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RESEARCH ARTICLE

Existence of Walrasian equilibria with discontinuous, non-ordered, interdependent preferences, without free disposal, and with an infinite-dimensional commodity space

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Abstract
A new proof of the existence of a Walrasian equilibrium with an infinite dimensional commodity space is provided, which allows agents' preferences to be discontinuous. The new theorems include as corollaries the existence results of Mas-Collel, Yannelis and Zame, Araujo and Monteiro, and Mas-Collel and Richard, among others.

Keywords Continuous inclusion property · Exchange economy · Infinite-dimensional commodity space · Existence of Walrasian equilibrium

JEL Classification C02 · C62 · D51 · D62

1 Introduction

In pioneering game theoretic papers Dasgupta and Maskin (1986) and Reny (1999) proved the existence of a Cournot-Nash equilibrium relaxing the continuity assumption on the payoff functions. Since then, several advances have been made; see for example Reny (2020). Recently, advances were also made in general equilibrium theory. In

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particular, the existence of a Walrasian equilibrium was proved for economies where consumers' preferences need not be continuous, need not be convex or monotone, and also preferences were allowed to be interdependent, non-ordered, and price dependent (see Podczeck and Yannelis 2022, and their references; see also Anderson et al., 2022). That work generalized all the previous finite dimensional results on the existence of a Walrasian equilibrium without ordered preferences, i.e., Gale and Mas-Colell (1975), Shafer (1976), among others. It should be noted that to prove the existence of a Walrasian equilibrium without ordered preferences, new methods of proofs were needed. Similarly, to allow for not necessarily continuous preferences, new methods were also introduced.

The two questions we wish to address on this paper are the following. Can one extend Podczeck and Yannelis (2022) to an infinite dimensional commodity space? If yes, can one obtain the infinite dimensional existence of a Walrasian equilibrium theorems of Mas-Colell (1986), Yannelis and Zame (1986), Araujo and Monteiro (1989), and Mas-Colell and Richard (1991) as corollaries? We provide a positive answer to both questions. Specifically, we prove the existence of a Walrasian equilibrium allowing for non-ordered, interdependent, and discontinuous preferences.

The standards proofs of the existence of Walrasian equilibrium in infinite dimensional commodity spaces typically trace the result in finite dimensions and then a limiting argument takes place to cover the whole space. Two main obstacles need to be overcome. First, in spaces whose positive cone has a non-empty interior the price simplex is similar with the one in finite dimensions, it is weak star compact and the limit price is always non-zero. Second, and more typical, in spaces whose positive cone has an empty interior, the closed unit ball contains zero and the limiting price could be zero. For this reason, cone conditions like properness or extreme desirability assumptions are used to make sure that the limiting price is different from zero. The continuity assumption on preferences plays an important role to obtain the limits. Any lack of this continuity assumption makes the limiting arguments more complicated. How do we deal with this problem? We suitably consider the set of all finite dimensional subspaces of our commodity space that are included in the order ideal generated by the initial endowments, and apply the finite dimensional existence theorem due to Podczeck and Yannelis (2022) to obtain an equilibrium there. At the last step, i.e. in the limit, the cone condition (proper preferences) plays an important role to show that the price is different from zero and has the designated continuity property. During the limiting process we use the assumption that preference correspondences have the continuous inclusion property, instead of the assumption of continuity of preferences, and this forces us to deviate from the standard arguments of the existing literature.[1]

The paper is organized as follows: Sect. 2 introduces the model, the assumptions, and the two main existence theorems. Section 3 contains the proofs of the theorems. Several remarks are collected in section 4. Finally, section 5 indicates how from our theorems we can derive as corollaries the results of Mas-Colell (1986), Yannelis and Zame (1986), Araujo and Monteiro (1989), and Mas-Colell and Richard (1991).

[1] In the theorems in which they assume the continuous inclusion property, Anderson et al. (2022) use, in addition, the assumption that consumption sets are compact.

2 The model and the result

We start with a definition (see He and Yannelis 2017).

Definition 1 Let X be a topological space and Y a linear topological space. A correspondence $\psi : X \to 2^Y$ is said to have the *continuous inclusion property at* x if there is an open neighborhood O of x and an $F : O \to 2^Y$ such that $F(z) \subseteq \psi(z)$ for any $z \in O$ and coF is upper hemicontinuous with non-empty compact values.

The content of this definition will serve as replacement of the usual notion of continuity of preferences which can be found in economic models of decision making.

In the economies we consider, the commodity-price duality is a dual system $\langle L, L^* \rangle$, where L is a Riesz space endowed with a locally convex Hausdorff topology τ,[2] and L^* is the topological dual of L, which is assumed to be a sublattice of the order dual $L^{\sim}$ of L. We always assume that L_+, the positive cone of L, is closed. (It is not required that the lattice operations in L are continuous.)

The hypothesis that the commodity space is a Riesz space goes back to Aliprantis and Burkinshaw (1983). Its present form, namely that the commodity space carries a topology that makes the topological dual a sublattice of the order dual, is taken from Mas-Colell and Richard (1991).

We write η for a Hausdorff vector space topology on L which is at least as weak as the original topology τ of L. We also consider the order ideal $L(e)$ in L, where e is any element of L_+.[3] There is a norm on $L(e)$ defined by the Minkowski functional of $[-e, e]$. This norm is denoted in the following by ρ. (See Lemma 3 below for the relationship between the topology induced on $L(e)$ by ρ and that induced on $L(e)$ by τ.) We always consider $L(e)^*$ as being endowed with the weak*-topology, coming from $L(e)$ viewed as being endowed with the norm ρ; in particular, in $L(e)^*$, the notion of "weak*-convergence" is always meant with respect to that topology.

