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RESEARCH ARTICLE

Marginal pricing equilibrium with externalities in Riesz spaces

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Abstract
The purpose of this paper is to prove the existence of a marginal pricing economic equilibrium in presence of increasing returns and externalities in a commodity space general enough as to encompass the vast majority of economic situations. This extends the existing literature on competitive equilibria in vector lattices by incorporating market failures, and it also generalises several non-competitive existence results to a larger class of commodity spaces. The key features are a suitable definition for the marginal pricing rule and an adaptation of the properness condition.

Keywords Riesz space · Marginal pricing rule · Non-competitive equilibrium · σ-Locally τ-Uniform properness or Properness condition

JEL Classification D51 · C62

1 Introduction

The purpose of this paper is to prove the existence of a marginal pricing economic equilibrium in presence of increasing returns and externalities in a commodity space general enough as to encompass the vast majority of economic situations. Our aim is twofold: reaching the same level of generality as competitive equilibrium existence theorems in Walrasian economies like Podczeck (1996), Tourky (1999), Florenzano and Marakulin (2001); generalising the previous existence results with a non-convex

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production sector like Bonnisseau (2002), Bonnisseau and Cornet (1990b), Bonnisseau and Meddeb (1999), Bonnisseau and Médecin (2001), Bonnisseau and Fuentes (2020) and Fuentes (2011, 2016). For the second aspect, this is achieved mainly by considering a larger class of commodity spaces. Indeed, as usual, we considered a Riesz space[1] but we remove the requirement that the topology be locally solid, or, equivalently that the lattice operations be continuous. This pursues the goal of encompassing relevant economic models of product differentiation or intertemporal allocation in continuous time.

Let us take, for instance, the space of measures on a compact set $(\mathcal{M}(K), \|\cdot\|)$, which has been usually considered to formalize commodity differentiation models. Even though it is a Banach lattice,[2] all compatible topologies with the commodity-price pairing $\langle \mathcal{M}(K), C(K) \rangle$ are not locally solid. Hence, the locally-solid hypothesis rules out models studied by Mas-Colell (1975) and Jones (1984) among others. Indeed, in these papers, the commodity space is equipped with the weak*-topology $\sigma(\mathcal{M}(K), C(K))$ so that $(\mathcal{M}(K), \sigma(\mathcal{M}(K), C(K)))$ is not a topological vector lattice.[3] In the same way, models with strong intertemporal substitution properties in consumption as the one studied by Hindy et al. (1992) where the commodity-price pairing is $\langle \mathcal{M}([0, 1]), \mathrm{Lip}([0, 1]) \rangle$,[4] makes use of the compatible topology $\sigma\ (M([0, 1]), \mathrm{Lip}([0, 1])$, which is not locally solid either, thus requiring to look beyond topological vector lattices.

Besides these locally-solid issue, all marginal pricing existence theorems assume that the positive cone of the commodity space has quasi-interior points for the stronger topology. This is an unpleasant restriction since $\mathcal{M}(K)_+$ has no quasi-interior points for the norm, unless K is countable.

The presence of increasing returns or more generally, non-convex production sets, is most often incompatible with a competitive equilibrium as already noticed in Debreu (1959). Consequently, we focus on the marginal pricing rule for which every producer sets on the market a price vector which fulfils the first order conditions for profit maximisation. This rule has a particular importance thanks to the Second Theorem of Welfare Economics stating that it is a necessary condition for the decentralization of Pareto optimal allocations (see, e.g., Bonnisseau and Cornet 1988; Cornet 1986; Florenzano et al. 2005). Furthermore, this rule coincides with the profit maximisation rule when the production set is actually convex.

A proper mathematical treatment of the marginal pricing rule in a finite dimensional setting was first provided in Guesnerie (1975) and then generalized in Cornet (1990): a price vector p satisfies the *marginal pricing rule* at a production y_j belonging to the production set Y_j if it belongs to the Clarke's normal cone to y_j at Y_j (see Clarke 1983). This rule associates, to each efficient production plan y_j at Y_j, the convex hull of the set of vectors that are perpendicular to y_j and those that are limit of perpendicular vector in a neighbourhood of it.

[1] That is, a partially ordered vector space with the lattice operations supremum and infimum.

[2] That is, a complete normed Riesz space.

[3] That is a Riesz space with a locally-solid topology.

[4] Lip([0, 1]) is the space of Lipschitz functions on [0, 1].

When the commodity space is infinite dimensional, Bonnisseau (2002) defines the marginal pricing rule based on a normal cone larger than the Clarke's one. This is justified by the fact that the graph of the correspondence from the set of weakly-efficient production to the price space associating to a production y_j the Clarke's normal cone at this point, is not closed for the relevant topologies. Later on, Bonnisseau and Fuentes (2020) introduced a new extension to cope with the emptiness of the interior of the positive cone by considering order intervals instead of open balls to measure the "distance" to the production set.

It is well known in the literature that externalities are a source of non-convexities with strong consequences on the optimality of equilibra (see, e.g., an example in Bonnisseau 1994). So, to cover a consistent and broad range of market failures, we take into account external effects between all economic agents, consumers as well as producers.

Like in the literature dealing with competitive equilibria, in this paper, the commodity space is a Riesz space equipped with a topology. We only assume that the topological dual is a sub-lattice of the order dual, which is weaker than the locally solid hypothesis and holds true in the above examples of commodity differentiation and intertemporal allocations. As in Podczeck (1996), Tourky (1999) and Florenzano and Marakulin (2001), our proof relies heavily on a reference commodity bundle e and its associated order ideal $L(e)$. Roughly speaking, in a measure space, it means that e is a reference measure, for example the Lebesgue measure, and the relevant economic bundles are absolutely continuous measures with respect to e. In exchange economy, it is coherent that e be the total initial endowment like in Podczeck (1996). In a production economy, the total initial endowment could belong to a small subspace of inputs, whereas the attainable consumptions are in a much larger space thanks to the production possibilities. For example, this is the case in a finite dimensional space when the total initial endowment is on the boundary of the positive cone. Then, in this setting, it would be very restrictive to assume that e be the initial endowments. Nevertheless, the initial endowments must be commensurable with respect to e.

As for the definition of the marginal pricing rule, we first provide a formula for the tangent cone, which is the set of inward directions at a given production. Then, the normal cone is defined by polarity, which means that a price belongs to it if the value of all inward directions is non-positive. In other words, the price maximises the profit on the tangent cone, which is the genuine formulation of the first order necessary condition for profit maximisation. For this definition, we follow the same methodology as in Bonnisseau (2002) and Bonnisseau and Fuentes (2020) except that we deal with a linear topology instead of a norm topology. The intuition is exposed in Sect. 3.1. The key difference comes from the fact that we consider only the inward directions in $L(e)$ or dominated for the order by such direction.

Then, most of the assumptions are standard in this framework but the properness assumption. Indeed, to cope with the non-convexity of the production sets, we borrow from Clarke (1983) the strategy to consider not only a production y_j and a given environment z but all productions in a neighbourhood compatible with an environment close to z. We mix this feature with the properness conditions of Florenzano and Marakulin (2001). Roughly speaking, the pre-technology set à la Mas-Colell is locally uniform.

Concerning the existence proof, we restrict the commodity space to $L(e)$ equipped with the standard norm. Thanks to a result in Bonnisseau and Fuentes (2020), an equilibrium exists in this auxiliary economy and the main task is to show that it can be extended to the original economy. As for the equilibrium price, we borrow techniques originally developed by Podczeck (1996) and extended by Florenzano and Marakulin (2001) crucially using the properness condition. Unlike many papers on competitive equilibria (e.g. Mas-Colell and Richards 1991; Richard 1989) we cannot follow the Negishi approach since it requires the aggregate production plan to be efficient which is not necessarily the case when production sets are non-convex.

Beyond the existence result, we show in detail that our result implies the existing ones in the literature when we have additional properties on the commodity space or on the production sets. We also prove that the marginal pricing rule is the profit maximising rule when the production set is convex under the Upper Order Boundedness Assumption. This means that the production set is close to its intersection with $L(e)$. Note that in Florenzano and Marakulin (2001), a stronger assumption derived from the properness assumption is necessary to get the existence of a competitive equilibrium. So, we are also able to get the existence of a competitive equilibrium with a convex production sector even in presence of externalities, which is, up to the best of our knowledge, the first one in the literature for such commodity spaces.

The paper is organised as follows: Sect. 2 deals with the mathematical structure. Section 3 presents the model and Assumptions together with the specification of the marginal pricing rule. In Sect. 4, we state an existence result in the intermediate economies which becomes the starting point for Sect. 5, where a general existence theorem is presented. In this section we also compare our results with previous ones both in competitive and non-competitive economies. In Sect. 6, we prove the existence theorem. With the exception of this section, all proofs are given in Appendix.

2 Terminology and notation

Let L be a Riesz space, i.e., an ordered vector space[5] such that for any x, x' in L the pair $\{x, x'\}$ has a supremum, $x \vee x'$, and an infimum, $x \wedge x'$, in L. The cone $L_+ = \{x \in L : x \geq 0\}$ is called the positive cone of L. For $x, x' \in L$ with $x \leq x'$, we denote by $[x, x']$ the order interval $\{y \in L : x \leq y \leq x'\}$. Let $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x^+ + x^-$ be respectively the positive part, the negative part and the absolute value of x.

We denote by $L^\sim$ the order dual of L, i.e., the vector space consisting of all linear functionals on L which map order intervals of L to bounded subsets of $\mathbb{R}$, ordered by the relation $f \geq g$ whenever $f(x) \geq g(x)$ for all $x \in L_+$. Let f and g in $L^\sim$ and let x in L_+. Then the Riesz–Kantorovich formula states $f \vee g(x) = \sup_{0 \leq x' \leq x} \{f(x') + g(x - x')\}$. In turn, $L(x) := \bigcup_{n \in \mathbb{N}} [-nx, nx]$ is the principal ideal in L generated by x.

[5] An ordered vector space is a (real) vector space endowed with a partial ordering $\geq$ such that $x \geq x'$ implies $x + y \geq x' + y$ for $x, x', y \in L$ and if $x \geq 0$ it entails $tx \geq 0$ for $t \in \mathbb{R}_{++}$. $x \leq x'$ means $x' \geq x$.

We endow L with a Hausdorff locally convex linear topology τ such that L_+ is closed and order intervals are bounded. Let L^* be the topological dual of L, i.e., the vector space of all τ-continuous linear functionals on L. For $x \in L$ and $\pi \in L^*$, $\pi \cdot x$ is the evaluation of x under π. Since every order interval of L is τ-bounded, it follows that $L^* \subset L^\sim$. Throughout this paper we shall assume that L^* is a vector sublattice of $L^\sim$. Note that if τ is locally solid, then both L_+ is τ-close and L^* is a vector sublattice of $L^\sim$.

We denote by $\mathcal{V}_\tau(0)$ the base of all convex, symmetric[6] τ- neighbourhoods of 0. According to Schaefer (1971) (1.2, p. 14), for each $W \in \mathcal{V}_\tau(0)$ there exists $W' \in \mathcal{V}_\tau(0)$ satisfying $W' + W' \subset W$. We shall also consider the weak topology $\sigma(L, L^*) = \sigma$ on L. This linear topology is also Hausdorff and locally convex. $\mathcal{V}_\sigma(0)$ is the base of weak neighbourhoods of 0 with the same properties as $\mathcal{V}_\tau(0)$.

For a subset A of L, we denote by $\tau - \text{int}A$ and $\tau - \text{cl}A$ (resp. $\sigma - \text{int}A$ and $\sigma - \text{cl}A$) the corresponding τ-interior and τ-closure of A (resp. the σ-interior and σ-closure of A). Let L^M be the Cartesian product of M copies of the space L. We denote by $\prod_{L^M} \tau$ (resp. $\prod_{L^M} \sigma$) the product topology on L^M when the topology on each L is τ (resp. σ). Hence, both $\left(L^M, \prod_{L^M} \tau\right)$ and $\left(L^M, \prod_{L^M} \sigma\right)$ are locally convex Hausdorff Riesz spaces.

Let $D : L^M \twoheadrightarrow L$ be a correspondence. We say that D has τ-closed values if for every $x \in L^M$, $D(x)$ is a τ-closed subset of L.

For further details on infinite dimensional spaces and Riesz spaces, we refer to Aliprantis and Border (2006) and Schaefer (1971).

3 The model

We consider an economy whose commodity space is the Riesz space L equipped with the τ-topology and such that L^* is a vector sublattice of $L^\sim$. There are finite sets of consumers and producers I and J respectively. We index each consumer by $i \in I$ and each producer by $j \in J$. An element $z = ((x_i)_{i \in I}, (y_j)_{j \in J})$ in $L^{I \cup J}$ is called an environment. Instead of consumption sets we have consumption correspondences, that is, the i-th consumer has a consumption correspondence $X_i : L^{I \cup J} \twoheadrightarrow L_+$ depending on the environment z. Given z, $X_i(z) \subset L_+$ is her consumption set. Preferences also depend upon the actions of the other economic agents. We shall denote by $\succeq_{i,z}$ the preference relation of consumer i given the environment z. This relation is assumed to be complete, reflexive and transitive on $X_i(z)$. The relation of strict preference $x \succ_{i,z} x'$ is then defined by $x \succeq_{i,z} x'$ and not $x' \succeq_{i,z} x$. We do not assume that we can compare two commodity bundles if they do not share the same environment. Let $\omega_i \in L_+$ be the initial endowment of the i-th agent such that $\omega_i \in X_i(z)$ for all $z \in L^{I \cup J}$. Let us denote the total initial endowment of the economy by $\omega = \sum_{i \in I} \omega_i \neq 0$.

