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Mechanisms and axiomatics for division problems with single-dipped preferences

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Abstract
A mechanism allocates one unit of an infinitely divisible commodity among agents reporting a number between zero and one. Nash, Pareto optimal Nash, and strong equilibria are analyzed for the case where the agents have single-dipped preferences. One main result is that when the mechanism satisfies anonymity, monotonicity, the zero–one property, and order preservation, then the Pareto optimal Nash and strong equilibria coincide and result in Pareto optimal allocations that are characterized by so-called maximal coalitions: members of a maximal coalition prefer an equal coalition share over obtaining zero, whereas the outside agents prefer zero over obtaining an equal share from joining the coalition. A second main result is an axiomatic characterization of the associated social choice correspondence as the maximal correspondence satisfying minimal envy Pareto optimality, equal division lower bound, and sharing index order preservation.

Keywords Division problems · Single-dipped preferences · Mechanisms · Nash equilibrium · Strong equilibrium

JEL Classification C72 · D71

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1 Introduction

We consider the problem of allocating one unit of an infinitely divisible commodity among agents with single-dipped preferences. A single-dipped preference has a worst point, the dip, and preference strictly increases in both directions away from the dip. Such a preference may arise from maximizing a strictly quasiconvex utility function on a (budget) line, and reflects that an agent prefers extremes over combinations.

Suppose, for instance, that the teaching load of a specific day has to be distributed among a few university employees who each prefer either only teaching or only research over a combination of the two, because it is more efficient to concentrate on just one task: each employee has a single-dipped preference on the interval (day) $[0, 1]$, with $t \in [0, 1]$ being teaching time and $1 - t$ research time. Another example is a common-pool resource allocation problem, as in Inoue and Yamamura (2023): a fixed amount of fish f can be caught from a lake by fishers who have linear revenue but concave cost functions, so that their profit functions are convex: maximizing these over $[0, f]$ results in single-dipped preferences, and the amount f has to be distributed among the fishers.[1]

Our approach in this paper is twofold: we consider a specific class of mechanisms, and we axiomatically characterize the main correspondence implemented by these mechanisms.

The mechanisms we consider, are as follows: each agent reports a number between zero and one, and a mechanism assigns an allocation of the commodity among the agents, which is evaluated by the agents according to their preferences. Under a number of conditions on mechanisms, we analyze the Nash, Pareto optimal Nash, and strong equilibria for each single-dipped preference profile and the resulting allocations in the induced game. These mechanisms are related to but simpler than social choice functions, which assign an allocation to each profile of preferences.[2] A mechanism can be interpreted as asking each agent what that agent wishes to have, and more generally it provides agents with actions to achieve their goals in a non-cooperative way. This is also reflected by the fact that, almost throughout, we assume that a mechanism is monotonic. This means that if an agent reports a higher (resp. lower) number, then that agent's share increases (resp. decreases), if possible. This condition makes sense especially given the interpretation of a mechanism as asking agents what they want. A monotonic mechanism provides the agents with ample possibilities to influence their shares—thus, making the mechanism sufficiently sensitive to their strategies. Besides monotonicity, we also impose anonymity of mechanisms almost throughout.

After preliminaries in Sects. 2, 3 we discuss the Nash equilibria of games induced by a mechanism and single-dipped preference profiles. A first insight here is that in every Nash equilibrium each agent plays 0 or 1, and we characterize all Nash equilibria (Theorem 3.5). If there are two agents then a Nash equilibrium always exists (Proposition 3.10), but this is no longer true for more than two agents. This part of our paper is closely related to Inoue and Yamamura (2023), who consider what they

[1] A related problem is the distribution of waste, having negative externalities, as in for instance Sakai (2012).

[2] Similar simple mechanisms are also used in You and Juarez (2021), who consider the distribution of a good under quasilinear preferences.

call the binary mechanism: agents report either 0 or 1, and the commodity is equally distributed among the agents who report 1, so that the agents who report 0 also receive 0. They characterize the Nash equilibria of the associated games in their Theorem 1. Our main result in this respect, Theorem 3.5, generalizes this since, first, we allow that agents who report 0 can receive more than 0, and, second, our mechanisms are not binary. As to the latter, although, indeed, in Nash equilibrium agents do report only 0 or 1, considering more general mechanisms allows for a comparison with mechanisms/rules for possibly different preferences (e.g. single-peaked preferences, see below).

The focus in our paper is not so much on Nash equilibrium but rather on Pareto optimal Nash equilibrium and on strong equilibrium. In Sect. 4 we consider Pareto optimal Nash equilibria, and we show that the additional condition on a mechanism imposed also by Inoue and Yamamura (2023), namely that when every agent plays 0 or 1 the agents who play 0 receive 0 and the agents who play 1 equally share the commodity, is necessary and sufficient for the existence of a Pareto optimal Nash equilibrium for *all* games, i.e., all preference profiles. We call this condition the 'zero–one property'. If a mechanism satisfies this property, then the Pareto optimal Nash equilibria are exactly those strategy profiles where agents in a so-called maximal coalition play 1 and the other agents play 0. 'Maximal' means that as many agents as possible (given the restrictions of best reply and Pareto optimality) play 1 and get a positive share. Equivalently, members of a maximal coalition prefer an equal coalition share over obtaining zero, whereas the outside agents prefer zero over obtaining an equal share from joining the coalition. Under the further condition of order preservation on a mechanism—meaning that playing a higher number than some other agent results in obtaining a higher share than that agent—these Pareto optimal Nash equilibria are, moreover, strong equilibria (Aumann 1959): no coalition can profitably deviate. As a consequence, under the mentioned conditions on a mechanism, a subcorrespondence M of the Pareto social choice correspondence is implemented in strong equilibrium: M picks those Pareto optimal allocations that are characterized by maximal coalitions.[3]

In Sect. 5 we provide an axiomatic characterization of this social choice correspondence M. We show that M is the maximal correspondence satisfying minimal envy Pareto optimality, equal division lower bound, and sharing index order preservation. The first mentioned condition requires that allocations are Pareto optimal and, within the set of Pareto optimal allocations, only those are selected from at which the number of envious agents—agents who prefer some one else's share over their own—is as small as possible. The second condition requires that each agent (weakly) prefers its share over an equal division of the commodity. The third condition says that if an agent i is willing to equally share the commodity with more other agents than some agent j does, then agent i does not receive less than agent j.

Sprumont (1991) shows that under a few natural conditions, the so-called uniform rule is the unique strategy-proof (Gibbard 1973; Satterthwaite 1975) rule for division problems with single-peaked preferences—a preference is single-peaked if there is a unique best point, the peak, and preference decreases in both directions away from

[3] Inoue and Yamamura (2015)—which is an earlier version of their 2023 paper—also consider strong equilibria for the binary mechanism and single-dipped preferences. In Sects. 4.2 and 5 we show how their results are related to ours. In particular our results about strong equilibria are closely related to their results.

this peak. Bochet et al. (2021)—combining work of Bochet and Sakai (2009) and Thomson (1994)—show that under similar assumptions as ours, equilibria (Nash, Pareto optimal Nash, strong) end up in the allocation assigned by the uniform rule—see our concluding Sect. 6. See also Thomson (1995, 2010). While the uniform rule for single-peaked preferences is strategy-proof, we show in Sect. 6 that no selection from the implemented correspondence M for single-dipped preferences is strategy-proof. Further, the uniform rule satisfies the first two of the three conditions characterizing M, as described above.

Single-dipped and single-peaked preferences were already studied by Inada (1964). For single-dipped preferences in division problems, see Klaus et al. (1997), who characterize Pareto optimal allocations (we use their result in Sect. 4), and study strategy-proofness of rules. For strategy-proofness in problems with indivisible goods and single-dipped preferences see Klaus (2001a, b) and Tamura (2023), and for probabilistic rules see Ehlers (2002). Doghmi (2013) considers Nash implementation of social choice correspondences when preferences are single-dipped.

There is a relatively large literature on single-dipped preferences and public goods (in this context also sometimes called public bads), including Peremans and Storcken (1999), Barberà et al. (2012), Bossert and Peters (2014), Öztürk et al. (2013, 2014), Manjunath (2014), Ayllón and Caramuta (2016), Tapki (2016), Yamamura (2016), Lahiri et al. (2017), and Feigenbaum et al. (2020).

2 Preliminaries

In this section we introduce allocations, preferences, mechanisms, rules, and equilibria.

2.1 Allocations, preferences, mechanisms, and equilibria

For $n \in \mathbb{N}$ with $n \geq 2$, let $N = \{1, \ldots, n\}$ be the set of *agents*. Among these agents one unit of a perfectly divisible commodity has to be distributed. The set of all *allocations* is denoted by $\mathcal{A} = \left\{x \in [0, 1]^N \;\middle|\; \sum_{i \in N} x_i = 1\right\}$. A subset of agents is also called a *coalition*.

An agent's *preference* is a transitive and complete binary relation R on the interval $[0, 1]$. We denote by P *strict* preference, and by I *indifference*: $\alpha P \beta$ if $\alpha R \beta$ and not $\beta R \alpha$, and $\alpha I \beta$ if $\alpha R \beta$ and $\beta R \alpha$, for $\alpha, \beta \in [0, 1]$. By $R_N = (R_i)_{i \in N}$ we denote a *profile* of preferences (for N).

An allocation $x \in \mathcal{A}$ is *Pareto optimal* at a preference profile R_N if there is no $x' \in \mathcal{A}$ such that $x'_i R_i x_i$ for all $i \in N$ and $x'_i P_i x_i$ for at least one $i \in N$.

In this paper we assume that there is a decision maker who does not know the preferences of the agents, and we focus on mechanisms in order to select allocations. A *mechanism* is a map $g : [0, 1]^N \to \mathcal{A}$. Such a mechanism, often also called game form, equips the agents with tools to solve the problem of dividing the commodity in a non-cooperative way. Indeed, a preference profile R_N and a mechanism g induce a non-cooperative game (R_N, g) in the usual way, as follows. Each agent $i \in N$ has *strategy set* $[0, 1]$. A *profile* of strategies $r = (r_i)_{i \in N} \in [0, 1]^N$ results in an allocation

$g(r) \in \mathcal{A}$, evaluated by each agent i via R_i. For a coalition S we denote by r_S the restriction of r to S, i.e., $r_S = (r_i)_{i \in S}$; if $S = N \backslash \{i\}$ for some $i \in N$, we also write r_{-i} instead of $r_{N \backslash \{i\}}$. A profile r^* is a *Nash equilibrium* of the game (R_N, g) if for all $i \in N$ and $r_i \in [0, 1]$,

$$g_i(r^*) R_i g_i(r_i, r^*_{-i}).$$

A Nash equilibrium r^* is a *Pareto optimal Nash equilibrium* of the game (R_N, g) if $g(r^*)$ is Pareto optimal at R_N. A profile r^* is a *strong equilibrium* of the game (R_N, g) if there are no $\emptyset \neq S \subseteq N$ and $r'_S \in [0, 1]^S$ such that

$$g_i(r'_S, r^*_{N \backslash S}) R_i g_i(r^*) \text{ for all } i \in S \text{ and } g_i(r'_S, r^*_{N \backslash S}) P_i g_i(r^*) \text{ for some } i \in S.$$

In most of this paper we focus on single-dipped preferences. A preference R is *single-dipped* if there is a *dip* $d(R) \in [0, 1]$ such that for all $\alpha, \beta \in [0, 1]$,

$$\alpha < \beta \leq d(R) \Rightarrow \alpha P \beta \text{ and } \alpha > \beta \geq d(R) \Rightarrow \alpha P \beta.$$

The set of all single-dipped preferences is denoted by $\mathcal{D}$, and $\mathcal{D}^N$ is the set of all single-dipped preference profiles.