Definition 2 An economy $\mathcal{E}$ is a family of triples $\{(X_i, P_i, e_i) : i \in I\}$ where

- I is a non-empty finite set of agents;
- $X_i \subseteq L$ is the consumption set of agent i, and $X = \prod_{i \in I} X_i$;
- $P_i : X \to 2^{X_i}$ is the preference correspondence of agent i;
- $e_i \in X_i$ is the endowment of agent i, and $e = \sum_{i \in I} e_i \neq 0$.

Definition 3 Let $\mathcal{E}$ be an economy.

(a) Given $\pi \in L^*$, the budget set of agent i is $B_i(\pi) = \{x_i \in X_i : \pi x_i \leq \pi e_i\}$, writing x_i for the projection of x onto its i-th coordinate.
(b) A (non-free disposal) Walrasian equilibrium is a pair (π, x), where $\pi \in L^* \setminus \{0\}$ and $x \in X$, such that

  (i) $x_i \in B_i(\pi)$ for each $i \in I$;
  (ii) $B_i(\pi) \cap P_i(x) = \emptyset$ for each $i \in I$;

[2] Unless something else is said, any topological notion appearing below in this paper is meant with respect to the topology τ.

[3] The order ideal $L(e)$ is the set $\bigcup_{n=1}^{\infty} [-ne, ne]$, $n \in \mathbb{N}$, where $[-ne, ne]$ is the order interval $\{z \in L : -ne \leq z \leq ne\}$.

(iii) $\sum_{i\in I} x_i = \sum_{i\in I} e_i$.

(c) A quasi-Walrasian equilibrium is a pair (π, x), where $\pi \in L^* \setminus \{0\}$ and $x \in X$, such that

(i) $x_i \in B_i(\pi)$ for each $i \in I$;
(ii) if $y \in P_i(x)$ then $\pi y \geq \pi x_i$ for each $i \in I$;
(iii) $\sum_{i\in I} x_i = \sum_{i\in I} e_i$.

Let $\mathcal{E}$ be an economy. For all $i \in I$, let $\psi_i : L(e)^* \times X \to 2^{X_i}$ be the correspondence defined by setting $\psi_i(p, x) = B_i(p) \cap P_i(x)$ for $(p, x) \in L(e)^* \times X$. Write $\mathcal{A}$ for the set of feasible allocations of $\mathcal{E}$; thus $\mathcal{A} = \{x \in X : \sum_{i\in I} x_i = \sum_{i\in I} e_i\}$. We consider the following assumptions: (A brief comment may be found after their statement.)

(A1) For each $i \in I$, $X_i = L_+$.
(A2) $\mathcal{A}$ is η^I-compact.
(A3) If $x \in \mathcal{A}$ and $p \in L(e)^*$ are such that the set $I' \subseteq I$ of all i with $\psi_i(p, x) \neq \emptyset$ is non-empty, then for some $i \in I'$, ψ_i has the continuous inclusion property at (p, x), where the topology on the domain side of ψ_i is the product of the weak*-topology of $L(e)^*$ (when $L(e)$ is given the norm ρ) with the topology η^I, and the topology on the codomain side of ψ_i is the original topology τ of L.

For the next assumption we need some preparation. Let H_0 be a finite-dimensional subspace of L containing the individual endowments, and $\mathcal{H}$ the set of all finite-dimensional linear subspaces H of L which include H_0 and are included in $L(e)$. Note that $\mathcal{H}$ is directed by inclusion.

(A4) If $H \in \mathcal{H}$ then, if $x \in \mathcal{A} \cap H^I$ and $p \in H^*$ are such that the subset I' of I consisting of the i with $\psi_i^H(p, x) \equiv \psi_i(p, x) \cap H \neq \emptyset$ is non-empty, then, for some $i \in I'$, the correspondence ψ_i^H has the continuous inclusion property at (p, x).

Actually we prove two theorems. One is based, besides of (A1)–(A4) on (A5) and (A7) below, the other on (A6) and (A8), besides of (A1)–(A4). The division corresponds to whether or not the order ideal $L(e)$ is dense in L. For (A5) and (A6), we call an allocation x Pareto efficient if it is feasible and there is no feasible allocation x' with $x'_i \in P_i(x)$ for all $i \in I$,[4] and call it individually rational if $e_i \notin P_i(x)$ for all $i \in I$.

(A5) The order ideal $L(e)$ is dense in L, and if x is any individually rational and Pareto efficient allocation, then for each $i \in I$, $P_i(x)$ is F-proper[5] at x, with $v_{i,x} \in L(e)_+$.
(A6) If x is an individually rational and Pareto efficient allocation, then for each $i \in I$, $P_i(x)$ is E-proper at x relative to $L(e)$,[6] with $v_{i,x} \in L(e)_+$.

[4] What we call Pareto efficient does not amount to its usual notion, if there are external effects.

[5] A preference relation $P_i : L_+^I \to L_+$ is *F-proper* at $x \in L_+^I$ if there exists a vector $v_{i,x} \in L$ and a neighborhood $U_{i,x}$ of $0 \in L$ such that $x_i + v_{i,x} \in L_+$, and if $u \in U_{i,x}$ then $x_i + \alpha v_{i,x} - \alpha u \in L_+$ implies $x_i + \alpha v_{i,x} - \alpha u \in P_i(x)$ for every real number α which is sufficiently small.