Each producer j has a production set which also depends upon the actions of the other agents. For each $j \in J$, $Y_j : L^{I \cup J} \twoheadrightarrow L$ is the production correspondence. For the environment $z \in L^{I \cup J}$, $Y_j(z) \subset L$ is the set of all feasible productions for the j-th producer. We denote the τ-boundary of $Y_j(z)$ by $\partial Y_j(z)$.

[6] For all $W \in \mathcal{V}_\tau(0)$, $W = -W$.

The wealth function of the i-th consumer is given by a function $r_i : \mathbb{R} \times \mathbb{R}^J \to \mathbb{R}$. If $\pi \in L^*$ and $(y_j)_{j\in J} \in \prod_{j\in J} Y_j(z)$, her wealth is $r_i\left(\pi \cdot \omega_i, \left(\pi \cdot y_j\right)_{j\in J}\right)$. This definition is general enough as to encompass the private ownership economy case, i.e., when $r_i\left(\pi \cdot \omega_i, \left(\pi \cdot y_j\right)_{j\in J}\right) = \pi \cdot \omega_i + \sum_{j\in J} \theta_{ij}\pi \cdot y_j$ for $\theta_{ij} \geq 0$ and $\sum_{i\in I} \theta_{ij} = 1$ for all $j \in J$.

For a given environment $z \in L^{I\cup J}$ and a given initial endowments $\omega' \in L_+$, we shall denote by $A(\omega', z)$ the set of attainable productions, that is,

$$A(\omega', z) := \left\{ (y'_j) \in \prod_{j\in J} \partial Y_j(z) : \sum_{j\in J} y'_j + \omega' \in L_+ \right\}$$

In order to consider only consistent situations, we define the set

$$Z := \left\{ z = ((x_i)_{i\in I}, (y_j)_{j\in J}) \in L^{I\cup J} : \forall i \in I,\ x_i \in X_i(z);\ \forall j \in J,\ y_j \in \partial Y_j(z) \right\}$$

The set of *weakly efficient attainable allocations* corresponding to a given total initial endowment $\omega' \in L_+$ is given by

$$A(\omega') := \left\{ z = ((x_i)_{i\in I}, (y_j)_{j\in J}) \in Z : \sum_{i\in I} x_i = \sum_{j\in J} y_j + \omega' \right\}$$

We fix a reference bundle $e \in L_+\backslash\{0\}$ and assume that ω is compatible with it in the sense that there is some $n_0 \in \mathbb{N}$ for which $\omega \leq n_0 e$ or, in other words, ω belongs to the principal order ideal $L(e)$. In most of the literature such a reference vector is given directly by ω, which is coherent in exchange economies. However, this is not always convenient in our setting since $L(\omega)$ may be very small whereas economic activities take place in a much larger space. Indeed, even in a ℓ-finite dimensional Euclidean space, ω may not be in $\mathbb{R}^\ell_{++}$ which makes $L(\omega)$ too small. That is why in previous works e has been suitably chosen depending on the commodity space L. For instance, when $L = \mathbb{R}^\ell$ as in Bonnisseau and Médecin (2001), e is the unit vector $(1, 1, \ldots, 1)$. When $L = L_\infty$ (Bewley 1972; Bonnisseau 2002) e is the constant function equal to 1 which obviously belongs to the $\|\cdot\|_\infty$-interior of L^+_∞. In Bonnisseau and Fuentes (2020), e is a quasi-interior point of L_+ satisfying $\|e\| = 1$. Actually, $L(e)$ must be large enough so that relevant consumptions and productions lie in the τ-closure of $L(e) - L_+$. It appears in the assumptions below that the larger is $L(e)$, the weaker are these assumptions.

3.1 Marginal pricing rule and equilibrium

The marginal pricing rule doctrine is not new. It comes at least from the thirties[7] when (Hotelling 1938) argued that when the firms exhibit increasing returns to scale, prices should be proportional to marginal costs to reach a Pareto optimal allocation. Hotelling also paid attention to the fact that, in some cases, a firm or even an industry which adopts marginal cost pricing will run at a loss if there are high fixed costs. Since "*taxes on commodities, including sales taxes, are more objectionable than taxes on incomes*", the deficit must then be financed from income taxes: "*the latter taxes* (income taxes) *might well be applied to cover the fixed costs of electric power plants, waterworks, railroads, and other industries in which the fixed costs are large, so as to reduce to the level of marginal cost the prices charged for the services and products of these industries*". In Hotelling's conception, the production sets are non-convex but with smooth boundaries. Hence, when the latter does not hold true, we replace the marginal cost condition by the first order necessary condition for profit maximisation (see Cornet 1990; Guesnerie 1975). This rule is called marginal pricing rule because when technologies are smooth, marginal productivities are proportional to the prices and, when the cost is well defined and differentiable, it coincides with the marginal cost à la Hotelling. Note also that a Pareto optimal allocation can be descentralized by a price satisfying the marginal pricing for all producers. The notion of marginal pricing rule extends to very general frameworks (see, e.g., Cornet 1986; Bonnisseau and Cornet 1988; Bonnisseau 2002; Florenzano et al. 2005) where it is defined by the fact that the price is an outward direction, meaning that it belongs to the normal cone. The later is the polar cone of the tangent cone, the set of inward or quasi-inward directions. In a finite dimensional space, the Clarke's normal cone as introduced in Cornet (1990) and extended in Bonnisseau and Médecin (2001) to encompass externalities, is the right concept and, in some sense, the smallest one for the existence of equilibrium. For a survey on the marginal pricing rule and its related notions we refer to the chapter on increasing returns of Brown (1991) and to the books of Villar (1996) and (2000). For an exhaustive study about the relationship between increasing returns and Pareto optimality we refer to the book of Quinzii (1992).

In infinite dimensional settings things change since the graph of the correspondence $y_j \twoheadrightarrow N_{Y_j}(y_j)$, where $N_{Y_j}(y_j)$ is the Clarke's normal cone, is not necessarily closed for the relevant topologies. These were the technical challenges in the works of Bonnisseau (2002) and Bonnisseau and Fuentes (2020) and this is what we face now but in a Riesz space. For $z \in Z$ and $y_j \in \partial Y_j(z)$, let us consider the following set

$$\mathscr{T}_{Y_j(z)}(y_j) = \bigcap_{\mathcal{C} \in \mathcal{V}_\tau(0)} \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$$

[7] We use the expression "at least" since in Hotelling's own opinion, the work of the French engineer Jules Dupuit (Dupuit 1952), positions him as the first marginal cost theorist. There is, however, some controversy as to whether he is the *progenitor* of the idea (Ekeland 1968).

where

$$\mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j) := \left\{ \nu \in L \;\middle|\; \begin{array}{c} \exists\, \eta > 0 \text{ such that } \forall\, r > 0, \\ \exists \;\; U \in \mathcal{V}_\sigma(0) \text{ and } \varepsilon > 0 \;\; | \\ \forall \;\; z' \in \left(\{z\} + (\mathcal{C} \cap U)^{I \cup J}\right), \\ \forall \;\; y'_j \in \left(\{y_j\} + \mathcal{C} \cap U\right) \cap Y_j(z') \\ \text{and } \forall \;\; t \in (0, \varepsilon), \;\; \exists\, \xi \in r[-e,\; e] \text{ such that} \\ y'_j + t(\nu + \eta(y_j - y'_j) + \xi) \in Y_j(z') \end{array} \right\} \tag{1}$$

Then, the set $\hat{\mathscr{T}}_{Y_j(z)}(y_j) := \mathscr{T}_{Y_j(z)}(y_j) \cap L(e) - L_+$ is the tangent cone at y_j in $Y_j(z)$.

Proposition 3.1 *$\hat{\mathscr{T}}_{Y_j(z)}(y_j)$ is a cone.*

For a better understanding on the intuition behind the above formula, we recall first that when $Y_j(z)$ is convex, a vector ν is inward at y_j if $y_j + t\nu \in Y_j(z)$ for all t small enough. In the non-convex case, according to the key contribution of Clarke, we consider in the formula (1) that a direction is inward if it is so for all productions in a neighbourhood of y_j. To deal with externalities, as in Bonnisseau and Fuentes (2020) and Bonnisseau and Médecin (2001), we also consider production plans consistent with an environment z' in a neighbourhood of z.

As for the intersection with $L(e)$ in the definition of $\hat{\mathscr{T}}_{Y_j(z)}(y_j)$, we remark that it is useless when $L(e)$ is equal to L or τ-dense in L as in the cases of Bonnisseau (2002) and Bonnisseau and Fuentes (2020) respectively. In our setting, $L(e)$ could be "small" and the only economically relevant inward directions are those which are in $L(e) - L_+$. Indeed, e is the reference commodity bundle and $-L_+$ is consistent with free-disposal of the technologies.

The *marginal pricing rule* is now formally defined: given $(y_j, z) \in \partial Y_j(z) \times Z$, Producer j chooses prices in $\hat{\mathscr{N}}_{Y_j(z)}(y_j) := \left[\hat{\mathscr{T}}_{Y_j(z)}(y_j)\right]^\circ$. In words, to set prices according to the marginal pricing rule, the producer takes into account the local shape of $Y_j(z)$ around y_j and the fact that her production set depends on external effects in the sense that the former changes when the latter varies. We note that this rule is equivalent to saying that the owner of the firm chooses the price π for which y_j maximizes profits on $\{y_j\} + \hat{\mathscr{T}}_{Y_j(z)}(y_j)$. Note that $\hat{\mathscr{N}}_{Y_j(z)}(y_j) \subset L^*_+$.

Finally, the set of *production equilibria* is

$$PE := \left\{(\pi, z) \in L^*_+ \times Z : \pi \in \cap_{j \in J} \hat{\mathscr{N}}_{Y_j(z)}(y_j) \backslash \{0\}\right\}$$

We are now able to state the definition of a marginal pricing equilibrium.

Definition 3.2 A marginal pricing equilibrium of the economy $\mathcal{E}$ is an element $(\bar{z} = (\bar{x}_i)_{i \in I}, (\bar{y}_j)_{j \in J}), \bar{\pi})$ in $Z \times L^*_+$ such that:

1. For all $i \in I$, $\bar{\pi} \cdot \bar{x}_i \leq r_i(\bar{\pi} \cdot \omega_i, (\bar{\pi} \cdot \bar{y}_j)_{j \in J})$ and $\bar{\pi} \cdot x'_i > r_i(\bar{\pi} \cdot \omega_i, (\bar{\pi} \cdot \bar{y}_j)_{j \in J})$ whenever $x'_i \succ_{i, \bar{z}} \bar{x}_i$.
2. For all $j \in J$, $\bar{\pi} \in \hat{\mathscr{N}}_{Y_j(\bar{z})}(\bar{y}_j) \backslash \{0\}$.

3. $\sum_{i \in I} \bar{x}_i = \sum_{j \in J} \bar{y}_j + \omega$.

Condition 1. says that every consumer maximises her preference under her budget constraint. Condition 2. says that every producer sets on the market the same equilibrium price vector $\bar{\pi}$ which satisfies the marginal pricing rule. Condition 3. says that all markets clear.

3.2 Assumptions

We now posit the following assumptions. Some of them became standard in the literature.

Assumption (C) For all $i \in I$

1. X_i is a convex-valued correspondence with a $(\prod_{L^{I \cup J}} \sigma, \sigma)$-closed graph. Furthermore, for all $z \in L^{I \cup J}$, $0 \in X_i(z)$ and for every $x \in X_i(z)$ the order interval $[0, x]$ is contained in $X_i(z)$.
2. For all $z \in L^{I \cup J}$, the half-line $\{\delta e : \delta > 0\}$ is included in $X_i(z)$. For all $x_i \in X_i(z)$ and for all $\delta > 0$ there exists a neighbourhood $V' \in \mathcal{V}_\sigma(0)$ such that $x_i + \delta e \in X_i(z')$ for all $z' \in \{z\} + (V')^{I \cup J}$.
3. For every $z \in L^{I \cup J}$, and all $x_i \in X_i(z)$ both sets $\{x'_i \in X_i(z) : x'_i \succeq_{i,z} x_i\}$ and $\{x'_i \in X_i(z) : x_i \succeq_{i,z} x'_i\}$ are τ-closed. For all $x'_i \in X_i(z)$ such that $x'_i \succ_{i,z} x_i$, for all $t \in (0, 1)$, $tx'_i + (1-t)x_i \succ_{i,z} x_i$.
4. The set $G_i = \{(x'_i, x_i, z) \in L^2 \times L^{I \cup J} : (x'_i, x_i) \in X_i(z)^2, x'_i \succeq_{i,z} x_i\}$ is a $(\tau \times \sigma \times \prod_{L^{I \cup J}} \sigma)$-closed subset of $L^2 \times L^{I \cup J}$.
5. The wealth function $r_i : \mathbb{R} \times \mathbb{R}^J \to \mathbb{R}$ is continuous, increasing and homogeneous of degree one with respect to the price vector. Furthermore, for all $((v_i), (v_j)) \in \mathbb{R}^{I \cup J}$, $\sum_{i \in I} r_i \left(v_i, (v_j)\right) = \sum_{i \in I} v_i + \sum_{j \in J} v_j$ and if $\sum_{i \in I} r_i(v_i, (v_j)) > 0$ then $r_i(v_i, (v_j)) > 0$ for all i.
6. Let $z = ((x_i), (y_j)) \in A(\omega) \cap L(e)^{I \cup J}$. Preferences are E-proper relative to $L(e)$ at x_i, i.e., there exists a τ-open convex cone Γ_i with vertex 0 and containing e, a set $R_{x_i} \subset L$ radial at x_i[8] and a sublattice $K_{x_i} \subset L(e)$ satisfying $x_i \in K_{x_i}$ and $K_{x_i} + L(e)_+ \subset K_{x_i}$ such that

$$\emptyset \neq \left(\left\{x'_i \in X_i(z) : x'_i \succ_{i,z} x_i\right\} + \Gamma_i\right) \cap K_{x_i} \cap R_{x_i} \subset \left\{x'_i \in X_i(z) : x'_i \succ_{i,z} x_i\right\} \tag{2}$$

and

$$\{x'_i \in X_i(z) : x'_i \succ_{i,z} x_i\} \cap R_{x_i} \subset \left(K_{x_i} + L_+\right) \tag{3}$$

Assumption (P) For every $j \in J$

1. $Y_j : L^{I \cup J} \twoheadrightarrow L$ has a $(\prod_{L^{I \cup J}} \sigma, \sigma)$-closed graph.
2. For every $z \in L^{I \cup J}$, $Y_j(z) \cap L_+ = \{0\}$ and Y_j satisfies the free-disposal condition, that is, $Y_j(z) - L_+ = Y_j(z)$.

