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Exposita Notes

The limit theorem on the core of a production economy in vector lattices with unordered preferences

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Summary. We prove Aliprantis, Brown, and Burkinshaw's (1987) theorem on the equivalence of Edgeworth production equilibria and pseudo-equilibria in a more general setting. We consider production economies with unordered preferences and general consumption sets in a vector lattice commodity space. We adapt the approach of Mas-Colell and Richard (1991) and prove our theorem by applying a separating hyperplane argument in the space of all allocations. We also generalize Podczeck's (1996) important result on the relationship between continuous and discontinuous equilibrium prices to the case of production.

Keywords and Phrases: Riesz space, Vector lattice, Uniform properness, Limit theorem on the core, Production economy.

JEL Classification: C62, C71, C51.

1 Introduction

In their papers, Mas-Colell and Richard [9] and Richard [11] identified vector lattices with lattice ordered topological duals as suitable mathematical settings for infinite dimensional general equilibrium analysis. They showed that subject to additional assumptions, termed uniform properness, economies modeled in such settings pass the existence of an equilibrium test: Mas-Colell and Richard [9] (see also [1]) considered the case of pure trade, Richard [11] allowed for production.

A main contribution of Mas-Colell and Richard is their use of the lattice structure of the price space to show the supportability of weakly optimal Pareto allocations. Indeed, they obtain this valuation equilibrium result by applying a separating hyperplane argument in the space of allocations and

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then calculating the supremum of the resulting prices. Recently, Tourky [12] showed that this technique can be quickly extended to obtain the classic theorem on the equivalence of Edgeworth and price equilibria in pure trade economies. Also recently, Podczeck [10] showed that the existence of discontinuous equilibrium prices in Mas-Colell and Richard's exchange model implies the existence of continuous equilibrium prices. Podczeck's result is an extension of the path-breaking methods of Yannelis and Zame [13], and is especially related to the theorem on discontinuous price equilibria in [13, Appendix].

The purpose of this note is two-fold. First, to extend the main theorem of Aliprantis-Brown-Burkinshaw [3], on the equivalence of Edgeworth's and Walras' notions of equilibrium for production economies in locally solid Riesz spaces, to the more general commodity-price space setting of Mas-Colell and Richard (related results include [2, 6]). Second, to extend Podczeck's important result to the case of economies with production.

Our proof extends the ideas in Tourky [12] and combines the proof of Aliprantis-Brown-Burkinshaw and that of Mas-Colell and Richard. We obtain the continuous equilibrium price by applying a separating hyperplane argument in a space larger that the space of allocations used by Richard [11]. Indeed, our separating hyperplane argument is applied in the space $L^{\Sigma_i(n_i+1)}$, where n_i is the number of firms that are at least partially owned by the i-th consumer and L is the commodity space. Furthermore, we identify as superfluous the extra cone-like assumption on pretechnology sets used in [11] but not in [3] (see the remark after Definition 2).

As in Tourky [12] we require that preferences satisfy a lattice theoretic assumption on the extendibility of upper-sections of preferences to convex sets with non-empty interior. When preferences are continuous preorderings this assumption is strictly more general than Mas-Colell's [7] ω-uniform properness (for related conditions see [4, 13]). Our properness assumption allows us to consider economies with unbounded consumption sets and may be useful in models of finance where some preferences with unbounded marginal utility are excluded by ω-uniform properness (see for example Duffie [5, p. 1639]). Indeed, Tourky [12] constructs an economy whose preferences are convex, strictly monotone, and continuous total preorderings. These preferences satisfy the alternative properness assumption, they have unbounded marginal rates of substitution, and they are not ω-uniformly proper.

The remainder of this note is organized as follows. Section 2 contains the model and the main result. The proof of the main result is in Section 3.

2 Model and results

We consider economies whose commodity space L is a topological vector space, which need not be Hausdorff or locally convex. L is assumed to be a vector lattice. We denote the positive cone L_+. The price space is the topological dual of L which is denoted L'. We require that L' be a sublattice of the order dual $L^\sim$ of L.