[6] A preference relation $P_i : L_+^I \to L_+$ is called *E-proper* at $x \in L_+^I$ relative to $K \subseteq L$ if there exists a vector $v_{i,x} \in L$, a neighborhood $U_{i,x}$ of $0 \in L_+$, and a set $A_{i,x} \subseteq K$, being radial at x_i (in K), such that $x_i + \alpha v_{i,x} \in P_i(x)$ for every sufficiently small real number $\alpha > 0$, and if $z \in A_{i,x} \cap L_+$ and $z \notin P_i(x)$, then $u \in U$ implies $z - \alpha v_{i,x} + \alpha u \notin P_i(x)$ for every real number $\alpha > 0$.

(A7) For each $i \in I$, each $p \in L(e)^*$, and each $x \in X$, $x_i \notin \psi_i(p, x)$.
(A8) For each $i \in I$ and each $x \in X$, $P_i(x)$ is convex and $x_i \notin P_i(x)$.

Remark 1 Assumptions (A1), (A2), (A5), (A6), and (A8) (together with the assumption that L_+ is closed in L) are standard in general equilibrium theory with infinitely many commodities (see, e.g., Podczeck 1996). New in this setting are (A3), (A4), and (A7). For the case of finitely many commodities, assumptions containing these can be found in He and Yannelis (2016) or in Podczeck and Yannelis (2022) (see also Remark 2).

Here are our results.

Theorem 1 *If the economy* $\mathcal{E} = \{(X_i, P_i, e_i) : i \in I\}$ *satisfies (A1)–(A5), and (A7) then it has a Walrasian equilibrium* (π, x) *in* $L^* \times L$.

Theorem 2 *If the economy* $\mathcal{E} = \{(X_i, P_i, e_i) : i \in I\}$ *satisfies (A1)–(A4), (A6), and (A8) then it has a Walrasian equilibrium* (π, x) *in* $L^* \times L$.

3 Proof of the theorems

The idea of proving infinite-dimensional theorems by tracing them to finite dimensions is rather standard in the mathematical literature. However, the arguments needed to approximate infinite-dimensional results by finite-dimensional ones are neither trivial nor routine. The lack of the continuity assumptions on preferences complicates the arguments considerably; in particular, we cannot rely on Bewley (1972). Our proof is based on the finite-dimensional existence of Walrasian equilibrium theorem of Podczeck and Yannelis (2022). Specifically, we construct a family of subeconomies satisfying the assumptions of this theorem and show that limit price-allocations constitute a Walrasian equilibria for the original infinite-dimensional economy. Assumptions (A5) and (A6) play an important role in that process. It should be noted that assumptions (A5) or (A6) are not needed for the finite-dimensional proof of the existence of a Walrasian equilibrium. Actually, instead of relying on continuity of preferences, we will make use of the assumption that the correspondences ψ_i have the continuous inclusion property. This assumption plays an important role not only in the establishment of equilibria for the approximating economies with finitely many commodities, but also after this, i.e., for the argument in the limit. Besides of this, the arguments in the limit are standard. Thus, the right combination of standard parts and the continuous inclusion property is what is important for the limit.

Below we provide the details of the proof. We prove the two theorems from above in the following way. After a common part, which is similar to the proof of the existence theorem in Podczeck and Yannelis (2022), we will indicate when the proof of the first theorem starts, and then when that of the second one does.

Let H_0 be a finite-dimensional subspace of L containing the individual endowments, and let $\mathcal{H}$ be the set of all finite-dimensional linear subspaces H of L which include H_0 and are included in $L(e)$. Note that $\mathcal{H}$ is directed by inclusion. Viewing $H \in \mathcal{H}$ as a finite-dimensional space in its own right, write H^* for its dual space.

Fix any $H \in \mathcal{H}$. Consider the restriction $\mathcal{E}^H$ of the economy $\mathcal{E}$ to H. Write $\mathbb{R}^\ell$ for the commodity space of $\mathcal{E}^H$, write X_i^H for $X_i \cap H$ and $\mathcal{A}^H$ for $\mathcal{A} \cap H^I$. Let

$\langle C_n\rangle_{n\in\mathbb{N}}$ be an increasing sequence of closed balls in $\mathbb{R}^l$, with $\bigcup_{n\in\mathbb{N}} C_n = \mathbb{R}^l$, and set $K_{i,n} = C_n \cap X_i^H$ for each i and each n. For each $n \in \mathbb{N}$, write $K_n = \prod_{i\in I} K_{i,n}$. Noting that $\mathcal{A}^H$ is compact, we can assume that $\mathcal{A}^H$ is included in K_n for each n. Write Δ for $\{p \in \mathbb{R}^\ell : \|p\| \le 1\}$, and for $p \in \Delta$, write $B_i^H(p)$ for $\{x \in X_i^H : px \le pe_i\}$, that is write $B_i^H(p)$ for $B_i(p) \cap X_i^H$, $i \in I$. Moreover, write $y = (p, x)$ for elements of $\Delta \times X^H$, where X^H denotes the product of the sets X_i^H over I. Note also that $K_{i,n}$ is non-empty, compact, and convex for each i and each n. From this it follows that the same is true of K_n for each n.