A preference R is *single-peaked* if there is a *peak* $p(R) \in [0, 1]$ such that for all $\alpha, \beta \in [0, 1]$,

$$p(R) \geq \alpha > \beta \Rightarrow \alpha P \beta \text{ and } p(R) \leq \alpha < \beta \Rightarrow \alpha P \beta.$$

The set of all single-peaked preferences is denoted by $\mathcal{P}$, and $\mathcal{P}^N$ is the set of all single-peaked preference profiles.

2.2 Conditions on mechanisms

In most of what follows, we impose the following additional conditions on a mechanism g:

- *anonymity*: $g_i(r^\pi) = g_{\pi(i)}(r)$ for all $r \in [0, 1]^N$ and every permutation π of N, where $r^\pi = (r_{\pi(i)})_{i \in N}$.
- *monotonicity*: for all $r \in [0, 1]^N$, $i \in N$ and $r'_i \in [0, 1]$,

$$r'_i > r_i \text{ and } g_i(r) < 1 \Rightarrow g_i(r'_i, r_{-i}) > g_i(r),$$
$$r'_i < r_i \text{ and } g_i(r) > 0 \Rightarrow g_i(r'_i, r_{-i}) < g_i(r).$$

The set of all anonymous and monotonic mechanisms is denoted by $\mathcal{G}$.

The monotonicity condition is closely related to the condition of 'strict own-peak monotonicity' in Bochet et al. (2021) when the latter is applied to social choice functions (assigning allocations to single-peaked preference profiles) that are peaks-only. The difference is that the condition in Bochet et al. (2021) allows that an agent i

receives 0 when that agent's strategy r_i is positive. Under our monotonicity condition this is not possible (see Lemma 3.2). Our monotonicity condition is also related to the 'respect for monotonic preferences' condition in Inoue and Yamamura (2015) when formulated for mechanisms. In a different vein, instead of monotonicity also the reverse would work, that is, reporting a higher number leads to a strict decrease in share when possible. While such variations are clearly possible, they do not substantially add to the results of the paper.

We conclude this section with two examples of mechanisms in $\mathcal{G}$.

Example 2.1 Let $N = \{1, 2\}$ and let $g : [0, 1]^N \to \mathcal{A}$ be defined by for each $r \in [0, 1]^N$,

$$g(r) = \left(\frac{1 + r_1 - r_2}{2}, \frac{1 - r_1 + r_2}{2} \right).$$

Then g is anonymous and monotonic, and thus $g \in \mathcal{G}$. △

Example 2.2 Let $g : [0, 1]^N \to \mathcal{A}$ be defined by for each $r \in [0, 1]^N$ and $i \in N$,

$$g_i(r) = \begin{cases} \frac{r_i}{\sum_{j \in N} r_j} & \text{if } \sum_{j \in N} r_j \geq 1 \\ 1 - \frac{(n-1)(1-r_i)}{\sum_{j \in N}(1-r_j)} & \text{if } \sum_{j \in N} r_j \leq 1. \end{cases}$$

This mechanism corresponds to the 'symmetrized proportional rule' in Bochet et al. (2021). Again, g is anonymous and monotonic, and therefore $g \in \mathcal{G}$. △

In the next two sections we analyze Nash equilibria, Pareto optimal Nash equilibria, and strong equilibria in games induced by single-dipped preference profiles and mechanisms in $\mathcal{G}$.

3 Nash equilibrium

Before stating the main results, we formulate two elementary lemmas concerning single-dipped preferences and mechanisms, respectively. The first lemma recalls the well-known fact (Inada 1964) that if an agent with a single-dipped preference prefers α to β in $[0, 1]$, then this agent prefers α to each γ between α and β. This will be used several times in the sequel.

Lemma 3.1 *Let $R \in \mathcal{D}$ and let $\alpha, \beta \in [0, 1]$ with $\alpha R \beta$. Then $\alpha R \gamma$ for all $\gamma \in [0, 1]$ with $\min\{\alpha, \beta\} \leq \gamma \leq \max\{\alpha, \beta\}$.*

Proof If $d(R) \leq \min\{\alpha, \beta\}$, then $d(R) \leq \beta \leq \alpha$, so $\alpha R \gamma$ for all $\beta \leq \gamma \leq \alpha$. If $d(R) \geq \max\{\alpha, \beta\}$, then $d(R) \geq \beta \geq \alpha$, so $\alpha R \gamma$ for all $\alpha \leq \gamma \leq \beta$. If $\min\{\alpha, \beta\} < d(R) < \max\{\alpha, \beta\}$, then we have $\alpha R \gamma$ for all $\min\{\alpha, d(R)\} \leq \gamma \leq \max\{\alpha, d(R)\}$, and $\alpha R \beta R \gamma$ for all $\min\{\beta, d(R)\} \leq \gamma \leq \max\{\beta, d(R)\}$. Therefore, $\alpha R \gamma$ for all $\min\{\alpha, \beta\} \leq \gamma \leq \max\{\alpha, \beta\}$. □

The next lemma shows that a monotonic mechanism assigns 0 to an agent only if its strategy is 0, and assigns 1 to an agent only if its strategy is 1.

Lemma 3.2 *Let g be a monotonic mechanism and let $r \in [0, 1]^N$. Then $r_i = 0$ for each $i \in N$ with $g_i(r) = 0$, and $r_i = 1$ for each $i \in N$ with $g_i(r) = 1$.*

Proof For each $i \in N$ with $g_i(r) = 0$, if $r_i \neq 0$, then $g_i(r_i', r_{-i}) = 0$ for all $0 \leq r_i' < r_i$, which contradicts monotonicity of g. For each $i \in N$ with $g_i(r) = 1$, if $r_i \neq 1$, then $g_i(r_i', r_{-i}) = 1$ for all $r_i < r_i' \leq 1$, which again contradicts monotonicity of g. □

The following two lemmas are about properties of Nash equilibria for single-dipped preference profiles. We first show that for a monotonic mechanism and a single-dipped preference profile, no agent receives its dip in a Nash equilibrium.

Lemma 3.3 *Let $R_N \in \mathcal{D}^N$ and let g be a monotonic mechanism. If a strategy profile $r^* \in [0, 1]^N$ is a Nash equilibrium of (R_N, g), then $g_i(r^*) \neq d(R_i)$ for all $i \in N$.*

Proof Let $i \in N$. Assume, to the contrary, that $r^* \in [0, 1]^N$ with $g_i(r^*) = d(R_i)$, is a Nash equilibrium of (R_N, g). Then we have $g_i(r_i, r_{-i}^*) = d(R_i)$ for all $r_i \in [0, 1]$, which is a contradiction to monotonicity of g. □

Next, we show that, in a Nash equilibrium, an agent's strategy is 0 if that agent receives less than its dip, and is 1 if that agent receives more than its dip.

Lemma 3.4 *Let $R_N \in \mathcal{D}^N$, let g be a monotonic mechanism, and let strategy profile $r^* \in [0, 1]^N$ be a Nash equilibrium of (R_N, g). Then $r_i^* = 0$ for all $i \in N$ with $g_i(r^*) < d(R_i)$, and $r_i^* = 1$ for all $i \in N$ with $g_i(r^*) > d(R_i)$.*

Proof Let $i \in N$ with $g_i(r^*) < d(R_i)$. If $g_i(r^*) = 0$, then $r_i^* = 0$ by Lemma 3.2. If $g_i(r^*) > 0$ with $r_i^* \neq 0$, then from monotonicity, we have $g_i(r_i, r_{-i}^*) < g_i(r^*) < d(R_i)$ for all $0 \leq r_i < r_i^*$. This implies that $g_i(r_i, r_{-i}^*) P_i g_i(r^*)$, which is a contradiction to the assumption that r^* is a Nash equilibrium. Hence, $r_i^* = 0$.

The case $g_i(r^*) > d(R_i)$ is analogous. □

We now introduce some additional notation for a mechanism $g \in \mathcal{G}$. For each $S \subseteq N$, define $e^S \in \mathbb{R}^N$ by $e_i^S = 1$ for all $i \in S$, and $e_j^S = 0$ for all $j \in N \backslash S$. Then, by anonymity we have $g_i(e^{\emptyset}) = g_i(e^N) = \frac{1}{n}$ for all $i \in N$, and there exist numbers $p^1(g), \ldots, p^{n-1}(g) \in [0, 1]$ such that for each $\emptyset \neq S \subsetneq N$ and $i \in S$,

$$g_i(e^S) = p^s(g),$$

where $s = |S|$. It follows that for each $\emptyset \neq S \subsetneq N$ and $j \in N \backslash S$,

$$g_j(e^S) = q^s(g),$$

where $sp^s(g) + (n - s)q^s(g) = 1$ for all $s = 1, \ldots, n - 1$. When no confusion arises, the notations $p^s(g)$ and $q^s(g)$ are abbreviated to p^s and q^s, respectively. For convenience, we denote $p^0 = p^n = q^0 = q^n = \frac{1}{n}$. Then, by monotonicity and Lemma 3.2, it holds that for each $i \in N$, $S \subseteq N\backslash\{i\}$, and $s = 0, \dots, n-1$

$$p^{s+1} = g_i(e^{S\cup\{i\}}) > g_i(e^S) = q^s.$$

The following theorem characterizes the Nash equilibria in games induced by single-dipped preference profiles and mechanisms in $\mathcal{G}$.

Theorem 3.5 *Let $R_N \in \mathcal{D}^N$, $g \in \mathcal{G}$, and $r^* \in [0,1]^N$. Then r^* is a Nash equilibrium of (R_N, g) if and only if $r^* = e^S$ for $S \subseteq N$ such that $p^s R_i q^{s-1}$ for all $i \in S$ and $q^s R_j p^{s+1}$ for all $j \in N\backslash S$.*

Proof For the if-part, assume that $r^* = e^S$ for $S \subseteq N$ such that $p^s R_i q^{s-1}$ for all $i \in S$ and $q^s R_j p^{s+1}$ for all $j \in N\backslash S$. We prove that r^* is a Nash equilibrium.

