Economic Theory **15**, 551–564 (2000)

**Economic Theory**


# Saving behavior in stationary equilibrium with random discounting*

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Received: December 28, 1998; revised version: May 19, 1999

**Summary.** We study the implications of random discount rates of future generations for saving behavior and capital holdings in a steady state competitive equilibrium with heterogeneous population. A well-known difficulty in deterministic economies with heterogeneous households is that in steady state only the most patient households hold capital. In this paper we state conditions under which this random discounting is sufficient for households other than the most patient ones to save. We thus provide a simple and natural way of overcoming the aforementioned difficulty.

**Keywords and Phrases:** Dynamic equilibrium with heterogeneous households, Random discounting, Saving.

**JEL Classification Numbers:** E13, E20.

## 1 Introduction

The analysis of infinite-horizon, deterministic, dynamic models in which individuals face borrowing constraints shows that in steady-state equilibrium only the most patient households hold capital. All other households have wages as their only source of income (Becker, 1980). This conclusion is contradicted by the most casual observations. To overcome this difficulty Becker and Zilcha (1997) analyze the saving behavior and the ownership of capital in an economy with

* Work on this paper began during a visit of the second author to the Department of Economics, Johns Hopkins University and was completed during a visit of the first author to the Berglas School of Economics, Tel Aviv University. We are grateful to both institutions for their hospitality and to an anonymous referee for useful comments.
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stochastic aggregate production. The random shocks to aggregate output imply that interest rate, the wage rate and, consequently, income are random variables. They show that, in stationary equilibrium, precautionary savings induces capital holdings by households other than the most patient.

In the present paper we examine the steady-state saving behavior and the ownership of capital when the rates of discount implicit in the preferences of future generations are random. We assume that these idiosyncratic variations in tastes average out in the aggregate so that the aggregate capital stock, output, wages and interest rates are *nonrandom.* We show that, in general, random variation in tastes is sufficient to assure that, in stationary equilibria, each generation's savings and asset holding are not restricted to households with the highest discount factor. We thus provide a way of overcoming the difficulty posed by the analysis of the deterministic model which is both simple and natural.

The apparent shortcomings of representative agent models has increased, in recent years, the interest in dynamic models with heterogenous agents (e.g., Kirman, 1992). For example, in macroeconomic models, the interest in stochastic heterogenous households economies stems from the persistent inconsistencies between the solutions attained in complete markets representative agent stochastic equilibria (based on the Brock-Mirman, 1972, framework) and the observations from macroeconomic and individual consumption data (Aiyagari, 1994, 1995). Incorporating precautionary savings and liquidity constraints into a life cycle model results in a framework which is more compatible with the existing empirical evidence (Deaton, 1992). We see the contribution of the present paper as a further step in this direction. In particular, our analysis emphasizes the importance of the variability of the rate of time preference, especially when we consider future generations (see Laibson, 1997; Schelling, 1995; Weitzman, 1994, 1998) and its impact on the economy.

Our main idea is rather simple. In steady-state equilibrium the aggregate capital stock is constant. However, in each generation there are individuals that inherit capital and whose rate of time preference is sufficiently high that they dissave. Therefore, to sustain the steady-state equilibrium, it is necessary that individuals whose rate of time preference is low save and that the resulting capital accumulation is sufficient to make up the decumulation of the capital stock by individuals of the first type. If the set of individuals with the lowest rate of time preference (the most patient households) is small they are unable, by themselves, to create sufficient additional capital to maintain the steady-state equilibrium. This implies that the interest rate must reach a level that will induce positive levels of saving by individuals whose rate of time preference is not the lowest. This logic implies that, in general, positive saving should take place among individuals with different rates of time preference. In addition, the stochastic process itself implies that, in each generation, capital is held by all types of individuals and that it is possible that individuals that have the same rate of time preference display different saving behavior. More specifically, given a rate of time preference, it is possible that individuals who inherit large fortunes dissave while individuals who inherit small fortunes save. We also show that the differences among discount factors play a role in determining which individuals save and which individuals dissave in a stationary equilibrium.

We would like to emphasize that our objective is limited to developing sufficient conditions for the existence of saving in steady-state equilibrium among individuals with different discount rates. Clearly, the actual process of capital accumulation is more complicated and involves additional factors not modeled in this paper.

The remainder of paper is organized as follows: The model appears in Section 2. In Section 3 we discuss the saving behavior of households who differ in their current rate of time preference in stochastic stationary equilibria. Concluding remarks appear in Section 4. The Appendix contains the proofs.

## 2 The model

### *2.1 Preliminaries*

Consider an infinite horizon, discrete-time, one-good economy consisting of a sequence of generations and suppose that in every generation each individual belongs to a distinct dynasty. Individuals in this economy live for one period during which they work, consume, and save. At the end of the period each individual's saving is bequeathed to a (single) offspring. There is continuum of dynasties represented by the unit interval $[0,1]$. Denote by $G_t$ the set of all the individuals living in period $t$ and let $\omega \in [0,1]$ denote the name of a dynasty. Since there is no population growth, $G_t$ is time invariant and may be identified with $[0,1]$. Generation $t$'s member of dynasty $\omega$ is denoted $\omega \in G_t$.