There are $m > 0$ consumers, let $I = \{1, \ldots, m\}$. Each consumption set is denoted X_i and $\omega_i \in X_i$ is the i-th consumer's endowment. Let $\omega = \sum_{i=1}^m \omega_i$. $P_i : X_i \to 2^{X_i}$ is the i-th consumer's preference map. There are $n > 0$ producers, let $J = \{1, \ldots, n\}$. Each production set is denoted Y_j.

A *feasible allocation* is a point in $(x, y) \in \prod_{i=1}^m X_i \times \prod_{j=1}^n Y_j$ such that $\sum_{i=1}^m x_i - \sum_{j=1}^n y_j - \omega = 0$. A *pseudo-equilibrium* for an economy is a pair $\{p, (x, y)\}$, where p is a non-zero linear functional on L, (x, y) is a feasible allocation, and

i. $p(\omega) > 0$;
ii. $\forall j$, $p(y_j) = \sup_{z \in Y_j} p(z)$;
iii. $\forall i$, $x_i \in \{z \in X_i : p(z) = p(\omega_i) + \sum_{j=1}^n \theta_{ij} p(y_j)\}$;
iv. $\forall i$, $P_i(x_i) \cap \{z \in X_i : p(z) < p(\omega_i) + \sum_{j=1}^n \theta_{ij} p(y_j)\} = \emptyset$.

$\{p, (x, y)\}$ is an *equilibrium* if it is a pseudo-equilibrium and

v. $\forall i$, $P_i(x_i) \cap \{z \in X_i : p(z) \le p(\omega_i) + \sum_{j=1}^n \theta_{ij} p(y_j)\} = \emptyset$.

A feasible allocation $(\tilde{x}, \tilde{y})$ is *blocked* by a non-empty coalition $S \subset I$ if there is an allocation $(x, y) \in \prod_{i=1}^m X_i \times \prod_{j=1}^n Y_j$ having the following properties

i. $\sum_{i \in S} x_i = \sum_{i \in S} \omega + \sum_{j \in J} (\sum_{i \in S} \theta_{ij}) y_j$;
ii. $\forall i \in S$, $x_i \in P_i(\tilde{x}_i)$.

A feasible allocation $(\tilde{x}, \tilde{y})$ is in the *core* of the economy if it is not blocked by any non-empty coalition $S \subset I$. A feasible allocation that is in the core of every r-replicated ($r = 1, 2, \ldots$) economy is an *Edgeworth* equilibrium.

We define the notion of M-proper upper sections, where the M refers to Mas-Colell [8].

Definition 1. P_i is *M-proper* at $x \in X_i$ if there are convex sets C_i and $\widehat{P}_i(x)$ such that

i. $\widehat{P}_i(x) \cap C_i = P_i(x)$;
ii. $x + \omega$ is an interior point of $\widehat{P}_i(x)$ and $P_i(x)$ is open in $C_i(x)$;
iii. $x, 0, \omega_i \in C_i$, and $L_+ + C_i = C_i$;
iv. if $y, z \in C_i$ then $y \wedge z \in C_i$;
v. $(1 + \alpha_i)x \in C_i$ for some $\alpha_i > 0$.

Remark. Observe that if $\omega > 0$ then $P_i(x) \subset \overline{\text{int } \widehat{P}_i(x) \cap C_i}$ and that $\text{int } \widehat{P}_i(x) \cap C_i \subset P_i(x)$. Hence, it is not difficult to see from the proof of our main theorem that the requirement that $P_i(x)$ be open in $C_i(x)$ can be dropped.

We define the notion of M-proper production sets.

Definition 2. Y_j is *M-proper* at $y \in Y_j$ if there are convex sets K_j and $\widehat{Y}_j$ such that

i. $\widehat{Y}_j \cap K_j = Y_j$;
ii. $y - \omega$ is in the interior of $\widehat{Y}_j$;
iii. $0 \in K_j$ and $-L_+ + K_j = K_j$;
iv. if $y, z \in K_j$ then $y \vee z \in K_j$.

