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Competitive pricing of information goods: Subscription pricing versus pay-per-use

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Summary. We investigate the existence and implications of competitive equilibria when two firms offer the same electronic goods under different pricing policies. One charges a fixed subscription fee per period; the other charges on a per-use basis. Two models are examined when firms' marginal costs are negligible and they can revise prices periodically. Both show that competition often leads to ruinous price wars in the absence of collusion. However, stable pricing equilibria exist in special cases. The findings are robust even when customers are willing to pay a fixed-subscription premium.

Keywords and Phrases: Competitive equilibria, Internet pricing, Subscription pricing, Price wars.

JEL Classification Numbers: C7, D4, G2.

1 Introduction

The purpose of this paper is to analyze competitive pricing for electronically transmitted information goods, such as news stories, software, entertainment items, and databases, that are offered by competing firms that have different pricing policies. With a few exceptions noted later, it is assumed that potential subscribers are cost minimizers, marginal costs to the firms are negligible, and each firm seeks to maximize its revenues from subscribers. We focus on the simple but revealing scenario of a single good or goods package offered by two firms, one of which charges a fixed subscription fee per unit time while the other charges a constant cost each time the good or package is accessed. The firms can announce new prices periodically, and subscribers

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are free to switch at the beginning of each period. Subscriber choices and competitive price dynamics will be considered within two models with somewhat different emphases that we describe shortly. Our primary concern is the existence of a competitive equilibrium at which neither firm gains an advantage by changing its price. Although situations that have competitive equilibria will be noted, most instances of the models lack this property and give rise to ruinous price wars in the absence of collusion.

Our analysis of competitive dynamics is set against a background of research on strategies for pricing electronic goods by a monopolist or single firm, for example [1, 2, 3, 8, 10, 11]. Summary discussions of that work appear in [4, 9]. The present research, which is overviewed in [4], was prompted by the question of what might transpire when competing firms offer similar products but use different pricing policies. A more specific motivating question arose from the observation [5, 6, 7] that subscribers often prefer fixed-fee or flat-rate pricing to per-use pricing, for reasons such as overestimation of usage and avoidance of worrying about occasional large bills or whether each usage is worth it, even when the fixed-fee option costs more over time. The specific question we address in this paper is whether fixed-fee has an advantage over per-use pricing when the above psychological factors are disregarded and both subscribers and firms make economically optimal decisions under conditions of complete information. A broad answer that we elaborate on later is that fixed-fee pricing sometimes enjoys a slight advantage, but that other considerations such as the lack of competitive equilibria and price wars tend to overwhelm any such advantage. Moreover, the latter finding persists when subscribers are willing to pay a premium for the fixed-fee option that is driven by psychological factors omitted in the main part of our analysis.

Almost all of the existing literature on trade in information goods is concerned with monopoly suppliers. The reason is that with zero marginal costs, classical theory predicts price wars that lead to zero prices. However, in practice we do observe seemingly stable competitive situations in a variety of markets, for example for certain types of software or news reports. Much more elaborate models than are available today will be required to understand how this happens, models that will incorporate dynamic elements of technological change and consumer behavior. Our work is an initial step in this direction.

We assume throughout that firms A and B offer the same information good or package. Firm A charges a subscription fee of a per period, and B charges a fee of b for each use or hit. Amounts a and b are fixed in each time period, whose length equals the time unit for A's fixed fee, but the firms can change their fees from period to period. We assume that such changes are announced prior to each new period, simultaneously and without collusion, so that the dynamic pricing situation can be viewed as a two-person non-cooperative repeated game.

Effects of collusion on the firms' revenues, which can be dramatic, are considered later in an example. Our basic assumption, that the two firms stick to fixed pricing policies, can also be thought of as a form of indirect collusion. We make that assumption in order to create the possibility of a stable equilibrium without unduly complicating the model.

When the firms' fees are announced for the next period, each potential subscriber has three options:choose A, choose B, choose neither. Thus, over a sequence of periods, a potential subscriber might choose A, then A again, then B, then neither A nor B, then B, and so forth.

We investigate the consequences of two models that characterize the population of potential subscribers and their period-to-period decisions. Both models assume that the potential subscriber population is described by a probability density function μ on $[0, \infty)$ with $\int_0^\infty \mu(x)dx = 1$, where $\mu(x)dx$ is the population proportion with usage rates between x and $x + dx$. We assume that μ is known by both firms and is invariant over time. The *usage rate* x for a potential subscriber is the number of times it would access the information provided by A and B in one period if it actually subscribed and did not curtail its usage for economic reasons. We assume that each potential subscriber knows its x for the next period, which can vary over time subject to the collective restriction that μ remains unchanged.

Our models differ in how a potential subscriber makes its choice from $\{A, B, \text{neither}\}$ in each period and whether it curtails its usage if it subscribes. Model 1 is a full-usage model in the sense that if a potential subscriber with usage rate x chooses A or B then it will access the information x times. The choice between A and B is made to minimize cost, and the decision to subscribe or not subscribe depends on whether that minimum is below a willingness-to-pay threshold. We model the latter feature by a probability function P on $x \geq 0$, where $P(y)$ denotes the probability that a potential subscriber will actually subscribe when it would pay y if it does so. We assume that P is known by each firm.

Model 1 may be appropriate when a third party (parent, company) pays for the usage of a consumer (teenager, employee) but does not control that usage. It neglects situations in which subscribers limit usages to less than their usage rates because of budget constraints or limits on their willingness to pay more than certain amounts for the service.

Model 2 accounts for the latter factor by assuming that each potential subscriber has a budget constraint w, which is the most it would pay for the service during each period. With x denoting usage rate, our second model assumes that (w, x) has a joint probability density function $f(w, x)$ over the population, with $\int_0^\infty \int_0^\infty f(w, x)dwdx = 1$. The function μ used in the first model can be thought of as the marginal of f on x. We assume for Model 2 that both firms know f.

Potential subscribers who choose neither firm in Model 2 can be characterized by a probability mass for f at $w = 0$, but as a technical convenience we will assume that f is continuous and applies only to potential subscribers who choose A or B. Hence the default option, modeled by $1 - P(t)$ in Model 1, is not used directly in Model 2 although it can be accommodated indirectly by probability mass in the neighborhood of $w = 0$. As in the first model, the choice between A and B in the second model is made by each subscriber to minimize its cost.

In both models, $A(a,b)$ denotes the average revenue per potential subscriber paid to firm A, and $B(a,b)$ denotes the average revenue per potential subscriber paid to B over one period in which charges (a,b) obtain. If there are N potential subscribers, A earns $NA(a,b)$ and B earns $NB(a,b)$ during any period in which a and b are in effect.

The rest of the paper is organized as follows. Section 2 elaborates on notions of dynamic behavior, competitive equilibria, and price wars. Sections 3 and 4 focus on Model 1. Section 3 outlines the basic structure of the model and discusses an extended example that has both stable competitive equilibria and price wars that are differentiated by the value of a parameter k in the definition of μ. Section 4 presents a modestly general result for the nonexistence of a competitive equilibrium, compares four pricing schemes designed to avert a price war, and notes an example of multiple equilibria. Section 5 outlines the basic structure of Model 2, proves a theorem that accounts in part for the predominance of price wars for the model, and gives an example of a situation with a competitive equilibrium. Section 6 concludes with a brief summary, remarks on fixed subscription fee premiums, and challenges for future research.

2 Dynamic behavior, equilibria, and price wars

As indicated earlier, we assume that A and B can change their fees prior to each period, but no generality would be lost if changes were allowed less often, say every fourth period. Because a firm could gain a competitive advantage if it knew the other firm's new fee before it set its own, we assume that new fees are announced simultaneously. Barring collusion, each firm must estimate or guess what the other will charge when it sets its new fee, so price-changing behavior is modeled as a noncooperative repeated game in which the firms' pricing strategies could have various forms.

One of these, referred to as a *naive strategy*, occurs when a firm sets its new fee to maximize its revenue under the assumption that the other firm will not change its fee for the next period. A naive strategy is clearly myopic and can result in very different revenues than anticipated when the other firm does in fact change its fee. More sophisticated strategies arise when the firms anticipate each other's change. Carried to an extreme, they might engage in a succession of changes and counterchanges 'on paper' before arriving at their to-be-announced new fees.

We will not assume explicit methods of new fee determination, but instead will base our analysis on changes and counterchanges to suggest how the firms' fees might evolve over time, or how they might be affected by sophisticated computation in a single period. Our approach begins with a fee pair (a_0, b_0) and determines a series of optimal new fees on an alternating basis under the assumption that the other firm retains its 'old price' for at least 'one more period.' Thus, if A goes first, it computes a_1 to maximize

$A(a, b_0)$, then B computes b_1 to maximize $B(a_1, b)$, then A computes a_2 to maximize $A(a, b_1)$, and so forth to produce a sequence

$$S(a_0, b_0) = a_0, b_0, a_1, b_1, a_2, b_2, \ldots$$

of potential changes and counterchanges.