In every generation individual preference relations are defined on the infinite "downstream" consumption corresponding to their dynasties. To formalize this idea assume that, for each $\omega \in G_t$, the discount factor takes one of a finite number of values, $\{\delta_1, \delta_2, \ldots, \delta_n\} \equiv D$, where $1 > \delta_1 > \delta_2 > \ \ldots\ > \delta_n > 0$ and $n > 2$. Let $\Delta = \Pi_{-\infty}^{\infty} D_t$, where $D_t = D$ for all $t$. Elements of $\Delta$, denoted by $\delta(\omega)$, $\omega \in [0,1]$, describe the dynastic evolutionary path of the discount factors and $\delta_t(\omega) \in D$, the $t$th coordinate of $\delta(\omega)$, is the discount factor of $\omega \in G_t$. We assume that $\delta(\omega)$ is a sequence of stochastically independent and identically distributed random variables. Formally, the random variables $\{\delta_t(\omega)\}_{t=-\infty}^{\infty}$ are distributed like a random variable $\tilde{\delta}$, for all $t$ and $\omega \in G_t$. For each $\delta(\omega) \in \Delta$, we denote by $\widehat{\delta}^t(\omega) = (\ldots.\delta_{t-2}(\omega), \delta_{t-1}(\omega))$ and $\delta^t(\omega) = (\delta_t(\omega), \delta_{t+1}(\omega), \ldots)$, respectively, the "history" and a "future" at time $t$ of dynasty $\omega$.

A *cylinder* in $\Delta$ is a subset of $\Delta$ of the form $C = \Pi_{-\infty}^{\infty} C_j$, where $C_j \subseteq D$ for all $j$ and $C_j = D$ except for finite number of $j$s. Let $\mathscr{F}$ be the Borel $\sigma$-field generated by the set of cylinders in $\Delta$. Denote by $\mathscr{F}_k$ the Borel $\sigma$- field generated by cylinder sets in $\Delta$ with $C_j = D$ for all $j > k$ (to be denoted by $\Delta_k$), namely all sequences $\delta(\omega)$ which depend upon histories up to date $k$ only. The distribution of $\tilde{\delta}$ induces a probability measure, $\mu$, on the Borel $\sigma$-field $\mathscr{F}$. Let $\mu_t$ be the probability measure induced by $\mu$ on ( $\Delta_t$, $\mathscr{F}_t$). Also denote by $Z_t$ the set of all $\mathscr{F}_t$−measurable functions defined on $\Delta$ into $\mathbf{R}$. Let

$C(\widehat{\delta^t}(\omega),.) \subset \Delta$ be a $\mathscr{F}-$measurable set containing all elements in $\Delta$ such that $\delta(\omega) = (\widehat{\delta^t}(\omega), ....)$. Let $T$ be a shift operator on $\Delta$ defined by: $T(\delta)_k = \delta_{k+1}$ for all $k$ and all $\delta \in \Delta$. Under our assumptions $T$ and $T^{-1}$ are measure preserving (Breiman, 1968, Proposition 6.18) and $T^{\kappa}(\hat{\delta}^t(\omega), \delta^t(\omega))_t$, $k = 1, 2, ...$, describes the evolution of the discount factors of dynasty $\omega$ from period $t$ onwards. Denote by $F$ the set of $\mathscr{F}$-measurable, real-valued functions on $\Delta$ and define $\hat{T} : F \to F$ by $\hat{T}(f)(\delta) = f(T(\delta))$ for all $f \in F$ and $\delta \in \Delta$.

Generation $t$'s member of dynasty $\omega$ is endowed with the history-dependent quantity, $b_{t-1}(\delta(\omega))$, where $b_{t-1} \in Z_{t-1}$, of a single good received from his parent and one unit of labor. The good (wheat) may be consumed or used as capital in the production process.

Given a dynastic history, $\hat{\delta}^t(\omega)$, the corresponding transfer, $b_{t-1}(\hat{\delta}^t(\omega), .)$, and the current realization of the random discount factor, $\delta_t(\omega)$, generation $t$'s member of dynasty $\omega$ chooses his current consumption $c_t(\omega)$ and, thereby, the level of saving $b_t$ which depends on $b_{t-1}(\hat{\delta}^t(\omega), .)$ and $\delta_t(\omega)$; in particular, $b_t \in Z_t$. In addition, he makes contingency consumption plans for the periods $t+1, t+2, ...$ that depend on the evolution of the discount factors in subsequent generations. To formalize this idea let $\Gamma : \mathbf{R} \to \mathbf{R}$ be a correspondence whose values, $\Gamma(b)$, describe the feasible bequests values of the next period if the current endowment is $b$. Then, given the dynastic history, $\hat{\delta}^t(\omega)$, and the corresponding bequest, $b_{t-1}(\hat{\delta}^t(\omega), \delta^t(\omega))$, a *consumption-saving plan as of period* $t$ of $\omega \in G_t$ is a sequence of $\mathscr{F}_{t+k}-$measurable functions $b_{t+k} : C(T^k(\widehat{\delta^t}(\omega), \delta^t(\omega))) \to \mathbf{R}$, $k = 1, 2, ....$ representing the intergenerational transfer that is chosen in period $t+k$ if the dynastic history from period $t$ to period $t+k$ is given by $\{T^{t+j}(\hat{\delta}^t(\omega), \delta^t(\omega))_t\}_{j=1}^k$. A consumption saving plan as of period $t$ of $\omega \in G_t$ is said to be *feasible given the bequest* $b_{t-1}(\hat{\delta}^t(\omega), .)$ if $b_{t+k}(T^k(\widehat{\delta^t}(\omega), \delta^t(\omega))) \in \Gamma(b_{t+k-1}(T^k(\widehat{\delta^t}(\omega), \delta^t(\omega))))$ for all $k = 1, 2, ...,$.