[8] That is, for all $x' \in L$, there exists a real number $\bar{\lambda} \in]0, 1[$ such that $(1-\lambda)x_i + \lambda x'$ belongs to A for every $\lambda \in]0, \bar{\lambda}[$.

3. For all $z \in L^{I \cup J}$, for all $y_j \in \partial Y_j(z)$ and for all $\delta > 0$, there exists $V' \in \mathcal{V}_\sigma(0)$, such that $y_j - \delta e \in Y_j(z')$ for all $z' \in \{z\} + (V')^{I \cup J}$.
4. For all $z \in L^{I \cup J}$, for all $y_j \in \partial Y_j(z)$, for all $\mathcal{C} \in \mathcal{V}_\tau(0)$ and for all $\delta > 0$ there exists $\hat{U} \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$ such that for all $z' \in \left(\{z\} + (\mathcal{C} \cap \hat{U})^{I \cup J}\right)$, for all $y'_j \in Y_j(z') \cap \left(\{y_j\} + \mathcal{C} \cap \hat{U}\right)$, for all $t \in (0, \varepsilon]$, $t(y_j - \delta e) + (1-t)y'_j \in Y_j(z')$.
5. Let $z = ((x_i), (y_j)) \in A(\omega) \cap L(e)^{I \cup J}$. The technology Y_j is σ-locally τ-uniformly-proper relative to $L(e)$ at y_j, i.e., there exists a τ-open convex cone with vertex 0, Γ_j, containing the vector e and an open convex neighbourhood $V_j \in \mathcal{V}_\sigma(0)$, such that for all $z' \in \left(\{z\} + V_j^{I \cup J}\right) \cap L(e)^{I \cup J}$

$$\left[\left(\left(\{y_j\} + V_j\right) \cap Y_j(z')\right) - \Gamma_j\right] \cap L(e) \cap \left(\{y_j\} + V_j\right) \subset Y_j(z') \tag{4}$$

Assumption B (Boundedness) The order interval $[0, e]$ is σ-compact. Furthermore, for all $\omega' \in L(e)$, $\omega' \geq \omega$, there exists $b \in \mathbb{R}_{++}$ such that, for all $z \in Z$, $A(\omega', z) \cap L(e)^J \subset [-be, be]^J$.

Assumption SA (Survival) For all $z \in Z$, for all $t \in \mathbb{R}_+$ and for all $(\pi, (y_j)) \in L^* \times A(\omega + te, z)$, if $\pi \in \cap_{j \in J} \hat{\mathcal{N}}_{Y_j(z)}(y_j) \setminus \{0\}$, it follows that $\pi \cdot \left(\sum_{j \in J} y_j + \omega + te\right) > 0$.

Before commenting the above assumptions, we refer to the topologies used in them in connection with the main objective of this paper. We remark that both the strong topology τ and the weak topology $\sigma(L, L^*)$ allow us to encompass interesting examples not covered by previous works that considered locally-solid topologies, that is to say, when lattice operations are continuous. We recall that two important characteristics of a (consistent) locally convex-solid topologies are: (i) the positive cone is closed and (ii) the topological dual space L^* of L is a sublattice of the order dual $L^\sim$ (Aliprantis and Border 2006, Theorems 8.43 and 8.48 respectively). The remarkable fact is that, in the motivational examples provided in Introduction concerning commodity differentiation and intertemporal preferences in continuous time, the topologies are not locally solid but the above characteristics are satisfied.

Regarding the first case, that is, models of commodity differentitation as in Mas-Colell (1975) and Jones (1984), the commodity space is $L = \mathcal{M}(K)$, the set of bounded, signed (Borel) measures on a compact set K. They consider the weak-star topology $\sigma\,(\mathcal{M}(K), C(K))$ on L. This topology is too weak for the lattice operations to be continuous, or, equivalently, $\sigma\,(\mathcal{M}(K), C(K))$ is not locally-solid. As for the second case, that is, those of intertemporal preferences such as Hindy et al. (1992), the authors consider the commodity space L of right continuous real-valued functions of bounded variation on [0, 1], with the positive cone L_+ of positive increasing right-continuous real-valued function on [0, 1]. A topology makes L a topological vector lattice with positive cone L_+ only if it is at least as strong as the total variation norm topology. However, such a topology does not satisfy desirable economic properties. Conversely, the topology that does fulfill such properties is $\sigma(L, L^*)$ for L^* the space of Lipschitz continuous functions (Hindy et al. 1992, Theorems 1 and 2, pp. 417–18). Under this topology, the lattice operations are not continuous (Hindy et al. 1992, Proposition 2, p. 413) but L^* is a vector lattice in the order dual (Hindy et al. 1992, Proposition 8, p. 418).

These examples justify that neither τ nor $\sigma(L, L^*)$ need to be locally-solid at the time that L_+ is τ-closed and L^* is a sublattice of $L^\sim$. In addition, we point out that in both models, attainable sets are weakly compact which agrees with Assumption (B).

Remark 3.1 Assumption (C) gathers standard conditions on the continuity and convexity of preferences together with the convexity and closedness of the consumption set. Assumption C(1) and C(3) are quite usual even though we are not imposing that the consumption sets equal to the positive cone L_+, which is commonly assumed in the literature. Assumption C(2) is a strong form of lower hemi-continuity that has been widely discussed in Fuentes (2011) and Bonnisseau and Fuentes (2020). We refer the reader to these works. We point out that homogeneity of r_i with respect to the price vector π is not necessary for getting equilibria but to deal with normalized prices.

The properness assumptions are well known in the literature on existence of equilibria in Riesz spaces. The main consequence of this condition is to get a continuous extension of supporting prices in $L(e)$ to the whole space L. Regarding preferences we use the one of Florenzano and Marakulin (2001) so we refer to it for further discussion on this condition.[9] Notice that (2) implies non-satiation of preferences on the attainable sets restricted to $L(e)$. Together with the convexity of preferences, we have local non-satiation on the subspace. This non-satiation in $L(e)$ rather than in L will be sufficient for our purposes since the relevant commodity bundles in the economy are close to $L(e)$.

On the production side, assumptions on closedness and free disposal are standard. Assumption P(3) is a stronger form of lower hemi-continuity and we refer the reader to Fuentes (2011) and Bonnisseau and Fuentes (2020) for details. We discuss now the remaining items of Assumption (P).

Remark 3.2 Properness condition means that if $y'_j \in Y_j(z')$ is close to the production vector $y_j \in Y_j(z)$ for the environment z' close to z, and we add to y_j the quantity e of inputs, then it is still producible if we add a vector small enough such that the resultant vector is again both sufficiently near to y_j and belongs to $L(e)$. Thus, marginal rates of substitution with respect to e are bounded away from zero in a neighbourhood of y_j.

Unlike Podczeck (1996) and Florenzano and Marakulin (2001), we consider an open neighbourhood V_j of y_j instead of a radial set at y_j. Even though this is a stronger requirement, in the sense that open neighbourhoods are radial but radial sets could have an empty interior, it is consistent with the idea of the tangent cone to a nonconvex set which, contrary to the convex case, not only focuses on y_j but also on productions in a neighbourhood.

When technologies are convex, we complement this assumption with condition UOB which is quite similar to the inclusion (2.4) in Florenzano and Marakulin (2001)

[9] We note that if the set V_x in the definition of E-properness in Florenzano and Marakulin (2001) is $\{x'_i \in X_i(z) : x'_i \succ_{i,z} x\} + \Gamma_i$, the inclusions (2.1) and (2.2) there, are exactly the same as (2) and (3) in C(6) since preferences are assumed to be convex.

and to the definition of M-properness in Tourky (1999). We shall define and discuss this Assumption on Sect. 5.1.

Remark 3.3 Boundedness assumption implies, first, that all order intervals in $L(e)$ are σ-compact. Since L^* is a subspace of $L^\sim$, such intervals are σ-bounded and then τ-bounded (Rudin 1991, Theorem 3.18, p. 70). Second, Assumption (B) says that, even if we increase the initial endowments, feasible production vectors belonging to $L(e)$ are order bounded which, in turn, implies that the attainable set in $L(e)^{I\cup J}$ is included in a $\prod_{L^{I\cup J}}$ σ-compact set. We point out that this means that properness condition is stated only on a compact subset of $L^{I\cup J}$. Note that this assumption is also necessary even if L is finite dimensional (see Bonnisseau and Cornet 1990b).

Remark 3.4 If $Y_j(\cdot)$ satisfies Assumption P(4), we say that Y_j is $\prod_{L^{I\cup J}}$ σ*-locally star-shaped with respect to* $y_j - \delta e$ whenever $y_j \in Y_j(z)$. If $Y_j(z)$ is convex for all z, Assumption P(4) is clearly a consequence of Assumption P(3) and, in turn, if $\varepsilon \geq 1$ in P(4), P(3) is a consequence of P(4). The next lemma shows first the equivalence between Assumption P(4) and the fact that the null vector belongs to the tangent cone to $Y_j(z)$ at y_j, and, second, that P(4) is a consequence of free-disposal in finite dimensional commodity spaces.

Lemma 3.3 (1) *Condition P(4) is equivalent to* $0 \in \mathscr{T}_{Y_j(z)}(y_j)$ *under Assumptions P(1)–P(3).*

(2) *Condition P(4) is a consequence of free disposal when L is finite dimensional.*

4 The restricted economy in $L(e)$

The purpose of this section is to show the existence of a marginal pricing equilibrium in the restricted economy, that is, the one with commodity space $L(e)$. This is the first step to get an equilibrium in the original economy. In $L(e)$, the order interval $[-e, e]$ is absorbing and its gauge induces a norm topology on $L(e)$ with $\|e\|_e = 1$. We call it the $\|\cdot\|_e$-topology. Actually, $[-e, e]$ is the closed unit ball on $L(e)$ while $B_1^e = \{x \in L(e) : \|x\|_e < 1\}$ is the open unit ball on $L(e)$. In general, for $\epsilon > 0$, $\epsilon[-e, e]$ and $B_\epsilon^e = \{x \in L(e) : \|x\|_e < \epsilon\}$ are respectively the closed and open balls of center 0 and radius ϵ on $L(e)$. We note that $[-e, e]$ is σ-compact and τ-bounded by Assumption (B). Consequently, there exists $\mathcal{C} \in \mathcal{V}_\tau(0)$ such that $B^e(0, 1) \subset \mathcal{C} \cap L(e)$. In turn, $L(e)^*$ denotes the $\|\cdot\|_e$-dual of $L(e)$. The relativization of the topologies τ and σ to $L(e)$ are weaker than the $\|\cdot\|_e$-topology. We point out that $(L(e), \|\cdot\|_e)$ is a normed lattice which is not necessarily complete.

