Economic Theory **11**, 457–464 (1998)

Economic Theory

On capital accumulation paths in a neoclassical stochastic growth model★

Kaushik Mitra**
Department of Economics, Uris Hall, Cornell University, Ithaca, NY 14853, USA

Received: June 21, 1996; revised version: October 31, 1996

Summary. Boldrin and Montrucchio [2] showed that any twice continuously differentiable function could be obtained as the optimal policy function for some value of the discount parameter in a deterministic neoclassical growth model. I extend their result to the stochastic growth model with non-degenerate shocks to preferences or technology. This indicates that one can obtain complex dynamics endogenously in a wide variety of economic models, both under certainty and uncertainty. Further, this result motivates the analysis of convergence of adaptive learning mechanisms to rational expectations in economic models with (potentially) complicated dynamics.

JEL Classification Numbers: D90, C61.

1 Introduction

Boldrin and Montrucchio [2] showed that an arbitary twice continuously differentiable function defined on a compact subset K of the positive orthant could be the optimal policy function for some value of the discount parameter in a neoclassical optimal accumulation model. Thus every kind of behavior is potentially possible in a neoclassical growth model. In particular, the assumption of maximizing behavior on the part of economic agents cannot rule out complicated dynamics (even chaos) in the time path of capital.

The result obtained by Boldrin and Montrucchio [2] is in the context of a deterministic model. In this paper I extend their result to a stochastic growth model with non-degenerate shocks to preferences or technology. This shows that the Boldrin-Montrucchio [2] result is actually true for a wider class of

★ I am grateful to David Easley, Tapan Mitra, and Mukul Majumdar for motivating me into finishing this paper and to an anonymous referee for improving the exposition of the paper.

** *Current address*: RUES6, Department of Economics, P.O.Box 10 (Snellmaninkatu 12B4.Kerros), FIN-00014, University of Helsinki, FINLAND

models than they considered. In particular, one can think of the Boldrin-Montrucchio [2] result as a special case of the more general result in this paper.

There is another reason to be interested in such a result. The literature on convergence of adaptive learning mechanisms to rational expectations equilibria in dynamic economic models has seen a tremendous growth over the last decade or so. In the beginning this issue was analyzed in linear economic models (see for instance Bray [3] and Marcet and Sargent [8, 9]). Learning in nonlinear models has been studied by Grandmont and Laroque [6]. More recently, Evans and Honkapohja [5] and Kuan and White [7] establish conditions for local convergence to rational expectations equilibria in stochastic nonlinear models.

In general, however, there is no guarantee that trajectories under learning converge to a rational expectations equilibrium. For instance, Grandmont and Laroque [6] mention "... adaptive learning might generate endogenously complex nonlinear trajectories, along which forecasting errors would never vanish". When the possibility of complex dynamics is shown in the context of some model, it is often (implicitly) assumed that one cannot converge to rational expectations.

The study of chaos in economic models till recently, to the best of my knowledge, has been exclusively confined to deterministic models. But a study of chaos in economic models with noise is also potentially interesting. Mitra ([10] and [11]) considers a model where the equilibrium law of motion of some variable is given by the quadratic map. The focus is on finite sample results. It is shown that it is possible that agents with a fixed sample size may be closer to rational expectations when the underlying dynamics is chaotic than when the underlying dynamics is converging to a fixed point.

Although the dynamics induced by learning may be complex, the reader can, nevertheless, ask whether such a phenomenon is robust or not? The possibility of complex dynamics (even chaos) has been shown in models of overlapping generations, both in a deterministic framework (for instance, see Benhabib and Day [1] and Carrera and Moran [4]) and in a stochastic framework (as in Mitra [10]). While the possibility of complicated dynamics in a neoclassical growth model has been shown in a deterministic framework, an extension of the result obtained in Boldrin and Montrucchio [2] to a stochastic framework indicates that there exist a wide variety of economic models (ranging from models of overlapping generations with finitely lived agents to infinite horizon growth models) where complicated dynamics can arise endogenously. The occurence of complex dynamics, therefore, seems to be a widespread phenomenon in economic models and cannot be dismissed as accidental.