Let $\mathcal{F}$ be the collection of correspondences F_y^i witnessing that (A4) is satisfied for the fixed H under consideration. Since in a Euclidean space, any neighborhood of any point includes a compact neighborhood of this point, and X^H is closed in its ambient Euclidean space, we can assume by Definition 1 that for each correspondence in $\mathcal{F}$ the domain O_y^i is bounded in $\Delta \times X^H$ and the image $F_y^i(O_y^i)$ is bounded in X_i^H (shrinking the domains of the members of $\mathcal{F}$ appropriately). Thus, each F_y^i in $\mathcal{F}$ is such that there is an $n_{i,y} \in \mathbb{N}$ such that for $n \ge n_{i,y}$, O_y^i is included in $\Delta \times K_n$, and $F_y^i(O_y^i)$ is included in $K_{i,n}$. For each $i \in I$ and $n \in \mathbb{N}$, write $\mathcal{C}_n^i$ for the collection of those O_y^i which are included in $\Delta \times K_n$ and are such that $F_y^i(O_y^i)$ is included in $K_{i,n}$.

Fix any $i \in I$ and any $n \in \mathbb{N}$. Let $V_n^i = \bigcup \mathcal{C}_n^i$. Note that $\mathcal{C}_n^i$ is an open cover of V_n^i. Being included in a Euclidean space, V_n^i is metrizable, therefore paracompact; see, e.g., Engelking (1989, p. 300, Theorem 5.1.3). Thus $\mathcal{C}_n^i$ has a closed locally finite refinement $\mathcal{F}_n^i = \{E_{j,n}^i : j \in J_n^i\}$, where J_n^i is an index set (and $E_{j,n}^i$ is closed in V_n^i); see Engelking (1989, p. 302, Theorem 5.1.11).[7]

For each $j \in J_n^i$ choose a $y_j \in \Delta \times K_n$ such that $E_{j,n}^i \subseteq O_{y_j}^i$, where $O_{y_j}^i$ belongs to $\mathcal{C}_n^i$. For each $y \in V_n^i$ let $I_n^i(y) = \{j \in J_n^i : y \in E_{j,n}^i\}$. Then $I_n^i(y)$ is finite for each $y \in V_n^i$. Let $\phi_n^i(y) = \big(\bigcup_{j\in I_n^i(y)} F_{y_j}^i(y)\big)$ for $y \in V_n^i$.

Define $H_n^i \colon \Delta \times K_n \to 2^{K_{i,n}}$ by setting, for each $y = (p, x) \in \Delta \times K_n$,

$$H_n^i(y) = \begin{cases} \phi_n^i(y) & \text{if } y \in V_n^i \\ B_i^H(p) \cap K_{i,n} & \text{otherwise.} \end{cases}$$

Evidently H_n^i has non-empty compact convex values and is upper hemicontinuous (note that V_n^i is open in $\Delta \times K_n$ and that $\phi_n^i(y) \subseteq B_i^H(p) \cap K_{i,n}$ for all $y = (p, x) \in V_n^i$; the fact saying that $y \mapsto \bigcup_{j\in I_n^i(y)} F_{y_j}^i(y) \colon V_n^i \to 2^{\mathbb{R}^l}$ is upper hemicontinuous can be easily checked; as $\big(\bigcup_{j\in I_n^i(y)} F_{y_j}^i(y)\big) = \big(\bigcup_{j\in I_n^i(y)} F_{y_j}^i(y)\big)$, it follows from Hildenbrand (1974, p. 26, Proposition 6) that $y \mapsto \big(\bigcup_{j\in I_n^i(y)} F_{y_j}^i(y)\big) \colon V_n^i \to 2^{\mathbb{R}^l}$ is upper hemicontinuous as well.)

[7] Recall that a topological space W is called *paracompact* if it is Hausdorff and every open cover of W has an open locally finite refinement. Recall also that a *refinement* of a cover $\mathcal{A}$ of a set W is a cover $\mathcal{B}$ of W such that every member of $\mathcal{B}$ is included in some member of $\mathcal{A}$. Finally, recall that a family $\mathcal{B}$ of subsets of a topological space W is called *locally finite* if every point of W has an open neighborhood which meets only finitely many members of $\mathcal{B}$.

Do this construction for each $i \in I$ and each $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$, define a correspondence $G_n : \Delta \times K_n \to 2^\Delta$ by setting

$$G_n(y) = \operatorname{argmax}_{q\in\Delta} q \sum_{i\in I}(x_i - e_i)$$

for each $y = (p, x) \in \Delta \times K_n$.

By Kakutani's fixed point theorem, for each n the correspondence $G_n \times \prod_{i\in I} H_n^i$ has a fixed point, (p_n^*, x_n^*) say. As in the proof of Theorem 4 in He and Yannelis (2016) it follows that $x_n^* \in \mathcal{A}^H$ for each n. To see this, fix any n. Write $z_n^* = \sum_{i\in I}(x_{n,i}^* - e_i)$. By the definition of G_n, we must have $p_n^* z_n^* \geq q z_n^*$ for each $q \in \Delta$, and by the definition of the correspondences H_n^i, we have $0 \geq p_n^* z_n^*$. Thus $0 \geq q z_n^*$ for each $q \in \Delta$. Because the definition of Δ implies that each non-zero excess demand vector can be given a positive value by an appropriate $q \in \Delta$, it follows that $z_n^* = 0$, i.e., $x_n^* \in \mathcal{A}^H$.