In each generation individuals choose their consumption and bequest (i.e., saving) as part of a feasible consumption-saving plan that maximizes the (expected) sum of discounted utilities from current consumption and the consumption of all forthcoming generations. In other words, each $\omega \in G_t$ evaluates the future consumption streams $(c_t, c_{t+1}, ...)$ by:

$$\sum_{\tau=t}^{\infty} \xi_{t,\tau}(\omega)\, u(c_\tau),$$

where $u(.)$ is a utility function, $\xi_{t,\tau}$ is defined for $t \leq \tau$ by $\xi_{t,\tau}(\omega) = \prod_{k=t}^{\tau-1} \delta_k(\omega)$ and $\xi_{t,t}(\omega) \equiv 1$, and $c_\tau$ denotes the consumption of $\omega \in G_\tau$. We assume throughout that $u(.)$ is monotone increasing, strictly concave and twice differentiable, and that $u'(0) = \infty$ and $u'(\infty) = 0$.

Given $b_{t-1}(\hat{\delta}^t(\omega), .)$ the set of feasible consumption streams available to $\omega \in G_t$ depends on the wage rates, $w^0 = (w_t)_{t=0}^{\infty}$, and the interest rates, $r^0 = (r_t)_{t=0}^{\infty}$. In other words, given $b_{t-1}(\hat{\delta}^t(\omega), .)$ and $\delta_t(\omega)$ individual $\omega \in G_t$ chooses the consumption-saving plan $\{b_\tau(.)\}_{\tau=t}^{\infty}$, where each $b_\tau \in Z_\tau$, so as to maximize his expected utility which is given by:

$$V\left(b_{t-1}\left(\hat{\delta}^{t}(\omega),.\right),\delta_{t}(\omega);w^{0},r^{0}\right)=$$

$$u\left(c_{t}\left(\hat{\delta}^{t}(\omega),.\right)\right)+\delta_{t}(\omega)\int_{C\left(\widehat{\delta^{t}}(\omega),.\right)}\sum_{\tau=t+1}^{\infty}\xi_{t+1,\tau}(\omega)u\left(c_{\tau}\left(\hat{\delta}^{\tau}(\omega),.\right)\right)d\mu(\delta(\omega)),$$

subject to the constraints: For all $\tau=t,t+1,t+2,...$

$$c_{\tau}\left(\hat{\delta}^{\tau}(\omega),.\right)=(1+r_{\tau})b_{\tau-1}\left(\hat{\delta}^{\tau}(\omega),.\right)+w_{\tau}-b_{\tau}\left(\hat{\delta}^{\tau+1}(\omega),.\right)\geq 0,$$

$$b_{\tau}\left(\hat{\delta}^{\tau+1}(\omega),.\right)\geq 0,\quad c_{\tau}\left(\hat{\delta}^{t}(\omega),.\right)\geq 0.$$

Note that the assumption that all individuals in each generation have the same utility function but may differ in their rate of time preference, introduced to simplify the exposition, strengthens the results below.

Firms in this economy produce a single good using capital and labor and are competitive. We take the production technology to satisfy the usual neoclassical assumptions. It is represented by a constant returns-to-scale, increasing, concave and continuously differentiable aggregate production function, $F(K,L)$, where $K$ denotes the aggregate capital stock and $L$ denotes the aggregate supply of labor. Assume that each factor is essential (i.e., $F(0,L)=F(K,0)=0$,) and that $F_K>0$, $F_L>0$, $F_{KK}<0$, $F_{LL}<0$, $F_{KL}>0$, $F_K(\infty,L)=0$, $F_K(0,L)=\infty$. The capital stock depreciates completely every period. Since the supply of labor is inelastic and there is no population growth the aggregate supply of labor, $\bar{L}$, is constant. We shall write $f(K)=F(K,1)$.

### *2.2 Consumption-saving behavior*

For the purpose of our analysis the individual optimization problem described in subsection 2.1 may be simplified due to the fact that, in each period, the relevant aspects of each dynastic history is summarized by the bequest. Thus, without risk of ambiguity we occasionally write $b_t(\omega)$ instead of $b_t(\hat{\delta}^{t+1}(\omega),.)$. Assuming that individuals are price takers the problem of choosing an optimal consumption-saving plan faced by individual $\omega\in G_0$, given the transfer received from his parent, $b_{-1}(\widehat{\delta^0}(\omega),.)$, the realization of his discount rate, $\delta_0(\omega)$, and the current and future wage rates and interest rate $(w^0,r^0)$ may be stated as follows: Choose $(c_\tau(\delta(\omega)))_{\tau=0}^{\infty}$, $c_\tau\in Z_\tau$ for all $\tau$, so as to maximize

$$u(c_0)+\delta_0(\omega)\int_{\Delta_0}\sum_{\tau=1}^{\infty}\xi_{1,\tau}(\omega)u(c_\tau)\,d\mu\left(\widehat{\delta^0}(\omega),\delta^1\right)\tag{2.1}$$

subject to

$$c_\tau=(1+r_\tau)b_{\tau-1}(\omega)+w_\tau-b_\tau(\omega)\geq 0,\tag{2.2}$$
$$b_\tau(\omega)\geq 0,\quad c_\tau(\omega)\geq 0\ ,\quad \tau=0,1,2,...$$

Assuming that $(w^0, r^0)$ are bounded (a property that holds in equilibrium under our assumptions) the solution for the optimization problem (2.2)-(2.3) exists and is unique. Henceforth we denote this solution by $(c_t^*, b_t^*)_{t=0}^{\infty}$. Note that, in the deterministic case (i.e., the discount factor is nonrandom for each dynasty) our model generalizes that of Becker (1980) and Becker and Foias (1987).