Remark. Aliprantis-Brown-Burkinshaw's [3] and Richard's [11] production sets are M-proper at every $y \in Y_j$. They assume that for each j there is a convex *preproduction* set $K_j \supset Y_j$ satisfying (iv). They also assume that there is an open cone $\Gamma \ni \omega$ such that $y \in Y_j$ and $z \in (-\Gamma + y) \cap K_j$ implies $z \in Y_j$. Letting $\widehat{Y} = Y_j - (\{0\} + \Gamma)$ we see that such production sets are M-proper at every point. Richard also assumes an extra cone like assumption on the preproduction sets: $(\alpha + 1)K_j = K_j$ for some $\alpha > 0$; which we don't need in this note.

Theorem 2.1. *Let $(\tilde{x}, \tilde{y})$ be an Edgeworth equilibrium and assume that*

i. $\omega > 0$;
ii. $\forall i \in I$, $L_+ + \tilde{x}_i \subset P_i(\tilde{x}_i) \cup \{\tilde{x}_i\}$;
iii. $\forall j \in J$, $-L_+ \subset Y_j$;
iv. $\forall i \in I$, P_i is *M-proper at $\tilde{x}_i$*;
v. $\forall j \in J$, Y_j is M-proper at $\tilde{y}_j$.

There is $\pi \in L'$ such that $(\pi, \tilde{x}, \tilde{y})$ is a pseudo-equilibrium.

The monotonicity assumption can be replaced by local non-satiation if we know that there exists an equilibrium price in the algebraic dual of L.

Theorem 2.2. *Let $(q, \tilde{x}, \tilde{y})$ be an equilibrium and assume that*

i. $\omega > 0$;
ii. $\forall i \in I$, $\tilde{x}_i \in \overline{P_i(\tilde{x}_i)}$;
iii. $\forall j \in J$, $-L_+ \subset Y_j$;
iv. $\forall i \in I$, P_i is *M-proper* at $\tilde{x}_i$;
v. $\forall j \in J$, Y_j is *M-proper* at $\tilde{y}_j$.

There is $\pi \in L'$ such that $(\pi, \tilde{x}, \tilde{y})$ is a pseudo-equilibrium.

3 Proof of Theorems

3.1 Proof of Theorem 2.1

The proof comprises three lemmas. First we identify several sets and points. For every consumer $i \in I$ define the index set $J_i = \{0\} \cup \{j \in J : \theta_{ij} > 0\}$ and the product space $M_i = \prod_{j \in J_i} (L)_j$. $z_{ij} \in L$ shall denote the j-th coordinate of the i-th coordinate of $z \in \prod_{i=1}^m M_i$.

For every $i \in I$ and $j \in J_i$ define the following sets and points:

$$\kappa_{ij} = \begin{cases} \tilde{x}_i - \omega_i & \text{if } j = 0, \\ -\theta_{ij}\tilde{y}_j & \text{otherwise} ; \end{cases} \qquad \Gamma_{ij} = \begin{cases} P_i(\tilde{x}_i) - \omega_i & \text{if } j = 0, \\ -\theta_{ij} Y_j & \text{otherwise} ; \end{cases}$$

$$\widehat{\Gamma}_{ij} = \begin{cases} \widehat{P}_i(\tilde{x}_i) - \omega_i & \text{if } j = 0, \\ -\theta_{ij}\widehat{Y}_j & \text{otherwise} ; \end{cases} \qquad \Psi_{ij} = \begin{cases} C_i - \omega_i & \text{if } \ j = 0, \\ -\theta_{ij} K_j & \text{otherwise} . \end{cases}$$

We can show by following the argument in Aliprantis-Brown-Burkinshaw [3, Proposition 5.4] that

$$0 \notin \mathrm{co}\left(\bigcup_{i=1}^{m}\left[\sum_{j\in J_i}\Gamma_{ij}\right]\right). \tag{1}$$