Because $\mathcal{A}^H$ and Δ are compact, we can assume, therefore, that the sequence $\langle(p_n^*, x_n^*)\rangle$ is convergent, say to (p^*, x^*). Thus $x^* \in \mathcal{A}^H$. By construction, $x_{n,i}^* \in B_i^H(p_n^*)$ for each $i \in I$ and each n, which implies that $x_i^* \in B_i^H(p^*)$ for each $i \in I$. Suppose there is an $i \in I$ such that $\psi_i^H(p^*, x^*) \neq \emptyset$. By (A4) we can assume that i is such that the continuous inclusion property holds for ψ_i^H at (p^*, x^*). Let $O^i_{(p^*,x^*)}$ and $F^i_{(p^*,x^*)}$ be chosen according to the fifth paragraph of this proof. We can pick an $n_0 \in \mathbb{N}$ so large that we have both $O^i_{(p^*,x^*)} \subseteq \Delta \times K_n$ and $F^i_{(p^*,x^*)}(O^i_{(p^*,x^*)}) \subseteq K_{i,n}$ for $n \geq n_0$. Thus $O^i_{(p^*,x^*)} \subseteq V_n^i$ for $n \geq n_0$. In addition, since $\langle(p_n^*, x_n^*)\rangle$ converges to (p^*, x^*) and $O^i_{(p^*,x^*)}$ is open in $\Delta \times X^H$, we can pick an $n_1 \in \mathbb{N}$ so that $(p_n^*, x_n^*) \in O^i_{(p^*,x^*)}$ whenever $n \geq n_1$. Set $\bar{n} = \max\{n_0, n_1\}$. Then $(p_{\bar{n}}^*, x_{\bar{n}}^*) \in V_{\bar{n}}^i$. Thus $H_{\bar{n}}^i(p_{\bar{n}}^*, x_{\bar{n}}^*) = \phi_{\bar{n}}^i(p_{\bar{n}}^*, x_{\bar{n}}^*)$. Consequently $x_{\bar{n},i}^* \in \phi_{\bar{n}}^i(p_{\bar{n}}^*, x_{\bar{n}}^*) \subseteq \psi_i(p_{\bar{n}}^*, x_{\bar{n}}^*)$, and we get a contradiction to (A7) as well as to (A8).

Setting $p^H = p^*$ and $x^H = x^*$, we have thus constructed a pair (p^H, x^H) satisfying

$$\sum_{i\in I} x_i^H = e; \text{ and}$$
$$\text{if } y \in P_i(x^H) \cap H \text{ then } p^H y > p^H e_i, \text{ for all } i \in I. \qquad (*)$$

Now let $H \in \mathcal{H}$ vary. Using the Hahn–Banach theorem, and the fact that the family $\mathcal{H}$ is directed by inclusion, we can view $\langle(p^H, x^H)\rangle$ as a net in $L(e)^* \times L^I$. Using Alaoglu's theorem, we can assume that $\langle(p^H, x^H)\rangle$ is such that $\langle p^H\rangle$ is weak*-convergent to some $p \in L(e)^*$, and, since $\mathcal{A}$ is η^I-compact by hypothesis, we can assume that $\langle(p^H, x^H)\rangle$ is such that $\langle x^H\rangle$ is η^I-convergent to some $x \in \mathcal{A} \subset L(e)^I$. In particular, $\langle x_i^H\rangle$ is η-convergent to $x_i \in L(e)$ for all $i \in I$. We have

$$\sum_{i\in I} x_i = e;$$

there exists no feasible allocation x' with $x'_i \in P_i(x) \cap L(e)$ for all $i \in I$;

$e_i \notin P_i(x)$ for all $i \in I$;

if $y \in P_i(x) \cap L(e)$ then $py > pe_i$, for all $i \in I$.

Indeed. The equality above is clear. The middle two claims follow from the last one. As for the last one, suppose by way of contradiction that there is an $i_0 \in I$ such that $py \leq pe_{i_0}$ but $y \in P_{i_0}(x) \cap L(e)$, so that $\psi_{i_0}(p, x) \neq \emptyset$. By (A3) we can assume that ψ_{i_0} has the continuous inclusion property at (p, x). Thus there is a neighborhood O_{i_0} of (p, x) in $L(e)^* \times X$ such that $\psi_{i_0}(p', x') \neq \emptyset$ for each $(p', x') \in O_{i_0}$. Because the net $\langle(p^H, x^H)\rangle$ is weak* $\times\, \eta^I$-convergent to $(p, x) \in L(e)^* \times \mathcal{A} \subseteq L(e)^* \times X$, there is an $H_1 \in \mathcal{H}$ such that $(p^H, x^H) \in O_{i_0} \cap (L(e)^* \times \mathcal{A})$ for $H \geq H_1$. But this implies, in view of (A4), that (p^H, x^H) cannot satisfy $(*)$ for the approximating economies $\mathcal{E}^H$, for large $H \in \mathcal{H}$, and we get a contradiction. Thus if $y \in P_i(x) \cap L(e)$, then $py > pe_i$, for all $i \in I$.

Identify p with an extension of it to a linear functional defined on all of L. Consider first the situation of Theorem 1. By (A5) and Lemma 1 below, we can choose, for every $i \in I$, an open and convex cone Γ_i, with $(\{x_i\} + \Gamma_i) \cap L(e)_+ \neq \emptyset$, such that $\gamma \in \Gamma_i$ and $x_i + \gamma \in L_+$ imply that $x_i + \lambda\gamma \in P_i(x)$ for every sufficiently small real number $\lambda > 0$. Since $x_i \in L(e)_+$, it is clear that $x_i + \gamma \in L(e)_+$ implies $x_i + \lambda\gamma \in L(e)_+$ if $0 \leq \lambda \leq 1$. Thus we must have $px_i \leq pz$ for all $z \in (\{x_i\} + \Gamma_i) \cap L(e)_+$ and all $i \in I$. Moreover, $(\{x_i\} + \Gamma_i)$ is open and convex, and, since Γ_i is a cone, x_i belongs to the closure of $\{x_i\} + \Gamma_i$, for all $i \in I$. Thus we can apply Lemma 2 below to choose, for every $i \in I$, an element p_i in L^* and a linear functional t_i on L such that all of the following hold: $p_i + t_i = p$, $p_i x_i \leq p_i z$ for all $z \in (\{x_i\} + \Gamma_i)$, and $t_i x_i \leq t_i z$ for all $z \in L(e)_+$. The latter inequality implies $t_i z \geq 0$ for all $z \in L(e)_+$ and $t_i x_i = 0$. Thus we must have, for each $i \in I$, $p_i x_i = px_i$ and $p_i z \leq pz$ for all $z \in L(e)_+$.