We consider the behavior of $S(a_0, b_0)$ as n for a_n and b_n gets large, and write $S(a_0, b_0) \to (a', b')$ if (a_n, b_n) *converges* to (a', b'). Thus, $S(a_0, b_0) \to (a', b')$ if for every $\epsilon > 0$ there is an $n(\epsilon)$ such that $|a_n - a'| + |b_n - b'| < \epsilon$ for all $n > n(\epsilon)$. When $S(a_0, b_0) \to (a', b')$ for a unique (a', b') that is the same for every initial position $(a_0, b_0) \geq (0, 0)$, we write $S \to (a', b')$ and say that S *converges uniquely* to (a', b'). Experience with an array of assumptions about μ and P for Model 1 and f for Model 2 indicates that unique convergence usually occurs although other behaviors are possible. We comment on exceptions later and focus here on unique convergence.

Two forms of unique convergence are possible. The first has

$$S \to (a^*, b^*) \quad \text{with} \quad a^* > 0 \quad \text{and} \quad b^* > 0 \ ,$$

and in this case we refer to (a^*, b^*) as a *strong equilibrium point*, or SEP for short. It typically occurs when

$$A(a^*, b^*) > A(a, b^*) \text{ for all } a \neq a^* \ ,$$
$$B(a^*, b^*) > B(a^*, b) \text{ for all } b \neq b^* \ ,$$

and (a^*, b^*) is the only fee pair with this property. If (a^*, b^*) is the initial position then neither firm has an incentive to change its fee and $S(a^*, b^*) = a^*, b^*, a^*, b^*, \ldots$; if $(a_0, b_0) \neq (a^*, b^*)$, then a succession of revenue-maximizing calculations typically drives (a_n, b_n) toward (a^*, b^*).

The second form of unique convergence is

$$S \to (0, 0) \ .$$

Under natural assumptions about μ and P for Model 1, or f for Model 2, the intermediate cases of $S \to (a^*, 0)$ with $a^* > 0$, and $S \to (0, b^*)$ with $b^* > 0$, are impossible. For example, $B(a^*, 0) = 0$ by definition since $b = 0$ means that B offers its service free, whereas $B(a^*, b) > 0$ for small positive b. For a similar reason, $S \to (0, 0)$ never identifies $(0, 0)$ as an SEP. We refer to $S \to (0, 0)$ as a *price war* because its typical behavior for $S(a_0, b_0)$ with positive a_0 and b_0 has $a_0 > a_1 > a_2 > \cdots$ and $b_0 > b_1 > b_2 > \cdots$ with $(a_n, b_n) \to (0, 0)$. In this case, each firm reduces its fee to increase market share and, hopefully, its revenue, but the long-run result is that $A(a_n, b_n)$ and $B(a_n, b_n)$ are driven toward zero.

To avoid the ruinous competitive result of $S \to (0, 0)$, the firms might revert to pricing schemes that bypass our presumption of competitive revenue maximization. This could involve covert or overt collusion, perhaps with a revenue-sharing agreement. We will not speculate on the legality of such schemes, but will note effects of collusion or ‘cooperation’ as an aside to our analysis of non-collusive competitive pricing.

An SEP but not a price war is an example of an equilibrium point in the pricing game. Following usual practice, we define (a^*, b^*) as an *equilibrium point* if, for all nonnegative (a, b),

$$A(a^*, b^*) \geq A(a, b^*) \quad \text{and} \quad B(a^*, b^*) \geq B(a^*, b) \ .$$

Our ensuing analyses of Models 1 and 2 generally assume smoothness properties for μ, P and f which imply that $A(a, b)$ is at least twice differentiable with respect to a, and $B(a, b)$ is at least twice differentiable with respect to b. Then the first-order conditions for an equilibrium point (a^*, b^*) are

$$\left.\frac{\partial A(a, b)}{\partial a}\right|_{(a^*, b^*)} = 0 \quad \text{and} \quad \left.\frac{\partial B(a, b)}{\partial b}\right|_{(a^*, b^*)} = 0 \ .$$

The usual second-order conditions for maxima require concavity, i.e., $\partial^2 A(a, b)/\partial a^2 < 0$ and $\partial^2 B(a, b)/\partial b^2 < 0$ at $(a, b) = (a^*, b^*)$, but to ensure that the defining inequalities for an equilibrium point hold globally and not just in the vicinity of (a^*, b^*), it may be necessary to look beyond local concavity.

3 Model 1: Formulation and example

For convenience here and later we refer to a potential subscriber as a *customer*. Calculations apply to a single period unless stated otherwise. Continuity and differentiability properties will be noted in context.

This section begins our analysis of Model 1. The next few paragraphs describe the essentials of the model. We then discuss an example with specific forms for μ and P in which both SEPs and price wars arise depending on a parameter k used in the definition of μ. The latter part of the section is devoted to proofs of the SEP and price war cases. Further observations on the model appear in the next section.

A customer with usage rate x in Model 1 pays a to firm A if it uses A's service, and pays bx to firm B if it uses B's service. Cost minimization implies that such a customer

$$\begin{array}{llllll} \text{pays} & a & \text{to} & A & \text{if} & a \leq bx, \quad \text{and} \\ \text{pays} & bx & \text{to} & B & \text{if} & bx < a \ , \end{array}$$

given that it uses the service. We assume that μ has no mass spikes, or is atomless, so that it makes no difference to the firms' revenues whether $a = bx$ is attributed to A, as done here, or to B.

In Model 1, $P(t)$ is the probability that a customer actually subscribes when it would pay t to do so. We assume that P is differentiable, and anticipate that the likelihood of subscribing decreases as the cost of doing so increases, in which case $P'(t) < 0$. It should be noted that P is defined independently of x, which is unrealistic when heavy-use customers are willing to pay more for the service. The possibility of such dependence is incorporated in Model 2.

A customer with usage rate x in Model 1 will subscribe with probability $P(\min\{a, bx\})$, and will not subscribe, hence pay nothing to A or B, with probability $1 - P(\min\{a, bx\})$. It follows that the average revenues to A and B for a period in which (a, b) applies are

$$A(a,b) = aP(a) \int_{x=a/b}^{\infty} \mu(x)dx \tag{1}$$

$$B(a,b) = \int_{x=0}^{a/b} bxP(bx)\mu(x)dx \ . \tag{2}$$

Assuming that the firms know μ and P, we are interested in their choices of the fee under their control – a for A, b for B – when they desire to maximize their own revenues.

Differentiation of $A(a, b)$ with respect to a and $B(a, b)$ with respect to b in (1) and (2) yields the following first-order conditions for an equilibrium point:

$$\frac{a}{b}P(a)\mu\left(\frac{a}{b}\right) = [P(a) + aP'(a)] \int_{x=a/b}^{\infty} \mu(x)dx \tag{3}$$

$$\left(\frac{a}{b}\right)^2 P(a)\mu\left(\frac{a}{b}\right) = \int_{x=0}^{a/b} [P(bx) + bxP'(bx)]x\mu(x)dx \ , \tag{4}$$

where $P'(x) = dP(x)/dx$. If (a^*, b^*) is an equilibrium point then (3) and (4) must hold when $(a, b) = (a^*, b^*)$.

We have considered a variety of specifications of μ and P for which (3) and (4) have no positive (a, b) solution. In most of these, $S \to (0, 0)$ for a price war. However, some specifications have equilibrium points that are SEPs. The following example is illuminating because it has both SEPs and price wars that are governed by a parameter in the specification of μ.

Example 1. Let $P(x) = e^{-cx}$ with $c > 0$. Then $P(0) = 1$ and $P' < 0$ with $P(x) \to 0$ as x gets large. Changes in c allow us to calibrate the probability that a customer will actually subscribe. For example, if $x = 10$, then $P(10) = 0.905$ when $c = 0.01$, and $P(10) = 0.607$ when $c = 0.05$.

Let μ be a negative power function with parameters k and α:

$$\mu(x) = \frac{(k-1)\alpha^{k-1}}{(\alpha + x)^k}, \alpha > 0 \quad \text{and} \quad k \geq 2 \ .$$

The case of $k = 2$ is of limited interest, for then the population's expected usage rate, defined by $E(x) = \int_0^{\infty} x\mu(x)dx$, is infinite. When $k > 2$, $E(x) = \alpha/(k - 2)$. For example, if $\alpha = 20$ and $k = 2.5$, the average number of hits per customer during a period is 40.