For each $i \in S$, we have $r_i^* = 1$ and $p^s R_i q^{s-1}$, which means that $g_i(1, r_{-i}^*) R_i g_i(0, r_{-i}^*)$. With monotonicity, it holds that $g_i(0, r_{-i}^*) \leq g_i(r_i, r_{-i}^*) \leq g_i(1, r_{-i}^*)$ for all $r_i \in [0,1]$. According to Lemma 3.1, we conclude that $g_i(r^*) R_i g_i(r_i, r_{-i}^*)$ for all $r_i \in [0,1]$. For each $j \in N\backslash S$, we have $r_j^* = 0$ and $q^s R_j p^{s+1}$, which means that $g_j(0, r_{-j}^*) R_j g_j(1, r_{-j}^*)$. From monotonicity and Lemma 3.1 again, it holds that $g_j(r^*) R_j g_j(r_j, r_{-j}^*)$ for all $r_j \in [0,1]$. So, $r^* = e^S$ is a Nash equilibrium.

For the only-if part, assume that r^* is a Nash equilibrium. From Lemmas 3.3 and 3.4, we have $r^* = e^S$ for some $S \subseteq N$. In view of $g_i(r^*) R_i g_i(0, r_{-i}^*)$ for all $i \in S$ and $g_j(r^*) R_j g_j(1, r_{-j}^*)$ for all $j \in N\backslash S$, it holds that $p^s R_i q^{s-1}$ for all $i \in S$ and $q^s R_j p^{s+1}$ for all $j \in N\backslash S$. □

Remark 3.6 In order to relate this result to the theory of implementation, as we will also do for Pareto optimal Nash equilibrium and strong equilibrium in the next section, we observe the following. For a given mechanism $g \in \mathcal{G}$ and the resulting numbers p^s associated to g one can define the correspondence, say K^g, assigning the Nash equilibrium allocations as in Theorem 3.5, and then by definition K^g is implemented in Nash equilibrium by g. In order to avoid the dependence on g in this formulation, one can also start from the numbers p^s for which there is a $g \in G$ giving these p^s. It is an open question for which choices of the numbers p^s this holds. Nevertheless, obviously there is not a unique correspondence that is Nash implemented by all $g \in \mathcal{G}$. △

Remark 3.7 Theorem 1 in Inoue and Yamamura (2023) can be retrieved from Theorem 3.5 by assuming that $p^s = \frac{1}{s}$ and $q^s = 0$ for every $s = 1, \dots, n-1$. △

The following example shows that a Nash equilibrium is not necessarily unique.

Example 3.8 Let $g \in \mathcal{G}$, $N = \{1,2,3\}$, $p^s = \frac{1}{s}$ and $q^s = 0$ for $s = 1, 2$ (for instance, take the symmetrized proportional rule from Example 2.2). Let $R_N \in \mathcal{D}^N$ be such that $d(R_1) = 0, 0 I_2 \frac{1}{2}$, and $d(R_3) = 1$. Then both $(1,0,0)$ and $(1,1,0)$ are Nash equilibria of (R_N, g). △

Theorem 3.5 can also be used to show that a Nash equilibrium does not have to exist, as for instance in the following example.

Example 3.9 Let $N = \{1, 2, 3\}$ and let $g \in \mathcal{G}$ satisfy that $p^2 > p^1$. By this assumption and monotonicity, it follows that $q^2 < q^1 < \frac{1}{3} < p^1 < p^2$. Consider $R_N \in \mathcal{D}^N$ such that $q^2 P_1 q^1 P_1 p^2 P_1 p^1 P_1 \frac{1}{3}$, $p^2 P_2 q^2 P_2 q^1 P_2 \frac{1}{3} P_2 p^1$ and $q^1 P_3 p^2$. Then $e^{\emptyset}$ is not a Nash equilibrium in view of $p^1 P_1 \frac{1}{3}$; $e^{\{1\}}$ and $e^{\{3\}}$ are not Nash equilibria in view of $p^2 P_2 q^1$; $e^{\{2\}}$ is not a Nash equilibrium in view of $\frac{1}{3} P_2 p^1$; $e^{\{1,2\}}$ and $e^{\{1,3\}}$ are not Nash equilibria in view of $q^1 P_1 p^2$; $e^{\{2,3\}}$ is not a Nash equilibrium in view of $q^1 P_3 p^2$; and e^N is not a Nash equilibrium in view of $q^2 P_1 \frac{1}{3}$. From Theorem 3.5 it follows that the game (R_N, g) has no Nash equilibrium.

A possible mechanism $g \in \mathcal{G}$ to which this example applies is as follows. For each $r \in [0, 1]^N$ and distinct $i, j, k \in N$ let

$$g_i(r) = \frac{8 + 2r_i - r_j - r_k + 2r_i r_j + 2r_i r_k - 4r_j r_k}{24}.$$

Since $g(1, 0, 0) = (\frac{10}{24}, \frac{7}{24}, \frac{7}{24})$ and $g(1, 1, 0) = (\frac{11}{24}, \frac{11}{24}, \frac{2}{24})$, we have $q^2 = \frac{2}{24} < q^1 = \frac{7}{24} < \frac{1}{3} < p^1 = \frac{10}{24} < p^2 = \frac{11}{24}$. △

We conclude this section with the result that for two agents a Nash equilibrium always exists. This result can also be derived from Yamamura (2016, Proposition 1), since our two-agent private good single-dipped case is equivalent to the two-person public good single-dipped case, studied in that paper. The "Appendix" provides an independent proof.

Proposition 3.10 *Let $N = \{1, 2\}$, $R_N \in \mathcal{D}^N$ and $g \in \mathcal{G}$. Then the game (R_N, g) has a Nash equilibrium.*

4 Pareto optimal Nash equilibrium, strong equilibrium, and implementation

In this section, we first consider Pareto optimal Nash equilibria, i.e., Nash equilibria resulting in Pareto optimal allocations. Next, we consider strong equilibria: no subset of agents can profitably deviate, in the sense that every member is at least as well off, and at least one member is better off. Third, we discuss the related issue of implementation: which social choice correspondence, i.e., multi-valued rule, collects exactly the Pareto optimal Nash equilibria or strong equilibria for a given mechanism?

4.1 Pareto optimal Nash equilibrium

Pareto optimal allocations for single-dipped preference profiles were characterized by Klaus et al. (1997). For each $R_N \in \mathcal{D}^N$, we denote by $N_+(R_N) = \{i \in N \mid 1 P_i 0\}$ the set of agents who strictly prefer 1 to 0, by $N_0(R_N) = \{i \in N \mid 0 I_i 1\}$ the set of agents who are indifferent between 0 and 1, and by $N_-(R_N) = \{i \in N \mid 0 P_i 1\}$ the set of agents who strictly prefer 0 to 1. The characterization by Klaus et al. (1997) is as follows.

Lemma 4.1 *Let $R_N \in \mathcal{D}^N$. An allocation $x \in \mathcal{A}$ is Pareto optimal at R_N if and only if*

(i) *If* $N_+(R_N) \neq \emptyset$, *then* $x_i = 0$ *for every* $i \in N \backslash N_+(R_N)$, *and for every* $i \in N_+(R_N)$ *either* $x_i = 0$ *or* $x_i P_i 0$.
(ii) *If* $N_+(R_N) = \emptyset$ *and* $N_0(R_N) \neq \emptyset$, *then* $x = e^{\{i\}}$ *for some* $i \in N_0(R_N)$.
(iii) *If* $N_-(R_N) = N$, *then for every* $i \in N$ *either* $x_i = 1$ *or* $x_i P_i 1$.

We first introduce so-called sharing indices and maximal coalitions, which are useful to describe Pareto optimal Nash equilibria.

Definition 4.2 (a) Let $i \in N$ and $R_i \in \mathcal{D}$. The *sharing index* of i at R_i is the number $m(R_i)$ defined by[4]

$$m(R_i) = \begin{cases} 0 & \text{if } 0 R_i 1 \\ \max\left\{k \in \mathbb{N} \mid \frac{1}{k} P_i 0\right\} & \text{if } 1 P_i 0 \text{ and } d(R_i) > 0 \\ \infty & \text{if } d(R_i) = 0. \end{cases}$$

(b) Let $R_N \in \mathcal{D}^N$. A coalition $S \subseteq N$ is a *maximal coalition* at R_N if the following holds.

(i) If $N_+(R_N) \neq \emptyset$, then $S \subseteq N_+(R_N)$ such that $m(R_i) \geq |S|$ for every $i \in S$ and $m(R_j) \leq |S|$ for every $j \in N \backslash S$.
(ii) If $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$, then $S = \{i\}$ for some $i \in N_0(R_N)$.
(iii) If $N_-(R_N) = N$, and $\{j \in N \mid 1 R_j \frac{1}{n}\} \neq \emptyset$, then $S = \{i\}$ for some $i \in N$ with $1 R_i \frac{1}{n}$.
(iv) If $N_-(R_N) = N$, and $\{j \in N \mid 1 R_j \frac{1}{n}\} = \emptyset$, then $S = \emptyset$.

The collection of all maximal coalitions at R_N is denoted by $\mathcal{M}(R_N)$. △

The sharing index of an agent i who strictly prefers 1 over 0, is the maximal number of agents, including i, with whom equally sharing the commodity is preferred by i over receiving 0; possibly unrestricted if i's dip is 0, in which case $m(R_i) = \infty$. Under the same interpretation, the sharing index of an agent $i \notin N_+(R_N)$ is zero. In Case (i) of (b), a maximal coalition consists of agents who strictly prefer 1 over 0 at R_N. Such a coalition is formed by starting with the agent(s) with maximal sharing index, next adding agent(s) with second maximal sharing index, etc., until the size of the coalition exceeds the sharing indices of the remaining agents. See Example 4.3 for an illustration. In a similar spirit, in Case (ii), a maximal coalition consists of any arbitrary single agent indifferent between 0 and 1. In Case (iii), where all agents strictly prefer 0 over 1, a maximal coalition consists of an arbitrary single agent who (weakly) prefers 1 over $\frac{1}{n}$. If there are no such agents, then Case (iv) applies and the only maximal coalition is the empty coalition.

Example 4.3 Let $N = \{1, 2, 3\}$ and let R_N satisfy $\frac{1}{3} P_1 0 P_1 \frac{1}{4}$, $\frac{1}{2} P_i 0 R_i \frac{1}{3}$ for $i = 2, 3$. Then $N_+(R_N) = N$, $m(R_1) = 3$, and $m(R_2) = m(R_3) = 2$. To construct a maximal coalition we start with agent 1 and then add either agent 2 or agent 3, to obtain $\{1, 2\}$ and $\{1, 3\}$ as maximal coalitions. Coalition $\{2, 3\}$ is not maximal since $m(R_1) = 3 >$

[4] In this definition, alternatively the last case can be left out if in the second case we take '$k \in \{1, \ldots, n\}$'. The present version, however, makes the sharing index independent of the number of agents.