With this motivation for the importance of analyzing learning in models with potentially complicated dynamics, we now proceed to extend the result of Boldrin and Montrucchio [2] to a stochastic growth model. The proof very closely follows the one in the deterministic case.

2 Statement of the problem

We let (K, ζ) and (Z, χ) be measurable spaces, and let $(S, \vartheta) = (K \times Z, \zeta \times \chi)$ be the product space. K is the set of possible values of the endogenous state variable, the capital stock, Z is the set of possible values for the exogeneous shock, and S is the set of possible states for the system. The evolution of the stochastic shocks is described by a stationary transition function Q on (Z, χ).

The neoclassical stochastic growth model with discounting is described by problem (P) and the subsequent assumption (A.1)–(A.3):

$$(\mathrm{P}) W_\delta(k_0, z_0) = \mathrm{Max} E \sum_{t=0}^{\infty} V(k, k', z)\delta^t \text{ subject to } (k, k', z) \in \Upsilon, t = 0, 1, 2, \ldots$$

$$(k_0, z_0) \text{ given in } K \times Z, \delta \in (0, 1).$$

(A.1) The set of feasible capital stocks $K \subset R^n_+$ is a convex, compact Borel set in R^n_+ with its Borel subsets ζ and with nonempty interior. Let $\Gamma : S \to K$ be a correspondence describing the technology constraints which is nonempty, compact-valued and continuous; let Υ be the graph of Γ, i.e., $\Upsilon = \{(k, k', z) : k' \in \Gamma(k, z)\}$ and $\Gamma(k, z) = \{k' \in K, \text{ s.t. } (k, k', z) \in \Upsilon\}$. Also, we assume that Z is a compact (Borel) set in R^k, with its Borel subsets χ, and the transition function Q on (Z, χ) has the Feller property.

(A.2) The return function $V : \Upsilon \to R$ is continuous and concave for each $z \in Z$; $V(k, ., z)$ is strictly concave for each $(k, z) \in (K, Z)$.

(A.3) $V(k, k', z)$ is strictly increasing in k, and strictly decreasing in k', for each $z \in Z$.

The functional (Bellman) equation is:

$$(1) W_\delta(k, z) = \mathrm{Max}\{V(k, k', z) + \delta E\{W_\delta(k', z') \mid (k, z)\} \text{ s.t. } k' \in \Gamma(k, z)\}$$

The optimal policy correspondence $\tau_\delta : K \times Z \to K$ is defined as

$$(2) \tau_\delta(k, z) = \mathrm{ArgMax}\{V(k, k', z) + \delta E\{W_\delta(k', z') \mid (k, z)\} \text{ s.t. } k' \in \Gamma(k, z)\}$$

Note: $E\{W_\delta(k', z') \mid (k, z)\} := \int_Z W_\delta(k', z') Q(z, dz')$

Under the above assumptions, it can be shown, for instance, that the function W is the unique fixed point of the Bellman equation and for each $z \in Z, \tau_\delta(., z) : K \to K$ is a continuous function.

Lemma 1. A map $\theta : K \times Z \to K$ is the policy function τ_δ of (P) under (A.1) and (A.2), for a fixed $\delta \in (0, 1)$, if and only if, the following two conditions are satisfied:

(i) There exists a real function, $W(k, k', z)$, (concave on $K \times K$, for each $z \in Z$), such that $\mathrm{Max}\{W(k, k', z), \text{ s.t. } k' \in \Gamma(k, z)\} = W(k, \theta(k, z), z)$
(ii) Setting $\Psi(k, z) = W(k, \theta(k, z), z)$, the real function $W(k, k', z) - \delta E\{\Psi \times (k', z') \mid (k, z)\}$, satisfies (A.2).