Because, by hypothesis, L^* is a sublattice of the order dual $L^\sim$ of L, the set $\{p_1, \ldots, p_I\}$ has a supremum π in L^*. Pick any $z \in L(e)_+$. We have

$$\pi z = \sup\left\{\sum_{i\in I} p_i z_i : z_i \in L_+ \cap L(e) \text{ and } \sum_{i\in I} z_i = z\right\}.$$

As $0 \leq z_i \leq z$ implies $z_i \in L(e)_+$, and since $pz' \geq p_i z'$ for all $z' \in L(e)_+$ and all $i \in I$, it follows that $pz \geq \pi z$. Moreover, since $x_i \geq 0$ and $px_i = p_i x_i$ for all $i \in I$, and since $\sum_{i\in I} x_i = e$, we have

$$pe = \sum_{i\in I} px_i = \sum_{i\in I} p_i x_i \leq \pi e.$$

Hence $pe = \pi e$. Finally, $z \in L(e)_+$ implies the existence of a number $\lambda > 0$ such that $\lambda e \geq z$, so $p(\lambda e - z) \geq \pi(\lambda e - z)$. We have shown that $\pi z = pz$ for all $z \in L(e)_+$. As $L(e)$ is a sublattice of L, it follows that π and p agree on $L(e)$.

Consider any $y \in L_+ \setminus L(e)_+$ and suppose $y \in P_{i_0}(x)$ but $\pi y \leq \pi e_{i_0}$ for some $i_0 \in I$. We can assume that the continuous inclusion property holds for ψ_{i_0} at (π, x), so there is a neighborhood O_{i_0} of $(\pi \restriction L(e), x)$ in $L(e)^* \times X$ (where $\pi \restriction L(e)$ denotes the restriction of π to $L(e)$) such that $F_{i_0}(\pi', x') \neq \emptyset$ for all $(\pi', x') \in O_{i_0}$. As $\langle(p^H, x^H)\rangle$ is weak* $\times\, \eta^I$-convergent to $(p, x) \equiv (\pi \upharpoonright L(e), x)$, there is an $H_1 \in \mathcal{H}$ such that $(p^H, x^H) \in O_{i_0} \cap (L(e)^* \times \mathcal{A})$ for $H \geq H_1$. But then, as above, (p^H, x^H) cannot satisfy $(*)$ for the approximating economies $\mathcal{E}^H$, for sufficiently large $H \in \mathcal{H}$, and we get a contradiction. Thus, for each $i \in I$, if $y \in P_i(x)$, then $\pi y > \pi e_i$.

We must have $x_i \in B_i(\pi)$ for each $i \in I$. Indeed, since x is a Pareto efficient and individually rational allocation, it follows from (A5), together with Lemma 1, that for each $i \in I$ there is a $v_{i,x} \in L(e)_+$ such that $x_i + \alpha v_{i,x} \in P_i(x)$ for every sufficiently small number $\alpha > 0$. Consequently, from above, for each $i \in I$, we have $\pi(x_i + \alpha v_{i,x}) > \pi e_i$ for every sufficiently small number $\alpha > 0$, and therefore, by feasibility of x, we have $\pi x_i = \pi e_i$. In particular, $x_i \in B_i(\pi)$ for each $i \in I$. It also follows from (A5) that $\pi \neq 0$. To see this, suppose, if possible, otherwise. Then $B_i(\pi) = X_i$ and $\psi_i(\pi, x) = \emptyset$ for each i, which is impossible in view of (A5), implying that $\alpha v_{i,x} \in P_i(x)$ for each $i \in I$ if $\alpha > 0$ is small enough. Thus (π, x) is a Walrasian equilibrium, with a price system π in L^*.

Now we come to the situation of Theorem 2. Consider any $i \in I$. Because of (A6) we can choose an open and convex cone Γ_i and a subset A_i of $L(e)$, being radial at x_i in $L(e)$, such that

$$\text{there is a } v_i \in \Gamma_i \text{ with } x_i + v_i \in L(e)_+ \text{ and } x_i + \lambda v_i \in P_i(x) \text{ if } \lambda > 0 \text{ is small,} \qquad (**)$$

and such that $y' \in A_i \cap L_+$ but $y' \notin P_i(x)$ implies $(y' - \Gamma_i) \cap P_i(x) = \emptyset$. In fact, since Γ_i is open we have, denoting by $cl\,P_i(x)$ the τ-closure of $P_i(x)$,

$$y' \in A_i \cap L_+ \text{ but } y' \notin P_i(x) \text{ implies } (y' - \Gamma_i) \cap \mathrm{cl}P_i(x) = \emptyset. \qquad (***)$$