The analysis below makes use of the value function corresponding to the optimization problem (2.1)-(2.2). Given $b_{t-1}(\omega)$, $\delta_t(\omega)$ and $(w^0, r^0)$ define, for each $t \geq 0$ and $\omega \in G_t$,

$$V_t(b_{t-1}(\omega), \delta_t(\omega); w^0, r^0) = Max \left\{ \int_{\Delta_t} \sum_{\tau=t}^{\infty} \xi_{t,\tau} u(c_\tau(\omega)) d\mu(\delta(\omega)) \mid (2.2) \text{ holds for all } \tau \geq t \right\} \quad (2.3)$$

Under our assumptions on $u(.)$ $V_t$ is well-defined. Moreover it is monotone increasing and strictly concave function of $b_{t-1}$. Since $w^0$ and $r^0$ are fixed, henceforth, when there is no room for ambiguity, we suppress them and write $V_t(b_{t-1}, \delta_t)$. Given the solution to the problem (2.1)-(2.2), $(c_t^*)_{t=0}^{\infty}$, and the corresponding bequests $(b_t^*)_{t=0}^{\infty}$, we use the Bellman equation to rewrite equation (2.3) as follows:

$$V_t(b_{t-1}^*(\omega), \delta_t(\omega)) = u(c_t^*(\omega)) + \delta_t E V_{t+1}(b_t^*(\omega), \delta_{t+1}(\omega)), \quad (2.4)$$

where $b_t^*(\omega) = (1+r_t) b_{t-1}^*(\omega) + w_t - c_t^*(\omega)$ and $E$ is the time invariant expectation operator with respect to the distribution of $\delta_{t+1}(\omega)$ (which is distributed like $\tilde{\delta}$).

### *2.3 Competitive equilibrium*

Define competitive equilibrium, CE, from given initial conditions.

**Definition 1.** *Given some initial endowments,* $\{b_{-1}(\omega)\}_{\omega\in[0,1]}$, *and discount factors,* $\{\delta_0(\omega)\}_{\omega\in[0,1]}$, $\{(b_t^*(\omega), c_t^*(\omega))_{t=0}^{\infty};\ (w_t^*, r_t^*)_{t=0}^{\infty}\}$, *is a* **competitive equilibrium** *if:*

*(i)* *Given* $(w^0, r^0)$, $b_{-1}(\omega)$, *and* $\delta_0(\omega)$, $\{(b_t^*(\omega), c_t^*(\omega))\}_{t=0}^{\infty}$ *is the solution to (2.1)-(2.2) for almost all* $\omega \in G_0$. *Moreover,* $b_t^*(\omega), c_t^*(\omega) \in Z_t$ *for all t.*

*(ii)* *The aggregate capital stock at date* $t$, *denoted* $K_t$, *equals the aggregate savings at date* $t-1$, *i.e.,*

$$K_t = \int_{\Delta} b_{t-1}^*(\delta(\omega)) d\mu(\delta(\omega)) \quad t = 0, 1, 2, \ldots. \quad (2.5)$$

*(iii)* *Given the aggregate capital stock in date* $t$, $K_t$, *the interest rate* $r_t^*$ *and wage rate* $w_t^*$ *are given by the marginal product of capital and the marginal product of labor, respectively, i.e.,*

$$(1 + r_t^*) = f'(K_t), \quad t = 1, 2, ... \tag{2.6}$$

*and*

$$w_t^* = f(K_t) - K_t f'(K_t), \quad t = 1, 2, ... \tag{2.7}$$

Note that labor income is equal across individuals since they are all endowed with the same amount of labor.

A competitive equilibrium is *stationary*, or a *steady-state*, if the optimal consumption-saving plan is generated by some fixed functions, namely, for some functions $b^*$ and $c^*$ in $Z_0$, $b_t^*(\omega) = b^*(T^t(\delta(\omega)))$ and $c_t^*(\omega) = c^*(T^t(\delta(\omega)))$ for all $\omega \in [0, 1]$ and, hence, for all $t$, $w_t^* = w^*$, $r_t^* = r^*$. This definition uses the fact that the only relevant aspect of a dynastic history is the current transfer.

**Definition 2.** *A* **steady-state** *(or a* **stationary** *CE) is a pair of functions $c^*, b^*$ in $Z_0$ and a pair of numbers $(w^*, r^*)$ such that the following is a competitive equilibrium: For all $t$,* (a) $c_t^*(\delta(\omega)) = c^*(T^t(\delta(\omega)))$, (b) $b_t^*(\delta(\omega)) = b^*\left(T^t(\delta(\omega))\right)$, (c) $w_t = w^*$, *and* (d) $r_t = r^*$.