Let $L(e)_+ = L_+ \cap L(e)$. Clearly $L(e)_+$ is $\|\cdot\|_e$-closed in $L(e)$ and has a non-empty $\|\cdot\|_e$-interior which contains e. Furthermore, $\omega \in L(e)_+$ since ω is commensurable to e. Let $Y_j^e : L(e)^{I\cup J} \twoheadrightarrow L(e)$ be the restricted production correspondence such that for all $z \in L(e)^{I\cup J}$, $Y_j^e(z) = Y_j(z) \cap L(e)$. Since we are looking for an equilibrium in the economy $\mathcal{E}$ when the commodity space is $L(e)$, we need the definition of the tangent cone of Bonnisseau and Fuentes (2020) in $L(e)$.

$$\hat{\mathcal{T}}_{Y_j^e(z)}(y_j) = \bigcap_{\rho>0} \hat{\mathcal{T}}^\rho_{Y_j^e(z)}(y_j)$$

where

$$\hat{\mathcal{T}}^{\rho}_{Y^e_j(z)}(y_j) := \left\{ \nu \in L(e) \;\middle|\; \begin{array}{c} \exists\ \eta > 0 \text{ such that } \forall\ r > 0, \\ \exists\ U \in \mathcal{V}_\sigma(0) \text{ and } \varepsilon > 0\ | \\ \forall\ z' \in \left(\{z\} + \left(B^e_\rho \cap U\right)^{I\cup J}\right), \\ \forall\ y'_j \in \left(\{y_j\} + B^e_\rho \cap U\right) \cap Y_j(z') \\ \text{and } \forall\ t \in (0, \varepsilon),\ \exists\ \xi \in r[-e,\ e] \text{ such that} \\ y'_j + t\left(\nu + \eta(y_j - y'_j) + \xi\right) \in Y_j(z') \cap L(e) \end{array} \right\}$$

In the restricted economy, the marginal pricing rule is defined thanks to the following normal cone in $L(e)^*$

$$\hat{\mathcal{N}}_{Y^e_j(z)}(y_j) = \left[\hat{\mathcal{T}}_{Y^e_j(z)}(y_j)\}\right]^\circ = \left\{p \in L(e)^* : p \cdot \nu \leq 0\ \forall\ \nu \in \hat{\mathcal{T}}_{Y^e_j(z)}(y_j)\right\}.$$

We remark that $\hat{\mathcal{T}}_{Y^e_j(z)}(y_j)$ is not the adaptation of $\hat{\mathscr{T}}_{Y_j(z)}(y_j)$ to $L(e)$. The former considers open balls $B^e(0, \rho)$ in $L(e)$ while the latter would take the neighbourhoods $\mathcal{C} \cap L(e)$. Consequently, $\hat{\mathcal{T}}_{Y^e_j(z)}(y_j)$ is equivalent to the definition of Bonnisseau (2002) with externalities (compare with Proposition 5.3 below). The following lemma provides important inclusions that we shall exploit later.

Lemma 4.1 *For all $(z, y_j) \in L(e)^{I\cup J} \times L(e)$,*

1. $\mathscr{T}_{Y_j(z)}(y_j) \cap L(e) \subset \hat{\mathcal{T}}_{Y^e_j(z)}(y_j)$.
2. $\left(-\Gamma_j\right) \cap L(e) \subset \hat{\mathcal{T}}_{Y^e_j(z)}(y_j)$.

We now define precisely the restricted economy $\mathcal{E}_{|L(e)}$. Let $X^e_i : L(e)^{I\cup J} \twoheadrightarrow L(e)_+$ be the restricted consumption correspondence such that for all $z \in L(e)^{I\cup J}$, $X^e_i(z) = X_i(z) \cap L(e)_+$. We suitably restrict the preference relation $\succeq_{i,z}$ to $X^e_i(z)$ by $\succeq^e_{i,z}$ for all $z \in L(e)^{I\cup J}$. We remark that $\partial Y^e_j(z) \subset \partial Y_j(z) \cap L(e)$ by free-disposal assumption and the fact that e belongs to L_+,[10] whence $Z^e \subset Z$ and then $A^e(\omega') = A(\omega') \cap Z^e \subset A(\omega')$. The revenue functions (r_i) are the same and

$$PE^e = \{(p, z) \in L(e)^* \times Z^e : p \in \cap_{j\in J} \hat{\mathcal{N}}_{Y^e_j(z)}(y_j) \backslash \{0\}\}$$

is the production equilibria set. Hence, the economy $\mathcal{E}_{|L(e)}$ is fully described by $\left((X^e_i, \succeq^e_{i,z}, r_i)_{i\in I}, (Y^e_j)_{j\in J}, \omega\right)$.

Theorem 4.1 in Bonnisseau and Fuentes (2020) implies that $\mathcal{E}_{|L(e)}$ has an equilibrium $((\bar{x}_i)_{i\in I}, (\bar{y}_j)_{j\in J}, \bar{p}) \in L(e)^{I\cup J} \times L(e)^*_+$ since all assumptions hold.[11] The only condition for which we need to provide a specific proof is (SA).

[10] Indeed, if y_j belongs to the $\|\cdot\|_e$ boundary of $Y^e_j(z)$, then $y_j + te$ does not belong to $Y^e_j(z)$ for all positive t since e belongs to the $\|\cdot\|_e$-interior of $L(e)_+$ and the free-disposal assumption holds. Then $y_j + te$ does not belong to $Y_j(z)$. Hence, y_j does not belong to the τ-interior of $Y_j(z)$ and then, it does belong to the boundary of $Y_j(z)$.

[11] We would like to emphasize that in Bonnisseau and Fuentes (2020) the commodity space of the intermediate economy is a Banach lattice (see Aliprantis and Border 2006, Theorem 9.28 and discussions on

Lemma 4.2 *The economy $\mathcal{E}_{|L(e)}$ satisfies the Survival Assumption* (SA).

Note that $\bar{p} \neq 0$ from the non-satiation of preferences and $\bar{p} \cdot e > 0$ since $e \in \| \cdot \|_e - \operatorname{int} L(e)_+$ and $\bar{p} \in L(e)^*_+$.

5 Existence of equilibria

We are now ready to state our main result

Theorem 5.1 *Let $\mathcal{E}$ be an economy. Under Assumptions* (C), (P), (B) *and* (SA) *there exists a bundle $(\bar{z}, \bar{\pi})$ in $Z \times L^*$ which is a marginal pricing equilibrium of $\mathcal{E}$.*

Remark 5.1 The proof of Theorem 5.1 provides an "additional information" on the equilibrium price, namely, $\bar{\pi}_{|L(e)} = \left[\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)\}\right]^{\circ}$.

We now compare with previous existence results.

5.1 The marginal pricing equilibria in special cases

The comparison of Theorem 5.1 with previous results in the literature is not obvious. Then, in order to perceive the scope of our existence result, we analyze the marginal pricing rule in three particular circumstances that are commonly assumed in economic theory, namely, when technologies are convex, i.e., the competitive case; when e belongs to the quasi-interior or the interior of the cone L_+ and when technologies are smooth, i.e., production sets may be described by differentiable transformation functions. The analysis will show that Theorem 5.1 encompasses competitive existence results as Theorems 3.1 and 2.7 of Florenzano and Marakulin (2001) and Tourky (1999) respectively (Proposition 5.2), non-competitive existence results as Theorem 5.1 in Bonnisseau and Fuentes (2020) (Proposition 5.3) and theorems on existence of equilibria under smooth technologies (Proposition 5.6).

Convex technologies and competitive equilibria

Competitive equilibria is related with a production sector the technologies of which are convex and producers' behaviour is profit maximisation. By comparing our properness condition with that of Florenzano and Marakulin (2001) and Tourky (1999), we note that their condition impose additional requirements on the structure of the production sets, namely that it is in the τ-closure of $L(e) - L_+$. So, we posit the following assumption:

Upper order bounded (UOB) For all $z \in Z$ and for all $y_j \in Y_j(z) \cap L(e)$, there exists a radial set R_{y_j} at y_j such that

$$Y_j(z) \cap R_{y_j} \subset \tau - \operatorname{cl}\left[Y_j(z) \cap L(e) - \Gamma_j - L_+\right] \cap \tau - \operatorname{cl}\left(L(e) - L_+\right) \qquad (5)$$

pages 357–58) which does not hold true for $(L(e), \| \cdot \|_e)$. Nonetheless, it is important to note that a careful reading of Bonnisseau and Fuentes (2020) reveals that the completeness property is not used in the proof of Theorem 4.1 in that paper.

where Γ_j is given by the properness condition (Assumption P(5)).

We claim that (UOB) means that for every $z \in Z$ and for every $y_j' \in Y_j(z)$, there exists ξ as close as we want to y_j' such that it is bounded from above (or *upper order bounded*) by a vector in $L(e)$: let y_j' belongs to $Y_j(z)$. Let $y_j \in Y_j(z) \cap L(e)$. Then there exists $\lambda_{y_j} \leq 1$ such that $(1-\lambda)y_j + \lambda y_j' \in Y_j(z) \cap R_{y_j}$ for every $\lambda \in (0, \lambda_{y_j})$ since $Y_j(z)$ is convex and R_{y_j} is radial at y_j. For any $W_j \in \mathcal{V}_\tau(0)$, there exists $u \in W_j$ such that $(1-\lambda)y_j + \lambda y_j' + \lambda u \in L(e) - L_+$ by Inclusion (5) and the fact that $\lambda W_j \in \mathcal{V}_\tau(0)$. This implies that $y_j' + u$ belongs to $L(e) - L_+$ and the claimed is proved with $\xi := y_j' + u$.

Notice that (5) is close to Condition (2.4) of Florenzano and Marakulin (2001); we recall it using our notation: Let $z \in Z$, for $y_j \in Y_j(z) \cap L(e)$ there exist a τ-open convex set F, a lattice $K_{y_j} \subset L(e)$ verifying $K_{y_j} - L(e)_+ \subset K_{y_j}$ and some subset R_{y_j} of L, radial at y_j, such that $y_j \in \tau - \mathrm{cl}\,(F) \cap K_{y_j}$ and

$$Y_j(z) \cap R_{y_j} \subset \tau - \mathrm{cl}\,(F) \cap \left(K_{y_j} - L_+\right) \tag{6}$$

In (5), we could take K_{y_j} as in (6) but for simplicity we left $L(e) = K_{y_j}$. In the way of comparison, we are taking $F = Y_j(z) \cap L(e) - \Gamma_j - L_+$ and we slightly weakened (6) by considering the τ-closure of $L(e) - L_+$ instead of $L(e) - L_+$.

On the other hand, (5) is also related to the M-properness condition of (Tourky 1999) that we transcript with our notation: $Y_j(z)$ is M-proper at $y_j \in Y_j(z)$ if there are convex sets $\hat{Y}_j(z)$ and $\hat{K}_{y_j}$ such that (i) $\hat{Y}_j(z) \cap \left(\hat{K}_{y_j} - L_+\right) = Y_j(z)$, (ii) $y_j - e \in \tau - \mathrm{int}\hat{Y}_j(z)$,[12] (iii) $0 \in \hat{K}_{y_j}$ and (iv) if y_j, y_j' belong to $\hat{K}_{y_j}$ then $y_j \vee y_j'$ belongs to $\hat{K}_{y_j}$.

It is clear that both $Y_j(z) \cap L(e) - \Gamma_j - L_+$ and $L(e)$ satisfy the requirements of $\hat{Y}_j(z)$ and $\hat{K}_{y_j}$ respectively. Then, the two conditions are similar. Nevertheless, our condition is stronger since, in Tourky (1999), $\hat{K}_{y_j}$ can be strictly included in $L(e)$. On the contrary, (5) only requires an inclusion on $Y_j(z) \cap R_{y_j}$ which is smaller than on $Y_j(z)$.

Proposition 5.2 *Let $z \in Z$ and $y_j \in \partial Y_j(z) \cap L(e)$. Assume that firm j has a convex technology and satisfies Assumptions (P), then:*

1. $\{\pi \in L^* : \pi \cdot y_j \geq \pi \cdot y_j' \ \forall\, y_j' \in Y_j(z)\} \subset \hat{\mathcal{N}}_{Y_j(z)}(y_j)$
2. *Furthermore, if Assumption UOB holds, then for every π_j in $\hat{\mathcal{N}}_{Y_j(z)}(y_j)$ such that $\sup\left\{\pi_j \cdot \left(-\gamma_j\right) : \gamma_j \in \Gamma_j\right\} \leq 0$, it follows that $\pi_j \cdot y_j' \leq \pi_j \cdot y_j$ for all $y_j' \in Y_j(z)$. Moreover, for $\pi_j' \geq \pi_j$ such that $\pi'_{j|L(e)} = \pi_{j|L(e)}$, $\pi_j' \cdot y_j' \leq \pi_j' \cdot y_j$ for all $y_j' \in Y_j(z)$.*

For convex technologies, the above proposition means, on the one hand, that a price for which the profit is maximum at y_j satisfies the marginal pricing rule at y_j and, on the other hand, the converse is true under Condition UOB. As a consequence, we stress that the marginal pricing equilibria of the Theorem 5.1 is a competitive equilibria if

[12] Actually, Tourky takes ω instead of e.

production sets are convex and Condition UOB holds. Indeed, thanks to the properness condition, the equilibrium price $\bar{\pi}$ satisfies the condition $\sup \{\pi_j \cdot (-\gamma_j) : \gamma_j \in \Gamma_j\} \leq 0$ (see Sect. 6).

Spaces with quasi-interior points in L_+

We provide now some properties of the marginal pricing rule when e is a quasi-interior point of L_+. The following result shows that all positive continuous linear functionals in $\hat{\mathscr{N}}_{Y_j(z)}(y_j)$, whose restriction to $L(e)$ belongs to $\hat{N}_{Y_j^e(z)}(y_j)$, belongs also to $[\mathscr{T}_{Y_j(z)}(y_j)]^\circ$. This fact implies that the marginal pricing rule of this paper coincides with the one in Bonnisseau and Fuentes (2020) when L is a Banach lattice.