$2 = |\{2, 3\}|$, and coalition N is not maximal since $m(R_2) = 2 < 3 = |N|$. Also singleton coalitions are not maximal: $\{1\}$ is not maximal since $m(R_2) = 2 > |\{1\}|$, $\{2\}$ is not maximal since $m(R_1) = 3 > |\{2\}|$, and $\{3\}$ is not maximal since $m(R_1) = 3 > |\{3\}|$. △

The basic reason why maximal coalitions play a role in our analysis, especially in the case where $N_+(R_N) \neq \emptyset$, is that a member of such a coalition prefers receiving an equal share over receiving 0 and therefore would not deviate and leave the coalition; on the other hand, there is no outside agent who would gain by joining the coalition. This will be made precise in Theorem 4.5.

We first formulate an additional property for a mechanism g:

- *zero–one property*: $g(e^S) = \frac{1}{|S|} e^S$ for every $\emptyset \neq S \subseteq N$.

If g satisfies the zero–one property, then $p^s = \frac{1}{s}$ and $q^s = 0$ for each $s = 1, 2, \ldots, n-1$. The mechanisms in Examples 2.1 and 2.2 satisfy this property, but the mechanism in Example 3.9 does not.[5]

We show that the zero–one property of a mechanism is a necessary and sufficient condition for *each* game based on this mechanism to have a Pareto optimal Nash equilibrium.

Lemma 4.4 *Let $g \in \mathcal{G}$ and suppose that (R_N, g) has a Pareto optimal Nash equilibrium for each $R_N \in \mathcal{D}^N$. Then g satisfies the zero–one property.*

Proof For each $S \in 2^N \backslash \{\emptyset, N\}$, we consider $R_N^S \in \mathcal{D}^N$ such that $d(R_i^S) = 0$ for all $i \in S$ and $d(R_j^S) = 1$ for all $j \in N \backslash S$. Then $N_+(R_N^S) = S$. From Lemmas 3.3 and 3.4, it follows that the only Nash equilibrium in the game (R_N^S, g) is $r^* = e^S$. From Lemma 4.1, we have $g_j(r^*) = 0$ for all $j \in N \backslash N_+(R_N^S) = N \backslash S$. It follows that $g_i(r^*) = \frac{1}{|S|}$ for all $i \in S$. Together with $g(e^N) = \frac{1}{|N|} e^N$, we conclude that $g(e^S) = \frac{1}{|S|} e^S$ for all $S \in 2^N \backslash \{\emptyset\}$. This implies that g satisfies the zero–one property. □

Lemma 4.4 says that the zero–one property of the mechanism is a necessary condition for a Pareto optimal Nash equilibrium to exist in every game induced by this mechanism. The sufficiency part follows from the following theorem, which is a main result of this paper.

Theorem 4.5 *Let $R_N \in \mathcal{D}^N$ and let $g \in \mathcal{G}$ satisfy the zero–one property. A strategy profile $r^* \in [0, 1]^N$ is a Pareto optimal Nash equilibrium of (R_N, g) if and only if $r^* = e^S$ for some $S \in \mathcal{M}(R_N)$.*

Proof For the if-part, let $S \in \mathcal{M}(R_N)$. We prove that $r^* = e^S$ is a Pareto optimal Nash equilibrium.

Case (i) $N_+(R_N) \neq \emptyset$.

Let $i \in S$, hence $m(R_i) \geq |S|$. Then $r_i^* = 1$ and $g_i(r^*) = \frac{1}{|S|}$. If $m(R_i) < \infty$ then $\frac{1}{m(R_i)} P_i 0$ and hence $\frac{1}{|S|} P_i 0$. If $m(R_i) = \infty$, i.e., $d(R_i) = 0$, then obviously $\frac{1}{|S|} P_i 0$.

[5] A weaker version of this property for social choice functions occurs under the name ‘least richness’ in Inoue and Yamamura (2015).

Hence in both cases $g_i(r^*)P_i g_i(0, r^*_{-i})$. Monotonicity then implies $g_i(0, r^*_{-i}) \leq g_i(r_i, r^*_{-i}) \leq g_i(r^*)$ for all $r_i \in [0, 1]$, and by Lemma 3.1, $g_i(r^*)R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0, 1]$.

If $i \in N_+(R_N)\backslash S$, then $r^*_i = 0$ and $g_i(r^*) = 0$. Since $m(R_i) \leq |S|$ it follows that $\frac{1}{|S|+1} \leq \frac{1}{m(R_i)+1}$. Together with $0R_i \frac{1}{m(R_i)+1}$, by Lemma 3.1, we have $0R_i \frac{1}{|S|+1}$, which implies that $g_i(r^*)R_i g_i(1, r^*_{-i})$. With monotonicity and Lemma 3.1 again, we can similarly verify that $g_i(r^*)R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0, 1]$.

For $i \in N\backslash N_+(R_N)$, in view of $S \subseteq N_+(R_N)$, we have $i \in N\backslash S$, $r^*_i = 0$ and $g_i(r^*) = 0$. In view of $0R_i 1$, by Lemma 3.1, we have $g_i(r^*)R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0, 1]$.

Thus, $g_i(r^*)R_i g_i(r_i, r^*_{-i})$ for all $i \in N$ and $r_i \in [0, 1]$, which implies that $r^* = e^S$ is a Nash equilibrium.

Case (ii) $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

Let $S = \{i\}$ with $i \in N_0(R_N)$. Then $g_i(r^*) = 1$ and $g_j(r^*) = 0$ for all $j \in N\backslash\{i\}$. For agent i, in view of $1R_i 0$, by Lemma 3.1, it holds that $g_i(r^*)R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0, 1]$. For each agent $j \in N\backslash\{i\}$, in view of $0R_j 1$, by Lemma 3.1 again, we have $g_j(r^*)R_j g_j(r_j, r^*_{-j})$ for all $r_j \in [0, 1]$. So, $r^* = e^{\{i\}}$ is a Nash equilibrium.

Case (iii) $N_-(R_N) = N$.

If $S = \{i\}$ for some $i \in N$, then $1R_i \frac{1}{n}$. Then $r^*_j = 0$ and $g_j(r^*) = 0$ for all $j \in N\backslash\{i\}$. In view of $0P_j 1$, by Lemma 3.1, we have $g_j(r^*)R_j g_j(r_j, r^*_{-j})$ for all $r_j \in [0, 1]$. With monotonicity, we have $g_i(r_i, r^*_{-i}) \geq \frac{1}{n}$ for all $r_i \in [0, 1]$. In view of $1R_i \frac{1}{n}$, by Lemma 3.1 again, we have $g_i(r^*)R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0, 1]$. So, $r^* = e^{\{i\}}$ is a Nash equilibrium.

If $S = \emptyset$, then $\frac{1}{n} P_i 1$ for all $i \in N$. For each $i \in N$, if $r_i > 0$, with monotonicity, we have $g_i(r_i, r^*_{-i}) > g_i(0, r^*_{-i}) = \frac{1}{n}$. Together with $\frac{1}{n} P_i 1$, by Lemma 3.1, we have $g_i(r^*)R_i g_i(r_i, r^*_{-i})$ for all $r_i \in [0, 1]$. So, $r^* = e^{\emptyset}$ is a Nash equilibrium.

Combining these three cases, we conclude that for each $S \in \mathcal{M}(R_N)$, $r^* = e^S$ is a Nash equilibrium. Lemma 4.1 implies that $g(r^*)$ is Pareto optimal at R_N.

For the only-if part, assume that r^* is a Pareto optimal Nash equilibrium. From Theorem 3.5, it follows that $r^* = e^S$ for some $S \in 2^N$. We prove that $S \in \mathcal{M}(R_N)$.

Case (i) $N_+(R_N) \neq \emptyset$.

Assume, to the contrary, that $S \notin \mathcal{M}(R_N)$. Let $T \in \mathcal{M}(R_N)$. First, we prove that $|S| = |T|$.

Since $g(e^S)$ and $g(e^T)$ are Pareto optimal at R_N, from Lemma 4.1, we have $\frac{1}{|S|} P_i 0$ for all $i \in S$, and $\frac{1}{|T|} P_i 0$ for all $i \in T$. Since e^S (by assumption) and e^T (from the if-part) are Nash equilibria of (R_N, g), we have $0R_j \frac{1}{|S|+1}$ for all $j \in N\backslash S$, and $0R_j \frac{1}{|T|+1}$ for all $j \in N\backslash T$. If $|S| < |T|$, then there exists $k \in T\backslash S$ such that $\frac{1}{|T|} P_k 0$ and $0R_k \frac{1}{|S|+1}$. However, in view of $|S| < |T|$, we have $|S|+1 \leq |T|$, i.e., $\frac{1}{|T|} \leq \frac{1}{|S|+1}$. From Lemma 3.1, it follows that $0R_k \frac{1}{|T|}$, which is a contradiction. If $|S| > |T|$, we similarly obtain a contradiction. Thus, $|S| = |T|$.

Then, since $S \notin \mathcal{M}(R_N)$ and $|S| = |T|$, there exist $i \in S$ and $j \in N\backslash S$ such that $m(R_i) < m(R_j)$. By Lemma 4.1, we have $\frac{1}{|S|} P_i 0$. It follows that $|S| \leq m(R_i)$. So, $|S| < m(R_j)$, hence $\frac{1}{|S|+1} \geq \frac{1}{m(R_j)}$. If $m(R_j) = \infty$, hence $d(R_j) = 0$, then

$\frac{1}{|S|+1} P_j 0$, and otherwise, since $\frac{1}{m(R_j)} P_j 0$, we also have $\frac{1}{|S|+1} P_j 0$. This implies that $g_j(1, e^S_{-j}) P_j g_j(e^S)$, which contradicts the assumption that e^S is a Nash equilibrium. Thus, $S \in \mathcal{M}(R_N)$.

Case (ii) $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

From Lemma 4.1, $g(e^T)$ is not Pareto optimal for all $T \in 2^N \setminus \mathcal{M}(R_N)$. Thus, $S \in \mathcal{M}(R_N)$.

Case (iii) $N_-(R_N) = N$.

If there exists $i \in N$ such that $1 R_i \frac{1}{n}$, then $e^\emptyset$ is not a Pareto optimal Nash equilibrium, hence $S \neq \emptyset$. Since $0 P_j 1$ for all $j \in N$, it follows that e^T is not a Nash equilibrium for each $T \in 2^N$ with $|T| \geq 2$. Hence, $|S| = 1$. For $j \in N$ such that $\frac{1}{n} P_j 1$, it is easily seen that $e^{\{j\}}$ is not a Nash equilibrium. Thus, $S \in \mathcal{M}(R_N)$.

Finally, suppose that $\{i \in N \mid 1 R_i \frac{1}{n}\} = \emptyset$, i.e., $\frac{1}{n} P_i 1$ for all $i \in N$. If $T \neq \emptyset$, then since $0 P_i 1$ and $\frac{1}{n} P_i 1$ for all $i \in N$, it follows that $g_i(e^{T \setminus \{i\}}) P_i g_i(e^T)$ for all $i \in T$, which implies that e^T is not a Nash equilibrium. So, $S = \emptyset \in \mathcal{M}(R_N)$, and the proof of the theorem is complete. □

Theorem 4.5 shows that for a mechanism satisfying the zero–one property, the Pareto optimal Nash equilibria are those strategy profiles in which all agents in a maximal coalition play 1 and all other agents play 0. Since there exists at least one maximal coalition for every single-dipped preference profile, Lemma 4.4 and Theorem 4.5 imply the result announced earlier.