Let $U = \left\{z \in \prod_{i=1}^{m} M_i : \sum_{i=1}^{m}\sum_{j\in J_i} z_{ij} = 0\right\}$, and for $h = 1,\ldots,m$ let $s^h \in \prod_{i=1}^{m} M_i$ have the following coordinates

$$s_{ij}^h = \begin{cases} 0 & \text{if } i = h, \\ \kappa_{ij} & \text{if } i \neq h. \end{cases}$$

Let $S = \cup_{h=1}^{m}\{s^h\}$ and let $Z = \mathrm{co}\{U \cup S\} \cap \prod_{i=1}^{m}\prod_{j\in J_i}\Psi_{ij}$. Clearly, $\kappa \in Z$.

Lemma 3.1. $Z \cap \prod_{i=1}^{m}\prod_{j\in J_i}\widehat{\Gamma}_{ij} = \emptyset$. *Also* $\forall i \in I, \forall j \in J_i$; *there is* $p_{ij} \in L'$ *such that*

i. $p_{ij} \geq 0$, *for some* $i \in I$ *and* $j \in J_i$;
ii. $\sum_{i=1}^{m} p_{i0}(\omega) + \sum_{i=1}^{m}\sum_{j\in J_i\setminus\{0\}} p_{ij}(\theta_{ij}\omega) > 0$;
iii. $\forall i \in I$, $\forall j \in J_i$; $p_{ij}[\Gamma_{ij}] \geq p_{ij}(\kappa_{ij})$;
iv. $\forall z \in Z$, $\sum_{i=1}^{m}\sum_{j\in J_i} p_{ij}(\kappa_{ij}) \geq \sum_{i=1}^{m}\sum_{j\in J_i} p_{ij}(z_{ij})$.

Proof. Suppose the contrary and that $z \in Z \cap \prod_{i=1}^{m}\prod_{j\in J_i}\widehat{\Gamma}_{ij}$, then $z_{ij} \in \Gamma_{ij}$ for every $i \in I$ and $j \in J_i$. Also z is the convex combination of a point $u \in U$ and the m points $s^h \in S$. That is $z = \alpha u + \sum_{h=1}^{m}\beta_h s^h$ and $\alpha + \sum_{h=1}^{m}\beta_h = 1$; with $\alpha \geq 0$ and $\beta_h \geq 0$, for $h = 1,\ldots,m$.

Since $u, \kappa \in U$ then $\sum_{i=1}^{m}\sum_{j\in J_i} u_{ij} = 0$ and $\sum_{i=1}^{m}\sum_{j\in J_i} s_{ij}^h = -\sum_{j\in J_h}\kappa_{hj}$, for $h = 1,\ldots,m$. Thus, $\sum_{i=1}^{m}\sum_{j\in J_i} z_{ij} = -\sum_{h=1}^{m}\beta_h\sum_{j\in J_h}\kappa_{hj}$.

Let $\gamma = \sum_{h=1}^{m}\beta_h \neq 0$. Since $z_{10} \nleq 0$ (by the monotonicity assumption) then $z_{10} \wedge 0 < z_{10}$; so a point $z'_{10} \in \Gamma_{10}$ can be chosen on open line segment joining $z_{10} \in \Psi_{10}$ and $z_{10} \wedge 0 \in \Psi_{10}$.

We get the convex combination

$$z'' = \frac{1}{\gamma+m}\left(z'_{10} + \sum_{j\in J_1\setminus\{0\}} z_{1j}\right) + \sum_{i=2}^{m}\frac{1}{\gamma+m}\left(\sum_{j\in J_i} z_{ij}\right) + \sum_{h=1}^{m}\frac{\beta_h}{\gamma+m}\left(\sum_{j\in J_h}\kappa_{hj}\right) < 0.$$