We simplify (3) and (4) by combining scale parameters c and α with the decision variables a and b to define p and q by

$$p = ca \quad \text{and} \quad q = \alpha cb \ .$$

Substitution of the specific forms for P and μ in (3), integration, and reduction, shows that (3) is equivalent to

$$q = p(p + k - 2)/(1 - p), \quad 0 < p < 1 \ . \tag{5}$$

Beginning from $B(a, b)$ given by (2), we make the variable substitution $y = bx$ to change the limits of integration to $y = 0$ to $y = a$, then differentiate with respect to b under the integral sign to obtain

$$\partial B(a, b)/\partial b = 0 \Leftrightarrow \int_{y=0}^{a} \frac{y[y(k-1) - \alpha b]}{(y + \alpha b)^{k+1}} \mathrm{e}^{-cy} dy = 0 \ .$$

Another change of variable to $z = cy$ shows that (4) is equivalent to

$$\int_{z=0}^{p} \frac{z[(k-1)z - q]}{(z + q)^{k+1}} \mathrm{e}^{-z} dz = 0 \ . \tag{6}$$

Hence the first-order conditions (3) and (4) for an equilibrium point reduce to (5) and (6), which leave k as the only free parameter. Its value determines whether there is an SEP or a price war.

The second-order conditions for maxima with (5) and (6) also hold for every $k \geq 2$. In particular, given $q > 0$, there is a unique p in $(0, 1)$ that satisfies (5), and $\partial^2 A(a, b)/\partial a^2 < 0$ for the corresponding $a = p/c$. Similarly, given $0 < p < 1$, there is a unique $q > 0$ that satisfies (6), and $\partial^2 B(a, b)/\partial b^2 < 0$ for the corresponding $b = q/(\alpha c)$. We shall not go into the details of the second-order conditions but will verify uniqueness of an equilibrium point (p, q) as a function of k when $2 \leq k < 3$.

Theorem 3.1 *Given P and μ as above,* (5) *and* (6) *have a unique joint positive solution* (p^*, q^*) *as a function of k when* $2 \leq k < 3$, *and this solution defines an SEP for each such k with* $S \to (p^*/c, q^*/(\alpha c))$. *If* $k \geq 3$, *then* (5) *and* (6) *have no positive solution and* $S \to (0, 0)$.

We comment on the effects of k in the SEP region $2 \leq k < 3$ before turning to a proof of the theorem. As k increases from 2 toward 3, firm A's equilibrium fee $a^* = p^*/c$ decreases from approximately 0.3/c to 0. This is shown on the left part of Figure 1. The equilibrium fee $b^* = q^*/(\alpha c)$ for B also decreases steadily via (5) as we approach the price-war region of $k \geq 3$. At the same time, the revenue ratio at equilibrium, $A(a^*, b^*)/B(a^*, b^*)$, favors firm A slightly but approaches parity as k approaches 3: see the right part of Figure 1. We note also that both firms' equilibrium revenues go to 0 as $k \to 3$. Specific calculations show that A's revenue at $k = 2.5$ is 42% of its revenue at $k = 2$; at $k = 2.75$, A's revenue is 22% of its revenue at $k = 2$.

The rest of this section outlines the proof of Theorem 3.1. We omit several algebraic and computational details but will pay close attention to the main steps in the proof.

We begin by substituting the value of q given by (5) into (6), and then change the variable z in (6) to $v = z/p$ so that the integral goes from $v = 0$ to $v = 1$. The integral in (6) then becomes $[(1 - p)^k/p^{k-2}]G(p, k)$, where

$$G(p, k) = \int_{v=0}^{1} \frac{v[v(k-1)(1-p) - (p + k - 2)]}{[v(1-p) + (p + k - 2)]^{k+1}} \mathrm{e}^{-pv} \, dv \ . \tag{7}$$

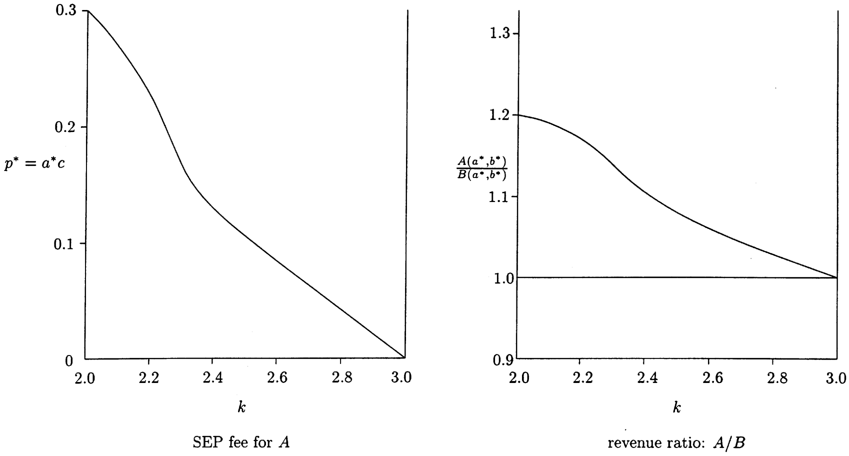

Figure 1. SEP illustrations for Example 1

Given k, (5) and (6) hold for some $0 < p < 1$ if and only if $G(p, k) = 0$ for some p in $(0, 1)$.

Because $k \geq 2$ and we want $0 < p < 1$, the bracketed part of the numerator in (7) is negative for $v < (p + k - 2)/[(k-1)(1-p)]$ and is positive for $v > (p + k - 2)/[(k-1)(1-p)]$. Hence $G(p, k) = 0$ only if $1 > (p + k - 2)/[(k-1)(1-p)]$, i.e., only if $p < \frac{1}{k}$, for otherwise $G(p, k) < 0$.

Since $G(\frac{1}{k}, k) < 0$, continuity of G in p assures the existence of an equilibrium p, given k, if $G(0, k) > 0$, where $G(0, k) = \lim_{p \to 0} G(p, k)$. At $k = 2$, where the integral of (6) equals $(1-p)^k G(p, 2)$, we have $G(0, 2) = \int_{v=0}^{1} dv/v = \ln v]_0^1 = \infty$. For $k > 2$, we set $p = 0$ in (7) and integrate to get

$$G(0, k) = \left[\frac{1}{(k-2)^{k-2}} - \frac{3k-5}{(k-1)^{k-1}}\right] \Big/ [(k-1)(k-2)], \quad k > 2 \ .$$

Let $\lambda = k - 2$, so

$$G(0, k) > 0 \Leftrightarrow \frac{1}{\lambda^{\lambda}} > \frac{1 + 3\lambda}{(\lambda + 1)^{\lambda+1}} \ .$$

The inequality here holds when $0 < \lambda < 1$, i.e. when $2 < k < 3$, but $G(0, 3) = 0$ and $G(0, k) < 0$ for $k > 3$. Consequently, we are assured of an equilibrium point when $2 \leq k < 3$ but not when $k \geq 3$.

Support for the nonexistence of an equilibrium when $k \geq 3$ comes from the derivative of G with respect to p near $p = 0$. Let $G_1(p, k) = \partial G(p, k)/\partial k$, and let $G_1(0, k) = \lim_{p \to 0} G_1(p, k)$. At $k = 2$, $G_1(0, 2) = -\infty$, and at $k = 3$, $G_1(0, 3) = 1.25 - 2ln2 = -0.136$. For other $k > 2$, we find that

$$G_1(0, k) = \frac{\left[\frac{8\lambda^2 + 6\lambda + 2}{(\lambda+1)^{\lambda+1}} - \frac{1 + 3\lambda}{\lambda^{\lambda}}\right]}{(\lambda - 1)\lambda(\lambda + 1)}, \quad \lambda \neq 1, \ \lambda \neq 0 \ .$$

When $0 < \lambda < 1$, the numerator is positive and the denominator is negative; when $\lambda > 1$, the numerator is negative and the denominator is positive. Therefore

$$G_1(0, k) < 0 \quad \text{for all} \quad k > 2 \ .$$

In particular, $G(p, k)$ begins at or below zero at $p = 0$ whenever $k \geq 3$, then decreases.

We confirm the nonexistence of an equilibrium point for $k \geq 3$ by considering a crude but informative approximation of the integral in (7) that ignores the exponential factor e^{-pv}, which equals 1 at $v = 0$ and decreases to e^{-p} at $v = 1$: for example, $\mathrm{e}^{-0.1} = 0.905$ and $\mathrm{e}^{-0.3} = 0.741$. Denote by $H(p, k)$ the integral obtained from (7) by replacing e^{-pv} by 1. Integration shows that $H(p, k) = 0$ with $k - 2 = \lambda > 0$ if and only if

$$(p + \lambda)^{\lambda}\left[\lambda\left(p^2 - 3p + 3\right) + 1\right] = (\lambda + 1)^{\lambda+1} \ . \tag{8}$$

This applies also to $k = 2$ by letting $\lambda \to 0$. Rewriting (8) as

$$\frac{\lambda + 1}{\lambda + p} = \left[\frac{\lambda\left(p^2 - 3p + 3\right) + 1}{\lambda + 1}\right]^{1/\lambda} = \left[1 + \frac{p^2 - 3p + 2}{n + 1}\right]^n, \quad n = 1/\lambda \ ,$$

and letting $\lambda \to 0$, we get $1/p = \mathrm{e}^{p^2 - 3p + 2}$. The unique solution to this equation for $0 < p < \frac{1}{2}$ is, approximately, $p = 0.3162$. We denote this solution value by $p_0(2)$ for $k = 2$.

