$ holds for all t, $\lim_{t\to+\infty} L(t)$ exists, and $\lim_{t\to+\infty} K(t)/L(t) = [\alpha/(\delta+\rho)]^{1/(1-\alpha)}$. Then there exists a unique measurable function $\nu : I \mapsto (0, +\infty)$ such that conditions (12) and (4) as well as the normalization $\int_0^1 \nu_i \, \mathrm{d}i = 1$ are satisfied.*

Proof. We define $\tilde{\nu}_i : I \mapsto (0, +\infty)$ by $\tilde{\nu}_i = \nu_i / \int_0^1 \nu_j \, \mathrm{d}j$. For a given trajectory $(K(\cdot), L(\cdot))$, equation (12) is a linear differential equation of the form $\dot{k}_i(t) = D(t)k_i(t) + E(t, \tilde{\nu}_i)$. From the assumptions of the lemma it follows that $\lim_{t\to\infty} D(t) = \rho$ and that $E(t, \tilde{\nu}_i)$ is strictly decreasing with respect to $\tilde{\nu}_i$. If $\tilde{\nu}_i$ is close to 0, then the unique solution of (12) diverges exponentially fast to $+\infty$. The rate of divergence is asymptotically equal to ρ. If $\tilde{\nu}_i$ is sufficiently large, then the unique solution of (12) diverges exponentially fast to $-\infty$ and the rate of divergence is asymptotically also equal to ρ. None of these solutions can satisfy the no-Ponzi-game condition (4). By continuity and by the monotonicity of $E(t, \tilde{\nu}_i)$ with respect to $\tilde{\nu}_i$ there exists a unique value $\tilde{\nu}_i \in (0, +\infty)$ such that $k_i(t)$ defined by (12) has a finite limit. This defines a unique function $\tilde{\nu} : I \mapsto (0, +\infty)$. Measurability of $\tilde{\nu}$ follows from the measurability of the initial wealth endowment k_{i0} with respect to i. Thus, the lemma is proved. □

Let $\mathscr{N}$ denote the set of positive and measurable functions defined on I which have the porperty $\int_0^1 \nu_i \, \mathrm{d}i = 1$. The above results can be used to define an operator $T : \mathscr{N} \mapsto \mathscr{N}$ in the following way. Starting from a given function $\nu \in \mathscr{N}$ one uses Lemma 5 to define the trajectory $(K(\cdot), L(\cdot))$ as the saddlepoint path converging to (K_ν, L_ν). The unique element of $\mathscr{N}$ defined by this trajectory via Lemma 6 is $T\nu$. Every fixed point of the operator T defines a equilibrium of the economy. Existence and uniqueness (determinacy) of an equilibrium could therefore be studied by investigating existence and uniqueness of a fixed point of the operator T. Instead of elaborating on this point, we now present a result which shows that, along every equilibrium, the ordering of the households, which is determined by their wealth level, is constant. In other words, in this model the rich stay rich and the poor stay poor (both in relative terms).

Theorem 4. *Consider any pair of households $(i, j) \in I \times I$. If $k_{i0} > k_{j0}$ then $k_i(t) > k_j(t)$ for all $t \in [0, +\infty)$.*

Proof. Assume the contrary. Then there must exist $t_0 \in (0, +\infty)$ such that $k_i(t_0) = k_j(t_0)$. Since households i and j have the same wealth at time t_0 and since they are identical in terms of their preferences, they must behave identically from time t_0 onwards. From (15) follows that $\nu_i = \nu_j$ must hold. But then the two households'

[13] The reason for this additional requirement is that there may not exist an equilibrium if a household is too indebted. For example, if one household has a large initial debt whereas all other households have positive initial wealth then the indebted household may not be able to satisfy the no-Ponzi-game condition (4) even if it would not consume anything at all.

behavior is identical not only from t_0 onwards but for all $t \in [0, +\infty)$. This is a contradiction to the assumption that they have different initial wealth levels. □

Despite the above result it is difficult to determine the evolution of the income (or wealth) distribution over time. In particular we were not able to derive interesting results on the wealth distribution similar to those in Chatterjee [13].