Lemma 5.3 *Let $(y_j, z) \in L(e) \times L(e)^{I \cup J}$ and $\pi \in \hat{\mathscr{N}}_{Y_j(z)}(y_j)$. Under Assumption (P), if e is a quasi-interior point of L_+ and $\pi_{|L(e)} \in [\hat{T}_{Y_j^e(z)}(y_j)]^\circ$, then $\pi \in [\mathscr{T}_{Y_j(z)}(y_j)]^\circ$.*

It follows from the very definition of the tangent sets that $[\mathscr{T}_{Y_j(z)}(y_j)]^\circ \subset \hat{\mathscr{N}}_{Y_j(z)}(y_j)$. So, the interesting result in the above lemma is just the converse inclusion. Then, we get the following consequence when the commodity spaces enjoy more properties:

Proposition 5.4 *Let $(L, \|\cdot\|)$ be a Banach lattice such that e is quasi-interior point of L_+. The marginal pricing equilibrium given by Theorem 5.1 is a marginal pricing equilibrium in the sense of* Bonnisseau and Fuentes (2020).[13] *If* $\text{int}L_+ \neq \emptyset$ *and e belongs to* $\text{int}L_+$, *then $\hat{\mathscr{N}}_{Y_j(z)}(y_j)$ is the normal cone of* Bonnisseau (2002) *with externalities and if $L = \mathbb{R}^\ell$, $\hat{\mathscr{N}}_{Y_j(z)}(y_j)$ is the marginal pricing rule of* Bonnisseau and Médecin (2001).

We close this section by showing that Condition UOB holds when L is a locally-solid Riesz space, e is a quasi-interior point of L_+ and technologies are convex. This explains why this condition is not necessary in Riesz spaces with quasi-interior points in the positive cone.

Lemma 5.5 *Let (L, τ) be a topological vector lattice such that e is a quasi-interior point of L_+. If Y_j is a convex-valued production correspondence satisfying σ-locally τ-uniformly properness relative to $L(e)$ for all $y_j \in Y_j(z)$ and all $z \in Z$, then condition* UOB *holds.*

Smooth technologies

Smooth technologies are a common assumption in economics. Indeed, the first results on equilibria with increasing returns are in this framework (see Mantel 1979; Beato 1982; Bonnisseau and Cornet 1990a) and even those with infinitely many commodities

[13] We remark that in the current paper, τ is the strong topology unlike in Bonnisseau and Fuentes (2020) where τ denotes the weak one. Proposition 5.4 takes the particular case where τ is the norm topology.

(Shannon 1996). It means that the production set is defined by an inequality involving a differentiable transformation function, that is, for all $z \in Z$ the technology is

$$Y_j(z) = \{\zeta_j \in L : f_j(\zeta_j, z) \leq 0\}$$

where $f_j : L \times L^{I \cup J} \to \mathbb{R}$ is a differentiable mapping. Let us consider the following assumption:

Assumption SB (Smooth boundary)

1. f_j is continuous for the $\sigma \times \prod_{L^{I \cup J}} \sigma$-topology on $L \times L^{I \cup J}$.
2. $f_j(\cdot, z)$ is Gateaux differentiable on L.
3. $\nabla_1 f_j(\zeta_j, z)$ belongs to $L^*_+ \backslash \{0\}$ if $\zeta_j \in \partial Y_j(z)$ where $\nabla_1 f_j(\zeta_j, z)$ is the gradient of f_j with respect to ζ_j.
4. $\nabla_1 f_j(\zeta_j, z) \cdot e > 0$ for every $\zeta_j \in \partial Y_j(z)$.
5. Let H be a $\sigma(L, L^*)$-equicontinuous[14] subset of L^*. For all $(\zeta_j, z) \in \partial Y_j(z) \times L^{I \cup J}$ and for all $\beta > 0$ there exists $U' \in \mathcal{V}_\sigma(0)$ such that $\nabla_1 f_j(\zeta'_j, z') - \nabla_1 f_j(\zeta_j, z) \in \beta H$ for all $(\zeta'_j, z') \in \left(\{\zeta_j\} + U'\right) \times \left(\{z\} + U'^{I \cup J}\right)$.

Proposition 5.6 *If the production set $Y_j(\cdot)$ is described by a transformation function f_j satisfying Assumption* SB*, then*

$$\hat{\mathcal{N}}_{Y_j(z)}(\zeta_j) \subset \{\nu \in \ \tau - \mathrm{cl}L(e) : \nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0\}^\circ$$

This means that for every $\pi_j \in \hat{\mathcal{N}}_{Y_j(z)}(y_j)$ there exists $\lambda_j > 0$ such that the restriction of π_j to $\tau - \mathrm{cl}L(e)$ equals the restriction of $\lambda_j \nabla_1 f_j(\zeta_j, z)$ to $\tau - \mathrm{cl}L(e)$. Of course, when e is a quasi-interior point of L_+, L equals $\tau - \mathrm{cl}L(e)$ so one can say that $\hat{\mathcal{N}}_{Y_j(z)}(\zeta_j)$ equals $\left\{\lambda \nabla_1 f_j(\zeta_j, z) : \lambda > 0\right\}$ which is the Clarke´s normal cone by Lemma 3.1 in Bonnisseau and Fuentes (2020).

6 Proof of the existence theorem

Let $((\bar{x}_i), (\bar{y}_j), \bar{p})$ be an equilibrium of $\mathcal{E}_{|L(e)}$, which exists from the result in Sect. 4. We shall prove that there exists a positive linear and continuous extension of $\bar{p}$ to the whole space L, $\bar{\pi}$, such that $((\bar{x}_i), (\bar{y}_j), \bar{\pi})$ is a marginal pricing equilibrium of the economy $\mathcal{E}$. Throughout the proof, we make use of the following Lemma whose first part comes from Podczeck (1996) and the second one is due to Florenzano and Marakulin (2001).

Lemma 6.1 (Lemma 2, Podczeck 1996; Lemma 2.1, Florenzano and Marakulin 2001) *Let K be a linear subspace of L. Let A and B be convex subsets of L with A τ-open, $B \subset K$ and such that $A \cap B \neq \emptyset$. Let f be any linear functional on K satisfying for some $x \in \tau - \mathrm{cl}(A) \cap B$, $f \cdot x \leq f \cdot x'$, for all $x' \in A \cap B$. Then,*

[14] That is, for every real number $a > 0$ there exists $U \in \mathcal{V}_\sigma(0)$ such that $g(U) \subset (-a, a)$ for all $g \in H$.

1. *There exist f_1 and f_2 with $f_1 \in L^*$ and f_2 a linear functional on L such that $f = f_{1|K} + f_{2|K}$ and*

$$f_1 \cdot x \leq f_1 \cdot x', \quad \forall x' \in A \quad \text{and} \quad f_2 \cdot x \leq f_2 \cdot x', \quad \forall x' \in B$$

2. *Let $K_+ = K \cap L_+$. If $B + K_+ \subset B$ then $f_{2|K} \geq 0$, $f_{1|K} \leq f$ and $f \cdot (x' - x) = f_1 \cdot (x' - x)$ for each $x' \in B$ such that $x' \leq x$.*

Claim 6.2 *For each $j \in J$, there exists $\pi_j \in L^*_+$ such that $\pi_{j|L(e)} = \bar{p}$.*

Proof From Lemma 4.1(2), $\left(-\Gamma_j\right) \cap L(e) \subset \hat{\mathcal{T}}_{Y^e_j(\bar{z})}(\bar{y}_j)$ and $-e \in \left(-\Gamma_j\right) \cap L(e) \neq \emptyset$. From the equilibrium conditions in $L(e)$, $\bar{p} \in \hat{\mathcal{N}}_{Y^e_j(\bar{z})}(\bar{y}_j)$, so, for all $\zeta \in \left(-\Gamma_j\right) \cap L(e)$, $-\bar{p} \cdot \zeta \geq 0$ since $\hat{\mathcal{N}}_{Y^e_j(z)}(\bar{y}_j) = \left[\hat{\mathcal{T}}_{Y^e_j(\bar{z})}(\bar{y}_j)\right]^\circ$. Taking $A = -\Gamma_j$, $B = K = L(e)$ and $x = 0$, we remark that the assumptions of Lemma 6.1 are satisfied, so there exist $\hat{\pi}_j$ and $\hat{\pi}'_j$ such that $\hat{\pi}_j$ is τ-continuous on L, $\bar{p} = \hat{\pi}_{j|L(e)} + \hat{\pi}'_{j|L(e)}$, $\hat{\pi}_j \cdot \zeta \geq 0$ for all $\zeta \in -\Gamma_j$ and $\hat{\pi}'_j \cdot x \geq 0$ for all $x \in L(e)$. From the last assertion, we deduce that $\hat{\pi}'_{j|L(e)} = 0$ since $L(e)$ is a linear subspace, so $\bar{p} = \hat{\pi}_{j|L(e)}$.

Let $\pi_j := \hat{\pi}^+_j = \hat{\pi}_j \vee 0$. π_j is continuous since L^* is a sub-lattice of $L^\sim$ by assumption. Since $\bar{p} \in L^*(e)_+$ we prove that $\pi_{j|L(e)} = \bar{p}$. Indeed, let $x \in L(e)_+$. By the Riesz–Kantorovich formula, $\pi_j \cdot x = \sup\left\{\pi_j \cdot \tilde{x} : 0 \leq \tilde{x} \leq x\right\}$. Since $0 \leq \tilde{x} \leq x$ implies $\tilde{x} \in L(e)_+$, $\pi_j \cdot x = \sup\{\bar{p} \cdot \tilde{x} : 0 \leq \tilde{x} \leq x\}$ and the supremum is $\bar{p} \cdot x$ since $\bar{p} \in L^*(e)_+$. For all $\zeta \in L(e)$, one has $\zeta = \zeta^+ - \zeta^-$ and both belong to $L(e)_+$, then one deduces that $\bar{p} \cdot \zeta = \pi_j \cdot \zeta$. Consequently π_j is a continuous extension of $\bar{p}$ belonging to L^*_+. □

Claim 6.3 *For all $i \in I$, there exists $\pi_i \in L^*$ such that $\pi_{i|L(e)} \leq \bar{p}$, $\pi_i \cdot \bar{x}_i \leq \pi_i \cdot x'_i$ for all $x'_i \in \tau - \text{cl}\left(\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i\right)$ and $\pi_i \cdot (\zeta_i - \bar{x}_i) = \bar{p} \cdot (\zeta_i - \bar{x}_i)$ for all $\zeta_i \in K_{\bar{x}_i}$*[15] *such that $\zeta_i \leq \bar{x}_i$.*

Proof By Assumption C(6) (1), there exists a set $R_{\bar{x}_i}$, radial at $\bar{x}_i$ such that $\left(\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i\right) \cap K_{\bar{x}_i} \cap R_{\bar{x}_i} \subset \{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\}$ hence, $\bar{p} \cdot \bar{x}_i < \bar{p} \cdot x_i$ for all $x_i \in \left(\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i\right) \cap K_{\bar{x}_i} \cap R_{\bar{x}_i}$ by the equilibrium conditions in $\mathcal{E}_{|L(e)}$. Since $R_{\bar{x}_i}$ is radial, $K_{\bar{x}_i}$ is convex[16] and $\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\}$ is radial at $\bar{x}_i$ from Assumption C(3), $\bar{p} \cdot \bar{x}_i < \bar{p} \cdot x_i$ for all $x_i \in \left(\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i\right) \cap K_{\bar{x}_i}$. Indeed, take $x_i \in \left(\left\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\right\} + \Gamma_i\right) \cap K_{\bar{x}_i}$. There exists $\lambda_{\bar{x}_i} > 0$ such that $(1-\lambda)\bar{x}_i + \lambda x_i$ belongs to $K_{\bar{x}_i} \cap R_{\bar{x}_i}$ for all $0 \leq \lambda \leq \lambda_{\bar{x}_i}$. Furthermore, there exists $\xi \in \Gamma_i$ and $x''_i \in \left\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\right\}$ such that $x_i = x''_i + \xi$. It follows that $(1-\lambda)\bar{x}_i + \lambda x_i = (1-\lambda)\bar{x}_i + \lambda x''_i + \lambda\xi \in \left\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\right\} + \Gamma_i$ by Assumption C(3). Hence, $(1-\lambda)\bar{x}_i + \lambda x_i$ belongs to $\left(\left\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\right\} + \Gamma_i\right) \cap K_{\bar{x}_i} \cap R_{\bar{x}_i}$. From the above inequality, $\bar{p} \cdot [(1-\lambda)\bar{x}_i + \lambda x_i] > \bar{p} \cdot \bar{x}_i$ so $\bar{p} \cdot x_i > \bar{p} \cdot \bar{x}_i$.

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15 $K_{\bar{x}_i} \subset L(e)$ comes from Assumption C(6).

16 $K_{\bar{x}_i}$ is convex since it is a sublattice of $L(e)$ and $K_{\bar{x}_i} + L(e)_+ \subset K_{\bar{x}_i}$. Indeed, let ξ and ξ' in $K_{\bar{x}_i} \subset L(e)$. $\xi \wedge \xi'$ belongs to $K_{\bar{x}_i}$ since $K_{\bar{x}_i}$ is a sublattice. For every $t \in (0, 1)$, $t\xi + (1-t)\xi' \geq \xi \wedge \xi'$ and thus $t\xi + (1-t)\xi'$ belongs to $K_{\bar{x}_i} + L(e)_+ \subset K_{\bar{x}_i}$.