We claim that (8) also has a unique p solution in $(0, 1/k)$ that we denote by $p_0(k)$ for every $2 < k < 3$, i.e., for every $0 < \lambda < 1$. Differentiation of the left side of (8) with respect to p shows that the derivative equals 0 at two points, namely $p = 1/k$ and $p = 1$. For $0 < \lambda < 1$, the left side of (8) equals $\lambda^{\lambda}[1 + 3\lambda]$ at $p = 0$, which we noted earlier is less than $(\lambda + 1)^{\lambda+1}$, increases to a maximum at $p = 1/k$, then decreases to $(\lambda + 1)^{\lambda+1}$ at $p = 1$. It follows for each such λ that there is a unique $p = p_0(k) = p_0(\lambda + 2)$ in $(0, 1/k)$ that satisfies (8) (see Fig. 2).

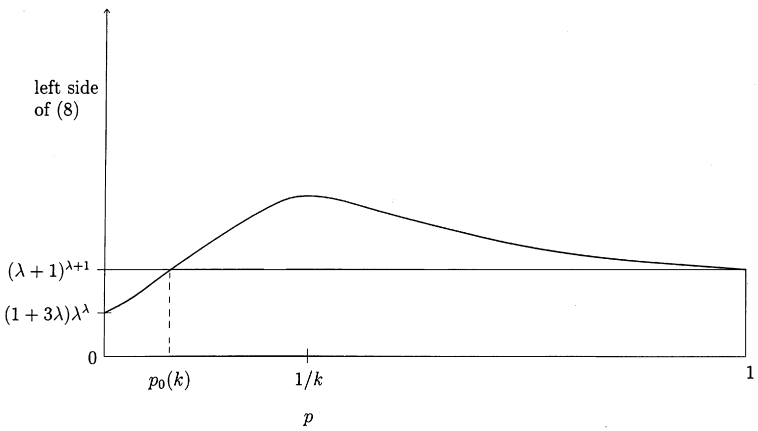

Figure 2. Illustration of solution to (8) for $0 < \lambda < 1$

It must be true for $2 \le k < 3$ that $G(p_0(k), k) < 0$ because our $H(p, k) = 0$ computation ignored e^{-pv} in (7), and the negative part of $G(p, k)$ for $v < (p + k - 2)/[(k - 1)(1 - p)]$ is weighted by larger values of e^{-pv} than the positive part for $v > (p + k - 2)/[(k - 1)(1 - p)]$. It follows that the actual equilibrium point $p^*(k)$ for $G(p^*(k), k) = 0$ is slightly less than $p_0(k)$ to correct for the exponential factor. Percentagewise, the correction is fairly modest, and is very small for k near 3, where p_0 is near to 0 and there is little variation in $e^{-p_0 v}$ as v goes from 0 to 1.

Suppose $k \ge 3$, where $G(0, k) \le 0$ and $G_1(0, k) < 0$. Then $H(p, k) = 0$ for no $0 < p < 1$. For example, when $k = 3$ ($\lambda = 1$), the left side of (8) increases from 4 at $p = 0$ up to a maximum and then decreases to $4 = (\lambda + 1)^{\lambda+1}$ at $p = 1$: alternatively, (8) reduces to $p(p - 1)^2 = 1$. When $k > 3$, the left side of (8) begins above $(\lambda + 1)^{\lambda+1}$, increases, and then decreases to $(\lambda + 1)^{\lambda+1}$ at $p = 1$. If $k \ge 3$ had an equilibrium p^* for $G(p, k) = 0$, $0 < p < 1/k$, then the inverse of the argument in the preceding paragraph shows that there must be a p_0 slightly larger than p^* at which $H(p_0, k) = 0$. Since no such p_0 exists, we conclude that (5) and (6) have no equilibrium point when $k \ge 3$.

For $2 \le k < 3$, we have seen that there is a unique $p = p_0(k)$ in $(0, 1/k)$ at which $H(p, k) = 0$. We have not yet proved, however, that $G(p, k) = 0$ for a *unique* $p = p^*(k)$ for $2 \le k < 3$. To prove uniqueness, we consider the ratio of the two main pieces of (7) as follows:

$$R(p, k) = \frac{\int_{v=0}^{1} \frac{v^2(k-1)(1-p)e^{-pv}}{[v(1-p)+p+k-2]^{k+1}}\, dv}{\int_{v=0}^{1} \frac{v(p+k-2)e^{-pv}}{[v(1-p)+p+k-2]^{k+1}}\, dv} \quad .$$

Point p is an equilibrium point if and only if $R(p, k) = 1$. We claim that $\partial R/\partial p < 0$ for $2 \le k < 3$ and $p \in [0, 1/k]$, which is clearly sufficient for uniqueness. Differentiation shows that $\partial R/\partial p < 0$ if and only if

$$\begin{aligned}
&\int_{v=0}^{1} v^2(1 - p)[v(1 - p) + s]V \int_{v=0}^{1} v[(k + 1)s(1 - v) + (v(1 - p) + s)(sv - 1)]V \\
&\quad < \int_{v=0}^{1} vs[v(1 - p) + s]V \int_{v=0}^{1} v^2[(v(1 - p) + s)(1 + v(1 - p)) \\
&\qquad + (k + 1)(1 - p)(1 - v)]V \ ,
\end{aligned}$$

where $s = p + k - 2$ and $V = \{e^{-pv}/[v(1 - p) + s]^{k+2}\}dv$. Omitting V, which is understood to be part of every integral [e.g., $\int v^3$ denotes $\int_{v=0}^{1} v^3 V$], the preceding inequality rearranges to

$$\begin{aligned}
&s(1 - p)^3 \int v^3 \int v^3 + s(1 - p)\left[k(3 - 2p) - (2 - p)^2\right] \int v^3 \int v \\
&\quad < s(1 - p)^3 \int v^4 \int v^2 + s^2(1 - p)^2 \int v^4 \int v \\
&\qquad + (1 - p)^2\left(k - 1 - s^2\right) \int v^3 \int v^2
\end{aligned}$$

$$+ s(1-p)\Big[k(3-2p) - (2-p)^2 + 2(k-1)\Big]$$
$$\times \int v^2 \int v^2 + s^2(k-1) \int v^2 \int v^1 \ .$$

All coefficient multipliers of the integrals are positive, except that $k - 1 - s^2 < 0$ when p is near $\frac{1}{k}$. Because

$$\int v^4 \int v^2 - \int v^3 \int v^3 = \int_{v=0}^{1} \int_{u=0}^{v} \frac{v^2 u^2 (v-u)^2 \mathrm{e}^{-p(u+v)}}{[(v(1-p)+s)(u(1-p)+s)]^{k+2}}\, du\, dv > 0 \ ,$$

we have $s(1-p)^3 \int v^3 \int v^3 < s(1-p)^3 \int v^4 \int v^2$. Hence $\partial R/\partial p < 0$ if it is true also that

$$s(1-p)\Big[k(3-2p) - (2-p)^2\Big] < s^2(1-p)^2 \frac{\int v^4}{\int v^3} + (1-p)^2\big(k-1-s^2\big)\frac{\int v^2}{\int v}$$
$$+ s(1-p)\Big[k(3-2p) - (2-p)^2 + 2(k-1)\Big]\frac{\int v^2 \int v^2}{\int v^3 \int v} + s^2(k-1)\frac{\int v^2}{\int v^3} \ ,$$

where $\int v^4 / \int v^3 < 1$, $\int v^2 / \int v < 1$, $\int v^2 / \int v^3 > 1$, and $\int v^2 \int v^2 / \int v^3 \int v < 1$. Analysis of the preceding inequality shows that it holds for $0 \le p \le \frac{1}{k}$ and $2 \le k < 3$. For example, when $p = \frac{1}{k}$, it reduces to

$$\big(3k^3 - 6k^2 + 4k - 1\big) + \big(k^3 - 4k^2 + 3k - 1\big)\frac{\int v^2}{\int v} < (k-1)^3 \frac{\int v^4}{\int v^3}$$
$$+ \big(5k^3 - 8k^2 + 4k - 1\big)\frac{\int v^2 \int v^2}{\int v^3 \int v} + \big(k^4 - 2k^3 + k^2\big)\frac{\int v^2}{\int v^3} \ .$$

It is not hard to show for this special case that $\int v^2 \int v^2 / \int v^3 \int v > \frac{1}{2}$ and $\int v^2 / \int v^3 > \frac{5}{4}$, which in conjunction with $\int v^2 / \int v < 1$ is enough to validate the inequality for all $k \in [2, 3]$. We omit further details.