Clearly $(**)$ implies $x_i \in \mathrm{cl}P_i(x)$, and therefore, since Γ_i is a cone, $x_i \in \mathrm{cl}(P_i(x)+\Gamma_i)$ too. Moreover, since $x_i + v_i = x_i + \lambda v_i + (1-\lambda)v_i$, we have $(P_i(x)+\Gamma_i) \cap L(e)_+ \neq \emptyset$. Further, $P_i(x)+\Gamma_i$ is open and, by (A8), convex. Finally, if $z \in (P_i(x)+\Gamma_i) \cap L(e)_+$, then $px_i \leq pz$. To see this, choose any $y \in P_i(x)$ and some $\gamma \in \Gamma_i$ and suppose $y+\gamma \in L(e)_+$. Since $x_i \in L(e)_+$ and because A_i is radial at x_i in $L(e)$, we can fix some λ, with $0 < \lambda \leq 1$, such that $(1-\lambda)x_i + \lambda(y+\gamma) \in A_i \cap L(e)_+$. Since, by (A8), $P_i(x)$ is convex and since $x \in \mathrm{cl}P_i(x)$, we have $(1-\lambda)x_i + \lambda y \in \mathrm{cl}P_i(x)$. Hence, because of $(***)$, $(1-\lambda)x_i + \lambda(y+\gamma) \in P_i(x) \cap L(e)_+$. Thus $px_i \leq p((1-\lambda)x_i + \lambda(y+\gamma))$, i.e., $px_i \leq p(y+\gamma)$.

It now follows from Lemma 2 again that, for each $i \in I$, there is an element p_i in L^* and a linear functional t_i on L such that all of the following hold: $p_i + t_i = p$, $p_i x_i \leq p_i z$ for all $z \in (P_i(x) + \Gamma_i)$, and $t_i x_i \leq t_i z$ for all $z \in L(e)_+$. As above, this implies that, for each $i \in I$, $p_i x_i = px_i$ and $p_i z \leq pz$ for all $z \in L(e)_+$. Moreover, since Γ_i is a cone, $p_i x_i \leq p_i y$ for all $y \in P_i(x)$ and all $i \in I$. Again since L^* is a sublattice of $L^\sim$, the set $\{p_1, \ldots, p_I\}$ has a supremum π in L^*. As above it follows that π and p agree on $L(e)$, and that if $y \in L_+ \backslash L(e)_+$ then $y \in P_i(x)$ implies $\pi y > \pi e_i$ for all $i \in I$. Finally, again as above, we must have $x_i \in B_i(\pi)$ for each $i \in I$, and $\pi \neq 0$, which this time follows by using (A6), instead of (A5). Thus (π, x) is a Walrasian equilibrium, also under (A6). This finishes the proof of the two theorems.

Lemma 1 *If $L(e)$ is dense in L, then $L_+(e)$ is dense in L_+.*

Proof See Podczeck (1996). □

Lemma 2 *Let Y be a (real) vector space, endowed with a Hausdorff, locally convex topology τ. Let A and B be convex subsets of Y, with A τ-open such that $A \cap B \neq \emptyset$. Let $y \in B \cap \mathrm{cl}A$ ($\mathrm{cl}A$ denotes the τ-closure of A), let f be a linear functional on Y and suppose $fy \leq fy'$ for all $y' \in A \cap B$. Then there exist linear functionals f_1 and f_2 on Y such that f_1 is τ-continuous, $f_1 y \leq f_1 a$ for all $a \in A$, $f_2 y \leq f_2 b$ for all $b \in B$, and $f_1 + f_2 = f$.*

Proof See Podczeck (1996). □

Lemma 3 *The topology induced on $L(e)$ by τ is weaker than that induced by ρ.*

Proof By hypothesis, L^*, the τ-dual of L, is a subspace of the order dual of L. Thus the order interval $[-e, e]$ is $\sigma(L, L^*)$-bounded, therefore also τ-bounded. Hence if V is any τ-neighborhood of zero in L, there is a real number $\varepsilon > 0$ such that $[-\epsilon e, \epsilon e] \subseteq V$. In particular, the topology induced on $L(e)$ by τ is weaker than that induced by ρ. □

4 Discussion

Remark 2 Assumptions (A5) or (A6) become superfluous if L is finite-dimensional. On the other hand, if L is finite-dimensional, then it is possible to let preferences be price dependent (see Podczeck and Yannelis 2022). The latter is not so in the present context, because, for instance, there is no canonical price space—look at the approximating economies $\mathcal{E}^H$.

Remark 3 The reader may ask why we don't prove first the existence of a quasi-equilibrium and then use some irreducibility condition to show that a quasi-equilibrium is a full Walrasian equilibrium. The reason is simply that the continuous inclusion property of the correspondences ψ_i can be used to check directly whether a certain price/allocation state is actually a Walrasian equilibrium of an economy (in the sense that if $(\pi, x) \in L^*\backslash\{0\} \times X$, with x being feasible and $x_i \in B(\pi)$ for all $i \in I$, then (π, x) is a Walrasian equilibrium if $\psi_i(\pi, x) = \emptyset$ for all $i \in I$). Note also that if the continuous inclusion property holds for some $i \in I$, then this i has, regardless of what are the ruling prices, local cheaper points in his budget set.

Remark 4 It was is stated in Remark 9 in He and Yannelis (2017) that the finite-dimensional proof of the existence of Walrasian equilibrium for economies with the continuous inclusion property, as the proof of their Theorem 4, goes through if consumption sets are assumed to be norm-compact. We note that this is irrelevant for the present paper as consumption sets are not assumed to be norm-compact.

5 Relation to the literature

The quasi-Walras equilibrium existence results in Mas-Colell (1986), Yannelis and Zame (1986, Theorem 1), Araujo and Monteiro (1989, Theorem 1), and Mas-Colell and Richard (1991) are corollaries of ours, under the following auxiliary assumption:

(B1) For some real number $k > 0$, $e_i^k = (1 - (1/k))e_i + (1/(k\#I))e$ for each $i \in I$.