Let $\kappa''_{h0} = \kappa_{h0} - \frac{\gamma+m}{\gamma}z''$, for $h = 1,\ldots,m$. From the monotonicity assumption every κ''_{h0} is in Γ_{h0}. But then we get the convex combination, which contradicts (1),

$$\frac{1}{\gamma+m}\left(z'_{10} + \sum_{j\in J_1\setminus\{0\}} z_{1j}\right) + \sum_{i=2}^{m}\frac{1}{\gamma+m}\left(\sum_{j\in J_i} z_{ij}\right) + \sum_{h=1}^{m}\frac{\beta_h}{\gamma+m}\left(\kappa''_{h0} + \sum_{j\in J_h\setminus\{0\}}\kappa_{hj}\right) = 0.$$

The set Z is convex and non-empty and $\prod_{i=1}^{m}\prod_{j\in J_i}\widehat{\Gamma}_{ij}$ is convex and has a non-empty interior. Thus, there is a $p \in \prod_{i=1}^{m}\prod_{j\in J_i}(L')_{ij}$, which is non-zero and separates the two sets. That is $\sum_{i=1}^m \sum_{j\in J_i} p_{ij}(x_{ij}) \geq \sum_{i=1}^m \sum_{j\in J_i} p_{ij}(z_{ij})$ for every $z \in Z$ and $x \in \prod_{i=1}^m \prod_{j\in J_i} \widehat{\Gamma}_{ij}$.

Since $\kappa \in Z$ and is a boundary point of $\prod_{i=1}^m \prod_{j\in J_i} \widehat{\Gamma}_{ij}$, then items (iii) and (iv) must hold. Since $\omega + \kappa_{i0}$ is in the interior of $\widehat{\Gamma}_{i0}$, for every $i \in I$, and $\theta_{ij}\omega + \kappa_{ij}$ is in the interior of $\widehat{\Gamma}_{ij}$, for every $i \in I$ and $j \in J_i \setminus 0$, then $\sum_{i=1}^m p_{i0}(\omega) + \sum_{i=1}^m \sum_{j\in J_i\setminus\{0\}} p_{ij}(\theta_{ij}\omega) > 0$. Item (i) is a consequence of the free disposability assumption and (iii). $\square$

Let $\pi = \bigvee_{i\in I \& j\in J_i} p_{ij}$, where p_{ij} are from Lemma 3.1. Evidently $\pi \in L'$, $\pi \geq 0$, and $\pi(\omega) > 0$ (because $\omega > 0$ and $\theta_{ij} \geq 0$).

Lemma 3.2. $\forall i \in I$, $\pi(\tilde{x}_i) = p_{i0}(\tilde{x}_i)$; and $\forall i \in I, \forall j \in J_i$, $\pi[\Gamma_{ij}] \geq \kappa_{ij}$.

Proof. We first show that for arbitrary $i' \in I, j' \in J$ if $g_{i'j'} \in \Psi_{i'j'}$ and $g_{i'j'} \leq \kappa_{i'j'}$, then $p_{i'j'}(g_{i'j'} - \kappa_{i'j'}) = \pi(g_{i'j'} - \kappa_{i'j'})$ (cf., [11]).

Let $g_{ij} = \kappa_{ij}$ if $(i,j) \neq (i',j')$; and $i \in I$, $j \in J_i$. Then $\sum_{i=1}^m \sum_{j\in J_i} g_{ij} \leq 0$. Let $z_{ij} \leq 0$ be arbitrarily chosen so that $\sum_{i=1}^m \sum_{j\in J_i} z_{ij} = \sum_{i=1}^m \sum_{j\in J_i} g_{ij}$. Then $\sum_{i=1}^m \sum_{j\in J_i} (g_{ij} - z_{ij}) = 0$ and $(g_{ij} - z_{ij}) \in \Psi_{ij}$; for all $i \in I$ and $j \in J_i$. Thus, $\sum_{i=1}^m \sum_{j\in J_i} p_{ij}(\kappa_{ij}) \geq \sum_{i=1}^m \sum_{j\in J_i} p_{ij}(g_{ij} - z_{ij})$ and $p_{i'j'}(g_{i'j'} - \kappa_{i'j'}) \leq \sum_{i=1}^m \sum_{j\in J_i} p_{ij}(z_{ij})$. We get $p_{i'j'}(g_{i'j'} - \kappa_{i'j'}) \leq \sum_{i=1}^m \sum_{j\in J_i} \pi(g_{ij}) - \sum_{i=1}^m \sum_{j\in J_i} \pi(\kappa_{ij})$ and $p_{i'j'}(g_{i'j'} - \kappa_{i'j'}) = \pi(g_{i'j'} - \kappa_{i'j'})$.