Proof. Necessity. Let the polity function τ_δ be equal to θ for a given return function $V(k,k',z)$ and a given technology set Υ satisfying (A.1) and (A.2). By the Bellman equation, we must have

(1′) $\text{Max}\{V(k,k',z) + \delta E\{W(k',z') \mid (k,z) \text{ s.t. } k' \in \Gamma(k,z)\} = w(k,z)$ and
(2′) $\theta(k,z) = \text{ArgMax}\{V(k,k',z) + \delta E\{W(k',z') \mid (k,z)\}\text{s.t. } k' \in \Gamma(k,z)\}$
Put: $W(k,k',z) = V(k,k',z) + \delta E\{W(k',z') \mid (k,z)\}$, to obtain
$\text{Max}\{W(k,k',z) \text{ s.t. } k' \in \Gamma(k,z)\} = W(k,\theta(k,z),z)$,

so that (i) is satisfied. Moreover, if we set $\Psi(k,z) = W(k,\theta(k,z),z)$, we obtain $\Psi(k,z) = W(k,z)$, by (1′). In this way we have

$$W(k,k',z) - \delta E\{\Psi(k',z') \mid (k,z)\} = V(k,k',z) + \delta E\{W(k',z') \mid (k,z)\} - \delta E\{W(k',z') \mid (k,z)\} = V(k,k',z)$$

which is continuous and concave in (k,k') and strictly concave in k' by (A.2). Hence (ii) is satisfied. Also, note that the concavity of W on $K \times K$, for each $z \in Z$, follows from the concavity of v on $k \times k$, for each $z \epsilon z$.

Sufficiency: Define the short-run return function of problem (P) as $V(k,k',z) = W(k,k',z) - \delta E\{\Psi(k',z') \mid (k,z)\}$, which satisfies (A.2) by (ii). Then $\text{Max}\{V(k,k',z) + \delta E\{\Psi(k',z') \mid (k,z) \text{ s.t. } k' \in \Gamma(k,z)\} = \text{Max}\{W(k,k',z) \text{ s.t. } k' \in \Gamma(k,z)\} = \Psi(k,z)$ and

$$\text{ArgMax}\{V(k,k',z) + \delta E\{\Psi(k',z') \mid (k,z) \text{ s.t. } k' \in \Gamma(k,z)\} = \text{ArgMax}\{W(k,k',z) \text{ s.t. } k' \in \Gamma(k,z)\} = \theta(k,z),$$

by hypothesis (i).

This, in turn, implies that θ is the optimal policy function of a problem (P) where we have $V(k,k',z) = W(k,k',z) - \delta E\{\Psi(k',z') \mid (k,z)\}$, as the short-run return function, the discount parameter δ is given and the value function defined in (1′) is exactly $\Psi(k,z) = \text{Max}\{W(k,k',z) \text{ s.t. } k' \in \Gamma(k,z)\}$. Q.E.D.

Corollary 1. A set of sufficient conditions in order to obtain a map $\theta : K \times Z \to K$ as the optimal policy function τ_δ of problem (P) under (A.1), (A.2) for a given $\delta \in (0,1)$ is:

(i) $\theta(k,z) \in \Gamma(k,z)$, for every $k \in K$, for each $z \in Z$, i.e, $\theta(k,z) \in \Gamma(k,z)$ for all $(k,z) \in K \times Z$
(ii) There exists a real function, $W(k,k',z)$, concave on $K \times K$, for each $z \in Z$, such that

$$\text{Max}\{W(k,k',z),\ \text{s.t. } k' \in K\} = W(k,\theta(k,z),z)$$

(iii) $\Psi(k,z) = W(k,\theta(k,z),z)$, the real function $W(k,k',z) - \delta E\{\Psi(k',z') \mid (k,z)\}$ is concave in (k,k'), for each $z \in Z$, and strictly concave in k', for each $(k,z) \in K \times Z$.

The family of functions corresponding to (3) in Boldrin and Montrucchio [2] is:

(3) $W(k, k', z), = -0.5\|k'\|^2 + < k' - k^-, \theta(k,z) > -0.5L\|k'\|^2$; where k^- is a given point in K.

Note: $\text{ArgMax}\{W(k, k', z), \text{s.t. } k' \in K\} = \theta(k, z)$

Definition 1. W is called $\alpha_{k'}$-concave if $W(k, k', z) + 0.5\alpha\|k'\|^2$ is concave over $K \times K$, for each $z \in Z$.