By applying Lemma 6.1(1) and (2) to $K = L(e)$, $A = \{x_i' \in X_i(z) \mid x_i' \succ_{i,z} \bar{x}_i\} + \Gamma_i$ and $B = K_{\bar{x}_i}$, we deduce the existence of $\pi_i \in L^*$ such that $\pi_{i|L(e)} \le \bar{p}$ and $\pi_i \cdot \bar{x}_i \le \pi_i \cdot x_i$ for all $x_i \in \tau - \mathrm{cl}\left(\left\{x_i' \in X_i(z) : x_i' \succ_{i,z} \bar{x}_i\right\} + \Gamma_i\right)$. Furthermore, $\pi_i \cdot (\zeta_i - \bar{x}_i) = \bar{p} \cdot (\zeta_i - \bar{x}_i)$ for all $\zeta_i \in K_{\bar{x}_i}$ such that $\zeta_i \le \bar{x}_i$. □

Claim 6.4 *Let* $\bar{\pi} = (\vee_{i\in I}\pi_i) \vee (\vee_j \pi_j)$. *Then* $\bar{\pi} \in L_+^*$ *and* $\bar{\pi}_{|L(e)} = \bar{p}$.

Proof $\bar{\pi} \in L^*$ since L^* is a sublattice of $L^\sim$. Furthermore, $\bar{\pi} > 0$ since $\pi_j > 0$ for all j. Let ζ be any vector in $L(e)$. Then ζ^+ and ζ^- belong to $L(e)$ since it is an ideal. Since $\bar{\pi} \ge \pi_j$ for some j, from Claim 6.3, one deduces that $\bar{\pi} \cdot \zeta^+ \ge \pi_j \cdot \zeta^+ = \bar{p} \cdot \zeta^+$.

From the Riesz–Kantorovich formula, $\bar{\pi} \cdot \zeta^+$ is the supremum of $\sum_{i\in I} \pi_i \cdot \zeta_i + \sum_{j\in J} \pi_j \cdot \zeta_j$ over the elements $((\zeta_i), (\zeta_j)) \in L_+^{I\cup J}$ such that $\sum_{i\in I} \zeta_i + \sum_{j\in J} \zeta_j \le \zeta^+$. From Claims 6.3 and 6.4, $\pi_j \cdot \zeta_j = \bar{p} \cdot \zeta_j$ and $\pi_i \cdot \zeta_i \le \bar{p} \cdot \zeta_i$. So, since $\bar{p} \in L^*(e)_+$, $\sum_{i\in I} \pi_i \cdot \zeta_i + \sum_{j\in J} \pi_j \cdot \zeta_j \le \bar{p} \cdot \left(\sum_{i\in I} \zeta_i + \sum_{j\in J} \zeta_j\right) \le \bar{p} \cdot \zeta^+$. Hence, $\bar{\pi} \cdot \zeta^+ \le \bar{p} \cdot \zeta^+$ and together with the inequality above, we conclude that $\bar{\pi} \cdot \zeta^+ = \bar{p} \cdot \zeta^+$. Analogously, $\bar{\pi} \cdot \zeta^- = \bar{p} \cdot \zeta^-$. Thus, $\bar{\pi} \cdot \zeta = \bar{p} \cdot \zeta$ and the proof is complete. □

Claim 6.5 $\bar{\pi} \in \bigcap_{j\in J} \hat{\mathscr{N}}_{Y_j(\bar{z})}(\bar{y}_j)$.

Proof This is a direct consequence of Lemma 4.1(1) and Claim 6.4 together with the fact that $\bar{p} \in \hat{N}_{Y_j^e(\bar{z})}(\bar{y}_j) = \left[\hat{T}_{Y_j^e(\bar{z})}(\bar{y}_j)\right]^\circ$. Indeed, let $\nu \in \hat{\mathscr{T}}_{Y_j(\bar{z})}(\bar{y}_j)$. Then, from the definition of the tangent cone and Lemma 4.1(1), $\nu = \nu_1 + \nu_2$ with $\nu_1 \in \mathscr{T}_{Y_j(z)}(y_j) \cap L(e) \subset \hat{T}_{Y_j^e(z)}(y_j)$ and $\nu_2 \in -L_+$. Since $\bar{\pi}_{|L(e)} = \bar{p} \in \hat{N}_{Y_j^e(z)}(y_j)$ and $\bar{\pi} > 0$, one gets $\bar{\pi} \cdot \nu_1 \le 0$ and $\bar{\pi} \cdot \nu_2 \le 0$. Hence, $\bar{\pi} \cdot \nu = \bar{\pi} \cdot \nu_1 + \bar{\pi} \cdot \nu_2 \le 0$ and the claim is proved. □

Claim 6.6 $\bar{\pi} \cdot \bar{x}_i = r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j))$ *and if* $x_i \succ_{i,\bar{z}} \bar{x}_i$ *then* $\bar{\pi} \cdot x_i > r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j))$.

Proof By the Survival Assumption (SA), $\bar{\pi} \cdot \left(\sum_{j\in J} \bar{y}_j + \omega\right) > 0$. From Assumption C(5), $\sum_{i\in I} r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j)) > 0$ and $r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j)) > 0$ for all $i \in I$. Furthermore, $\bar{\pi} \cdot \bar{x}_i = r_i(\bar{p} \cdot \omega, (\bar{p} \cdot \bar{y}_j)) = r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j))$ follows since $((\bar{x}_i)_{i\in I}, (\bar{y}_j)_{j\in J}, \bar{p})$ is an equilibrium in $\mathcal{E}_{|L(e)}$, $\bar{\pi}_{|L(e)} = \bar{p}$ and the preferences $\preceq_{i,\bar{z}}^e$ are locally nonsatiated thanks to Assumptions C(3) and C(6).

Let $x_i \in X_i(\bar{z})$ such that $x_i \succ_{i,\bar{z}} \bar{x}_i$ and $R_{\bar{x}_i}$ and $K_{\bar{x}_i}$ as given by Assumption C(6). Since $R_{\bar{x}_i}$ is radial at $\bar{x}_i$ and $X_i(\bar{z})$ is convex, there exists $\bar{\lambda} \in (0, 1)$ such that for $\lambda \in (0, \bar{\lambda}]$, $\tilde{x}_i = (1 - \lambda)\bar{x}_i + \lambda x_i \in X_i(\bar{z}) \cap R_{\bar{x}_i}$. By Assumption C(3), $\tilde{x}_i \succ_{i,\bar{z}} \bar{x}_i$. Due to E-properness relative to $L(e)$ (C (6.2)), $\tilde{x}_i \in K_{\bar{x}_i} + L_+$, that is, there exists $\zeta_i \in K_{\bar{x}_i}$ such that $\zeta_i \le \tilde{x}_i$. Let $\zeta_i' = \bar{x}_i \wedge \zeta_i$. $\zeta_i' \in K_{\bar{x}_i}$ since $K_{\bar{x}_i}$ is a sublattice and $\bar{x}_i \in K_{\bar{x}_i}$. By Claim 6.3, it follows that $\pi_i \cdot \tilde{x}_i \ge \pi \cdot \bar{x}_i$ and $\pi_i \cdot (\bar{x}_i - \zeta_i') = \bar{p} \cdot (\bar{x}_i - \zeta_i')$. By Claim 6.4, $\bar{\pi}_{|L(e)} = \bar{p}$, so $\bar{\pi} \cdot (\bar{x}_i - \zeta_i') = \bar{p} \cdot (\bar{x}_i - \zeta_i')$. Since $\bar{\pi} \ge \pi_i$, we get

$$\begin{aligned} \bar{\pi} \cdot \left(\tilde{x}_i - \zeta_i'\right) &\ge \pi_i \cdot \left(\tilde{x}_i - \zeta_i'\right) \ge \pi_i \cdot \left(\bar{x}_i - \zeta_i'\right) \\ &= \bar{p} \cdot \left(\bar{x}_i - \zeta_i'\right) = \bar{\pi} \cdot (\bar{x}_i - \zeta_i') \end{aligned}$$

Consequently, $\bar{\pi} \cdot \tilde{x}_i \ge \bar{\pi} \cdot \bar{x}_i$ whence, $\bar{\pi} \cdot x_i \ge \bar{\pi} \cdot \bar{x}_i$.

Let us suppose that $\bar{\pi} \cdot x_i = \bar{\pi} \cdot \bar{x}_i$. Since $\bar{\pi} \cdot \bar{x}_i > 0, \bar{\pi} \cdot (\gamma x_i) < \bar{\pi} \cdot \bar{x}_i$ for all $\gamma \in (0, 1)$. Notice that $(\bar{x}_i, x_i, \bar{z}) \notin G_i$ and γx_i belongs to $X_i(z)$ because this set is convex and 0 belongs to it. Hence, Assumption C(4) implies that for γ close enough to 1, $(\bar{x}_i, \gamma x_i, \bar{z}) \notin G_i$ and thus, $\gamma x_i \succ_{i,\bar{z}} \bar{x}_i$. By the previous result, $\bar{\pi} \cdot (\gamma x_i) \geq \bar{\pi} \cdot \bar{x}_i$ which contradicts the above converse inequality. Consequently, $\bar{\pi} \cdot x_i > r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j))$ and the proof is complete. □

Proofs

Proposition 3.1

It suffices to show that $\mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$ is a cone for all $\mathcal{C} \in \mathcal{V}_\tau(0)$ since $L(e)$ is a cone. Let $\mathcal{C} \in \mathcal{V}_\tau(0)$ and $v \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$. Then there exists $\eta > 0$ such that for all $r > 0$ there exists $U \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$ such that for all $z' \in \left(\{z\} + (\mathcal{C} \cap U)^{I \cup J}\right)$, for all $y'_j \in \left(\{y_j\} + \mathcal{C} \cap U\right) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$ there exists $\xi \in r[-e, e]$ such that $y'_j + t(v + \eta(y_j - y'_j) + \xi) \in Y_j(z')$.

Let $\kappa > 0$, $\eta_\kappa = \kappa\eta$ and U_κ and ε_κ be the open neighbourhood and the parameter associated for v to $\frac{r}{\kappa}$. Hence, for all $z' \in \left(\{z\} + (\mathcal{C} \cap U_\kappa)^{I \cup J}\right)$, for all $y'_j \in \left(\{y_j\} + \mathcal{C} \cap U_\kappa\right) \cap Y_j(z')$ and for all $t \in (0, \varepsilon_\kappa)$, there exists $\xi \in \frac{r}{\kappa}[-e, e]$ such that $y'_j + t(v + \eta(y_j - y'_j) + \xi) \in Y_j(z')$. Then, for all $t \in \left(0, \frac{\varepsilon_\kappa}{\kappa}\right)$, $\kappa t \in (0, \varepsilon_\kappa)$, so, for all $z' \in \left(\{z\} + (\mathcal{C} \cap U_\kappa)^{I \cup J}\right)$, for all $y'_j \in \left(\{y_j\} + \mathcal{C} \cap U_\kappa\right) \cap Y_j(z')$, there exists $\xi \in \frac{r}{\kappa}[-e, e]$ such that $y'_j + \kappa t(v + \eta(y_j - y'_j) + \xi) = y'_j + t(\kappa v + \kappa\eta(y_j - y'_j) + \kappa\xi) \in Y_j(z')$. Since $\kappa\xi \in r[-e, e]$, this shows that $\kappa v \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$ with the parameter η_κ and U_κ and $\frac{\varepsilon_\kappa}{\kappa}$ associated to r. □

Lemma 3.3

1. Let $z \in Z$ and $y_j \in \partial Y_j(z)$. Let us assume that Condition P(4) is satisfied. Let $\mathcal{C} \in \mathcal{V}_\tau(0)$ and $\eta = 1$. Let $r > 0$ and choose $\delta > 0$ smaller than r. Let $\hat{U}$ be the σ-open set and $\varepsilon > 0$ as given by Assumption P(4). Then, for all $z' \in \left(\{z\} + (\mathcal{C} \cap \hat{U})^{I \cup J}\right)$, for all $y'_j \in \left(\{y_j\} + \mathcal{C} \cap \hat{U}\right) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$, the vector $t(y_j - \delta e) + (1 - t)y'_j = y'_j + t(y_j - y'_j - \delta e) \in Y_j(z')$, which means $0 \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$. Since this is true for all neighbourhood $\mathcal{C}$, we conclude that $0 \in \mathscr{T}_{Y_j(z)}(y_j)$.
   For the converse, let $z \in Z$, $y_j \in \partial Y_j(z)$ and let us assume that $0 \in \mathscr{T}_{Y_j(z)}(y_j)$. Let $\mathcal{C} \in \mathcal{V}_\tau(0)$ and $\delta > 0$. So, using the free-disposal Assumption on $Y_j(z')$ and the fact that $0 \in \mathscr{T}_{Y_j(z)}(y_j)$, there exists $\eta > 0$ and for $r = \delta\eta > 0$, there exists $U \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$, such that for all $z' \in \left(\{z\} + (\mathcal{C} \cap U)^{|I|+|J|}\right)$, for all $y'_j \in (\{y\} + \mathcal{C} \cap U) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$

$$y'_j + t(\eta(y_j - y'_j) - re) = (1 - t\eta)y'_j + t\eta(y_j - \delta e) \in Y_j(z')$$

So Assumption P(4) is satisfied with $\hat{U} = U$ and $\varepsilon' = \eta\varepsilon$.