Notice that the value function in steady-state is defined as follows:

$$V\left(b^*\left(T^{-1}(\delta(\omega))\right), \delta_0(\omega); w^*, r^*\right) = E\sum_{t=0}^{\infty} \xi_{0,t}(\omega)\, u\left(c^*\left(T^t(\delta(\omega))\right)\right).$$

**Theorem 1.** *Under the above assumptions there exists a steady-state competitive equilibrium.*

The uniqueness of the steady-state equilibrium involves two arguments: The first concerns the uniqueness of the invariant distributions of bequest and consumption, and the second the uniqueness of the steady state capital stock. The uniqueness of the invariant distribution functions $b^*(.)$ and $c^*(.)$ , is guaranteed by the assumptions in the hypothesis of Theorem 1 (see the proof). The uniqueness of the steady-state capital stock requires additional restrictions. For instance, to guarantee the uniqueness of the fixed point capital stock $K$ we need to assure that the mapping describing the evolution of the capital stock is monotone nondecreasing and that some "mixture" property of this process holds as well (see, for example, Theorem 12.12 in Stokey and Lucas, 1989). Further study of this issue is beyond the scope of this paper.

## 3 Stationary saving behavior

To study the properties of stationary CE we begin by characterizing the stationary consumption behavior. Let $w^*$ and $r^*$ denote steady-state equilibrium wage and interest rates. Then, using equation (2.4), we can derive a stationary optimal consumption function. Write,

$$\begin{aligned} V(b, \delta_i;\ w^*, r^*) \quad = \quad & \max_{0 \le c \le (1+r^*)b + w^*} \left\{u(c) + \delta_i EV\left[(1 + r^*)b \right.\right. \\ & \left.\left. + w^* - c, \tilde{\delta}; w^*, r^*\right]\right\} \end{aligned} \tag{3.1}$$

Note that the right-hand side of equation (3.1) is a strictly concave function of $c$. Thus, for each given $b \geq 0$ the maximum is attained at some $c^*$ and it is unique. The optimal consumption function, $g$, is defined by:

$$c^* = g(b, \delta_i; w^*, r^*) \tag{3.2}$$

The optimal saving function, $H$, in this stationary equilibrium is defined by:

$$H(b, \delta_i \ ; \ w^*, r^*) = b(1 + r^*) + w^* - g(b, \delta_i; w^*, r^*).$$

Since $w^*$ and $r^*$ are fixed we suppress them in the sequel.

**Lemma 1.** *For a given $b > 0$ and $\delta_i \in D$, $V_1(b, \delta_i) = \frac{\partial}{\partial b} V(b, \delta_i)$ exists and*

$$V_1(b, \delta_i) = (1 + r^*)u'(g(b, \delta_i)). \tag{3.3}$$

The proof of Lemma 1 follows the same arguments as in Mirman and Zilcha (1975, Lemma 1) and is omitted.

If $c^* < (1 + r^*)b + w^*$ (i.e., the optimal level of saving is positive) then differentiating equation (3.1) and using (3.3) we obtain for $i = 1, ..., n$,

$$u'(g(b, \delta_i)) = \delta_i (1 + r^*)Eu'(g((1 + r^*)b + w^* - g(b, \delta_i), \tilde{\delta} )). \tag{3.4}$$

If $c^* = (1 + r^*)b + w^*$ (i.e., boundary solution) the equality in equation (3.4) is replaced by inequality $\geq$ .

Next we derive some properties of the optimal consumption function, $g(b, \delta_i)$, using (3.4). In particular we show that, regardless of the rate of time preference, both consumption and saving increase with the level of bequest received. Moreover, except of the case in which individuals consume their entire income, consumption is a decreasing function of the discount factor.

**Proposition 1.** *Given $(w^*, r^*)$ then, under our assumptions:*

(a) *$g(b, \delta_i)$ is strictly monotone increasing in b and $H(b, \delta_i)$ is monotone increasing in b, for all $\delta_i$.*
(b) *Unless $g(b, \delta) \equiv b(1 + r^*) + w^*$, $g(b, \delta)$ is monotone decreasing in $\delta$.*

*Remark 1.* By Proposition 1(a) $H(b, \delta_i)$ is monotone increasing in $b$ and by Proposition 1(b) it is monotonic increasing in $\delta_i$. Thus, except of the case in which individuals consume all their income, we have:

$$H(b, \delta_1) > H(b, \delta_2) \ > ... \ > \ H(b, \delta_n).$$

To present our next result we introduce the following additional notations and definitions: For each $t$, let $\left(A_{i,t}\right)_{i=1}^{n}$ be a partition of $[0, 1]$ where,

$$A_{i,t} = \{\omega \in G_t \mid \delta_t(\omega) = \delta_i\}.$$

Given a steady state $(c^*, b^*;\ w^*, r^*)$ let $\overline{b}_{i,t}$ the average transfer received by individuals of generation $t$ with $\delta_t(\omega) = \delta_i$. This is equal, for all i, (due to the i.i.d. assumption about the random discounting) to the average transfer to generation $t$, denoted $\overline{b}_t$. Similarly, let $\overline{b}_t^*$ be the average transfer made by generation $t$ to generation $t+1$ (i.e., $\overline{b}_t^* = \int H(b^*(T^t(\delta(\omega)))d\mu(\delta(\omega)))$. Next observe that, since the stochastic process $\delta_t(\omega),\ \omega \in G_t,$ is *i.i.d* across dynasties and over time, we have $\overline{b}_{i,t} = \overline{b}_t^* = \overline{b}^*$, for all $i$ and $t$.