Lemma 2. Consider the family (3) for a given map $\theta : K \times Z \to K, \theta$ is of class C^2. Then for any $\alpha \in [0, 1]$, there exists a positive constant L, such that the corresponding W is $\alpha_{k'}$-concave. More precisely, it is enough to put $L \geq \mu\sigma + \gamma^2/(1-\alpha)$,

where: $\text{Max}\{\| D\theta(k,z) \| \text{s.t. } (k,z) \in K \times Z\} = \gamma$, $\text{Max}\{\| D^2\theta(k,z) \| \text{s.t. } (k,z) \in K \times Z\} = \sigma, \text{Max}\{\|k_1 - k_2\|, k_1, k_2 \in K\} = \mu$, where D is the derivative operator.

Note: Since θ is C^2 and defined on a compact domain, the maximum values above are finite.

Proof. The proof exactly follows the one in Boldrin and Montrucchio [2] by simply (re)defining $W^*(x, y, z) = -0.5(1-\alpha)\ \|y\|^2 + \ < y - y^-, \theta(k,z) > -(L/2)\ \|x\|^2$, and

$$f(t) = W^*(x_0 + tx_1, y_0 + ty_1, z_0)$$

with x_0, y_0 fixed in K and z_0 fixed in Z and $(x_0 + tx_1), (y_0 + ty_1) \in K, t \in R$.

Theorem 1. Let $\theta : K \times Z \to K$ be any C^2-map as in Lemma 2. Then for every given $\alpha \in [0, 1)$, there exists a C^2-function $W : K \times K \times Z \to R$ such that

(i) $\text{Max}\{W(k, k', z) \text{ s.t. } k' \in K\} = W(k, \theta(k,z), z)$, for each $z \in Z$.
(ii) $W(k, k', z)$ is $\alpha_{k'}$-concave over $K \times K$, for each $z \in Z$.

Proof. Let $W(k, k', z)$ be defined by the family (3). Then (i) follows by the first order conditions and (ii) has been proved in Lemma 2. Q.E.D.

Definition 2. Ψ is called concave-β if $\Psi(k,z) + 0.5\beta\|k\|^2$ is convex over K, for each $z \in Z$.

Lemma 3. If W is the same as in (3) then, for each $z \in Z$, the function:

$$\Psi(k,z) = \text{Max}\{W(k, k', z) \text{ s.t. } k' \in K\} = W(k, \theta(k,z), z)$$

is concave-β for all $\beta \geq L + \mu\sigma$.
Proof. In analogy with Lemma 2 in Boldrin and Montrucchio [2], just redefine

$$F(k,z) = \Psi(k,z) + 0.5\beta\|k\|^2 = 0.5\|\theta(k,z)\|^2 - < k^-, \theta(k,z) > + 0.5(\beta - L)\|k\|^2.$$

and $f(t) = F(k_0 + tk_1, z_0)$.

Lemma 4. Let $W(k, k', z)$ be $\alpha_{k'}$-concave (for each fixed $z \in Z$) with $\beta \leq \alpha$. Then $W(k, k', z) - E\{\Psi(k', z') \mid (k, z)\}$ is $(\alpha - \beta)_{k'}$-concave on $K \times K$, for each fixed $z \in Z$.

Proof. By assumption, $W(k, k', z) + 0.5\alpha\|k'\|^2$ is concave over $K \times K$, for each fixed $z \in Z$.

$\Psi(k, z) = W(k, \theta(k, z), z)$, where W is C^2 defined on a compact domain. By Lemma 3, we know that Ψ is concave-β for all $\beta \geq L + \mu\sigma$ which in turn implies that $E\{\Psi(k', z') \mid (k, z)\}$ is concave-β, too, under the same condition, for each $z \in Z$. (By Lemma 9.5 of Stokey and Lucas [12]).

Hence, $W(k, k', z) - E\{\Psi(k', z') \mid (k, z)\} + 0.5(\alpha - \beta)\|k'\|^2$ is concave over $K \times K$, for each $z \in Z$. Q.E.D.