To summarize, we have a unique equilibrium point $(p^*(k), q^*(k))$ for each $2 \le k < 3$, where $G(p^*(k), k) = 0$ and q^* is computed from p^* in (5). There is no equilibrium for (5) and (6) when $k \ge 3$. The proof of Theorem 3.1 will be completed by showing that (p^*, q^*) is an SEP in terms of p and q for $k < 3$, and that a price war occurs if $k \ge 3$.

Given q, the unique positive solution in p for (5) is

$$p = \frac{\sqrt{(\lambda+q)^2 + 4q} - (\lambda+q)}{2} < 1 \ . \qquad (9)$$

This is the value of p that maximizes $A(a, b)$ when q is given. As before, $\lambda = k - 2$. When $\lambda = 1$ ($k = 3$), it is easily seen that $p < q$ with p given by (9). The same thing is true for $\lambda > 1$, but not for $\lambda < 1$ in the $2 \le k < 3$ range.

Similarly, given p in $(0, 1)$, there is a unique $q > 0$ that satisfies (6), and this is the value of q that maximizes $B(a, b)$ when p is given. When $k = 3$, (6) is

$$\int_{z=0}^{p} \frac{z(2z-q)}{(z+q)^4} \mathrm{e}^{-z} dz = 0 \ . \qquad (10)$$

If we set $q = p$ here and replace e^{-z} by 1, the resulting integral equals 0, and because e^{-z} decreases in z it follows that the left side of (10) is negative when $q = p$. The sign of the left side of (10) is the same as the sign of $\partial B(a,b)/\partial b$, so when q is near p we have already exceeded the value of q that satisfies (10). Consequently

$$k = 3 \Rightarrow q < p \text{ when p is given and (10) holds} \quad .$$

The same conclusion holds for $k > 3$ and, as proved later, for $2 \le k < 3$.

Suppose $k = 3$, and assume without loss of generality that firm B begins our iterative process described for S in Section 2 with $q = q_0$. Then firm A's best response is $p_1 < q_0$, where p_1 is given by (9) when $q = q_0$. Given p_1, B's best response via (10) is $q_1 < p_1$. Then A's best response to $q = q_1$ is $p_2 < q_1$, where p_2 is given by (9) when $q = q_1$. Continuation yields the best response and counter-response sequence $q_0 > p_1 > q_1 > p_2 > q_2 > p_3 > \cdots$. This sequence converges either to $(p^*, q^*) > (0,0)$ or to $(0,0)$. However, it cannot converge to (p^*, q^*), for suppose that $p^* > 0$ and, at some n, $p_n = p^* + \epsilon$ with $0 < \epsilon < 2(p^*)^2/(1-p^*)$. Then $q_n < p^* + \epsilon$, so

$$p_{n+1} < \left\{ \left[(1 + p^* + \epsilon)^2 + 4(p^* + \epsilon) \right]^{1/2} - (1 + p^* + \epsilon) \right\} \Big/ 2 \quad .$$

But the right side of this inequality is less than p^*, so $p_{n+1} < p^*$, a contradiction.

We conclude for $k = 3$ that $S \to (0,0)$. Similar reasoning yields the same conclusion for every $k > 3$, so there is a price war whenever $k \ge 3$.

Assume henceforth that $2 \le k < 3$. We will work with a fixed $k = \lambda + 2$ and denote its unique equilibrium pair by (p^*, q^*).

Let $f_1(q)$ be the value of $p < 1$ given by (9) that satisfies (5):

$$f_1(q) = \frac{\sqrt{(\lambda + q)^2 + 4q} - (\lambda + q)}{2} \quad .$$

It is easily checked that $f_1(0) = 0$, $f_1(\frac{1-\lambda}{2}) = \frac{1-\lambda}{2}$, and f_1 is concave increasing with slope $1/\lambda$ at 0:

$$f_1'(0) = \frac{1}{\lambda} \quad .$$

Figure 3 shows f_1 as the bowed curve through $(\frac{1-\lambda}{2}, \frac{1-\lambda}{2})$.

Let $f_2(p)$ be the unique q that satisfies (6). Clearly, $f_2(0) = 0$ and f_2 is increasing and continuous as shown in Figure 3. We verify three other properties of f_2:

[1] $f_2(p) < p$ for $p > 0$;
[2] $f_2'(0) = 1/r$, where $r > 1$ is the unique positive solution to

$$\ln(r+1) = \frac{r(2r+1)}{(r+1)^2} \quad \text{when} \quad k = 2 \ , \tag{11}$$

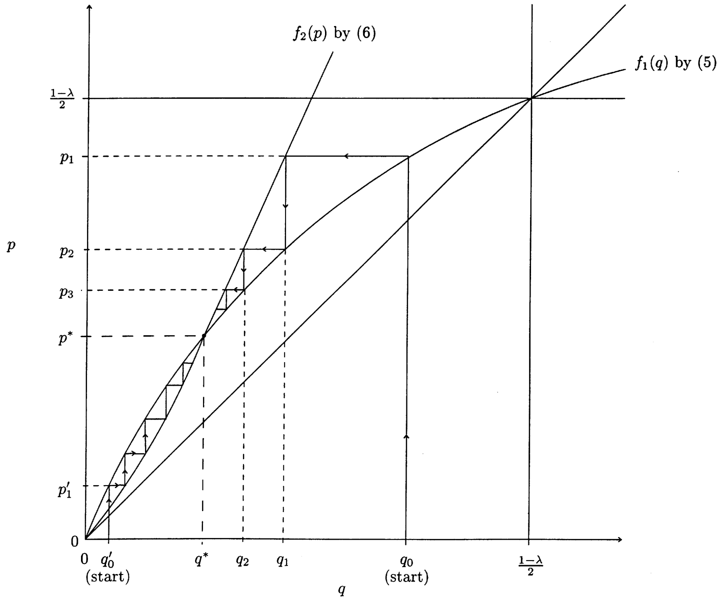

Figure 3. Zigzag approaches to SEP (p^*, q^*) from above (q_0) and below (q'_0), for a fixed k in $(2, 3)$

$$\frac{1}{(r+1)^k}\left[\frac{k-1}{k-2}(r+1)^2 - \frac{2k-1}{k-1}(r+1)+1\right] = \frac{1}{(k-1)(k-2)} \quad \text{when } k > 2 \ ; \tag{12}$$

[3] $r < \frac{1}{\lambda}$.

Property [1] says that f_2 stays to the left of the 45° line on Figure 3. Property [3] shows that the slope of f_2 (viewed in the usual Cartesian orientation) is less than the slope of f_1 at the origin. It follows that f_2 begins below f_1 going out from the origin, then crosses f_1 at (p^*, q^*), and lies above f_1 thereafter. Because the equilibrium point is unique, there is exactly one point above the origin at which f_1 and f_2 cross (our case) or touch without crossing.

Property [1] is proved as follows. Replace the upper limit of integration p by q and replace e^{-z} by 1 in (6), then integrate to get

$$\int_{z=0}^{q} \frac{z[(k-1)z-q]}{(z+q)^{k+1}}\,dz = \frac{2^k - k^2 + k - 2}{q^{k-2}2^k(k-1)(k-2)}$$

when $k > 2$. (We omit the calculation for $k = 2$.) The preceding fraction is negative for $2 < k < 3$, i.e. $2^k < k^2 - k + 2$, and when e^{-z} is reconsidered, we conclude that

$$\int_{z=0}^{q} \frac{z[(k-1)z-q]}{(z+q)^{k+1}} e^{-z}\,dz < 0 \ .$$

It follows that the upper limit of integration must be increased for the integral to vanish, or that $f_2(p) < p$.

To compute $f_2'(0)$ for [2], we disregard e^{-z} in (6) because it is virtually constant in $[0, p]$ when p is near 0 and has no effect on the limit as we take p to 0. A change of variable from z to $v = z/q$ gives

$$\int_{z=0}^{p} \frac{z[(k-1)z-q]}{(z+q)^{k+1}}\, dz = \frac{1}{q^{k-2}} \int_{v=0}^{r} \frac{v[(k-1)v-1]}{[v+1]^{k+1}}\, dv, \quad r = \frac{p}{q} \ .$$

The integral on the right, when set equal to 0, yields (11) for $k = 2$ and (12) for $k > 2$. The unique solution to (11) is, approximately, $r = 2.16$. Differentiation of the left side of (12) with respect to r shows that it decreases in $r \geq 1$ and hence gives a unique $r > 1$ at which the left side equals $[(k-1)(k-2)]^{-1}$. Because the r solution approximates $p/f_2(p)$, or $f_2(p) \doteq p/r$, which becomes exact in the limit, $f_2'(p)$ is approximated by $1/r$ and we have $f_2'(0) = 1/r$.

Property [3] is trivial for $k = 2$ ($\lambda = 0$). For $k > 2$, the decreasing aspect of the left side of (12) implies [3] if that side is less than $[(k-1)(k-2)]^{-1}$ when r is replaced there by $1/\lambda$. The desired inequality reduces to $1/\lambda^{\lambda} > (1+3\lambda)/(\lambda+1)^{\lambda+1}$. As noted earlier, this holds for $0 < \lambda < 1$.