Denote by $\hat{b}_{i,t}$ the average transfer that individuals who draw $\delta_i$ in period $t$ (i.e., members of dynasties $\omega$ belonging to $A_{i,t}$) bequeath to their offsprings in period $t$. Formally,

$$\hat{b}_{i,t} = \int_{A_{i,t}} H(b^*(\widehat{\delta}^t(\omega)), \delta_i, \delta^{t+1}(\omega))d\mu((\widehat{\delta}^{t+1}(\omega), .)). \tag{3.5}$$

Next we show that, in each generation, individuals with the highest discount factor transfer on average more than they received from their parents, and individuals with the lowest discount factor transfer less than they received from their parents. Formally, assuming that $n > 1$,

**Proposition 2.** *In a steady state* $\hat{b}_{1,t} > \overline{b}^* > \hat{b}_{n,t}$ *holds for all* $t$.

It is worth noting the differences in steady-state saving behavior between the deterministic models and the present stochastic model. In the deterministic model (e.g., Becker and Foias, 1987), given the steady state wage rate and interest rate, the saving function $b^*$, satisfies the following:

(a) $H(b^*, \delta_1) = b^*$.
(b) $H(b, \delta_i) < b$ for $b > 0$ and $H(0, \delta_i) = 0,\ i = 2, 3, ..., n$.

Proposition 2 establishes that (a) does not hold in our model. In other words, it is not true that in any given period individuals that have the highest discount factor transfer to their respective offspring the exact amount of capital they inherited from their parents. Theorem 2 below asserts that (b) may not hold either. Put differently, not all the saving in any given generation is done by members of the dynasty with the highest discount factor. The proof of this assertion is based on the fact that, in our model, the steady-state rate of interest, $r^*$, satisfies $(1 + r^*)E\tilde{\delta} > 1$. Hence, unlike in the *deterministic* model in which the equality $(1 + r^*)\delta_1 = 1$ holds in steady-state and, consequenlty, $(1 + r^*)\delta_2 < 1$, in the present model it may well be the case that $(1 + r^*)\delta_2 > 1$. Consequently, individuals in $G_t$ whose discount factor is $\delta_2$ may save. The following Lemma formalizes this feature of our model.

**Lemma 2.** *Given a steady-state equilibrium,* $(c^*, b^*; w^*, r^*)$, *we have*

$$\left(1 + r^*\right) E\tilde{\delta} > 1.$$

In addition to playing a key role in the proof of Theorem 2 the inequality in Lemma 2 is analogous to the inequality (for a representative agent without random discounting) that is used in Deaton's (1992) study of consumption behavior.[1]

**Theorem 2.** *Let* $\{\delta_i\}_{i=1}^n$, $n \geq 3$, *be the given set of discount factors. Then in each of the following cases individuals belonging to dynasties in* $A_{2,t}$ *are net savers:*

(a) *if* $\Pr\left[A_{1,t}\right]$ *is sufficiently small,*
(b) *if* $\Pr\left[A_{1,t}\right]$ *is not too large and* $\delta_2$ *is sufficiently close to* $\delta_1$.

*Remark 2.* Theorem 2 implies that, in general, asset accumulation in any given period is not confined to the most patient individuals. If $\Pr[A_{1,t} \bigcup A_{2,t}]$ is sufficiently small, additional types of individuals will be included in the population which has net positive saving. Notice that this claim is stronger than just showing that some individuals with $\delta_i,\ i \geq 2$, have positive capital holdings in steady-state equilibrium, which in our model, is an immediate implication of the random process governing the evolution of the discount factors.

## 4 Concluding remarks

In this paper we examined the implications of random rate of time preference for saving behavior and capital accumulation in steady-state competitive equilibrium. We developed a framework in which individuals face only one source of uncertainty, namely, the time preference of future generations. We showed that this is sufficient to guarantee that saving takes place among households other than the most patient ones. Our analysis is based on the assumption that the discount factors of future generations are independent and identically distributed random variables. A widely held view of human nature suggests that rich families "spoil" their children. If being spoiled is tantamount to being impatient then the random process governing the evolution of the rates of discount will have a regressive factor. High rate of discount will lead to accumulation of wealth which increases the probability of low discount rate of future generations. Thus, the observed rates of discount will display negative correlation. We conjecture that this tendency will not alter our conclusions.

The main thrust of this paper was the study of the effect of random discounting on saving behavior. In the process we established the existence of steady-state equilibrium. We did not investigate the uniqueness of this equilibrium since this issue is secondary to the main point of this paper. We believe, however, that under mild assumptions regarding consumer preferences the steady-state equilibrium is unique.

[1] We owe this observation to our referee.

## Appendix

*(A.1) Proof of Theorem 1.* We shall not spell out all the minor details in this proof. By our assumptions about the production function there exists a constant $M \in (0,\infty)$ such that $F(M,1) = M$. Hence for all $K > M$ we have $F(K,1) < K$. Consider capital stocks $K \in [0,M]$. For each $K$ in this interval there is a corresponding interest rate $r(K)$ and wage rate $w(K)$. Denote the function $H(b,\delta_i,w(K),r(K))$ by $H^K(b,\delta_i)$. We shall choose $M$ large enough such that for each $K > \frac{M}{2}$ and $b > M$ we have $H^K(b,\delta_i) < b$.