Theorem 2. Take any θ which is C^2 on $K \times Z$ such that $\theta(k, z) \in \Gamma(k, z)$ for all $k \in K$, for each $z \in Z$. Then there exists a discount parameter $\delta^* \in (0, 1)$, the value of which depends on θ, such that for every fixed $0 < \delta < \delta^*$, we can construct a return function $V_\delta(k, k', z)$ satisfying (A.2) and with the following property: the optimal policy function τ_δ solving (P) under (A.1), (A.2) with $V = V_\delta$ is the map θ. Moreover, a lower bound for δ^* can be estimated as

$$\delta^* \geq \delta^{**} = \{\gamma^2 + \mu\sigma - \gamma(\gamma^2 + 2\mu\sigma)^{1/2}\}/(2\mu^2\sigma^2) > 0.$$

Proof. Take any θ which is C^2 such that $\theta(k, z) \in \Gamma(k, z)$, for every $k \in K$, for each $z \in Z$. By Theorem 1, there exists a $W(k, k', z)$ such that for each $z \in Z$.

$$\text{Max}\{W(k, k', z),\ \text{s.t. } k' \in K\} = W(k, \theta(k, z), z).$$

By Lemma 2, we know it is sufficient to take $W(k, k', z)$ as defined in (3). The same lemma implies that this function turns out to be $\alpha_{k'}$-concave over $K \times K$ for each $z \in Z$, when

$$(4)\ L \geq \gamma^2/(1 - \alpha) + \mu\sigma, \alpha \in [0, 1)$$

Let $\Psi(k, z) = \text{Max}\{W(k, k', z), \text{s.t. } k' \in K\}$, then Ψ is concave-β, for each $z \in Z$, by Lemma 3, when

$$(5)\ \beta \geq L + \mu\sigma$$

Corollary 1 will imply that the function, $V_\delta(k, k', z) = W(k, k', z) - \delta E \times\{\Psi(k', z') \mid (k, z)$, is our desired return function if we can prove it is strictly concave in k'. But Lemma 4 tells us that $W(k, k', z) - \delta E\{\Psi(k', z') \mid (k, z)\}$ is $(\alpha - \delta\beta)_{k'}$-concave on $K \times K$, for each $z \in Z$. So we need the three parameters to satisfy

$$(6)\ \alpha - \delta\beta > 0$$

to have $V_\delta(k, k', z)$ concave on $K \times K$, for each $z \in Z$, and strictly concave in k', for each $(k, z) \in K \times Z$.

Summing up: The theorem is proved if the conditions (4), (5), and (6) are simultaneously satisfied by some values of the parameters L, α, β such that $\delta \in (0,1)$. It can be shown that the set of solutions to the system (4)–(6) is not empty and that the value:

$$\delta^{**} = \text{Max}\{\alpha/\beta, \text{s.t.}\gamma^2/(1-\alpha) = \beta - 2\mu\sigma\}$$

is the largest value of the discount parameter which assures a nonempty solution. Also, the solution of the latter maximization exercise is exactly the value δ^{**} given in the theorem. Q.E.D.

Till now the monotonicity conditions (A.3) on the return function V have been ignored. This question is tackled in the following extension of Theorem 2.

Theorem 3. Assume θ is as defined in Theorem 2. Then for every $\delta' \in (0, \delta^*)$, with δ^* as given in Theorem 2, there exists a return function $V_{\delta'}$ depending on δ' and satisfying (A.2), (A.3), such that θ is the optimal policy function τ_δ of the associated problem (P) with $V = V_{\delta'}$ when $\delta = \delta'$.

Proof. Consider the following modified version of the family (3): $(3')W(k,k',z) = -0.5\|k'\|^2 + < k' - k^-, \theta(k,z) > -0.5L\|k'\|^2 + < a, k >$ where a is a strictly positive n-dimensional vector and k^- is a given point in K.