Corollary 4.6 *Let $g \in \mathcal{G}$. There exists a Pareto optimal Nash equilibrium of (R_N, g) for every $R_N \in \mathcal{D}^N$ if and only if g satisfies the zero–one property.*

The next example shows that for a game based on a mechanism satisfying the zero–one property, besides Pareto optimal Nash equilibria, there may exist Nash equilibria without Pareto optimal outcomes, or Pareto optimal outcomes, not obtained in any Nash equilibrium.

Example 4.7 Let $N = \{1, 2\}$ and let $g \in \mathcal{G}$ be as in Example 2.1.

(a) Consider $R_N \in \mathcal{D}^N$ such that $1 P_1 0 P_1 \frac{1}{2}$ and $0 P_2 1 P_2 \frac{1}{2}$. Then, we have $g_1(0, 1) P_1 g_1(1, 1)$ and $g_2(0, 1) P_2 g_2(0, 0)$. With monotonicity and Lemma 3.1, it follows that $g_1(0, 1) R_1 g_1(r_1, 1)$ and $g_2(0, 1) R_2 g_2(0, r_2)$ for all $r_1, r_2 \in [0, 1]$. So, $e^{\{2\}} = (0, 1)$ is a Nash equilibrium. However, $g(e^{\{2\}}) = (0, 1)$ is not Pareto optimal at R_N. In fact, Theorem 4.5 implies that the unique Pareto optimal Nash equilibrium is $e^{\{1\}} = (1, 0)$.

(b) Consider $R_N \in \mathcal{D}^N$ such that $d(R_1) = d(R_2) = 0$. Then $x = g(\frac{1}{2}, \frac{1}{3}) = (\frac{7}{12}, \frac{5}{12})$ is Pareto optimal at R_N, but there is no $S \in 2^N$ such that $g(e^S) = x$. Thus, Theorem 3.5 implies that there is no Nash equilibrium r^* such that $g(r^*) = x$. In fact, $m(R_1) = m(R_2) = \infty$, and hence the unique maximal coalition is N. From Theorem 4.5 (or direct inspection), the unique Pareto optimal Nash equilibrium is $e^N = (1, 1)$. △

4.2 Strong equilibrium

In this subsection we consider a further strengthening of Pareto optimal Nash equilibrium, namely strong equilibrium (Aumann 1959): no coalition can profitably deviate. We show that the Pareto optimal Nash equilibria and strong equilibria coincide if, besides anonymity, monotonicity, and the zero–one property, the mechanism g satisfies:

- *order preservation*: $g_i(r) \geq g_j(r)$ for all $r \in [0,1]^N$ and $i, j \in N$ with $r_i \geq r_j$.[6]

Theorem 4.8 *Let $R_N \in \mathcal{D}^N$ and let $g \in \mathcal{G}$ satisfy the zero–one property and order preservation. Then a strategy profile is a Pareto optimal Nash equilibrium of (R_N, g) if and only if it is a strong equilibrium.*

Proof We start with the only-if part. Let $S \in \mathcal{M}(R_N)$. By Theorem 4.5, it is sufficient to verify that e^S is a strong equilibrium. Assume, to the contrary, that there exist $T \in 2^N \backslash \{\emptyset\}$ and $r_T \in [0,1]^T$ such that $g_i(r_T, e^S_{N \backslash sT}) R_i g_i(e^S)$ for all $i \in T$ and $g_j(r_T, e^S_{N \backslash T}) P_j g_j(e^S)$ for some $j \in T$. We consider three cases.
Case (i) $N_+(R_N) \neq \emptyset$.

If $S \cap T \neq \emptyset$, then for each $i \in S \cap T$, it holds that $g_i(r_T, e^S_{N \backslash T}) \geq \frac{1}{|S|}$ in view of $\frac{1}{|S|} P_i 0$ from Theorem 4.5 and $g_i(r_T, e^S_{N \backslash T}) R_i g_i(e^S)$ by assumption. By order preservation, it follows that $g_j(r_T, e^S_{N \backslash T}) \geq g_i(r_T, e^S_{N \backslash T}) \geq \frac{1}{|S|}$ for all $j \in S \backslash T$. So, we have $g_i(r_T, e^S_{N \backslash T}) = \frac{1}{|S|} = g_i(e^S)$ for all $i \in S$, and $g_j(r_T, e^S_{N \backslash T}) = 0 = g_j(e^S)$ for all $j \in N \backslash S$, i.e., $g_k(r_T, e^S_{N \backslash T}) I_k g_k(e^S)$ for all $k \in T$, contradicting our assumption.

If $S \cap T = \emptyset$, then we claim that $g_i(r_T, e^S_{N \backslash T}) \leq \frac{1}{|S|+1}$ for each $i \in T$. If not, take $i \in T$ with $g_i(r_T, e^S_{N \backslash T}) > \frac{1}{|S|+1}$. Then $g_j(r_T, e^S_{N \backslash T}) \geq g_i(r_T, e^S_{N \backslash T}) > \frac{1}{|S|+1}$ for all $j \in S$. It follows that $\sum_{k \in T \cup S} g_k(r_T, e^S_{N \backslash T}) > 1$, which is not possible. In view of $0 R_i \frac{1}{|S|+1}$ for each $i \in T$ from Theorem 4.5, together with Lemma 3.1, we have $g_i(e^S) R_i g_i(r_T, e^S_{N \backslash T})$ for all $i \in T$, which contradicts our assumption.
Case (ii) $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

In this case, $S = \{i\}$ for some $i \in N_0(R_N)$. Then $g(e^S) = e^{\{i\}}$. Since $1 I_i 0$ and $0 R_j 1$ for all $j \in N \backslash \{i\}$, by Lemma 3.1 we have $g_k(e^S) R_i g_k(r_T, e^S_{N \backslash T})$ for all $k \in T$, which is a contradiction to our assumption.
Case (iii) $N_-(R_N) = N$.

If $S = \{i\}$ for some $i \in N$, then $1 R_i \frac{1}{n}$. It follows that $g_j(e^S) = 0$ for all $j \in N \backslash \{i\}$. Since $e^{\{i\}}$ is a Nash equilibrium, it holds that $T \neq \{i\}$. For each $k \in T \backslash \{i\}$, we have $g_k(e^S) R_k g_k(r_T, e^S_{N \backslash T})$ from $0 P_k 1$ and Lemma 3.1. Together with our assumption, it follows that $g_k(r_T, e^S_{N \backslash T}) = g_k(e^S) = 0$ for all $k \in T \backslash \{i\}$. By order preservation, we have $g_j(r_T, e^S_{N \backslash T}) = 0$ for all $j \in N \backslash \{i\}$. So, $g(r_T, e^S_{N \backslash T}) = g(e^S)$, which is a contradiction.

[6] A similar condition also occurs in Bochet et al. (2021) under the same name. It is an open problem whether order preservation is implied by anonymity and monotonicity.

If $S = \emptyset$, then $\frac{1}{n} P_i 1$ for all $i \in N$. For each $k \in T$, in view of $g_k(r_T, e^S_{N\setminus T}) R_k g_k(e^S)$ and $0 P_k \frac{1}{n} P_k 1$, we have $g_k(r_T, e^S_{N\setminus T}) \leq \frac{1}{n}$. By order preservation, it holds that $g_j(r_T, e^S_{N\setminus T}) \leq g_k(r_T, e^S_{N\setminus T}) \leq \frac{1}{n}$ for all $j \in N\setminus T$ and $k \in T$. So, $g_k(r_T, e^S_{N\setminus T}) = g_k(e^S) = \frac{1}{n}$ for all $k \in T$, which is a contradiction. This concludes the proof of the only-if part.

For the if-part, suppose that r^* is a strong equilibrium of (R_N, g). Obviously, r^* is a Nash equilibrium. By Theorem 3.5, there is a coalition S such that $r^* = e^S$. Since g satisfies the zero–one property, we have $g(e^S) = \frac{1}{|S|} e^S$ if $S \neq \emptyset$. If $S = \emptyset$, then $g(e^S) = \frac{1}{n} e^N$.

If $S = \emptyset$, then, if $1 R_i \frac{1}{n}$ for some $i \in N$, then N can deviate to $e^{\{i\}}$, contradicting that $e^{\emptyset}$ is a strong equilibrium. Hence, $\frac{1}{n} P_i 1$ for all $i \in N$, so that $g(e^{\emptyset}) = \frac{1}{n} e^N$ is Pareto optimal by Lemmma 4.1.

If $|S| \geq 2$, then, since e^S is a Nash equilibrium, $\frac{1}{|S|} R_i 0$ for all $i \in S$; in this case, if $x_i R_i g_i(e^S)$ for some $x \in \mathcal{A}$ and all $i \in N$, then in particular $x_i \geq \frac{1}{|S|}$ for all $i \in S$, which implies $x = g(e^S)$ and, thus, $g(e^S)$ is Pareto optimal.

Finally, suppose that $|S| = 1$, say $S = \{n\}$.

If $1 P_n 0$ then clearly $g(e^S) = (0, \ldots, 0, 1)$ is Pareto optimal.

If $1 I_n 0$ and there is some $j \neq n$ with $1 P_j 0$, then $\{j, n\}$ can profitably deviate by $r_j = 1$ and $r_n = 0$, contradicting that e^S is a strong equilibrium; hence, $0 R_j 1$ for all $j \neq n$, so that $g(e^S) = (0, \ldots, 0, 1)$ is Pareto optimal.

If $0 P_n 1$ and there is some $j \neq n$ with $1 R_j 0$, then $\{j, n\}$ can profitably deviate by $r_j = 1$ and $r_n = 0$, contradicting that e^S is a strong equilibrium; hence, $0 P_j 1$ for all $j \neq n$, so that $g(e^S) = (0, \ldots, 0, 1)$ is Pareto optimal. This concludes the proof of the if-part. □

From Theorems 4.5 and 4.8 we obtain the following corollary.

Corollary 4.9 *Let $R_N \in \mathcal{D}^N$ and let $g \in \mathcal{G}$ satisfy the zero–one property and order preservation. A strategy profile $r^* \in [0, 1]^N$ is a strong equilibrium of (R_N, g) if and only if $r^* = e^S$ for some $S \in \mathcal{M}(R_N)$.*

We conclude this part with pointing out the relation between this result and a result in Inoue and Yamamura (2015), in the following remark.