To simplify the notations, instead of defining the bequest function, $H$, on $D$ we define it directly on $[0,1]$. Choose a function $b_0(\omega)$ such that $b_0(\omega) \in [0,M]$ a.s. and $Eb_0(\omega) = K$. Define the following stochastic process staring from this $b_0(\omega)$ :

$$b_{t+1}(\omega) = H^K[b_t(\omega),\delta_{t+1}(\omega)], \quad t = 0,1,2,... \tag{A.1}$$

Since each $\delta_{t+1}(\omega) = \widetilde{\delta}$ the existence of invariant distribution for this type of dynamic stochastic process of our model follows from Stokey and Lucas [(1989), Theorem 12.10.; see also Razin and Yahav (1979)]. This process maps functions from $C[0,M]$ into itself. Thus, for the given $K$, let $F_t(.)$ be the c.d.f. of $b_t(\omega)$, defined by the process (A.1), $t = 0,1,2,...$ then this sequence of c.d.f.'s converges uniformly on $[0,M]$ to some $F^*(.)$ as $t \to \infty$ . Hence there exists some $b_K^* : [0,1] \to [0,M]$ which is mapped into itself by the dynamic process described above. By our assumptions we have :$b_K(\omega) \leq M$ for all $\omega$.

There exists some $M^* > 0$ such that for any $0 < K < \frac{M}{2}$ and any $0 < b < M^*$ we have:$g(b,\delta_n,w(K),r(K)) < b$ (since as $K$ declines $w(K)$ declines while $r(K)$ increases). Thus *in this range* for $K$ and $b$ we have: $H(b,\delta_i) > 0$ for all $i$. Moreover, if $M^*$ is sufficiently small there exists some $\epsilon^* >0$ such that : $H(b,\delta_i) \geq b$ for $0 < b < \epsilon^*$ for $i = 1,2,.....,n$. It is easy to see that this implies that for any $0 < K < M^*$ the fixed point corresponding to the above process $H^K$, denoted by $b_K^*(\omega)$ , is different from 0.

It can be shown that there exists a unique invariant probability distribution corresponding to each $b_K^*(\omega)$. This follows by direct verification of all the conditions in Proposition 1 in Razin and Yahav (1979).

Consider now the mapping $\phi$ :$[0,M^*] \longrightarrow [0,M^*]$ defined as follows:

$$\phi(K) = \sum_{i=1}^{n} \int_{A_i} H^K(b_K^*(\omega),\delta_i)d\mu(\delta(\omega))$$

where $A_i = \{\omega \mid \delta_0(\omega) = \delta_i\}$, $i = 1,2,...,n$. We claim that the mapping $\phi$ is continuous on $[0,M^*]$. To verify this claim one needs the following observations: The optimal consumption function $g(b,\delta_i;w,r)$ is a continuous function of $w$ and $r$. Since $1 + r = f'(K)$ and $w = f(K) - Kf'(K)$ this implies that $g(b,\delta_i;w,r)$ is continuous in $K$. Hence, $H^K(b,\delta_i;w,r)$ is a continuous function of $K$ on $[0,M^*]$ . By Brouwer's fixed point theorem there exist $\widehat{K}$ such that $\phi(\widehat{K}) = \widehat{K}$. In particular, the invariant function for the stochastic process in (A.1)

$b^*(.)$ which corresponds to $\widehat{K}$ satisfies : $\int_0^1 b^*(\omega)d\omega = \widehat{K}$. Thus, the aggregate intergenerational transfer equals the aggregate capital stock and, at the same time, the stationary wage rate and the interest rate correspond to $\widehat{K}$. Now, it is clear that a steady state with $\widehat{K}$ and $b^*(.)$ exists. Moreover, starting from $\mathrm{b}_0(\omega) > 0$ a.s. the above arguments imply that $\widehat{K} > 0$. $\square$

*(A.2) Proof of Proposition 1.* (a) If $g(b, \delta_i) = b(1+r^*)+w^*$, then it is clear that it is increasing in $b$. Assume, therefore, that $g(b, \delta_i) < b(1+r^*)+w^*$. Thus equation (3.4) holds. By the strict concavity of $V$ in $b$, $V_1$ is a decreasing function of $b$ for each $\delta_i$. Since $u'$ is a decreasing function, equation (3.3) implies that $g(b, \delta_i)$ is strictly increasing in $b$.

To show that $b(1 + r^*) - g(b, \delta_i)$ increases in $b$ as well (unless $g(b, \delta_i) \equiv b(1 + r^*) + w^*$) we use equation (3.4). Suppose that $b' > b$ and $(1 + r^*)b' - g(b', \delta_i) \leq (1 + r^*)b - g(b, \delta_i)$. Since $g(b', \delta_i) > g(b, \delta_i)$ for all $\delta_i$ and $u$ is concave $u'(g(b', \delta_i)) < u'(g(b, \delta_i))$, while $Eu'(g((1+r^*)b' + w^* - g(b', \delta_i), \tilde{\delta})) \geq Eu'(g((1 + r^*)b + w^* - g(b, \delta_i), \tilde{\delta}))$.These inequalities contradict equation (3.4). This completes the proof of part (a).