Note that all the arguments on α-concavity and concavity-β we have been using in Theorems 1 and 2 hold true even after the addition of the linear term <a, k>. Thus, for $\delta < \delta^*$, $V_\delta(k,k',z) = W(k,k',z) - \delta E\{\Psi(k',z') \mid (k,z)\}$ satisfies (A.2).

$$(\partial/\partial k)[V_\delta(k,k',z)] = \{D\theta(k,z)\}^T(k' - k^-) - Lk + a$$

were $\{.\}^T$ is the transpose operator and

$$(\partial/\partial k')[V_\delta(k,k',z)] = -(1-\delta L)k' - \delta a + \theta(k,z) + \delta E \times \left[\{D\theta(k',z')\}^T(k^- - \theta(k',z')) \mid (k,z)\right]$$

Let $N = \text{Max}\{\|k'\| \text{ s.t. } k \in K\}$ and

$M = \text{Max}\{\|E[\{D\theta(k',z')\}^T(k^- - \theta(k',z')) \mid (k,z)]\| \text{ s.t.}(k,k',z) \in K \times K \times Z\}$ (the maximum exists by an application of Lemma 9.5 of Stokey and Lucas [12]).

So,

$$a_i > LN + \mu\gamma + N/\delta + M, \text{for all } i = 1,2,..n$$

is sufficient to make $(\partial/\partial k)[V_\delta(k,k',z)] > 0$ and $(\partial/\partial k')[V_\delta(k,k',z)] < 0$.

Finally because $(\partial/\partial k')[W(k,k',z)] = 0$ implies that $k' = \theta(k,z)$, in view of the Bellman equation (2), we can conclude that θ is the policy function τ_δ of this optimal growth model. Q.E.D.

3 Concluding remarks

In this paper I have extended the result of Boldrin and Montrucchio [2] to the case when allows for non-degenerate stochastic shocks to preferences or technology. This confirms that one can indeed obtain complex dynamics (even chaos) in a wide variety of economic models both under certainty and uncertainty. However, one need not necessarily get non-convergence to rational expectations in models with complex dynamics. One should probably pay more attention as to when one can get convergence results in such models rather than simply assume non-convergence as is often (implicitly) done.

References

1. Benhabib, J., Day, R.H.: A characterization of Erratic Dynamics in the Overlapping Generations Model. Journal of Economic Dynamics and Control **4**, 37–55 (1982)
2. Boldrin, M., Montrucchio, L.: On the indeterminacy of capital accumulation paths. Journal of Economic Theory **40**, 26–39 (1986)
3. Bray, M.: Learning, Estimation and the Stability of Rational Expectations Equilibria. Journal of Economic Theory **26**, 318–339 (1982)
4. Carrera C., and Moran M.: General dynamics in overlapping generations models. Journal of Economic Dynamics and Control **19**, 813–830 (1995)
5. Evans, G.W., Honkapohja, S.: Local Convergence of Recursive Learning to Steady States in Stochastic Nonlinear Models. Econometrica **63**, 195–206 (1995)
6. Grandmont, J.M., Laroque G.: Economic Dynamics with Learning: Some Instability Examples. In: Barnett, W.A., et al. (eds.) Equilibrium Theory and Applications. Proceedings of the Sixth International Symposium in Economic Theory and Econometrics. pp. 247–273 Cambridge: Cambridge Unversity Press (1991)
7. Kuan, C-M., White, H.: Adaptive learning with nonlinear dynamics driven by dependent processes. Econometrica **62**, 1087–1114 (1994)
8. Marcet, A., Sargent, T.J.: The Fate of Systems with Adaptive Expectations. American Economic Review, Papers and Proceedings **78**, 168–172 (1988)
9. Marcet, A., Sargent, T.J.: Convergence of Least Squares Learning Mechanisms in Self-Referential Stochastic Models. Journal of Economic Theory **48**, 337–368 (1989)
10. Mitra, K.: Least Squares Prediction and Non Linear Dynamics under Non-Stationarity. Revision of Accurate Prediction in a Chaotic Framework. Mimeo, Cornell University (1996)
11. Mitra, K.: Least Squares and Non Linear Dynamics: Implications for Prediction. Mimeo. Cornell University (1996)
12. Stokey, N.L., Lucas, R.E., with Prescott, E.C.: Recursive Methods in Economic Dynamics (1989)