Remark 4.10 Corollary 4.9 is closely related to Theorem 6 in Inoue and Yamamura (2015). In that theorem, social choice functions are considered—assigning allocations to preference profiles—satisfying a number of conditions, but this setting and the conditions can be reformulated in terms of mechanisms in our sense. More precisely, these conditions are symmetry, least richness, and respect for monotonic preferences (these are weaker versions of anonymity, the zero–one property, and monotonicity), plus a property called ‘others oriented resource monotonicity’. The resulting set of mechanisms characterized by these conditions is different from the set of mechanisms to which Corollary 4.9 applies: neither of the two is contained in the other one. For instance, the uniform rule (Sprumont 1991; see also Sect. 6 below), reformulated as a mechanism, satisfies the conditions following from Inoue and Yamamura (2015), and satisfies our conditions except for monotonicity. On the other hand, consider the following mechanism g, which is a variation on the 'symmetrized proportional rule' in Example 2.2 (cf. Bochet et al. 2021). Let $N = \{1, 2, 3\}$ and (w.l.o.g.) $(0 \leq)\ r_1 \leq r_2 \leq r_3\ (\leq 1)$, then

$$g(r) = \begin{cases} \frac{(r_1(1+r_2-r_1), r_2, r_3)}{r_1(1+r_2-r_1)+r_2+r_3} & \text{if } r_1(1+r_2-r_1)+r_2+r_3 \geq 1 \\ (1,1,1) - 2\frac{(1-r_1(1+r_2-r_1), 1-r_2, 1-r_3)}{3-(r_1(1+r_2-r_1)+r_2+r_3)} & \text{if } r_1(1+r_2-r_1)+r_2+r_3 < 1. \end{cases}$$

The mechanism g is anonymous, monotonic, order-preserving, and satisfies the zero–one property, i.e., it satisfies all the conditions in Corollary 4.9. It is obtained by adapting the lower claim (strategy) in the direction of the median claim without, indeed, changing the order. But g is not 'others oriented resource monotonic', the condition mentioned above, which is related to 'others oriented peak monotonicity' in Bochet et al. (2021), and states that if one agent gets more, then all other agents get (weakly) less. For instance, $g(1/4, 1/2, 7/8) = (5/27, 8/27, 14/27)$, whereas $g(1/4, 3/4, 7/8) = (3/16, 6/16, 7/16)$: agent 2's strategy has changed, agent 2 gets more, but also agent 1 gets more.

Mechanisms satisfying either of the two sets of conditions result in the same strong equilibrium allocations but these sets of conditions are, thus, logically independent. The examples additionally show that neither (our) monotonicity nor the others oriented resource monotonicity specifically drives this result. △

4.3 Implementation

In this subsection we reformulate our main results in terms of implementation. A *social choice correspondence* F is a map assigning to each preference profile $R_N \in \mathcal{D}^N$ a nonempty set of allocations. If this set always consists of exactly one allocation, then F is a rule, as defined earlier in Sect. 2. We say that a mechanism g *implements* F *in Pareto optimal Nash equilibrium* if

$$F(R_N) = \{g(r) \in \mathcal{A} \mid r \text{ is a Pareto optimal Nash equilibrium of } (R_N, g)\}$$

for every preference profile $R_N \in \mathcal{D}^N$. Mechanism g *implements* F *in strong equilibrium* if

$$F(R_N) = \{g(r) \in \mathcal{A} \mid r \text{ is a strong equilibrium of } (R_N, g)\}$$

for every preference profile $R_N \in \mathcal{D}^N$. For each $S \subseteq N$ define the allocation $\hat{e}^S \in \mathcal{A}$ by

$$\hat{e}^S = \begin{cases} \frac{1}{|S|} e^S & \text{if } S \neq \emptyset \\ (\frac{1}{n}, \ldots, \frac{1}{n}) & \text{if } S = \emptyset. \end{cases}$$

Define the social choice correspondence M on $\mathcal{D}^N$ by

$$M(R_N) = \{\hat{e}^S \in \mathcal{A} \mid S \in \mathcal{M}(R_N)\}$$

for every $R_N \in \mathcal{D}^N$. We now have the following consequence of Theorem 4.5, Corollary 4.6, and Corollary 4.9.

Corollary 4.11 *Let $g \in \mathcal{G}$. Then g satisfies the zero–one property if and only if g implements M in Pareto optimal Nash equilibrium. If g satisfies the zero–one property and order preservation, then g implements M in strong equilibrium.*

It is an open question whether the condition of order preservation can be left out in this corollary, in particular since it is an open question whether order preservation is implied by monotonicity and anonymity (cf. footnote 6).

5 An axiomatic characterization of the correspondence *M*

In this section we present an axiomatic characterization of the correspondence M, i.e., the correspondence implemented in Pareto optimal Nash or strong equilibrium as in Corollary 4.11.

Unless stated otherwise, F is a social choice correspondence defined on $\mathcal{D}^N$. In order to formulate the first axiom we define the concept of an envious agent. Let $R_N \in \mathcal{D}^N$ and $x \in \mathcal{A}$. An agent $i \in N$ is (an) *envious* (*agent*) *at* R_N *and* x if $x_j P_i x_i$ for some $j \in N$. We denote by $E(R_N, x)$ the set of all envious agents at R_N and x. By $PO(R_N)$ we denote the set of all Pareto optimal allocations at R_N.

The first axiom requires that F assigns Pareto optimal allocations and, among those, only allocations with a minimal number of envious agents.[7]

Minimal envy Pareto optimality $x \in PO(R_N)$ and $|E(R_N, x)| \le |E(R_N, y)|$ for all $R_N \in \mathcal{D}^N$, $x \in F(R_N)$, and $y \in PO(R_N)$.

The second condition requires that every agent (weakly) prefers an assigned allocation over equal division. Under the same name, this condition occurs in Thomson (2010) for the case of single-peaked preferences; it also occurs already in Pazner (1977) under the name 'per-capita-fairness'.

Equal division lower bound $x_i R_i \frac{1}{n}$ for all $R_N \in \mathcal{D}^N$, $x \in F(R_N)$, and $i \in N$.

The third and final condition requires that a higher sharing index cannot result in a lower share. A higher sharing index expresses more eagerness to receive a nonzero share, and this should not result in a lower share.

Sharing index order preservation $x_i \ge x_j$ for all $R_N \in \mathcal{D}^N$, $x \in F(R_N)$, and $i, j \in N$ with $m(R_i) > m(R_j)$.

Our characterization result says that M is the maximal correspondence with these three properties.

Theorem 5.1 *A social choice correspondence F satisfies minimal envy Pareto optimality, equal division lower bound, and sharing index order preservation, if and only if $F(R_N) \subseteq M(R_N)$ for all $R_N \in \mathcal{D}^N$.*

7 A somewhat related minimal envy condition occurs in Combe (2023) in the problem of assigning objects to individuals.

The proof of this theorem is based on a number of lemmas. We first show that M satisfies the three axioms in the theorem.

Lemma 5.2 *The social choice correspondence M on $\mathcal{D}^N$ is minimal envy Pareto optimal.*

Proof Let $R_N \in \mathcal{D}^N$ and $S \in \mathcal{M}(R_N)$. Then, by Corollary 4.11, $\hat{e}^S$ is Pareto optimal. Denote

$$\mu(R_N) = \min_{y \in PO(R_N)} |E(R_N, y)|.$$

We have to show that $|E(R_N, \hat{e}^S)| = \mu(R_N)$. To this end, we distinguish four cases.
Case (i) $N_+(R_N) \neq \emptyset$.

Denote

$$S^* = \{i \in N_+(R_N) \backslash S \mid m(R_i) = |S|\}.$$

We claim that $E(R_N, \hat{e}^S) = S^*$.

To prove this claim, first observe that there is no envious agent in S since $\frac{1}{|S|} R_i \frac{1}{m(R_i)} P_i 0$ for all $i \in S$ with $d(R_i) > 0$, $\frac{1}{|S|} P_i 0$ for all $i \in S$ with $d(R_i) = 0$, and there is also no envious agent in $N \backslash N_+(R_N)$ since $0 R_i \frac{1}{|S|}$ for all $i \in N \backslash N_+(R_N)$. For the agents in $N_+(R_N) \backslash S$, we consider the following three subcases.

(i.a) $N_+(R_N) \backslash S = \emptyset$.

In this case, $S = N_+(R_N)$ and $S^* = \emptyset$, and therefore there is no envious player, i.e., $E(R_N, \hat{e}^S) = \emptyset = S^*$.

(i.b) $N_+(R_N) \backslash S \neq \emptyset$ and $S^* = \emptyset$.

Let $i \in N_+(R_N) \backslash S$. Then $m(R_i) < |S|$, hence $\frac{1}{m(R_i)+1} \geq \frac{1}{|S|} > 0$. This and $0 R_i \frac{1}{m(R_i)+1}$, imply that $0 R_i \frac{1}{|S|}$, hence there is no envious agent in $N_+(R_N) \backslash S$. Hence, again $E(R_N, \hat{e}^S) = \emptyset = S^*$.

(i.c) $N_+(R_N) \backslash S \neq \emptyset$ and $S^* \neq \emptyset$.

Every agent $i \in S^*$ is an envious agent in view of $\frac{1}{|S|} P_i 0$. If $i \in N_+(R_N) \backslash (S \cup S^*)$, then $m(R_i) < |S|$, and similarly as in case (i.b), i is not envious. Hence, also in this case, $E(R_N, \hat{e}^S) = S^*$.

Hence, $E(R_N, \hat{e}^S) = S^*$ in all three subcases.

To complete the proof for Case (i), we show that $\mu(R_N) = |S^*|$.

Assume, to the contrary, that there exists $x \in PO(R_N)$ such that $|E(R_N, x)| < |S^*|$. If there exists $j \in S \cup S^*$ such that $x_j > \frac{1}{|S|}$, then

$$|\{i \in S \cup S^* \mid x_i = \max_{k \in N} x_k\}| \leq |S| - 1.$$

Since $\frac{1}{|S|} P_i 0$ for all $i \in S \cup S^*$, it follows that $(\max_{k \in N} x_k) P_i x_i$ for all $i \in S \cup S^*$ with $x_i \neq \max_{k \in N} x_k$. This means that at least $|S^*| + 1$ agents in $S \cup S^*$ are envious at x, which is a contradiction. Hence, $x_i \leq \frac{1}{|S|}$ for all $i \in S \cup S^*$.

Suppose there exists $j \in N_+(R_N)\backslash(S \cup S^*)$ with $x_j > 0$. Since $m(R_j) < |S|$ and $0 R_j \frac{1}{m(R_j)+1}$, we have $\frac{1}{m(R_j)+1} \geq \frac{1}{|S|}$ and $0 R_j \frac{1}{|S|}$. Hence, since $x \in PO(R_N)$, $x_j > \frac{1}{|S|}$. It follows that

$$\left|\left\{i \in S \cup S^* \;\middle|\; x_i \geq x_j\right\}\right| \leq |S| - 1,$$

which implies that at least $|S^*| + 1$ agents in $S \cup S^*$ are envious at x, which is a contradiction. Thus, $x_j = 0$ and therefore $\sum_{i \in S \cup S^*} x_i = 1$.