(b) Suppose that $\delta_i' > \delta_i$ while $g(b, \delta_i') > g(b, \delta_i)$. Thus,

$$u'(g((1 + r^*)b + w^* - g(b, \delta_i'), \delta_j)) \geq u'(g((1 + r^*)b + w^* - g(b, \delta_i), \delta_j))$$

for all $j = 1, ..., n$ with strict inequality for some $j$. Hence,

$$Eu'(g((1 + r^*)b + w^* - g(b, \delta_i'), \tilde{\delta})) > Eu'(g((1 + r^*)b + w^* - g(b, \delta_i), \tilde{\delta})).$$

But $\delta_i'(1 + r^*) > \delta_i(1 + r^*)$. Hence, if equation (3.4) holds with equality for $\delta_i$ then it does not hold for $\delta_i'$ since the right-hand side is larger for $\delta_i'$ while the left-hand side is smaller for $\delta_i'$.

If $g(b, \delta_i') = b(1 + r^*) + w^*$ then $g(b, \delta_i) = b(1 + r^*) + w^*$, by the argument above, $g(b, \delta)$ is constant in $\delta$. $\square$

*(A.3) Proof of Proposition 2.* Suppose that $\hat{b}_{1,t} \leq \overline{b}^*$ for some $t$ (and hence for all $t$). Then, since $\overline{b}^* > 0$, Remark 1 implies that $\hat{b}_{i,t} < \overline{b}^*$, $i = 2, 3, ..., n$. But, in steady state, $\overline{b}^* = \sum_{i=1}^n \hat{b}_{i,t} \Pr(\tilde{\delta} = \delta_i)$. A contradiction. A similar argument applies to the second inequality. $\square$

*(A.4) Proof of Lemma 2.* To begin with observe that, by definition of the optimal saving function we have:

$$b^*(\delta(\omega)) = H\left(b^*\left(T^{-1}(\delta(\omega)), \delta_0(\omega); w^*, r^*\right)\right).$$

Using these notations and equations (3.3) and (3.4) we can write, for steady state,

$$V_1\left(b^*(T^{-1}(\delta(\omega)), \delta_0(\omega)\right) = \delta_0(\omega)\left(1 + r^*\right) E_1 V_1\left(H(b^*(T^{-1}(\delta(\omega)), \delta_0(\omega)), \widetilde{\delta}\right)$$

which holds almost surely. Taking expectations on both sides of the last equation conditional on $t = 0$ we get,

$$E_0 V_1\left(b^*(T^{-1}(\delta(\omega)), \delta_0(\omega)\right) =$$

$$E_0\left[\delta_0(\omega)\left(1+r^*\right)E_1V_1\left(H(b^*(T^{-1}(\delta(\omega)),\delta_0(\omega)),\widetilde{\delta}\right)\right]$$

But the last expression is equivalent to:

$$\left(1+r^*\right)E_0\delta_0(\omega)E_0V_1\left(H(b^*(T^{-1}(\delta(\omega)),\delta_0(\omega)),\widetilde{\delta}\right)+$$

$$Cov\left[\delta_0(\omega),E_1V_1\left(H(b^*(T^{-1}(\delta(\omega)),\delta_0(\omega)),\widetilde{\delta}\right)\right].$$

Since $H(.,.)$ is monotone increasing in both arguments and $V_1(\cdot,\cdot)$ is decreasing in its first argument the covariance in the equation above is negative (or zero in the case of a corner solution.) Thus,

$$\begin{aligned}&E_0V_1\left(b^*(T^{-1}(\delta(\omega)),\delta_0(\omega)\right)\\&\quad<\left(1+r^*\right)E_0\delta_0(\omega)E_0V_1(H(b^*(T^{-1}(\delta(\omega)),\delta_0(\omega)),\widetilde{\delta}).\end{aligned}\qquad (A.2)$$

But $T$ and $T^{-1}$ are measure preserving hence,

$$E_0V_1\left(b^*(T^{-1}\delta(\omega)),\delta_0(\omega)\right)=E_0V_1\left(b^*(\delta(\omega)),\widetilde{\delta}\right).\qquad (A.3)$$

Thus, $(1+r^*)E_0\delta_0(\omega)>1$. Since $E_0\delta_0(\omega)=E\widetilde{\delta}$, the last inequality implies the conclusion. □

*(A.5) Proof of Theorem 2.* (a) By Lemma 2, in steady-state,

$$(1+r^*)E\tilde{\delta}=\left(1+r^*\right)[\delta_1\Pr\{A_1\}+\sum_{i=2}^{n}\delta_i\Pr\{A_i\}]>1.$$

Thus, if $\Pr\{A_1\}$ is sufficiently small then $(1+r^*)\delta_2>(1+r^*)E\tilde{\delta}$. Hence, (A.2) and (A.3) imply that

$$E_0V_1\left(b^*(T^{-1}(\delta(\omega)),\delta_0(\omega)\right)<\left(1+r^*\right)\delta_2EV_1(H(b^*(T^{-1}(\delta(\omega)),\delta_0(\omega)),\widetilde{\delta}).$$

Consequently, $H(b,\delta_2)>b$, (i.e., individuals in $A_{2,t}$ are net savers). By the same logic if $(1+r^*)\delta_i>(1+r^*)E\tilde{\delta}$ then $H(b,\delta_i)>b$. Hence, for such $i$, individuals in $A_{i,t}$ are net savers.

(b) Assume that $\delta_2$ is sufficiently close to $\delta_1$. By a similar argument as in the proof of part (a), if $\Pr\{A_{1,t}\}$ is not large then $(1+r^*)\delta_2>(1+r^*)E\tilde{\delta}$ and $H(b,\delta_2)>b$. Hence, individuals in $A_{2,t}$ are net savers. □

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