Setting $g_{i0} = \tilde{x}_i \wedge 0 - \omega_i$ we get $p_{i0}(-\tilde{x}_{i0}^+) = \pi(-\tilde{x}_{i0}^+)$. Setting $g_i = (1+\alpha_i)\tilde{x}_{i0} \wedge 0 + \tilde{x}_{i0}^+ - \omega_{i0}$, for the $\alpha_i > 0$ from Definition 1, we get $p_{i0}(-\alpha_i \tilde{x}_{i0}^-) = \pi(-\alpha_i \tilde{x}_{i0}^-)$. Thus, $p_{i0}(\tilde{x}_{i0}) = \pi(\tilde{x}_{i0})$.

Let $z \in \widehat{\Gamma}_{ij} \cap \Psi_{ij}$ and let $g_{ij} = z \wedge \kappa_{ij}$. Then $g_{ij} \in \Psi_{ij}$ and $g_{ij} \leq \kappa_{ij}$. Thus,

$$\pi(g_{ij}) = \pi(\kappa_{ij}) + p_{ij}(g_{ij} - \kappa_{ij}) \geq \pi(\kappa_{ij}) + p_{ij}(g_{ij} - z) \geq \pi(\kappa_{ij} + g_{ij} - z),$$

(we used (iii) of Lemma 3.1 and $\pi \geq 0$) which implies that $\pi(z) \geq \pi(\kappa_{ij})$. $\square$

Lemma 3.3. *$\forall i \in I$, $\pi(\tilde{x}_i) = \pi(\omega_i) + \sum_{j=1}^m \theta_{ij}\pi(\tilde{y}_j)$, and $(\pi, \tilde{x}, \tilde{y})$ is a pseudo-equilibrium.*

Proof. We take $i = 1$ as an arbitrary representative. Let $z_{ij} \geq 0$ be arbitrarily chosen so that $\sum_{i=1}^m \sum_{j\in J_i} z_{ij} = \omega_1^+ + \sum_{j\in J_1\setminus\{0\}} \kappa_{1j}^-$.

Define the point $x' \in U \subset \prod_{i=1}^m M_i$ as follows

$$x'_{ij} = \begin{cases} \kappa_{10} - \omega_1^+ + z_{10} & \text{for } i = 1 \;\&\; j = 0, \\ \kappa_{1j} - \kappa_{1j}^- + z_{1j} & \text{for } i = 1 \;\&\; j \in J_1 \setminus \{0\}, \\ \kappa_{ij} + z_{ij} & \text{otherwise.} \end{cases}$$

We want to show that $x'' = \frac{1}{2}x' + \frac{1}{2}s^1 \in Z$. This is implied if that $x'' \in \prod_{i=1}^m \prod_{i\in J_i} \Psi_{ij}$. But

$$x''_{ij} = \begin{cases} \frac{1}{2}\kappa_{10} - \frac{1}{2}\omega_1^+ + \frac{1}{2}z_{10} & \text{for } i = 1 \;\&\; j = 0, \\ \frac{1}{2}\kappa_{1j} - \frac{1}{2}\kappa_{1j}^- + \frac{1}{2}z_{1j} & \text{for } i = 1 \;\&\; j \in J_1 \setminus \{0\}, \\ \kappa_{ij} + \frac{1}{2}z_{ij} & \text{otherwise.} \end{cases}$$

Since $-\omega_1^+ = -\omega_1 \wedge 0 \in \Psi_{10}$ then $x''_{10} \in \Psi_{10}$. Since, for $j \in J_1 \setminus \{0\}$, $-\kappa_{1j}^- = \kappa_{1j} \wedge 0 \in \Psi_{1j}$ then $x''_{1j} \in \Psi_{1j}$. Evidently, $x''_{ij} \in \Psi_{ij}$ when $i \neq 1$.