(Also recall that $L(e)^*$ means the dual of $L(e)$, where $L(e)$ is viewed as carrying the norm ρ, giving $L(e)^*$ the corresponding weak*-topology; see Sect. 2.)

In the quasi-Walrasian equilibrium existence result of Yannelis and Zame (1986, Theorem 1) the following assumptions were made:

(1) The commodity space (L, τ) is a Banach lattice.
(2) There is a Hausdorff topology Ξ on L such that
   (a) Ξ is weaker than the norm topology of L;
   (b) Ξ is a vector space topology;
   (c) all order intervals in L are Ξ-compact.
(3) $X_i = L_+$ and $e_i \in X_i$ for all $i \in I$.
(4) The aggregate endowment is strictly positive (i.e., the order ideal spanned by the aggregate endowment is τ-dense in L).
(5) Each preference relation P_i is (Ξ, norm)-continuous for all $i \in I$.
(6) $x_i \notin P_i(x)$ for all $i \in I$ and all $x \in X$.
(7) For each i, there is a commodity $v_i \in L_+$ which is strongly desirable on the set $\mathcal{A}$.

Our assumptions made at the beginning of Sect. 2, together with (A2), are implied by their (1) and their (2). Assumption (A1) and their (3) amount to the same. Our (A5) follows from their (7), together with their (4). Assumption (A7) is implied by their (6). Together, their 5) and our (B1) imply our (A3) and (A4). (Note that if $|\lambda|$ is close to 1, then, still, $\lambda y \in P_i(x)$ as well as there is a neighborhood O_x of x such that $\lambda y \in P_i(x')$ for all $x' \in O_x$, and that, by (B1), $\lambda y \in B(p')$ for all p' out of some neighborhood O_p of p in $L(e)^*$, for $\lambda < 1$ or $\lambda > 1$; define the correspondence F_i by setting $F_i(p', x') = \{\lambda y\}$ for $(p', x') \in O_p \times O_x$.) Letting $k = n$ for some $n \in \mathbb{N}$, and letting $n \to \infty$, we get from (2) and (5) a quasi-Walrasian equilibrium for the original economy, i.e., the economy without the auxiliary assumption (B1).

Araujo and Monteiro (1989, Theorem 1) also proved the existence of a quasi-equilibrium and considered preferences P_i coming from a complete pre-order $\succcurlyeq_i$ on X_i, $i \in I$, without any interdependence, so that the usual notion of Pareto efficiency applies; it was assumed that:

(1) The commodity space (L, τ) is a locally convex lattice.
(2) (L, τ) is Dedekind complete and order continuous.
(3) $X_i = L_+$ and $e_i \in X_i$ for all $i \in I$.
(4) P_i is convex (in the sense that if $y \in P_i(x_i)$ then $\lambda y + (1 - \lambda)x_i P_i(x_i)$ for all $0 < \lambda < 1$), τ-continuous, and non-satiable on $[0, e]$, for all $i \in I$.
(5) If x is any individually rational and Pareto efficient allocation, then, for each $i \in I$, $P_i(x)$ is F-proper, with $v_{i,x} \in L(e)_+$.
(6) $L(e)$ is τ-dense in L.

The assumptions made at the beginning of Sect. 2 are implied by (1) of Araujo and Monteiro (1989, Theorem 1). Their Proposition 1 says that (2) is equivalent to the fact that all order intervals are $\sigma(L, L^*)$-compact, which implies our (A2). Their (3) and our (A1), together with the last point in Definition 2, are the same. Moreover, their (4) implies our (A7). Their (5) and (6) imply our (A5). Using 4) and (B1), we can see that (A3) and (A4) are true (use the fact that $\{z \in X_i : y \succ_i z\} = X_i \backslash \{z \in X_i : z \succcurlyeq_i y\}$ and that $\{z \in X_i : z \succcurlyeq_i y\}$, being τ-closed and convex by 4), is also $\sigma(L, L^*)$-closed, for any $i \in I$ and any $y \in X_i$, together with the fact noted in the discussion of Yannelis and Zame (1986)). As above, we can get a quasi-Walrasian equilibrium for the original economy (of course, this time from other assumptions).

Mas-Colell and Richard (1991), who generalized Mas-Colell (1986), proved the existence of a quasi-equilibrium and considered preferences P_i coming from a complete pre-order $\succcurlyeq_i$ on X_i, $i \in I$, without any interdependence, too. The following assumptions were made:

(1) The commodity space (L, τ) is a Riesz space endowed with a locally convex Hausdorff topology τ.
(2) L^*, the topological dual of L, is a sublattice of the order dual $L^\sim$ of L.
(3) The positive cone L_+ of L is τ-closed.
(4) $X_i = L_+$ and $e_i \in X_i$ for all $i \in I$.
(5) P_i is convex, τ-continuous, and monotone on L_+, for all $i \in I$.
(6) The order interval $[0, e] = \{z \in L : 0 \leq z \leq e\}$ is $\sigma(L, L^*)$-compact.
(7) For each $i \in I$, $P_i(x)$ is uniformly E-proper on L_+, with $v_{i,x} = e$.

What is at the beginning of Sect. 2 is implied by (1), (2), and (3) of Mas-Colell and Richard (1991). Their (4) and our (A1), together with the last point in Definition 2, are the same. Their (6) implies our (A2), and their (7) our (A6). Their (5) implies our (A8). Finally, (A3) and (A4) follow from (5) and our (B1) (use the same facts as above). As before, we can get a quasi-Walrasian equilibrium for the original economy.

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