2. Let $z \in L^{I\cup J}$ and $y_j \in \partial Y_j(z)$. Since L is finite dimensional, the two topologies τ and σ coincide with the standard norm topology and L_+ has a nonempty interior containing e. Let $\mathcal{C}$ be a neighbourhood of 0 and $\delta > 0$. Let us consider the neighbourhood of 0, $\hat{U} = \{-\delta e\} + \text{int}L_+$ and $\varepsilon = 1$. Thus, for all $z' \in \left(\{z\} + (\mathcal{C} \cap \hat{U})^{I\cup J}\right)$, for all $y'_j \in Y_j(z') \cap \left(\{y_j\} + (\mathcal{C} \cap \hat{U})\right)$ and for all $t \in (0, 1)$, the vector $t(y_j - \delta e) + (1-t)y'_j = y'_j - t(y'_j - y_j + \delta e)$ belongs to $Y_j(z')$ from the free-disposal assumption since $y'_j - y_j + \delta e$ belongs to L_+. □

Lemma 4.1

1. Let $\nu \in \mathscr{T}_{Y_j(z)}(y_j) \cap L(e)$ then $\nu \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j) \cap L(e)$ for all $\mathcal{C} \in \mathcal{V}_\tau(0)$. Let $\rho > 0$. Since order intervals are τ-bounded, there exists $\mathcal{C}' \in \mathcal{V}_\tau(0)$ such that $B^e_\rho \subset \mathcal{C}'$. Hence, since $\nu \in \mathscr{T}^{\mathcal{C}'}_{Y_j(z)}(y_j)$, there exists $\eta > 0$ such that for all $r > 0$, there exists $U \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$ such that for all $z' \in \left(\{z\} + (B^e_\rho \cap U)^{I\cup J}\right) \subset \left(\{z\} + (\mathcal{C}' \cap U)^{I\cup J}\right)$, for all $y'_j \in \left(\{y_j\} + B^e \cap U\right) \cap Y_j(z') \subset \left(\{y_j\} + \mathcal{C}' \cap U\right) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$, there exists $\xi \in r[-e, e]$ such that $y'_j + t(\nu + \eta(y_j - y'_j) + \xi) \in Y_j(z') \cap L(e)$. Hence, $\nu \in \hat{\mathcal{T}}^\rho_{Y^e_j(z)}(y_j)$ and since it is true for all $\rho > 0$, $\nu \in \hat{\mathcal{T}}_{Y^e_j(z)}(y_j)$.
2. By Lemma 3.3(1), 0 belongs to $\mathscr{T}_{Y_j(z)}(y_j)$ and then to $\hat{\mathcal{T}}_{Y^e_j(z)}(y_j) = \bigcap_{\rho>0} \hat{\mathcal{T}}^\rho_{Y^e_j(z)}(y_j)$ by the first part of the proof. Let $\rho > 0$. Let $\eta > 0$ be the parameter associated to 0 in the definition of $\hat{\mathcal{T}}^\rho_{Y_j(z)}(y_j)$. Let $r > 0$ and let $U \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$ be the neighbourhood and the parameter associated to r in the definition of $\hat{\mathcal{T}}^\rho_{Y_j(z)}(y_j)$. Let V_j as given by Assumption P(5) at $y_j \in Y_j(z)$. Finally, let $V'_j \in \mathcal{V}_\sigma(0)$ such that $V'_j + V'_j + V'_j \subset V_j$.

Let $U' = U \cap V'_j$. We note that there exists $n_0 \in \mathbb{N}$ such that $\eta(y_j - y'_j) + \xi \in n_0[-e, e]$ for all $y'_j \in \left(\{y_j\} + B^e_\rho \cap U'\right) \cap Y_j(z')$ and $\xi \in r[-e, e]$. Since $[0, e]$ is σ-compact and therefore σ-bounded, there exists $\lambda > 0$ such that $n_0[-e, e] \subset \lambda V'_j$. Note that $y'_j \in \left(\{y_j\} + B^e_\rho \cap U'\right)$ implies that $y'_j \in \{y_j\} + V'_j$ and, for all $0 < t < \frac{1}{\lambda}$, $t(\eta(y_j - y'_j) + \xi) \in V'_j$ since V'_j is circled.

Now, $0 \in \mathscr{T}_{Y_j(z)}(y_j)$ implies that for all $z' \in \left(\{z\} + \left(B^e_\rho \cap U'\right)^{|I|+|J|}\right)$, for all $y'_j \in \left(\{y_j\} + B^e_\rho \cap U'\right) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$, there exists $\xi \in r[-e, e]$ such that $y'_j + t(\eta(y_j - y'_j) + \xi) \in Y_j(z') \cap L(e)$.

Let $\zeta \in \left(-\Gamma_j\right) \cap L(e)$. There exists $\varepsilon' > 0$ such that $t\zeta \in V'_j$ for all $t \in (0, \varepsilon')$. Consequently, for all $0 < t < \varepsilon'' < \min\left\{\varepsilon, \varepsilon', \frac{1}{\lambda}\right\}$, we have $y'_j + t(\eta(y_j - y'_j) + \xi + \zeta) \in \left(\{y_j\} + V'_j + V'_j + V'_j\right) \cap L(e) \subset \left(\{y_j\} + V_j\right) \cap L(e)$. Furthermore, since $y'_j + t(\eta(y_j - y'_j) + \xi) + t\zeta \in \left(Y_j(z') \cap \left(\{y_j\} + V'_j + V'_j\right)\right) - \Gamma_j \subset$

$\big(Y_j(z') \cap \big(\{y_j\} + V_j\big)\big) - \Gamma_j$, we deduce by Assumption P(5) that $y'_j + t(\zeta + \eta(y_j - y'_j) + \xi) \in Y_j(z') \cap L(e)$. So, for the parameter ρ taken η as given above, and for all $r > 0$, taken U' and ε'', we have shown that $\zeta \in \mathcal{T}^{\rho}_{Y^e_j(z)}(y_j)$. Since this holds for all $\rho > 0$, we have the desired result. □

Lemma 4.2

We want to prove that for all $(p, z, t) \in L(e)^* \backslash \{0\} \times Z^e \times \mathbb{R}_+$, if $(y_j) \in A^e(\omega + te, z)$ and $p \in \cap_{j \in J} \hat{\mathcal{N}}_{Y^e_j(z)}(y_j)$, it follows that $p \cdot \left(\sum_{j \in J} y_j + \omega + te\right) > 0$. By Claim 6.2, there exists $\pi_j \in L^*_+$ such that $\pi_{j|L(e)} = p$. Using the same argument as in the proof of Claim 6.4, $\pi = \vee_{j \in J} \pi_j \in L^*_+$ and $\pi_{|L(e)} = p$. From Claim 6.5, $\pi \in \cap_{j \in J} \hat{\mathscr{N}}_{Y_j(z)}(y_j)$. Hence, $(\pi, z, t) \in L^*_+ \backslash \{0\} \times Z \times \mathbb{R}_+$ and $(y_j) \in A(\omega + te, z)$. So Assumption SA implies that $0 < \pi \cdot \left(\sum_{j \in J} y_j + \omega + te\right) = p \cdot \left(\sum_{j \in J} y_j + \omega + te\right)$ and the lemma is proved. □

Proposition 5.2

Let

$$PM(y_j, z) = \{\pi \in L^* : \pi \cdot y_j \geq \pi \cdot y'_j \ \forall \ y'_j \in Y_j(z)\}$$

be the profit maximisation rule.

1. Let $\pi \in PM(y_j, z)$. By free disposal, it follows that $\pi \geq 0$. If $v \in \mathscr{T}_{Y_j(z)}(y_j)$ then, from the definition of the tangent cone with $z' = z$ and $y'_j = y_j$, for all $r > 0$ there exists $\varepsilon > 0$ such that for $t \in (0, \varepsilon)$, $y_j + t(v - re) \in Y_j(z)$. Consequently $\pi \cdot (y_j + t(v - re)) \leq \pi \cdot y_j$, which implies $\pi \cdot v \leq 0$ when r tends to 0. Consequently, for every $v' \in \hat{\mathscr{T}}_{Y_j(z)}(y_j) \cap L(e) - L_+$ we get $\pi \cdot v' \leq 0$ whence, $\pi \in \hat{\mathscr{N}}_{Y_j(z)}(y_j)$ and $PM(y_j, z) \subset \hat{\mathscr{N}}_{Y_j(z)}(y_j)$.
2. We first prove that for all $\pi \in \hat{\mathscr{N}}_{Y_j(z)}(y_j)$, $\pi \cdot y'_j \leq \pi \cdot y_j$ for all $y'_j \in Y_j(z) \cap L(e)$. Indeed, let $\zeta_j \in Y_j(z)$. We prove that for all $\mathcal{C} \in \mathcal{V}_\tau(0)$, $\zeta_j - y_j \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$. Let $\eta = 1$, $r > 0$, $0 < \delta < r$, $\varepsilon = 1$. By Assumption P(3), there exists $V' \in \mathcal{V}_\sigma(0)$ such that $\zeta_j - \delta e \in Y_j(z')$ for all $z' \in \{z\} + (V')^{I \cup J}$. Then, for $z' \in \left(\{z\} + \left(\mathcal{C} \cap V'\right)^{I \cup J}\right)$, for $y'_j \in \left(\{y_j\} + \mathcal{C} \cap V'\right) \cap Y_j(z')$ and $t \in (0, 1)$, $t(\zeta_j - \delta e) + (1 - t)y'_j \in Y_j(z')$ since $Y_j(z')$ is convex. But this means that $y'_j + t(\zeta_j - \delta e - y_j + (y_j - y'_j)) \in Y_j(z')$. Since $-\delta e \in r[-e,\ e]$, we deduce that $\zeta_j - y_j \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$. Since this is true for all $\mathcal{C} \in \mathcal{V}_\tau(0)$, $\zeta_j - y_j \in \bigcap_{\mathcal{C} \in \mathcal{V}_\tau(0)} \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$ and thus $(Y_j(z) - \{y_j\}) \cap L(e) \subset \bigcap_{\mathcal{C} \in \mathcal{V}_\tau(0)} \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j) \cap L(e) \subset \hat{\mathscr{T}}_{Y_j(z)}(y_j)$. Since y_j belongs to $L(e)$ this implies that for every $\pi \in \hat{\mathscr{N}}_{Y_j(z)}(y_j)$, $\pi \cdot y'_j \leq \pi \cdot y_j$ for all $y'_j \in Y_j(z) \cap L(e)$.

We proceed now to show that for every π_j in $\hat{\mathcal{N}}_{Y_j(z)}(y_j)$ such that $\pi_j \cdot \gamma_j \geq 0$ for all $\gamma_j \in \Gamma_j$, it follows that $\pi_j \cdot y'_j \leq \pi_j \cdot y_j$ for all $y'_j \in Y_j(z)$. Let $y'_j \in Y_j(z)$. Let R_{y_j} as given by Assumption (UOB). There exists $0 < \lambda \leq 1$ such that $(1-\lambda)y_j + \lambda y'_j \in R_{y_j}$ whence in $Y_j(z) \cap R_{y_j}$ since $Y_j(z)$ is convex. Let W be a τ-neighbourhood of 0. By Assumption (UOB), there exist $w \in W$, $\zeta_j \in Y_j(z) \cap L(e)$ and $\xi \in \Gamma_j$ such that $(1-\lambda)y_j + \lambda y'_j + \lambda w \leq \zeta_j - \xi$. By the above result, $\pi_j \cdot \big(\zeta_j - \xi\big) \leq \pi_j \cdot y_j$ since $\pi_j \cdot (-\xi) \leq 0$. Hence, $\pi_j \cdot \Big((1-\lambda)y_j + \lambda y'_j + \lambda w\Big) \leq \pi_j \cdot y_j$, which implies $\lambda \pi_j \cdot \Big((y'_j - y_j) + w\Big) \leq 0$. Since $\lambda > 0$ and W is an arbitrary neighbourhood of 0, this implies $\pi_j \cdot y'_j \leq \pi_j \cdot y_j$. Then we proved that $\pi_j \in PM(y_j, z)$.

We finish the proof by considering π'_j such that $\pi'_j \geq \pi_j$ and $\pi'_{j|L(e)} = \pi_{j|L(e)}$. Let $y'_j \in Y_j(z)$. Assumption (UOB) implies that $Y_j(z) \subset \tau - \mathrm{cl}\,(L(e) - L_+)$, so, for every W in $\mathcal{V}_\tau(0)$ there exists $w \in W$ such that $y'_j + w \leq \zeta$ for some $\zeta \in L(e)$. Without any loss of generality we may assume that $\zeta \geq y_j$. Hence,

$\pi'_j \cdot (y'_j + w - y_j) + \pi_j \cdot (y_j - \zeta) = \pi'_j \cdot (y'_j + w - y_j) + \pi'_j \cdot (y_j - \zeta) = \pi'_j \cdot (y'_j + w - \zeta) \leq \pi_j \cdot (y'_j + w - \zeta) \leq \pi_j \cdot (y_j + w - \zeta)$.

Consequently, $\pi'_j \cdot (y'_j + w - y_j) \leq \pi_j \cdot w$, which implies $\pi'_j . y'_j \leq \pi'_j . y_j$ since W is an arbitrary neighbourhood of 0. □

Lemma 5.3

Let $\pi \in L^* \backslash \{0\}$ such that $\pi_{|L(e)} \in [\hat{\mathcal{T}}_{Y^e_j(z)}(y_j)]^\circ$. We prove below that $\mathcal{T}_{Y_j(z)}(y_j)$ is a subset of the τ-closure of $\hat{\mathcal{T}}_{Y^e_j(z)}(y_j)$. Then, for all $\nu \in \mathcal{T}_{Y_j(z)}(y_j)$, $\pi \cdot \nu \leq 0$, which implies that $\pi \in \big[\mathcal{T}_{Y_j(z)}(y_j)\big]^\circ$.