It remains to show that (p^*, q^*) is an SEP, i.e. that $q_0, p_1, q_1, p_2, q_2, \ldots$ for any $q_0 > 0$ is such that $p_n \to p^*$ and $q_n \to q^*$ where

$$p_n = f_1(q_{n-1}) \quad \text{and} \quad q_n = f_2(p_n) \quad \text{for} \quad n = 1, 2, \ldots \ .$$

Suppose $q_0 > \frac{1-\lambda}{2}$. We have $p_n < 1$ by (9) for all n, and $q_n < p_n$ by [1] for all n. As long as $\frac{1-\lambda}{2} < q_n$, $p_{n+1} < q_n$. After a finite number of iterations, which depends on q_0 and k, p_n is sufficiently close to $\frac{1-\lambda}{2}$ so that $q_n < \frac{1-\lambda}{2}$. It suffices henceforth to suppose that $0 < q_0 < \frac{1-\lambda}{2}$.

If $q_0 = q^*$ then $p_n = p^*$ and $q_n = q^*$ for all n. If $q^* < q_0 < \frac{1-\lambda}{2}$, we get the arrowed zigzag pattern between f_1 and f_2 pictured above (p^*, q^*) on Figure 3. If $0 < q_0 < q^*$, we get the arrowed zigzag pattern between f_1 and f_2 pictured below (p^*, q^*) on Figure 3. We claim that $p_n \to p^*$ and $q_n \to q^*$ for the $q_0 \neq q^*$ cases. The only way convergence to (p^*, q^*) could fail is to get stuck at some $(p', q') \neq (p^*, q^*)$. Suppose this happens above (p^*, q^*) so that $p_1 > p_2 > p_3 > \cdots$ converges to $p' > p^*$ and $q_0 > q_1 > q_2 > \cdots$ converges to $q' > q^*$. Then $f_1(q_n) \downarrow p' > p^*$ and $f_2(p_n) \downarrow q' > q^*$, so continuity implies that $f_1(q') = p'$ and $f_2(p') = q'$. This implies that (p', q') is an equilibrium pair, contrary to (p^*, q^*) as the unique equilibrium. A similar contradiction obtain below (p^*, q^*), and we conclude in all cases that $(p_n, q_n) \to (p^*, q^*)$.

4 Model 1: Price wars and collusion

Example 1 demonstrates the sensitivity of equilibrium existence to the forms assumed for μ and P. The present section elaborates on this and related issues in three ways.

First, we show that there is a general condition for P that invariably implies a price war for forms of μ. In particular, if $xP(x)$ is concave increasing up to a maximum and decreases thereafter, and if μ is a negative exponential density, then $S \to (0, 0)$.

Second, we describe four pricing schemes that avoid a ruinous price war when $S \to (0, 0)$. In two of these, one firm announces and maintains a fixed fee while the other maximizes its revenue given the announcement. The other two schemes involve cooperative and perhaps covert collusion. We compare the four schemes' revenues and note that a collusive scheme maximizes overall revenues.

Third, we discuss implications of Model 1 that can occur when P is positive and constant. Although unrealistic in practice, this allows us to illustrate interesting cases of multiple equilibria.

Price wars

The following theorem shows that a price war can obtain under fairly general conditions. Its assumptions for P hold for a wide variety of specific forms, including that of Example 1. We let $Q(x) = xP(x)$ for all $x \geq 0$.

Theorem 4.1 *Suppose P is nonincreasing and twice differentiable, and Q is concave increasing up to a maximum at x^* and then decreases. If $\mu(x) = \gamma e^{-\gamma x}$ for some $\gamma > 0$, then $S \to (0, 0)$.*

Proof. The theorem's assumptions for P and Q imply that $Q(x)/Q'(x)$, whose derivative for $x \neq x^*$ is $1 - Q(x)Q''(x)/Q'(x)^2$, increases from 0 at the origin toward ∞ as $x \to x^*$, and is negative thereafter. It follows from (1) that $A(a, b)$ is maximized at $a < x^*$ when a satisfies

$$\frac{Q(a)}{Q'(a)} = \frac{b}{\gamma} \ .$$

We denote this a value by $f_1(b)$. Clearly, $b < b' \Rightarrow f_1(b) < f_1(b')$ and $f_1(b) \to 0$ as $b \to 0$. In addition, our assumptions imply $Q(x)/Q'(x) > x$ for $x < x^*$, so $f_1(b) < b/\gamma$ for all b. Thus

$$f_1(b) < \min\{x^*, b/\gamma\} \ .$$

Equation 2 implies that $\partial B(a, b)/\partial b = 0$ if and only if

$$\int_{y=0}^{a} Q(y)(\gamma y/b - 1)e^{-\gamma y/b}\, dy = 0 \ , \tag{13}$$

where the sign of $\partial B(a, b)/\partial b$ is the same as the sign of the left side of (13). The derivative of the integral in (13) with respect to b equals $1/b$ times

$$\int_{y=0}^{a} Q(y)(\gamma y/b)(\gamma y/b - 2)e^{-\gamma y/b}\, dy \ .$$

This approaches 0 as $b \to 0$, it is positive for small $b > 0$, and it is negative for $b/\gamma \geq a/2$ because $\gamma y/b - 2 \leq 0$ for all $y \in [0, a]$ when $\gamma a/b \leq 2$. We shall not prove that there is a unique b that satisfies (13), given a, but note that some such b maximizes $B(a, b)$ when a is given. Moreover, such a b satisfies

$$\frac{b}{\gamma} < \frac{2a}{3} \tag{14}$$

for the following reasons, assuming that $a < x^*$. The argument $(\gamma y/b - 1)e^{-\gamma y/b}$ in (13) increases from -1 at $y = 0$ up to $y = 2b/\gamma$, passing through 0 at $y = b/\gamma$. Because Q is concave increasing on $[0, x^*]$, we consider the limit of the case for Q that forces b/γ to be as close to a as possible when b/γ is the (largest) solution to (13). This limit is $Q(y) = c_0 y$, where c_0 is a positive constant, because it makes the negative part of the integral (y from 0 to b/γ) as absolutely small as possible while maximizing the positive part for $y > b/\gamma$, given a fixed value of $Q(b/\gamma)$. Set $c_0 = 1$ with no loss of generality. Then, with $Q(y) = y$, the left side of (13) is

$$\int_{y=0}^{a} \left[y^2\gamma/b - y\right] e^{-\gamma y/b}\, dy \ .$$

When $e^{-\gamma y/b}$ is replaced by 1, the resulting integral equals 0 if and only if $b/\gamma = 2a/3$. The derivative of the preceding integral is negative for all $b/\gamma > a/2$, and because $e^{-\gamma y/b}$ is larger for its negative part than its positive part, it follows that the integral can equal 0 only if $b/\gamma < 2a/3$. Hence (14) holds when b maximizes $B(a, b)$ for a given a.

Now suppose that firm B begins our iterative process with $b = b_0 > 0$, firm A counters with $a_1 = f_1(b_0)$ that maximizes $A(a, b_0)$, firm B counters this with a $b = b_1$ that maximizes $B(a_1, b)$, and so forth. We have $a_n < b_{n-1}/\gamma$ and $b_n/\gamma < 2a_n/3$ for $n = 1, 2, \ldots$, so $(a_n, b_n) \to (0, 0)$. Hence the conditions of the theorem induce a price war. □

Four pricing schemes

A particular case of Theorem 4.1 occurs when $P(x) = e^{-cx}$, as in the preceding section. We outline four pricing schemes for the firms that circumvent a price war, and note their revenue implications for this P and $\mu = \gamma e^{-\gamma x}$.

Scheme 1. Firm A chooses a fixed subscription fee per period and announces that it will not deviate from this fee. When A's fee is a, firm B chooses $b = f_2(a)$ to maximize $B(a, b)$ and, knowing this, A chooses a to maximize $A(a, f_2(a))$. Computations show that a is approximately $(0.7)/c$ and $f_2((0.7)/c) \doteq \gamma/(2.86c)$. To illustrate these fees, suppose $c = 0.05$ and $\gamma = 1/10$. Then $a = 14$ and $b = 0.7$, so if revenue in measurement in dollars, A charges \$14 per period and B charges 70 cents per hit. As one would expect, firm B gets the lions share of the business: the revenue ratio at the solution point is $A(a, f_2(a))/B(a, f_2(a)) = 0.325$.

Scheme 2. Firm B chooses a fixed per-use fee and announces that it will not deviate from this fee. When B's fee is b, firm A chooses $a = f_1(b)$ to maximize $A(a, b)$ and, knowing this, B chooses b to maximize $B(f_1(b), b)$. In this case b is approximately γ/c and $f_1(b) \doteq (0.5)/c$. When $c = 0.05$ and $\gamma = 1/10$, A

charges \$10 per period and B charges \$2 per hit. The revenue ratio at the solution point is $A(f_1(b), b)/B(f_1(b), b) = 2.784$.