6 Concluding remarks

In this paper we have demonstrated that in a simple one-sector growth model with endogenous labor supply there is a relation between the distribution of income (or wealth) and the level of per-capita output. Income inequality goes along with higher output levels if the elasticity of intertemporal substitution is larger than 1, and it goes along with lower income inequality if this elasticity is smaller than 1. Therefore, countries with identical production technologies and identical preferences can have different per-capita output levels if wealth is distributed differently in each country. This is a result which, to our best knowledge, has not yet been proved in this framework.

Whereas the specific features of our results depend on the particular functional forms chosen in our analysis, one can replicate the general features also in other settings. In particular it seems that neither the assumption of a Cobb-Douglas technology nor the specific form of the utility function matters much for the existence of a continuum of stationary equilibria with different per-capita output levels. The dynamic properties of equilibria for more general production and utility functions, however, are likely to change considerably. This can be expected from the results in de Hek [14].

One parametric example for which the analysis of the present paper can easily be repeated is the case where the utility function has the form[14]

$$U(c,\ell) = \begin{cases} \dfrac{c^{1-1/\theta}-1}{1-1/\theta} - \dfrac{\ell^{1+\beta}}{1+\beta} & \text{if } \theta \in (0,1) \cup (1,+\infty), \\ \ln c - \dfrac{\ell^{1+\beta}}{1+\beta} & \text{if } \theta = 1. \end{cases} \tag{23}$$

In this case labor has a slightly different effect on utility than in our model. Assuming that preferences are described by (23) only a positive relation between income inequality and per-capita output can be derived. That is, the results for this utility function resemble those for the case $\theta \in (1, +\infty)$ in our analysis. This follows from the fact that, independently of θ, the consumption rate of a household is in equilibrium always a strictly convex function of its labor supply.

In the economy described in this paper the marginal productivity of capital goes to 0 as the capital-labor ratio approaches infinity. Since there are no other produced input factors, permanent growth of per-capita output at a positive rate is not possible. One can, however, easily extend the model to include endogenous growth. The simplest way to do this is to introduce technological progress via

[14] This utility function (with $\theta = 1$) has been used for example by Benhabib and Farmer [8].

learning-by-investing in the sense of Arrow [4] and Romer [25]. In such an extension one would naturally be interested in balanced growth paths, i.e. equilibria along which all endogenous variables grow at constant (but not necessarily identical) rates. In the parametrization that was chosen in the present paper, balanced growth can only occur if $\theta = 1$, which is exactly the borderline case between a positive and a negative relation between income inequality and output. Using the utility function (23), balanced growth can also only occur if $\theta = 1$ but, in that case, there is a positive relation between income inequality and growth of per-capita output. More specifically, there exists a continuum of balanced growth paths with different growth rates of per-capita output. The relation between the equilibrium *growth rates* and the equilibrium income distribution resembles the relation between equilibrium *output levels* and the equilibrium income distribution described in Theorems 2 and 3 above for the case $\theta \in (1, +\infty)$. Details of this analysis can be found in Sorger [26].

It is also possible to include a government in the model of the present paper and to study issues concerning taxation and redistribution. In Sorger [27] the government levies a proportional income tax and distributes the entire tax revenue to the households in the form of lump-sum transfers. The government's budget is balanced in every point of time. The qualitative properties of the relation between the per-capita output level and inequality are not changed by this modification. Moreover, it can be shown that in the case of a positive relation between output and inequality ($\theta > 1$), increasing the tax rate cannot reduce income inequality without reducing per-capita output at the same time. For the case $\theta < 1$, the situation is more complicated and the possibility of growth-enhancing and inequality-reducing tax policies depends in a non-trivial way on the model parameters.

The model we have used is the deterministic, continuous-time version of a standard real-business-cycle model (see, e.g. Romer [24, Chapter 4]). Our results suggests that real-business-cycle models with heterogeneous households might have solutions which differ in a substantial way from the solutions in the homogeneous household case. The business cycle characteristics of income distributions have only recently been studied by Castañeda et al. [12], whereby their theoretical models do not allow for endogenous labor supply. We think that the generalization of this approach to a model with endogenous labor supply is an interesting topic for future research.

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