Let $\nu \in \mathcal{T}_{Y_j(z)}(y_j)$. We prove that for all $\beta > 0$ and for all $W \in \mathcal{V}_\tau(0)$, there exists $w \in W$ such that $\nu - \beta e + w \in \hat{\mathcal{T}}_{Y^e_j(z)}(y_j)$. Let $W' \in \mathcal{V}_\tau(0)$ such that $-\beta e + W' \subset -\Gamma_j$. Let $\rho > 0$. There exists $\mathcal{C} \in \mathcal{V}_\tau(0)$ such that $[-\rho e, \rho e] \subset \mathcal{C}$. Since $\nu \in \mathcal{T}_{Y_j(z)}(y_j) \subset \mathcal{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$, there exists $\eta > 0$ such that for all $r > 0$, there exists $U \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$ such that for all $z' \in \big(\{z\} + (\mathcal{C} \cap U)^{I \cup J}\big)$ for all $y'_j \in \big(\{y_j\} + \mathcal{C} \cap U\big) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$, there exists $\xi \in r[-e,\ e]$ such that

$$y'_j + t(\nu + \eta(y_j - y'_j) + \xi) \in Y_j(z') \tag{7}$$

Let $U' \in \mathcal{V}_\tau(0)$ such that $U' + U' + U' + U' + U' \subset V_j \cap U$ where V_j comes from Properness Assumption P(5). Since $L(e)$ is τ-dense in L, there exists $w \in W \cap W' \cap U'$ such that $\nu + w \in L(e)$. Furthermore, $t(-\beta e + w) \in -\Gamma_j$ for all $t > 0$. Since U' is absorbing, there exist $\alpha > 0$ and $\gamma > 0$ such that $\nu - \beta e \in \alpha U'$ and $\nu \in \gamma U'$. Since order intervals are τ-bounded, there exists $\delta > 0$ such that $-r[e, e] \subset \delta U'$. Let $\varepsilon' > 0$ strictly smaller than 1, ε, $\frac{1}{\alpha}$, $\frac{1}{\gamma}$, $\frac{1}{\eta}$ and $\frac{1}{\delta}$. Hence, by invoking (7) above, we deduce that for all $z' \in \Big(\{z\} + \big(\mathcal{C} \cap U'\big)^{I \cup J}\Big)$ for all $y'_j \in \big(\{y_j\} + \mathcal{C} \cap U'\big) \cap Y_j(z')$ and for all $t \in (0, \varepsilon')$, there exists $\xi' \in r[-e,\ e]$ such that $y'_j + t(\nu + \eta(y_j - y'_j) + \xi') \in Y_j(z')$.

We remark that $y'_j \in \{y_j\} + U'$ and from the choice of ε', $t\nu$, $t\eta(y_j - y'_j)$ and $t\xi'$ belong to U'. So, $y'_j + t(\nu + \eta(y_j - y'_j) + \xi') \in \{y_j\} + U' + U' + U' + U' \subset \{y_j\} + V_j$. Then, $y'_j + t(\nu + \eta(y_j - y'_j) + \xi') + t(-\beta e + w) \in \big((\{y_j\} + V_j) \cap Y_j(z')\big) - \Gamma_j$.

Again, from the choice of ε', $t(\nu - \beta e)$ and tw belong to U'. So, $y'_j + t(\nu + \eta(y_j - y'_j) + \xi') + t(-\beta e + w) = y'_j + t(\nu - \beta e + w + \eta(y_j - y'_j) + \xi')$ belongs to $\{y_j\} + U' + U' + U' + U' + U' \subset \{y_j\} + V_j$.

So, if $y'_j \in L(e)$

$$y'_j + t(\nu + \eta(y_j - y'_j) + \xi') + t(-\beta e + w) \in \Big[\big((\{y_j\} + V_j\big) \cap Y_j(z')\big) - \Gamma_j\Big] \cap \big(\{y_j\} + V_j\big) \cap L(e)$$

Hence, by σ-locally τ-uniformly properness relative to $L(e)$ at y_j, we conclude that

$$y'_j + t(\nu - \beta e + w + \eta(y_j - y'_j) + \xi') \in Y^e_j(z')$$

which means that $\nu - \beta e + w \in \hat{\mathcal{T}}^{\rho}_{Y^e_j(z)}(y_j)$. Since $\hat{\mathcal{T}}_{Y^e_j(z)}(y_j) = \bigcap_{\rho > 0} \hat{\mathcal{T}}^{\rho}_{Y^e_j(z)}(y_j)$, we get that $\nu - \beta e + w \in \hat{\mathcal{T}}_{Y^e_j(z)}(y_j)$. □

Proposition 5.4

The results follow since if we consider the norm-topology on L, the collection $\mathcal{V}_\tau(0)$ is the collection of open ball of center 0 and radius $\rho > 0$. Hence, the normal cone defined in Bonnisseau and Fuentes (2020) coincides with $\big[\mathscr{T}_{Y_j(z)}(y_j)\big]^\circ$. From Lemma 5.3, the equilibrium given by Theorem 5 is then a marginal pricing equilibrium in the sense of Bonnisseau and Fuentes (2020). The proof of the remaining items is straightforward. □

Lemma 5.5

It suffices to show that $Y_j(z) \subset \tau - \mathrm{cl}\big[Y_j(z) \cap L(e) - \Gamma_j\big]$ since $\tau - \mathrm{cl}\,(L(e) - L_+) = L$. Let $y_j \in Y_j(z)$ and let V_j the 0-neighbourhood given by the definition of σ-locally τ-uniformly-properness relative to $L(e)$ at $y_j \in Y_j(z)$. Let W_j be a neighbourhood in $\mathcal{V}_\tau(0)$. Let W'_j be a neighbourhood in $\mathcal{V}_\tau(0)$ such that $W'_j + W'_j \subset W_j \cap V_j$ and $\{-e\} + W'_j \subset -\Gamma_j$. Since W'_j is absorbing and circled, there exists $0 < \delta < 1$ such that $-\delta e$ belongs to W'_j. Since $L(e)$ is τ-dense in L there exists a vector u in W'_j such that $y_j - \delta e + \delta u$ belongs to $L(e)$. Since $-\delta e + \delta u$ belongs to $-\Gamma_j$, we conclude that $y_j - \delta e + \delta u \in \big[\{y_j\} - \Gamma_j\big] \cap L(e) \cap \Big(\{y_j\} + W'_j + W'_j\Big) \subset \Big[\big((\{y_j\} + V_j) \cap Y_j(z)\big) - \Gamma_j\Big] \cap L(e) \cap (\{y_j\} - V_j)$. Hence, by Assumption P(5) on properness with $z' = z$, it follows that $y_j - \delta e + \delta u \in Y_j(z)$. Since $W'_j + W'_j \subset W_j$, we have $y_j - \delta e + \delta u \in \{y_j\} + W_j$, hence $\big(\{y_j\} + W_j\big) \cap Y_j(z) \cap L(e) \neq \emptyset$. Since this is true for any neighbourhood W_j in $\mathcal{V}_\tau(0)$, $y_j \in \tau - \text{cl}\left[Y_j(z) \cap L(e)\right] \subset \tau - \text{cl}\left[Y_j(z) \cap L(e) - \Gamma_j\right]$. □

Proposition 5.6

We show first that $\{\nu \in L : \nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0\} \subset \mathscr{T}_{Y_j(z)}(\zeta_j)$. By SB (2) and (3), $\nabla_1 f_j(\zeta_j, z) \in L_+^* \backslash \{0\}$ and $\nabla_1 f_j(\zeta_j, z) \cdot e > 0$ respectively. Let $\nu \in L$ such that $\nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0$, $\mathcal{C} \in \mathcal{V}_\tau(0)$, $\eta = 1$ and $r > 0$. Since H is equicontinuous, there exists $U \in \mathcal{V}_\sigma(0)$ such that $\bigcup_{g \in H} g(U) \subset (-1, 1)$ (Schaefer 1971, Theorem 4.1 (c), p. 83). Since U is absorbing, there exists $\alpha > 0$ such that $\nu - \frac{r}{2}e \in \alpha U$. Let $\beta > 0$ such that $\beta < \frac{r \nabla_1 f_j(\zeta_j, z) \cdot e}{4(\alpha+1)}$. By SB (5), there exists $U' \in \mathcal{V}_\sigma(0)$ such that for all $(\zeta_j', z') \in \left(\{\zeta_j\} + U'\right) \times \left(\{z\} + (U')^{I \cup J}\right)$, $\nabla_1 f_j(\zeta_j', z') - \nabla_1 f_j(\zeta_j, z) \in \beta H$.

Let the σ-neighbourhood of 0 defined by

$$U'' = \left\{\zeta_j' \in L \mid \nabla_1 f_j(\zeta_j, z) \cdot \zeta_j' < \frac{r \nabla_1 f_j(\zeta_j, z) \cdot e}{4}\right\}$$

We claim that there exists $U''' \in \mathcal{V}_\sigma(0)$ such that $U''' + U''' \subset U'$ and $U''' \subset U \cap U''$. Indeed, let $U_0 \in \mathcal{V}_\sigma(0)$ such that $U_0 \subset U \cap U' \cap U''$. Since U_0 is convex, $U_0 = \frac{1}{2}U_0 + \frac{1}{2}U_0$. Then it suffices to take $U''' = \frac{1}{2}U_0$.

Since U''' is absorbing, there exists $\gamma > 0$ such that $\nu - \frac{r}{2}e \in \gamma U'''$. Let $\varepsilon > 0$ such that $\varepsilon < \frac{1}{\gamma+1}$. From the Mean Value Theorem for topological vector spaces (Khan 1997, Theorem 1, p. 2) for all $z' \in \left(\{z\} + \left(\mathcal{C} \cap U'''\right)^{I \cup J}\right)$, for all $\zeta_j' \in \left(\{\zeta_j\} + \mathcal{C} \cap U'''\right) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$, there exists $\theta \in (0, 1)$ such that $\zeta_j'' = \zeta_j' + \theta t \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right)$ belongs to $\left[\zeta_j', \zeta_j' + t\left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right)\right]$ and satisfies

$$f_j\left(\zeta_j' + t\left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right), z'\right) = f_j(\zeta_j', z') + t \nabla_1 f_j(\zeta_j'', z') \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right)$$

Since $\theta < 1$, from our choice of ε, it follows that $\theta t(\nu - \frac{r}{2}e) \in U'''$ and $\zeta_j' = \theta t(\zeta_j - \zeta_j') = \zeta_j + (1 - \theta t)(\zeta_j' - \zeta_j) \in U'''$. So, $\zeta_j'' \in \{\zeta_j\} + U''' + U''' \subset \{\zeta_j\} + U'$. Hence, $\nabla_1 f_j(\zeta_j'', z') - \nabla_1 f_j(\zeta_j, z) \in \beta H$. We remark that:

$$\nabla_1 f_j(\zeta_j'', z') \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right) = (\nabla_1 f_j(\zeta_j'', z') - \nabla_1 f_j(\zeta_j, z)) \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right) + \nabla_1 f_j(\zeta_j, z) \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right)$$

Since $\zeta_j - \zeta_j' \in U'''$ and $\nu + \zeta_j - \zeta_j' - \frac{r}{2}e \in \alpha U$, $\nu + \zeta_j - \zeta_j' - \frac{r}{2}e \in \alpha U + U''' \subset \alpha U + U = (\alpha + 1)U$ since U is convex.[17] Consequently, since $\nabla_1 f_j(\zeta_j'', z') - \nabla_1 f_j(\zeta_j, z) \in \beta H$, $(\nabla_1 f_j(\zeta_j'', z') - \nabla_1 f_j(\zeta_j, z)) \cdot (\nu + \zeta_j - \zeta_j' - \frac{r}{2}e) < \beta(1 + \alpha) < \frac{r\nabla_1 f_j(\zeta_j, z)\cdot e}{4}$. In addition, since $\zeta_j - \zeta_j' \in U''' \subset U''$ and $\nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0$, $\nabla_1 f_j(\zeta_j, z) \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right) \leq \frac{r\nabla_1 f_j(\zeta_j, z)\cdot e}{4} - \frac{r\nabla_1 f_j(\zeta_j, z)\cdot e}{2} = -\frac{r\nabla_1 f_j(\zeta_j, z)\cdot e}{4}$. Consequently, $\nabla_1 f_j(\zeta_j'', z') \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right) \leq 0$. Since $f(\zeta_j', z') \leq 0$, we get $f\left(\zeta_j' + t\left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right), z'\right) \leq 0$, that is $\zeta_j' + t\left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right) \in Y_j(z')$. Hence, $\nu \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(\zeta_j)$. Since this is true for all $\mathcal{C} \in \mathcal{V}_\tau(0)$, we have that $\nu \in \mathscr{T}_{Y_j(z)}(\zeta_j)$. Thus, $\{\nu \in L(e) : \nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0\} \subset \hat{\mathscr{T}}_{Y_j(z)}(\zeta_j)$ whence, $\hat{\mathscr{N}}_{Y_j(z)}(\zeta_j) \subset \{\nu \in L(e) : \nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0\}^\circ$. Since $\nabla_1 f_j(\zeta_j, z) \cdot e < 0$, $\{\nu \in L(e) : \nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0\}^\circ = \{\nu \in \tau - \mathrm{cl}L(e) : \nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0\}^\circ$. □

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[17] Let u and u' be two vectors in U. Let $u'' = \frac{\alpha}{1+\alpha}u + \frac{1}{1+\alpha}u'$. It is clear that u'' belongs to U since it is convex. Hence, $\alpha U + U = (\alpha + 1)U$.

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