Next, suppose there exists $j \in S^*$ with $0 < x_j < \frac{1}{|S|}$. By Lemma 4.1(i), and since $m(R_j) = |S|$, we have $x_j > \frac{1}{|S|+1}$. Hence,

$$|\{i \in S \cup S^* \mid x_i = \max_{k \in N} x_k\}| \leq |S|,$$

and the number of envious agents at x is at least $|S^*|$, contradicting the assumption $|E(R_N, x)| < |S^*|$. Thus, $\mu(R_N) = |S^*|$, and the proof of Case (i) is complete.

Case (ii) $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

In this case, $S = \{i\}$ for some $i \in N_0(R_N)$, and $|E(R_N, \hat{e}^S)| = \mu(R_N) = 0$.

Case (iii) $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} \neq \emptyset$.

In this case, $S = \{j\}$ for some $j \in N$ such that $1 R_j \frac{1}{n}$. Since $E(R_N, \hat{e}^S) = \{j\}$, it follows that $\mu(R_N) \leq 1$. We show that $\mu(R_N) = 1$.

Consider any $x \in PO(R_N)$. If $x_i = 1$ for some agent i, then i is an envious agent. Otherwise, from Lemma 4.1(iii), we have $x_j P_j 1$ for all $j \in N$. Since $(\frac{1}{n}, \ldots, \frac{1}{n}) \notin PO(R_N)$, there exist distinct $k, l \in N$ such that $x_k < x_l$. It follows that $x_k P_l x_l P_l 1$, which means that l is an envious agent. Hence, $\mu(R_N) = 1$.

Case (iv) $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} = \emptyset$.

Then $S = \emptyset$, and $|E(R_N, \hat{e}^S)| = |E(R_N, (\frac{1}{n}, \ldots, \frac{1}{n}))| = 0 = \mu(R_N)$.

This completes the proof of the lemma. □

Lemma 5.3 *The social choice correspondence M on $\mathcal{D}^N$ satisfies equal division lower bound.*

Proof Let $R_N \in \mathcal{D}^N$ and $S \in \mathcal{M}(R_N)$. We show that $\hat{e}_i^S R_i \frac{1}{n}$ for all $i \in N$, by considering four cases.

Case (i) $N_+(R_N) \neq \emptyset$.

For each $i \in S$, if $m(R_i) = \infty$, i.e., $d(R_i) = 0$, then $\frac{1}{|S|} R_i \frac{1}{n}$; and otherwise this holds since $\frac{1}{m(R_i)} P_i 0$, $\frac{1}{m(R_i)} \leq \frac{1}{|S|}$, and $0 < \frac{1}{n} \leq \frac{1}{|S|}$. For each $i \in N_+(R_N)\backslash S$, we have $m(R_i) < n$, and thus $0 R_i \frac{1}{n}$ since $0 R_i \frac{1}{m(R_i)+1}$ and $0 < \frac{1}{n} \leq \frac{1}{m(R_i)+1}$. For each $i \in N\backslash N_+(R_N)$, $0 P_i \frac{1}{n}$. Thus, $\hat{e}_i^S R_i \frac{1}{n}$ for all $i \in N$.

Case (ii) $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

In this case, $S = \{i\}$ for some $i \in N_0(R_N)$. Clearly, $\hat{e}_i^S R_i \frac{1}{n}$ for all $i \in N$.

Case (iii) $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} \neq \emptyset$.

In this case, $S = \{j\}$ for some $j \in N$ such that $1 R_j \frac{1}{n}$. Since $0 R_i 1$ for all $i \in N\backslash\{j\}$, we have $\hat{e}_i^S R_i \frac{1}{n}$ for all $i \in N$.

Case (iv) $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} = \emptyset$.

Then $S = \emptyset$, and $\hat{e}_i^S I_i \frac{1}{n}$ for all $i \in N$. □

Lemma 5.4 *The social choice correspondence M on $\mathcal{D}^N$ satisfies sharing index order preservation.*

Proof Let $R_N \in \mathcal{D}^N$ and $S \in \mathcal{M}(R_N)$. Let $i, j \in N$ with $m(R_i) > m(R_j)$. Then $i \in N_+(R_N)$. If $i \in S$, then $\hat{e}_i^S = \frac{1}{|S|} \geq \hat{e}_j^S \in \{0, \frac{1}{|S|}\}$. If $i \in N_+(R_N) \backslash S$, then also $j \notin S$, and hence $\hat{e}_i^S = 0 = \hat{e}_j^S$. □

Proof of Theorem 5.1 The if-part of the theorem follows from Lemmas 5.2–5.4 and the observation that these lemmas also hold for any F with $F(R_N) \subseteq M(R_N)$ for all $R_N \in \mathcal{D}^N$.

For the only-if part, assume that F satisfies minimal envy Pareto optimality, equal division lower bound, and sharing index order preservation. Let $R_N \in \mathcal{D}^N$ and $x \in F(R_N)$. We show that $x \in M(R_N)$, by distinguishing four cases.

Case (i) $N_+(R_N) \neq \emptyset$.

Let $S \in \mathcal{M}(R_N)$, and $S^* = \{i \in N_+(R_N) \backslash S \mid m(R_i) = |S|\}$. By Lemma 4.1(i), $x_i = 0$ for all $i \in N \backslash N_+(R_N)$. If $x_j > 0$ for some $j \in N_+(R_N) \backslash (S \cup S^*)$, then by Pareto optimality and $m(R_j) < |S|$ it follows that $x_j > \frac{1}{|S|}$. By sharing index order preservation, this implies $x_i > \frac{1}{|S|}$ also for all $i \in S \cup S^*$, but then $\sum_{i \in N} x_i > 1$, a contradiction. Therefore, $\sum_{i \in S \cup S^*} x_i = 1$.

Denote $S^+ = \{i \in S \cup S^* \mid x_i > 0\}$. If $|S^+| > |S|$, then Pareto optimality, sharing index order preservation and $0 R_i \frac{1}{|S|+1}$ for all $i \in N_+(R_N)$ with $m(R_i) = |S|$, imply that $\sum_{i \in S \cup S^*} x_i > 1$, which is a contradiction. So, $|S^+| \leq |S|$, and therefore $\max_{i \in S \cup S^*} x_i \geq \frac{1}{|S|}$. Since $\mu(R_N) = |S^*|$, with $\mu(R_N)$ as defined in the proof of Lemma 5.2, minimal envy Pareto optimality requires that $|S^+| = |S|$ and $x_i = x_j$ for all $i, j \in S^+$. In turn, this implies that $x \in M(R_N)$.

Case (ii) $N_+(R_N) = \emptyset$ and $N_0(R_N) \neq \emptyset$.

Since in this case $PO(R_N) = M(R_N)$, we have $x \in M(R_N)$.

Case (iii) $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} \neq \emptyset$.

Equal division lower bound implies $x_i \leq \frac{1}{n}$ for each $i \in N$ with $\frac{1}{n} P_i 1$. Equal division lower bound and Pareto optimality imply $x_i \leq \frac{1}{n}$ or $x_i = 1$ for each $i \in N$ with $1 R_i \frac{1}{n}$. Since $(\frac{1}{n}, \ldots, \frac{1}{n}) \notin PO(R_N)$, this implies that $x_j = 1$ for some $j \in N$ such that $1 R_j \frac{1}{n}$. Hence, $x \in M(R_N)$.

Case (iv) $N_-(R_N) = N$ and $\{j \in N \mid 1 R_j \frac{1}{n}\} = \emptyset$.

In this case, $M(R_N) = \{(\frac{1}{n}, \ldots, \frac{1}{n})\}$, and therefore minimal envy Pareto optimality implies that there are no envious agents at x. If $y \in PO(R_N)$ with $y \neq (\frac{1}{n}, \ldots, \frac{1}{n})$, then there are $i, j \in N$ such that $y_i < \frac{1}{n} < y_j$. Since $0 P_j \frac{1}{n} P_j 1$, we have $y_i P_j y_j$, which means that player j is envious at R_N and y, and thus $y \neq x$. Thus, $x = (\frac{1}{n}, \ldots, \frac{1}{n}) \in M(R_N)$. This completes the proof of the theorem. □

The following example shows that the axioms in Theorem 5.1 are logically independent.

Example 5.5 (a) Let $n = 3$ and let $\tilde{R}_N \in \mathcal{D}^N$ be a preference profile with $d(\tilde{R}_i) = \frac{1}{3}$ and $1\tilde{P}_i\frac{1}{2}\tilde{P}_i 0$ for all $i \in N$. Define F by $F(\tilde{R}_N) = \{(1, 0, 0)\}$ and $F(R_N) = M(R_N)$ for all $R_N \in \mathcal{D}^N \backslash \{\tilde{R}_N\}$. Then F satisfies equal division lower bound and, since $m(\tilde{R}_1) = m(\tilde{R}_2) = m(\tilde{R}_3) = 2$, also sharing index order preservation. However, $E(\tilde{R}_N, (1, 0, 0)) = \{2, 3\}$, $E(\tilde{R}_N, (\frac{1}{2}, \frac{1}{2}, 0)) = \{3\}$, and both $(1, 0, 0)$ and $(\frac{1}{2}, \frac{1}{2}, 0)$ are Pareto optimal at $\tilde{R}_N$, so that F violates minimal envy Pareto optimality. Note that $M(\tilde{R}_N) = \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$, so that $F \not\subseteq M$.
(b) Let $n = 2$ and let $\tilde{R}_N \in \mathcal{D}^N$ be a preference profile with $d(\tilde{R}_1) = 1$, $d(\tilde{R}_2) = \frac{1}{2}$, and $0\tilde{P}_2 1$. Define F by $F(\tilde{R}_N) = \{(1, 0)\}$ and $F(R_N) = M(R_N)$ for all $R_N \in \mathcal{D}^N \backslash \{\tilde{R}_N\}$. Then F does not satisfy equal division lower bound. Since $m(\tilde{R}_1) = m(\tilde{R}_2) = 0$, F satisfies sharing index order preservation. Since $(1, 0)$ is Pareto optimal at $\tilde{R}_N$, $E(\tilde{R}_N, (1, 0)) = \{1\}$, and at every Pareto optimal allocation at $\tilde{R}_N$ there is exactly one envious player, F satisfies minimal envy Pareto optimality. Note that $M(\tilde{R}_N) = \{(0, 1)\}$, so that $F \not\subseteq M$.
(c) Let $n = 4$ and let $\tilde{R}_N \in \mathcal{D}^N$ be a preference profile with $d(\tilde{R}_i) = \frac{1}{4}$ and $1\tilde{P}_i\frac{1}{3}\tilde{P}_i 0$ for $i = 1, 2$; and $d(\tilde{R}_i) = \frac{1}{4}$, $1\tilde{P}_i\frac{1}{2}\tilde{P}_i 0$, and $0\tilde{P}_i\frac{1}{3}$ for $i = 3, 4$. Then $m(\tilde{R}_1) = m(\tilde{R}_2) = 3$ and $m(\tilde{R}_3) = m(\tilde{R}_4) = 2$. Define F by $F(\tilde{R}_N) = \{(0, 0, \frac{1}{2}, \frac{1}{2})\}$ and $F(R_N) = M(R_N)$ for all $R_N \in \mathcal{D}^N \backslash \{\tilde{R}_N\}$. Then F is not sharing index order preserving. Note that $M(\tilde{R}_N) = \{(\frac{1}{2}, \frac{1}{2}, 0, 0)\}$. Since $|E(\tilde{R}_N, (\frac{1}{2}, \frac{1}{2}, 0, 0))| = |\{3, 4\}| = |\{1, 2\}| = |E(\tilde{R}_N, (0, 0, \frac{1}{2}, \frac{1}{2}))|$, and $(0, 0, \frac{1}{2}, \frac{1}{2})$ is Pareto optimal at $\tilde{R}_N$, we have that F satisfies minimal envy Pareto optimality. Also, F satisfies equal division lower bound, but $F(\tilde{R}_N) \not\subseteq M(\tilde{R}_N)$. △