The *sum* of the firms' revenues per customer is $(0.19)/c$ for scheme 1 and $(0.25)/c$ for scheme 2, or \$3.80 and \$5.00 respectively when $c = 0.05$. Greater sums are possible when A and B collude, to the detriment of customers. This is illustrated by two collusion schemes.

Scheme 3. The firms agree to set (a, b) so that their revenues are equal and as large as possible. That is, they maximize

$$A(a,b) + B(a,b) = a\mathrm{e}^{-ca} \int_{x=a/b}^{\infty} \gamma \mathrm{e}^{-\gamma x}\, dx + \int_{x=0}^{a/b} \gamma b x \mathrm{e}^{-(cb+\gamma)x}\, dx$$

subject to $A(a, b) = B(a, b)$. The (a, b) solution here is approximately $(1.38/c, \gamma/(0.6c))$ with $A(a, b) + B(a, b) = (0.3034)/c$.

Scheme 4. The firms collude to maximize $A(a, b) + B(a, b)$, which would be the monopolist solution if A and B were the same firm, and agree to split the total revenue equally. The monopolistic maximum occurs when b is effectively ∞ and $a = 1/c$. The total revenue per customer, all of which comes from A's fixed subscription fee, is $\mathrm{e}^{-1}/c = (0.368)/c$, which is a 21% increase over scheme 3 and a 94% increase over scheme 1. When $c = 0.05$, firm A charges \$20 per period in scheme 4.

The following table shows the revenues per customer per period for the firms when $(c, \gamma) = (0.05, 1/10)$.

	Firm A	Firm B	Total
Scheme 1	\$0.93	\$2.87	\$3.80
Scheme 2	\$3.68	\$1.32	\$5.00
Scheme 3	\$3.03	\$3.03	\$6.06
Scheme 4	\$3.68	\$3.68	\$7.36

Constant P and multiple equilibria

We conclude our present remarks for Model 1 by considering the unrealistic but analytically informative case in which P is constant and positive. Assume that $P(x) = c_0 > 0$ for all x. Then the first-order conditions (3) and (4) for an equilibrium reduce to

$$t\mu(t) = \int_{x=t}^{\infty} \mu(x) dx \ , \tag{15}$$

$$t^2\mu(t) = \int_{x=0}^{t} x\mu(x) dx \ , \tag{16}$$

where $t = a/b$. The proof of Theorem 4.1 shows that if μ is a negative exponential with $\mu(x) = \gamma \mathrm{e}^{-\gamma x}$ then $S \to (0, 0)$. When μ is a negative power function as in Example 1 with parameters $\alpha > 0$ and $k = 2$, a succession of fee changes and counterchanges drives (a_i, b_i) toward (∞, ∞).

In some cases, (15) and (16) have a unique solution $t^* > 0$. An example arises when μ is a particular convex combination of a negative exponential μ_1 and a negative power function μ_2 with $k = 2$, say $\mu = \lambda^* \mu_1 + (1 - \lambda^*)\mu_2$, with $0 < \lambda^* < 1$. The particular λ^* defines a stable knife-edge situation, for if $\lambda > \lambda^*$ then the combination $\lambda\mu_1 + (1 - \lambda)\mu_2$ yields $S \to (0, 0)$, and if $\lambda < \lambda^*$ then $\lambda\mu_1 + (1 - \lambda)\mu_2$ forces $S \to (\infty, \infty)$.

We consider further the particular case of $\mu = \lambda^*\mu_1 + (1 - \lambda^*)\mu_2$, where (a, b) is an equilibrium point if and only if $a/b = t^*$, with $A(bt^*, b) = B(bt^*, b)$. This case has a continuum of equilibrium points. If (a_0, b_0) is not an equilibrium point, then a revenue-maximizing change by one firm but not the other creates an equilibrium point at which neither firm benefits by a further unilateral change. However, if both firms change naively and simultaneously in every period, we get an alternating pattern in which every other period has $(a, b) = (a_0, b_0)$ and the in-between periods have $(a, b) = (b_0 t^*, a_0/t^*)$. Finally, because t^* is fixed and $A(a, b) = a\int_{t^*}^{\infty} \mu(x)dx$ when $a/b = t^*$, both firms have an incentive to collude and make a and b arbitrarily large.

5 Model 2

Model 2 has two features not shared by Model 1. First, it allows direct interdependence between a willingness-to-pay budget constraint w and usage rate x in its joint probability density function $f(w, x)$. Second, it allows restricted usage, as when a usage-rate-x customer makes only $y < x$ hits during a period because its cost otherwise would exceed w.

We assume that f is continuous and there is no explicit default option, so f accounts for all customers. Our cost-minimization assumption implies that a customer with parameter pair (w, x) in a period with fee pair (a, b) will

$$\begin{aligned}&\text{choose } A \text{ and pay } a \text{ to } A \text{ if } a \le \min\{bx, w\}, \text{ and}\\&\text{choose } B \text{ and pay } \min\{bx, w\} \text{ to } B \text{ if } \min\{bx, w\} < a\ .\end{aligned}$$

Because of continuity, it makes no difference whether $a = \min\{bx, w\}$ is attributed to A or B. If either the willingness-to-pay amount w is less than a, or the full-usage per-hit-basis cost bx is less than a, then and only then will the customer subscribe to firm B. If it does, and if $w < bx$, then the customer limits its hits to y such that $by = w$. Alternatively, a customer with $a \le \min\{bx, w\}$ subscribes to A, uses its full hit rate x, and pays a. Figure 4 describes a customer's choice and payment under Model 2

It follows from Model 2 that the average revenues per customer to A and B for a period in which (a, b) applies are

$$A(a, b) = a\int_{w=a}^{\infty}\int_{x=a/b}^{\infty} f(w, x)dx\,dw \tag{17}$$

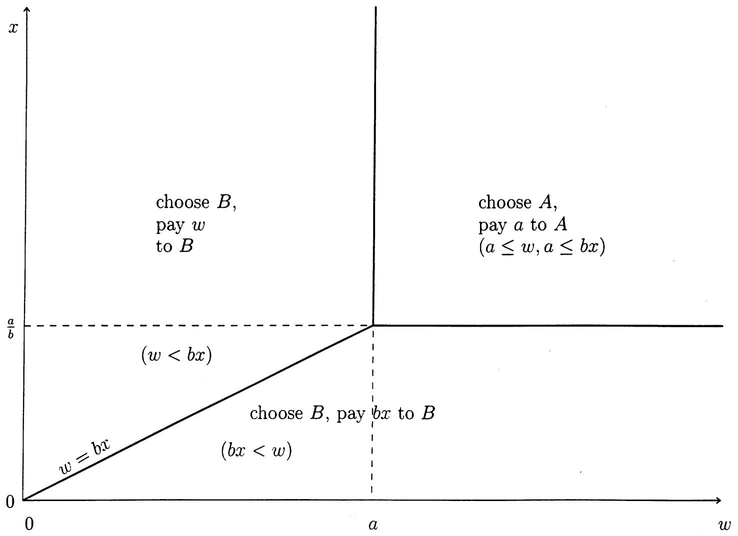

Figure 4. Customer behavior for Model 2

$$B(a,b) = \int_{w=0}^{a} \int_{x=w/b}^{\infty} wf(w,x)dx\,dw + \int_{x=0}^{a/b} \int_{w=bx}^{\infty} bxf(w,x)dw\,dx \ . \tag{18}$$

We assume that each firm knows f and wants to maximize its own revenue by choice of the fee under its control.

Differentiation of $A(a,b)$ with respect to a and $B(a,b)$ with respect to b in (17) and (18) gives the following first-order conditions for equilibrium:

$$\int_{w=a}^{\infty} \int_{x=a/b}^{\infty} f(w,x)dx\,dw = a \int_{x=a/b}^{\infty} f(a,x)\,dx + \frac{a}{b} \int_{w=a}^{\infty} f\left(w, \frac{a}{b}\right) dw \tag{19}$$

$$\int_{x=0}^{a/b} \int_{w=bx}^{\infty} xf(w,x)dw\,dx = \frac{a^2}{b^2} \int_{w=a}^{\infty} f\left(w, \frac{a}{b}\right) dw \ . \tag{20}$$

We have examined many specific forms of f, both when w and x are bounded above and when they are unbounded, and found in most cases that (19) and (20) have no positive (a,b) solution. The predominant result is $S \to (0,0)$. The simplest examples of a price war in the normalized bounded case of $(w,x) \in [0,1]^2$ are $f(w,x) = 1$, $f(w,x) = wx$, and $f(w,x) = c(1 - wx)$. An unbounded example is $f(w,x) = c_1 c_2 e^{-c_1 w - c_2 x}$.