We therefore know that $\sum_{i=1}^m \sum_{j \in J_i} p_{ij}(\kappa_{ij}) \geq \sum_{i=1}^m \sum_{j \in J_i} p_{ij}(x''_{ij})$ and

$$\sum_{j \in J_1} p_{1j}(\kappa_{1j}) \geq \frac{1}{2} \sum_{j \in J_1} p_{1j}(\kappa_{1j}) - \frac{1}{2} \sum_{j \in J_1 \setminus \{0\}} p_{1j}(\kappa_{1j}^-) - \frac{1}{2} p_{10}(\omega_1^+) + \frac{1}{2} \sum_{i=1}^m \sum_{j \in J_i} p_{ij}(z_{ij}).$$

Because the points z_{ij} where taken arbitrarily:

$$\sum_{j \in J_1} p_{1j}(\kappa_{1j}) + \sum_{j \in J_1 \setminus \{0\}} p_{1j}(\kappa_{1j}^-) + p_{10}(\omega_1^+) \geq \pi \left(\omega_1^+ + \sum_{j \in J_1 \setminus \{0\}} \kappa_{1j}^- \right),$$

$$\begin{aligned} \Rightarrow \quad \pi(\tilde{x}_1) = p_{10}(\tilde{x}_1) &\geq -p_{10}(\omega_1^-) - \sum_{j \in J_1 \setminus \{0\}} p_{1j}(\kappa_{1j}^+) + \pi \left(\omega_1^+ + \sum_{j \in J_1 \setminus \{0\}} \kappa_{1j}^- \right) \\ &\geq -\pi(\omega_1^-) - \sum_{j \in J_1 \setminus \{0\}} \pi(\kappa_{1j}^+) + \pi \left(\omega_1^+ + \sum_{j \in J_1 \setminus \{0\}} \kappa_{1j}^- \right) \\ &\geq \pi(\omega_1) - \sum_{j \in J_1 \setminus \{0\}} \pi(\kappa_{1j}) \\ &\geq \pi(\omega_1) + \sum_{j \in J_1 \setminus \{0\}} \theta_{1j} \pi(\tilde{y}_{1j}). \end{aligned}$$

Since $(\tilde{x}, \tilde{y})$ is feasible and $\sum_{i=1}^m \sum_{j=1}^n \theta_{ij} = 1$, the first part of the lemma is proved.

It is easy to check that Lemma 3.2 implies that $\pi(\tilde{y}_j) = \max_{z \in Y_j} \pi(z)$, $\forall j$; and $\pi[P_i(\tilde{x}_i)] \geq \pi(\tilde{x}_i)$, $\forall i$. Consequently $(\pi, \tilde{x}, \tilde{y})$ is a pseudo equilibrium. $\square$

3.2 Proof of Theorem 2.2

The monotonicity assumption on preferences is needed at only one point in the proof of Theorem 2.1: to show that $Z \cap \prod_{i=1}^m \prod_{j \in J_i} \widehat{\Gamma}_{ij} = \emptyset$ in Lemma 3.1.

Suppose the contrary and that $z \in Z \cap \prod_{i=1}^m \prod_{j \in J_i} \widehat{\Gamma}_{ij}$ then as before $z_{ij} \in \Gamma_{ij}$ for every $i \in I$ and $j \in J_i$. Also, $\sum_{i=1}^m \sum_{j \in J_i} z_{ij} = -\sum_{h=1}^m \beta_h \sum_{j \in J_h} \kappa_{hj}$. But this is a contradiction since $q(\sum_{i=1}^m \sum_{j \in J_i} z_{ij}) > 0$ and $q(\sum_{j \in J_h} \kappa_{hj}) = 0$, for $h = 1, \ldots, m$.

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