We conclude this section with pointing out the main connections with results in Inoue and Yamamura (2015), besides the relation already discussed in Remark 4.10 between their Theorem 6 and our Corollary 4.9. They consider the binary mechanism (agents can report only 0 or 1) which has the zero–one property: the commodity is shared equally between the agents who report 1. In their Theorem 3, they show that strong equilibria are Pareto optimal. This is consistent with our Theorem 5.1, which implies (by minimal envy Pareto optimality) the same. They also show (their Theorem 4) that if there exist Pareto optimal allocations such that no player envies any other player (i.e., there exist envy-free Pareto optimal allocations), then these are exactly the strong equilibrium allocations. This is, indeed, also true for the correspondence M, as is not hard to verify. If there are no envy-free Pareto optimal allocations, however, then it is not necessarily true that M picks all minimal envy Pareto optimal allocations. For instance, in part (b) of Example 5.5, $\{2\}$ is the unique maximal coalition, $M(\tilde{R}_N) = \{(0, 1)\}$, but $(1, 0)$ is also a Pareto optimal allocation with one envious agent. In this example, no agent strictly prefers 1 over 0. The following example shows that also if there are agents who strictly prefer 1 over 0, then still M does not necessarily pick all minimal envy Pareto optimal allocations.

Example 5.6 Let $N = \{1, 2\}$ and let R_N be a preference profile with $\frac{1}{3}P_1 0 P_1\frac{1}{4}$ and $1P_2 0 P_2\frac{1}{2}$. Then $m(R_1) = 3$, $m(R_2) = 1$, and therefore $\{1\}$ is the unique maximal coalition, $M(R_N) = \{(1, 0)\}$, agent 2 is envious, and $(0, 1)$ is also Pareto optimal with one envious agent, namely agent 1. △

6 Concluding remarks

We have shown that in division problems with single-dipped preferences, the Pareto optimal Nash and strong equilibria of games induced by a reasonable class of mechanisms, result in Pareto optimal allocations characterized by maximal coalitions.

An obvious counterpart, the case of single-peaked preferences, is extensively studied in Bochet et al. (2021). The result that is most closely related to our approach is their Theorem 2, which applies to peaks-only rules—these are analogous to mechanisms in our sense. Under conditions on rules (mechanisms g), partly similar to ours, they show that the Pareto optimal Nash equilibria and strong equilibria in a game (R_N, g) coincide and result in the uniform allocation, for every $R_N \in \mathcal{P}^N$. An allocation $x \in \mathcal{A}$ is the *uniform allocation at* $R_N \in \mathcal{P}^N$ if there is a $\lambda \in [0, 1]$ such that

$$x_i = \begin{cases} \min\{p(R_i), \lambda\} & \text{if } \sum_{i \in N} p(R_i) \geq 1 \\ \max\{p(R_i), \lambda\} & \text{if } \sum_{i \in N} p(R_i) \leq 1. \end{cases}$$

The uniform allocation is the allocation assigned by the uniform rule (social choice function), characterized by Sprumont (1991). At the uniform allocation, either all agents obtain at most their peaks or all agents obtain at least their peaks, or both, and thus the uniform allocation is indeed Pareto optimal (it is 'same-sided').

For single-peaked preferences, Sprumont (1991) shows that the uniform rule is the unique anonymous, Pareto optimal, and strategy-proof social choice function. Recall that a social choice function F is strategy-proof if $F_i(R_N) R_i F_i(R'_i, R_{N\setminus\{i\}})$ for every preference profile R_N, agent $i \in N$, and preference R'_i, where preferences are chosen within a specific domain, for instance $\mathcal{P}$ or $\mathcal{D}$. The following example shows that, in the single-dipped case, social choice functions obtained by selecting from M are not strategy-proof. Let $F : \mathcal{D}^N \to \mathcal{A}$ such that $F(R_N) \in M(R_N)$ for every $R_N \in \mathcal{D}^N$.

Example 6.1 Let $R_N \in \mathcal{D}^N$ such that $0 P_i 1 R_i \frac{1}{n}$ for all $i \in N$. Then $M(R_N) = \{\{i\} \mid i \in N\}$, and therefore $F(R_N) = e^{\{j\}}$ for some $j \in N$ (cf. Theorem 4.5). Consider $R'_j \in \mathcal{D}$ such that $0 P'_j 1$ and $\frac{1}{n} P'_j 1$. Then $M(R'_j, R_{-j}) = \{\{i\} \mid i \in N\setminus\{j\}\}$, and therefore $F_j(R'_j, R_{-j}) = 0$, so that $F_j(R'_j, R_{-j}) P_j F_j(R_N)$. Hence, F is not strategy-proof. △

The uniform rule for single-peaked preference profiles is Pareto optimal and envy-free (at the uniform allocation no agent envies any other agent), thus trivially satisfies minimal envy Pareto optimality formulated for single-peaked preferences. It also satisfies equal division lower bound (Thomson 2010). It is not hard to see that these conditions do not uniquely characterize the uniform rule, but it is not obvious what an analogue of sharing index order preservation for single-peaked preferences would be. In this respect, also observe that M depends on preferences between 0, 1, and the point in between indifferent to 0 or 1, whereas the uniform rule depends only on the peaks.

Uniform rules for division problems with single-dipped preferences are studied by Yamamura (2023), however under the assumption that the resource to be distributed is freely disposible: an allocation x satisfies $\sum_{i \in N} x_i \leq 1$. The main result in that paper is that these uniform rules are envy-free, weakly Pareto optimal, and strategy-proof.

A natural extension of our analysis and the analysis in Bochet et al. (2021) is to other domains of preferences, notably if both single-dipped and single-peaked preferences in a profile are allowed, cf. Thomson (2023). Recently, this problem has been studied for the public good/bad case by Alcalde-Unzu et al. (2023).

Appendix: Proof of Proposition 3.10

By $p^{s+1} > q^s$ for all $s = 0, 1, \ldots, n-1$, we have $p^1 > \frac{1}{2} > q^1$. We consider three cases.

(a) Suppose that $p^1 P_1 q^1$. Then, by Lemma 3.1, $p^1 R_1 \frac{1}{2}$.

(a1) First suppose that $q^1 R_2 p^1$. Then, by Lemma 3.1, $q^1 R_2 \frac{1}{2}$. It follows that $g_1(1,0) R_1\, g_1(0,0)$ and $g_2(1,0) R_2 g_1(1,1)$. With monotonicity and Lemma 3.1 again, it holds that $g_1(1,0) P_1 g_1(r_1, 0)$ and $g_2(1,0) P_2 g_2(1, r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,0)$ is a Nash equilibrium.

(a2) Second, suppose that $p^1 R_2 q^1$. Then, by Lemma 3.1, $p^1 R_2 \frac{1}{2}$.

(a2.1) If $\frac{1}{2} P_1 q^1$ and $\frac{1}{2} P_2 q^1$, then $g_1(1,1) P_1 g_1(0,1)$ and $g_2(1,1) P_2 g_2(1,0)$. From monotonicity and Lemma 3.1, it holds that $g_1(1,1) R_1 g_1(r_1, 1)$ and $g_2(1,1) R_2 g_2(1, r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,1)$ is a Nash equilibrium.

(a2.2) If $q^1 R_1 \frac{1}{2}$, together with $p^1 R_2 \frac{1}{2}$, then $g_1(1,0) R_1 g_1(0,0)$ and $g_2(1,0) R_2 g_2 (1,1)$. With monotonicity and Lemma 3.1 again, it holds that $g_1(1,0) R_1 g_1(r_1, 0)$ and $g_2(1,0) R_2 g_2(1, r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,0)$ is a Nash equilibrium.

(a2.3) If $q^1 R_2 \frac{1}{2}$, then similar to (a2.2), we can prove that $r^* = (0,1)$ is a Nash equilibrium.

(b) Suppose that $p^1 I_1 q^1$. Then, $p^1 R_1 \frac{1}{2}$ and $q^1 R_1 \frac{1}{2}$.

(b1) If $p^1 P_2 q^1$, then $p^1 R_2 \frac{1}{2}$. So, $g_1(0,1) P_1 g_1(1,1)$ and $g_2(0,1) P_2 g_2(0,0)$. With monotonicity and Lemma 3.1, it holds that $g_1(0,1) R_1 g_1(r_1, 1)$ and $g_2(0,1) R_2 g_2(0, r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (0,1)$ is a Nash equilibrium.

(b2) If $q^1 R_2 p^1$, then $q^1 R_2 \frac{1}{2}$. Similar to (b1), we can prove that $r^* = (1,0)$ is a Nash equilibrium.

(c) Suppose that $q^1 P_1 p^1$. Then, by Lemma 3.1, $q^1 R_1 \frac{1}{2}$.

(c1) First, suppose that $p^1 R_2 q^1$, then similar to (a1), it follows that $r^* = (0,1)$ is a Nash equilibrium.

(c2) Second, suppose that $q^1 P_2 p^1$. Then, $q^1 R_2 \frac{1}{2}$.

(c2.1) If $\frac{1}{2} P_1 p^1$ and $\frac{1}{2} P_2 p^1$, then $g_1(0,0) P_1 g_1(1,0)$ and $g_2(0,0) P_2 g_1(0,1)$. With monotonicity and Lemma 3.1, it holds that $g_1(0,0) R_1 g_1(r_1, 0)$ and $g_2(0,0) R_2 g_2(0, r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (0,0)$ is a Nash equilibrium.

(c2.2) If $p^1 R_1 \frac{1}{2}$, together with $q^1 R_2 \frac{1}{2}$, we have $g_1(1,0) R_1 g_1(0,0)$ and $g_2(1,0) R_2 g_1(1,1)$. With monotonicity and Lemma 3.1, it holds that $g_1(1,0) R_1 g_1 (r_1, 0)$ and $g_2(1,0) P_2 g_2(1, r_2)$ for all $r_1, r_2 \in [0,1]$. So, $r^* = (1,0)$ is a Nash equilibrium.

(c2.3) If, finally, $p^1 R_2 \frac{1}{2}$, then similar to (c2.2), it can be proved $r^* = (0, 1)$ is a Nash equilibrium. □

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