The propensity of Model 2 to induce a price war is explained in part by considering situations in which f is separable. We say that f is *separable* if there are probability density functions g for w and h for x such that

$$f(w,x) = g(w)h(x) \ .$$

Separability has the defect that the expected usage rate given w, $E(x|w)$, is independent of w. We normally expect $E(x|w)$ to increase in w since it seems likely that customers who are willing to pay more for the service will, on average, have greater usage rates. However, we assume separability in what follows because it simplifies the analysis and still allows us to demonstrate key points.

When f is separable, (19) and (20) reduce to

$$[1-G(a)]\left[1-H\left(\frac{a}{b}\right)\right] = ag(a)\left[1-H\left(\frac{a}{b}\right)\right] + \frac{a}{b}[1-G(a)]h\left(\frac{a}{b}\right) \ ,$$

$$\int_{x=0}^{a/b} x[1-G(bx)]h(x)\,dx = \frac{a^2}{b^2}[1-G(a)]h\left(\frac{a}{b}\right) \ ,$$

where G and H are the cumulative distribution functions of g and h, respectively. Rearrangement of the first equation and a change of variable to $v = bx$ in the second yield

$$\left[1-H\left(\frac{a}{b}\right)\right][1-G(a)-ag(a)] = \frac{a}{b}[1-G(a)]h\left(\frac{a}{b}\right) \ , \tag{21}$$

$$\int_{v=0}^{a} v[1-G(v)]h\left(\frac{v}{b}\right)dv = a^2[1-G(a)]h\left(\frac{a}{b}\right) \ , \tag{22}$$

which are tantamount to (19) and (20), respectively, under separability. We refer to separable f as *regular* if for each $b > 0$, (22) holds for at most one $a > 0$. When regularity holds and a' is the unique solution to (22) for a given b', the left side of (22) is less than the right side for $a < a'$ and exceeds the right side for $a > a'$. Regularity holds for many cases of g and h although it is certainly possible to specify g and h that violate it.

The following theorem gives fairly general conditions under which (21) and (22) have no positive (a, b) solution and hence no equilibrium. In most such cases, $S \to (0, 0)$. Apart from regularity, the theorem focuses on the usage rate density h.

Theorem 5.1 *Suppose f is separable and regular, and h is differentiable with derivative h'. If, in addition,*

$$[h(x)]^2 + h'(x)[1-H(x)] \geq 0 \tag{23}$$

for all x for which $h(x) > 0$, then (21) *and* (22) *have no positive* (a, b) *solution.*

Proof. Given the initial hypotheses of the theorem, we show that if (21) and (22) have a positive (a, b) solution, then (23) is contradicted.

Assume that f is separable and regular, h is differentiable, and $(a, b) > (0, 0)$ satisfies (21) and (22). We parameterize the difference of the two sides of (22) by replacing a by z and write

$$T(z) = \int_{v=0}^{z} v[1-G(v)]h\left(\frac{v}{b}\right)dv - z^2[1-G(z)]h\left(\frac{z}{b}\right) \ .$$

Clearly, $T(0) = T(a) = 0$ and $h\left(\frac{a}{b}\right) > 0$. By regularity, T does not vanish elsewhere. It is easily seen that $T(\epsilon) < 0$ for small $\epsilon > 0$ so, as z approaches a from the left, $T(z)$ increases with derivative $T'(z) > 0$ at $z = a$, where

$$T'(z) = z\left\{-[1-G(z)]\left[h\left(\frac{z}{b}\right) + \frac{z}{b}h'\left(\frac{z}{b}\right)\right] + zg(z)h\left(\frac{z}{b}\right)\right\} \ .$$

Then $T'(a) > 0$ if and only if

$$-h\left(\frac{a}{b}\right)[1 - G(a) - ag(a)] > \frac{a}{b}[1 - G(a)]h'\left(\frac{a}{b}\right) \ .$$

According to (21), this inequality is tantamount to

$$0 > \left[h\left(\frac{a}{b}\right)\right]^2 + h'\left(\frac{a}{b}\right)\left[1 - H\left(\frac{a}{b}\right)\right] \ .$$

However, this contradicts (23). □

When x is bounded with domain $[0, K]$, we have found that (23) holds for a variety of h densities, and it takes some imagination to formulate an h that violates (23) on a subdomain of $[0, K]$. Even then there is no assurance that (21) and (22) have a positive solution. In fact, we have no explicit example of an SEP for Model 2 in which f is separable and w and x are bounded.

Plausible failures of (23) are easier to imagine when x is unbounded. A case in point appears in our concluding example.

Example 2. Assume that

$$f(w,x) = \frac{(1.5)^2}{(1+w)^{2.5}(1+x)^{2.5}} \quad \text{for all} \quad w,x \geq 0 \ ,$$

where x and w have been scaled so that the additive constants in the denominator are both 1. Then $E(x) = E(w) = 2$ in the units used for x and w. For example, if each w unit represents \$10, and each x unit represents 7 hits, then the average willingness to pay is \$20 per period and the mean usage rate prior to budget-induced reductions is 14 hits per period.

The preceding f admits a unique positive solution for (21) and (22), at approximately $(a^*, b^*) = (0.15, 0.13)$, that is an SEP. In the present case,

$$A(a,b) = \frac{a}{(1+a)^{1.5}(1+a/b)^{1.5}} \ ,$$

which increases for fixed b from 0 at $a = 0$ up to a maximum when a satisfies

$$b = \frac{a(1+4a)}{2-a}$$

and then decreases. The preceding equation is (21). Equation (22) is

$$\int_{v=0}^{a} \frac{(1.5)v}{(1+v)^{1.5}(1+v/b)^{2.5}}\, dv = \frac{(1.5)a^2}{(1+a)^{1.5}(1+a/b)^{2.5}} \ ,$$

which substitution from (21) reduces to

$$\int_{v=0}^{a} \frac{v}{(1+v)^{1.5}(1+v(2-a)/[a(1+4a)])^{2.5}}\, dv = \frac{a^2(1+4a)^{2.5}}{3^{2.5}(1+a)^4} \ .$$

The unique solution to this is approximately $a = 0.15$, and we then obtain $b = 0.13$ from (21). A proof that is similar to aspects of the proof of Theorem 3.1 shows that the solution is an SEP.

With a unit of w representing \$10 and a unit of x representing 7 hits, the equilibrium solution $(a^*, b^*) = (0.15, 0.13)$ puts A's fee at \$1.50 per period and B's fee at about 19 cents per hit. The average revenues are $A(a^*, b^*) = 0.0385$ and $B(a^*, b^*) = 0.0365$, so A has a slight edge over B.

These revenues translate into an average of 38.5 cents per customer for A and 36.5 cents per customer for B. The total of 75 cents per customer seems low in view of the average willingness to pay of \$20, but is a consequence of competition. If firm B stopped offering the service, leaving A without competition, A would change a from 0.15 to 2, or \$20, and realize a 10-fold increase in revenue to \$3.85 per customer on average. In other words, about 19% of the original customers would pay A the new \$20 fee and the other 81% would stop using the service altogether.

6 Discussion

Our purpose has been to analyze competition between two firms that offer the same information service but use different fee arrangements. Two models were employed to investigate competitive pricing modeled as a noncooperative repeated game. A predominant finding for both models was that competition often leads to a ruinous price war. However, there are situations that have stable equilibrium prices for the firms. When that occurs, the total revenues of the firms are well below a monopolist's revenues, and the firm which uses a fixed subscription fee per period tends to do slightly better than a firm which charges on a per-hit basis.

We noted earlier that potential subscribers often have a preference for the fixed subscription fee arrangement, even when they would pay more this way than for a per-hit arrangement. This can be factored into our models by introducing subscription fee premiums. For example, a 20% premium in Model 1 would change the customer's decision rule in Section 3 to

$$\begin{array}{llllll} \text{pay} & a & \text{to} & A & \text{if} & a \le (1.2)bx, \quad \text{and} \\ \text{pay} & bx & \text{to} & B & \text{if} & (1.2)bx < a \ , \end{array}$$

given that the service is used. Revenue equations (1) and (2) would then be changed by replacing $x = a/b$ by $x = a/(1.2b)$ in the integration limits. Such changes appear to have relatively little effect on the status of competitive equilibrium although they obviously alter the revenue ratio in A's favor when an SEP exists.

We conclude with several challenges that could be addressed by future research:

1. Extend Theorem 4.1 to usage rate densities other than the negative exponential;

2. Construct a plausible separable f for Model 2 that has a bounded domain and an SEP;

3. Construct an interesting nonseparable f for Model 2 of bounded or unbounded domain that has an SEP;

4. Determine for either model if it is possible to have $A(a^*, b^*) < B(a^*, b^*)$ at an SEP (a^*, b^*). The SEP examples in the paper all have $A(a^*, b^*) > B(a^*, b^*)$, and if f in Example 2 is replaced by $1/[(1+w)(1+x)]^2$, we get an SEP for which $A(a^*, b^*) = B(a^*, b^*)$. But we know of no case in which $A(a^*, b^*) < B(a^*, b^*)$.

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